ESSAYS ON MATCHING MARKETS

BY

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DISSERTATION

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ABSTRACT

Matching markets are common methods to allocate resources around the world. There are two kinds of matching market: centralized matching market and decentralized matching market. In a centralized matching market, there is a clearing house that functions to collect information from market participants and uses the information to determine the allocation among the participants. In a decentralized matching market, market participants contact one another and possibly exchange information, and the allocation is determined among the participants based on their agreement.

In my thesis, I study the matching markets that are *two-sided*. In a two-sided matching market, participants are separated into two groups. For example, men and women, schools and students, and so on. Such a market is first studied by Gale and Shapley (1962). They study matching markets between men and women and between colleges and students. Consider the simples matching problem: the marriage problem, the problem of matching between men and women. Each of them has a preference over the agents on the opposite side and the option of remaining single. A central question is: does a matching that is individual rational and no pair of agents from different sides who are not matched together but rather be together exist? (Such a matching is referred as a *stable* matching.) Gale and Shapley (1962) show that the answer is affirmative with the introduction of the *deferred acceptance algorithm*. Since Gale and Shapley (1962), the concept of stability and the application of deferred acceptance algorithm are widely used in the literature of two-sided matching.

My thesis consists of four individual papers. Chapter 1, with the title "When is the Boston Mechanism Dominance-Solvable?", Chapter 2, with the title "Undominated Strategies and the Boston Mechanism", and Chapter 3, with the title "Promoting Diversity of Talents: A Market Design Approach", are my own work. In these three chapters, I study the matching problems between schools and students. Chapter 4, with the title "Platform Markets

and Matching with Contracts", is a joint work of Juan Fung and I. In this chapter, we propose a model to study platform markets.

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Chapter 1

When is the Boston Mechanism Dominance-Solvable?

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Abstract: This paper considers weaker notions of strategy-proofness and use them to study the Boston mechanism. I propose a new solution concept, *dominance solvability**, which is stronger than dominance solvability while weaker than strategy-proofness. First, I show that the Boston mechanism satisfies the *nonbossy* condition and this makes the order of deleting weakly dominated strategies irrelevant to the outcomes produced from the survived strategy profiles. Then, I design the *acyclic priority structure* and show that the Boston mechanism is dominance-solvable* or dominance-solvable if and only if the priority structure is acyclic. Despite the dominance-solvable* or dominance-solvable outcome under the Boston mechanism is stable and efficient, I find the acyclic priority structure is very restrictive, which does

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not bring justification to the use of the Boston mechanism.

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Key Words: Dominance solvability*, dominance solvability, dominant strategy, acyclic priority structure, nonbossy condition, Boston mechanism

1.1 Introduction

Centralized matching programs are commonly used for allocating students to schools. The *school choice problem* by Abdulkadiroğlu and Sönmez (2003) and Blinski and Sönmez (1999) studies such matching problems. In this literature, the centralized matching mechanisms use students' submitted preferences and their priorities in schools to compute the outcome. In addition, the schools are viewed as objects to consume and the welfare consideration is on students.

Strategy-proofness has been advocated as a design objective instead of an incentive compatibility constraint in the setting of school choice problem (Abdulkadiroğlu et al., 2006; Pathak and Sönmez, 2013). This consideration plays an important role of replacing the *Boston mechanism* with the *Student-Optimal Stable Mechanism* (SOSM) of Gale and Shapley (1962) in the public school matching program in Boston, since one important feature of the Boston mechanism is that it is not strategy-proof (Abdulkadiroğlu et al., 2006). On the other hand, the SOSM is strategy-proof (Dubins and Freedman, 1981; Roth, 1982).

Pathak and Sönmez (2013) show that the replacement of a manipulable mechanism in Boston is not a single event. They show that there is a trend of replacing manipulable mechanisms in US and UK. Despite the mechanisms that replace the old mechanisms are still manipulable, Pathak and Sönmez (2013) show that the new mechanisms are *less manipulable*.[2]

[2]Note that the reasons that the new mechanisms are not strategy-proof is that there are constraints on the number of schools choices students can declare in their submitted preferences. As long as there is such a constraint, a strategy-proof mechanism does not

Having a strategy-proof mechanism also prevents a situation that some students' welfares are sacrificed due to their unawareness of the strategic properties of the mechanisms. Pathak and Sönmez (2008) consider an environment where there are sincere students and sophisticated students and compare the welfare consequences of the Boston mechanism and the SOSM. Sincere students are not aware of the strategic properties of the Boston mechanism and always submit their true preferences, while sophisticated students are fully aware of those properties and plan their strategies accordingly. Pathak and Sönmez (2008) show that all sophisticated students weakly prefer the Pareto-dominant Nash equilibrium outcome under the Boston mechanism to the dominant-strategy outcome under the SOSM. In addition, Abdulkadiroğlu et al. (2006) find empirical evidence that shows students have different levels of sophistication in the public school matching program in Boston during 2000 and 2004, when the Boston mechanism is still in use.

Despite the importance of strategy-proofness, there are some situations when the market designers might want to consider a weaker notion of strategy-proofness. Kesten (2010) shows that there does not exist an efficient and strategy-proof mechanism that Pareto dominates the dominant-strategy outcome of the SOSM. Abdulkadiroğlu et al. (2009) consider an environment where there are ties in students' priorities in schools and some random tie-breaking rule is used to break the tie. They show for any tie-breaking rule, there does not exist a mechanism that is strategy-proof and *dominates* the SOSM.[3] Both Kesten (2010) and Abdulkadiroğlu et al. (2009) show that it

exist. See also Haeringer and Klijn (2009) for the analysis of the SOSM, the TTCM and the Boston mechanism in such an environment and Calsamglia et al. (2010) for an experimental study.

[3]By Abdulkadiroğlu et al. (2009), a mechanism ν dominates another mechanism ϕ, if when all students submit the true preferences under both ν and ϕ, all students weakly prefer the outcome under ν to the outcome under ϕ, and some students strictly prefer the outcome under ν to the outcome under ϕ.

costs strategy-proofness to have a mechanism that dominates the SOSM.

Another situation that calls for a weaker notion of strategy-proofness is when there is a constraint on the number of schools that a student can declare in his or her submitted preference. In such an environment, a strategy-proof mechanism does not exist. As shown in Pathak and Sönmez (2013), many school districts in US and UK have such a constraint.[4] Although Pathak and Sönmez (2013) propose an important measure to compare the vulnerability to manipulation between mechanisms, they did not provide any design guideline regarding incentives when strategy-proofness is impossible to achieve.

This paper considers weakening the requirement of strategy-proofness. A natural candidate would be *dominance solvability*, since it gives an unique prediction on the strategies used by the participants.[5] Dominance solvbaility is a natural candidate when one relaxes strategy-proofness. In particular, in p. 80 of Moulin (1983), "dominance solvability is a generalization of strategy-proofness."

This paper proposes a new solution concept, the *dominance solvability**. Dominance sovability* requires that in the process of deleting weakly dominated strategies, one can delete strategies only if there exists a dominant strategy in that subgame. In other words, if a game is dominance-solvable*, then there exits an ordering over the players so that each player can find some dominant strategy in the subgame in a sequential manner starting from the player with a highest ordering. Note that dominance solvability* is stronger than dominance solvability, while it is weaker than strategy-proofness.

[4]For the school districts that Pathak and Sönmez (2013) survey, the only school district that uses strategy-proof mechanism and does not have a constraint on the number of school choices that one can declare is Boston. See Table 1 in Pathak and Sönmez (2013).

[5]Dominance solvbaility, introduced by Moulin (1979), requires the following: a game is dominance-solvable if the outcomes produced by the strategy profiles that survive iterative deletion of weakly dominated strategies are the same.

This paper applies the solution concepts, dominance solvability* and dominance solvability, to study the Boston mechanism. The Boston mechanism works in the following way.

- In the first round, each school uses its priority rule and admits the students who rank it as their first choices up to its capacity constraint. A student who is assigned to some school is inactive from the next round on; otherwise he is active. Each school reduces its capacity constraint by the number of students who are assigned to it after this round.

- In the t-th round, each school uses its priority rule and admits the active students who rank it as the t-th choices up to its capacity constraint. A student who is assigned to some school is inactive from the next round on; otherwise he is active. Each school reduces its capacity constraint by the number of students who are assigned to it after this round.

The mechanism terminates when each student is either matched to some school or all his acceptable schools[6] reach their maximum capacity constraints.

Ergin and Sönmez (2006) analyze the equilibrium properties of the preference revelation game induced by the Boston mechanism.[7] In particular, Ergin and Sönmez (2006) show that all Nash equilibrium outcomes in the Boston mechanism are stable matchings. Since the dominant strategy outcome of the SOSM Pareto dominates all other stable matchings, it Pareto dominates all (except one) Nash equilibrium outcomes in the Boston mechanism. Moreover, despite the Boston mechanism is efficient, efficiency is not

[6]A school is an acceptable school for a student, if attending that school is preferred to the outside option.

[7]In the rest of the paper, I will simply refer the preference revelation game induced by the Boston mechanism as the Boston mechanism.

guaranteed in equilibrium in general.[8]

This paper observes that the multiplicity of Nash equilibrium outcomes in the Boston mechanism in many cases results from using weakly dominated strategies. In particular, if one performs iterative deletion of weakly dominated strategies, the outcome resulted from the survived strategy profiles may be a desirable one. We use Example 2 in Ergin and Sönmez (2006) to illustrate this point.

Example 1. There are three students i_1, i_2, i_3 and three schools s_1, s_2, s_3. Each school can admit at most one student. Students' preferences and their priorities in schools are as follows.[9]

$$\begin{array}{ll} P_{i_1} : s_1, s_2, s_3 & \quad f_{s_1} : i_3 - i_2 - i_1 \\ P_{i_2} : s_2, s_1, s_3 & \quad f_{s_2} : i_1 - i_3 - i_2 \\ P_{i_3} : s_2, s_3, s_1 & \quad f_{s_3} : i_2 - i_1 - i_3 \end{array}$$

The setting of the school choice problem is the same as in Example 2 in Ergin and Sönmez (2006). As shown in Ergin and Sönmez (2006), there are two Nash equilibrium outcomes:[10]

$$\mu_1 = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}; \qquad \mu_2 = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_3 & s_1 \end{pmatrix}.$$

Note that μ_1 Pareto dominates μ_2, and μ_1 is the dominant strategy outcome in the SOSM.

We show that the unique dominance-solvable outcome is μ_1. For each student, the submitted preference that does not rank the least preferred

[8] Note the student-optimal stable matching Pareto dominates other stable matchings, and the student-optimal stable matching is not efficient in general. Therefore, efficiency in equilibrium in the Boston mechanism is not guaranteed.

[9] It means that student i_1 prefers school s_1 to school s_2, school s_2 to school s_3, and school s_3 to not being admitted to any schools. In addition, student i_1 has the highest priority in school s_1, student i_2 has the second highest priority in school s_1, and student i_1 has the third highest priority in school s_1.

[10] It means that student i_1 is matched to school s_2 in the matching μ_1.

school as the third school choice is weakly dominated. This is because the sum of capacity constraints over all schools equals the total number of students. Ranking all three schools as acceptable schools guarantees that he will receive at least one school. Ranking the least preferred school as the third school choice guarantees that he will receive a school that is at least as good as the least preferred school. A strategy that ranks the least preferred school as the first or the second school choice is weakly dominated by a strategy that ranks the least preferred school as the third school choice. After deleting all the strategies that do not rank the least preferred school as the third school choice, we have the resulting payoff matrix in Figure 2. For the ease of comparison, let a student obtain utility of 2 when he receives his favorite school and utility of 1 with the second favorite school, and utility of 0 with the third favorite school. Note that student i_1 is the row player, student i_2 is the column player, and student i_3 is the box player.

	s_2, s_1, s_3	s_1, s_2, s_3
s_1, s_2, s_3	$0, 1, 2$	$2, 0, 2$
s_2, s_1, s_3	$1, 1, 1$	$1, 1, 1$

s_2, s_3, s_1

	s_2, s_1, s_3	s_1, s_2, s_3
s_1, s_2, s_3	$1, 1, 1$	$2, 2, 1$
s_2, s_1, s_3	$1, 1, 1$	$1, 1, 1$

s_3, s_2, s_1

Figure 1.1: *The payoff matrix after deleting the strategies that change the ranking of the least preferred schools for all students.*

From Figure 1.1, if one performs iterative deletion of weakly dominated strategies for the remaining strategies, there is an unique outcome, which is μ_1. Therefore, the game is dominance-solvable. Note that μ_1 is the matching produced by the SOSM when all students submit the true preferences as dominant strategies. Also note that one can find in the $6 \times 6 \times 6$ payoff matrix presented in Ergin and Sömnez (2006, Fig. 2) that there are more than one path of deleting weakly dominated strategies, and that all result in the same outcome.

The first result in this paper is that the Boston mechanism satisfies the *nonbossy* condition.[11] A mechanism is nonbossy if no student can affect other students' assignments without changing his own assignment by submit a different preference. This condition prevents a version of manipulation.

This paper shows the nonbossy condition of the Boston mechanism creates a useful property in terms of investigating equilibrium strategies. It is known that different orders of deleting weakly dominated strategies could create different set of outcomes produced by the survived strategy profiles. Marx and Swinkels (1997) propose the *Transference Decisionmaker Indifference* (TDI) condition and show that if a game satisfies the TDI condition, the order of deleting weakly dominated strategies does not affect the set of outcomes produced by the survived strategy profiles.[12] It turns out the TDI condition is equivalent to the nonbossy condition in the setting of school choice problem.[13] Therefore, the order of deleting weakly dominated strategies in the Boston mechanism does not matter.

It might be tempting to conclude that the Boston mechanism is dominance-solvable. However, it is not true in general. In the following example, the Boston mechanism is not dominance-solvable.

Example 2. There are three students i_1, i_2, i_3 and three schools s_1, s_2, s_3. Each school can admit at most one student. The preference profile and priority structure are as follows:

$$\begin{array}{ll} P_{i_1} : s_2, s_3, s_1 & f_{s_1} : i_1 - i_3 - i_2 \\ P_{i_2} : s_1, s_2, s_3 & f_{s_2} : i_2 - i_1 - i_3 \\ P_{i_3} : s_3, s_1, s_2 & f_{s_3} : i_1 - i_3 - i_2 \end{array}$$

[11] The nonbossy condition is proposed by Satterthwaite and Sonnenschein (1981).
[12] This result holds when no further weakly dominated strategies can be deleted.
[13] See also footnote 5 in Marx and Swinkels (1997).

Using a similar reasoning as deleting the weakly dominated strategies in Example 1, the remaining strategies and the associated payoff matrix are in Figure 1.2. Again, a student obtains utility of 2, 1 and 0, if he receives his first, second and third favorite schools, respectively. Student i_1 is the row players; student i_2 is the column player; student i_3 is the box player. It is clear that there is no other weakly dominated strategies that can be deleted. Therefore, the game is not dominance-solvable.

	s_1, s_2, s_3	s_2, s_1, s_3
s_2, s_3, s_1	$2, 2, 2$	$0, 1, 2$
s_3, s_2, s_1	$1, 2, 0$	$1, 1, 1$

s_3, s_1, s_2

	s_1, s_2, s_3	s_2, s_1, s_3
s_2, s_3, s_1	$2, 0, 1$	$1, 1, 1$
s_3, s_2, s_1	$1, 1, 1$	$1, 1, 1$

s_1, s_3, s_2

Figure 1.2: *The payoff matrix after deleting the strategies that change the ranking of the least preferred schools for all students.*

This paper studies the environment in which the Boston mechanism is dominance-solvable. In particular, I characterize the priority structure so that the Boston mechanism is dominance-solvable if and only if the characterization holds. This approach is along the line with the approach in the literature of designing priority structure starting from Ergin (2002). In this literature, the focus is designing the priority structure so that the deficiencies of the mechanism under study can be circumvented.

A related paper in this literature is Kumano (2013). Kumano (2013) shows that the Boston mechanism is stable or strategy-proof if and only if the priority structure is *Kumano-acyclic*. Since dominance solvability is a generalization of strategy-proofness, a result in this paper is a generalization of a result in Kumano (2013). Another related paper is Haeringer and Klijn (2009). Haeringer and Klijn (2009) show that all Nash equilibrium outcomes in the Boston mechanism are efficient if and only if the priority structure is *strongly X-acyclic*. I show that there is no logic relationship between the strongly X-acyclic priority structure and the priority structure characterized in this paper.[14]

This paper uses dominance solvability* to establish the main results. The following example shows how dominance solvability* can be applied to the Boston mechanism.

Example 3. There are three students i_1, i_2, i_3 and three schools s_1, s_2, s_3. Each school can admit at most one student. The students' preferences and priorities in schools are as follows.

$$\begin{array}{ll} P_{i_1} : s_1, s_2, s_3 & \quad f_{s_1} : i_1 - i_2 - i_3 \\ P_{i_2} : s_2, s_3, s_1 & \quad f_{s_2} : i_1 - i_2 - i_3 \\ P_{i_3} : s_1, s_2, s_3 & \quad f_{s_3} : i_2 - i_3 - i_1 \end{array}$$

In the beginning of the game, only student i_1 has dominant strategies, which are the strategies that rank school s_1 as the first school choice. After student i_1 removes her weakly dominated strategies, student i_2 finds dominant strategies in the subgame, which are the strategies that rank school s_2 as the first school choice. Note that student i_3 still cannot find dominant strategies in this subgame. After student i_2 removes her weakly dominated strategies, student i_3 finds dominant strategies, which are the strategies that rank s_3 as the first school choice.

I characterize the *acyclic priority structure* and show that the Boston mechanism is dominance-solvable* or dominance-solvable, if and only if the priority structure is acyclic. To do so, I characterize an order of finding dominant strategies and then characterize the priority structure so that it is guaranteed that each student can find his dominant strategy in some stage. In other words, when the characterization holds, the Boston mechanism is dominance-solvable*. On the other hand, I show that when this characterization does not hold, the Boston mechanism is not dominance-solvable. In addition, I also consider an environment where the recourses are scarce

[14]See Figure 3 for this comparison.

and characterize the priority structure so that the Boston mechanism is dominance-solvable* or dominance-solvable if and only if the characterization holds.

On the policy side, I find that despite the acyclic priority structure is weaker than the Kumano-acyclic priority structure, it is still restrictive. In particular, the acyclic priority structure is more restrictive than the Ergin-priority structure. Therefore, the findings do not support the use of the Boston mechanism.

1.1.1 Related Literature

Literature on Designing Priority Structure

Ergin (2002) shows that the SOSM is Pareto efficient, or group strategy-proof, or consistent if and only if the priority structure is *Ergin-acyclic*. Kesten (2006) shows that the TTCM and the SOSM are equivalent, or the TTCM is stable, or *resource monotonic*, or *population monotonic* if and only if the priority structure is *Kesten-acyclic*. Haeringer and Klijn (2009) study an environment where there is a maximum number of school choices, the *quota*, that students can declare in their submitted preferences. They show that for any quota, all the Nash equilibrium outcomes under the TTCM are Pareto efficient if and only if the priority structure is *X-acyclic*. Moreover, they show that for any quota, the priority structure is *strongly X-acyclic* if and only if the set of stable matchings is a singleton, or the Nash equilibrium outcomes under the Boston mechanism are Pareto efficient, or the Nash equilibrium outcomes under the SOSM are Pareto efficient. Ehlers and Erdil (2010) consider an environment where there are ties in students' priorities and show that the SOSM is efficient or consistent if and only if the priority structure is *strongly acyclic*. Kojima (2011) defines robust stability and shows that the SOSM is robust stable if and only if the priority structure is Ergin-acyclic. Hatfield et al. (2011) show the equivalence between existence of a stable or Pareto efficient mechanism for students that respects improvements of school qualities and the *virtually homogeneous* school preference profile when there is at least one school that has a capacity constraint that is larger than one. Kesten (2012) shows that the SOSM is immune to manipulation via capacities if and only if the priority structures is Ergin-acyclic.[15] Kojima (2013) studies the environment where agents have multi-unit demands and shows the equivalence of the existence of stable and efficient mechanism, the existence of stable and strategy-proof mechanism, and the priority structure being *essentially homogenous*.

Literature on Nonbossy Condition

The nonbossy condition is proposed by Satterthwaite and Sonnenschein (1981). The study of the nonbossy condition in the literature of resource allocation problem is the following. Svensson (1999) shows that an assignment rule is strategy-proof, nonbossy and neutral if and only if it is a serial dictatorship. Pàpai (2000) proposes hierarchical exchange rules and shows that an assignment rule is group strategy-proof, Pareto efficient, and reallocation-proof if and only if it is a hierarchical exchange rule. In addition, Pàpai (2000) shows that an assignment rule is group strategy-proof if and only if it is nonbossy and strategy-proof. Pycia and Ünver (2014) propose the Trading Cycles (TC) mechanism and show that a mechanism is group strategy-proof and Pareto efficient if and only if it is a TC mechanism.

The study of nonbossy condition in the literature of two-sided matching is the following. Kojima (2010) shows that there does not exist a matching mechanism that is both stable and nonbossy. In school choice problem, Ergin (2002) shows that if the SOSM is consistent, then it is nonbossy.[16] Kumano and Watabe (2012) show that the SOSM satisfies a weaker version of nonbossy condition, the *weak nonbossiness*. Haeringer and Klijn (2008) point out that by Pàpai (2000) the TTCM is nonbossy.[17]

[15]Manipulation via capacities is proposed by Sönmez (1997).

[16]By Ergin (2002), when the priority structure is Ergin-acyclic, the SOSM is nonbossy.

[17]See footnote 27 in Haeringer and Klijn (2008).

The rest of the paper is organized in the following way. Section 1.2 presents the model. Section 1.3 presents the results. Section 1.4 contains discussions. All proofs are presented in Section 1.5. Section 1.6 contains example that show the relationships among different characterizations of the priority structure.

1.2 Model

A **school choice problem** is a tuple (I, S, q, P, f) where

- $I = \{i_1, ..., i_n\}$ is the set of students.
- $S = \{s_1, ..., s_m\}$ is the set of schools.
- $q = (q_{s_1}, ..., q_{s_m})$ is the vector of capacity constraints for each school.
- $P = (P_{i_1}, ..., P_{i_n})$ is the preference profile, where P_i is student i's preference ranking over $S \cup \{\varnothing\}$, and $\varnothing$ is the outside option. We will use R_i as the weak preference relation associated with P_i, i.e., sR_is' if and only if sP_is' or $s = s'$. A school s is **acceptable** for student i if and only if $sP_i\varnothing$. For student i, let $P_i(s)$ denote the ranking of school s in P_i.
- $f = (f_{s_1}, ..., f_{s_m})$ is the priority structure, where $f_s : I \to \{1, ..., |I|\}$ is school s's ranking over I. For any two students $i, j \in I$, $f_s(i) < f_s(j)$ means student i has a higher priority than than j in school s.

We assume that each student has a strict preference over $S \cup \{\varnothing\}$ and each school's priority over I is also strict. We will treat S, I, q, f as fixed, so we will call P as a school choice problem as long as it is clear.

A **matching** $\mu : I \to S \cup \{\varnothing\}$ is a function that assigns a student to a school or the outside option such that each school is not assigned more students than its capacity constraint.

A matching μ **Pareto dominates** another matching ν if $\mu(i)R_i\nu(i)$ for all $i \in I$ and $\mu(i)P_i\nu(i)$ for some $i \in I$. A matching μ is **Pareto efficient** if it is not Pareto dominated by another matching.

Definition 1. A matching μ is **stable** if it has the following three properties.[18]

- Individual rationality: $\mu(i)R_i\varnothing$ for each student $i \in I$.
- No blocking: there does not exist a student i and a school s such that sP_ii and $f_s(i) < f_s(j)$ for some student $j \in \mu^{-1}(s)$.
- Non-wastefulness: $sP_i\mu(i)$ implies that $|\mu^{-1}(s)| = q_s$ for each student $i \in I$.

The **student-optimal stable matching** is a stable matching that Pareto dominates all other stable matchings.

Given I, S, and q, let $\mathcal{P}$ be the set of all possible preference profiles and $\mathcal{M}$ be the set of all possible matchings. A **mechanism** $\gamma : \mathcal{P} \to \mathcal{M}$ is a function that specifies a matching for each submitted preference profile. For student i, let $\gamma_i : \mathcal{P} \to S \cup \{\varnothing\}$ be the function that indicates the school or the outside option that student i receives for each submitted preference profile. In this paper, we will let φ denote the Boston mechanism.

For each student, the set of strategies is equal to the set of possible submitted preferences. We will use Q_i as a generic strategy for student i and Q_{-i} as the generic strategy profile for students other than i. Denote $Q = (Q_i, Q_{-i})$ as a generic strategy profile. Let $\mathcal{Q}_i$ be the set of all possible strategies for student i and let $\mathcal{Q} = \bigcup_{i \in I} \mathcal{Q}_i$. Let the game induced by mechanism γ with a school choice problem (I, S, q, P, f) be $(\mathcal{Q}, P, \gamma)$.

Next, we define several solution concepts. Given a game $(\mathcal{Q}, P, \gamma)$, a strategy profile $Q = (Q_i, Q_{-i})$ is a **Nash equilibrium** if

$$\gamma_i(Q_i, Q_{-i})R_i\gamma_i(Q'_i, Q_{-i})$$

[18] This definition is given by Blinski and Sönmez (1999).

for all $Q'_i \in \mathcal{Q}_i$ and for all $i \in I$.

We need some more notation for the analysis of how the set of undeleted strategies evolves over iterations of deleting weakly dominated strategies. Let $\mathcal{W}$ be a subset of $\mathcal{Q}$. We say that $\mathcal{W}$ is a **restriction of** $\mathcal{Q}$ if the set of each student's available strategies in $\mathcal{W}$ is not empty, i.e., $\mathcal{W} \bigcap \mathcal{Q}_i \neq \emptyset$ for all $i \in I$.[19]

Consider a mechanism γ. Given a restriction $\mathcal{W}$, we say that Q_i **weakly dominates** Q'_i **on** $\mathcal{W}$ if

$$\gamma_i(Q_i, Q_{-i}) R_i \gamma_i(Q'_i, Q_{-i})$$

for all $Q_{-i} \in \mathcal{W}_{-i}$ and

$$\gamma_i(Q_i, Q_{-i}) P_i \gamma_i(Q'_i, Q_{-i})$$

for some $Q_{-i} \in \mathcal{W}_{-i}$. We say that Q_i is a **dominant strategy on** $\mathcal{Q}$ if

$$\gamma_i(Q_i, Q_{-i}) R_i \gamma_i(Q'_i, Q_{-i})$$

for all $Q'_i \in \mathcal{W}_i$ and for all $Q_{-i} \in \mathcal{W}_{-i}$.

Given a restriction $\mathcal{W}$, define $D_\gamma(\mathcal{W})$ to be the subset of $\mathcal{W}$ obtained by removing strategies from some student that are weakly dominated on $\mathcal{W}$, when mechanism γ is used.

Definition 2. The game $(\mathcal{Q}, P, \gamma)$ is **dominance-solvable**, if there exists a sequence $\mathcal{Q}^1, \mathcal{Q}^2, ..., \mathcal{Q}^k$, where $\mathcal{Q}^1 = \mathcal{Q}$ and $\mathcal{Q}^t = D_\gamma(\mathcal{Q}^{t-1})$, such that for any two strategy profiles $Q, Q' \in \mathcal{Q}^k$ we have $\gamma(Q) = \gamma(Q')$.

Note that by Moulin (1979), if a game is dominance-solvable, the survived strategy profile is also a Nash equilibrium.

1.3 Results

The results in this section are primarily in three groups. Section 1.3.1 and Section 1.3.2 establish the result that the order of deleting weakly dominated

[19] I will use "a restriction" instead of "a restriction of $\mathcal{Q}$" as long as it is clear.

strategies does not affect the outcomes produced by the survived strategy profiles. Section 1.3.3 and Section 1.3.4 characterize a particular order of deleting weakly dominated strategies. Section 1.3.5 and Section 1.3.6 characterize and study the priority structure such that it is dominance-solvable* or dominance-solvable if and only if the characterization holds.

1.3.1 Nonbossiness

A mechanism γ is **nonbossy**, if $\forall i \in I$, $\forall Q_i, \forall Q'_i \in \mathcal{Q}_i$ and $\forall Q_{-i} \in \mathcal{Q}_{-i}$, we have

$$\gamma_i(Q_i, Q_{-i}) = \gamma_i(Q'_i, Q_{-i}) \Rightarrow \gamma(Q_i, Q_{-i}) = \gamma(Q'_i, Q_{-i}).$$

A mechanism is nonbossy, if no student can change the assignments to other students without affecting his own assignment by submitting a different preference.

Theorem 1. *The Boston mechanism is nonbossy.*

The nonbossy condition prevents a version of manipulation. With the nonbossiness of the Boston mechanism, we have the following result.

Theorem 2. *The Boston mechanism is group strategy-proof if and only if it is strategy-proof.*

The proof of Theorem 2 is identical to Lemma 1 in Pàpai (2000) and is thus omitted. Theorem 2 generalizes part of results in Kumano (2013) regarding strategy-proofness and group strategy-proofness.[20]

[20] Kumano (2013) shows the equivalence of Kumano-acyclic priority structure, strategy-proofness, and group strategy-proofness in the sense that under the Kumano-acyclic priority structure, the Boston mechanism is strategy-proof or group strategy-proof for all possible preferences, and when the Kumano-acyclic structure is not satisfied, there exists a preference profile such that the Boston mechanism is not strategy-proof or group strategy-proof. (See Theorem 1 and Corollary 1 in Kumano (2013).) Theorem 2 says that for any school choice problem, if the Boston mechanism is strategy-proof, then it is group strategy-proof, and vice versa. (Therefore, it is possible to have a school choice problem

1.3.2 Order Independence

Marx and Swinkels (1997) show that if a game satisfies the TDI condition, or the nonbossy condition in our setting, the order of deleting weakly dominated strategies does not affect the outcomes produced by the survived strategy profiles. Two definitions are introduced before we present a main result in Marx and Swinkels (1997).

Definition 3 (Definition 3 of Marx and Swinkels (1997))**.** Let $\mathcal{Q}'$ be a restriction of $\mathcal{Q}$ and let $\mathcal{Q}''$ be a restriction of $\mathcal{Q}'$.

- $\mathcal{Q}''$ is a **reduction of $\mathcal{Q}'$ by weak dominance** if $\mathcal{Q}'' = \mathcal{Q}' \setminus \mathcal{Q}^1, ..., \mathcal{Q}^m$, where $\forall k \in \{1, ..., m\}$, $\mathcal{Q}^k \subset \mathcal{Q}$, and $\mathcal{Q}' \setminus \mathcal{Q}^1, ..., \mathcal{Q}^k = D_\gamma(\mathcal{Q}' \setminus \mathcal{Q}^1, ..., \mathcal{Q}^{k-1})$, where $\mathcal{Q}^0 = \emptyset$.
- $\mathcal{Q}''$ is a **full reduction of $\mathcal{Q}'$ by weak dominance** if $\mathcal{Q}''$ is a reduction of $\mathcal{Q}'$ by weak dominance and there does not exist a student who has strategies in $\mathcal{Q}''$ that are weakly dominated on $\mathcal{Q}''$.

Definition 4 (Definition 5 of Marx and Swinkels (1997))**.** Let $\mathcal{Q}'$ be a restriction of $\mathcal{Q}$, and let $Q_i, Q'_i \in \mathcal{Q}'_i$ be two strategies of student i.

- Q_i **is redundant to Q'_i on $\mathcal{Q}'$** if $\varphi_i(Q_i, Q_{-i}) = \varphi_i(Q'_i, Q_{-i}), \forall Q_{-i} \in \mathcal{Q}'_{-i}$.
- Q_i **is redundant on $\mathcal{Q}'$** if there is a strategy $Q'_i \in \mathcal{Q}'_i \setminus \{Q_i\}$ such that Q_i is redundant to Q'_i on $\mathcal{Q}'$.

One of the main theorems in Marx and Swinkels (1997) is rephrased as follows.

Theorem 3 (Corollary 1 of Marx and Swinkels (1997))**.** *Let $(\mathcal{Q}, P, \gamma)$ satisfy the nonbossy condition, and let $\mathcal{Q}'$ and $\mathcal{Q}''$ be full reduction of $\mathcal{Q}$ by weak*

(I, S, q, P, f) where f is not Kumano-acyclic, and the Boston mechanism is strategy-proof. In such a case, there exists P', such that the Boston mechanism is not strategy-proof in (I, S, q, P', f).)

dominance. Then, $\mathcal{Q}'$ and $\mathcal{Q}''$ are the same up to the addition or removal of redundant strategies and a renaming of strategies.

The theorem says that if the game satisfies the TDI condition (or the nonbossy condition), the sets of outcomes produced by two different paths of iteratively deleting weakly dominated strategies are the same. In particular, if players obtain two different sets of surviving strategies as a result of two different paths of iteratively deleting weakly dominated strategies, then after substituting strategies with redundant strategies or naming strategies in one set of surviving strategies after the names of the strategies in the other set (because they produce the same outcome), these two sets of strategies are equivalent.

By Theorem 1, the Boston mechanism is nonbossy. With Theorem 3, we have the following corollaries.

Corollary 1. *The order of deleting weakly dominated strategies in the Boston mechanism does not affect the outcomes produced by the survived strategy profiles.*

Corollary 2. *When the Boson mechanism is dominance-solvable, regardless of the order of deleting weakly dominated strategies, the outcomes produced by the survived strategy profiles are the same.*

Since the order of deletion in the Boston mechanism does not matter, I will focus on one order of deletion introduced in the following sections.

1.3.3 Dominant Strategy

The first step in charactering the order of deleting weakly dominated strategies considered in this paper is to characterize the the dominant strategies. In this section, I characterize the dominant strategy in the Boston mechanism.

Let

$$U(f_s, k) = \{i \in I : |\{j \in I : f_s(j) < f_s(i)\}| < k\}.$$

This is the set of students who are ranked higher than or equal to the k-th position in f_s. The following theorem identifies the three types of dominant strategies in the game $(\mathcal{Q}, P, \varphi)$.

Theorem 4. *A dominant strategy Q_i for a student i in the game $(\mathcal{Q}, P, \varphi)$ takes one of the following three forms.*

1. *The first school choice in Q_i is s and s is the only acceptable school for i according to P_i. Moreover, $i \notin U(f_s, q_s)$.*

2. *The first school choice in Q_i is s and s is the favorite school choice for i according to P_i. Moreover, $i \in U(f_s, q_s)$.*

3. *The first two school choices in Q_i are the same as the first two school choices in P_i. Moreover, $i \notin U(f_s, q_s)$ and $q_s + q_{s'} \geq |I|$, where s and s' are the first and second favorite schools, respectively.*

I will refer these three dominant strategies as type-1, type-2, and type-3 dominant strategies, respectively. When a student has a type-2 dominant strategy, he will receive the first school choice in his dominant strategy for sure. When a student has a type-3 dominant strategy, he will receive one of the first two school choices in his dominant strategy.

Remark 1. The characterization of the dominant strategies requires *no* information about other students' preferences.

Remark 2. Suppose there is a constraint on the number of school choices that a student can declare. As long as the constraint is larger than or equal to two, Theorem 4 holds.

A corollary of Theorem 4 is the following.

Corollary 3. *Consider a school choice problem (I, S, q, P, f). If $q_s + q_{s'} \geq |I|$ for any two schools $s, s' \in S$, then the Boston mechanism is strategy-proof in the school choice problem (I, S, q, P', f) for any $P' \in \mathcal{P}$.*

Corollary 3 can be used to strengthen Proposition 1 in Kumano (2013).

Theorem 5 (Proposition 1 in Kumano (2013))**.** *Consider a school choice problem (I, S, q, P, f). If $|I| \geq 3$ and $|S| \geq 2$ and there are two schools, s and s', such that $q_s + q_{s'} \leq |I| - 1$, then f is not Kumano-acyclic.*

Since Kumano (2013) shows that the Boston mechanism is strategy-proof if and only if the priority structure is Kumano-acyclic, with Corollary 3 and Theorem 5, we have the following corollary.[21]

Corollary 4. *Consider a school choice problem (I, S, q, P, f). Suppose $|I| \geq 3$ and $|S| \geq 2$. Then the priority structure f is Kumano-acyclic if and only if $q_s + q_{s'} \geq |I|$ for any two schools $s, s' \in S$.*

1.3.4 Dominance Solvability*

I propose a new solution concept, dominance solvability*. Roughly speaking, dominance solvability* sequentially identifies dominant strategies for the players. I propose the *iterative deletion of weakly dominated strategies only if there exists some dominant strategy* as follows. Given a restriction $\mathcal{W}$, define $D^*_\gamma(\mathcal{W})$ to be the subset of $\mathcal{W}$ obtained by removing weakly dominated strategies on $\mathcal{W}$ by some student who has dominant strategies on $\mathcal{W}$, when mechanism γ is used.

Definition 5. The game $(\mathcal{Q}, P, \gamma)$ is **dominance-solvable***, if there exists a sequence $\mathcal{Q}^1, \mathcal{Q}^2, ..., \mathcal{Q}^k$, where $\mathcal{Q}^1 = \mathcal{Q}$ and $\mathcal{Q}^t = D^*_\gamma(\mathcal{Q}^{t-1})$, such that for any two strategy profiles $Q, Q' \in \mathcal{Q}^k$ we have $\gamma(Q) = \gamma(Q')$.

It is easy to see that dominance solvability* is stronger than dominance solvabilty but is weaker than strategy-proofness.

Remark 3. If a game is dominance-solvable*, then it is dominance-solvable. But the other direction may not be true.

[21] Since the establishment of Corollary 4 does not require the knowledge of the characterization of the priority structure, I postpone the description of the Kumano-acyclic priority structure to Section 1.3.5.

This paper considers the following order of deleting weakly dominated strategies. Consider an arbitrary stage t in the process of deleting weakly dominated strategies. We remove a student in the beginning of stage t, if the following happens. If in stage $t-1$, there is some student who finds a dominant strategy and the dominant strategy ensures that he will be matched with the first school choice in the dominant strategy no matter what strategies that the remaining students use, remove this student in stage t. In addition, reduce the capacity constraint of the school that is the first school choice in this student's dominant strategy.

There are three situations that a student would delete his dominated strategies in stage t. First, if he finds the most preferred school among the schools that still have seats is the only acceptable school, than it is a dominant strategy to report this school as the only acceptable school. Second, if he finds that the most preferred school among the schools that still have seats considers him as the top choice among the remaining students, then ranking this school as the first school choice is a dominant strategy. Third, if he finds that the sum of the remained seats in the most and second preferred schools among the schools that still have seats is greater than or equal to the number of remained students, then it is a dominant strategy to ranks these two schools as the first and second school choice with the same relative ranking as in the true preference.

The above process can be formally described with the following sequence.

$$\{\Gamma^t\}_{t=1}^T \equiv \{(I^t, S^t, q^t, P^t, f^t)\}_{t=1}^T.$$

Let $\Gamma^1 = (I, S, q, P, f)$. Each component in the sequence is defined as follows.

- $I^t = I^{t-1} \setminus \{i \in I^{t-1} : i \in U(f_s^{t-1}, q_s^{t-1}) \text{ and } P_i^{t-1}(s) = 1\}$, for $t = 2, ..., T$.

- $q_s^t = q_s^{t-1} - |\{i \in I^{t-1} : i \in U(f_s^{t-1}, q_s^{t-1}) \text{ and } P_i^{t-1}(s) = 1\}|$, for $s \in S^{t-1}$ and $t = 2, ..., T$.

- $S^t = S^{t-1} \setminus \{s \in S^{t-1} : q_s^t = 0\}$, for $t = 2, ..., T$.
- $f^t = (f_s)_{s \in S^t}$, where (1) $f_s^t(i) < f_s^t(j)$ if and only if $f_s(i) < f_s(j)$ for $i, j \in I^t$, and (2) $f_s^t(i) < f_s^t(j)$ for $i \in I^t$ and $j \in I \setminus I^t$, for $t = 2, ..., T$.
- $P^t = (P_i^t)_{i \in I^t}$, where (1) $sP_i^t s'$ if and only if $sP_i s'$ for $s, s' \in S^t$, and (2) $\varnothing P_i^t s$ if $s \in S \setminus S^t$.

We have the following observations.

Observation 1. For $i \in I^t$ and $s \in S^t$, if $i \in U(f_s^{t'}, q_s^{t'})$, then $i \in U(f_s^{t''}, q_s^{t''})$, for $t' < t''$ and $t'' \in \{1, ..., t\}$.

Observation 2. For $s, s' \in S^t$, if $q_s^{t'} + q_{s'}^{t'} \leq |I^{t'}|$, then $q_s^{t''} + q_{s'}^{t''} \leq |I^{t''}|$, for $t' < t''$ and $t'' \in \{1, ..., t\}$.

Let $\mathcal{Q}_i^t = \mathcal{Q}_i$, $\mathcal{Q}_{-i}^t = \bigcup_{i \in I \setminus \{i\}} \mathcal{Q}_i^t$, and $\mathcal{Q}^t = \bigcup_{i \in I^t} \mathcal{Q}_i$.

Lemma 1. *For $i \in I^t$ and $s \in S^t$, if $sP_i^t s'$ and $\varnothing P_i^t s'$ for all $s' \in S^t \setminus \{s\}$, then it is a dominant strategy for i to rank s as the only acceptable school choice in $\Gamma^{t'}$, for $t' \in \{t, ..., T\}$. Moreover, if for $i \in I^t$, $\varnothing P_i^t s''$ for all $s'' \in S^t$, then it is a dominant strategy to rank $\varnothing$ as the first school choice in $\Gamma^{t'}$ for $t' \in \{t, ..., T\}$.*

Lemma 1 says that if there is a student i who finds that there is only one acceptable school in the school choice problem Γ^t, then it is a dominant strategy to rank that school as the only acceptable school in the school choice problem $\Gamma^{t'}$, for $t' \in \{t, ..., T\}$. If a student finds that all schools in S^t are not acceptable, then it is a dominant strategy to rank $\varnothing$ as the first school choice in $\Gamma^{t'}$, for $t' \in \{t, ..., T\}$.

Lemma 2. *For $i \in I^t$ and $s \in S^t$, if $sP_i^t s'$ for all $s' \in S^t$ and $i \in U(f_s^t, q_s^t)$, then it is a dominant strategy for student i to rank s as the first school choice in the school choice problem $\Gamma^{t'}$, for $t' \in \{t, ..., T\}$. Moreover, $\varphi_i(Q_i, Q_{-i}) = s$ for all $Q_{-i} \in \mathcal{Q}_{-i}^{t'}$ and for $t' \in \{t, ..., T\}$, where Q_i is a strategy that ranks s as the first school choice.*

Lemma 2 says that if there is a student i who finds her most preferred school according to P_i^t considers her as the top choice according to f_i^t in the school choice problem Γ^t, then it is a dominant strategy for her to rank s as the first school choice in the school choice problem $\Gamma^{t'}$, for $t' \in \{t, ..., T\}$. Moreover, as long as she uses a strategy that ranks s as the first school choice, she will obtain a seat in s for sure in the school choice problem $\Gamma^{t'}$ for $t' \in \{t, ..., T\}$.

Lemma 3. *For $i \in I^t$ and $s, s' \in S^t$, if $sP_i^t s' P_i^t s''$ for all $s'' \in S^t \setminus \{s, s'\}$, and $q_s^t + q_{s'}^t \geq |I^t|$, then it is a dominant strategy for student i to rank s as the first school choice and s' as the second school choice in the school choice problem $\Gamma^{t'}$, for $t' \in \{t, ..., T\}$. Moreover, $\varphi_i(Q_i, Q_{-i}) \in \{s, s'\}$ for all $Q_{-i} \in \mathcal{Q}_{-i}^{t'}$ and for $t' \in \{t, ..., T\}$, where Q_i is a strategy that ranks s as the first school choice and s' as the second school choice.*

Lemma 3 says that if there is student i who finds her most preferred school s and second preferred school s' according to P_i^t have the property that the sum of q_s^t and $q_{s'}^t$ is greater than or equal to $|I_t|$, then it is a dominant strategy to rank s as the first school choice and s' as the second school choice. Moreover, as long as she ranks s as the first school choice and s' as the second school choice, she will obtain a seat in one of these two schools in the school choice problem $\Gamma^{t'}$ for $t' \in \{t, ..., T\}$.

The following theorem gives a general description of the dominant strategies in school chocie problem Γ^t.

Theorem 6. *Consider the sequence of school choice problems $\{\Gamma^t\}_{t=1}^T$. If a student i finds a dominant strategy Q_i in the school choice problem Γ^t, it takes one of the following three forms.*

1. *The first school choice in Q_i is s, and s is the only acceptable school for i according to P_i^t. Moreover, $i \notin U(f_s^t, q_s^t)$.*

2. *The first school choice in Q_i is s, and s is the favorite school choice for i according to P_i^t. Moreover, $i \in U(f_s^t, q_s^t)$.*

3. *The first school choice in Q_i is s, and the second school choice in Q_i is s'. The first school choice in P_i^t, and the second school choice in P_i^t is s'. Moreover, $i \notin U(f_s^t, q_s^t)$ and $q_s^t + q_{s'}^t \geq |I^t|$.*

Refer the above three dominant strategies as the type-1, type-2, and type-3 dominant strategy in the school choice problem Γ^t. If Q_i is a type-1 or type-3 dominant strategy for student i in the school choice problem Γ^t, then it is a dominant strategy for student i in the school choice problem $\Gamma^{t'}$, for $t' \in \{t, ..., T\}$. If Q_i is a type-2 dominant strategy for student i in the school choice problem Γ^t, $\varphi_i(Q_i, Q_{-i}) = s$ for all $Q_{-i} \in \mathcal{Q}_{-i}^{t'}$ for $t' \in \{t, ..., T\}$.

Remark 4. Suppose there is a constraint on the number of school choices that a student can declare. As long as the constraint is larger than or equal to two, Theorem 19 holds.

One property of the dominance-solvable* outcome when the Boston mechanism is used is the following.

Theorem 7. *When a game $(\mathcal{Q}, P, \varphi)$ is dominance-solvable*, the dominance-solvable* outcome is Pareto efficient.*

1.3.5 Acyclic Priority Structure

Let $U_s^f(i) = \{j \in I : f_s(j) < f_s(i)\}$. This is the set of students who are ranked higher than student i in f_s.

An α-**cycle** is defined as follows. There exist two students i, j and two schools s, s' such that the following two conditions hold:

- Cycle condition 1: $f_s(i) < f_s(j)$ and $f_{s'}(j) < f_{s'}(i)$.
- Scarcity condition 1: there exist two disjoint sets $I_i \subseteq I \setminus \{j\}$ and $I_j \subseteq I_j \setminus \{i\}$ such that $I_i \subseteq U_{s'}^f(i)$, $I_j \subseteq U_s^f(j)$, $|I_i| = |q_{s'} - q_s|$ and $|I_j| = \min\{q_s, q_{s'}\} - 1$.

A priority structure f is α-**acyclic** if there is no α-cycle.

A **quasi-cycle**[22] is defined as follows. There exist three students i, j, k and two schools s, s' such that the following two conditions hold:

- Cycle condition 2: $f_s(i) < f_s(j)$ and $f_{s'}(j) < f_{s'}(k)$.
- Scarcity condition 2: there exist two disjoint sets $I_i, I_j \subseteq I \setminus \{i, j, k\}$ such that $I_j \subseteq U_s^f(j)$, $I_k \subseteq U_{s'}^f(k)$, $|I_j| = q_s - 1$ and $|I_k| = q_{s'} - 1$.

A priority structure f is **Kumano-acyclic** if there is no quasi-cycle.

We are ready to define the acyclic priority structure. A priority structure f has a **cycle**, if we perform the following two-step procedure to any two schools and find that there is a quasi-cycle in Step 2.

- Step 0: Consider two schools s, s'.
- Step 1: If there exist two students i, j such that together with s, s' they constitute an α-cycle, then proceed to Step 2. If there does not exist any two students that can form an α-cycle with s, s', stop at this step.
- Step 2: If there exist three students i, j, k such that together with s, s' they constitute a quasi-cycle, then there is a cycle in the priority structure f.

A priority structure f is **acyclic**, if there is no cycle. In other words, a priority structure f is acyclic, if we perform the above two-step procedure to any two schools and find that either the procedure stops at Step 1 or there is no quasi-cycle at Step 2.

The following example gives an example in which the priority structure is acyclic.

Example 4. Let the set of schools be $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, the set of students be $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}\}$, and the vector of capacity

[22] This is defined in Kumano (2013).

constraints be $q = (2, 3, 4, 8, 8, 9)$. The following diagram represents the possible construction of the priority structure f. Each "$\bullet$" in the diagram represents a student.

$$f_{s_6}: \underbrace{\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet}_{A_6}$$

$$f_{s_5}: \underbrace{\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet}_{A_5}$$

$$f_{s_4}: \underbrace{\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet}_{A_4}$$

$$f_{s_3}: \underbrace{\bullet - \bullet - \bullet - \bullet}_{A_3} \underbrace{- \bullet - \bullet - \bullet - \bullet - \bullet - \bullet}_{B_3}$$

$$f_{s_2}: \underbrace{\bullet - \bullet - \bullet}_{A_2} \underbrace{- \bullet}_{B_2} \underbrace{- \bullet - \bullet - \bullet - \bullet - \bullet - \bullet}_{C_2}$$

$$f_{s_1}: \underbrace{\bullet - \bullet}_{A_1} \underbrace{- \bullet}_{B_1} \underbrace{- \bullet}_{C_1} \underbrace{- \bullet - \bullet - \bullet - \bullet - \bullet - \bullet}_{D_1}$$

The priority structure f is acyclic if the following conditions hold: (1) $A_1 \subset A_2 \subset A_3$; (2) for any $i \in B_1$, we have $i \in A_2$; (3) for any $i \in C_1$, we have $i \in B_2$; (4) for any $i \in D_1$, we have $f_{s_1}(i) = f_{s_2}(i) = f_{s_3}(i)$; (5) there is no restriction on the ordering of students in A_4, A_5, and A_6.

Note that $|A_1| = q_{s_1}$, $|A_2| = q_{s_2}$, $|A_3| = q_{s_3}$, $|A_4| = q_{s_1} + q_{s_4}$, $|A_5| \geq q_{s_1} + q_{s_4}$, and $|A_6| \geq q_{s_1} + q_{s_4}$. In particular, notice that $q_{s_1} + q_{s_4} = |I|$.

A priority structure that satisfies the above requirement is the following.

$$f_{s_6} : i_3 - i_8 - i_1 - i_5 - i_7 - i_4 - i_1 - i_{10} - i_2 - i_9$$

$$f_{s_5} : i_7 - i_6 - i_5 - i_4 - i_{10} - i_9 - i_8 - i_3 - i_2 - i_1$$

$$f_{s_4} : i_{10} - i_9 - i_8 - i_7 - i_6 - i_5 - i_4 - i_3 - i_1 - i_2$$

$$f_{s_3} : i_4 - i_3 - i_1 - i_2 - i_5 - i_6 - i_7 - i_8 - i_9 - i_{10}$$

$$f_{s_2} : i_3 - i_1 - i_2 - i_4 - i_5 - i_6 - i_7 - i_8 - i_9 - i_{10}$$

$$f_{s_1} : i_1 - i_2 - i_3 - i_4 - i_5 - i_6 - i_7 - i_8 - i_9 - i_{10}$$

The following theorem says that the Boston mechanism is dominance-solvable* or dominance-solvable if and only if the priority structure is acyclic.

Theorem 8. *The following statements are equivalent.*

1. *The priority structure f is acyclic.*
2. *For every P, the Boston mechanism is dominance-solvable*.*
3. *For every P, the Boston mechanism is dominance-solvable.*

We have the following corollary.

Corollary 5. *When the priority structure is acyclic, the Boston mechanism is dominance-solvable* or dominance-solvable. Moreover, the dominance-solvable* or dominance-solvable outcome is stable and Pareto efficient.*

Consider an environment where $|S| \geq 3$ and $\sum_{s \in S} q_s \leq |I|$, i.e., the resources are very scarce. In such an environment, the sufficient and necessary condition of the priority structure for the Boston mechanism to be dominance-solvable is reduced to the α-acyclic priority structure.

Theorem 9. *Suppose $|S| \geq 3$ and $\sum_{s \in S} q_s \leq |I|$. The following statements are equivalent.*

1. *The priority structure f is α-acyclic.*

2. *For every P, the Boston mechanism is dominance-solvable*.*

3. *For every P, the Boston mechanism is dominance-solvable.*

Using a similar reasoning as in Corollary 5, we have the following corollary.

Corollary 6. *Suppose $|S| \geq 3$ and $\sum_{s \in S} q_s \leq |I|$. When the priority structure is α-acyclic, the Boston mechanism is dominance-solvable. Moreover, the dominance-solvable outcome is stable and Pareto efficient.*

Notice that $|S| \geq 3$ and $\sum_{s \in S} q_s \leq |I|$ imply that for any two schools s, s', we have $q_s + q_{s'} < |I|$. By Kumano (2013), the priority structure will never be Kumano-acyclic in such an environment.

1.3.6 Comparison with Other Characterizations of the Priority Structure

In this section, we compare the acyclic priority structure with several other types of priority structure related to the SOSM and the Boston mechanism.

A priority structure f has an **Ergin cycle**[23] if there exist three students i, j, k and two schools s, s' such that the following conditions hold:

- Cycle condition 3: $f_s(i) < f_s(j) < f_s(k) < f_{s'}(i)$.

- Scarcity condition 3: there exist disjoint sets of agents $I_i, I_j \subseteq I \setminus \{i, j, k\}$ such that $I_j \subseteq U^f_s(j)$, $I_i \subseteq U^f_{s'}(i)$, $|I_j| = q_s - 1$, and $|I_i| = q_{s'} - 1$.

A priority structure f is **Ergin-acyclic** if f does not have an Ergin cycle.

[23] This is defined in Ergin (2002).

A priority structure f has a **weak X-cycle**[24] if there exist two students i, j and two schools s, s' such that the following conditions hold.

- Cycle condition 4: $f_s(i) < f_s(j)$ and $f_{s'}(j) < f_{s'}(i)$.
- Scarcity condition 4: there exist disjoint sets $I_i \subseteq I\backslash\{j\}$ and $I_j \subseteq I\backslash\{i\}$ such that $I_i \subseteq U^f_{s'}(i)$, $I_j \subseteq U^f_s(j)$, $|I_i| = q_{s'} - 1$ and $|I_j| = q_s - 1$.

A priority structure f is **strongly X-acyclic** if f does not have a weak X-cycle.

The following theorem shows the restrictiveness of acyclic priority structure and α-acyclic priority structure.

Theorem 10. *When the priority structure f is acyclic or α-acyclic, the priority structure f is also Ergin-acyclic.*

Figure 1 depicts the relationship among the four domains of priority structure.

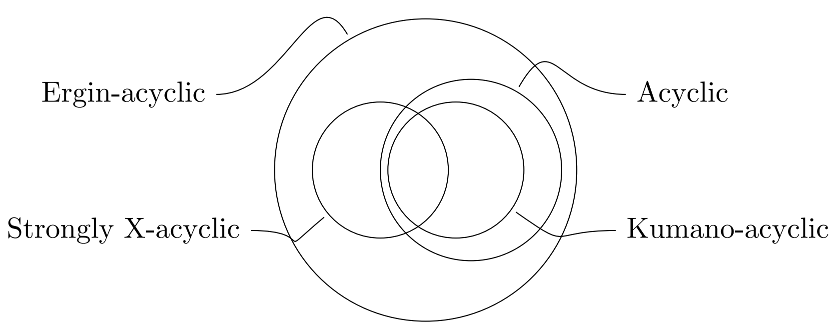

Figure 1.3: *The relationship among four domains of priority structure*

The examples that show the relationship in Figure 1 are contained in Section 1.6.

[24] This is defined in Haeringer and Klijn (2009).

Both acyclic and α-acyclic priority structures turn out to be very restrictive. For a priority structure to be acyclic, it has the following requirement. First, schools are separated into two disjoint sets S_1 and S_2 such that (1) any school in S_2 has a larger capacity constraint than any school in S_1, and (2) the sum of capacity constraints of the school with the smallest capacity constraint in S_1 and the school with the smallest capacity constraint in S_2 is greater than or equal to $|I|$. The requirement is that for any two schools s, s' in S_1 with $q_{s'} \geq q_s$, a student who is ranked lower than the $q_{s'}$-th position in one of the schools has the same ranking in both schools. Note that if there does not exist two schools such that the sum of their capacity constraints is greater than or equal to $|I|$, then all schools are treated as in S_1. The proof of this characterization is in Lemma 5 in Section 1.5. For a priority structure to be α-acyclic, it requires that for any two schools s, s' with $q_{s'} \geq q_s$, a student who is ranked lower than the $q_{s'}$-th position in one of the school has the same ranking in both schools. The proof of this characterization is in Lemma 4 in Section 1.5. Therefore, both types of priority structure are very restrictive.

Table 1 shows the performance of the SOSM and the Boston mechanism under four different domains of priority structure. With appropriately chosen solution concepts and domains of priority structure, there is no trade-off between stability and efficiency in equilibrium for the SOSM and the Boston mechanism.

If the designer wants to achieve both stability and efficiency in equilibrium, the SOSM gives the designer the greatest freedom of designing the priority structure. The SOSM in this sense outperforms the Boson mechanism.

1.4 Discussions

In this paper, I propose a new solution concept, dominance solvability*, and use the solution concepts, dominance solvability* and dominance solvability,

	Priority Structure	Solution Concept			S.E.	P.E.
		SP	NE	DS*/DS		
SOSM	Ergin acyclic	$\surd$			$\surd$	$\surd$
BM	Strongly X-acyclic		$\surd$		$\surd$	$\surd$
	Kumano-acyclic	$\surd$			$\surd$	$\surd$
	Acyclic			$\surd$	$\surd$	$\surd$

Table 1.1: *A comparison of the equilibrium performance of the SOSM and the Boston mechanism under four different domains of priority structure. Note that BM stands for Boston mechanism, S.E. for stability in equilibrium, and P.E. for Pareto efficiency in equilibrium. In addition, SP stands for strategy-proofness, NE for Nash equilibrium, DS* for dominance solvability*, and DS for dominance solvability. The mark "$\surd$" represents the solution concept that is applied and whether stability or Pareto efficiency in equilibrium is satisfied.*

to analyze the widely-used Boston mechanism. I find the Boston mechanism is nonbossy, and this property makes the order of deleting weakly dominated strategies irrelevant to the outcomes produced by the survived strategy profiles. Then, I characterize the acyclic priority structure, so that the Boston mechanism is dominance-solvable* or dominance-solvable, if and only if the priority structure is acyclic. I also consider an environment where the resources are scarce and characterize the priority structure, so that the Boston mechanism is dominance-solvable* or dominance-solvable, if and only if the priority structure is α-acyclic. Moreover, when the priority structure is acyclic or α-acyclic, the dominance-solvable outcome is stable and efficient. However, both the acyclic priority structure and the α-acyclic priority structure are very restrictive, and such restrictions make it difficult to achieve in reality. In other words, the findings do not bring justification to the use the Boston mechanism.

Despite the results from weakening strategy-proofness does not justify the use of the Boston mechanism, the methodology developed in this paper could be useful for designing matching mechanisms or investigating the properties of the matching mechanisms. As mentioned in the introduction, there are some situations that make any matching mechanism impossible to be strategy-proof. I argue that if the first best solution concept, strategy-proofness, cannot be fulfilled in any matching mechanism, we could develop the second best solution concept, so that the incentives properties of the matching mechanisms in such environment still have some tractability. The solution concept proposed in this paper, dominance solvability*, could be a suitable candidate in such an environment.

1.5 Proofs

This section contains all the proofs. Note that the proof of Theorem 9 is presented before the proof of Theorem 8, because the proof of Theorem 8 uses some of the proof of Theorem 9.

1.5.1 Proof of Theorem 1

Consider any two strategy profiles Q and Q'. Suppose there is a student j who submits the same preference Q_j in both Q and Q'. If student j receives different schools under Q and Q', then the algorithm of the Boston mechanism implies that there must be another student who also receives different schools in Q and Q'. Therefore, if $\varphi_i(Q_i, Q_{-i}) = \varphi_i(Q'_i, Q_{-i})$, since student i is the only student who submits different preferences in both strategy profiles, then other students also receive the same schools in both strategy profiles.

1.5.2 Proof of Theorem 4

The proof has two parts.

Part 1: I show that the strategies of each form listed in the statement of Lemma 4 are dominant strategies.

When student i's favorite school is s, and either s is the only acceptable school, or $i \in U(f_s, q_s)$, then it is clear that the dominant strategy is to report s as the first school choice.

Next, suppose there are more than two acceptable schools for student i. Let s and s' be his first and second favorite schools, respectively. We further assume as in the statement that $i \notin U(f_s, q_s)$ and $q_s + q_{s'} \geq |I|$.

Consider Q_{-i}. Q_{-i} belongs to one of the two cases.

Case 1: the number of students other than i who report s as their first school choices and have higher priority than i in f_s is less than q_s. Then $\varphi_i(Q_i, Q_{-i}) = s$. Since s is student i's favorite school, there does not exist any other strategy that makes student i strictly better off.

Case 2: the number of students other than i who reports s as the first school choices and have higher priority than i in f_s is greater than or equal to q_s. Then $\varphi_i(Q_i, Q_{-i}) = s'$. This is because $q_s + q_{s'} \geq |I|$. If the number of students who report s as the first school choices is greater than or equal to q_s, then the number of students who can report s' as the first or second school choice must be less than $q_{s'}$. Therefore, student i will receive s'. Since (1) there does not exist another strategy that can make student i receive s and (2) s' is student i's second favorite school, submitting Q_i is weakly better than all other strategies given Q_{-i}.

Therefore, Q_i is a dominant strategy.

Part 2: I show that there does not exist a dominant strategy that does not take one of the forms listed in Lemma 4.

Let student i's favorite school to be s. If s is the only acceptable school, or $i \in U(f_s, q_s)$, it is a weakly dominated strategy to report a school other than s as the first school choice.

Next, consider the case that student i has at least two acceptable schools. If $i \in U(f_s, q_s)$, then student i has a type-2 dominant strategy by reporting s as the first school choice.

Let s' be student i's second favorite school. If $q_s + q_{s'} \geq |I|$, then student i has a type-3 dominant strategy by reporting s and s' truthfully.

Assume that student i has at least two acceptable schools, $i \notin U(f_s, q_s)$, and $q_s + q_{s'} < |I|$. I show that there is no dominant strategy for student i.

Suppose on the contrary there exists a dominant strategy, and the dominant strategy takes the form

$$Q_i : s_1, s_2, s_3, ...$$

There are three sub-cases to consider: (1): s_1 is student i's favorite school and s_2 is student i's second favorite school; (2): s_1 is student i's favorite school but s_2 is not student i's second favorite school; (3): s_1 is not student i's favorite school.

In Case 1, by assumption, $i \notin U(f_{s_1}, q_{s_1})$ and $q_{s_1} + q_{s_2} < |I|$. Construct Q_{-i} is the following way. Let students in $U(f_{s_1}, q_{s_1})$ submit s_1 as the first school choice and let students not in $U(f_{s_1}, q_{s_1}) \cup \{i\}$ submit s_2 as the first school choice. Since $i \notin U(f_{s_1}, q_{s_1})$ and $q_{s_1} + q_{s_2} < |I|$, we have $\varphi_i(Q_i, Q_{-i}) \notin \{s_1, s_2\}$. However, if we consider Q_i', such that

$$Q_i' : s_2,$$

then $\varphi_i(Q_i', Q_{-i}) = s_2$. This contradicts Q_i being a dominant strategy.

In Case 2, construct Q_{-i} such that students other than i submit s_1 as the only school choice. Since $i \notin U(f_{s_1}, q_{s_1})$, we have $\varphi_i(Q_i, Q_{-i}) = s_2$. However, if we consider Q_i', such that

$$Q_i' : s',$$

then $\varphi_i(Q_i', Q_{-i}) = s'$, which is a contradiction.

In Case 3, construct Q_{-i} such that students other than i do not report s_1 or s as the acceptable schools. Then $\varphi_i(Q_i, Q_{-i}) = s_1$. Consider the strategy Q_i' such that

$$Q_i' : s.$$

Then, $\varphi_i(Q_i, Q_{-i}) = s$. This is a contradiction.

It is easy to verify the above reasoning holds, if we replace any of s_1, s_2, or s_3 with $\varnothing$ in Q_i.

1.5.3 Proof of Lemma 1

Consider student i. By construction of $\{\Gamma^t\}_{t=1}^T$, if there exists $\Gamma^{t'}$ with $t' \in \{1, ..., T\}$ where $sP_i^{t'}\varnothing P_i^{t'}s'$ for some $s \in S^{t'}$ and for all $s' \in S^{t'} \setminus \{s\}$, then there does not exist $\Gamma^{t''}$ with $t'' \in \{t'+1, ..., T\}$ where $s'P_i^{t''}\varnothing$ for some $s' \in S^{t''}$ and $s' \neq s$. Therefore, if there exists $\Gamma^{t'}$ such that $s \in S^{t'}$ is the only acceptable school according to $P_i^{t'}$, then it is a dominant strategy to rank s as the only acceptable school choice in $\Gamma^{t''}$ for $t'' \in \{t', ..., T\}$.

1.5.4 Proof of Lemma 2

Consider student i. By construction of $\{\Gamma^t\}_{t=1}^T$, if there exists $\Gamma^{t'}$ with $t' \in \{1, ..., T\}$ where $sP_i^{t'}\varnothing$ and $sP_i^{t'}s'$ for some $s \in S^{t'}$ and for all $s' \in S^{t'}\setminus\{s\}$ and $i \in U(f_s^{t'}, q_s^{t'})$, then there does not exist $\Gamma^{t''}$ with $t'' \in \{t'+1, ..., T\}$ where $s'P_i^{t''}s$ for some $s \in S^{t''}$. Therefore, if there exists $\Gamma^{t'}$ where $sP_i^{t'}\varnothing$ and $sP_i^{t'}s'$ for some $s' \in S^{t'}$ and for all $s' \in S^{t'} \setminus \{s'\}$ and $i \in U(f_s^{t'}, q_s^{t'})$, then it is a dominant strategy for i to rank s as the first school choice in $\Gamma^{t'}$. Note that when such $\Gamma^{t'}$ exists, there are at most $q_s^{t'} - 1$ number of students in $I^{t'}$ who have higher priority than i in $f_s^{t'}$ and f_s^1. Also note that by construction of $\{\Gamma^t\}_{t=1}^T$, for any $j \in I \setminus \{I^{t'}\}$, we have $\bar{s}P_j^{\bar{t}}\varnothing$ and $\bar{s}P_j^{\bar{t}}\hat{s}$ for some $\bar{s} \in S^{\bar{t}}$ and for all $\hat{s} \in S^{\bar{s}} \setminus \{\hat{s}\}$ and $j \in U(f_{\bar{s}}^{\bar{t}}, q_{\bar{s}}^{\bar{t}})$ in some $\Gamma^{\bar{t}}$ where $\bar{t} \in \{1, ..., t'-1\}$. Therefore, $\varphi_i(Q_i, Q_{-i}) = s$ for all $Q_{-i} \in \mathcal{Q}_{-i}^{t''}$ with $t'' \in \{t', ..., T\}$, where Q_i is a strategy that ranks s as the first school choice.

1.5.5 Proof of Lemma 3

Consider $i \in I^t$ and $s, s' \in S^t$. Suppose $sP_i^t s' P_i^t s''$ for all $s'' \in S^t \setminus \{s, s'\}$ and $q_s^t + q_{s'}^t \geq |I^t|$. By Observation 2, $q_s^{t'} + q_{s'}^{t'} \geq |I^{t'}|$ for $t' \in \{t, ..., T\}$. With a similar argument as in the proof of Theorem 4, submitting a strategy that ranks s and s' as the first and second choices, respectively, is a dominant strategy in $\Gamma^{t'}$ for $t' \in \{t, ..., T\}$. Moreover, $q_s^{t'} + q_{s'}^{t'} \geq |I^{t'}|$ implies that $\varphi_i(Q_i, Q_{-i}) \in \{s, s'\}$ for all $Q_{-i} \in \mathcal{Q}_{-i}^{t'}$ for $t' \in \{t, ..., T\}$, where Q_i is a strategy that ranks s and s' as the first and second choices, respectively.[25]

1.5.6 Proof of Theorem 7

Observe that when a student i finds his dominant strategy, all schools that are strictly better than the first school choice in student i's dominant strat-

[25] Note that it there exists $t'' \in \{t, ..., T\}$ such that $q_s^{t''} = 0$, submitting a strategy that ranks s and s' as the first and second choices, respectively, is still a dominant strategy in $\Gamma^{t''}$. This is because in this case $q_{s'}^{t''} \geq |I^{t''}|$ and $\varphi_i(Q_i, Q_{-i}) = s'$ for all $Q_{-i} \in \mathcal{Q}_{-i}^{t''}$, where Q_i is a strategy that ranks s, s' as the first and second choices, respectively.

egy do not have vacant seat. In other words, those seats are received by other students who have type-2 dominant strategies and have that school as the first school choice. Notice that the first school choice in the dominant strategy is the favorite school among those schools that have vacant seat and the second school choice in the type-3 dominant strategy is the second favorite school among those schools that have vacant seats. Suppose the first school choice in the dominant strategy is s.

If a student has a type-1 dominant strategy, then he will receive either the first school choice of the dominant strategy or the outside option. When he receives the outside option, it means that there are a number of q_s students who rank school s as the first school choice and have higher priority than student i in f_s.

If a student has a type-2 dominant strategy, then he will receive the first school choice of the dominant strategy for sure.

If a student has a type-3 dominant strategy, then he/she will receive one of the first school choice and the second school choice. When he receives the second school choice, it means there are a number of q_s students who rank school s as the first school choice and have higher priority than student i in f_s.

Therefore, for a student to receive a strictly better school than the school he receives in the dominance-solvable* outcome, there must be a student who receives a strictly worse school than his dominance-solvable* outcome. In other words, the dominance-solvable* outcome is Pareto efficient.

1.5.7 Proof of Theorem 9

(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3):

I prove that when the priority structure f is α-acyclic, the game is dominance-solvable*. Thus, it is dominance-solvable.

Lemma 4. *A priority structure f is α-acyclic, if and only if the following* *two statements hold for any two schools s, s' with $q_s \le q_{s'}$: (1) for $i \in U(f_s, q_s)$, we have $i \in U(f_{s'}, q_{s'})$; (2) for $j \notin U(f_s, q_{s'})$ or $j \notin U(f_{s'}, q_{s'})$, we have $f_s(j) = f_{s'}(j)$.*

Proof. *If* part: Suppose the priority structure f has an α-cycle. Let i, j, s, s' be the students and schools that constitute the α-cycle and let them play the same roles as in the definition of an α-cycle in the text. Observe that the Cycle condition 2 in the definition of an α-cycle implies that either $f_s(i) \neq f_{s'}(i)$, $f_s(j) \neq f_{s'}(j)$, or both. For part (1), assume that $i \in U(f_s, q_s)$. We prove that $i \notin U(f_{s'}, q_{s'})$. Since $I_i \subseteq I \setminus \{j\}$, $f_{s'}(j) < f_{s'}(i)$, $I_i \subseteq U^f_{s'}(i)$, if $i \in U(f_{s'}, q_{s'})$, there cannot be disjoint sets $I_i, I_j \subseteq I \setminus \{i, j\}$ that make $|I_i| = |q_s - q_{s'}|$ and $|I_j| = q_s - 1$ happen. Note that (1) implies that $U(f_s, q_{s'}) = U(f_{s'}, q_{s'})$. For part (2), suppose $i \notin U(f_s, q_{s'})$. If $f_s(i) = f_{s'}(i)$, then the Cycle condition 2 implies that $f_s(j) \neq f_{s'}(j)$. Therefore, one of i and j has different ranking in f_s and $f_{s'}$.

Only if part: First, suppose there is a student $i \in U(f_s, q_s)$ and $i \notin U(f_{s'}, q_{s'})$. Then there exists another student $j \in U(f_{s'}, q_{s'})$ and $j \notin U(f_s, q_s)$. Then i, j, s, s' constitute an α-cycle. Second, suppose there is a student $j \notin U(f_s, q_{s'})$ and $f_s(j) \neq f_{s'}(j)$. Without loss of generality, assume that $f_{s'}(j) < f_s(j)$. There exists another student i such that $f_s(i) < f_s(j)$, $f_{s'}(j) < f_{s'}(i)$ and $i \notin U(f_{s'}, q_{s'})$. Then i, j, s, s' constitute an α-cycle. □

Re-index the schools so that for any two schools $s_t, s_{t'}$, we have $t \le t'$ if and only if $q_{s_t} \le q_{s_{t'}}$. Notice that the largest index is m. After the re-indexing, Lemma 4 implies the following corollary.

Corollary 7. *If the priority structure f is α-acyclic, then the following statements hold:*

1. *$U(f_{s_t}, q_{s_t}) \subseteq U(f_{s_{t'}}, q_{s_{t'}})$ for any $t' > t$.*
2. *$U(f_{s_t}, q_{s_{t'}}) \setminus U(f_{s_t}, q_{s_t}) \subseteq U(f_{s_{t'}}, q_{s_{t'}})$ for any $t' > t$.*
3. *For $i \notin U(f_{s_1}, q_{s_m})$, we have $f_{s_t}(i) = f_{s_{t'}}(i)$ for any t, t'.*

We are ready to prove the main theorem.

Re-index students in the following way. Divide students into m disjoint sets $\{I_r\}_{r=1}^m$ so that $I = \bigcup_{r=1}^m I_r$ and the following conditions hold.

- $|I_1| = q_{s_1}$.
- $|I_r| = q_{s_r} - q_{s_{r-1}}$ for $r = 2, ..., m-1$.
- $|I_m| = |I| - q_{s_{m-1}}$.
- Consider student i.
  - If $f_{s_1}(i) \in \{1, ..., q_{s_1}\}$, then $i \in I_1$.
  - If $f_{s_1}(i) \in \{q_{s_r} - q_{s_{r-1}} + 1, ..., q_{s_r}\}$ for $r = 2, ..., m-1$, then $i \in I_r$.
  - If $f_{s_1}(i) \in \{|I| - q_{s_{m-1}} + 1, ..., |I|\}$, then $i \in I_m$.
- For two students i, j in the set I_r, i has a smaller index than j if i has a higher priority than j in f_{s_1}.

Note that I_r could be empty for some $r \in \{2, ..., m\}$.

The order of deleting weakly dominated strategies only if there exists dominant strategies are as follows.

- Step 1: Consider students in J_1.
  - Consider an arbitrary student $i \in J_1$. Suppose his most preferred school is s_t. Then it is a dominant strategy to rank s_t as the first school choice. (By Corollary 7, $f_{s_t}(i) \leq q_{s_t}$. By Lemma 2, the strategy specified is a dominant strategy.)
  - Simultaneously delete the weakly dominated strategies for students in J_1.
  - For each school $s \in \{s_1, ..., s_{k-1}\}$, reduce its capacity q_s by the number of students who rank it as the first school choice. Call the new vector of capacity constraint as $q_{2,1}$. Let $S_{2,1}$ be the set of schools that still have positive capacities.

- Step 2: Consider students in J_2. (If J_2 is empty, move to Step 3.)

  Consider the students in J_2 sequentially according to their indexes from low to high.

  – Step 2.1: Consider the student with the lowest index in J_2. Call this student $i_{2,1}$.

    * Suppose all schools in $S_{2,1}$ are not acceptable, then it is a dominant strategy to submit $\varnothing$ as the first school choice.
    * Suppose his most preferred acceptable schools over $S_{2,1}$ is s_t. Then it is a dominant strategy to rank s_t as the first school choice. (Suppose $t \in \{2, ..., m\}$. By Corollary 7, $f_{s_t} \leq q_{s_t}$. By Lemma 2, the strategy specified is a dominant strategy. Suppose $t = 1$. The fact that s_1 still has a positive capacity means that the number of students who have lower indexes than i and rank s_1 as the first choice is smaller than q_{s_1}. By Lemma 2, the strategy specified is a dominant strategy.)
    * Deleted all the weakly dominated strategies.
    * If the most preferred acceptable school in $S_{2,1}$ is s_t, then reduce the capacity of school s_t in $q_{2,1}$ by one. Call the new vector of capacity constraints as $q_{2,2}$. Let $S_{2,2}$ be the set of schools that still have positive capacities.

  – Step 2.l: Consider the student with the l-th lowest index in J_2. Call this student $i_{2,l}$.

    * Repeat the procedure in Step 2.1 for student $i_{2,l}$ with the replacement of $S_{2,1}$ with $S_{2,l}$, $S_{2,2}$ with $S_{2,l+1}$, and $q_{2,2}$ with $q_{2,l+1}$ in the statement of the procedure.

  – Suppose the end of this stage is Step 2.p. At the end of Step 2.p, call $S_{2,p}$ as $S_{3,1}$ and $q_{2,p}$ as $q_{3,1}$.

- Step r: Consider students in J_r. (If J_r is empty, move to Step $r + 1$.)

Consider the students in J_r sequentially according to their indexes from low to high.

- Step $r.1$: Consider the student with the lowest index in J_r. Call this student $i_{r,1}$.
  * Suppose all schools in $S_{r,1}$ are not acceptable, then it is a dominant strategy to submit $\varnothing$ as the first school choice.
  * Suppose his most preferred acceptable schools over $S_{r,1}$ is s_t. Then it is a dominant strategy to rank s_t as the first school choice. (Suppose $t \in \{r+1, ..., m\}$. By Corollary 7, $f_{s_t} \leq q_{s_t}$. By Lemma 2, the strategy specified is a dominant strategy. Suppose $t \in \{1, ..., r\}$. The fact that s_t still has a positive capacity means that the number of students who have lower indexes than i and rank s_t as the first choice is smaller than q_{s_t}. By Lemma 2, the strategy specified is a dominant strategy.)
  * Deleted all the weakly dominated strategies.
  * If the most preferred acceptable school in $S_{r,1}$ is s_t, then reduce the capacity of school s_t in $q_{r,1}$ by one. Call the new vector of capacity constraints as $q_{r,2}$. Let $S_{r,2}$ be the set of schools that still have positive capacities.
- Step $r.l$: Consider the student with the l-th lowest index in J_r. Call this student $i_{r,l}$.
  * Repeat the procedure in Step $r.1$ for student $i_{r,l}$ with the replacement of $S_{r,1}$ with $S_{r,l}$, $S_{r,2}$ with $S_{r,l+1}$, and $q_{r,2}$ with $q_{r,l+1}$ in the statement of the procedure.
- Suppose the end of this stage is Step $r.p$. At the end of Step $r.p$, call $S_{r,p}$ as $S_{r+1,1}$ and $q_{r,p}$ as $q_{r+1,1}$.

The above procedure shows that the Boston mechanism is dominance-solvable* and thus dominance-solvable.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1):

Suppose $|S| \geq 3$ and $\sum_{s \in S} q_s \leq |I|$. Given a priority structure f such that it is not α-acyclic, I construct a preference profile P such that the game induced by the Boston mechanism is neither dominance-solvable nor dominance-solvable*.

Re-index schools so that for any two schools $s_t, s_{t'}$, we have $t \leq t'$ if and only if $q_{s_t} \leq q_{s_{t'}}$. Then, re-index students so that

$$\{i_1, ..., i_{q_{s_2}}\} = U(f_{s_2}, q_{s_2}), \text{ and}$$
$$\{i_{q_{s_2}+1}, ..., i_k\} = U(f_{s_1}, k) \setminus U(f_{s_2}, q_{s_2}),$$

where $k = q_{s_1} + q_{s_2}$.

When the priority structure f is not α-acyclic, there are two cases to consider. In the first case, there is a student in $U(f_{s_1}, q_{s_1})$ but not in $U(f_{s_2}, q_{s_2})$. In the second case, $U(f_{s_1}, q_{s_1}) \subseteq U(f_{s_2}, q_{s_2})$.

Consider the first case, where there is a student in $U(f_{s_1}, q_{s_1})$ but not in $U(f_{s_2}, q_s)$. Let this student be i_k. The case also implies that there is a student in $U(f_{s_2}, q_{s_2})$ but not in $U(f_{s_1}, q_{s_1})$. Let this student to be $i_{q_{s_2}}$. Construct P as follows:

P_{i_1}	$\cdots$	$P_{i_{q_2-1}}$	$P_{i_{q_2}}$	$P_{i_{q_2+1}}$	$\cdots$	$P_{i_{k-1}}$	P_{i_k}	$P_{i_{k+1}}$	$P_{i_{k+2}}$	$\cdots$	P_{i_n}
s_2	$\cdots$	s_2	s_1	s_1	$\cdots$	s_1	s_2	s_1	$\varnothing$	$\cdots$	$\varnothing$
s_1	$\cdots$	s_1	s_2	s_2	$\cdots$	s_2	s_1	s_2			

Notice that the construction requires that the number of students is at least $q_{s_1} + q_{s_2} + 1$. This is guaranteed by $|S| \geq 3$ and $\sum_{s \in S} q_s \leq |I|$, which implies that $q_{s_1} + q_{s_2} < |I|$.

Consider the following iterations of deleting weakly dominated strategies. In the first iteration, all students in $U(f_{s_2}, q_{s_2}) \setminus \{i_{q_{s_2}}\}$ have dominant strategies of reporting s_2 as the first school choice. All students in $\{i_{q_{s_2}+1}, ..., i_{k-1}\}$ have dominant strategies of reporting s_1 as the first school choice. All students in $\left(U(f_{s_2}, q_{s_2}) \setminus \{i_{q_{s_2}}\}\right) \cup \{i_{q_{s_2}+1}, ..., i_{k-1}\}$ will be matched with the favorite schools in their true preferences, regardless of the strategies that other students submit.

In the second iteration, all students but $i_{q_{s_2}}, i_k, i_{k+1}$ find dominant strategies. Each of schools s_1 and s_2 has one seat remained from the second iteration. The priority of $i_{q_{s_2}}, i_k, i_{k+1}$ in schools s_1 and s_2 are as follows:

$$f_{s_1}(i_k) < f_{s_1}(i_{q_{s_2}}) \text{ and } f_{s_1}(i_{k+1});$$
$$f_{s_2}(i_{q_{s_2}}) < f_{s_2}(i_k) \text{ and } f_{s_2}(i_{k+1}).$$

The payoff matrix of these three students are as follows. Note that $i_{q_{s_2}}$ is the row player, i_k is the column player, and i_{k+1} is the box player.

	s_1, s_2	s_2, s_1
s_1, s_2	$s_2, s_1, \varnothing$	$s_1[\varnothing], s_2, \varnothing[s_1]$
s_2, s_1	$s_2, s_1, \varnothing$	$s_2, \varnothing, s_1$

s_1, s_2

	s_1, s_2	s_2, s_1
s_1, s_2	$\varnothing, s_1, s_2$	$s_1, s_2(\varnothing), \varnothing(s_2)$
s_2, s_1	$s_2, s_1, \varnothing$	$s_2, s_1, \varnothing$

s_2, s_1

The entries are based on $f_{s_1}(i_{q_{s_2}}) < f_{s_1}(i_{k+1})$ and $f_{s_2}(i_k) < f_{s_2}(i_{k+1})$. The entries in $[\cdot]$ are based on $f_{s_1}(i_{k+1}) < f_{s_1}(i_{q_{s_2}})$. The entries in $(\cdot)$ are based on $f_{s_2}(i_{k+1}) < f_{s_2}(i_k)$.

There are no weakly dominated strategies for these students. Therefore, the game is not dominance-solvable.

In the second case, we have $U(f_{s_1}, q_{s_1}) \subseteq U(f_{s_2}, q_{s_2})$. Since f is not α-acyclic, there are two students i_r, i_t such that $i_r \notin U(f_{s_1}, q_{s_1})$, $f_{s_1}(i_r) \neq f_{s_2}(i_r)$, $f_{s_1}(i_r) < f_{s_1}(i_t)$ and $f_{s_2}(i_t) < f_{s_2}(i_r)$.

If $i_r \in \{i_{q_{s_2}+1}, ..., i_k\}$ and $i_t \in U(f_{s_2}, q_{s_2})$, then denote i_r as i_k and denote i_t as $i_{q_{s_2}}$. The follow-up analysis is the same as in the first case.

If $i_r \notin \{i_{q_{s_2}+1}, ..., i_k\}$ and $i_t \in U(f_{s_2}, q_{s_2})$, then denote i_t as $i_{q_{s_2}}$. Construct P' as follows. All students other than i_1 and $i_{q_{s_2}+1}$ have the same preferences as in P. The follow-up analysis is the same as in the first case.

$P'_{i_{q_{s_2}+1}}$	P'_{i_r}
$\varnothing$	s_2
	s_1

If $i_r \in \{i_{q_{s_2}+1}, ..., i_k\}$ and $i_t \notin U(f_{s_2}, q_{s_2})$, then denote i_r as i_k. Construct P'' as follows. All students other than i_1 and i_t have the same preferences as in P. The follow-up analysis is the same as in the first case.

P''_{i_1}	P''_{i_t}
$\varnothing$	s_1
	s_2

If $i_r \notin \{i_{q_{s_2}+1}, ..., i_k\}$ and $i_t \notin U(f_{s_2}, q_{s_2})$. Construct P''' as follows. All students other than $i_1, i_{q_{s_2}+1}, i_r, i_t$ have the same preferences as in P. The follow-up analysis is the same as in the first case.

P_{i_1}	$P_{i_{q_{s_2}+1}}$	P_r	P_t
$\varnothing$	$\varnothing$	s_2	s_1
		s_1	s_2

The follow-up analyses are the same as in the first case. Therefore, the game is neither dominance-solvable nor dominance-solvable*.

1.5.8 Proof of Theorem 8

(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3):

I prove that when the priority structure f is acyclic, then the game is dominance-solvable*. It implies that the game is dominance-solvable.

Re-index the schools so that for any two schools $s_t, s_{t'}$, we have $q_{s_t} \leq q_{s_{t'}}$ if and only if $t < t'$.

Lemma 5. *When the priority structure f is acyclic, either: (1) there exists a school s_k such that for any index $k' \geq k$, we have $q_{s_1} + q_{s_{k'}} \geq |I|$ and for school $s_{k''}$ with $k'' < k$, there is no α-cycle between s_1 and $s_{k''}$, or (2) the priority structure f is α-acyclic.*

Proof. Suppose the priority structure f is acyclic. Consider a school s_t such that $q_{s_1} + q_{s_t} \geq |I|$. Find the smallest t with this property and let $t = k$. Then, for any two schools $s_r, s_{r'}$ with $r, r' \geq k$, we have $q_{s_r} + q_{s_{r'}} \geq |I|$. Moreover, there is no Kumano cycle between schools s_r and $s_{r'}$. This is because a Kumano cycle requires at least $q_s + q_{s'} + 1$ number of students. For $k'' < k$, we have $q_{s_1} + q_{s_{k''}} < |I|$. To make the priority structure f to be acyclic, it is required that there is no α-cycle in schools $s_1, s_{k''}$.

If for any two schools, Kumano cycles always exist, then for the priority structure f to be acyclic, f must be α-acyclic. □

Lemma 6. *Suppose there exists a school s_k such that for $k' \geq k$, we have $q_{s_1} + q_{s_{k'}} \geq |I|$. The priority structure f is acyclic, if and only if for t, t' such that $t < t' < k$, we have (1) for $i \in U(f_{s_t}, q_{s_t})$, we have $i \in U(f_{s_{t'}}, q_{s_{t'}})$; (2) for $j \notin U(f_{s_t}, q_{s_{t'}})$ or $j \notin U(f_{s_{t'}}, q_{s_{t'}})$, we have $f_{s_t}(j) = f_{s_{t'}}(j)$.*

Proof. As pointed out in the proof of Lemma 5, for two schools $s_r, s_{r'}$ with $r, r' \geq k$, there is no Kumano cycle between these two schools. The priority structure f is acyclic if and only if for $t < t' < k$, there is no α-cycle for schools $s_t, s_{t'}$. The remaining proof is the same as the proof in Lemma 4. □

Lemma 6 implies the following corollary.

Corollary 8. *Suppose there exists a school s_k such that for $k' \geq k$ we have $q_{s_1} + q_{s_{k'}} \geq |I|$. If the priority structure f is acyclic, we have the following:*

1. *$U(f_{s_t}, q_{s_t}) \subseteq U(f_{s_{t'}}, q_{s_{t'}})$ for any $t < t' < k$.*

2. $U(f_{s_t}, q_{s_{t'}}) \setminus U(f_{s_t}, q_{s_t}) \subseteq U(f_{s_{t'}}, q_{s_{t'}})$ *for any* $t < t' < k$.

3. *For* $i \notin U(f_{s_1}, q_{s_m})$, *we have* $f_{s_t}(i) = f_{s_{t'}}(i)$, *for any* $t, t' < k$.

We are ready to prove the main theorem. There are two cases to consider. Case 1: there exists a school s_k such that for $k' \geq k$, $q_{s_1} + q_{s_{k'}} \geq |I|$; Case 2: there does not exist such a school.

We first consider Case 1. We have the following observation.

Observation 3. Suppose there exists a school s_k such that for $k' \geq k$ we have $q_{s_1} + q_{s_{k'}} \geq |I|$. Then, for any two schools $s_r, s_{r'}$ such that $r' \geq k$ we have $q_{s_r} + q_{s_{r'}} \geq |I|$.

Re-index students in the following way. Divide students into k disjoint sets $\{I_r\}_{r=1}^{k}$ so that $I = \bigcup_{r=1}^{k} I_r$ and the following conditions hold.

- $|I_1| = q_{s_1}$.
- $|I_r| = q_{s_r} - q_{s_{r-1}}$ for $r = 2, ..., k-1$.
- $|I_k| = |I| - q_{s_{k-1}}$.
- Consider student i.
  - If $f_{s_1}(i) \in \{1, ..., q_{s_1}\}$, then $i \in I_1$.
  - If $f_{s_1}(i) \in \{q_{s_r} - q_{s_{r-1}} + 1, ..., q_{s_r}\}$ for $r = 2, ..., k-1$, then $i \in I_r$.
  - If $f_{s_1}(i) \in \{|I| - q_{s_{k-1}} + 1, ..., |I|\}$, then $i \in I_k$.
- For two students i, j in the set I_r, i has a smaller index than j if i has a higher priority than j in f_{s_1}.

Note that I_r could be empty for some $r \in \{2, ..., k\}$.

The order of deleting weakly dominated strategies only if there exists dominant strategies are as follows.

- Step 1: Consider students in J_1.

- Consider an arbitrary student $i \in J_1$. Suppose his most preferred school is s_t and his second most preferred school is $s_{t'}$. (Or, suppose his only acceptable school is s_t.)
  * If $t \geq k$, then it is a dominant strategy to rank s_t as the first and second school choice and $s_{t'}$ as the second school choice. (By Lemma 3, the strategy specified is a dominant strategy.) (Or, it is a dominant strategy to rank s_t as the only acceptable school, if s_t is the only acceptable school. By Lemma 1, the strategy specified is a dominant strategy.)
  * If $t < k$, then it is a dominant strategy to rank s_t as the first school choice. (By Corollary 8, $f_{s_t}(i) \leq q_{s_t}$. By Lemma 2, the strategy specified is a dominant strategy.)
- Simultaneously delete the weakly dominated strategies for students in J_1.
- For each school $s \in \{s_1, ..., s_{k-1}\}$, reduce its capacity q_s by the number of students who rank it as the first school choice. Call the new vector of capacity constraint as $q_{2,1}$. Let $S_{2,1}$ be the set of schools that still have positive capacities.

• Step 2: Consider students in J_2. (If J_2 is empty, move to Step 3.)

Consider the students in J_2 sequentially according to their indexes from low to high.

- Step 2.1: Consider the student with the lowest index in J_2. Call this student $i_{2,1}$.
  * Suppose all schools in $S_{2,1}$ are not acceptable, then it is a dominant strategy to submit $\varnothing$ as the first school choice.
  * Suppose his most preferred acceptable schools over $S_{2,1}$ is s_t and the second most preferred acceptable school over $S_{2,1}$ is $s_{t'}$. (Or, suppose the only acceptable school in $S_{2,1}$ is s_t.)

- If $t \geq k$, then it is a dominant strategy to rank s_t as the first school choice and $s_{t'}$ as the second school choice. (Or, it is a dominant strategy to rank s_t as the only acceptable school choice, if s_t is the only acceptable school in $S_{2,1}$.) (By Lemma 3, the strategy specified is a dominant strategy.)
- If $t \in \{2, ...k-1\}$, then it is a dominant strategy to rank s_t as the first school choice. (By Corollary 8, $f_{s_t}(i) \leq q_{s_t}$. By Lemma 2, the strategy specified is a dominant strategy.)
- If $t = 1$, then it is a dominant strategy to rank s_t as the first school choice. (The fact that s_1 still has a positive capacity means that the number of students who have lower indexes than i and rank s_1 as the first choice is smaller than q_{s_1}. By Lemma 2, the strategy specified is a dominant strategy.)

* Deleted all the weakly dominated strategies.
* If the most preferred acceptable school in $S_{2,1}$ is s_t and $t \leq k$, then reduce the capacity of school s_t in $q_{2,1}$ by one. Call the new vector of capacity constraints as $q_{2,2}$. Let $S_{2,2}$ be the set of schools that still have positive capacities.

– Step 2.l: Consider the student with the l-th lowest index in J_2. Call this student $i_{2,l}$.

* Repeat the procedure in Step 2.1 for student $i_{2,l}$ with the replacement of $S_{2,1}$ with $S_{2,l}$, $S_{2,2}$ with $S_{2,l+1}$, and $q_{2,2}$ with $q_{2,l+1}$ in the statement of the procedure.

– Suppose the end of this stage is Step 2.p. At the end of Step 2.p, call $S_{2,p}$ as $S_{3,1}$ and $q_{2,p}$ as $q_{3,1}$.

- Step r: Consider students in J_r. (If J_r is empty, move to Step $r+1$.)

Consider the students in J_r sequentially according to their indexes from low to high.

- Step $r.1$: Consider the student with the lowest index in J_r. Call this student $i_{r,1}$.
  * Suppose all schools in $S_{r,1}$ are not acceptable, then it is a dominant strategy to submit $\varnothing$ as the first school choice.
  * Suppose his most preferred acceptable schools over $S_{r,1}$ is s_t and the second most preferred acceptable school over $S_{r,1}$ is $s_{t'}$. (Or, suppose the only acceptable school in $S_{r,1}$ is s_t.)
    · If $t \geq k$, then it is a dominant strategy to rank s_t as the first school choice and $s_{t'}$ as the second school choice. (Or, it is a dominant strategy to rank s_t as the only acceptable school choice, if s_t is the only acceptable school in $S_{r,1}$.) (By Lemma 3, the strategy specified is a dominant strategy.)
    · If $t \in \{r, ...k-1\}$, then it is a dominant strategy to rank s_t as the first school choice. (By Corollary 8, $f_{s_t}(i) \leq q_{s_t}$. By Lemma 2, the strategy specified is a dominant strategy.)
    · If $t \in \{1, ..., r-1\}$, then it is a dominant strategy to rank s_t as the first school choice. (The fact that s_t still has a positive capacity means that the number of students who have lower indexes than i and rank s_t as the first choice is smaller than q_{s_t}. By Lemma 2, the strategy specified is a dominant strategy.)
  * Deleted all the weakly dominated strategies.
  * If the most preferred acceptable school in $S_{r,1}$ is s_t and $t \leq k$, then reduce the capacity of school s_t in $q_{r,1}$ by one. Call the new vector of capacity constraints as $q_{r,2}$. Let $S_{r,2}$ be the set of schools that still have positive capacities.

- Step $r.l$: Consider the student with the l-th lowest index in J_r. Call this student $i_{r,l}$.
  - Repeat the procedure in Step $r.1$ for student $i_{r,l}$ with the replacement of $S_{r,1}$ with $S_{r,l}$, $S_{r,2}$ with $S_{r,l+1}$, and $q_{r,2}$ with $q_{r,l+1}$ in the statement of the procedure.
- Suppose the end of this stage is Step $r.p$. At the end of Step $r.p$, call $S_{r,p}$ as $S_{r+1,1}$ and $q_{r,p}$ as $q_{r+1,1}$.

The above procedure shows that the Boston mechanism is dominance-solvable* and thus dominance-solvable.

Next, consider the Case 2, where there does not exist a school s_k such that for $k' \geq k$, $q_{s_1} + q_{s_{k'}} \geq |I|$. Then the priority structure must also be α-acyclic. By Theorem 4, the game is dominance-solvable* and thus dominance-

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1):

Suppose the priority structure f is not acyclic, I show that there exists a preference profile P such that the game is neither dominance-solvable nor dominance-solvable*.

Lemma 7. *When the priority structure f is not acyclic, then there exists at least two schools s and s' such that $q_s + q_{s'} < |I|$.*

Proof. Suppose for any two schools s, s', we have $q_s + q_{s'} \geq |I|$, then as pointed out in the proof of Lemma 5 the priority structure is Kumano-acyclic, and thus acyclic. Therefore, the statement holds. $\square$

Lemma 7 implies that $q_{s_1} + q_{s_2} < |I|$, since there is no other school that has a smaller capacity constraint than school s_1 or school s_2. This allows constructing P in the same way as the construction of P in the part of

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) in the proof of Theorem 9.[26] The remainder of the proof is the same as in the part of (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) in the proof of Theorem 9. Therefore, the game is neither dominance-solvable nor dominance-solvable*.

1.5.9 Proof of Theorem 10

By Theorem 2 of Ergin (2002), a priority structure is Ergin-acyclic if and only if for any two schools s, s' the priority of the students who are ranked lower than $q_s + q_{s'}$ in one of the schools cannot differ by more than one in these two schools.

By Lemma 4, when the priority structure is α-acyclic, for any two schools s, s', a student who is ranked lower than $\max\{q_s, q_{s'}\}$ in one of the schools need to have the same priority ranking in both schools. Therefore, when a priority structure is α-acyclic, it is also Ergin-acyclic.

Suppose a priority structure is acyclic. Re-index the schools as in the proof of Theorem 8. Suppose there exists a school s_k such that for $k' \geq k$, $q_{s_1} + q_{s_{k'}} \geq |I|$. This implies that for any two schools $s_t, s_{t'}$ such that $t, t' \geq k$, we have $q_{s_t} + q_{s_{t'}} \geq |I|$. Therefore, there does not exist any student who is ranked lower than $q_{s_t} + q_{s_{t'}}$ in one of the schools. By Lemma 5, for any two schools $s_r, s_{r'}$ such that $r, r' < k$, the student who is ranked lower than $\max\{q_{s_r}, q_{s_{r'}}\}$ has the same priority in both schools. If there does not exist such s_k, then the priority structure is α-acyclic. Therefore, when the priority structure is acyclic, it is also Ergin-acyclic.

1.6 Examples

Example 5 (A priority structure that is Ergin-acyclic, strongly X-acyclic, Kumano-acyclic and acyclic.)**.** The example is from Kumano (2013).

[26] Recall that the construction of P in the proof of Theorem 9 requires that the number of students is at least $q_{s_1} + q_{s_2} + 1$.

$I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$ and $q_{s_1} = q_{s_2} = 1$. The priority structure is as follows.

$$f_{s_1} : i_1 - i_2,$$
$$f_{s_2} : i_1 - i_2.$$

Example 6 (A priority structure that is Ergin-acyclic, strongly X-acyclic and acyclic, but is not Kumano-acyclic.)**.** The example is from Kumano (2013).

$I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, $q_{s_1} = q_{s_2} = 1$ and $q_{s_3} = 3$. The priority structure is as follows.

$$f_{s_1} : i_1 - i_2 - i_3,$$
$$f_{s_2} : i_1 - i_2 - i_3,$$
$$f_{s_3} : i_3 - i_1 - i_2.$$

Example 7 (A priority structure that is Ergin-acyclic and strongly X-acyclic, but is not Kumano-acyclic nor acyclic.)**.** $I = \{i_1, i_2, i_3, i_4, i_5\}$, $S = \{s_1, s_2\}$ and $q_{s_1} = q_{s_2} = 2$. The priority structure is as follows.

$$f_{s_1} : i_1 - i_2 - i_3 - i_4 - i_5,$$
$$f_{s_2} : i_1 - i_3 - i_2 - i_4 - i_5.$$

Example 8 (A priority structure that is Ergin-acyclic, Kumano-acyclic and acyclic, but is not strongly X-acyclic.)**.** The example is from Haeringer and Klijn (2008).

$I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$, $q_{s_1} = 1$ and $q_{s_2} = 1$. The priority structure is as follows.

$$f_{s_1} : i_1 - i_2,$$
$$f_{s_2} : i_2 - i_1.$$

Example 9 (A priority structure that is Ergin-acyclic and acyclic but is not strongly X-acyclic nor Kumano-acyclic.)**.** $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2\}$, $q_{s_1} = q_{s_2} = 1$ and $q_{s_3} = 3$. The priority structure is as follows.

$$f_{s_1} : i_1 - i_2 - i_3 - i_4,$$
$$f_{s_2} : i_1 - i_2 - i_3 - i_4,$$
$$f_{s_3} : i_1 - i_2 - i_4 - i_3.$$

Example 10 (A priority structure that is Ergin-acyclic but is not strongly X-acyclic nor Kumano-acyclic nor acyclic.)**.** The example is from Kumano (2013).

$I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2\}$, $q_{s_1} = 1$ and $q_{s_2} = 1$. The priority structure is as follows.

$$f_{s_1} : i_1 - i_2 - i_3,$$
$$f_{s_2} : i_1 - i_3 - i_2.$$

Chapter 2

Undominated Strategies and the Boston Mechanism

Chia-Ling Hsu[1]

Abstract: This paper studies the (weakly) undominated strategies in the widely-used Boston mechanism in the school choice problem. Two important components in characterizing the undominated strategies are the characterization of the most preferred guaranteed school and the characterization of the rank-preserving monotonic transformation of the most preferred guaranteed school. The characterization requires no information about other students' preferences. Finally, I provide analysis on undominated Nash equilibria. The findings suggest that in any undominated Nash equilibrium, it is unlikely that all students obtain their most preferred guaranteed schools, the worst possible school choices students could obtain if they use undominated strategies.

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Key words: Boston mechanism, Most preferred guaranteed school, Rank

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preserving monotonic transformation, γ-algorithm, School choice problem.

2.1 Introduction

Centralized matching mechanisms are frequently employed when it comes to allocating students to schools. The matching problem is formulated as the *student placement problem* by Blinski and Sönmez (1999) or the *school choice problem* by Abdulkadiroğlu and Sönmez (2003).[2] In this problem, students submit their preferences over schools and the option of not being matched to the schools participating in the program. Then, the mechanism computes the matching between students and schools by using students' submitted preferences and their priorities in schools.[3]

Since Abdulkadiroğlu and Sönmez (2003) introduce the school choice problem, there has been debates for what is the best mechanism for allocating students to schools. Primarily, the debates focus on three mechanisms: the Student Optimal Stable Mechanism (SOSM) proposed by Gale and Shapley (1962), the Top-Trading Cycles Mechanism (TTCM) proposed by Abdulkadiroğlu and Sönmez (2003), and the Boston mechanism introduced by Abdulkadiroğlu and Sönmez (2003). The SOSM and the TTCM have their own theoretic grounds for using them, while the Boston mechanism has been used in practice around the world. In these debates, a significant emphasis is on the Boston mechanism.

Before 2005, the Boston mechanism was used in Boston and was replaced by the SOSM. (See Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu et al. (2005), and Abdulkadiroğlu et al. (2006) for the description of the program and the transition.) Abdulkadiroğlu and Sönmez (2003) point out three important criteria to evaluate a matching mechanism: *stability*,

[2]The difference is that in student placement problem the priories of students are determined by their exam scores, while in school choice problem, the priorities may depend on other factors.

[3]In this paper, I use submitted preferences and strategies interchangeably.

efficiency, and *strategy-proofness*. They show that the SOSM is stable and strategy-proof, and the TTCM is efficient and strategy-proof. In addition, they show that the Boston mechanism does not satisfy any of these three criteria. Ergin and Sönmez (2006) show that despite the Boston mechanism is not stable, the Nash equilibrium outcomes are stable.[4] However, they show that there are usually multiple Nash equilibrium in the Boston mechanism, and the dominant-strategy outcome of the SOSM Pareto dominates all Nash equilibrium outcomes except the outcome that is identical to it. This result gives rationale for replacing the Boston mechanism with the SOSM from the welfare perspective.

In the transition in Boston, Abdulkadiroğlu et al. (2006) indicate the importance of considering incentive constraints as a design goal instead of merely a constraint. Pathak and Sönmez (2008) indicate the criterion, strategy-proofness, "levels the playing field." They show that in the Boston mechanism, the students who are not aware of the manipulation opportunity, i.e., the students who always submit the true preferences, may suffer welfare loss. Pathak and Sönmez (2013) show that there is a trend in the US and the UK that the mechanisms that are vulnerable to manipulation are replaced with mechanisms that are less vulnerable to manipulation.

In this paper, instead of showing whether the Boston mechanism is a better or worse mechanism, I study the strategic properties of the Boston mechanism. Since the Boston mechanism is a popular mechanism, its popularity suggests further investigation is worthwhile.

I characterize the undominated strategies and the weakly dominated strategies in the Boston mechanism. The characterization has two steps. In the first step, I define a critical school choice for each student, the *most preferred guaranteed school*. A *guaranteed school* for a student is defined to be a school such that he will never be matched to a school that is strictly worse than that school, if he submits the true preference. The most preferred

[4] Note that the definition of a stable mechanism says that a mechanism is stable if it produces a stable outcome if all participants submit the true preferences.

guaranteed school is the most preferred school among all the guaranteed schools.

In the second step, I define the *rank preserving monotonic transformation*. A strategy that is a rank preserving monotonic transformation of a school s if the ranking of school s in the strategy is the same as in the true preference, while all the schools that are preferred to school s according to the true preference have higher ranking than school s in the strategy.

I show that if a student's strategy satisfies (1) rank preserving monotonic transformation of the most preferred guaranteed school and (2) the condition that if this student is considered as the top choice[5] for the first school choice in this strategy, there does not exist another school that is preferred to the first school choice in the strategy and also considers him as the top choice, then it is an undominated strategy. On the other hand, if a school as the first school choice in a student's strategy does not consider her as the top choice and it is not a rank preserving monotonic transformation of most preferred guaranteed school, then the strategy is a weakly dominated strategy.

In addition, I show that it requires little information to identify whether a strategy is an undominated strategy. The information required is (1) whether a student is a top choice for her acceptable schools and (2) the total number of students. Note that the identification does not require the information of other students' preferences.

The characterization of the undominated strategies also have implications for equilibrium analysis. I show that in any Nash equilibrium, a student will not be matched to a school that is strictly worse than the most preferred guaranteed school. I further show that in any undominated Nash equilibrium, it is unlikely for all students to be matched with the most preferred guaranteed schools. To show this, I show that all students are matched to the most preferred guaranteed schools in each undominated Nash equilib-

[5] A school is said to consider a student as the top choice if the student's ranking in the school's priority is within its capacity.

rium if and only if the school choice problem is nested.[6] A nested school choice problem is a very restrictive requirement, which makes it difficult to satisfy in reality.

The rest of the paper is organized as follows. Section 2.2 presents the model. Section 2.3 contains the main results. Section 2.4 concludes. An example of the γ-algorithm is in Section 2.5. Omitted proofs in the main text are in Section 2.6.

2.2 Model

There are five components in a school choice problem (S, I, q, P, f). S is the set of schools and the outside option denoted as s_0.[7] I is the set of students. $q = (q_s)_{s \in S}$ is the vector of capacity for each school, where q_s is the maximum number of students that school $s \in S$ can admit. Assume $q_{s_0} \geq |I|$. $P = (P_i)_{i \in I}$ is the preference profile of students, where P_i is the strict preference of student $i \in I$ over S. A school s is acceptable for student i, if sP_is_0. $f = (f_s)_{s \in S}$ is the priority structure, where f_s denotes the priority rule of school $s \in S$ that is used to assign the priority of each student in school $s \in S \setminus \{s_0\}$, such that $f_s(i) \in \{1, ..., |I|\}$ and $f_s(i) \neq f_s(j)$ for any $i, j \in I$. We say student i has a higher priority than student j in a school $s \in S \setminus \{s_0\}$, if and only if $f_s(i) < f_s(j)$. For student i, let R_i be the weak preference relation associated with P_i, so that sR_is' if and only if sP_is' or $s = s'$. For $S' \subseteq S$, let $Ch_i(S')$ be the most preferred school in S' according to P_i.

Consider student i. Denote Q_i a generic strategy of i and $\mathcal{Q}_i$ the set of all strategies. Similarly, denote $\mathcal{Q}_{-i}$ be the set of all strategy profiles of

[6] This result holds under an assumption. As discussed in Section 2.3.5, if this assumption does not hold, then some students could obtain schools that are better than their most preferred guaranteed schools in undominated Nash equilibria.

[7] To reduce the amount of notation, I include s_0 in S. In the rest of the paper, when I refer some school $s \in S$, s could be some real school or the outside option.

student i's opponents. $Q = (Q_i, Q_{-i})$ is a strategy profile when i submits Q_i and all students other than i submit Q_{-i}.

A **matching** μ is a function $\mu : I \to S$ such that no schools in $S \setminus \{s_0\}$ admit more students than its capacity. Denote $\mu(i)$ as the school that is matched to student i under μ. A **stable matching** μ is a matching such that (1) $\mu(i) R_i s_0$ for all $i \in I$, (2) there does not exist two students i, j and a school s such that $s P_i \mu(i)$ and $f_s(i) < f_s(j)$ and (3) $s P_i \mu(i)$ implies $|\mu^{-1}(s)| = q_s$ for all $i \in I$. A matching μ **Pareto dominates** another matching ν if $\mu(i) R_i \nu(i)$ for all $i \in I$ and $\mu(i) P_i \nu(i)$ for some $i \in I$.

Let φ be the **Boston mechanism**, which is described as follows.

- In the first iteration, each school uses its priority rule and admits the students who consider it as the first choice up to the capacity. A student who is assigned to some school is inactive from the next iteration; otherwise she is active. Each school reduces its capacity by the number of students who are assigned to it from the next iteration.

- In the t-th iteration, each school uses its priority rule and admits the active students who consider it as the t-th choice up to the capacity A student who is assigned to some school is inactive from the next iteration; otherwise she is active. Each school reduces its capacity by the number of students who are assigned to it from the next iteration.

The algorithm terminates when all students are assigned to some real schools or the outside option.

Denote $\varphi(Q_i, Q_{-i})$ as the matching produced by the Boston mechanism with the strategy profile (Q_i, Q_{-i}). Denote $\varphi(Q_i, Q_{-i})(i)$ the assignment to student i with the strategy profile (Q_i, Q_{-i}).

Consider student i. We say that strategy Q_i **(weakly) dominates** Q'_i, if

$$\varphi(Q_i, Q_{-i})(i) R_i \varphi(Q'_i, Q_{-i})(i)$$

for all $Q_{-i} \in \mathcal{Q}_{-i}$, and

$$\varphi(Q_i, Q_{-i})(i) P_i \varphi(Q_i', Q_{-i})(i)$$

for some $Q_{-i} \in \mathcal{Q}_{-i}$. We say Q_i **very weakly dominates** Q_i', if

$$\varphi(Q_i, Q_{-i})(i) R_i \varphi(Q_i', Q_{-i})(i)$$

for all $Q_{-i} \in \mathcal{Q}_{-i}$. A strategy Q_i is a **(weakly) undominated strategy** if there does not exist another strategy that weakly dominates Q_i.

A strategy profile Q is a **Nash equilibrium** if

$$\varphi(Q_i, Q_{-i})(i) R_i \varphi(Q_i, Q_{-i})(i), \forall i \in I.$$

A strategy profile Q is a **undominated Nash equilibrium** if no student uses dominated strategy in Q.

The following notation is frequently used in the rest of the paper. Let $P_i(s)$ as the ranking of s in P_i and let $Q_i(s)$ as the ranking of s in Q_i. Let $P_i^{-1}(t)$ be the t-th ranked school choice in P_i and let $Q_i^{-1}(t)$ be the t-th ranked school choice in Q_i, for $t = 1, ..., |S|$. Let $U^{P_i}(s) = \{s' \in S : s' R_i s\}$. Similarly, let $U^{Q_i}(s) = \{s' \in S : Q_i(s') \leq Q_i(s)\}$. In addition, let $V^{P_i}(s) = \{s' \in S : s' P_i s\}$ and let $V^{Q_i}(s) = \{s' \in S : Q_i(s') < Q_i(s)\}$. In other words, $U^{P_i}(s) = V^{P_i}(s) \cup \{s\}$ and $U^{Q_i}(s) = V^{Q_i} \cup \{s\}$.

2.3 Results

2.3.1 Two Strategic Properties of the Boston Mechanism

The following two lemmas are about the properties of the Boston mechanism.

Lemma 8. *Consider a student i and two of her strategies $\overline{Q}_i$ and $\widetilde{Q}_i$. Fix a strategy profile of her opponents Q_{-i}. Suppose $\varphi(\overline{Q}_i, Q_{-i})(i) = \widetilde{s}$. Denote $\widehat{S} = \{s \in S : \overline{Q}_i(s) < \overline{Q}_i(\widetilde{s})$ and $\overline{Q}_i(s) = \widetilde{Q}_i(s)\}$. Then $\varphi(\widetilde{Q}_i, Q_{-i})(i) \notin \widehat{S}$.*

Consider a strategy Q_i such that $(Q_i, Q_{-i})(i) = s$ for some Q_{-i}. Suppose $Q_i(s) = k$. If we replace Q_i with another strategy Q'_i, then under (Q'_i, Q_{-i}), student i will not be matched to any school s' such that it is ranked higher than the k-th position and it has the same ranking in both Q_i and Q'_i. For example, consider a strategy Q_i:

$$Q_i : s_1, s_2, s_3, s_4, s_5, s_6, s_0.$$

Suppose $\varphi(Q_i, Q_{-i})(i) = s_5$. Consider another strategy Q'_i:

$$Q'_i : s_6, s_2, s_3, s_5, s_4, s_1, s_0.$$

Then $\varphi(Q'_i, Q_{-i})(i) \notin \{s_2, s_3\}$.

Lemma 9. *Consider a student i and two of her strategies $\overline{Q}_i$ and $\widetilde{Q}_i$. Fix a strategy profile of her opponents Q_{-i}. Suppose $\varphi(\overline{Q}_i, Q_{-i})(i) = \widehat{s}$. If $\widetilde{Q}_i(\widehat{s}) \leq \overline{Q}_i(\widehat{s})$ and $\varphi(\widetilde{Q}_i, Q_{-i})(i) \notin U^{\widetilde{Q}_i}(\widehat{s}) \setminus \{\widehat{s}\}$, then we have $\varphi(\widetilde{Q}_i, Q_{-i})(i) = \widetilde{s}$.*

Consider a strategy Q_i such that $\varphi(Q_i, Q_{-i})(i) = s$. Suppose $Q_i(s) = k$. Next, consider another strategy Q'_i such that $Q'_i(s) = k'$ with $k' \leq k$. Then, under (Q'_i, Q_{-i}), if student i is not matched with any school that is ranked higher than s in Q'_i, she will be matched with s. For example, consider a strategy Q_i:

$$Q_i : s_1, s_2, s_3, s_4, s_5, s_6, s_0.$$

Suppose $\varphi(Q_i, Q_{-i})(i) = s_5$. Consider another strategy Q'_i:

$$Q'_i : s_2, s_6, s_5, s_4, s_3, s_1, s_0.$$

If $\varphi(Q'_i, Q_{-i})(i) \notin \{s_2, s_6\}$, then $\varphi(Q_i, Q_{-i})(i) = s_5$.

Lemma 8 and Lemma 9 are crucial for proving the γ-algorithm produces a strategy that very weakly dominates another strategy as the input.[8]

[8] This refers to the proofs in Theorem 13, Theorem 14 and Theorem 15.

2.3.2 The Most Preferred Guaranteed School

In this section, we define and characterize a critical school, the most preferred guaranteed school. First, we define a guaranteed school.

Definition 6. A school s is a **guaranteed school** for student i if

$$\varphi(P_i, Q_{-i})(i) R_i s, \qquad \forall Q_{-i} \in \mathcal{Q}_{-i}.$$

In other words, if a school s is a guaranteed school for student i, then i would obtain a school that is weakly preferred to school s no matter what other students' submitted preferences, as long as i submit the true preference P_i.

Clearly, there are more than one guaranteed school in general. A school is the **most preferred guaranteed school** if it is the most preferred school among all guaranteed schools. For the rest of the paper, I will use $s^*(i)$ to denote the most preferred guaranteed school. The following theorem characterizes this critical school choice.

Theorem 11. *The most preferred guaranteed school $s^*(i)$ can be characterized in the following way.*

$$s^*(i) = \begin{cases} P_i(1) & \text{if } f_{P_i^{-1}(1)}(i) \leq q_{P_i^{-1}(1)}; \\ Ch_i(\{s \in S : \sum_{s' \in U^{P_i}(s)} q_{s'} \leq |I| - 1\}) & \text{otherwise.} \end{cases}$$

Note that to characterize a student i's most preferred guaranteed school, it only requires the information of whether she is the top choice for her most preferred school, the capacity of each school, and the total number of students. It does not require the information of other students' preferences.

Next, consider a special case where the number of students is equal to the total capacities of all real schools and all real schools are acceptable. The characterization of the most preferred guaranteed school can be simplified as in the following theorem.

Theorem 12. *Suppose* $\sum_{s\in S\setminus\{s_0\}} q_s = |I|$ *and* sR_js_0 *for all* $s \in S$ *and for all* $j \in I$. *Consider student* i. *Let the least preferred school choice other than* s_0 *to be* $\widehat{s}$. *Then*

$$s^*(i) = \begin{cases} P_i(1) & \text{if } f_{P_i^{-1}(1)}(i) \leq q_{P_i^{-1}(1)}; \\ \widehat{s} & \text{otherwise.} \end{cases}$$

With the setting considered in Theorem 12, the characterization of a student i's most preferred guaranteed school only requires the information of whether he is considered as the top choice for his most preferred school and the knowledge of his worst acceptable school.

2.3.3 Rank Preserving Monotonic Transformation

In this section, we consider a class of strategies, the strategies that satisfy the rank-preserving monotonic transformation of the most preferred guaranteed school. First, we define the strategy that is a rank-preserving monotonic transformation of a school s.

Definition 7. Given a strategy Q_i, we say that Q_i satisfies the **rank preserving monotonic transformation of** s if

$$(\forall s' \in S \setminus \{s\}) \qquad Q_i(s') < Q_i(s) \Leftrightarrow P_i(s') < P_i(s). \tag{2.1}$$

If a strategy Q_i satisfies equation (2.1), then the rankings of s in Q_i and P_i are the same and sets of the school choices that are ranked higher than s in Q_i and P_i are the same, i.e., $Q_i(s) = P_i(s)$ and $U^{Q_i}(s) = U^{P_i}(s)$.

In the following analysis, we focus on the strategies that are the rank preserving monotonic transformation of the most preferred guaranteed school. In other words, we focus on the strategy Q_i such that the following holds.

$$(\forall s \in S \setminus \{s^*(i)\}) \qquad Q_i(s) < Q_i(s^*(i)) \Leftrightarrow P_i(s) < P_i(s^*(i)). \tag{2.2}$$

Lemma 10. *Suppose Q_i makes satisfies rank-preserving monotonic transformation of the most preferred guaranteed school, then $\varphi(Q_i, Q_{-i})(i) R_i s^*(i)$ for all $Q_{-i} \in \mathcal{Q}_{-i}$.*

Note that Lemma 10 does not hold, if we replace $s^*(i)$ with an arbitrary school s in the statement.

2.3.4 Undominated Strategies and Weakly Dominated Strategies

In this section, we present the main results.

Theorem 13. *Consider student i and one of his strategies Q_i. If Q_i is not a rank-preserving monotonic transformation of the most preferred guaranteed school, then there exists another strategy Q_i' that very weakly dominates Q_i.*

To prove Theorem 13, I develop the following algorithm, the γ-algorithm. For a strategy Q_i, the γ-algorithm creates a strategy that very weakly dominates Q_i. The algorithm is the following.

γ-**algorithm**: Consider some strategy Q_i. Let the most preferred guaranteed school be $s^*(i)$. Denote the strategy created by this algorithm to be Q_i'. This algorithm specifies a school choice to Q_i' step by step.

- For $r = 1$,
  - Let $A(r) = U^{P_i}(s^*(i))$.
  - Let
  $$Q_i'(r) = \begin{cases} Q_i(r), & \text{if } Q_i(r) \in A(r); \\ Ch_i(A(r)), & \text{otherwise.} \end{cases}$$
- In general, for $r = 2, ..., P(s^*(i))$,
  - Let $A(r) = U^{P_i}(s^*(i)) \setminus U^{Q_i'}(Q_i'^{-1}(r-1))$.

– Let

$$Q'_i(r) = \begin{cases} Q_i(r), & \text{if } Q_i(r) \in A(r); \\ Ch_i(A(r)), & \text{otherwise.} \end{cases}$$

- In general, for $r = P(s^*(i)) + 1, ..., |S|$,
  - Let $A(r) = S \setminus U^{Q'_i}(Q'^{-1}_i(r-1))$.
  - Let $Q'_i(r) = Ch_i(A(r))$.

Note that if Q_i satisfies equation (2.2), the γ-algorithm creates a strategy that is identical to Q_i. An example of how the γ-algorithm works is in Section 2.5.

The next theorem characterizes the undominated strategies.

Theorem 14. *Consider student i and one of his strategies Q_i. Suppose the first school choice in Q_i is $\bar{\bar{s}}$. If Q_i satisfies both of the following two conditions, then Q_i is an undominated strategies.*

1. *It is either (1) $f_{\bar{\bar{s}}}(i) > q_{\bar{\bar{s}}}$ or (2) $f_{\bar{\bar{s}}}(i) \leq q_{\bar{\bar{s}}}$ and there does not exist another school s such that $sP_s\bar{\bar{s}}$ and $f_s \leq q_s$.*

2. *Q_i is a rank-preservation monotonic transformation of the most preferred guaranteed school.*

Suppose a student is considered as the top choice for the first school choice in a strategy, we have the following corollary.

Corollary 9. *Consider a strategy Q_i. Let the first school choice in Q_i be s_1. Suppose $f_{s_1}(i) \leq q_{s_1}$. If one of the following conditions holds: (1) $s_1 P_i s^*(i)$ and there does not exists another school s such that $sP_i s_1$ and $f_s(i) \leq q_s$ or (2) $s_1 = s^*(i)$, then Q_i is an undominated strategy.*

The above corollary says that if the first school choice in a strategy satisfies the condition in the statement, then it is an undominated strategy no matter how other school choices are ranked in the strategy.[9]

[9] The reasoning of this corollary is the following. Suppose condition (1) in the statement holds. Consider another strategy Q'_i such that its first school choice is s_1 and it satisfies

The next theorem characterizes the weakly dominated strategies.

Theorem 15. *Consider student i and one of his strategies Q_i. Suppose the first school choice in Q_i is $\bar{\bar{s}}$. If Q_i satisfies one of the following two conditions, then Q_i is a weakly dominated strategies.*

1. *$f_{\bar{\bar{s}}}(i) \leq q_{\bar{\bar{s}}}$ and there exists another school s such that $sP_is^*(i)$, $sP_i\bar{\bar{s}}$ and $f_s(i) \leq q_s$.*

2. *$f_{\bar{\bar{s}}}(i) > q_{\bar{\bar{s}}}$ and Q_i is not a rank-preserving monotonic transformation of the most preferred guaranteed school.*

2.3.5 Equilibrium Analysis

In this section, we provide some equilibrium analysis. The following theorem says that the most preferred guaranteed school provides a lower bound for the school that a student may obtain in any Nash equilibrium.

Theorem 16. *Consider student i. In any Nash equilibrium, she would not be matched with a school that is worse than $s^*(i)$.*

The implication is that we should not expect a student obtain a school that is worse than her most preferred guaranteed school in Nash equilibria. The following analysis shows that if we consider undominated Nash equilibria instead of Nash equilibria, then the implication would be even stronger.

It is easy to see that a student might obtain a school that is better than his most preferred guaranteed school if he uses an undominated strategy if the following condition holds. Suppose this student is a top choice for a school s that is better than his most preferred guaranteed school and there

equation (2.2), then by Theorem 14, Q'_i is an undominated strategy. Since $\varphi(Q_i, Q_{-i})(i) = \varphi(Q'_i, Q_{-i})(i) = s_1$ for all $Q_{-i} \in \mathcal{Q}_{-i}$, Q_i is also an undominated strategy. If condition (2) in the statement holds, by Theorem 14, it is straightforward that it is an undominated strategy.

is no other school that is preferred to this school and also considers him as the top choice.[10] Clearly, if this student submits a strategy that ranks s as the first school choice, then he will obtain s no matter how other students submit their preferences. To rule out this situation, we impose the following assumption, Assumption 1, for the rest of the analysis in this section.[11]

Assumption 1. For every student i, one of the following conditions holds: (1) $f_{P_i^{-1}(1)}(i) \leq q_{P_i^{-1}(1)}$ or (2) $f_s(i) > q_s$ for sP_is_0.

I show that under Assumption 1, for all students to obtain their most preferred guaranteed schools in undominated Nash equilibria, a very restrictive requirement, the nested school choice problem, must be satisfied. In other words, such requirement is difficult to be satisfied in reality. Since by Theorem 16, students cannot obtain schools that are worse than their most preferred guaranteed schools in any Nash equilibrium, this suggests in reality a fraction of students would obtain schools that are preferred to their most preferred guaranteed schools in undominated Nash equilibria.

The definition of the nested school choice is as follows.

Definition 8. A school choice problem is **nested**, if for any student i, the following holds. Let s_t be $P_i^{-1}(t)$ for $t = 1, ..., |S|$. There exists q_t number of students (other than i) who consider s_t as their most preferred guaranteed schools, for $t = 1, ..., P_i(s^*(i)) - 1$.

For a school choice problem to be nested, it requires that there is a precise number of students who consider a particular school as their most preferred guaranteed school. Therefore, the requirement is very restrictive.

The following theorem states some property of a nested school choice problem.

Theorem 17. *Suppose the school choice problem is nested, then there is an unique undominated Nash equilibrium outcome, but not vice versa.*

[10] Note that this implies that s is not his most preferred guaranteed school.

[11] This assumption is used in Theorem 17 and Theorem 18.

The next theorem says that all students obtain their most preferred guaranteed schools if and only if the school choice problem is nested.

Theorem 18. *The following two statements are equivalent.*

1. *The school choice problem is nested.*

2. *All students are matched with their most preferred guaranteed schools in every undominated Nash equilibrium.*

2.3.6 A General Characterization of the Most Preferred Guaranteed School

In this section, we give a general characterization of the most preferred guaranteed school. The characterization of the most preferred guaranteed school in Section 2.3.2 is based on an assumption that there is no restriction on the preferences that students can submit. In other words, if $|S| = n$, then each student has $n!$ ways of submitting his preference in the original model. In reality, this might not be the case. One example is that a student can declare a school as acceptable only if he has some qualification for that school. Another example is that there is a geographical restriction on students' submitted preferences, i.e., students can only declare schools that are close to their residence as acceptable schools. In both examples, students cannot declare arbitrary set of schools as the set of acceptable schools in their submitted preferences.

Let a **restriction** $\mathcal{W}$ be a subset of $\mathcal{Q}$. In addition, let $\mathcal{W}_i = \mathcal{Q}_i \cap \mathcal{W}$ and $\mathcal{W}_{-i} = \mathcal{Q}_{-i} \cap \mathcal{W}$. Formally, a **school choice problem with restriction** $\mathcal{W}$ is a school choice problem such that the submitted preference profile can only be in $\mathcal{W}$.

We impose an additional assumption for the following analysis. For every student i, we have $P_i \in \mathcal{W}_i$.[12]

[12]One can read this interpretation in the following way. Consider student i and his

Given Q_{-i}, define the set of student i's **actual competitors to school** s **for** Q_i to be

$$I^{Q_i}(s|Q_{-i}) = \{j \in I : \varphi(Q_i, Q_{-i})(i) = s\}.$$

Theorem 19. *Consider student i. The most preferred guaranteed school for student i is*

$$s^*(i) = Ch_i\Big[\Big(\max_{Q_{-i}\in\mathcal{Q}_{-i}}\Big|\bigcup_{s'\in U^{P_i}(s)} I^{P_i}(s'|Q_{-i})\Big|\Big) < \sum_{s'\in U^{P_i}(s)} q_{s'}\Big].$$

2.4 Conclusion

In this paper, I study the undominated strategies and the weakly dominated strategies in the Boston mechanism. Two important components in this characterization are the characterization of the most preferred guaranteed school and the characterization of the rank-preserving monotonic transformation of the most preferred guaranteed school. I further show that the characterization of the undominated strategies and the weakly dominated strategies requires little information. In particular, it does not require the information of other students' preferences in order to identify whether a strategy is an undomianted strategy for a student. Finally, I provide equilibrium analysis and show that it is unlikely that all students obtain their most preferred guaranteed schools in the undominated Nash equilibria. In other words, a fraction of students would obtain schools that are strictly preferred to their most preferred guaranteed schools in the undominated Nash equilibria.

preference P_i. Remove all schools such that student i is not qualified for or not close to from the set of acceptable schools. Call the set of remained acceptable schools as $\bar{S}$. Create an updated preference $\bar{P}_i$ so that the relative ordering over schools in $\bar{S}$ is the same as in P_i. Then we treat $\bar{P}_i$ as if it is the true preference.

2.5 An Example of the γ-Algorithm

There are 11 schools. Consider student i and her strategy Q_i.

$$P_i : s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s^*(i), s_9, s_{10};$$
$$Q_i : s_7, s_9, s_2, s_1, s_{10}, s_6, s_8, s_4, s^*(i), s_3, s_5.$$

The iterations of the γ-algorithm with Q_i as the input are as follows.

- $r = 1$,
  - $A(1) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s^*(i)\}$.
  - Since $Q_i^{-1}(1) \in A(1)$, we have $Q_i'^{-1}(1) = Q_i^{-1}(1) = s_7$.
- $r = 2$,
  - $A(2) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_8, s^*(i)\}$.
  - Since $Q_i^{-1}(1) \notin A(2)$, we have $Q_i'^{-1}(2) = Ch_i(A(2)) = s_1$.
- $r = 3$,
  - $A(3) = \{s_2, s_3, s_4, s_5, s_6, s_8, s^*(i)\}$.
  - Since $Q_i^{-1}(1) \in A(3)$, we have $Q_i'^{-1}(3) = Q_i^{-1}(3) = s_2$.
- $r = 4$,
  - $A(4) = \{s_3, s_4, s_5, s_6, s_8, s^*(i)\}$.
  - Since $Q_i^{-1}(4) \notin A(4)$, we have $Q_i'^{-1}(4) = Ch_i(A(4)) = s_3$.
- $r = 5$,
  - $A(5) = \{s_4, s_5, s_6, s_8, s^*(i)\}$.
  - Since $Q_i^{-1}(5) \notin A(5)$, we have $Q_i'^{-1}(5) = Ch_i(A(5)) = s_4$.

- $r = 6$,
  - $A(6) = \{s_5, s_6, s_8, s^*(i)\}$.
  - Since $Q_i^{-1}(6) \in A(6)$, we have $Q_i'^{-1}(6) = Q_i^{-1}(6) = s_6$.
- $r = 7$,
  - $A(7) = \{s_5, s_8, s^*(i)\}$.
  - Since $Q_i^{-1}(7) \in A(7)$, we have $Q_i'^{-1}(7) = Q_i^{-1}(7) = s_8$.
- $r = 8$,
  - $A(8) = \{s_5, s^*(i)\}$.
  - Since $Q_i^{-1}(8) \notin A(8)$, we have $Q_i'^{-1}(8) = Ch_i(A(8)) = s_5$.
- $r = 9$,
  - $A(9) = \{s^*(i)\}$.
  - Since $Q_i^{-1}(9) \in A(9)$, we have $Q_i'^{-1}(9) = Q_i^{-1}(9) = s^*(i)$.

Therefore, we have Q_i'. Strategy Q_i' and the original strategy Q_i are as follows.

$$Q_i = s_7, s_9, s_2, s_1, s_{10}, s_6, s_8, s_4, s^*(i), s_3, s_5;$$
$$Q_i' = s_7, s_1, s_2, s_3, s_4, s_6, s_8, s_5, s^*(i), s_9, s_{10}.$$

2.6 Omitted Proofs in the Main Text

2.6.1 Proof of Lemma 8

Choose an arbitrary school $\widehat{s} \in \widehat{S}$. Assume $\overline{Q}_i(\widehat{s}) = k$.

Consider the strategy profile $(\overline{Q}_i, Q_{-i})$. The event that $\varphi(\overline{Q}_i, Q_{-i})(i) \neq \widehat{s}$ means under $(\overline{Q}_i, Q_{-i})$ either (1) before iteration k, the capacity of $\widehat{s}$ is full, or (2) when iteration k begins, there are $\widetilde{q}_{\widehat{s}}$ capacity remained, with $\widetilde{q}_{\widehat{s}} \leq q_{\widehat{s}}$, and the number of students such that (i) she ranks $\widehat{s}$ as the k-th choice, and that (ii) she is not matched with any school when iteration k begins, and that (iii) she has a higher priority than i for $\widehat{s}$, is greater than or equal to $\widetilde{q}_{\widehat{s}}$.

Now, consider the strategy profile $(\widetilde{Q}_i, Q_{-i})$. Suppose i is not matched with $\varphi(\widetilde{Q}_i, Q_{-i})(i)$ in the first $k-1$ iterations. Then one of condition (1) and condition (2) from above still holds.

Suppose i is matched with $\varphi(\widetilde{Q}_i, Q_{-i})(i)$ in the first $k-1$ iterations. Since i ranks $\widehat{s}$ as the k-th choice, i cannot be matched with $\widehat{s}$ under $(\widetilde{Q}_i, Q_{-i})$. Therefore, $\varphi(\widetilde{Q}_i, Q_{-i})(i) \neq \widehat{s}$. Since the above argument holds for all $\widehat{s} \in \widehat{S}$, we have $\varphi(\widetilde{Q}_i, Q_{-i})(i) \notin \widehat{S}$.

2.6.2 Proof of Lemma 9

Suppose $\overline{Q}_i(s) = t$ and $\widetilde{Q}'_i(s) = t'$, with $t' \leq t$.

Consider the strategy profile $(\overline{Q}_i, Q_{-i})$. The event that $\varphi(\overline{Q}_i, Q_{-i})(i) = \widehat{s}$ means that (1) i is not matched with any school in the first $t-1$ iterations, and that (2) suppose when iteration t begins, the remaining capacity for s is $\tilde{q}_s$, the number of other students such that (i) she ranks $\widehat{s}$ as the t-th choice, and that (ii) she is not matched with any school in the first $t-1$ iterations, and that (iii) she has a higher priority than i for $\widehat{s}$, is less than $\widetilde{q}_{\widehat{s}}$.

Now, consider the strategy profile $(\widetilde{Q}_i, Q_{-i})$. If i is not matched with any school in the first $t'-1$ iterations, then condition (2) from above still holds if t is replaced with t' in the argument. Call this new condition as condition (3).

The event that i is not matched with any school in the first $t'-1$ iterations and condition (3) make it sufficient for i to be matched with $\widehat{s}$ under $(\widetilde{Q}_i, Q_{-i})$.

2.6.3 Proof of Theorem 11

It is easy to see the statement holds when $f_{P_i(1)}(i) \leq q_{s_{P_i(1)}}$. Suppose $f_{P_i(1)}(i) > q_{s_{P_i(1)}}$. Let $P_i(1) = \widehat{s}$. Let $\widehat{I}$ be the set of students whose priority in $f_{\widehat{s}}$ is smaller than or equal to $q_{\widehat{s}}$, i.e., $\widehat{I} = \{j \in I : |k \in I : f_{\widehat{s}}(k) < f_{\widehat{s}}(j)| < q_{\widehat{s}}\}$. Let $I' = I \setminus (\widehat{I} \cup \{i\})$. For a school $s_t \in S$, call t as the index of school s_t. Re-index the schools in $S \setminus \{\widehat{s}, s_0\}$, so that (1) no two schools has the same index and (2) any index is smaller than or equal to $|S|-2$ and is larger than zero. Pick q_{s_1} number of students from I' and call them the set of students I_1. Pick q_{s_2} number of students from $I' \setminus I_1$ and call them the set of students I_2. In general, pick q_{s_t} number of students from $I' \setminus \bigcup_{t'=1,\ldots,t-1} I_{t'}$ and call them the set of students I_t. If there are any remaining students, call them the set of such students I_0. Let $j \in \widehat{I}$ submit $\widehat{s}$ as the first school choice in Q_{-i}. Let $k \in I_t$ submit s_t as the first school choice in Q_{-i}. For students in I_0, let they submit some arbitrary preferences. It is clear that $\varphi(P_i, Q_{-i})(i) = Ch_i(s \in S : \sum_{s' \in U^{P_i(s)}} q_{s'} \leq |I|-1)$. Since this is the worst school choice that i can obtain when he submits P_i, this is the most preferred guaranteed school.

2.6.4 Proof of Theorem 12

Consider student i. Let $\widehat{s}$ be her least preferred acceptable school. With the setting considered in the statement, $Ch_i(\{s \in S : \sum_{s' \in U^{P_i(s)}} q_{s'} \leq |I|-1\}) = \widehat{s}$. Therefore, Theorem 12 is a direct result of Theorem 11.

2.6.5 Proof of Lemma 10

Since Q_i satisfies equation (2.2), the total capacities in the set of schools $U^{Q_i}(s^*(i))$ is the same as the total capacities in the set of schools $U^{P_i}(s^*(i))$. By the definition of $s^*(i)$, $\varphi(P_i, Q_{-i})(i) R_i s^*(i)$ for all Q_{-i}, it means that student i takes one of the seat in the total seats in the set of schools $U^{Q_i}(s^*(i))$ no matter what strategies other student use.

Since $\sum_{s \in U^{Q_i}(s^*(i))} q_s = \sum_{s \in U^{P_i}(s^*(i))} q_s$, then i must have a seat in the seats of the set of schools $U^{Q_i}(s^*(i))$ by the algorithm of the Boston mechanism.

2.6.6 Proof of Theorem 13

Claims 1 and 2

Given Q_i. Let Q'_i be the strategy that is created by the γ-algorithm. The following two claims state the properties of Q'_i. Both are direct consequences of the γ-algorithm.

Claim 1. *Q'_i satisfies equation* (2.2)*.*

Claim 2. *Compare Q_i and Q'_i. Let $\widetilde{S} = \{s \in S : s \in U^{P_i}(s^*(i))$ and $Q'_i(s) \neq Q_i(s)\}$. First, for $s \in \widetilde{S}$, we have $Q'_i(s) < Q_i(s)$. Second, for $s \in \widetilde{S}$ and $s' \in U^{Q'_i}(s^*(i))$, we have $Q'_i(s) < Q'_i(s')$ implies sP_is'. Third, for $s, s' \in \widetilde{S}$, we have $Q'_i(s) < Q'_i(s')$ if and only if sP_is'.*

The Proof

Consider a strategy Q_i.

Let Q'_i be the strategy produced by the γ-algorithm. The proof is to analyze each possible Q_{-i} and show that $\varphi(Q'_i, Q_{-i})(i) R_i \varphi(Q_i, Q_{-i})(i)$.

Case 1: Consider Q_{-i} such that $\varphi(Q_i, Q_{-i})(i) \in S \setminus U^{P_i}(s^*(i))$. By Claim 1 and Lemma 10, we have $\varphi(Q'_i, Q_{-i})(i) \in U^{P_i}(s^*(i))$. Therefore, $\varphi(Q'_i, Q_{-i})(i) P_i \varphi(Q_i, Q_{-i})(i)$.

Case 2: Consider Q_{-i} such that $\varphi(Q_i, Q_{-i})(i) := s \in U^{P_i}(s^*(i))$. Note that if $Q'_i(s) \neq Q_i(s)$, by Claim 2, we have $Q'_i(s) < Q_i(s)$. In other words, $Q'_i(s) \leq Q_i(s)$. Let $\varphi(Q'_i, Q_{-i})(i) := s'$. By Lemma 9, $s' \in U^{Q'_i}(s)$.

Let $\widehat{S} = \{\widehat{s} \in S : Q'_i(\widehat{s}) < Q'_i(s) \text{ and } Q_i(\widehat{s}) \neq Q'_i(\widehat{s})\}$. Claim 2 implies that for $s' \in \widehat{S}$, we have $s' P_i s$. Let $\widetilde{S} = \{\widetilde{s} \in S : Q'_i(\widetilde{s}) < Q'_i(s) \text{ and } Q_i(\widetilde{s}) = Q'_i(\widetilde{s})\}$. By Lemma 8, $s' \notin \widetilde{S}$. By Claim 2, if $s' \in \widehat{S}$, then $s' P_i s$. Therefore, if $s' \neq s$, then $s' \in \widehat{S}$. In other words, $\varphi(Q'_i, Q_{-i})(i) R_i \varphi(Q_i, Q_{-i})(i)$.

This completes the proof of Theorem 13.

Proof of Theorem 14

Lemmas 11 and 12

Before proving Theorem 14, we need the following two lemmas.

We say that a school s is **achievable by strategy** Q_i, if there exists Q_{-i} such that $\varphi(Q_i, Q_{-i})(i) = s$.

Lemma 11. *Consider Q_i. If s is achievable by Q_i, then schools in $U^{Q_i}(s)$ are achievable by Q_i.*

Proof. Suppose Q_{-i} is such that $\varphi(Q_i, Q_{-i})(i) = s$. Consider a school $s' \in V^{Q_i}(s)$. Then another Q'_{-i} can be constructed so that all students other than i submit their preferences in the same way as in Q_{-i} except they make s' as an unacceptable school in their submitted preferences. Then $\varphi(Q_i, Q'_{-i})(i) = s'$. □

Lemma 12. *Consider Q_i. Suppose there is a strategy Q'_i that weakly dominates Q_i. Then if there is a school s that is achievable by Q'_i and $Q_i(s) \neq Q'_i(s)$, we have*

$$Q'_i(s) < Q_i(s) \Rightarrow s P_i Q_i^{-1}(Q'_i(s)). \tag{2.3}$$

Proof. Let $\widehat{S}$ be the set of schools such that schools in this set have different rankings in Q_i and Q'_i, and that they are achievable by Q'_i. Suppose school

$s \in \widehat{S}$ is the school that has the highest ranking in Q'_i among all schools in $\widehat{S}$.

Suppose equation (2.3) does not hold. Let $Q_i^{-1}(Q'_i(s)) := s'$. The school s' has the following properties: s' is the school with the highest ranking in Q_i that has different rankings in Q_i and Q'_i. Since if it does not hold, then there exists another school s'' that has a higher ranking than s' in Q_i, and s'' has different ranking in Q_i and Q'_i. This implies $Q_i'^{-1}(Q_i(s'')) := s''' \neq s''$. By this construction, $Q'_i(s''') < Q_i(s)$. By Lemma 11, since s is achievable by Q'_i, then s''' is achievable by Q'_i. However, this violates the construction that s has the highest ranking in Q'_i among all other schools in $\widehat{S}$.

Since $s \neq s'$, this implies that $Q'_i(s) < Q_i(s)$. The statement that equation (2.3) does not hold implies that $s' P_i s$.

Since all school that have higher rankings than s in Q_i have the same ranking in Q_i and Q'_i, the assumption that Q'_i weakly dominates Q_i implies that there exists Q_{-i} such that $\varphi(Q_i, Q_{-i})(i)$ and $\varphi(Q'_i, Q_{-i})(i)$ do not belong to those schools. Denote $\overline{S} = \{\overline{s} \in S : Q_i(\overline{s}) < Q_i(s)\}$. Construct Q'_{-i} by transforming Q_{-i} in the following way. Let all students who are matched with schools in $\overline{S}$ submit the schools they are matched with under (Q_i, Q_{-i}) as the first school choice. All other students rank s and s' as unacceptable. Therefore, $\varphi(Q_i, Q'_{-i})(i) = s'$ and $\varphi(Q'_i, Q'_{-i})(i) = s$. Therefore, the statement that Q'_i weakly dominates Q_i does not hold. $\square$

The Proof

Let Q_i be a strategy such that make equation (2.2) hold. Suppose there is another strategy Q'_i that weakly dominates Q_i.

There are two cases to consider.

Case 1: $Q_i^{-1}(1)$ is such that $f_{Q_i^{-1}(1)}(i) > q_{Q_i^{-1}(1)}$.

Let s be the lowest ranked achievable school by Q'_i.

Suppose $Q'_i(s) > Q_i(s^*(i))$. Then there exists some school $s' \in U^{Q'_i}(s)$

such that $s^*(i)P_is'$. By Lemma 11, s' is an achievable school. Therefore, Q_i' cannot weakly dominate Q_i.

Suppose $Q_i'(s) = Q_i(s^*(i))$. If there is a school $s' \in U^{Q_i'}(s)$ such that $s^*(i)P_is'$. Then Q_i' cannot weakly dominate Q_i by Lemma 11. If $U^{Q_i'}(s) = U^{Q_i}(s^*(i))$, it is impossible for equation (**??**) to hold. Therefore, Q_i' cannot weakly dominate Q_i.

Suppose $Q_i'(s) < Q_i(s^*(i))$. Further, suppose there is a school $s' \in U^{Q_i'}(s)$ such that $\widehat{s}P_is'$ for some $\widehat{s} \in U^{Q_i}(Q_i^{-1}(Q_i(s)))$. Then by Lemma 12, Q_i' cannot weakly dominate Q_i. Therefore, for $s' \in U^{Q_i'}(s)$ and $\widehat{s} \in U^{Q_i}(Q_i^{-1}(Q_i(s)))$, we must have $s'R_i\widehat{s}$. This implies that $s^*(i) \notin U^{Q_i}(s)$. Note that $s \neq P_i^{-1}(1)$, since otherwise $P_i^{-1}(1)$ would be the most preferred guaranteed school, and there cannot exist any other strategy that weakly dominate Q_i. Construct Q_{-i} in the following way. Denote $s_t = Q_i(t)$ for $t = 1, ..., |S|$. Pick q_{s_1} number of students who have higher priority than i in school s_1. Call this set of student I_1. Then, pick q_{s_2} number of students from $I \setminus I_1$ and call them I_2. Then, pick q_{s_3} number of students from $I \setminus \cup_{t=1,2} I_t$ and call them I_3. Proceed this assignment until all students in $I \setminus \{i\}$ belong to some group I_t. For a student j in I_t, let her submit s_t as the first school choice. The definition of the most preferred school $s^*(i)$ implies that there are at least $\sum_{l=1}^{P_i(s^*(i))} q_{s_l - 1}$ number of students. Denote $\widehat{S} = \{s \in S : sR_is^*(i)\}$. The above construction implies that $U^{Q_i'}(s) \subseteq \widehat{S} \setminus \{s^*(i)\}$. Since $\sum_{s'' \in \widehat{S}} q_{s''} > \sum_{s'' \in \widetilde{S}} q_{s''}$ for any $\widetilde{S} \subset \widehat{S}$, the construction of Q_{-i} implies $\varphi(Q_i', Q_{-i})(i) \notin U^{Q_i}(s)$. This contradicts s is the least preferred achievable school by Q_i'.

Case 2: $Q_i^{-1}(1)$ is such that $f_{Q_i^{-1}(1)}(i) \leq q_{Q_i^{-1}(1)}$.

Let $Q_i^{-1}(1) = s_1$ and $Q_i'^{-1}(1) = s_1'$. Note that in this case, we have $\varphi(Q_i, Q_{-i})(i) = s_1$ for all $Q_{-i} \in \mathcal{Q}_{-i}$. If $f_{s_1'}(i) \leq q_{s_1'}$, then by condition (2) in condition 1 in the statement of the theorem, $s_1P_is_1'$. Therefore, Q_i' cannot weakly dominate Q_i, since $\varphi(Q_i', Q_{-i})(i) = s_1'$ for all $Q_{-i} \in \mathcal{Q}_{-i}$. Then, $f_{s_1'}(i) > q_{s_1'}$. Note that with the condition (1) in the statement of the theorem, if $P_i^{-1}(s^*(i)) \neq 1$, then $s_1 \neq s^*(i)$.

Let the lowest ranked achievable school in Q_i' be s. If for any school $\bar{s} \in U^{Q_i'}(s)$, we have $\bar{s} R_i s_1$, then s must be the $s^*(i)$. However, this possibility is ruled out as indicated before. Therefore, there exists $\hat{s} \in U^{Q_i}(s)$ such that $s_1 P_i \hat{s}$. Since s is achievable by Q_i', then by Lemma 11, $\hat{s}$ is achievable by Q_i'. Therefore, Q_i' cannot weakly dominate Q_i.

This completes the proof of Theorem 14.

Proof of Theorem 15

Lemmas 13 and 14

Lemma 13. *Consider student i. There exists a strategy profile of her opponent Q_{-i} such that $\varphi(P_i, Q_{-i})(i) = s^*(i)$.*

Proof. Suppose this is not true. Since $s^*(i)$ is one of the guaranteed schools, this means that for any Q_{-i}, $\varphi(P_i, Q_{-i})(i) P_i s^*(i)$. Pick Q_{-i} that produces the less preferred school for i under P_i and call this school as s', i.e., $\varphi(P_i, Q_{-i})(i) = s'$. Then it means s' instead of $s^*(i)$ is the most preferred guaranteed school. □

Lemma 14. *Consider a strategy Q_i that satisfies the rank-preserving monotonic transformation of the most preferred guaranteed school, Let the first school choice in Q_i be s. Suppose $f_s(i) > q_s$. Then there exists Q_{-i} such that $\varphi(Q_i, Q_{-i})(i) = s^*(i)$.*

Proof. By Lemma 13, there exists Q_{-i} such that $\varphi(P_i, Q_{-i})(i) = s^*(i)$. Construct Q'_{-i} in the following way. Let $s_t = Q_i(t)$ for $t = 1, ..., |S|$. Pick q_{s_1} number of students who have higher priority than i in f_{s_1} and call them I_1. Pick q_{s_2} number of students and call them as I_2, and so on. Let j in I_t submit s_t as the first choice. Call this strategy profile of i's opponents Q'_{-i}. Then $\varphi(Q_i, Q'_{-i})(i) = s^*(i)$. □

The Proof

It is straightforward to see that if Q_i satisfies condition 1 in the statement, Q_i is a weakly dominated strategy.

Suppose Q_i satisfies condition 2 in the statement. I follow the structure of the proof in Theorem 13. Recall that Q_i' is a strategy created by the γ-algorithm. Also recall that $\varphi(Q_i, Q_{-i})(i) := s$ and $\varphi(Q_i', Q_{-i})(i) := s'$. By Theorem 13, $\varphi(Q_i', Q_{-i})(i) R_i \varphi(Q_i, Q_{-i})(i)$ for all $Q_{-i} \in \mathcal{Q}_{-i}$. I show that there always exists some Q_{-i} such that $\varphi(Q_i', Q_{-i})(i) P_i \varphi(Q_i, Q_{-i})(i)$.

It is clear that in Case 1, $\varphi(Q_i', Q_{-i})(i) P_i \varphi(Q_i, Q_{-i})(i)$. In Case 2, I show that there always exists Q_{-i} such that $\varphi(Q_i', Q_{-i})(i) P_i \varphi(Q_i, Q_{-i})(i)$. There are two sub-cases to consider.

Sub-case 1: There exists $\overline{s} \in U^{Q_i}(s)$ such that $s^*(i) P_i \overline{s}$. By Lemma 14, there exists Q_{-i} such that $\varphi(Q_i', Q_{-i})(i) = s^*(i)$. Construct Q_{-i}' in the following way. For students other than i who are not matched with $\overline{s}$ under (Q_i', Q_{-i}), let them submit their match as the first school choices in Q_{-i}'. For students other than i who are matched with $\overline{s}$ under (Q_i, Q_{-i}), let them submit s_0 as the first choices in Q_{-i}'. Since $f_{s_1}(i) > q_{s_1}$, we have $\varphi(Q_i', Q_{-i}')(i) = s^*(i)$ and $\varphi(Q_i, Q_{-i}')(i) = \overline{s}$. Therefore, $\varphi(Q_i', Q_{-i}')(i) P_i \varphi(Q_i, Q_{-i}')(i)$.

Sub-case 2: There does not exists any school $\overline{s} \in U^{Q_i}(s)$ such that $s^*(i) P_i \overline{s}$. By construction, this implies $Q_i(s^*(i)) \neq Q_i'(s^*(i))$, otherwise both Q_i and Q_i' satisfy equation (3). This further implies that there exists $\widetilde{s} \in U^{Q_i'}(s^*(i))$ such that $\widetilde{s} \notin U^{Q_i}(s^*(i))$. By Lemma 14, there exists Q_{-i} such that $\varphi(Q_i', Q_{-i})(i) = s^*(i)$. Construct Q_{-i}' in the following way. For students other than i who are not matched with $s^*(i)$ or $\widetilde{s}$ under (Q_i', Q_{-i}), let them submit the schools they are matched with as the first school choices in Q_{-i}'. For students other than i who are matched with $s^*(i)$ or $\widetilde{s}$ under (Q_i', Q_{-i}), let them submit s_0 as the the first school choice in Q_{-i}'. Since $f_{s_1}(i) > q_{s_1}$, we have $\varphi(Q_i', Q_{-i}')(i) = \widetilde{s}$ and $\varphi(Q_i, Q_{-i}')(i) = s^*(i)$. Therefore, $\varphi(Q_i', Q_{-i}')(i) P_i \varphi(Q_i, Q_{-i}')(i)$.

2.6.7 Proof of Theorem 16

Suppose there is a Nash equilibrium Q such that student i obtains a school that is worse than $s^*(i)$, then by Lemma 10, Q_i does not satisfy equation (2.2). Then, student i can profitably deviate to another strategy that satisfies equation (2.2).

2.6.8 Proof of Theorem 17

Suppose the school choice problem is nested. Let $I(t)$ be the set of students whose most preferred guaranteed schools are ranked as the t-th positions in their true preferences for $t = 1, ..., |S|$. Students in $I(1)$ will obtain their most preferred schools if they submit strategies that satisfy equation (2.2). Students in $I(2)$ will obtain their most preferred guaranteed schools if they submit strategies that satisfy equation (2.2), since all the schools that are better than their most preferred guaranteed schools are obtained by students in $I(1)$. Students in $I(k)$ will obtain their most preferred guaranteed schools if they submit strategies that satisfy equation (2.2), since all schools that are better than their most preferred guaranteed schools are obtained by students in $\bigcup_{t=1,...,k-1} I_t$. Therefore, all students obtain their most preferred guaranteed schools.

On the other hand, note that the school choice problem in Example 1 is not nested, but there is an unique undominated Nash equilibrium.

2.6.9 Proof of Theorem 18

Lemma 15

The following lemma is a direct result of the definition of nested school choice problem.

Lemma 15. *Suppose the school choice problem is nested. Consider student* *i. The following holds for $s \in V^{P_i}(s^*(i))$.*

$$\sum_{s' \in U^{P_i}(s)} q_{s'} = \left|\left\{ j \in I \setminus \{i\} : s^*(j) \in U^{P_i}(s) \right\}\right|.$$

The Proof

(1) $\Rightarrow$ (2): Consider student i. By Lemma 15, the total number of students who consider schools that i prefers to $s^*(i)$ as the most preferred guaranteed schools are equal to the total capacities of those schools. Therefore, if all students use undominated strategies, student i cannot obtain the school that is preferred to $s^*(i)$. Let $I^t = \{j \in I : P_j(s^*(j)) = t\}$. With the above reasoning, all students in I^t obtains their most preferred guaranteed schools if they submit any undominated strategies.

(2) $\Rightarrow$ (1): Pick one of the first students in the algorithm of the Boston mechanism who obtain schools other than their most preferred guaranteed schools. Call this student i. Clearly, i obtains a school that is better than his most preferred guaranteed school. Since there does not exist other students who obtain a school different from his most preferred guaranteed school earlier than i in the algorithm, all students who obtain schools earlier than i in the algorithm are matched to their most preferred guaranteed schools. Therefore, the only possibility that i obtains a school better than his most preferred guaranteed school is that there are fewer students who obtains schools as their most preferred guaranteed schools in earlier iteration than the total quotas of all schools that are weakly better than the school that i obtains. However, this violates the equation in Lemma 15. In other words, the school choice problem is not nested.

2.6.10 Proof of Theorem 19

Consider a school s such that the following holds for some Q_{-i}.

$$\left| \bigcup_{s' \in U^{P_i}(s)} I^{P_i}(s'|Q_{-i}) \right| \geq \sum_{s' \in U^{P_i}(s)} q_{s'} \tag{2.4}$$

Then there exists some Q_{-i} such that $\varphi(P_i, Q_{-i})(i) \notin U^{P_i}(s)$. In other words, school s is not a guaranteed school for i. This means that if there is a school s such that the following holds, then this school s is a guaranteed school.

$$\max_{Q_{-i} \in \mathcal{Q}_{-i}} \left| \bigcup_{s' \in U^{P_i}(s)} I^{P_i}(s'|Q_{-i}) \right| < \sum_{s' \in U^{P_i}(s)} q_{s'} \tag{2.5}$$

After choosing from the most preferred school that satisfies equation (2.5), we have the desired outcome.

Chapter 3

Promoting Diversity of Talents: A Market Design Approach

Chia-Ling Hsu[1]

Abstract: I model a centralized matching problem between students and schools where students' priorities in schools are endogenous in terms of their efforts. I show that the consideration of endogenous aspects of students' priorities opens a new dimension of evaluating a matching system. An ill designed system could discourage students from pursuing the knowledge/skill in the skill categories that they are talented at. I propose four criteria of promoting diversity of talents: *respecting versatility of talents*, *respecting versatility of talents with minimum efforts*, *respecting unique talents*, and *respecting unique talents with minimum efforts*. These criteria are used to evaluate whether a matching system encourages students to pursue the knowledge/skill in the skill categories that they are talented at. Then, I

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propose four matching systems that accomplish these four criteria, respectively. The model in this paper could be applied to Taiwanese High School Match, Specialized High School matching program in New York City, Selective Enrollment High Schools matching program in Chicago, and Exam Schools matching program in Boston.

In 2014, the Taiwanese High School Match uses a new centralized matching program to allocate students to schools. I show that the unique features in this program either makes a matching system lose some desirable properties or fails the criteria of promoting diversity of talents defined in this paper.

JEL: C72, C78

Key Words: Diversity of talents, endogenous priority structure, school choice problem.

3.1 Introduction

Centralized matching mechanisms are commonly used for allocating students to schools. The *school choice problem*, formulated by Abdulkadiroğlu and Sönmez (2003) and Blinski and Sönmez (1999), considers several criteria to evaluate the performances of such mechanisms.[2] Most criteria surround two important issues: the properties of the matching outcomes of the mechanisms and the incentives students face when they submit their preferences over schools. Despite the importance of all these criteria, there is one important aspect that has been missing in this literature: the analysis of the distribution of students' effort inputs across different categories, which are the aspects where students are evaluated. In this paper, I propose several criteria to evaluate the performances of a matching system[3] in terms of

[2]Notable, these criteria include stability, efficiency and strategy-proofness. See Section 3.4 for formal definitions.

[3]I refer a matching system as a matching process which involves students' decision on the distribution of effort inputs and a matching mechanism that uses students' submitted

whether it induces a desirable distribution of the effort inputs from students.

I consider an environment in which students' performances are evaluated in several categories. A simple example is a set of exam subjects. Students have limited time endowments that they can allocate on different categories. A student receives a higher evaluation in a category, if she spends more time in that category. At the meantime, she faces trade-offs among different categories, because spending more time on one category means less time could be spent on others. Moreover, it is common that a student is more talented in one category but less talented in others. Under this framework, there are several possible ideologies that an education planner could have. The followings are the ideologies considered in this paper. First, a student could be encouraged to pursue the knowledge/skill in the category that she is most talented at. Second, a student could be encouraged to pursue the maximum of knowledge/skill across different categories. Third, a student could be encouraged to have minimum knowledge/skill across all categories. In addition to the proposition of criteria that captures the above ideologies, I propose several matching systems that satisfy these criteria and use these criteria to evaluate the Taiwanese High School Match.

The framework of this paper is built upon the basic framework of the school choice problem. Students care about which high schools they will attend after middle school.[4] A matching mechanism then uses students' stated preferences over schools and their priorities in schools to determine a matching between students and schools. The priorities of students in a school are used to determine which student is more entitled for being admitted to a school than other students. An example of determining priority is using exam scores so that a student with a higher score has a higher priority in a school. Since in this paper, students' priorities depend on their effort inputs,

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preferences and priorities resulted from their effort inputs to create a matching. See Section 3.4 for the formal definition of a matching system. Also see footnote 27.

[4]In the rest of the paper, I will simply call a high school as a school.

the priorities of students are endogenous.[5] The choice of a matching mechanism also plays an important role. There are three matching mechanisms that are widely studied: the Student-Optimal Stable Mechanism (SOSM), the Top-Trading Cycles Mechanism (TTCM) and the Boston mechanism.[6] The way I design the matching systems that satisfy the criteria proposed in this paper is by designing the priority system and choosing a matching mechanism that *respects improvement of priority*,[7] so that a student whose distribution of effort inputs fits better into the education planner's ideology has a higher priority in all schools and, as a result of a higher priority, obtains a weakly better school.

In 2014, the Taiwanese High School Match employs a new centralized matching program. The priorities of students are computed by summing up the numeric values from several categories. Those values either depend on how a student ranks schools in her submitted preference or a student's performances in middle school and exams. A student with higher total numeric values has a higher priority. After students submit their preferences, the Boston mechanism is used to create a matching. In 2014, there are at least 200,347 students who participate in this matching program.[8]

[5]There is another dimension of endogenicity in Taiwanese High School Match. That is, if a student ranks a school higher, then her priority in that school is weakly improved. To my knowledge, this is the first paper that considers the environment where the priorities of students are endogenous in terms of their efforts and the way they rank the schools. Dur (2011) considers a dynamic school choice problem, where parents may have more than one child and they could strategically place one child in a school so that the priority of the second child is improved in that school.

[6]The Student-Optimal Stable Mechanism is proposed by Gale and Shapley (1962). The Top-Trading Cycles Mechanism is proposed by Abdulkadiroğlu and Sönmez (2003). See Section 3.4 for descriptions of these three mechanisms.

[7]A mechanism respects improvement of priority if a student receives a weaker better school when her priority in all schools improves.

[8]This makes the Taiwanese High School Match the largest matching program that uses the Boston mechanism to allocate students to schools. Note that by Abdulkadiroğlu et al. (2005), the largest district that uses a priority mechanism at the time is Hillsborough

The followings are the key features in the Taiwanese High School Match.

Feature 1 If a student ranks a school higher, she has a (weakly) higher priority in that school.

Feature 2 The numeric value that a student receives in each category is "coarse" in terms of her effort.[9]

Feature 3 The tie-breaking rule (essentially) uses a lexicographic ordering of categories to break a tie.[10]

Feature 1 says that students' submitted preferences influence a matching system in two ways. First, a mechanism uses the reported preferences to compute the outcome. Second, if a student has higher priority in a school if she ranks a school higher. It is tempting to think that a design with Feature 1 could improve the performance of a matching system, since it intensifies students' claims for desired schools. However, I find that such design actually undermines a matching system in terms of fairness, stability, efficiency, and strategy-proofness, when the mechanisms is one of the SOSM, the TTCM, and the Boston mechanism.

I show that Feature 2 and Feature 3 influence students' decisions on allocating efforts among different categories. Feature 2 and Feature 3 create an environment where the priority of a student is determined by (1) the number of better letter grades that she has comparing to other students[11] and (2) the better scores in the categories with a higher ordering in the

County School District in Tampa-St. Petersburg, where there are 170,000 students. Also note that the Boston mechanism belongs to the class of priority mechanisms.

[9]Some categories are exams in different subjects. Note that the letter grades instead of the original exam scores are used for determining the numeric values for the categories.

[10]Suppose there are only two categories t_1 and t_2, and t_1 has a higher ordering than t_2 in the tie-breaking rule. When there is a tie between two students, the tie-breaking rule first compares the scores in the category t_1 and then the scores in the category t_2. The student with a higher score in t_1 has a higher priority. If there is a tie in the scores in t_1, then the student with a higher score in t_2 has a higher priority.

[11]Suppose there are six categories. A student with six As has a higher priority than

lexicographic tie-breaking rule comparing to other students. Since Feature 2 could frequently create ties, such environment gives students incentives to allocate more time in the category with a higher ordering in the lexicographic tie-breaking rule.

Beside the Taiwanese High School Match, the model in this paper can also be applied to Specialized High School matching program in New York City, Selective Enrollment High Schools matching program in Chicago, and Exam Schools matching program in Boston. In this three matching programs, the priorities of students are based on exam scores and grades. Brief descriptions of these matching programs are as follows. The Specialized High School matching program in New York City uses students' exam scores in Specialized High School Admissions Test (SHSAT) to determine students' priorities and then uses the serial dictatorship mechanism to determine the assignments.The Selective Enrollment High Schools matching program in Chicago uses students' exam scores in Selective Enrollment High Schools admissions test and their grades to determine students' priorities and then use the serial dictatorship mechanism to determine the assignments. The Exam Schools matching program in Boston uses students' exam scores in the Independent Schools Entrance Exam (ISEE), their grade point average (GPA), and their submitted school choices to determine the assignments. Note that all three exams have more than one skill categories.

The results of this paper are divided into two parts. In the first part, I propose four criteria of promoting diversity of talents. These four criteria are categorized in two dimensions: (1) whether a matching system respects *versatility of talents* or *unique talents* and (2) whether a matching system promotes *minimum efforts* or not. The first dimension captures an education planner's ideology on whether a student should be encour-

another students with five As and one B, if both students rank the schools in the same way.

aged to pursue the "breadth" or "depth" of her knowledge. A matching system respects versatility of talents if it encourages students to explore every learning possibilities among all categories. A matching system respects unique talents if it encourages students to pursue the knowledge/skill in the category that they are most talented at. The second dimension captures an education planner's ideology on encouraging students to have minimum knowledge/skill across all categories. Note that the combination of these two dimensions creates four criteria. With these criteria, I design four matching systems that achieve each criteria, respectively.

In the second part, I show that a matching system that has the features in the Taiwanese High School Match could produce undesirable outcomes in terms of the distribution of students' effort inputs. In particular, such a matching system could discourage students from pursuing knowledge/skill in the categories that they are most talented at. First, I analyze a priority system that has Feature 1. My analysis suggests such feature should be removed completely. Then, I study a priority system that has Feature 2 and Feature 3. I show that a matching system with such a priority system does not achieve any of the four criteria of promoting diversity of talents mentioned above, when the mechanism is strategy-proof and stable or the TTCM.

The most relevant paper to this paper is Hatfield et al. (2014), who consider an important problem in which agents value differently on the outcome of the game, or the alternative, which is partly determined by agents' investments on their human capitals. They study the mechanism that induces efficient ex ante investment on human capitals and achieves ex post efficient. In the setting of labor market matching, the agents are firms and workers and the outcome of the game is the matching between firms and workers.[12] There are many differences in their paper and this paper. The primary difference is the following. The welfare consideration in the design

[12]See Section II in Hatfield et al. (2014).

goal of a mechanism in Hatfield et al. (2014) is to maximize the ex post social welfare, which is the sum of each agent's valuation on the realized alternative. In this paper, the welfare consideration in this paper has two parts. From students' point of view, the welfare consideration depends on the schools that are matched to them.[13] On the other hand, the education planner has her ideology on the desirable distribution of effort inputs across different categories from the students. The ideology of an education planner might reflect the ideology of the society. Since students' valuations on the education outcome may not be consistent with the society's valuation on the distribution of students' effort inputs, such separation of welfare consideration is necessary in the education setting.

Other related literatures include the choice of mechanism, effort distortion resulted from the design of mechanisms, and designing priority. These literatures are contained in Section 3.8.

The rest of the paper is organized in the following way. In Section 3.2, I describe the Taiwanese High School Match. In Section 3.3, I present a motivating example. Section 3.4 presents the model. Section 3.5 presents the first part of the main results. Section 3.6 presents the second part of the main results. Section 3.8 contains the related literatures that are not contained in the introduction. Section 3.9 presents the details of the Taiwanese High School Match that are not contained in Section 3.2. Section 3.10 presents the proofs omitted in the main text.

3.2 Taiwanese High School Match

In 2014, the Taiwanese High School Match employs a new centralized matching program. The whole area is divided into fifteen school zones. Each school zone holds its own matching program. Similar to the school matching in

[13] Such welfare consideration is the standard welfare consideration in school choice problem following Abdulkadiroğlu and Sönmez (2003) and Blinski and Sönmez (1999).

Boston, the schools have no control on the priority of students. The education administration in each school zone decides the method that determines the priority to students in schools.[14]

The Taiwanese High School Match includes a centralized matching program and individual entrance held by individual schools.[15] The schools need to announce the capacity for the centralized matching and the capacity for the individual entrance before the matching takes place. The centralized matching program has two identical stages. Before the first-stage centralized matching program, there is a centralized exam that consists of several subjects. The exam scores are parts of the components that are used to determine students' priority. In the first-stage centralized matching program, students submit their preferences overs schools and the matching mechanism uses the submitted preferences and students' priority to compute the matching. After the first-stage centralized matching program, students who are admitted by some schools can either accept the admissions or reject them. Once a student accepts an admission, she cannot participate the following individual entrance and the second-stage centralized matching program. Students who accept admissions from the individual entrance cannot participate the second-stage centralized matching program. Schools can participate the second-stage centralized matching program only if there is some residual capacities left from the first-stage centralized matching program. The second-stage centralized matching operates in the same way as the first-stage centralized matching. Students submit their preferences over schools and the mechanism uses their preferences and the priority in the schools to compute the matching.[16]

The method of determining students' priorities in Keelung and Taipei district is described as follows. There are three grand categories that de-

[14]There are some differences in the methods for determining students' priority in different school zones. Within a school zone the method applies uniformly to all schools.

[15]In this paper, I focus on the centralized matching process.

[16]To my knowledge, the centralized matching program will only have one stage in 2015.

termine a student's priority in schools: the way a student ranks schools, the performances of students in middle school and the exam scores.[17] Each grand category gives a student a numeric value,[18] and a student's priority is determined by the sum of these three numeric values, so that a student with a higher total numeric value has a higher priority. In the first grand category, a school choice receives a (weakly) higher numeric value if a student ranks that school higher.[19] The second grand category consists of the evaluations of students' performances in middle school. The third grand category is the scores from a centralized exam that consists of several subjects. The scores in each subject are then transformed to letter grades, $A^{++}, A^{+}, A, B^{++}, B^{+}, B, C$. Each letter grade has its own numeric value. The numeric value from the third grand category is computed by summing up the numeric values from the letter grades. If there is a tie, the tie-breaking rule uses a pre-determined ordering over numeric values in different grand categories or categories withing these three grand categories, so that the priority is given to the students with a higher numeric value in the grand category or category with a higher ordering. The details of how students' priority is determined are in Section 3.9.

3.3 A Motivating Example

Let the set of students be $I = \{i_1, i_2\}$ and the set of schools be $S = \{s_1, s_2\}$. Suppose there are two categories that are used to evaluate students, mathematics and literature, which are denoted as t_m and t_l, respectively. The students' preferences over schools are the following.[20]

[17]Within the second and third grand categories, there are several categories.

[18]The numeric value from the second and third grand categories is computed by summing up the numeric value in each category within the grand category.

[19]For most school choices, the school that is ranked higher receives a strictly higher higher numeric value.

[20]This notation represents that student i_1 prefers school s_1 to s_2.

$$P_{i_1} : s_1, s_2$$
$$P_{i_2} : s_1, s_2$$

Suppose the time endowment is $\bar{L} = 10$. Each student can choose how to allocate the time endowment. The learning curves of the students are as follows.

$$\mathcal{L}_1^m(l) = 3\sqrt{l}$$
$$\mathcal{L}_1^l(l) = 2\sqrt{l}$$
$$\mathcal{L}_2^m(l) = \sqrt{l}$$
$$\mathcal{L}_2^l(l) = \frac{5}{3}\sqrt{l}$$

The superscripts represent the categories and the subscripts represent the identities of students. If a student $i \in \{i_1, i_2\}$ spends time $l \in [0, 10]$ on subject $t \in \{t_m, t_l\}$, he will receive a grade $\mathcal{L}_i^t(l)$ for category t. Assume that the maximum score in both categories is 10.

The priorities of the students are determined in the following way. The priority system sets thresholds in both subjects so that a student receives an A if his grade in that subject passes the threshold and a B if otherwise. An A worths 10 points and a B worths 5 points. Let the threshold in both subjects be 3. The student who has a higher total points from both subjects has a higher priority. If there is a tie on total points, use the following rule to break the tie. The priority system first compares the grades in subject t^l and whoever has a higher grade in t^l is given the higher priority. If there is still a tie, then the system compares the grades in subject t^m and the priority is given to the student with a higher grade in t^m. If there is still a tie, then use the students' age to break the tie. Assume the threshold in subject t_m is 1 and the threshold in subject t_l is 3.

After the students decide their time allocations, they submit their preferences over schools. Let the matching mechanism be the Student-Optimal Stable Mechanism. Therefore, there are two components of a student's strategy: the time allocation and the submitted preference over schools. Note that we consider an environment where students only care about the schools that they will be admitted at.

The following strategy profile is a Nash equilibrium. Student i_1's time endowment is $(\frac{15}{4}, \frac{25}{4})$,[21] and she submits a preference that s_1 is preferred to s_2. Student i_2's time allocation is $(1, 3)$, and she submits a preference that s_1 is preferred to s_2. The resulted matching is that student i_1 is matched to school s_1, her favorite school, and student i_2 is matched to school s_2, her less favored school. Although there are other Nash equilibria, there does not exist a Nash equilibrium where student i_1 spends more than $\frac{15}{4}$ on subject t_m, the subject that she is most talented at.

Next, consider two scenarios. In the first scenario, the students are encouraged to allocate their time so that they pass the thresholds and pursue the knowledge in the subject that they are talented at. In the second scenario, the students are encouraged to allocate their time so that they pass the minimum thresholds and maximize their knowledge across both subjects. Assume that in both scenarios, the students do not fully understand their time allocations would have an effect on the schools that they will be admitted at.[22] In the first scenario, student i_1 will have the time allocation $(\frac{31}{4}, \frac{9}{4})$, and student i_2 will have the time allocation $(1, 9)$. In the second scenario, student i_1 will have the time allocation $(\frac{90}{13}, \frac{40}{13})$, and student i_2 will have the time allocation $(\frac{90}{34}, \frac{250}{34})$. Let both students submit their true preferences. In both scenarios, student i_1 is matched to school s_2, and stu-

[21] This notation represents that student i_1 spends $\frac{15}{4}$ on subject t^m and $\frac{25}{4}$ on subject t^l.

[22] This assumption is along the line with Pathak and Sönmez (2008). Pathak and Sönmez (2008) analyze the Boston mechanism and show that the welfare of students who are not aware of the properties of the Boston mechanism would be undermined.

dent i_2 is matched to school s_2. Despite student i_1 is more productive in both subjects, in both scenario, student i_1 is matched to her less preferred school, s_2.

3.4 General Framework

Let $I = \{i_1, ..., i_n\}$ be the set of students and $S = \{s_1, ..., s_m\}$ the set of schools. Each student i has a strict preference P_i over $S \cup \{\varnothing\}$, where $\varnothing$ represents the outside option. Let $P = (P_i)_{i \in I}$ be the **preference profile**. Let R_i be the weak preference relation associated with preference P_i, so that sR_is' if and only if sP_is' or $s = s'$. A school s is **acceptable** for student i if $sP_i\varnothing$. Each school s has a capacity q_s. Let $q = (q_s)_{s \in S}$ be the **vector of capacities**. A student i is assigned with her priority $f_s(i)$ in school s. For two students i, j, if $f_s(i) < f_s(j)$, then student i has a higher priority in school s. Let $f = (f_s)_{s \in S}$ be the **priority structure**. Let $T = \{t_1, ..., t_r\}$ be the **set of skill categories**. Let $\tau_i = (\tau_i^{t_1}, ..., \tau_i^{t_r})$ be the **vector of scores** for student i, where $\tau_i^{t_k} \in [0, \bar{\tau}]$ is the student i's score for skill category t_k. Let τ_i^T be the total scores of student i when her score vector is τ_i. Let $\tau = (\tau_i)_{i \in I}$ be the **score profile**. For most parts of analysis in this paper, the score profile is not directly used for determining students' priorities. There could be some transformation of a score profile that determines students' priorities. A **grading rule**, $\xi : [0, \bar{\tau}]^{|I||T|} \to [0, \bar{\bar{\tau}}]^{|I|}$, is a function that transforms the score profile into final grades. The final grade of a student partly determines her priority in a school. I assume that if a student's score in one skill category is improved, then her final grade in each school is weakly improved.[23] In order to have more flexibility and wider applications of this model, I assume that each student i is endowed with a **merit score** m_i^s

[23] In other words, consider a student i with two of her score profiles $\tau_i = (\tau_i^{t_1}, ..., \tau_i^{t_k}, ..., \tau_i^{t_r})$ and $\hat{\tau}_i = (\hat{\tau}_i^{t_1}, ..., \hat{\tau}_i^{t_k}, ..., \hat{\tau}_i^{t_r})$ where $\hat{\tau}_i^{t_k} > \tau_i^{t_k}$ and $\hat{\tau}_i^t = \tau_i^t$ for $t \in T \setminus \{t_k\}$. Then for any grading rule ξ and score profile of her opponents τ_{-i}, we have $\xi_i(\hat{\tau}) \geq \xi_i(\tau)$, where $\tau = (\tau_i, \tau_{-i})$ and $\hat{\tau} = (\hat{\tau}_i, \tau_{-i})$.

for school s, which represents how much student i is entitled to school s. This merit score is fixed and cannot be changed by students or schools. Let $m_i = (m_i^s)_{s \in S}$ be the vector of merit scores for student i.[24]

A **matching** $\mu : I \to S \cup \{\varnothing\}$ is a function that assigns students to schools or the outside option so that the number of students that are assigned to a school does not exceed its capacity. Let $\mathcal{M}$ be the set of all matchings.

A matching μ is **fair** if there does not exist two students i, j and a school s such that $sP_i\mu(i)$, $\mu(j) = s$, and $\tau_i^T + m_i^s > \tau_j^T + m_j^s$. A matching μ is **individually rational** if $\mu(i)R_i\varnothing$ for all $i \in I$. A matching μ **respects priority** if there does not exist two students i, j and a school s such that $sP_i\mu(i)$, $\mu(j) = s$, and $f_s(i) < f_s(j)$.[25] A matching μ is **nonwasteful** if $sP_i\mu(i)$ implies $|\mu^{-1}(s)| = q_s$. A matching μ is **stable** if it is individually rational, respects priority and is nonwasteful. A matching μ is **efficient** if there does not exist another matching ν such that $\nu(i)R_i\mu(i)$ for all $i \in I$ and $\nu(i)P_i\mu(i)$ for some $i \in I$.

[24] I introduce this construction merely for the flexibility of the model and wider applications. One possible situation that this merit score applies is when the proximity of students' residence to schools is used for determining student's priority. In such a case, the priority of students is a hybrid of their proximity and final grades. Note that all results in this paper, unless mentioned otherwise, hold when there is no such merit scores.

There are two categories for merit score in Taiwanese High School Match: the merit score for minority students and the merit score for students who live in the same district as the district that he or she participates in the matching program. Not all the districts have the feature of merit scores. The districts that use merit score for minority students include Tsinchu and Miaoli district, Taichung and Nantou district, Changhua district, Yunlin district, Pingtung district, and Hualien district. The districts that use merit score for students who live in the same district as the district that he or she participates the matching program include Taoyuan district, Tsinchu and Miaoli district, Taichung and Nantou district, Changhua district, Yunlin district, Tainan district, Yilan district, and Kinmen district.

[25] In the following sections except Section 3.6.1, I assume that $f_s(i) < f_s(j)$ if and only if $\xi_i(\tau) + m_i^s > \xi_j(\tau) + m_j^s$. In Section 3.6.1, I assume that $f_s(i) < f_s(j)$ if and only if $\tau_i^T + m_i^s + v_t > \tau_j^T + m_j^s + v_{t'}$, when i ranks s as the t-th choice and j ranks s as the t'-th choice.

Let $\mathcal{Q}_i$ be the set of all submitted preferences for student i and let $\mathcal{Q}_{-i}$ be the set of all submitted preference profile for students other than i. Let $\mathcal{Q}$ be the set of all submitted preference profiles.

A **mechanism** ω is a systematic method that uses P, f and q to create a matching. Let $\omega(Q)$ be the matching created by mechanism φ when the strategy profile is Q. Let $\omega_i(Q)$ be the school or the outside option that is matched to student i when the strategy profile is Q. A mechanism φ is stable (efficient, or far) if $\varphi(P)$ is stable (efficient, or fair).[26]

Next, I describe three mechanisms. When there is a tie between two students' priorities, a tie-breaking rule is used. I delay the introduction of several tie-breaking rules until next two sections. The **Student-Optimal Stable Mechanism** (SOSM) is described as follows.

Step 1 Each student proposes to her most preferred acceptable school. Each school tentatively holds the set of students with highest priorities up to its capacity out of the students who propose to it. The students that are not tentatively held by any school are rejected.

Step t Each student that is not tentatively held by any school at step $t-1$ proposes to her most preferred acceptable school out of the schools that have not rejected her. Each school tentatively holds the set of students with highest priorities out of the students who propose to it at this set together with the students that are tentatively held in step $t-1$. The students that are not tentatively held by any school are rejected.

The algorithm terminates when each student is tentatively held by some schools or rejected by all acceptable schools.

[26] With some abuse of notion, the strategy profile (P_i, Q_{-i}) refers to the profile where i submits the true preference and other students submit Q_{-i}. The strategy profile (P_i, P_{-i}) refers to the profile where every student submits the true preference.

The **Top-Trading Cycles Mechanism** (TTCM) is described as follows.

Step 1 Each student points to her most preferred acceptable school. Each school points to the student with the highest priority. When there is a cycle among students and schools, the student in the cycle is matched to the school that she points to, and that school reduces its capacity by one.

Step t Each student that is not matched in any previous steps points to the most preferred acceptable school that still has a positive capacity. Each school that still has a positive capacity points to the student with the highest priority out of the students that are not matched with any schools in the previous steps. When there is a cycle among students and schools, the student in the cycle is matched to the school that she points to, and that school reduces its capacity by one.

The algorithm terminates when each student is matched to some school or rejected by all acceptable schools.

The **Boston mechanism** is described as follows.

Step 1 Each student proposes to her most preferred acceptable school. Each school accepts the set of students with highest priorities up to its capacity out of the students who propose to it. Moreover, each school reduces its capacity by the number of students it accepts. Students that are not accepted by any school are rejected.

Step t Each student that is not accepted by any school in previous steps proposes to her most preferred acceptable school out of the schools that have not rejected her. Each school accepts the set of students with highest priorities up to its capacity out of the students who propose to it. Moreover, each school reduces its capacity by the number of students it accepts. Students that are not accepted by any school are rejected.

The algorithm terminates when each student is accepted by some school or rejected by all acceptable schools.

A mechanism ω is **strategy-proof**, if

$$\omega_i(P_i, Q_{-i}) R_i \omega_i(Q_i, Q_{-i}),$$

for any $i \in I$, $Q_i \in \mathcal{Q}_i$, and $Q_{-i} \in \mathcal{Q}_{-i}$.

A **matching system** is a strategy space $S = S_{i_1} \times, \cdots, \times S_{i_n}$ associated with an outcome function $g : S \to \mathcal{M}$.[27]

Each student is endowed with an amount of time $\bar{L}$ for their disposal. They decide the time allocation (or, equivalently, their efforts) on each skill category. The effort on each skill category is directly reflected on the score in that skill category. Let $\mathcal{L}_i^t : [0, \bar{L}] \to [0, \bar{\tau}]$ be student i's **learning curve on the skill category** t.[28] If student i spends time $l \in [0, L]$ on the skill category t, then her score on t would be $\mathcal{L}_i^t(l)$. Assume that $\frac{d}{dl}\mathcal{L}_i^t(l) \geq 0$ and $\frac{d^2}{dl^2}\mathcal{L}_i^t(l) \leq 0$ for $l \in (0, L)$ and for all $t \in T$ and $i \in I$. Moreover, assume that $\mathcal{L}_i^t(0) = 0$ for all $i \in I$ and $t \in T$. Let $L_i = (L_i^t)_{t \in T}$ be student i's **time allocation** such that $L_i^t \in [0, \bar{L}]$ for all $t \in T$ and $\sum_{t \in T} L_i^t \leq \bar{L}$, where L_i^t is the amount of time that student i spends on subject t. Let time allocation profile be $L = (L_i)_{i \in I}$. Let Π be the set of all possible time allocations for each student. With slightly abuse of notation, I write $\mathcal{L}_i^t(L_i)$, where L_i is a time allocation, to represent $\mathcal{L}_i^t(L_i^t)$.

Let $\theta_i^t(L_i)$ be the score on subject t for student i when the time allocation is L_i. Let $\theta_i^T(L_i)$ be the total scores for student i when the time allocation is L_i.

[27]The definition of a matching system here is equivalent to the definition of a mechanism in Sönmez and Switzer (2013). In this paper, I use the terminology "matching system" instead of "mechanism" in order to avoid confusion, since I reserve the terminology "mechanism" for the SOSM, the TTCM, and the Boston mechanism.

[28]Note that although I model the learning cures in a deterministic manner, one can imagine a learning curve to be an estimation of the output when a student makes the decision of how much time to allocate on a skill category.

Let a **priority system with endogenous efforts** $\mathcal{F}$ be a systematic method to create a priority structure f with input of L. Let $\varphi_\omega^{\mathcal{F}} : [0, \bar{L}]^{|I||T|} \times \mathcal{Q} \to \mathcal{M}$ be a **matching system with endogenous efforts** to be a systematic method to create a matching with the procedure that it first creates a priority structure by $\mathcal{F}$ and then the mechanism ω uses the priority structure and the submitted preferences to produce a matching.[29]

Let $\varphi_{\omega,i}^{\mathcal{F}}[(L,Q)]$ be the school that student i obtains when the strategy profile is (L,Q).

A **dominant strategy** (L_i, Q_i) for student i is a strategy such that

$$\varphi_{\omega,i}^{\mathcal{F}}[(L_i, Q_i), (L_{-i}, Q_{-i})] R_i \varphi_{\omega,i}^{\mathcal{F}}[(L_i', Q_i'), (L_{-i}, Q_{-i})],$$

for all $(L_i', Q_i') \in [0, \bar{L}]^{|T|} \times \mathcal{Q}_i$ and for all $(L_{-i}, Q_{-i}) \in [0, \bar{L}]^{|T|(|I|-1)} \times \mathcal{Q}_{-i}$.

Definition 9. A time allocation L_i is a **partial dominant strategy** for student i under $\varphi_\omega^{\mathcal{F}}$ if (L_i, P_i) is a dominant strategy under $\varphi_\omega^{\mathcal{F}}$.

In other words, a time allocation L_i is a partial dominant strategy, if the time allocation along with submitting the true preference is a dominant strategy.

A **Nash equilibrium** (L,Q) is a strategy profile such that

$$\varphi_{\omega,i}^{\mathcal{F}}[(L_i, Q_i), (L_{-i}, Q_{-i})] R_i \varphi_{\omega,i}^{\mathcal{F}}[(L_i', Q_i'), (L_{-i}, Q_{-i})],$$

for all $i \in I$ and all $(L_i', Q_i') \in [0, \bar{L}]^{|T|} \times \mathcal{Q}_i$.

3.5 Promoting Diversity of Talents

In this section, I introduce four criteria of promoting diversity of talents: *respecting versatility of talents*, *respecting versatility of talents with minimum efforts*, *respecting unique talents*, and *respecting unique talents with minimum efforts.*

[29] Since there is no confusion, in Section 3.5 and Section 3.6.2, I will refer the priority system with endogenous efforts and the matching system with endogenous efforts as the priority system and matching system, respectively.

As mentioned in the introduction, these four criteria belong to two groups: promoting versatility of talents and promoting unique talents. I first introduce the notion of promoting versatility of talents. This notion captures the idea of encouraging the "breadth" of students' study. It says that the marginal improvement on each skill category is the same. In particular, it means that all the learning possibilities are fulfilled. Consider a student i with a time allocation L_i. If L_i makes the marginal improvement on each skill category is the same, the time allocation L_i is the solution to the following problem.

$$\begin{aligned} &\max_{\hat{L}_i \in \Pi} \theta_i^T(\hat{L}_i) \\ \text{s.t. } &\sum_{t \in T} \hat{L}_i^t \leq \bar{L} \end{aligned}$$

In other words, when a student's time allocation respects versatility of talents, then this time allocation maximizes the total scores. The following is the first notion of promoting diversity of talents.

Definition 10. A matching system **respects versatility of talents** if it is a partial dominant strategy for each student to allocate time so that the total scores is maximized.

It is not unusual that a student is more talented in one skill category than others. Denote such skill category as her unique talent.

Definition 11. A student i has an **unique talent** on skill category t, if $\mathcal{L}_i^t(\bar{L}) \geq \mathcal{L}_i^{t'}(\bar{L})$ for all $t' \in T \setminus \{t\}$.

The next group, promoting unique talents, captures the idea of encouraging the "depth" of students' study.

Definition 12. A matching system **respects unique talents** if it is a partial dominant strategy for each student i to allocation all time to the skill category of (one of) her unique talents.

One can see that it may not be the best to a society's interest, if all students put most of their efforts on pursing the knowledge/skill in the skill categories that they are relatively talented at but are ignorant in other skill categories. An education planner might have an ideology that each student should be encouraged to have minimum knowledge/skill in all (or a subset of) skill categories. Let η_t be the **threshold on skill category** t, which represents the minimum performance, or score, that an education planner wants to induce a student to have. Denote $\eta = (\eta_t)_{t \in T}$ as the **threshold**.

Definition 13. A matching system **promotes minimum effort** η if it is a partial dominant strategy for each student i to allocate time such that she maximizes the number of skill categories that achieve the threshold.

The following two criteria are the combinations of respecting versatility of talents (or respecting unique talents) and promoting minimum efforts.

Definition 14. A matching system **promotes versatility of talents with minimum efforts** η if it is a partial dominant strategy for each student i to allocate time L_i such that (1) it maximizes the number of skill categories that achieve the thresholds and (2) there does not exist another time allocation L_i' such that it results in the same number of skill categories that achieve the thresholds and a higher total exam scores than the total exam scores resulted from L_i.

The above criterion encourage students to first maximize the number of skill categories that achieve the thresholds and then maximize the total scores.

Definition 15. A matching system **promotes unique talents with minimum efforts** η if it is a partial dominant strategy for each student i to allocate time L_i such that (1) it maximizes the number of subject that achieves the threshold and (2) there does not exist another time allocation L_i' that results in the same number of skill categories that achieve the thresholds and higher scores in every subject of her unique talents than the scores resulted from L_i.

The above criterion encourages students to first maximize the number of skill categories that achieve the thresholds and then maximize the score in the skill category that she is talented at.

In the following, I propose several matching systems that achieves the criteria mentioned above. A **total score-based priority system** is a priority system that uses the total scores to determine students' priorities so that a student with higher total scores has a higher priority.

Denote φ^T_{SOSM} (or φ^T_{TTCM}) as the matching system where the priority system is the total score-based priority system and the mechanism is the SOSM (or the TTCM).

Theorem 20. *The matching systems φ^T_{SOSM} and φ^T_{TTCM} respect versatility of talents.*

Next, I propose the Sequential Eating Algorithm (SEA) in order to create a priority system that is used in a matching system that promoting unique talents.

For any score profile τ, the **Sequential Eating Algorithm** works in the following way.

Step 1 For each student i, pick the highest score from all categories. Denote this score as $\tau(i)$.

Step 2 For each student, let $\pi^s_i = \tau(i) + m^s_i$, for all $s \in S$.

In other words, for any τ, this algorithm creates $(\pi^s_i)_{i\in I, s\in S}$. A **deterministic tie-breaker** $\succ_d$ is a complete ordering over students. One example of such tie-breaking rule is an ordering of students based on students' age.

The **SEA priority system** is a priority system that uses $(\pi^s_i)_{i\in I, s\in S}$ and a deterministic tie-breaker $\succ_d$ to create a priority structure f^{SEA} such that $f_s^{SEA}(i) < f_s^{SEA}(j)$ if and only if (1) $\pi_i^s > \pi_j^s$, or (2) $\pi_i^s = \pi_j^s$ and $i \succ_d j$.[30]

Denote φ_{SOSM}^{SEA} (or φ_{TTCM}^{SEA}) as the matching system where the priority system is the SEA priority system and the mechanism is the SOSM (or the TTCM).

Theorem 21. *The matching systems φ_{SOSM}^{SEA} and φ_{TTCM}^{SEA} respect unique talents.*

Before I introduce a construction that will be used in the design of matching systems that promotes minimum efforts, I show that the total score-based priority system does not promote minimum efforts. Denote $\varphi_\omega^{T,d}$ as a matching system where the priority system is the total score-based priority system, the tie-breaking rule is $\succ_d$, and the mechanism is ω.

Theorem 22. *A matching system $\varphi_\omega^{T,d}$, where ω is strategy-proof and stable (or is the TTCM), does not promote minimum effort η.*

A **letter grade priority system** is a priority system that it first transforms a student's score in each subject into letter grade, which has a numeric value, and then sum up all the numeric values to determine student's total numeric values, so that a student with higher total numeric values has a higher priority.

For simplicity, I assume that there are only two letter grades: A and B. With the construction of two letter grades, a letter grade priority system essentially gives priority to the students with more skill categories with letter grade A. The actual numeric value that is assigned to the letter grade A or

[30]For example, consider two students i, j with two subjects t_1, t_2, and their scores are $\tau_i = (10, 90)$ and $\tau_j = (80, 20)$. Assume $m_i^s = m_j^s = 0$ for all $s \in S$. The priority created by the SEA priority system is

$$f_s^{SEA} : i, j.$$

B does not matter. Therefore, in the following analysis, I will not be specific on the numeric values assigned to the letter grades.

A **letter grade priority system with threshold** η is a letter grade priority system where the a student with a score on skill category $\tau_i^t \geq \eta_t$ receives an A in the skill category t and gives priority to the students who have more As than other students. As mentioned in the introduction, such a system could easily create ties and requires a tie-breaking rule.

Denote $\varphi_\omega^{\eta,d}$ as a matching system where the priority system is a letter grade priority system with threshold η, the tie-breaking rule is $\succ_d$, and the mechanism is ω.

Theorem 23. *A matching system $\varphi_\omega^{\eta,d}$, where ω is strategy-proof and stable (or is the TTCM), promotes minimum effort η.*

Finally, I incorporate this construction into the matching systems that respects versatility of talents and respects unique talents, respectively, which are introduced earlier.

A **total score-based tie breaker** is a tie-breaking rule hat uses the total scores to break a tie.

Denote $\varphi_{SOSM}^{\eta,\overline{T}}$ (or $\varphi_{TTCM}^{\eta,\overline{T}}$) as a matching system where the priority system is a letter grade priority system with threshold η, the tie-breaking rule is the total score-based tie breaker, and the mechanism is the SOSM (or the TTCM).

Theorem 24. *The matching systems $\varphi_{SOSM}^{\eta,\overline{T}}$ and $\varphi_{TTCM}^{\eta,\overline{T}}$ respects versatility of talents with minimum effort η.*

A **SEA tie breaker** is a tie-breaking rule that uses the priority in f^{SEA} to break a tie.

Denote $\varphi_{SOSM}^{\eta,\overline{SEA}}$ (or $\varphi_{TTCM}^{\eta,\overline{SEA}}$) be a matching system, where the priority system is the letter grade system with threshold η, the tie breaking rule is SEA tie breaker, and the mechanism is the SOSM (or the TTCM).

I make the following assumption for the next theorem.

Assumption 2. For each student i, $\mathcal{L}_i^t(l) \neq \mathcal{L}_i^{t'}(l)$ for all $l \in (0, \bar{L}]$ and for all $t, t' \in T$. Moreover, for each student i and for all $t, t' \in T$, if $\mathcal{L}_i^t(l) > \mathcal{L}_i^{t'}(l)$, then $\mathcal{L}_i^t(l) > \mathcal{L}_i^{t'}(l)$ for all $l \in (0, \bar{L}]$.

Theorem 25. *Under Assumption 2, the matching systems* $\varphi_{SOSM}^{\eta,\overline{SEA}}$ *and* $\varphi_{TTCM}^{\eta,\overline{SEA}}$ *respect unique talents with minimum effort* η.

3.6 The Analysis on the Taiwanese High School Match

In this section, I analyze the Taiwanese High School Match. This section is divided into two parts. In the first part, I analyze Feature 1 mentioned in the introduction. In the second part, I analyze Feature 2 and Feature 3 mentioned in the introduction.

3.6.1 The Floating Priority System

I first analyze the effect of having a priority system that gives students different priority in schools if they rank the schools differently. Throughout this section, I assume that students' score profile τ is given. In other words, the strategy for a student in this section is the preference that she submits. The priority of a student is determined by the sum of the total exam scores, the merit score, and the numeric value given to the school choice according to their rankings. If there is a tie, a deterministic tie breaker $\succ_d$ is used.

Definition 16. A **floating priority system** $v = \{v_1, ..., v_{|S|}\}$ is a sequence of numbers such that $v_i \geq v_j$ for $i < j$ and there exists $v_i, v_j \in v$ such that $v_i \neq v_j$.

A student i with a total exam scores τ_i^T who ranks school s as the r-th school choice in her submitted preference will have a numeric value of $\tau_i^T + m_i^s + v_r$ for school choice s. A student i has a higher priority than student j in school s if and only if (1) $\tau_i^T + m_i^s + v_r > \tau_j^T + m_j^s + v_{r'}$ or (2)

$\tau_i^T + m_i^s + v_r = \tau_j^T + m_j^s + v_{r'}$ and $i \succ_d j$, where student i ranks school s as the r-th school choice and student j ranks school s as the r'-th choice.

A **mechanism** ω **associated with a floating priority system** v is a matching system where the matching is determined in the following sequence: (1) students submit their preferences, (2) the priority structure is computed so that each student's priority in schools is based on the sum of her total exam scores, the merit score, and the numeric value given to each school choice according to v, and (3) the mechanism ω uses the computed priority structure and the submitted preference profile to determine the matching.

Denote $\varphi_\omega^{v,d}$ as a matching system, where it uses a floating priority system v, the tie-breaking rule is $\succ_d$, and the mechanism is ω. The next theorem says that it is impossible for a mechanism associated with a floating priority system to have some desirable properties. Theorem 26 shows that there is a conflict between fairness and stability for a mechanism.

Theorem 26. *There does not exist a mechanism ω associated with a floating priority system v, $\varphi_\omega^{v,d}$, that is fair and stable.*

Next, I analyze the effect of having a floating priority system on the three mechanisms: the SOSM, the TTCM, and the Boston mechanism. Denote φ_{SOSM}^d, φ_{TTCM}^d, and φ_{BM}^d, as the matching system that does not use a floating priority system, where the tie-breaking rule is $\succ_d$, and the mechanism is the SOSM, the TTCM, and the Boston mechanism, respectively.

We say that a matching system φ is **equivalent** to another matching system φ' if $\varphi(Q) = \varphi'(Q)$ for any $Q \in \mathcal{Q}$. In other words, if two matching systems are equivalent, then for any submitted preference profile, both matching systems produce the same matching. The following theorem says that having a floating priority system does not have any impact on the Boston mechanism.

Theorem 27. *$\varphi_{BM}^{v,d}$ is equivalent to φ_{BM}^d.*

As a result of Theorem 27, φ^v_{BM} has the properties of efficiency, instability, and nonstrategy-proffness as φ_{BM} does.

Theorem 28. *$\varphi^{v,d}_{BM}$ is efficient, but it unfair, unstable, nonstrategy-proof.*

The next two theorems show that when the SOSM and TTCM are associated with a floating priority system, some desirable properties are lost.

Theorem 29. *$\varphi^{v,d}_{SOSM}$ is stable, but it is unfair, inefficient and nonstrategy-proof.*

Theorem 30. *$\varphi^{v,d}_{TTCM}$ is efficient, but it is unfair, unstable, and nonstrategy-proof.*

The following table summarizes the properties of all three mechanisms when they are associated with and without a floating priority system.

Table 3.1: Comparison of the Six Matching Systems

	Fair	Stable	Efficient	Strategy-proof
φ^d_{SOSM}	Yes	Yes	No	Yes
$\varphi^{v,d}_{SOSM}$	No	Yes	No	No
φ^d_{TTCM}	No	No	Yes	Yes
$\varphi^{v,d}_{TTCM}$	No	No	Yes	No
φ^d_{BM}	No	No	Yes	No
$\varphi^{v,d}_{BM}$	No	No	Yes	No

Table 1 shows that the SOSM and the TTCM lose some of their desirable properties when they are associated with a floating priority system, but they do not gain any other desirable properties. Therefore, a policy suggestion would be removing the floating priority system entirely, regardless of which of the three mechanisms is used.

3.6.2 Incentive Distortion

In this section, I analyze a priority system that is letter grade system with a lexicographic tie-breaker.[31] Such priority system captures Feature 2 and Feature 3 in the Taiwanese High School Match.

A **lexicographic tie-breaker** $\succ_T$ is a tie-breaking rule, which is a complete ordering over T, so that a tie is broken in the way that the priority is given to the student who has a higher score in the subject with a higher ordering. For example, suppose $\succ_T = t_1, t_2, t_3$. If there is a tie between students i, j, the student with a higher score in subject t_1 has a higher priority. If there is a tie on the score in subject t_1, the student with a higher score in subject t_2 has a higher priority, and so on. I assume that if after comparing scores in all skill categories, a tie still exists, then a deterministic tie-breaking rule $\succ_d$ is used.

Denote $\varphi_\omega^{\eta, \succ_T}$ as a matching system, where the priority system is the letter grade system with threshold η, and the tie-breaking rule is $\succ_T$, and the mechanism is ω.

The following theorem says that when the mechanism is strategy-proof and stable (or the TTCM), the matching system does not respect versatility of talents.

Theorem 31. *A matching system $\varphi_\omega^{\eta, \succ_T}$, where ω is strategy-proof and stable (or is the TTCM), does not respect versatility of talents.*

With a letter grade system, the matching system gives students incentives to allocate time so that it maximizes the number of subjects that achieve thresholds. Thus, it could cause some students to sacrifice the subjects that they are comparatively good at and allocate more time on the subjects that they could achieve minimum thresholds.

The following theorem says that when the mechanism is strategy-proof and stable (or the TTCM) and the priority system is a letter grade sys-

[31] Note that the setting of the model is the same as in Section 3.5.

tem with a lexicographic tie-breaker, the matching system does not respect versatility of talents with minimum efforts.

Theorem 32. *A matching system $\varphi_{\omega}^{\eta,\succ_T}$, where ω is strategy-proof and stable (or is the TTCM), does not respect versatility of talents with minimum efforts η.*

Although $\varphi_{\omega}^{\eta,\succ_T}$ promotes minimum efforts η, it gives students incentives to allocate time on the subject with a higher order in the lexicographic tie-breaker. In other words, consider a student i with two time allocations L_i and L_i' such that both the time allocations give the same number of subjects that achieve the thresholds. Student i would favor the time allocation that results in a higher score in the subject with a higher order in the lexicographic tie-breaker, regardless of whether that subject is the subject that she is comparatively good at.

The following theorem says that when the mechanism is strategy-proof and stable (or is the TTCM) and the priority system is a letter grade system with a lexicographic tie-breaker, the matching system does not respect unique talents.

Theorem 33. *A matching system $\varphi_{\omega}^{\eta,\succ_T}$, where ω is strategy-proof and stable (or is the TTCM), does not respect unique talents.*

When there is a tie between students, the lexicographic tie-breaker gives students incentives to allocate time on the subject with a higher order, regardless of whether that subject is of their unique talents.

The following theorem says that when the mechanism is strategy-proof and stable (or is the TTCM) and the priority system is a letter grade system with a lexicographic tie-breaker, the matching system does not respect unique talents with minimum efforts.

Theorem 34. *A matching system $\varphi_{\omega}^{\eta,\succ_T}$, where ω is strategy-proof and stable (or is the TTCM), does not respect unique talents with minimum efforts η.*

Despite φ^{η} promotes minimum efforts η, the lexicographic tie-breaker gives students incentive to allocate on the time on the subject with a higher order in the lexicographic tie-breaker. Consider a student i with two time allocations L_i and L'_i that give student i the same number of subjects that achieve thresholds. If L_i results in a higher score in the subject with a higher order in the lexicographic tie-breaker, then student i has an incentive to choose L_i, regardless of whether that subject is of her unique talents.

3.7 Conclusion

In this paper, I consider a matching problem between students and schools with an environment where students' priority in schools is endogenous in terms of students' efforts or the way they rank schools. I show that the consideration of the endogenous aspects of students' priority has profound effects on the performance of a matching system. In the first part of the results, I propose four criteria of promoting diversity of talents. Then, I propose four matching systems that achieve these four criteria. In the second part of the results, I analyze the Taiwanese High School Match. First, I find that a floating priority system does not have any benefits, regardless of which of the SOSM, the TTCM or the Boston mechanism is used. Second, I show that a matching system with the features of Feature 2 and Feature 3 mentioned in the introduction fails all four criteria of promoting diversity of talents.

3.8 Related Literature

In this section, I present three related literatures: choice of mechanism, effort distortion and designing priority.

3.8.1 Choice of Mechanism

The Student-Optimal Stable Mechanism (SOSM) is proposed by Gale and Shapley (1962). Gale and Shapley (1962) show that this mechanism produces a stable matching. Dubins and Freedman (1981) and Roth (1982) show that it is strategy-proof for students. Note that this mechanism is not efficient.

The Top-Trading Cycles Mechanism (TTCM) is proposed by Abdulkadiroğlu and Sönmez (2003). Abdulkadiroğlu and Sönmez (2003) show that it is efficient, strategy-proof, but it is not stable.

The Boston mechanism is a widely used mechanism. Abdulkadiroğlu and Sönmez (2003) show that this mechanism is efficient, but it is not stable nor strategy-proof. Ergin and Sönmez (2006) show that the matching produced by the dominant-strategy outcome Pareto dominates the matching produced by the Nash equilibrium outcomes in the Boston mechanism. Pathak and Sönmez (2008) study an environment where some students are unaware of the properties of the Boston mechanism and find that such students' welfare would be hurt due to the nonstrategy-proofness of the Boston mechanism. Abdulkadiroğlu et al. (2006) provide empirical evidence to show that some parents do not response to the Boston mechanism strategically in Boston. Pathak and Sönmez (2013) propose a method to measure the vulnerability to manipulation and find many places in the world are replacing more manipulable mechanisms to less manipulable mechanisms. Kumano (2013) shows that for the Boston mechanism to be strategy-proof or stable, it has a very restrictive requirement on the priority structure, which is very difficult to be satisfied in reality. Hsu (2013) relaxes strategy-proofness to dominance solvability and finds that for the Boston mechanism to be dominance-solvable, the requirement on the priority structure is still restrictive and suggests replacement of the Boston mechanism. Despite the deficiencies of the Boston mechanism, it remains a very popular mechanism and therefore, it is worthwhile to understand the properties of the mechanism and how participants behave in the mechanism. Kojima and Ünver (2014) use an axiomatic approach to characterize the Boston mechanism. Hsu (2014) characterizes the undominated strategies and weakly dominated strategies in the Boston mechanism.

3.8.2 Effort Distortion

The design of a mechanism has effects on how much effort a participant is willing to provide. A desirable mechanism should give incentives to participants for them to improve their standings. For example, in the context of student placement problem, students should not be punished for better exam scores. This property is called *respecting improvements*.[32]

Many real-world mechanisms, however, do not have such properties. Blinski and Sönmez (1999) show that the mechanism used in Turkey college admission does not respect improvements. Sönmez and Switzer (2013) and Sönmez (2013) propose the *cadet-branch matching problem* that matches cadets to branches, while cadets have options to provide additional service years for increasing their priority in branches. Sönmez and Switzer (2013) and Sönmez (2013) show that the mechanisms that are currently used in the United States Military Academy (USMA) and the Reserved Officer Training Corps (ROTC), respectively, do not respect improvements.

It is worthwhile to point out that the deficiency in the ROTC pointed out in Sönmez (2013) is resulted from the design of the priority, while the deficiency in the Turkey college admission pointed out in Blinski and Sönmez (1999) and the deficiency in the USMA pointed out in Sönmez and Switzer (2013) are due to the design of mechanism.

[32]This property is introduced by Blinski and Sönmez (1999).

In above cases, the effort distortion refers to the situation that participants may have incentives to reduce their exam scores or their standings in the order-of-merit list[33]. In this paper, I introduce another kind of effort distortion, which is the incentive distortion on the distribution of participants' efforts on different subjects.

On the other hand, the choice of a matching mechanism also has an effect on schools' incentives. Hatfield et al. (2011) propose the criterion of *respecting improvement of school quality* and show that any stable mechanism approximately respects improvements of school quality when the school district is large.

3.8.3 Designing Priority

As pointed out in Section 3.8.1, there are deficiencies in all three mechanisms. One way to circumvent the deficiencies of a mechanism is to design the priority structure so that the shortcomings would never occur. Ergin (2002) shows the SOSM is efficient if and only if the priority structure is Ergin-acyclic. Kesten (2006) shows that the TTCM is stable if and only if the priority structure is Kesten-acyclic. Haeringer and Klijn (2009) consider an environment where students' submitted preferences have a limit on the number of schools that can be included in the submitted preferences and show that the Nash equilibrium outcomes in the TTCM are efficient if and only if the priority structure is X-acyclic and that the Nash equilibrium outcomes in the Boston mechanism (or the SOSM) are efficient if and only if the priority structure is strongly X-acyclic. Kojima (2011) shows that the SOSM is robustly stable if and only if the priority structure is Ergin-acyclic. Hatfield et al. (2011) show the equivalence of the existence of a stable (or efficient) mechanism and virtually homogeneous school preference profile. Kumano (2013) shows that the Boston mechanism is stable and strategy-proof if and only if the priority structure is Kumano-acyclic. Kojima (2013)

[33]The order-of-merit list is used to determine cadets' priority in branches.

considers an environment where agents have multi-unit demands and shows the equivalence of existence of a stable and efficient mechanism, the existence of a stable and strategy-proof mechanism, and the essentially homogeneous priority structure. Hsu (2013) shows that the Boston mechanism is dominance-solvable if and only if the priority structure is acyclic.

3.9 The Priority System in the Taiwanese High School Match

In this section, I present the priority system in Keelung and Taipei district. Students' priority in the centralized matching process is computed from the sum of the numeric values in three grand categories: the Ranking Score, the Diversified Learning Score, and the Exam Score. The maximum value in each category is 30. The components in each category are as follows.

- Category 1: Ranking Score
  - For school choices ranked from 1st to 10th: the first school choice receives a numeric value of 30, the second 29, and so on.
  - For school choices ranked from 11th to 20th: each school receives a numeric value of 20.
  - For school choices ranked from 21st to 30th: each school receives a numeric value of 18.
- Category 2: Diversified Learning Score
  - For the following criteria, only the last three semesters count.
  - Each of the fields, sports, literature and general, receives a numeric value of 6, if that field passes the minimum requirement for three semesters.

- Each of the three semester receives a numeric value of 4, if a student provides campus or community service with a total hours of 6 or more in that semester.

- Category 3: Exam Score
  - There are five subjects. The raw (numerical) scores are transformed into letter grades with A, B, and C.
  - A subject with a letter grade A receives a numeric value of 6.
  - A subject with a letter grade B receives a numeric value of 4.
  - A subject with a letter grade C receives a numeric value of 2.

Notice that an important feature in the above system is that the evaluation of students in Category 2 and Category 3 is "coarse" in terms of students' efforts. For example, consider the evaluation of providing campus or community service in the Category 2. A student who provides her campus or community service with six hours receives a numeric value of 4. However, if she provides her campus or community service with five and a half hours, instead of receiving a fraction of 4, she receives zero. This observation holds for all categories in Category 2 and Category 3.

If there is a tie, the following steps are used to break a tie.

Step 1 Use the numeric value from Diversified Learning Score to break the tie.

Step 2 If there is still a tie, use the following order of subjects to break tie: total, Chinese, Math, English, Social Science, Nature Science, Writing. (Comparison is done with letter grades, A, B, and C.)

Step 3 If there is still a tie, use the Ranking Score to break the tie.

Step 4 If there is still a tie, the priority is given to the students who have skill categories with grades $A^{++}, A^{+}, B^{++}, B^{+}$. The priority is given to the students with more grades A^{++}, then A^{+}, then B^{++}, and finally B^{+}.

Step 5 If there is still a tie, break the tie by comparing the grades in letter grades with A^{++}, A^{+}, A, B^{++}, B^{+}, B, and C in the skill categories with the following order: Chinese, Math, English, Social Science, Natural Science.

Step 6 If there is still a tie, use the following method for enrollment.

- For the schools that also hold individual entrant exams, transfer the capacity for admission by individual entrant exams by at most 5% to the capacity for admission by centralized matching process. If there are still more students than the capacity for admission by centralized matching process, the jurisdiction is on the education administration.
- For the schools the do not hold individual entrant exams, the jurisdiction is on the education administration.

Notice that the above tie-breaking rule essentially employs a pre-determined ordering over several skill categories so that a tie is broken in favor of a student who has a higher evaluation in the skill category with a higher ordering.

3.10 Omitted Results and Proofs in the Main Text

3.10.1 Respecting Improvement

A desirable property of a mechanism is that it should not punish a student if her priority is improved.

Definition 17 (Blinski and Sönmez (1999))**.** A mechanism **respects improvement of priority** if a student's priority in some schools improves, then she will obtain a weakly better school.

Blinski and Sönmez (1999) show that the SOSM respects improvements of priority. The following theorem says that if a mechanism is strategy-proof and stable, then it respects improvement of priority.

Theorem 35. *If a mechanism φ is strategy-proof and stable, then it respects improvement of priority.*

Proof. Consider a mechanism φ that is strategy-proof and stable. Suppose it does not respect improvement of priority. Then there exists a student i and two priority structures f and f', where (1) $f'_s(i) \leq f_s(i)$ for all $s \in S$ and $f'_{s'}(i) < f_{s'}(i)$ for some $s' \in S$ and (2) the relative rankings of all students other than i are the same in both f and f', such that i obtains a worse school under f' than under f. Denote the matching produced by φ under (P, f, q) be μ_1 and the matching under (P, f', q) be μ_2. Let $\mu_1(i) = s_1$ and $\mu_2(i) = s_2$. By construction, $s_1 P_i s_2$. Since φ is a stable matching, then μ_1 is stable in the problem (P, f, q). Moreover, by construction, $f'_{s_1}(i) \leq f_{s_1}(i)$. Therefore, μ_1 is stable in the problem (P, f', q). Consider the strategy Q_i, where

$$Q_i : s_1.$$

Note that the matching μ_1 is stable in the problems (P', f, q) and (P', f', q), where $P' = (Q_i, P_{-i})$. In the problem (P', f', q), the mechanism will match i to s_1. This is because of the rural hospital theorem[34] and the fact that μ_1 is stable in the problem (P', f', q). Therefore, when the priority structure is f', student i has incentive to deviate from reporting the true preference P_i. This contradicts the assumption that the mechanism φ is strategy-proof. $\square$

The following theorem says that the TTCM respects improvement of priority.

Theorem 36. *The Top-Trading Cycles Mechanism respects improvement of priority.*

Proof. Consider a student i and two priority structures f and f', where (1) $f'_s(i) \leq f_s(i)$ for all $s \in S$ and $f'_{s'}(i) < f_{s'}(i)$ for some $s' \in S$ and (2) the

[34] The rural hospital theorem says that for a given problem, a student who is matched to some school in one stable matching will be matched to some school in all stable matchings. See Theorem 5.12 in Roth and Sotomayor (1990).

relative rankings of all students other than i are the same in both f and f'. Assume that i is in a cycle at the iteration t in the problem (P, f, q), and assume i is in a cycle at the iteration t' in the problem (P, f', q). Since in the problem (P, f', q), the number of schools that point to i before iteration t' increases comparing to the problem (P, f, q), we have $t' \leq t$. Therefore, the school that i obtains in the problem (P, f', q) weakly improves comparing to the problem (P, f', q). $\square$

3.10.2 Proof of Theorem 20

By Blinski and Sönmez (1999), the SOSM respects improvement of priority. By Theorem 36, the TTCM respects improvement of priority. Since both the SOSM and TTCM respects improvement of priority, it is a partial dominant strategy for each student to maximize the final grade. Under φ^T_{SOSM} or φ^T_{TTCM}, it is equivalent to maximize their total scores.

3.10.3 Lemma 16

Lemma 16. *Consider i and two of her time allocations L_i and L'_i. Let $t \in argmax_{t' \in T} \mathcal{L}_i^{t'}(L_i)$ and $\hat{t} \in argmax_{t' \in T} \mathcal{L}_i^{t'}(L'_i)$. Let f^{SEA} be the priority structure generated by (L_i, L_{-i}), and let $\widetilde{f}^{SEA}$ be the priority structure generated by (L'_i, L_{-i}). If $\mathcal{L}_i^t(L_i) = \mathcal{L}_i^{\hat{t}}(L'_i)$, then $f_s^{SEA}(i) = \widetilde{f}_s^{SEA}(i)$ for all $s, \in S$. Moreover, if $\mathcal{L}_i^t(L_i) > \mathcal{L}_i^{\hat{t}}(L_i)$, then $f_s^{SEA}(i) \leq \widetilde{f}_s^{SEA}(i)$ for all $s \in S$.*

Proof. Let the highest score resulted from L_i be $\tau(i)$ and the highest score resulted from L'_i be $\tau(i)'$. If $\mathcal{L}_i^t(L_i) = \mathcal{L}_i^{\hat{t}}(L'_i)$, then $\tau(i) + m_i^s = \tau(i)' + m_i^s$. Therefore, $f_s^{SEA}(i) = \tilde{f}_s^{SEA}(i)$. If $\mathcal{L}_i^t(L_i) > \mathcal{L}_i^{\hat{t}}(L'_i)$, then $\tau(i) + m_i^s > \tau(i)' + m_i^s$. Therefore, $f_s^{SEA}(i) \leq \tilde{f}_s^{SEA}(i)$. $\square$

3.10.4 Proof of Theorem 21

Suppose each student has one unique talent. By Lemma 16, for any given time allocation of her opponents L_{-i}, if a student i allocates time L_i such that she maximizes the score on the subject of her unique talent, then she has the highest possible priority comparing to other time allocation L_i'. By Blinski and Sönmez (1999), the SOSM respects improvement of priority. By Theorem 36, the TTCM respects improvement of priority. Since the SOSM and the TTCM respect improvement of priority, the time allocation L_i is a partial dominant strategy for student i. Suppose some students have more than one unique talent. With a similar argument, it is a partial dominant strategy for such a student to spend all her time one a skill category of her unique talents.

3.10.5 Proof of Theorem 22

Consider the problem. Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$, and $q = (1, 1)$. Let $m_i^s = 0$ for all $i \in I$ and $s \in S$. The preference profile P is the following.

$$P_{i_1} : s_1, s_2$$

$$P_{i_2} : s_1, s_2$$

Let $T = \{t_1, t_2, t_3\}$, $\bar{L} = 1$ and $\eta = (0.12, 0.18, 0.3)$. The learning curves of students are the following.

$$
\begin{aligned}
\mathcal{L}_{i_1}^{t_1}(l) =& \frac{2}{10}\sqrt{l} \\
\mathcal{L}_{i_1}^{t_2}(l) =& \frac{3}{10}\sqrt{l} \\
\mathcal{L}_{i_1}^{t_3}(l) =& \frac{5}{10}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_1}(l) =& \frac{2}{10}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_2}(l) =& \frac{2}{10}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_3}(l) =& \frac{2}{10}\sqrt{l}
\end{aligned}
$$

Let $L_1 = (\frac{15}{31}, \frac{10}{31}, \frac{6}{31})$ and $L_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Note that L_1 and L_2 are the unique time allocations that maximize total scores for i_1 and i_2, respectively. Also note that i_1 has a higher total scores than i_2. The generated priority structure is the following.

$$
\begin{aligned}
f_{s_{i_1}} :& i_2, i_1 \\
f_{s_{i_2}} :& i_2, i_1
\end{aligned}
$$

The unique stable matching μ is the following. (Note that it is also the matching created by the TTCM.)

$$
\mu = \begin{pmatrix} s_1 & s_2 \\ i_2 & i_1 \end{pmatrix}.
$$

Based on L_{i_1} and L_{i_2}, i_1 has no subjects that achieve the thresholds. If i_1 has a time allocation $L'_{i_1} = (0.5, 0.5, 0)$, then he will have two subjects that achieve the thresholds. The priority generated by (L'_{i_1}, L_{i_2}) becomes the following.

$$f_{s_1} : s_1, s_2$$
$$f_{s_2} : s_1, s_2$$

The unique stable matching μ' is the following. (This is also the matching created by the TTCM.)

$$\mu' = \begin{pmatrix} s_1 & s_2 \\ i_1 & i_2 \end{pmatrix}.$$

Since i_1's welfare improves, L_{i_1} is not a partial dominant strategy for her. Therefore, $\varphi_{\omega}^{T,d}$ does not promote minimum efforts η.

3.10.6 Proof of Theorem 23

By Theorem 35 and Theorem 36, the mechanism ω respects improvement of priority. Therefore, it is a partial dominant strategy to maximize the final grade. Under $\varphi_{\omega}^{\eta,d}$, it is equivalent to maximize the number of subjects that achieve the thresholds.

3.10.7 Proof of Theorem 24

By Blinski and Sönmez (1999), the SOSM respects improvement of priority. By Theorem 36, the TTCM respects improvement of priority. Since both the SOSM and the TTCM respect improvement of priority, it is a partial dominant strategy for each student to maximize her final grade. Under $\varphi_{SOSM}^{\eta,\overline{T}}$ or $\varphi_{TTCM}^{\eta,\overline{T}}$, it is equivalent for a student to allocate time such that it maximizes the number of subjects that achieve the thresholds and there does not exist another time allocation that results in the same number of subjects achieving thresholds and has a higher total scores. Note that whenever there is a tie in the number of subjects that achieve the thresholds, the score-based tie-breaker breaks the tie in favor of the student who has a higher total scores. Therefore, it is a partial dominant strategy to allocate time as described above.

3.10.8 Lemma 17

Lemma 17. *Under Assumption 2, consider a student i with her unique talent t. Let L_i and $\widetilde{L}_i$ be two time allocations such that $\sum_{t\in T} L_i^t + \sum_{t\in T} \widetilde{L}_i^t \leq \bar{L}$. For any given L_i, the time allocation $\widetilde{L}_i$ suh that $\widetilde{L}_i^t = \bar{L} - \sum_{t\in T} L_i^t$ gives i the highest score in subject t, comparing to other time allocation $\widehat{L}_i$ such that $\widehat{L} \neq \widetilde{L}$ and $\sum_{t\in T} L_i^t + \sum_{t\in T} \widetilde{L}_i^t \leq \bar{L}$.*

Proof. By Assumption 2, student i only has one unique talent. Moreover, by Assumption 2, $\mathcal{L}_i^t(l) > \mathcal{L}_i^{t'}(l)$ for all $l \in (0, \bar{L}]$ and $t' \in T \setminus \{t\}$. Therefore, if we fix L_i and let $L_i^t = \bar{L} - \sum_{t\in T} L_i^t$, the score in subject t is maximized. □

3.10.9 Proof of Theorem 25

Under Assumption 2, each student has only one unique talent. Consider student i with her unique talent t. Let L_i be a time allocation such that (1) it maximizes the number of subjects that achieve the thresholds and (2) there does not exists another time allocation $\widetilde{L}_i$ such that it achieves the same number of subjects that achieve the thresholds and $\sum_{t\in T} \widetilde{L}_i^t < \sum_{t\in T} L_i^t$. With Assumption 2, such time allocation L_i is unique.

Let $\widehat{L}_i^t = \bar{L} - \sum_{t\in T} L_i^t$. Then, let $\check{L}_i^t = L_i^t + \widetilde{L}_i^t$. Let $\check{L}_i = (\check{L}_i^t, L_i^{-t})$. By Lemma 17, the time allocation $\check{L}_i$ maximizes the score in subject t, given L_i.

By Blinski and Sönmez (1999), the SOSM respects improvement of priority. By Theorem 36, the TTCM respects improvement. Since the SOSM and the TTCM respect improvement of priority, then it is a partial dominant strategy to have time allocation $\check{L}_i$. Therefore, $\varphi_{SOSM}^{\eta,\overline{SEA}}$ and $\varphi_{TTCM}^{\eta,\overline{SEA}}$ respects unique talents with minimum efforts.

3.10.10 Proof of Theorem 26

Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, $q = (1, 1, 1)$. Students' preferences are the following.

$$P_{i_1} : s_1, s_2, s_3$$
$$P_{i_2} : s_1, s_2, s_3$$
$$P_{i_3} : s_2, s_1, s_3$$

Let $\tau^T = (30, 20, 19)$ and $v = (30, 20, 10)$. Moreover, let $m_i^s = 0$ for all $i \in I$ and $s \in S$. The generated priority structure of students is the following.

$$f_{s_1} : i_1, i_2, i_3$$
$$f_{s_2} : i_1, i_3, i_2$$
$$f_{s_3} : i_1, i_2, i_3$$

The unique stable matching μ is the following.

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_3 & i_2 \end{pmatrix}.$$

However, μ is not fair, since i_2 has a higher total scores than i_3 and prefers s_2 to s_3, but i_3 is matched to s_2.

3.10.11 Proof of Theorem 27

Note that a student i is proposing to a school at iteration t in the both φ_{BM}^d and $\varphi_{BM}^{v,d}$ only if she is rejected by all schools that she ranks as the first $t-1$ schools. Therefore, for any two students i, j who are proposing to the same school s at iteration t in either φ^d_{BM} or $\varphi^{v,d}_{BM}$, whether i or j is rejected by school s depends on their priorities, which are $\tau^T_i + m^s_i + v_t$ and $\tau^T_j + m^s_j + v_t$. It is equivalent to comparing $\tau^T_i + m^s_i$ and $\tau^T_j + m^s_j$. In other words, φ^d_{BM} is equivalent to $\varphi^{v,d}_{BM}$.

3.10.12 Proof of Theorem 28

By Theorem 27, $\varphi^{v,d}_{BM}$ is efficient, but it is not stable nor strategy-proof.

To see that $\varphi^{v,d}_{BM}$ is not fair, consider the following example. Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$ and $q = (1, 1, 1)$. The preferences of students are the following.

$$P_{i_1} : i_1, i_2, i_3$$
$$P_{i_2} : i_1, i_2, i_3$$
$$P_{i_3} : i_2, i_2, i_3$$

Let $\tau^T = (30, 20, 19)$ and $v = (30, 20, 10)$. Moreover, let $m^s_i = 0$ for $i \in I$ and $s \in S$. The generated priority structure is the following.

$$f_{s_1} : i_1, i_2, i_3$$
$$f_{s_2} : i_1, i_3, i_2$$
$$f_{s_3} : i_1, i_2, i_3$$

The matching μ created by $\varphi^{v,d}_{BM}$ is the following.

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_3 & i_2 \end{pmatrix}.$$

Since i_2 has a higher total scores than i_3 and i_3 is admitted to a school that is preferred to i_2's match, this is not a fair match.

3.10.13 Proof of Theorem 29

To see that the $\varphi^{v,d}_{SOSM}$ is stable, note that the only different between $\varphi^{v,d}_{SOSM}$ and φ^{d}_{SOSM} is that the priority of students in these two problems are changed. In addition, Gale and Shapley (1962) shows that for any given preferences of students and their priority, the φ^{d}_{SOSM} produces a stable matching. Therefore, $\varphi^{v,d}_{SOSM}$ produces a stable matching.

Tee see $\varphi^{v,d}_{SOSM}$ is not fair, consider a problem where $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, and $q = (1, 1, 1)$. The preferences of students are the following.

$$P_{i_1} : s_1, s_2, s_3$$
$$P_{i_2} : s_1, s_2, s_3$$
$$P_{i_3} : s_2, s_1, s_3$$

Let $\tau^T = (30, 20, 19)$ and $v = (30, 25, 20)$. Moreover, let $m_i^s = 0$ for all $i \in I$ and $s \in S$. The generated priority structure is the following.

$$f_{s_1} : i_1, i_2, i_3$$
$$f_{s_2} : i_1, i_3, i_2$$
$$f_{s_3} : i_1, i_2, i_3$$

Then the matching μ created by $\varphi^{v,d}_{SOSM}$ is the following.

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_3 & i_2 \end{pmatrix}.$$

The matching μ is not fair, because student i_2 has higher total scores than i_3, while i_3 is admitted to school s_2, and i_2 prefers s_2 to her matched school s_3.

Moreover, it also shows that $\varphi^{v,d}_{SOSM}$ is not strategy-proof. If i_2 submits

$$Q_{i_2} : s_2, s_1, s_3,$$

then the matching becomes

$$\mu' = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}.$$

Since i_2 is better off by submitting Q_{i_2}, $\varphi^{v,d}_{SOSM}$ is not strategy-proof.

To see $\varphi^{v,d}_{SOSM}$ is not efficient, consider another problem. Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$ and $q = (1, 1)$.

The preferences of students are the following.

$$\begin{aligned} P_{i_1} :& s_1, s_2 \\ P_{i_2} :& s_2, s_1 \end{aligned}$$

Let $\tau^T = (20, 20)$ and $v = (20, 10)$. Moreover, let $m_{i_1} = (0, 20)$ and $m_{i_2} = (20, 0)$. The matching μ'' created by $\varphi^{v,d}_{SOSM}$ is the following.

$$\mu'' = \begin{pmatrix} s_1 & s_2 \\ i_2 & i_1 \end{pmatrix}.$$

However, μ'' is Pareto dominated by the following matching μ'''.

$$\mu''' = \begin{pmatrix} s_1 & s_2 \\ i_1 & i_2 \end{pmatrix}.$$

Therefore, $\varphi^{v,d}_{SOSM}$ is not efficient.

3.10.14 Proof of Theorem 30

To see that $\varphi^{v,d}_{TTCM}$ is efficient, note that the difference between $\varphi^{v,d}_{TTCM}$ and φ^{d}_{TTCM} is that the priority of students in these two problems are different.

Since Abdulkadiroğlu and Sönmez (2003) show that φ^{d}_{TTCM} is efficient for any preferences and priority of students, $\varphi^{v,d}_{TTCM}$ is efficient.

To see $\varphi^{v,d}_{TTCM}$ is not strategy-proof, consider the following problem. Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$ and $q = (1, 1, 1)$. The preferences of students are the following.

$$\begin{aligned} P_{i_1} :& s_1, s_2, s_3 \\ P_{i_2} :& s_1, s_2, s_3 \\ P_{i_3} :& s_1, s_2, s_3 \end{aligned}$$

Let $\tau = (30, 20, 19)$ and $v = (30, 25, 20)$. Moreover, let $m_i^s = 0$ for all $i \in I$ and $s \in S$.

The matching produced by $\varphi^{v,d}_{TTCM}$ when all students report the true preferences is the following.

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}.$$

However, if i_3 reports Q_{i_3}, where

$$Q_{i_3} : s_2, s_1, s_3,$$

then the matching becomes

$$\mu' = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_3 & i_2 \end{pmatrix}.$$

Since i_3 is better off by reporting Q_{i_3}, $\varphi^{v,d}_{TTCM}$ is not strategy-proof.

To see $\varphi^{v,d}_{TTCM}$ is not fair, consider another problem. Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $q = (1, 1, 1, 1)$. The preferences of students are the following.

$$P_{i_1}: s_1, s_2, s_3, s_4$$
$$P_{i_2}: s_1, s_2, s_3, s_4$$
$$P_{i_3}: s_2, s_1, s_3, s_4$$
$$P_{i_4}: s_3, s_1, s_2, s_4$$

Let $\tau = (30, 29, 28, 27)$ and $v = (30, 25, 20, 15)$. Moreover, let $m_i^s = 0$ for all $i \in I$ and $s \in S$.

The matching produced by $\varphi^{v,d}_{TTCM}$ when students submit the true preferences is the following.

$$\mu'' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_1 & i_3 & i_4 & i_2 \end{pmatrix}.$$

Since i_2 has a higher total scores than i_3 and i_3 obtains a school that i_2 prefers to her match, the matching is not fair.

To see $\varphi^{v,d}_{TTCM}$ is not stable, consider the following example. Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$, and $q = (1, 1)$. The preferences of the students are the following.

$$P_{i_1}: s_1, s_2$$
$$P_{i_2}: s_2, s_1$$

Let $\tau^T = (20, 20)$ and $v = (20, 10)$. Moreover, let $m_{i_1} = (0, 20)$ and $m_{i_2} = (20, 0)$. The matching μ''' created by $\varphi^{v,d}_{TTCM}$ is the following.

$$\mu''' = \begin{pmatrix} s_1 & s_2 \\ i_1 & i_2 \end{pmatrix}.$$

However, μ''' is not stable.

3.10.15 Proof of Theorem 31

Consider the problem in the proof of Theorem 22. Note that if $\varphi_{\omega}^{\eta,\succ_T}$ promotes versatility of talents, then (L_{i_1}, P_{i_1}) and (L_{i_2}, P_{i_2}) are the dominant strategies for students i_1 and i_2, respectively. However, if student i_1 uses the time allocation $L'_{i_1} = (0.5, 0.5, 0)$, then the generated priority structure becomes the following.

$$f_{s_1} : i_1, i_2$$

$$f_{s_2} : i_1, i_2$$

The unique stable matching μ' is the following. (Note that it is also the matching created by the TTCM.)

$$\mu' = \begin{pmatrix} s_1 & s_2 \\ i_1 & i_2 \end{pmatrix}.$$

Therefore, (L_{i_1}, P_{i_1}) is not a dominant strategy for i_1. In other words, $\varphi_{\omega}^{\eta,\succ_T}$ does not respect versatility of talents.

3.10.16 Proof of Theorem 32

Consider the case that ω is a strategy-proof and stable mechanism.

Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$ and $q = (1, 1)$. The preference profile P is the following.

$$P_{i_1} : s_1, s_2$$

$$P_{i_2} : s_1, s_2$$

Let $T = \{t_1, t_2, t_3\}$, $\bar{L} = 10$ and $\eta = (0.4, 0.6, 0.9)$. Moreover, let $m_i^s = 0$ for all $i \in I$ and for all $s \in S$. Let the lexicographic tie-breaker to be $\succ^T = t_1, t_2, t_3$. Let the learning curves of students to be the following.

$$\begin{aligned}
\mathcal{L}_{i_1}^{t_1}(l) =& \frac{2}{10}\sqrt{l} \\
\mathcal{L}_{i_1}^{t_2}(l) =& \frac{3}{10}\sqrt{l} \\
\mathcal{L}_{i_1}^{t_3}(l) =& \frac{5}{10}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_1}(l) =& \frac{2}{10}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_2}(l) =& \frac{3}{10}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_3}(l) =& \frac{1}{10}\sqrt{l}
\end{aligned}$$

The time allocation $L_{i_1} = (4, 4, 2)$ and $L_{i_2} = (4, 6, 0)$ are the unique time allocations that give i_1 and i_2, respectively, the maximal number of subjects that achieve the thresholds and there does not exist another time allocation that have the same number of subjects that achieve the thresholds and results in a higher total scores.

The generated priority structure is the following.

$$\begin{aligned}
f_{s_1} :& i_1, i_2 \\
f_{s_2} :& i_1, i_2
\end{aligned}$$

The unique stable matching μ_1 is the following. (Note that this is also the matching created by the TTCM.)

$$\mu_1 = \begin{pmatrix} s_1 & s_2 \\ i_1 & i_2 \end{pmatrix}.$$

If $\varphi_{\omega}^{\eta,\succ_T}$ respects versatility of talents with minimum efforts, then (L_{i_1}, P_{i_1}) and (L_i, P_{i_2}) are the dominant strategies for i_1 and i_2, respectively. However, if i_2 uses the time allocation $L'_{i_2} = (6, 4, 0)$, the generated priority structure is the following.

$$f_{s_1} : s_2, s_1$$
$$f_{s_2} : s_2, s_1$$

The unique stable matching μ_2 is the following. (Note that it is also the matching created by the TTCM.)

$$\mu_2 = \begin{pmatrix} s_1 & s_2 \\ i_2 & i_1 \end{pmatrix}.$$

Therefore, $\varphi_{\omega}^{\eta,\succ_T}$ does not respect versatility of talents with minimum efforts.

3.10.17 Proof of Theorem 33

Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2\}$ and $q = (1, 1)$. The preference profile of students is the following.

$$P_{i_1} : s_1, s_2$$
$$P_{i_2} : s_1, s_2$$

Let $T = \{t_1, t_2\}$, $\bar{L} = 1$ and $\eta = (\frac{1}{3}, \frac{2}{5})$. Moreover, let $m_i^s = 0$ for all $i \in I$ and for all $s \in S$. Let the lexicographic tie-breaker to be $\succ^T = t_1, t_2$. The learning curves of students are the following.

$$
\begin{aligned}
\mathcal{L}_{i_1}^{t_1}(l) =& \frac{1}{3}\sqrt{l} \\
\mathcal{L}_{i_1}^{t_2}(l) =& \frac{1}{5}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_1}(l) =& \frac{11}{30}\sqrt{l} \\
\mathcal{L}_{i_2}^{t_2}(l) =& \frac{2}{5}\sqrt{l}
\end{aligned}
$$

Consider the time allocation $L_{i_1} = (1, 0)$ and $L_{i_2} = (0, 1)$. If $\varphi_{\omega}^{\eta, \succ_T}$ respect unique talents, then (L_{i_1}, P_{i_1}) and (L_{i_2}, P_{i_2}) are dominant strategies for student i_1 and i_2, respectively.

The generated priority structure is the following.

$$
\begin{aligned}
f_{s_1} :& i_1, i_2 \\
f_{s_2} :& i_1, i_2
\end{aligned}
$$

The unique stable matching μ is the following. (Note that it is also the matching created by the TTCM.)

$$
\mu = \begin{pmatrix} s_1 & s_2 \\ i_1 & i_2 \end{pmatrix}.
$$

However, if i_2 uses the time allocation $L'_{i_2} = (1, 0)$, the generated priority structure becomes the following.

$$
\begin{aligned}
f_{s_1} :& i_2, i_1 \\
f_{s_2} :& i_2, i_1
\end{aligned}
$$

The unique stable matching μ' is the following. (Note that it is also the matching generated by the TTCM.)

$$\mu' = \begin{pmatrix} s_1 & s_2 \\ i_2 & i_1 \end{pmatrix}.$$

Therefore, (L_{i_2}, P_{i_2}) is not the dominant strategy for i_2. In other words, $\varphi_\omega^{\eta,\succ_T}$ does not respect unique talents.

3.10.18 Proof of Theorem 34

Consider the problem in the proof of Theorem 33. If $\varphi_\omega^{\eta,\succ_T}$ respects unique talents with minimum efforts, then the time allocations (L_{i_1}, P_{i_1}) and (L_{i_2}, P_{i_2}) are the dominant strategies for i_1 and i_2, respectively, where $L_{i_1} = (1,0)$ and $L_{i_1} = (0,1)$. However, as shown in the proof in Theorem 33, (L_{i_2}, P_{i_2}) is not a dominant strategy for i_2.

Chapter 4

Platform Markets and Matching with Contracts

Juan Fung[1] and Chia-Ling Hsu[2]

Abstract: We introduce a new application of two-sided matching. Agents are separated into two groups, and coordinate interactions across sides through their choice of a platform. The solution concept of stability captures successful coordination by eliminating switching in equilibrium. We use the structure of stable matching to derive two key properties of the equilibrium market structure: the seesaw principle and market tipping. Finally, we study an environment in which agents who join the same platform must share the same attributes. Such constrained stable allocations may not exist under the standard assumption of substitutable preferences. Under the additional requirement of lexicographic preferences, we use an algorithm to show a constrained stable allocation exists.

JEL: C78, D43, D47

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4.1 Introduction

In many markets, intermediaries facilitate interactions between participants across different sides. For example, dating websites such as eHarmony and Match.com serve as matchmakers for men and women. Such markets are characterized by "network effects": users value participation by the other side, and generate value to the other side from participating themselves. The intermediaries in these markets, or *platforms*, capture value by enabling coordination among users from each side.

We model platform markets in a two-sided matching setting. Agents are separated into two groups, and interactions across groups require an intermediary. A fundamental aspect of matching is that agents cannot simply choose, but must also be chosen. Agents care about the identities of potential match partners, not just the number. In other words, users value the quality of interactions and not just the quantity. Our approach thus complements the existing literature on platform markets, in which users only value the quantity of interactions.

Our analysis is based on the matching with contracts setting of Hatfield and Milgrom (2005). Agents not only choose one another, but must also agree on the terms of their relationship (e.g., wage, tasks, length of relationship). A *contract* summarizes this relationship. In our model, the contract terms specify an intermediary through which agents agree to interact, possibly coupled with membership terms.[3] Agents on each side mutually choose one another and a platform.[4] Platforms are thus objects of choice rather

[3]The "terms" are used loosely to mean the role an agent plays at a platform, and are quite general. They may include standard membership terms, such as interaction fees, length of contract, and cancellation penalties, as well as traits, such as political ideology, religious zeal, and education, chosen by participants.

[4]We focus on markets with single-homing, in which agents choose at most one platform.

than players, highlighting the role of a platform as a coordination device.

The solution concept of stability guarantees that no other matching is mutually preferred by any group of agents. This property is desirable for successful coordination, as it eliminates switching in equilibrium. Under the standard assumption of substitutable preferences, a stable allocation of contracts is guaranteed to exist. Moreover, the set of stable allocations has a particular structure that we exploit to derive two key properties of the equilibrium market structure: the seesaw principle and market tipping.

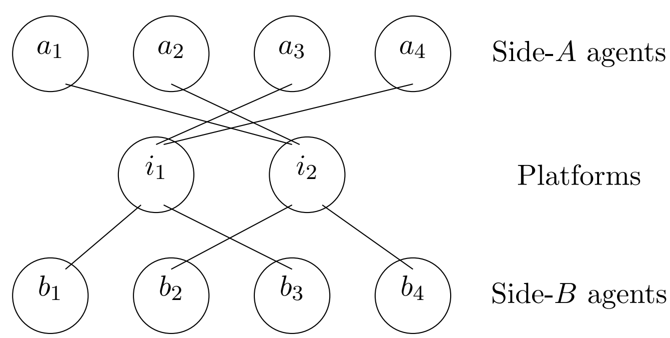
The seesaw principle roughly states that, in equilibrium, platforms subsidize one side in order to attract the other side at a higher price. In our setting, we use the lattice structure of the set of stable allocations to show that side-optimal allocations "favor" that side, in the sense of providing as many agents on that side with their most favorable terms, and as many agents on the opposite side with their least favorable terms. Consider the example of a nightclub: at the woman-optimal stable allocation, more women enter for free and more men pay the highest cover than at any other stable allocation. The result exploits the opposition of interests across sides with respect to stable allocations, and in particular the side-optimality of the allocations at the extreme points of the lattice.

Market tipping concerns the equilibrium balance of market power across platforms. The competition for users and the strength of network effects tends to result in most users allocated to one or a handful of platforms which dominate the market. We extend Hatfield and Milgrom (2005)'s law of aggregate demand (which roughly states that more contracts are better) to derive results on market tipping. Roughly, if all agents prefer more interactions with agents sharing similar attributes, then either agents with those traits all sort into one platform or every platform has equal market power. Consider, as an example, social media sites for sharing content such as photos, videos, and blogging with a general audience. Early on, users were scattered across many sites. As adoption of social media increased, users sorted into a dominant platform (for example, YouTube), or into specialized platforms (such as Tumblr and Pinterest).

Finally, we consider an environment in which agents who join the same platform must agree to the same terms. An interpretation of such an environment is that agents who join the same platform must share the same attributes. For example, an agent who wants to join a political interest group must agree with the agenda the group advocates. In this example, the preferences of agents are lexicographic: the political agenda is the most important component in the preferences, followed by the group (or platform) and the identities of the agents. When preferences are substitutable, a stable allocation is not guaranteed to exist. We show that a stable allocation exists with the additional assumption of lexicographic preferences.

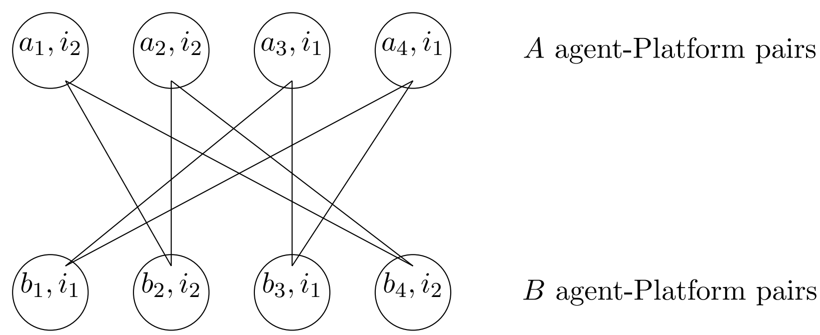
The setup is as follows. A two-sided market consists of disjoint sets of users (e.g., men and women), platforms (e.g., dating websites), and subscription terms (e.g., membership fees). Users directly choose a platform (and terms), but indirectly choose users on the other side; see Figure 4.1.

Figure 4.1: Many-to-one-to-many matching



We make this implicit choice explicit by defining contracts directly between users, with platforms and terms as objects of choice agreed upon in the contract. Focusing on user choice of such contracts frames the problem in a many-to-many matching with contracts setting; see Figure 4.2.[5] A key difference from the standard two-sided matching setting is that users are assumed to single-home (i.e., may only subscribe to one platform). Thus, an allocation must bind a user to the same platform at every contract.

Figure 4.2: Many-to-many matching



The next section reviews the related literature. Section 4.3 introduces the model. Section 4.4 presents the main results. Section 4.5 concludes. Proofs are in Section 4.6.

4.2 Literature Review

4.2.1 Two-Sided Markets

The literature on platform markets is extensive. In general, the focus is on platform pricing (rather than user interaction). The key feature of all these models is that agents typically care about the number of agents from the other side that join a platform and not about the identities of those agents.

[5]In our framework, agents can join a platform without interacting with all agents on the other side of market. Thus platforms provide access but not necessarily interaction with the other side. If we insist on interactions then stable allocations may not exist; see Example 12 in the Section 4.7. This assumption is standard in the platform markets literature; see Rochet and Tirole (2006), Weyl (2010).

Our approach, on the other hand, focuses on users with interaction-specific values rather than quantity-based values, taking platform behavior as given. The following is a selective review of several representative papers and is by no means exhaustive.

Rochet and Tirole (2003) model platforms competing for users that are heterogeneous in their value for interactions. They argue that markets with network effects are two-sided in the sense that interactions and profits depend on the structure of prices charged to each side rather than on the total price itself. In this sense, platforms compete by pricing to coordinate users. Armstrong (2006) also models competing platforms, but with users differentiated by how they value the platforms rather than interactions. Under single homing, Armstrong (2006) shows there is a continuum of equilibria when platforms use two-part tariffs that additionally charge users on one side a marginal price for additional users on the other. Rochet and Tirole (2006) present a 'canonical' monopoly platform model that incorporates both types of user heterogeneity. A general version of the seesaw principle is observed, arising from a linkage between both sides through the price.

Rather than analyzing pricing strategies explicitly, Weyl (2010) uses an allocation approach in which a platform chooses participation rates on each side. The prices, or "insulating tariffs," support the allocation by coordinating participation on each side so as to guarantee the platform its desired network size. Each network size corresponds to a unique profit and welfare, and so a flexible pricing mechanism is employed to sustain the network as an equilibrium. The use of prices as a means to sustain allocations is one of Weyl (2010)'s main contributions. White and Weyl (2012) extends Weyl (2010)'s framework to a setting with an unrestricted number of imperfectly competing platforms. They propose the solution concept of *insulated equilibrium*, the natural extension of Weyl (2010)'s allocation approach for competing platforms. The model includes Rochet and Tirole (2003) and Armstrong (2006) as special cases.

In so far as we focus on the resulting allocation rather than pricing, our approach shares some similarity to that of Weyl (2010) and White and Weyl (2012). The similarities end there, however, as we focus on allocations determined by agents' incentives to switch platforms rather than by platform objectives, and take prices as exogenous (i.e., stable allocations). The trade-off from increasing the scope of user heterogeneity is that we reduce the role of the platforms.

4.2.2 Two-Sided Matching

Hatfield and Milgrom (2005) and Hatfield and Kojima (2008) formally introduce the idea of a matching through biltaeral *contracts*, which fully summarize the relationship between two agents at a match. The matching with contracts framework generalizes preceding models of matching, including models with endogenous salaries (e.g., Kelso and Crawford (1982), Roth (1984)), and without, as special cases. Klaus and Walzl (2009) explore many-to-many matching with contracts further, examining the relationships between different notions of stability and substitutability in this context. One important result is that under substitutability, pairwise and setwise stability are equivalent.[6] Hatfield and Kominers (2012a) extends many of the results of Hatfield and Milgrom (2005) to the setting of many-to-many matching with contracts. Hatfield and Kojima (2010) propose weaker conditions that substitutes for recovering existence and other results in many-to-one matching with contracts. In Fung and Hsu (2014), we propose weaker conditions for existence in many-to-many matching with contracts.

[6]Echenique (2012) addresses the relationship between contracts and salaries in many-to-one matching, establishing a one-to-one mapping between Hatfield and Milgrom (2005)'s model of matching and contracts into Kelso and Crawford (1982)'s model of matching with salaries. Substitutable preferences over contracts is sufficient for the mapping, as it allows a global pairwise mapping of contracts into salaries. Kominers (2012) extends Echenique (2012)'s embedding to many-to-many matching with contracts, with a slightly stronger sufficient condition.

Ostrovsky (2008) extends the scope of (many-to-many) two-sided matching to a supply chain setting, in which several tiers of agents may interact with other agents, directly upstream or downstream or through intermediaries. Under a suitable notion of substitutable preferences, Ostrovsky (2008) shows the set of chain stable allocations is a nonempty finite lattice, extending classical results to more general setting. Hatfield and Kominers (2012b) generalizes Ostrovsky (2008) further by extending the supply chain structure to any network structure satisfying substitutes and an "acyclicity" condition. Moreover, they show chain stability is equivalent to stability. Hatfield and Kominers (2013) explores the welfare consequences of exit by agents at different tiers on stable allocations. Hatfield et al. (2013) dispenses with network structure altogether by introducing a numeraire good.

Recently, Sönmez and Switzer (2013) and Sönmez (2013) provide two of the first real-world applications of the matching with contracts framework, illustrating its usefulness in environments that fail the substitutes condition but nevertheless admit stable allocations.[7] The matching with contracts framework thus represents a significant step in the literature on matching.

4.3 Model

Let $S = A \cup B$ be the finite set of all agents in the market, and I the finite set of intermediaries, or platforms. Let a, b, and s denote generic agents in the sets A, B, and S, respectively, and let i denote a generic platform in the set I. Agents from A can only interact with agents from B through a platform in I. In this paper we assume that each agent can participate in at most one platform. In the language of the two-sided markets literature, both sides are **single-homing**. Moreover, an agent can choose not to participate in any platform.

[7] These settings satisfy Hatfield and Kojima (2010)'s weaker notion of unilateral substitutes, and are the first matching with contracts models that are not equivalent to Kelso and Crawford (1982).

When an agent $s \in S$ joins a platform, she may be required to agree to terms in a contract. The terms we consider here are rather broad. They may be positive or negative membership or transaction fees, length of relationship, or any other membership details specified by the platform. We call t_s the contract terms, or simply **terms**, imposed by a platform to agent s. Let $T \equiv T_A \times T_B$ denote the finite set of possible terms, where T_A are the terms for side A and T_B are the terms for side B. The sets T_A and T_B may be identical, disjoint, or have elements in common. We also assume that the set of available terms is the same for all platforms.

In the next subsections, we first describe how we incorporate platforms into the contract terms. We then discuss allocations and the equilibrium notion of stability. Finally, we introduce three classes of preferences: substitutable preferences, preferences that satisfy the law of aggregate demand, and a class of lexicographic preferences.

4.3.1 Platforms as objects

Consider an agent's choice of platform. When an agent chooses a platform directly, she indirectly chooses agents on the other side. In this paper, we treat platforms as objects of choice—that is, platforms do not have preferences. As such, the only relevant choices concern agents $s \in S$.

A **contract** x associates an agent from one side of the market to an agent on the other side through a platform and associated contract terms. Let X be the set of all contracts. Formally, a typical contract $x \in X$ between agents $a \in A$ and $b \in B$ takes the form

$$x = (a, b, i, t_a, t_b) \in X \equiv A \times B \times I \times T_A \times T_B.$$

For a contract $x \in X$, let $x_A \in A$ denote the side A agent, $x_B \in B$ denote the side B agent, $x_I \in I$ denote the platform associated with contract x, and $x_T = (x_{T_A}, x_{T_B}) \in T_A \times T_B$ denote the pair of terms associated with contract x. Agents may sign any number of contracts in general but may not sign multiple contracts with the same agent. Kominers (2012) calls such environments **unitary**. Moreover, agents are free to choose an outside option instead of joining any platform, represented by the **null contract**, $\varnothing$. Thus, we are in a many-to-many matching with contracts setting.

For a given set of contracts $Z \subseteq X$, let $Z^a \subseteq Z$ denote the contracts in Z associated with agent $a \in A$. Analogous definitions hold for $Z^b \subseteq Z$ and $Z^i \subseteq Z$. Also, let $Z_A \subseteq A$ denote the set of side A agents associated with Z. Analogous definitions hold for $Z_B \subseteq B, Z_I \subseteq I$, and $Z_T = Z_{T_A} \times Z_{T_B} \subseteq T_A \times T_B$.

We write $(Z^a)_B \subseteq B$ to indicate the side B agents with contracts in Z^a. A similar definition applies for $(Z^a)_I \subseteq I, (Z^a)_T \subseteq T_A \times T_B$, and so on.

Before formally introducing agent preferences, several issues that are unique to our model must be discussed.

Consider an agent $a \in A$. If a signs multiple contracts then by single-homing, the platforms must be the same across all those contracts.

Formally, a set of contracts $Z \subseteq X$ is **feasible** if

1. $\forall a \in A$: (i) $z_B = z'_B \Rightarrow z = z', \forall z, z' \in Z^a$, and (ii) $(Z^a)_I = \{i_a\}$
2. $\forall b \in B$: (i) $z_A = z'_A \Rightarrow z = z', \forall z, z' \in Z^b$, and (ii) $(Z^b)_I = \{i_b\}$

In other words, Z is feasible if each agent with contracts in Z holds contracts that are (i) unitary and (ii) single-homing.

To illustrate feasibility, consider an agent $a \in A$ and suppose $B = \{b_1, b_2\}$, $I = \{i_1, i_2\}$, and the terms are $T_A = \{t_L, t_H\}$ and $T_B = \emptyset$. The set $Z = \{(a, b_1, i_1, t_L), (a, b_1, i_1, t_H)\}$ is not unitary for any agent but satisfies single-homing for both agents, while the set $Z' = \{(a, b_1, i_1, t_L), (a, b_2, i_2, t_L)\}$ is unitary for both agents but violates single-homing for a. On the other hand, the set $Z^* = \{(a, b_1, i_2, t_L), (a, b_2, i_2, t_L)\}$ is feasible.

Let P_s denote s's **preference relation** over feasible sets of contracts. For ease of analysis, we assume this ordering is strict. Moreover, for an agent s choosing between two contracts that only differ with respect to the terms for the agent on the other side, we assume s breaks ties with respect to the other agent's preferences. This is reasonable since the terms are set to facilitate coordination along a platform, and are not internalized by the agents themselves. For instance, if we interpret terms as membership fees, then these are paid to the platform and not the agent on the other side. Let $P \equiv (P_s)_{s \in S}$ denote the profile of strict preferences over sets contracts. For a given set of available contracts $Z \subseteq X$, agent s's **choice set** is defined as

$$C_s(Z) \equiv \{Z' \subseteq Z : Z' P_s Z'', \forall Z'' \subseteq Z \text{ feasible and } Z' \text{ feasible}\}$$

In other words, when agent $s \in A \cup B$ is offered a set of contracts Z, she chooses the feasible subset from Z that she most prefers with respect to her preferences, P_s.

It is also useful to define s's **rejected set**,

$$R_s(Z) \equiv Z \setminus C_s(Z).$$

Let $C_A(Z) \equiv \cup_{a \in A} C_a(Z)$ and $C_B(Z) \equiv \cup_{b \in B} C_b(Z)$ denote the set of contracts from Z chosen by all $a \in A$ and all $b \in B$, respectively. The rejected sets for each side are then $R_A(Z) \equiv Z \setminus C_A(Z)$ and $R_B(Z) \equiv Z \setminus C_B(Z)$.

4.3.2 Stable allocations

An **allocation** $Y \subseteq X$ is a feasible set of contracts. That is, each $s \in S$ signs unitary contracts associated with the same platform. Preferences over allocations correspond directly to the underlying preferences over sets of contracts.

The fundamental solution concept for matching is stability. Our notion of stability follows from the notion of weak setwise stability in the matching with contracts literature (Klaus and Walzl (2009), Hatfield and Kominers (2012a)).

An allocation Y is **stable** if the following holds:

1. **Individual rationality**: $C_s(Y) = Y^s$, for $s \in S$.

2. **No blocking set of contracts**: $\nexists Y' \subseteq X$ such that $Y^s \subseteq C_s(Y \cup Y')$, for all $s \in (Y)_S$.

Individual rationality requires that an agent s voluntarily participates in allocation Y by accepting all available contracts in Y. Intuitively, an agent s who drops some or all of the contracts in Y^s is dissatisfied with some of the agents on the other side or his platform. A set Y' is a blocking set for Y if some or all of the contracts in Y are replaced by Y' for some group of users. Intuitively, users on opposite sides of a platform mutually agree to switch to another platform, to update their subscription terms, or to swap match partners.

Let $\mathcal{S}(P)$ be the set of all stable allocations with respect to preferences P. An allocation Y is **pairwise stable** if it is individually rational and $\nexists z \in X \setminus Y$ such that $z \in C_{z_A}(Y \cup \{z\}) \cap C_{z_B}(Y \cup \{z\})$. The term "pairwise" highlights the fact that an agent pair $(a, b) \in A \times B$ associated with z can block an allocation Y. Let $\mathcal{PS}(P)$ denote the set of pairwise stable allocations.

A desirable property for successful coordination in equilibrium is that agents are not constantly switching, be it their platform, subscription terms, or match partners. Stability captures this very notion and is thus a reasonable criterion for allocations in platform markets.

4.3.3 Preferences

Substitutable preferences

In order to guarantee existence of a stable allocation, it is standard in the matching literature to assume a preference domain that rules out certain complementarities over contracts. Preferences of $s \in S$ over contracts are **substitutable** if

$$Z \subseteq Z' \subseteq X \Rightarrow R_s(Z) \subseteq R_s(Z').$$

In other words, offering new contracts does not create complementarities with previously rejected contracts. The latter continue to be rejected. An equivalent definition is that $z \in C_s(Y \cup \{x, z\}) \Rightarrow z \in C_s(Y \cup \{z\}), \forall x, z,$ and $Y \subseteq X$. In other words, the choice of a contract z does not depend on the availability of another contract x. Under substitutability, weak setwise stability coincides with pairwise stability.[8] Checking pairwise deviations is enough. Moreover, substitutes guarantees existence, and a unique structure, of the set of stable allocations:

Result 1 (Hatfield and Kominers (2012a)). *When preferences are substitutable, the set of stable allocations is nonempty and forms a lattice.*

The lattice structure implies that there exists a partial ordering over stable allocations, where stable allocations more preferred by one side are less preferred by the other side. Moreover, the extreme points of the lattice are optimal in this sense. In particular, there exists a stable allocation that side A agents prefer to all other stable allocations, the **A-optimal stable allocation**, and this allocation is the least preferred stable allocation for side B agents, the **B-pessimal stable allocation**. Similarly, the most preferred stable allocation for side B agents, the **B-optimal stable allocation**, is the least preferred stable allocation for side A agents, the **A-pessimal stable allocation**.

Law of Aggregate Demand

Following Hatfield and Milgrom (2005) for many-to-one matching markets and Hatfield and Kominers (2012a) for many-to-many matching markets, we say that preferences satisfy the **law of aggregate demand**, if for agent $s \in S$, and $X' \subseteq X'' \subseteq X$, we have $|C_s(X')| \leq |C_s(X'')|$.

Intuitively, if more contracts become available, agent s demands weakly more contracts. The law of aggregate demand has a natural interpretation

[8] See Theorem 2(i) in Klaus and Walzl (2009), and Lemma 7 of Hatfield and Kominers (2012a).

in two-sided markets, which is the network externality generated purely by having more agents to interact with. In other words, more contracts are better. In section 4.4.2, we introduce an extension of the law of aggregate demand that is defined only with respect to the subscription terms.[9]

Lexicographic Preferences

We introduce a class of preferences that is particularly relevant to platform markets: the class of lexicographic preferences. We say that preferences are lexicographic if agents consider the desirability of contracts in a particular order with respect to the components of the contracts: the agents from the other side, the choice of platform, and the subscription terms. In particular, we consider an environment where each agent first considers the set of agents on the other side, followed by the platform, and finally the terms.[10]

Definition 18. Consider an agent $a \in A$. Preferences of a are **lexicographic** if whenever any of the following conditions hold,

1. $B_1 = (Y_1^a)_B = (\tilde{Y}_1^a)_B \neq (Y_2^a)_B = (\tilde{Y}_2^a)_B = B_2$,
2. $B_1 = B_2$ and $i_1 = (Y_1^a)_I = (\tilde{Y}_1^a)_I \neq (Y_2^a)_I = (\tilde{Y}_2^a)_I = i_2$,
3. $(B_1, i_1) = (B_2, i_2)$ and $t_1 = (Y_1^a)_T = (\tilde{Y}_1^a)_T \neq (Y_2^a)_T = (\tilde{Y}_2^a)_T = t_2$.

for any sets $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \subseteq X$, then

$$C_a(Y_1 \cup Y_2) = Y_1 \quad \Rightarrow \quad C_a(\tilde{Y}_1 \cup \tilde{Y}_2) = \tilde{Y}_1. \tag{4.1}$$

Moreover, for any $\hat{Y} \subseteq X$ such that $(\hat{Y}^a)_{B,I} = (\hat{B}, \hat{i})$,

$$C_a(\hat{Y})_{B,I} = (\hat{B}, \hat{i}) \Rightarrow C_a(Y)_{B,I} = (\hat{B}, \hat{i}), \quad \forall Y \subseteq X \text{ such that } (Y^a)_{B,I} = (\hat{B}, \hat{i}). \tag{4.2}$$

[9] The law of aggregate demand is used to prove a version of the rural hospitals theorem for matching with contracts; see Section 4.7.1 for details and a related result.

[10] Our results also hold if we define the lexicographic ordering with platforms as most important, followed by the agents from the other side, and finally the contract terms.

The first equation requires that an agent evaluates any set of contracts lexicographically, first in terms of the set of agents on the other side, followed by the platform, and finally by the contract terms. The second equation requires that if an agent finds a $(\hat{B}, \hat{i})$-combination acceptable at some set of contracts $\hat{Y}$, then the agent finds this combination acceptable at any set of contracts Y; that is, at any $t \in T$.

Note that it is possible that $C_a(Y_1 \cup Y_2) = \varnothing$ in equation (4.1). In particular, we do not require that every (B', i', t') combination be acceptable to a, only that any $t' \in T$ is acceptable.

Lexicographic preferences may be relevant in several natural settings. Consider the example of a nightclub for men and women. Men might reasonably consider the group of women in the club above the club itself or the cover charge. Similarly, agents joining dating websites, religious organizations, or political groups, may evaluate choices lexicographically based on a desire for shared attributes. Lexicographic preferences play an important role in section 4.4.3 when stable allocations have an additional constraint.

4.4 Main Results

In this section, we use the structure of the set of stable allocations to study equilibrium market structure. In particular, we exploit the lattice structure to derive two key properties in platform markets: the seesaw principle and market tipping. Later, we consider an environment in which agents who join the same platform must agree to the same subscription terms.

4.4.1 The seesaw principle

Consider platform markets where the only subscription terms are prices, and suppose there are only two prices: high and low. Rochet and Tirole (2006) observe a tendency for all agents on one side to pay the low price while agents on the other side pay the high price. The intuition is that platforms may subsidize one side in order to attract the other and thus capture a larger overall market share. This result is sometimes called the "seesaw principle." In this section, we use our framework to capture this phenomenon.

Nightclubs sometimes promote "Ladies' Night" events, which allow women to enter for free while men pay a cover charge. Dating websites provide another example where this is observed. What's Your Price allows men ("Generous Members") to bid for dates with women ("Beautiful Members"), who may accept incoming bids or make explicit asks. Men must purchase credits, in addition to having an offer accepted, in order to read and send messages. Ashley Madison, which caters to married men and women, requires men to buy credits to initiate contact with women on the site. Women may also buy credits, but in addition can initiate contact and have the *men* pay.

Suppose the set of subscription terms is $T_A = T_B = \{t_1, \dots, t_n\}$, where $t_1 < \cdots < t_n$. We interpret such ordered terms as "fees" and assume agents prefer low fees t_r to high fees t_s, where $r < s$, all else equal. The next result shows that when preferences are substitutable, the number of side-A agents who sign the lowest fee t_1 at the A-optimal stable allocation is weakly larger than in any other stable allocation. The opposite holds for the side-B agents.

Theorem 37. *Suppose preferences are substitutable. Then at the A-optimal stable allocation,*

1. *the number of side-A agents that sign t_1-contracts is weakly higher than at any stable allocation, and*
2. *the number of side-B agents that sign t_n-contracts is weakly higher than at any stable allocation.*

By symmetry, the number of side-A agents that sign t_n-contracts is weakly higher at the B-optimal stable allocation than at any other stable allocation, and so on. The proof, in Section 4.6, exploits the lattice structure and in particular side optimality.

In the example of dating websites, the result is interpreted as saying that at the woman-optimal stable allocation more women join for free and more men pay for premium membership (or message credits) than at any other stable allocation. This seems to be reflected in practice for some sites.

4.4.2 Market Tipping

In some situations, an agent may prefer to join a platform that draws agents from the other side with similar interests or attributes. Consider the example of social media sites for sharing content, such as Instagram for photos, YouTube for videos, and WordPress for blogging. Such sites not only host content, but provide a general audience for users generating their own content. While YouTube has come to dominate video sharing, photo sharing sites such as Instagram, Imgur, and Pinterest, each draw different types of users. Thus some markets may "tip" toward a single platform, while others may share a portion of the market based on niches. Another interpretation is that tipping occurs other platforms fail differentiate themselves with respect to "network effects" driven by user traits.

In this section, we interpret subscription terms as "traits." Let $x = (a, b, i, t_a, t_b)$, where t_a is the **trait** of agent a and t_b is the trait of agent b. Let T_A be the set of traits for agents on side A and T_B be the set of traits for agents on side B. Assume that $(X^a)_{T_A} = t_a$ for all $a \in A$ and $(X^b)_{T_B} = t_b$ for all $b \in B$. In other words, each agent has only one trait. Moreover, assume that $T_A = T_B$. The following condition on preferences says that agents prefer agents from other side with the same trait.

Definition 19. We say that preferences satisfy the **law of aggregate demand for similar traits**, if for an agent a with trait t_a and two sets of contracts Y and Y', we have the following.

$$|\{x \in Y : x_{T_B} = t_a\}| \geq |\{x \in Y' : x_{T_B} = t_a\}| \Rightarrow Y R_a Y';$$
$$|\{x \in Y : x_{T_B} = t_a\}| > |\{x \in Y' : x_{T_B} = t_a\}| \Rightarrow Y P_a Y'.$$

Theorem 38. *Suppose preferences satisfy substitutes and the law of aggregate demand for similar traits. Consider a trait t. In any stable allocation, either (1) all agents with trait t are associated with one platform or (2) they are associated with multiple platforms such that each platform has the same number of agents with trait t from each side.*

In the example of social media, all agents with the same trait t will join a dominant platform or else the market share of t agents will be distributed uniformly across platforms.

4.4.3 Trait-Constrained Stable Allocations

Our setting allows two agents on the same side to join the same platform at different subscription terms. In this section, we consider an environment in which agents from the same side joining the same platform must agree to the same subscription terms.

A natural interpretation of such environment is based on traits, as in the previous section. Recall each agent had a single trait at any contract. Suppose now each agent may be associated with several traits. The constraint on subscription terms is interpreted as requiring that agents who join the same platform have the same trait in any allocation.

Consider the example of political interest groups. Agents on one side are staff and active members, leading lobbying, fundraising, and outreach activities, while agents on the other side are potential donors and sponsors, such as businesses. Here, the platform is a group advocating a shared political agenda, which may be very narrow or consist of several related issues. For example, it may be an environmental organization focused specifically on promoting ethanol as a biofuel, or more broadly on advocating for green energy. The traits may specify such distinctions over issues, as well intensity of participation within the platform, say high and low.

The example has two important features that distinguish it from the previous examples. First, there is an additional restriction on an allocation that requires agents joining the same platform to advocate the same issue or set of issues, that is, the group's political agenda. Second, agents' preferences naturally express lexicographic order on several components. The most important component in their preferences is the political agenda, which we call a trait. The second most important component may be the platform, followed by the agents on the other side. This may occur if the interest group has a recognizable name, or is well established.

Note that the lexicographic ordering may be different. For instance, members and donors may be most drawn to platforms and issues advocated by individuals with strong political ties and businesses with a large public presence. In this case, agents are most important, followed perhaps by the agenda and the platform.

To allow flexibility of this interpretation, we again refer to subscription terms as **traits** in this section. In the example above, the traits are the various sets of issues a group advocates. A shared political agenda is thus a constraint on traits. Let T_A be the set of traits for side A and let T_B be the set traits for side B. In contrast to the previous section, we do not assume agents have only one trait. That is, $\exists s \in A \cup B$ and $z, z' \in X^s$ such that $t_s \neq t'_s$, where t_s, t'_s are the traits associated with agent s at contracts z, z', respectively.

We define the notion of a trait-constrained stable allocation as follows.

Definition 20. A set of contracts $Y \subseteq X$ is **trait-constrained feasible** if for $x, x' \in Y$, if $x_I = x'_I$ then $x_{T_A,T_B} = x'_{T_A,T_B}$. Let $\widetilde{X}$ denote the trait-constrained feasible set. An allocation Y is **trait-constrained stable**, if the following holds:

1. **Feasibility**: $Y \in \widetilde{X}$.
2. **Individual rationality**: $C_s(Y) = Y^s$, for $s \in A \cup B$.
3. **No feasible blocking set of contracts**: $\nexists Y' \in \widetilde{X}$, such that $Y'^s = C_s(Y \cup Y')$, for all $s \in (Y')_S$.

The requirement of individual rationality is the same, but now we impose feasibility on both the allocation Y and any potential blocking set Y'.

The following result states that a trait-constrained stable may not exist when the preferences of agents are substitutable.

Theorem 39. *When preferences are substitutable, the set of trait-constrained stable allocations may be empty.*

The proof is by the following example.

Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $I = \{i\}$, $T_A = \{t_1, t_2\}$, and $T_B = \emptyset$. Suppose preferences over contracts are as follows:[11]

$$\begin{aligned}
P_{a_1} &: \{(a_1, b_1, t_1)\}, \{(a_1, b_2, t_1)\} \\
P_{a_2} &: \{(a_2, b_2, t_2)\}, \{(a_2, b_1, t_2)\} \\
P_{b_1} &: \{(a_2, b_1, t_2)\}, \{(a_1, b_1, t_1)\} \\
P_{b_2} &: \{(a_1, b_2, t_1)\}, \{(a_2, b_2, t_2)\}
\end{aligned}$$

Note the preferences satisfy substitutes. When agents joining the same side must share the same trait, there are only four possible feasible and individual rational allocations, and one can easily check that none of these is stable:

$$\begin{aligned}
Y_1 &= \{(a_2, b_1, t_2)\} &&\rightarrow \text{blocked by } Y_2 \\
Y_2 &= \{(a_2, b_2, t_2)\} &&\rightarrow \text{blocked by } Y_3 \\
Y_3 &= \{(a_1, b_2, t_1)\} &&\rightarrow \text{blocked by } Y_4 \\
Y_4 &= \{(a_1, b_1, t_1)\} &&\rightarrow \text{blocked by } Y_1
\end{aligned}$$

Thus, there is no trait-constrained stable allocation. It is worth noting that these preferences also (trivially) satisfy the law of aggregate demand. Intuitively, the latter assumption would not help because it makes no demands on the contract terms.

[11] Since there is one platform and side B agents do not face a trait, we suppress the notation of platform i and the trait for side B in the listed contracts.

However, if we allow agents who join the same platform to have different traits, then there are two (unconstrained) stable allocations:

$$Y_5 = \{(a_1, b_1, t_1), (a_2, b_2, t_2)\}$$
$$Y_6 = \{(a_1, b_2, t_1), (a_2, b_1, t_2)\}$$

As mentioned above, in the example of a political interest group, it is natural that agents' preferences express a lexicographic order over several components. Other examples include charities, religious organizations, and professional associations. Under the additional assumption of lexicographic preferences, we have the following.

Theorem 40. *When preferences are substitutable and lexicographic, a trait-constrained stable allocation exists.*

4.5 Concluding remarks

The solution concept of stability is a reasonable equilibrium criterion as it eliminates switching. We use the structure of the set of stable allocations to analyze equilibrium market structure. The seesaw principle and market tipping arise as a consequence of users having match-specific values for agents on the other side. Finally, we show that under substitutable preferences, trait-constrained stable allocations may not exist. Using a modified cumulative offer algorithm, we show existence under the additional assumption of lexicographic preferences.

The main contribution of this paper is a new application of two-sided matching that provides a complementary perspective for analyzing certain platform markets. Our focus is on users rather than platforms, and the "microstructure" of interactions at platforms. This allows users to have preferences over the quality of interactions with agents on the other side, and not just the quantity. As Weyl (2010) points out, there is a gulf between the standard platform ("two-sided") markets literature, and the two-sided matching (and more broadly, market design) literature, though it would seem their intersection may be quite fruitful. This paper is a first attempt at bridging that gap.

4.6 Proofs

4.6.1 Proof of Theorem 37

Without loss of generality, let the set of terms be $T_A = T_B = \{t_L, t_H\}$, with $t_L < t_H$. The extension to $T_A = T_B = \{t_1, \ldots, t_n\}$ is straightforward.

Let $\overline{Y}$ denote the A-optimal stable allocation. Suppose $|\{a \in A : (\overline{Y}^a)_{T_A} = t_L\}| < |\{a \in A : (Y^a)_{T_A} = t_L\}|$, for some stable allocation Y. Then there is some agent $\overline{a} \in A$ that has either: (i) no contracts at $\overline{Y}$, or (ii) only t_H-contracts at $\overline{Y}$.

Consider case (i). If some $b \in (Y^{\overline{a}})_B$ has no contracts at $\overline{Y}$ then (a, b) form a blocking pair for $\overline{Y}$. So all $b \in (Y^{\overline{a}})_B$ have contracts with some side A agents in $\overline{Y}$. Moreover, these contracts must be preferred to those in Y or they can block $\overline{Y}$ with $\overline{a}$. But this contradicts $\overline{Y}$ as the B-pessimal stable allocation.

Consider case (ii). Let $y \in Y^{\overline{a}}$ such that $y_{T_A} = t_L$. Suppose $y_B \in (\overline{Y}^{\overline{a}})_B$, and let $\overline{y} \in \overline{Y}^{\overline{a}}$ be the associated contract. Suppose $y_I = \overline{y}_I$. By the assumption that all agents prefer t_L-contracts to t_H-contracts, and that ties with respect to terms on the other side are broken with respect to that side's preferences, $y \in C_{\overline{a}}(Y \cup \overline{Y}) \neq \overline{Y}^{\overline{a}}$, regardless of $y_{T_B}, \overline{y}_{T_B}$. This contradicts A-optimality of $\overline{Y}$. So $y_I \neq \overline{y}_I$, and there exists $z \in X^{\overline{a}}$ that coincides with $\overline{y}$ except $z_{T_A,T_B} = (t_L, t_L)$. Then $\overline{a}$ and y_B mutually prefer z to $\overline{y}$, contradicting stability of $\overline{Y}$.

Therefore $y_B \in Y^{\overline{a}} \Rightarrow y_B \notin (\overline{Y}^{\overline{a}})_B$. Moreover, such agents y_B must have contracts in $\overline{Y}$ or they can form a blocking pair for $\overline{Y}$ with $\overline{a}$. Thus such agents y_B have contracts with different side-A agents at $\overline{Y}$. These contracts are preferred by y_B to y, otherwise they can block $\overline{Y}$ with $\overline{a}$. But this contradicts $\overline{Y}$ as B-pessimal stable.

An analogous argument shows that the number of side-B agents that sign t_H-contracts at $\overline{Y}$ is weakly higher than at any other stable allocation. $\square$

4.6.2 Proof of Theorem 38

Consider an allocation X'. Without loss of generality, assume agents are associated with two platforms i and i' such that $|\{a \in A : (X'^a)_I = i \text{ and } (X'^a)_{T_A} = t\}| > |\{a \in A : (X'^a)_I = i' \text{ and } (X'^a)_{T_A} = t\}| > 0$. Consider an agent $b \in B$ such that $(X^b)_{T_B} = t$. Suppose $(X'^b)_I \neq i$. Let the set of side A agents whose trait is t and who are associated with platform i be A'. Let $Y = \{x \in X : x_B = b, x_A \in A', \text{ and } x_I = i\}$. Then $Y' = X'_{A'} \cup Y$ forms a blocking set of contracts.

Therefore, X' is not a stable allocation. Next, suppose all side B agents whose traits are t are associated with platform i. Using a similar argument as above, we can construct a blocking set of contracts in which a side A agent whose trait is t and who is associated with platform i' is included. Therefore, when agents with trait t are associated with multiple platforms, the number of agents with trait t is the same across these platforms on each side. $\square$

4.6.3 Proof of Theorem 40

The proof below incorporates the following algorithm, an extension of Hatfield and Milgrom (2005)'s cumulative offer algorithm to the many-to-many setting introduced in Fung and Hsu (2014).

The many-to-many cumulative offer algorithm

The (side A-proposing) many-to-many cumulative offer algorithm works as follows.

- Step 1: Let $O_a(1) = X_a$ for all $a \in A$. An arbitrary side A agent a_1

proposes $Y(1) = C_{a_1}(O_{a_1}(1))$. Side B agents hold $C_b(Y_b(1))$ for all $b \in B$.

  - Let $O_b(1) = Y_b(1)$ for all $b \in B$.
  - Let $O_B(1) = \cup_{b \in B} O_b(1) = Y(1)$.

- Step t: Let $O_a(t) = X_a - R_B(O_B(t-1))$ for all $a \in A$. An arbitrary side A agent a_t such that $C_{a_t}(O_{a_t}(t)) \not\subset C_B(O_B(t-1))$ proposes $Y(t) = C_{a_t}(O_{a_t}(t))$. Hospitals hold $C_b(O_b(t-1) \cup Y_b(t))$ for all $b \in B$.

  - Let $O_b(t) = O_b(t-1) \cup Y_b(t)$ for all $b \in B$.
  - Let $O_B(t) = \cup_{b \in B} O_b(t) = O_B(t-1) \cup Y(t)$.

The algorithm terminates in some step T such that $C_a(O_a(T)) \subset C_B(O_B(T-1))$ for all $a \in A$. The final allocation is $X' \equiv C_A(O_A(T))$.

In Fung and Hsu (2014), we show the algorithm produces a stable allocation in unitary many-to-many matching with contracts markets, under substitutes as well as under weaker preference assumptions. We now prove Theorem 40.

Proof. Suppose the preferences are lexicographic. Consider the following construction.

For any X, construct a reduced problem $\bar{X}$ such that for any $\bar{x} \in \bar{X}$ we have

$$\bar{x} = (a, b, i),$$

where $a \in A$, $b \in B$ and $i \in I$. For any agent $a \in A$, we can construct a reduced preference $\bar{P}_a$ such that if $\bar{x}\bar{P}_a\bar{x}'$ for any $\bar{x}, \bar{x}' \in \bar{X}$, then xP_ax' for some $x, x' \in X$, where $\bar{x}_A = \bar{x}'_A = x_A = x'_A = a$, $\bar{x}_B = x_B$, $\bar{x}'_B = x'_B$, $\bar{x}_I = x_I$, and $\bar{x}'_I = x'_I$.

We use the following algorithm to produce a trait-constrained stable allocation. There are three steps.

- Step 1: For the reduced problem $\bar{X}$ and reduced preferences $\bar{P}_{s\in A\cup B}$, run the many-to-many cumulative offere algorithm. The outcome is $\bar{X}'$.

- Step 2: For each platform i that has a positive number of agents associated with it, let the side B agent with the smallest index, say b, chooses his favorite trait among X. In other words, let the trait assocaited with a platform i be $(C_b(X))_{T_A,T_B}$.

  For each platform i that has a positive number of agents associated with it, let $i(t_A, t_B)$ be the trait that is produced by the above method.

- Step 3: Let $X' \subseteq X$ be such that

  1. $(X'^a)_B = (\bar{X}'^a)_B$, $(X'^a)_I = (\bar{X}'^a)_I := i$, and $(X'^a)_{T_A,T_B} = i(t_A, t_B), \forall a \in X'_A$;
  2. $(X'^b)_A = (\bar{X}'^b)_A$, $(X'^b)_I = (\bar{X}'^b)_I := j$, and $(X'^b)_{T_A,T_B} = j(t_A, t_B), \forall b \in X'_B$.

We claim that X' is a trait-constrained stable allocation. Note that if the preference is not substitutable nor lexicogrpahic, the algorithm may not produce a feasible allocation.

In the following, we show that X' is trait-constrained stable.

1. The allocation X' is feasible: By lexicographic preference, for any allocaiton $\bar{X}' \subseteq \bar{X}$, if we create an allocation $Y \subseteq X$ by assigning some traits (t_A, t_B) to each platform i, so that for agents associated with platform i, they also have the traits in their contracts, then the allocation Y is feasible. This is precisely what the three-step algorithm does.

2. The allocation X' is IR: Implied by lexicogrpahic preference, for any allocation $\bar{X}' \subseteq \bar{X}$, if we create an allocation $Y \subseteq X$ by assigning some traits (t_A, t_B) to each paltform i, so that for agents assocaited with platform i, they also have the traits in their contracts, then the allocation Y is IR. Therefore, X' is IR.

3. The allocation does not have feasible blocking set: Since preferences are substitutable, $\bar{X}'$ is a stable allocation in the reduced problem. In other words, no agents would jointly change the contracts so that they can switch platforms or be matched with different agents in the reduced problem.

Since preferences are lexicographic, for each agent, the set of agents on the other side and the platform in the contracts are more important than the traits. No agents would jointly change the contracts so that they can switch platforms or be matched with different agents. Within a platform i, the trait is chosen by the side B agent with the smallest index. In other words, agents associated with i cannot jointly change the contracts so that they can have different traits in their contracts, since there is at least one agent who would disagree.

Based on the above argument, the allocation X' is trait-constrained stable allocation. □

4.7 Miscellaneous results

4.7.1 Rural Hospitals Theorem

The rural hospitals theorem is a classic result in two-sided matching concerning the distribution of agents at stable allocations. The interpretation for our setting is that if agent s does not sign any contracts at some stable allocation, and thus does not join a platform, then s does not join any platform at any other stable allocation. Consider the example of mobile phone carriers. If a user chooses no mobile carrier at some stable allocation then *at every stable allocation* the user chooses no mobile carrier. A platform's objective is thus not to get a user to increase interactions at stable allocations, but to entice a user to switch from another platform (perhaps through more attractive terms).

Result 2 (Hatfield and Kominers (2012a))**.** *Suppose preferences are substitutable and satisfy the law of aggregate demand. Then at every stable alloca-*

tion Y, Y' and for every $a \in A$ and every $b \in B$, we have $|C_a(Y)| = |C_a(Y')|$ and $|C_b(Y)| = |C_b(Y')|$ that is, each agent signs the same number of contracts, at every stable allocation.

Result 2 follows directly from substitutability and the law of aggregate demand.

An immediate question is, if a platform has no agents associated with it in one stable allocation, does the platform have no agents associated with it in any other stable allocation? The answer is negative. The following example shows that if a platform i is unmatched at a stable allocation Y, in the sense that contracts with i are not in Y, then it is not the case that i is unmatched at every stable allocation.

Example 11. Let $A = \{a_1, a_2\}, B = \{b_1, b_2\}, I = \{i, i'\}, F = \emptyset$, and consider the following contracts,

$$x_1 = (a_1, b_1, i) \qquad y_1 = (a_2, b_2, i)$$
$$x_2 = (a_1, b_1, i') \qquad y_2 = (a_2, b_2, i')$$

Agents have the following preferences over the contracts:

$$P_{a_1} : \{x_1\}, \{x_2\} \qquad P_{b_1} : \{x_2\}, \{x_1\}$$
$$P_{a_2} : \{y_1\}, \{y_2\} \qquad P_{b_2} : \{y_2\}, \{y_1\}$$

The A-optimal stable allocation is $Y_1 = \{x_1, y_1\}$, and the B-optimal stable allocation is $Y_2 = \{x_2, y_2\}$. Preferences are clearly substitutable and trivially satisfy the law of aggregate demand. All agents subscribe to i at Y_1, and all agents subscribe to i' at Y_2. While i' is unmatched in the former, i' is not unmatched in the latter. Similarly for i.

In the following proposition, we show that when preferences are subsitutable and lexicographic, then if there is a platform that is not matched with any agents in a stable allocation, then it is not matched with any agent in any stable allocation.

Theorem 41. *Suppose preferences for all agents are substitutable and lexicographic. Consider a platform $i \in I$ such that:*

1. *agents prefer contracts with other platforms over contracts with i, or*

2. *contracts with i are unacceptable.*

Then $i \notin Y_I, \forall Y \in \mathcal{S}(P)$. That is, contracts with i are not part of a stable allocation.

Proof. Suppose a profile P of preferences is substitutable and lexicographic. Suppose $\exists Y \in \mathcal{S}(P)$ such that $i \in Y_I$. Then $\exists A' \subseteq A$ and $B' \subseteq B$ such that $(Y^i)_A = A'$ and $(Y^i)_B = B'$. By stability, contracts in Y must be individually rational, so condition 2 holds.

Consider $x \notin Y$ such that $(x_A, x_B) = (a, b) \in A' \times B'$. By stability and pairwise blocking, $x \notin C_a(Y \cup \{x\}) \cap C_b(Y \cup \{x\})$. That is, $x \in \big(X \setminus C_a(Y \cup \{x\})\big) \cup \big(X \setminus C_b(Y \cup \{x\})\big)$. So $x \notin C_s(Y \cup \{x\})$ for some $s \in \{a, b\}$. Since $x \notin Y$ and $x_A = a$ and $x_B = b$, it follows that $x_I \neq i$. Then s prefers i to x_I, contradicting condition 1. $\square$

Theorem 41 and Example 11 establish necessary and sufficient conditions for the existence of a platform with no agents associated with it in any stable allocation.

4.7.2 Access versus interaction at stable allocations

Example 12. The following example illustrates that, at a stable allocation, an agent may join a platform and gain access to users on the other side but only interact with a subset. This assumption is standard in the platform markets models of Rochet and Tirole (2006) and Weyl (2010).

Let $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2\}, I = \{i\}, F = \emptyset$. Consider the following contracts,

$$\begin{aligned} x_1 &= (a_1, b_1, i) \qquad & x_2 &= (a_1, b_2, i) \\ y_1 &= (a_2, b_1, i) \qquad & y_2 &= (a_2, b_2, i) \\ z_1 &= (a_3, b_1, i) \qquad & z_2 &= (a_3, b_2, i) \end{aligned}$$

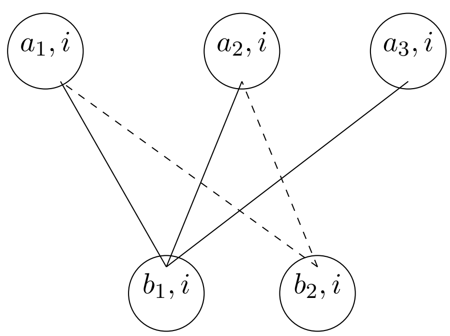
and preferences over sets of these contracts,

$$P_{a_1} : \{x_1, x_2\}, \{x_1\}, \{x_2\}$$
$$P_{a_2} : \{y_1, y_2\}, \{y_1\}, \{y_2\}$$
$$P_{a_3} : \{z_1, z_2\}, \{z_1\}, \{z_2\}$$
$$P_{b_1} : \{x_1, y_1, z_1\}, \{x_1, y_1\}, \{x_1, z_1\}, \{y_1, z_1\}\{x_1\}, \{y_1\}, \{z_1\}$$
$$P_{b_2} : \{x_2, y_2\}, \{x_2\}, \{y_2\}$$

Note that preferences are substitutable.

The A-optimal stable allocation is $Y = \{x_1, y_1, z_1, x_2, y_2\}$. Note that at Y, b_1 is linked to a_1, a_2, and a_3 through i, while b_2 only links to a_1 and a_2. However, b_2 has access to a_3 via platform i. Thus at a stable allocation, users may share a platform without actually interacting. For example, a mobile phone user does not contract with every mobile device offered by the carrier. See Figure 4.3.

Figure 4.3: Access versus Interaction: b_2 has access to a_3 but only interacts with a_1, a_2.



Now, suppose we require full interaction at stable allocations. Then substitutes does not guarantee the set of "full-interaction stable" allocations is empty. In particular, b_2 does not want to interact with a_3, and so a_3 cannot be part of an allocation. However, a_3 and b_1 mutually prefer interacting to not interacting, blocking full-interaction stability.

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