A NEW COMPUTATION OF THE BERGMAN KERNEL AND RELATED TECHNIQUES

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2016

Urbana, Illinois

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Abstract

We introduce a technique for obtaining the Bergman kernel on certain Hartogs domains. To do so, we apply a differential operator to a known kernel function on a domain in lower dimensional space. We rediscover some known results and we obtain new explicit formulas. Using these formulas, we analyze the boundary behavior of the kernel function on the diagonal. Our technique also leads us to results about a cancellation of singularities for generalized hypergeometric functions and weighted Bergman kernels. Finally, we give an alternative approach to obtain explicit bases for complex harmonic homogeneous polynomial spaces.

Acknowledgement

This project would not have been possible without the support of many people. Many thanks to my adviser Prof. John D'Angelo for his patience, encouragement, and valuable advice. Also thanks to my committee members, Prof. Richard Laugeson, Prof. Alexander Tumanov, and Prof. Jeremy Tyson, who offered guidance and suggestions. Thanks to Ming Xiao, Luke Edholm, and Prof. Jeff McNeal for helpful conversations. I acknowledge NSF grant DMS 13-61001 of D'Angelo for partial research support. And finally, thanks to my parents for their love and support.

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Chapter 1

Introduction

Let Ω denote an open connected set in complex Euclidean space $\mathbb{C}^n$. Let $A^2(\Omega)$ denote the space of square-integrable holomorphic functions on Ω. The orthogonal projection $P : L^2(\Omega) \to A^2(\Omega)$ is called the Bergman projection. The Bergman kernel function K_Ω on Ω is the reproducing kernel associated with the Bergman projection P.

Since it was first introduced by Stefan Bergman [Ber70], the Bergman kernel has been an important tool in the study of several complex variables and differential geometry. The boundary behavior of the Bergman kernel is closely related to the boundary regularity of holomorphic mappings. The Bergman metric, defined by $\partial\bar{\partial}\log K_\Omega(z;\bar{z})$, is a Kähler metric of major importance in complex geometry.

Explicit formulas for the Bergman kernel are available in only a few cases. One method to obtain an explicit formula is to sum an orthonormal series. Let $\{\phi_\alpha\}$ be a complete orthonormal system for $A^2(\Omega)$. Since it reproduces every element in $A^2(\Omega)$, the Bergman kernel $K_\Omega(z;\bar{\zeta})$ satisfies:

$$K_\Omega(z;\bar{\zeta}) = \sum_\alpha \phi_\alpha(z)\overline{\phi_\alpha(\zeta)}. \tag{1.1}$$

Another method for computing K_Ω applies when Ω (for example, the unit ball) has a transitive holomorphic automorphism group. Suppose the automorphism group of domain Ω is transitive and $K_\Omega(z_0;\bar{z}_0)$ is known at some fixed z_0. For any $z \in \Omega$, there is a holomorphic automorphism sending z_0 to z. Then $K_\Omega(z;\bar{z})$ can be obtained by applying the biholomorphic transformation rule for the Bergman kernel to $K_\Omega(z_0;\bar{z}_0)$.

In Chapter 2, we provide details about these two methods and use both of them to obtain the explicit formula of the Bergman kernel on the unit disk. We also review previous results where explicit formulas for the kernel functions are obtained.

Our approach for computing the kernel function is different. We start with a domain $\Omega \subseteq \mathbb{C}^{n+m}$ with certain symmetry properties in the first

n variables. Rescaling the first n components of Ω by a multi-parameter $\mathbf{t}$ yields a family of domains $\{\Omega_{\mathbf{t}}\}$ with each $\Omega_{\mathbf{t}}$ biholomorphic to Ω. By regarding the rescaling parameters $\mathbf{t}$ as two different kinds of functions in k new complex variables, we obtain domains U^α and V^γ in $\mathbb{C}^{n+m+k}$. For $\eta \in \mathbb{C}^k$, let U^α_η and V^γ_η denote the slice of domains U^α and V^γ with the last k variables equal to η. We obtain the kernel function on U^α and V^γ by first computing $K_{U^\alpha_\eta}$ and $K_{V^\gamma_\eta}$, evaluating these kernel functions at certain points off the diagonal, and then applying a differential operator to them. This technique combines the method of equation (1.1) and biholomorphic transformation with the idea of variation of parameters, and gives explicit formulas for the Bergman kernel functions in many new cases. We introduce the notion of n-star-shaped Hartogs for those symmetry properties Ω requires, and illustrate the construction of U^α and V^γ in Chapter 2.

Theorems 3.1 and 3.2, two of our main results, give formulas that relate K_{U^α} and K_{V^γ} to $K_{U^\alpha_\eta}$ and $K_{V^\gamma_\eta}$. In each case, the kernel function on the target domain is obtained explicitly from the kernel function on the base domain by the three step process described above. First one computes the kernel function on a domain biholomorphic to the base domain. Then one evaluates this kernel off the diagonal. Finally one applies a differential operator to this expression, obtaining the kernel on the target domain. See formulas (3.5) and (3.6). The differential operators used for U^α and for V^γ differ, but each is also completely explicit.

Example 3.1 gives a case where the known kernel function can be rediscovered using our method. Examples 3.2 and 3.3 give new cases where explicit formulas for the kernel function are obtained. In Example 3.4, we apply Theorems 3.1 and 3.2 repeatedly, obtaining explicit formulas for the kernel function on rather elaborate domains. Using our technique, we also rediscover a recent result of Edholm [Edh15], who found an explicit formula for the Bergman kernel on some generalized Hartogs triangles in $\mathbb{C}^2$.

We use our explicit formulas to describe the boundary behaviors of K_{U^α} and K_{V^γ}. In [Fef74], Fefferman gave an asymptotic expansion of the Bergman kernel on the diagonal when the domain is bounded, smooth, and strongly pseudoconvex. He used this result to prove that a biholomorphic mapping between two bounded smooth strongly pseudoconvex domains can be extended smoothly to the closure. Later, S. Bell and E. Ligocka [BL80] extended Fefferman's result to smooth bounded pseudoconvex domains satisfying condition R. This condition means that the Bergman projection P associated with the domain Ω maps elements of $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$. This regularity condition is equivalent to an inequality on the derivatives of the

Bergman kernel near the boundary. Because of its close connection to the boundary regularity of holomorphic mappings, the behavior of the Bergman kernel near the boundary has been studied for many decades. In the weakly pseudoconvex case, the boundary behavior is difficult to analyze. Near a weakly pseudoconvex point of finite type, certain estimates on the Bergman kernel were obtained by McNeal [McN89, McN94]. Fu [Fu94] gave a sharp lower bound estimate for the Bergman kernel on a bounded pseudoconvex domain with C^2 boundary. Less is known about the behavior of the Bergman kernel near non-smooth boundary points. It is worth noting that our calculations produce many domains whose kernel functions have no log terms.

In Chapter 4, we discuss the pseudoconvexity of U^α and V^γ provided the domain Ω is pseudoconvex. We determine the boundary behavior of the Bergman kernels in Examples 3.2 and 3.3 using explicit formulas and admissible approach regions. Then we combine Theorems 3.1 and 3.2 with Fefferman's asymptotic expansion for the Bergman kernel to obtain the boundary behavior of the Bergman kernel on U^α and V^γ when Ω is bounded, smooth and strongly pseudoconvex. In Theorems 4.2 and 4.3, the boundaries of U^α and V^γ need not be smooth.

While the Bergman kernel tends to infinity when approaching a pseudoconvex boundary point along the diagonal, the behavior of the kernel function near the boundary off the diagonal is different. Kerzman first showed in [Ker72] that, for smooth, bounded strongly pseudoconvex domains, the Bergman kernel is C^∞-smooth up to the boundary off the diagonal. Later, Bell [Bel86] and Boas [Boa87] independently generalized Kerzman's result to cases where boundary points are of finite type. The simplest examples where differentiability results hold for the Bergman kernel are complex ovals. Francsics and Hanges, in [FH96, FH97], expressed the Bergman kernel on complex ovals as a sum of some generalized hypergeometric functions. Despite the smooth extension of the kernel function on complex ovals off the diagonal, each generalized hypergeometric function in the sum tends to infinity. This phenomenon suggests a cancellation of singularities of these generalized hypergeometric functions. In Theorem 5.5, we give a smooth extension result for these hypergeometric functions at the boundary of their domains of convergence.

Let $f : \Omega_1 \to \Omega_2$ be a surjective proper holomorphic mapping. Bell, in [Bel81], gave a transformation formula of f that relates K_{Ω_2} to K_{Ω_1}. Unlike the biholomorphic transformation formula, Bell's formula is explicit for K_{Ω_2} but implicit for K_{Ω_1}, i.e. K_{Ω_1} may not be obtained even if an explicit formula for K_{Ω_2} is given. In Theorem 5.8, we give, for domains with some symmetry properties and special proper mappings f, an explicit transformation formula for K_{Ω_1} that relates K_{Ω_1} to weighted Bergman kernel functions on Ω_2. Combining this result with our observation about generalized hypergeometric functions, we obtain, in Theorem 5.9, a cancellation of singularities for weighted Bergman kernels.

The idea of our technique for computing the Bergman kernel is to relate a complete orthonormal system in one space to another. Such an idea can be applied in different settings. Ikeda and Kayama [IK67] first gave an explicit basis for complex harmonic homogeneous polynomial space $H_{m,n}(\mathbb{C}^k)$ using complex harmonic homogeneous polynomials of lower degree in fewer variables. As consequences, their result implied the existence of mappings between elements in $H_{a,b}(\mathbb{C}^{k-1})$ and $H_{m,n}(\mathbb{C}^k)$ with $a \leq m$ and $b \leq n$, and an orthogonal decomposition of the space $H_{m,n}(\mathbb{C}^k)$. Koornwinder [Koo72] gave another proof of the result of Ikeda and Kayama. In [IK67], the authors solved the Laplace equation using special coordinates and separation of variables to obtain their results. Koornwinder's approach involves zonal harmonics and Jacobi polynomials.

In this thesis, we recover these results using neither special coordinates nor special functions. Instead, we use the methods of undetermined coefficients and separation of variables. In Chapter 6, we introduce an inner product $\prec \cdot, \cdot \succ$ in which the Laplacian is the adjoint of multiplication by $\|z\|^2$. We prove an orthogonal decomposition of the space of complex homogeneous polynomials. Using the methods of undetermined coefficients and separation of variables, we obtain multiplication operators sending complex harmonic homogeneous polynomials with $k-1$ variables into $H_{m,n}(\mathbb{C}^k)$, and prove the decomposition theorem for $H_{m,n}(\mathbb{C}^k)$. We also give a higher dimensional analogue of our argument where our mappings send $H_{a,b}(\mathbb{C}^k)$ to $H_{m,n}(\mathbb{C}^{k+r})$ for $r \geq 2$.

The common theme in the thesis is to analyze norm preserving operators between a Hilbert space $\mathcal{H}_1$ of functions in several variables and another Hilbert space $\mathcal{H}_2$ of functions in more variables. When $\mathcal{H}_1$ and $\mathcal{H}_2$ are spaces of complex harmonic homogeneous polynomials, we use the method of separation of variables to compute the operator and obtain a decomposition for $\mathcal{H}_2$. When $\mathcal{H}_1$ and $\mathcal{H}_2$ are $A^2(U^\alpha_\eta)$ and $A^2(U^\alpha)$, these norm-preserving mappings suggest certain relations between the kernel functions on U^α_η and U^α and finally lead to our explicit formula for K_{U^α}.

Chapter 2

Background information

We recall some definitions, basic facts, and well-known results about the Bergman kernel.

2.1 Preliminaries

Let Ω be a domain (open, connected set) in complex Euclidean space $\mathbb{C}^n$. Let dV denote Lebesgue measure on $\mathbb{C}^n$. Let $L^2(\Omega)$ denote the space of square-integrable functions with the inner product $\langle\cdot,\cdot\rangle$:

$$\langle f, g\rangle = \int_\Omega f\bar{g}\, dV.$$

Let $A^2(\Omega)$ denote the subspace of $L^2(\Omega)$ that consists of holomorphic functions. It is closed in $L^2(\Omega)$ and hence is itself a Hilbert space.

Consider $z \in \Omega$. The map δ_z from $A^2(\Omega)$ to $\mathbb{C}$ defined by

$$\delta_z(f) = f(z) \tag{2.1}$$

is a bounded linear functional. By Riesz's representation theorem, there exists a unique $K_z \in A^2(\Omega)$ such that

$$f(z) = \delta_z(f) = \langle f, K_z\rangle = \int_\Omega \overline{K_z(\zeta)} f(\zeta) dV(\zeta). \tag{2.2}$$

The Bergman kernel function K_Ω is defined by $K_\Omega(z;\bar{\zeta}) = \overline{K_z(\zeta)}$. By this definition, $K_\Omega(\cdot;\cdot)$ is defined on $\Omega \times \Omega^*$ where $\Omega^* = \{z : \bar{z} \in \Omega\}$. For simplicity of our notion, we consider $K_\Omega(z;\bar{\zeta})$ as a function on $\Omega \times \Omega$ by regarding $\bar{\zeta}$ as a function of ζ. Let P denote the orthogonal projection from $L^2(\Omega)$ to $A^2(\Omega)$. Then, for each $f \in L^2(\Omega)$, we have

$$Pf(z) = \int_\Omega K_\Omega(z;\bar{\zeta}) f(\zeta) dV(\zeta).$$

We call P the Bergman projection.

These considerations lead to the following lemma (See, e.g., Prop. 1.4.6 in [Kra01]):

Lemma 2.1. *A function $K : \Omega \times \Omega \to \mathbb{C}$ is the Bergman kernel function on Ω if and only if the following three properties hold:*

1. *For each $\zeta \in \Omega$, the map $z \mapsto K(z; \bar{\zeta})$ is in $A^2(\Omega)$.*

2. *$\overline{K(z; \bar{\zeta})} = K(\zeta; \bar{z})$. (Hermitian symmetry)*

3. *$\int_\Omega K(z; \bar{\zeta}) f(\zeta) dV(\zeta) = f(z)$ for all $f \in A^2(\Omega)$. (reproducing prop.)*

If $\{\phi_\alpha\}$ is an complete orthonormal system for $A^2(\Omega)$, then the Bergman kernel K_Ω satisfies

$$K_\Omega(z; \bar{\zeta}) = \sum_\alpha \phi_\alpha(z) \overline{\phi_\alpha(\zeta)}. \tag{2.3}$$

Let $F : \Omega_1 \to \Omega_2$ be a biholomorphic map. Let JF be the holomorphic Jacobian determinant of F. Then we have the transformation formula:

$$K_{\Omega_1}(z; \bar{\zeta}) = JF(z) \overline{JF(\zeta)} K_{\Omega_2}(F(z); \overline{F(\zeta)}). \tag{2.4}$$

Explicit formulas for the Bergman kernel on the polydisk $\mathbb{D}^n \subseteq \mathbb{C}^n$ and unit ball $\mathbb{B}^n \subseteq \mathbb{C}^n$ can be obtained using either (2.3) or (2.4). Here we demonstrate these two different approaches using the kernel function on the unit disk $\mathbb{B}$ as an example.

To use formula (2.3) for K_Ω, we need to find a complete orthogonal system for $A^2(\Omega)$. Since $\mathbb{B}$ is bounded, all monomials are square-integrable on $\mathbb{B}$. Consider the inner product of monomial z^a and z^b:

$$\langle z^a, z^b \rangle = \int_\mathbb{B} z^a \bar{z}^b dV.$$

Using polar coordinates, we have

$$\int_\mathbb{B} z^a \bar{z}^b dV = \int_0^{2\pi} e^{i(a-b)\theta} d\theta \int_0^1 r^{a+b+1} dr = \begin{cases} 0 & \text{if } a \neq b \\ \frac{\pi}{a+1} & \text{if } a = b \end{cases}.$$

Thus $\{z^a\}$ forms an orthogonal system for $A^2(\Omega)$. Since every holomrphic function in $A(\mathbb{B})$ has a power series expansion with normal convergence on $\mathbb{B}$, the system $\{z^a\}$ is complete. Therefore (2.3) yields

$$K_\mathbb{B}(z; \bar{\zeta}) = \sum_{a=0}^\infty \frac{(a+1)(z\bar{\zeta})^a}{\pi} = \frac{1}{\pi(1 - z\bar{\zeta})^2}.$$

The other approach uses the transitivity of automorphisms on the unit disk. For arbitrary $z \in \mathbb{B}$, the Möbius transformation

$$f_z : \zeta \mapsto \frac{\zeta - z}{1 - \bar{z}\zeta}$$

maps the point z to the origin and is an automorphism on $\mathbb{B}$. Formula (2.4) implies that

$$K_{\mathbb{B}}(z; \bar{z}) = |Jf_z(z)|^2 K_{\mathbb{B}}(0, 0).$$

Since $|Jf_z(z)|^2 = \frac{1}{(1-|z|^2)^2}$ and $K_{\mathbb{B}}(0, 0) = \frac{1}{\|1\|^2} = \frac{1}{\pi}$, we have

$$K_{\mathbb{B}}(z; \bar{z}) = \frac{1}{\pi(1 - |z|^2)^2}.$$

By Lemma 2.1, the function $K_{\mathbb{B}}$ is Hermitian symmetric. Therefore

$$K_{\mathbb{B}}(z; \bar{\zeta}) = \frac{1}{\pi(1 - z\bar{\zeta})^2}.$$

Similarly, computations using either the complete orthonormal system for $A^2(\mathbb{B}^n)$ or the automorphisms on $\mathbb{B}^n$ yield the explicit formula for the kernel function on the unit ball in $\mathbb{C}^n$:

$$K_{\mathbb{B}^n}(z; \bar{\zeta}) = \frac{n!}{\pi^n(1 - \langle z, \zeta \rangle)^{n+1}}.$$

Applying (2.4) to $K_{\mathbb{B}^n}$, we can also obtain the kernel functions on those domains that are biholomorphic to $K_{\mathbb{B}^n}$. Take $\mathbb{B}_r = \{z \in \mathbb{C} : |z| < r\}$ as an example. $\mathbb{B}_r$ is biholomorphic to $\mathbb{B}$ via the mapping $F : z \mapsto \frac{z}{r}$. Therefore applying (2.4) to $K_{\mathbb{B}}$ yields:

$$K_{\mathbb{B}_r}(z; \bar{\zeta}) = \frac{r^2}{\pi(r^2 - z\bar{\zeta})^2}.$$

If we regard the parameter r as a positive function $f(w; \bar{w})$ of a new complex variable w on the domain $\mathcal{D} \subseteq \mathbb{C}$, then we can construct a new domain in $\mathbb{C}^2$ using the function f and the unit disk $\mathbb{B}$:

$$\{(z, w) \in \mathbb{C} \times \mathcal{D} : |z| < f(w, \bar{w})\}.$$

Similarly, for a domain $\Omega \subseteq \mathbb{C}^n$, if we start with a multi-parameter family of domains $\{\Omega_{\mathbf{r}}\}$ with each $\Omega_{\mathbf{r}}$ biholomorphic to Ω and replace r_j by some function f_j, then we can construct a new domain $\mathcal{U}$ in a higher dimensional space. It is natural to ask whether the Bergman kernel on $\mathcal{U}$ can be obtained if K_Ω is known. Our technique provides, for certain kinds of Ω and f_j, an explicit formula that connects the Bergman kernel on $\mathcal{U}$ with the kernel function on Ω. We first introduce the class of domains Ω where our technique works.

Let Ω be a domain in $\mathbb{C}^{n+m}$. Let $z_1, \ldots, z_n$ and $\zeta_1, \ldots, \zeta_m$ denote the first n and last m coordinates in $\mathbb{C}^{n+m}$. Our method of obtaining Bergman kernels requires the space $A^2(\Omega)$ to have a complete orthogonal system of the form $\{z^{\mathbf{a}}\phi_{\mathbf{a}}(z')\}$. This consideration leads us to a class of domains with a symmetry property in the z coordinates. We call these domains n-star-shaped Hartogs domains. Before defining them, we recall the definitions of Reinhardt domain and Hartogs domain.

Definition 2.1. *A domain $\Omega \subseteq \mathbb{C}^n$ is called Reinhardt if $(z_1, \ldots, z_n) \in \Omega$ implies $(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \in \Omega$ for any real θ_j's.*

Definition 2.2. *A domain $\Omega \subseteq \mathbb{C}^n$ is called Hartogs with symmetric plane $\{z_j = 0\}$ if $(z_1, \ldots, z_n) \in \Omega$ implies the containment*

$$\{(z_1, \ldots, e^{i\theta}z_j, \ldots, z_n) : \theta \in \mathbb{R}\} \subseteq \Omega.$$

We introduce in this thesis the following class of domains.

Definition 2.3. *A domain $\Omega \subseteq \mathbb{C}^{n+m}$ is called n-star-shaped Hartogs in $(z_1, \ldots, z_n)$ if $(z_1, \ldots, z_n, \zeta) \in \Omega$ implies that*

$$\{(\lambda_1 z_1, \ldots, \lambda_n z_n, \zeta) : |\lambda_j| \leq 1 \text{ for } 1 \leq j \leq n\} \subseteq \Omega.$$

A Reinhardt domain in $\mathbb{C}^n$ containing the origin is automatically n-star-shaped Hartogs.

Example 2.1. *By their definitions, the unit ball*

$$\{z \in \mathbb{C}^n : \|z\| < 1\},$$

and polydisk

$$\{z \in \mathbb{C}^n : |z_j| < 1 \text{ for } 1 \leq j \leq n\}$$

are Reinhardt domains containing the origin. Therefore, they are also n-star-shaped Hartogs domains.

Example 2.2. *The Hartogs triangle*

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$$

is a Reinhardt domain. Since $z_2 \neq 0$ on this domain, the domain is not star shaped in the z_2 coordinate. Therefore it is a 1-star-shaped Hartogs domain in the z_1 coordinate.

If Ω is a Reinhardt domain in $\mathbb{C}^n$ containing the origin, then every holomorphic function on Ω has a power series expansion with normal convergence in Ω. The following lemma is an analogue for the general n-star-shaped Hartogs domain.

Lemma 2.2. *Let $\Omega \subseteq \mathbb{C}^{n+m}$ be n star-shaped Hartogs. Let f be a holomorphic function on Ω. Then f has a unique expansion*

$$f(z, \zeta) = \sum_{\mathbf{a}} \phi_{\mathbf{a}}(\zeta) z^{\mathbf{a}},$$

where $\phi_{\mathbf{a}}(\zeta)$ is holomorphic in ζ and the series converges normally in Ω.

Proof. The uniqueness is obvious. We show the existence. Let ${}_{\epsilon}\Omega$ denote the set

$$\left\{ z \in \Omega : \operatorname{dist}(z, \complement\Omega) > \epsilon \sum_{j=1}^{n} |z_j|^2 \right\}.$$

Let $Y = \{(z, \zeta) \in \Omega : z = 0\}$. Since Ω is connected and n-star-shaped, Y is connected. Fix $(0, \zeta_0) \in Y$. For small ϵ, the point $(0, \zeta_0)$ is contained in ${}_{\epsilon}\Omega$. Let ${}_{\epsilon}\tilde{\Omega}$ denote the connected component of ${}_{\epsilon}\Omega$ that contains $(0, \zeta_0)$. Then $\bigcup\limits_{\epsilon} {}_{\epsilon}\tilde{\Omega} = \Omega$. For any compact set $K \subset \Omega$, we can therefore choose a small ϵ so that $K \subseteq_{\epsilon} \tilde{\Omega}$. For $(z, \zeta) \in {}_{\epsilon}\tilde{\Omega}$, we set

$$g(z, \zeta) = \frac{1}{(2\pi i)^n} \int_{|t_1|=1+\epsilon} \cdots \int_{|t_n|=1+\epsilon} \frac{f(z_1 t_1, \ldots, z_n t_n, \zeta_1, \ldots, \zeta_n)}{\prod_{j=1}^{n} (t_j - 1)} dt_1 \cdots dt_n.$$

The function is well-defined by the construction of ${}_{\epsilon}\tilde{\Omega}$. It defines an analytic function on ${}_{\epsilon}\tilde{\Omega}$ and equals $f(z, \zeta)$ when $\|z\|$ is small. Therefore, $f = g$ on ${}_{\epsilon}\tilde{\Omega}$. Note that

$$\frac{1}{\prod_{j=1}^{n} (t_j - 1)} = \sum_{\mathbf{a}} t_1^{-a_1 - 1} \cdots t_n^{-a_n - 1}, \tag{$\star$}$$

where this Laurent series converges when $|t_j| = 1 + \epsilon$ for all $1 \leq j \leq n$. Substituting ($\star$) into our formula in g yields

$$g(z, \zeta) = \sum_{\mathbf{a}} \int_{|t_1|=1+\epsilon} \cdots \int_{|t_n|=1+\epsilon} \frac{f(z_1 t_1, \ldots, z_n t_n, \zeta_1, \ldots, \zeta_n)}{(2\pi i)^n \prod_{j=1}^{n} t_j^{a_j + 1}} dt_1 \cdots dt_n,$$

with normal convergence in ${}_\epsilon\tilde{\Omega}$. Since

$$\frac{z^{\mathbf{a}}}{\mathbf{a}!}\partial_z^{\mathbf{a}} f(0,\zeta) = \int_{|t_1|=1+\epsilon} \cdots \int_{|t_n|=1+\epsilon} \frac{f(z_1t_1,\dots,z_nt_n,\zeta_1,\dots,\zeta_n)}{(2\pi i)^n \prod_{j=1}^n t_j^{a_j+1}} dt_1 \cdots dt_n,$$

we conclude that

$$f(z,\zeta) = \sum_{\mathbf{a}} \frac{\partial_z^{\mathbf{a}} f(0,\zeta)}{\mathbf{a}!} z^{\mathbf{a}}.$$

The proof is complete. □

When $\Omega \subseteq \mathbb{C}^n$ is bounded and Reinhardt, the monomial z^α is orthogonal to the monomial z^γ in $L^2(\Omega)$ if $\alpha \neq \gamma$. When Ω is an n-star-shaped Hartogs domain, we have a similar orthogonality property for elements in $A^2(\Omega)$.

Lemma 2.3. *Let $\Omega \subseteq \mathbb{C}^{n+m}$ be n-star-shaped Hartogs. Let $z^\alpha f(\zeta)$ and $z^\gamma g(\zeta)$ be square-integrable functions on Ω with $\alpha \neq \gamma$. Then $z^\alpha f(\zeta)$ is orthogonal to $z^\gamma g(\zeta)$ in $A^2(\Omega)$.*

Proof. Let Ω^+ denote the set $\{(r,\zeta) : (r,\zeta) \in \Omega \text{ and } r_j \geq 0 \text{ for all } j\}$. Note that Ω is n-star-shaped Hartogs. In polar coordinates, $\Omega = \Omega^+ \times [0,2\pi]^n$. Hence we have

$$\int_\Omega z^\alpha f(\zeta)\overline{z^\gamma g(\zeta)} dV = \int_{\Omega^+} r^{\alpha+\gamma+n} f(\zeta)\overline{g(\zeta)} dV(r,\zeta) \prod_{j=1}^n \int_0^{2\pi} e^{i(\alpha_j-\gamma_j)\theta_j} d\theta_j.$$

If $\alpha_j \neq \gamma_j$, then $\int_0^{2\pi} e^{i(\alpha_j-\gamma_j)\theta_j} d\theta_j$ equals 0. Therefore for $\alpha \neq \gamma$, we have $\langle z^\alpha f(\zeta), z^\gamma g(\zeta)\rangle = 0$. □

Let $\pi : (z,\zeta) \mapsto \zeta$ denote the projection from $\mathbb{C}^n \times \mathbb{C}^m$ to $\mathbb{C}^m$. For $\zeta \in \pi(\Omega)$, we set $\Omega_\zeta = \{z \in \mathbb{C}^n : (z,\zeta) \in \Omega\}$. The following lemma is a version of Ligocka's result in [Lig89]. A related idea has also been used by Forelli and Rudin in [FR75]. In our version, there is no boundedness assumption on the domain Ω. For convenience, we provide a complete proof below.

Lemma 2.4. *Let Ω be an n-star-shaped Hartogs domain. Then*

1. *Let f be an element in $A^2(\Omega)$. Then f can be expanded as follows:*

$$f(z,\zeta) = \sum_{\mathbf{a}} \phi_{\mathbf{a}}(\zeta) z^{\mathbf{a}}.$$

For each multi-index $\mathbf{a}$, $\phi_{\mathbf{a}}$ is a square-integrable holomorphic function on $\pi(\Omega)$ with respect to the measure $\|z^{\mathbf{a}}\|^2_{\Omega_\zeta} dV(\zeta)$.

2. *If* $\{\phi_{\mathbf{a},\mathbf{b}}\}$ *is a complete orthogonal system for* $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$*, then* $\{\phi_{\mathbf{a},\mathbf{b}} z^{\mathbf{a}}\}$ *forms a complete orthogonal system for* $A^2(\Omega)$*.*

Proof. Let $\{\Omega^k\}$ denote an increasing sequence of compact n-star-shaped Hartogs domains exhausting Ω. Thus $\Omega^k \subset\subset \Omega^{k+1}$ for all k and $\bigcup^\infty \Omega^k = \Omega$. Since $\sum_{\mathbf{a}} \phi_{\mathbf{a}}(\zeta) z^{\mathbf{a}}$ converges normally on Ω, the series converges uniformly on Ω^k. Lemma 2.3 implies that $\phi_{\mathbf{a}}(\zeta)z^{\mathbf{a}} \perp \phi_{\mathbf{b}}(\zeta)z^{\mathbf{b}}$ in $L^2(\Omega)$ when $\mathbf{a} \neq \mathbf{b}$. Hence, for all k and $\phi_{\mathbf{a}}(\zeta)z^{\mathbf{a}} \in A^2(\Omega^k)$,

$$\|f(z,\zeta)\|^2_{A^2(\Omega)} \geq \sum_{\mathbf{a}} \|\phi_{\mathbf{a}}(\zeta)z^{\mathbf{a}}\|^2_{A^2(\Omega^k)},$$

and therefore $\phi_{\mathbf{a}}(\zeta)z^{\mathbf{a}}$ is square-integrable on Ω. Since

$$\begin{aligned} \|\phi_{\mathbf{a}}(\zeta)z^{\mathbf{a}}\|^2_{A^2(\Omega)} &= \int_D |\phi_{\mathbf{a}}(\zeta)|^2 \int_{\Omega_\zeta} |z^{\mathbf{a}}|^2 dV(z)\, dV(\zeta) \\ &= \int_D |\phi_{\mathbf{a}}(\zeta)|^2 \|z^{\mathbf{a}}\|^2_{\Omega_\zeta} dV(\zeta), \end{aligned} \tag{2.5}$$

we have $\phi_{\mathbf{a}} \in A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$. We claim that $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$ inherits its completeness from $A^2(\Omega)$: Consider an arbitrary compact set $K \subseteq \pi(\Omega)$. Since Ω is n-star-shaped Hartogs, the compact set $\{0\} \times K$ is in Ω. Thus there exists a constant $r_K > 0$ such that for any point $(0,\zeta) \in \{0\} \times K$, the $(n+m)$-ball $B((0,\zeta); r_K)$ is contained in Ω. Let $r = r_K/3$. Let B_1 denote the n-ball centered at the point $z_r = (\frac{r}{n}, \ldots, \frac{r}{n})$ with radius $\frac{r}{2n}$. For $\zeta \in K$, let B_ζ denote the m-ball centered at the ζ with radius r. Then we have $B_1 \times B_\zeta \subseteq B((0,\zeta); r_K) \subseteq \Omega$. Let $g(\zeta)$ be an element of $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$. By the mean value property and Hölder's inequality,

$$|g(\zeta)| = \left|\frac{z_r^{\mathbf{a}} g(\zeta)}{z_r^{\mathbf{a}}}\right| \leq \frac{\int_{B_1 \times B_\zeta} |z^{\mathbf{a}} g(w)| dV(z,w)}{Vol(B_1 \times B_\zeta)|z_r^{\mathbf{a}}|} \leq C_K \|g(\zeta)\|_{A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})}.$$

Taking the supremum of $|g(\zeta)|$ on K, we have

$$\sup_{\zeta \in K} |g(\zeta)| \leq C_K \|g(\zeta)\|_{A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})}.$$

L^2 convergence in $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$ implies normal convergence in $A(\pi(\Omega))$, and hence $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$ is closed.

Let $\{\phi_{\mathbf{a},\mathbf{b}}\}$ be a complete orthogonal system of $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$. We finish the proof by showing that $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta)\}$ forms a complete orthogonal system of $A^2(\Omega)$. For any $f \in A^2(\Omega)$,

$$f(z,\zeta) = \sum_{\mathbf{a},\mathbf{b}} c_{\mathbf{a},\mathbf{b}} z^{\mathbf{a}} \phi_{\mathbf{a},\mathbf{b}}(\zeta).$$

To show the completeness of $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}\}$, we assume $f \in A^2(\Omega)$ and

$$\langle f(z,\zeta), z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta)\rangle = 0$$

for all $\mathbf{a},\mathbf{b}$. We verify that $f = 0$.

Let $\{\Omega^k\}$ be the domains used above. For arbitrary $\mathbf{a}$ and $\mathbf{b}$,

$$\int_{\Omega^k} f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV + \int_{\Omega-\Omega^k} f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV = 0.$$

We therefore have

$$\Big|\int_{\Omega^k} f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big| = \Big|\int_{\Omega-\Omega^k} f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big|.$$

By Hölder's inequality,

$$\Big|\int_{\Omega-\Omega^k} f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big| \le \|z^{\alpha}\phi_{\mathbf{a},\mathbf{b}}(\zeta)\|_{A^2(\Omega)}\Big(\int_{\Omega-\Omega^k}|f|^2dV\Big)^{\frac{1}{2}}.$$

Since $f \in A^2(\Omega)$ and Ω^k exhausts Ω,

$$\lim_{k\to\infty}\int_{\Omega-\Omega^k}|f|^2dV = 0.$$

Therefore

$$\lim_{k\to\infty}\Big|\int_{\Omega^k} f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big| = 0.$$

Using Hölder's inequality again yields $f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)} \in L^1(\Omega)$. The compactness of Ω^k and polar coordinates imply that

$$\begin{aligned}
&\Big|\int_{\Omega^k} f(z,\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big| \\
=&\Big|\int_{\Omega^k}\sum_{\alpha,\beta} c_{\alpha,\beta}z^{\alpha}\phi_{\alpha,\beta}(\zeta)\bar{z}^{\mathbf{a}}\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big| \\
=&\Big|\int_{\Omega^k}\sum_{\beta} c_{\mathbf{a},\beta}|z^{\mathbf{a}}|^2\phi_{\mathbf{a},\beta}(\zeta)\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big|.
\end{aligned}$$

By the Dominated Convergence Theorem,

$$
\begin{aligned}
&\lim_{k\to\infty}\Big|\int_{\Omega^k}\sum_{\beta} c_{\mathbf{a},\beta}|z^{\mathbf{a}}|^2\phi_{\mathbf{a},\beta}(\zeta)\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big|\\
=&\Big|\int_{\Omega}\lim_{k\to\infty}\chi_{\Omega^k}(z,\zeta)\sum_{\beta} c_{\mathbf{a},\beta}|z^{\mathbf{a}}|^2\phi_{\mathbf{a},\beta}(\zeta)\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big|\\
=&\Big|\int_{\Omega}\sum_{\beta} c_{\mathbf{a},\beta}|z^{\mathbf{a}}|^2\phi_{\mathbf{a},\beta}(\zeta)\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big|\\
=&\Big|\int_{\Omega} c_{\mathbf{a},\mathbf{b}}|z^{\mathbf{a}}|^2\phi_{\mathbf{a},\mathbf{b}}(\zeta)\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta)}dV\Big|\\
=&|c_{\mathbf{a},\mathbf{b}}|\|z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta)\|^2_{A^2(\Omega)}.
\end{aligned}
$$

Therefore $c_{\mathbf{a},\mathbf{b}} = 0$ for all $\mathbf{a}$, $\mathbf{b}$ and $f \equiv 0$. $\square$

Corollary 2.1. *Let $\{\phi_{\mathbf{a},\mathbf{b}}(\zeta)\}$ be a complete orthogonal system for the space $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{\Omega_\zeta})$. Then*

$$K_\Omega(z, z'; \bar\zeta, \bar\zeta') = \sum_{\mathbf{a},\mathbf{b}} \frac{(z\bar\zeta)^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}}{\|z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\|^2_{L^2(\Omega)}}. \tag{2.6}$$

In particular, when $\Omega \subseteq \mathbb{C}^n$ is a Reinhardt domain containing the origin, the square-integrable monomials form a complete orthogonal system. Let $\mathfrak{I}$ denote the set of multi-indices $\{\mathbf{a} : z^{\mathbf{a}} \in L^2(\Omega)\}$. Then the Bergman kernel K_Ω has the following expansion:

$$\sum_{\mathbf{a}\in\mathfrak{I}} \frac{(z\bar\zeta)^{\mathbf{a}}}{\|z^{\mathbf{a}}\|^2_{L^2(\Omega)}}. \tag{2.7}$$

2.2 Previous results

Explicit formulas for the Bergman kernel are available in only a few cases. Among them, most results have been obtained on domains with symmetries since fairly simple systems $\{\phi_\alpha\}$ can be chosen on these domains. The kernels for the unit ball and polydisk have been long known. D'Angelo [D'A78,D'A94] gave an explicit formula of the Bergman kernel on the domain

$$\{(z, w) \in \mathbb{C}^{n+m} : \|z\|^2 + \|w\|^{2p} < 1\},$$

where p is positive. See Example 3.1. On the complex ovals

$$\{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2a_j} < 1\},$$

where a_j's are positive integers, Francsics and Hanges [FH96, FH97] expressed the Bergman kernel in terms of generalized hypergeometric functions. See Theorem 5.4. Boas, Fu, and Straube [BFS99] introduced a method for obtaining the Bergman kernel. They differentiate the kernel function on $\{(z,w) \in \mathbb{C}^2 : |z| < p(w)\}$ for $z \in \mathbb{C}$ to obtain the kernel for $z \in \mathbb{C}^n$ on $\{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \|z\| < p(w)\}$. More recent results can be found in [Par08, Par13, Yam13, Beb15, Edh15].

Our method rediscovers some of the formulas mentioned above and also yields some new explicit formulas. See Chapter 3 for more details.

2.3 Settings and notations

Let $\Omega \subseteq \mathbb{C}^{n+m}$ be an n-star-shaped Hartogs domain. Let K_Ω denote the Bergman kernel on Ω. Set $\mathbf{t} = (t_1, \ldots, t_n)$ with $t_j > 0$ for all j. Let $\Omega_{\mathbf{t}}$ denote the set

$$\{(z,\zeta) \in \mathbb{C}^{n+m} : (t_1 z_1, \ldots, t_n z_n, \zeta) \in \Omega\}.$$

Let $\mathcal{D}$ be a conjugate invariant domain in $\mathbb{C}^k$. For $1 \le j \le n$, let ψ_j be a Hermitian symmetric function on $\mathcal{D} \times \mathcal{D}$ satisfying $\psi_j(w,\bar{w}) > 0$ on $\mathcal{D}$. By regarding each parameter t_j as the value of ψ_j on the diagonal $(w,\bar{w})$, we construct a new domain $\mathcal{U}$ in a higher dimensional space:

$$\{(z,\zeta,w) \in \mathbb{C}^{n+m+k} : (\psi_1(w,\bar{w})z_1, \ldots, \psi_n(w,\bar{w})z_n, \zeta) \in \Omega \text{ and } w \in \mathcal{D}\}.$$

We call $\mathcal{U}$ the "target" domain and Ω its "base" domain. The main concern in this thesis is obtaining explicit formulas for the Bergman kernel on $\mathcal{U}$ when ψ_j is chosen in the following two ways:

(i) $\psi_j(w,\bar{\eta}) = (1 - \langle w, \eta\rangle)^{-\frac{\alpha_j}{2}}$ where $\|w\| < 1$ and $\alpha_j \ge 0$.

(ii) $\psi_j(w,\bar{w}) = \exp\{\frac{\gamma_j}{2}\langle w, \eta\rangle\}$ where $w \in \mathbb{C}^k$ and $\gamma_j > 0$.

To avoid confusion, we use $\mathcal{U}$ to denote the domain constructed by the general ψ_j's and use U^α and V^γ to denote the domains where ψ_j is chosen in (i) and (ii):

- $U^\alpha = \{(z,z',w) \in \mathbb{C}^{n+m} \times \mathbb{C}^k : (f_\alpha(z,w), z') \in \Omega, \|w\| < 1\}$
  where
  $$f_\alpha(z,w) = \left(\frac{z_1}{(1-\|w\|^2)^{\frac{\alpha_1}{2}}}, \ldots, \frac{z_n}{(1-\|w\|^2)^{\frac{\alpha_n}{2}}}\right)$$
  and α_j's are positive numbers.

- $V^\gamma = \{(z, z', w) \in \mathbb{C}^{n+m} \times \mathbb{C}^k : (g_\gamma(z, w), z') \in \Omega\}$
  where
  $$g_\gamma(z, w) = \left(e^{\frac{\gamma_1 \|w\|^2}{2}} z_1, \ldots, e^{\frac{\gamma_n \|w\|^2}{2}} z_n \right)$$
  and γ_j's are positive numbers.

Remark. *In our definition of U^α and V^γ, we avoid the cases when all α_j and γ_j equal 0 since they are not interesting. When $\alpha = \mathbf{0}$, $U^{\mathbf{0}}$ becomes $\Omega \times \mathbb{B}^k$ and $K_{U^{\mathbf{0}}}$ equals the product of the Bergman kernels on Ω and the unit disk $\mathbb{B}$. When $\gamma = \mathbf{0}$, $V^{\mathbf{0}} = \Omega \times \mathbb{C}^k$. Since $A^2(V^{\mathbf{0}}) = \{0\}$, the kernel function $K_{V^{\mathbf{0}}}$ is identically zero. These results are consistent with Theorems 3.1 and 3.2.*

Since $e^{\|w\|^2}$ and $(1 - \|w\|^2)^{-1}$ are increasing in $\|w\|$ and invariant under the rotation map $w_j \mapsto e^{i\theta} w_j$ for $\theta \in \mathbb{R}$ and $1 \le j \le k$, the slice domains of U^α and V^γ with z and z' coordinates fixed are Reinhardt domains containing the origin in $\mathbb{C}^k$. This observation yields the following:

Lemma 2.5. *If Ω is n-star-shaped Hartogs in the variables $(z_1, \ldots, z_n)$, then U^α and V^γ are $(n+k)$-star-shaped Hartogs in the variables $(z_1, \ldots, z_n, w)$.*

By Lemma 2.4, a complete orthogonal system of the form $\{z^{\mathbf{a}} \phi_{\mathbf{a},\mathbf{b},c}(z') w^{\mathbf{c}}\}$ can be chosen for $A^2(U^\alpha)$ and $A^2(V^\gamma)$. The next lemma implies that for $\mathbf{c} \in \mathbb{N}^k$, $\{z^{\mathbf{a}} \phi_{\mathbf{a},\mathbf{b}}(z') w^{\mathbf{c}}\}$ is a complete orthogonal system for both $A^2(U^\alpha)$ and $A^2(V^\gamma)$ if $\{z^{\mathbf{a}} \phi_{\mathbf{a},\mathbf{b}}(z')\}$ is a complete orthogonal system for $A^2(\Omega)$.

Lemma 2.6. *The function $z^{\mathbf{a}} \phi(z')$ is square-integrable on Ω if and only if for all $\mathbf{c} \in \mathbb{N}^k$, the function $z^{\mathbf{a}} \phi(z') w^{\mathbf{c}}$ is square-integrable on U^α $\big($or $V^\gamma\big)$.*

Proof. Suppose $z^{\mathbf{a}} \phi(z') w^{\mathbf{c}} \in A^2(U^\alpha)$. Then

$$\int_{U^\alpha} |z^{\mathbf{a}}|^2 |\phi(z')|^2 |w^{\mathbf{c}}|^2 dV(z, z', w) = \|z^{\mathbf{a}} \phi(z') w^{\mathbf{c}}\|^2_{L^2(U^\alpha)} < \infty. \tag{2.8}$$

Substituting $t_j = z_j (1 - \|w\|^2)^{-\frac{\alpha_j}{2}}$ for $1 \le j \le n$ and applying Fubini's theorem to the integral in (2.8) yield:

$$\begin{aligned} &\int_{U^\alpha} |z^{\mathbf{a}}|^2 |\phi(z')|^2 |w^{\mathbf{c}}|^2 dV(z, z', w) \\ &= \int_{\mathbb{B}^k} |w^{\mathbf{c}}|^2 (1 - \|w\|^2)^{\alpha \cdot (\mathbf{a}+\mathbf{1})} dV(w) \int_\Omega |t^{\mathbf{a}}|^2 |\phi(z')|^2 dV(t, z') \\ &= \int_{\mathbb{B}^k} |w^{\mathbf{c}}|^2 (1 - \|w\|^2)^{\alpha \cdot (\mathbf{a}+\mathbf{1})} dV(w) \big\| t^{\mathbf{a}} \phi(z') \big\|^2_{L^2(\Omega)} < \infty. \end{aligned} \tag{2.9}$$

Since $\int_{\mathbb{B}^k} |w^{\mathbf{c}}|^2(1-\|w\|^2)^{\alpha\cdot(\mathbf{a}+\mathbf{1})}dV(w)$ is a constant, $\|z^{\mathbf{a}}\phi(z')\|^2_{L^2(\Omega)} < \infty$ and hence $z^{\mathbf{a}}\phi(z')$ is in $A^2(\Omega)$. By (2.9), the converse is also true. A similar argument proves the statement for V^γ. We omit the details. $\square$

The definitions of U^α and V^γ also imply that the slices of U^α and V^γ, with the w coordinate fixed, are biholomorphic to Ω. For fixed $w \in \mathbb{B}^k$ and $\eta \in \mathbb{C}^k$, let U^α_w denote the slice domain $\{(z,z') \in \mathbb{C}^{n+m} : (z,z',w) \in U^\alpha\}$ of U^α and let V^γ_η denote the slice domain $\{(z,z') \in \mathbb{C}^{n+m} : (z,z',\eta) \in V^\gamma\}$ of V^γ. Applying the mappings $f_\alpha(\cdot,w)$ and $g_\gamma(\cdot,\eta)$ to U^α_w and V^γ_η yields:

Lemma 2.7. *U^α_w and V^γ_η are biholomorphic to Ω.*

We illustrate our technique of obtaining K_{U^α} and K_{V^γ} using the following special case of Example 3.1:

Example. *Let Ω be the unit disk in $\mathbb{C}$ and $\mathcal{U}$ be the complex oval*

$$\{(z,w) \in \mathbb{C}^2 : |z|^{2a} + |w|^2 < 1\}.$$

Regarding w above as a parameter, we obtain a family of domains in $\mathbb{C}$:

$$\mathcal{U}_w = \Big\{z \in \mathbb{C} : \frac{|z|^2}{(1-|w|^2)^{\frac{1}{a}}} < 1\Big\}.$$

For each $\eta \in \mathbb{C}$ with $|\eta| < 1$, $\mathcal{U}_\eta$ is biholomorphic to the unit disk. Applying the biholomorphic transformation rule to the Bergman kernel $K_{\mathcal{U}_\eta}$ on $\mathcal{U}_\eta$ yields:

$$K_{\mathcal{U}_\eta}(z;\bar\zeta) = \frac{(1-|\eta|^2)^{\frac{1}{a}}}{\pi\big((1-|\eta|^2)^{\frac{1}{a}} - z\bar\zeta\big)^2}. \tag{2.10}$$

Replacing z in (2.10) by $z(\frac{(1-|\eta|^2)}{(1-w\bar\eta)})^{\frac{1}{a}}$ and multiplying the right hand side of (2.10) by $(1-|\eta|^2)^{\frac{1}{a}}$ yield a Hermitian symmetric function K_1 on $\mathcal{U}\times\mathcal{U}$:

$$K_1(z,w;\bar\zeta,\bar\eta) = \frac{(1-w\bar\eta)^{\frac{2}{a}}}{\pi\big((1-w\bar\eta)^{\frac{1}{a}} - z\bar\zeta\big)^2}. \tag{2.11}$$

Let I denote the identity operator. Applying the first order differential operator

$$D_{\mathcal{U}} = \frac{1}{\pi(1-w\bar\eta)^{2+\frac{1}{a}}}\Big((1+\frac{1}{a})I + \frac{1}{a}z\frac{\partial}{\partial z}\Big),$$

to K_1, we obtain

$$\frac{(1+a)(1-w\bar\eta)^{\frac{1}{a}} + (1-a)z\bar\zeta}{\pi^2 a(1-w\bar\eta)^{2-\frac{1}{a}}\big((1-w\bar\eta)^{\frac{1}{a}} - z\bar\zeta\big)^3}.$$

We then can verify that this function is the Bergman kernel on $\mathcal{U}$ (It agrees with the formula in [D'A78].)

As in the example, the procedure of obtaining $K_{\mathcal{U}}$ can be summarized as follows:

1. compute the Bergman kernel $K_{\mathcal{U}_w}$.
2. evaluate $K_{\mathcal{U}_w}$ at a certain point off the diagonal.
3. obtain a Hermitian symmetric function K on $\mathcal{U}$ by multiplying the result in Step (2) by a certain function.
4. apply a differential operator $D_{\mathcal{U}}$ to K.
5. verify that the result in Step 4 is the Bergman kernel on $\mathcal{U}$.

When $\mathcal{U}$ is U^α or V^γ, Lemmas 2.3 and 2.6 imply that a complete orthogonal system for $A^2(\mathcal{U})$ of the form $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^{\mathbf{c}}\}$ can be chosen. Therefore the Bergman kernel on $\mathcal{U}$ has the expansion:

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{c}} \frac{(z\bar{\zeta})^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}(w\bar{\eta})^{\mathbf{c}}}{\|z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^{\mathbf{c}}\|^2_{L^2(\mathcal{U})}}. \tag{2.12}$$

The function K we obtained in Step 3 is defined on $\mathcal{U}\times\mathcal{U}$ and has the expansion:

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{c}} c_{\mathbf{a},\mathbf{b},\mathbf{c}}(z\bar{\zeta})^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}(w\bar{\eta})^{\mathbf{c}}. \tag{2.13}$$

After applying $D_{\mathcal{U}}$ to (2.13) in Step 4 and verifying that

$$\int_{\mathcal{U}} D_{\mathcal{U}}K(z,z',w;\bar{\zeta},\bar{\zeta}',\bar{\eta})\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta')\eta^{\mathbf{c}}dV = z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^{\mathbf{c}}$$

in Step 5, we conclude that the Bergman kernel $K_{\mathcal{U}}$ equals $D_{\mathcal{U}}K$.

Chapter 3

Computation of the Bergman kernel

In this chapter, we state and prove our formulas for K_{U^α} and K_{V^γ}. Recall $\mathcal{U}$, K and $D_{\mathcal{U}}$ from Section 2.3. If ψ_j satisfies inequality (3.1) below, then one can always construct an appropriate Hermitian symmetric function K on $\mathcal{U} \times \mathcal{U}$. It is our particular choice of ψ_j for U^α and V^γ that enables us to obtain $D_{\mathcal{U}}$ explicitly as the differential operator in Step 4 such that $K_{\mathcal{U}} = D_{\mathcal{U}} K$.

3.1 The construction of K

Let $\Omega \subseteq \mathbb{C}^{n+m}$ be an n-star-shaped Hartogs domain. Let ψ_j, $\mathcal{U}$ and $\mathcal{U}_w$ be the same as in Section 2.3. Let ψ denote the function

$$\psi(z, w, \eta) = \left(z_1 \left(\frac{\psi_1(w, \bar{\eta})}{\psi_1(\eta, \bar{\eta})} \right)^2, \ldots, z_n \left(\frac{\psi_n(w, \bar{\eta})}{\psi_n(\eta, \bar{\eta})} \right)^2 \right).$$

Then we have the following lemma.

Lemma 3.1. *Suppose for all j, the function ψ_j satisfies the Cauchy-Schwarz inequality*

$$|\psi_j(w, \bar{\eta})|^2 \leq |\psi_j(w, \bar{w})||\psi_j(\eta, \bar{\eta})| \tag{3.1}$$

where $w, \eta \in \mathcal{D}$. Then for $(z, z', w; \zeta, \zeta', \eta) \in \mathcal{U} \times \mathcal{U}$, the function

$$\frac{1}{\prod_{j=1}^n (\psi_j(\eta, \bar{\eta}))^2} K_{\mathcal{U}_\eta}(\psi(z, w, \eta), z'; \bar{\zeta}, \bar{\zeta}') \tag{3.2}$$

is defined, holomorphic in (z, z', w) and anti-holomorphic in (ζ, ζ', η).

Proof. By its definition, $\mathcal{U}_\eta$ contains (ζ, ζ'). To prove $(\psi(z, w, \eta), z') \in \mathcal{U}_\eta$, it suffices to show that

$$\left(z_1 \frac{\psi_1^2(w, \bar{\eta})}{\psi_1(\eta, \bar{\eta})}, \ldots, z_n \frac{\psi_n^2(w, \bar{\eta})}{\psi_n(\eta, \bar{\eta})}, z' \right) \in \Omega.$$

Note that by $(z, z', w) \in \mathcal{U}$, we have

$$\left(z_1\psi_1(w, \bar{w}), \dots, z_n\psi_n(w, \bar{w}), z'\right) \in \Omega.$$

Since Ω is n-star-shaped Hartogs, the containment $(z, z', w) \in \mathcal{U}$ and inequality (3.1) imply the containment $(\psi(z, w, \eta), z') \in \mathcal{U}_\eta$. Hence (3.2) is defined. Consider the biholomorphic map F from $\mathcal{U}_\eta$ to Ω:

$$F : (z_1, \dots, z_n, z') \mapsto (z_1\psi_1(\eta, \bar{\eta}), \dots, z_n\psi_n(\eta, \bar{\eta})).$$

By the transformation rule (2.4), we have

$$\frac{K_{\mathcal{U}_\eta}\left(\psi\left(z, w, \eta\right), z'; \bar{\zeta}, \bar{\zeta}'\right)}{\prod_{j=1}^n \left(\psi_j\left(\eta, \bar{\eta}\right)\right)^2} = K_\Omega\left(F\left(\psi\left(z, w, \eta\right), z'\right); \overline{F\left(\zeta, \zeta'\right)}\right).$$

Recall $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}\}$ in Corollary 2.1. By applying (2.6) to the right-hand side of the equality, (3.2) becomes

$$\sum_{\mathbf{a},\mathbf{b}} \frac{(z\bar{\zeta})^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}\prod_{j=1}^n \psi_j^{2a_j}(w, \bar{\eta})}{\|z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\|_{L^2(\Omega)}^2}.$$

Since ψ_j is holomorphic in w and $\bar{\eta}$, the series is holomorphic in (z, z', w) and anti-holomorphic in (ζ, ζ', η). □

When $\mathcal{U}$ is U^α or V^γ, the function $\psi_j^{2a_j}$ has a power series expansion on $\mathcal{D}$. Thus (3.2) can be expressed as

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{c}} c_{\mathbf{a},\mathbf{b},\mathbf{c}}(z\bar{\zeta})^{\mathbf{a}}(w\bar{\eta})^{\mathbf{c}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}. \tag{3.3}$$

Note that $f(z, z') \in A^2(\Omega)$ if and only if $f(z, z')w^{\mathbf{c}} \in A^2(U^\alpha)$ (or $A^2(V^\gamma)$) for all multi-index $\mathbf{c}$. Corollary 2.1 implies that the Bergman kernels K_{U^α} and K_{V^γ} also have expansions

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{c}} c'_{\mathbf{a},\mathbf{b},\mathbf{c}}(z\bar{\zeta})^{\mathbf{a}}(w\bar{\eta})^{\mathbf{c}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}. \tag{3.4}$$

In the next section, we introduce the differential operator which "corrects" the coefficient $c_{\mathbf{a},\mathbf{b},\mathbf{c}}$ in (3.3) to $c'_{\mathbf{a},\mathbf{b},\mathbf{c}}$ in (3.4) and conclude the proof of our main theorems.

3.2 Main theorems

Let I denote the identity operator. Let $\mathbf{1}$ denote the vector $(1,\ldots,1)\in\mathbb{N}^n$. Let D_{U^α} denote the differential operator defined by

$$D_{U^\alpha}=\frac{(1-\|\eta\|^2)^{\alpha\cdot\mathbf{1}}}{\pi^k(1-\langle w,\eta\rangle)^{1+k+\alpha\cdot\mathbf{1}}}\prod_{i=1}^{k}\left(iI+\sum_{j=1}^{n}\alpha_j\left(I+z_j\frac{\partial}{\partial z_j}\right)\right).$$

Let D_{V^γ} denote the differential operator defined by

$$D_{V^\gamma}=\frac{e^{(\gamma\cdot\mathbf{1})(w\bar\eta-|\eta|^2)}}{\pi^k}\left(\sum_{j=1}^{n}\gamma_j\left(I+z_j\frac{\partial}{\partial z_j}\right)\right)^k.$$

Let $h(z,w,\eta)$ denote $\psi(z,w,\eta)$ when $h_j(w,\bar\eta)=(1-\langle w,\eta\rangle)^{-\frac{\alpha_j}{2}}$, i.e.

$$h\left(z,w,\eta\right)=\left(z_1\left(\frac{1-\|\eta\|^2}{1-\langle w,\eta\rangle}\right)^{\alpha_1},\ldots,z_n\left(\frac{1-\|\eta\|^2}{1-\langle w,\eta\rangle}\right)^{\alpha_n}\right).$$

Let $l(z,w,\eta)$ denote $\psi(z,w,\eta)$ when $h_j(w,\bar\eta)=\exp\{\frac{\gamma_j\langle w,\eta\rangle}{2}\}$, i.e.

$$l(z,w,\eta)=\left(z_1e^{\gamma_1(\langle w,\eta\rangle-\|\eta\|^2)},\ldots,z_ne^{\gamma_n(\langle w,\eta\rangle-\|\eta\|^2)}\right).$$

Then our main results can be expressed as follows:

Theorem 3.1. *For $(z,z',w;\zeta,\zeta',\eta)\in U^\alpha\times U^\alpha$, let $h(z,w,\eta)$ and D_{U^α} be as above. Then*

$$K_{U^\alpha}\left(z,z',w;\bar\zeta,\bar\zeta',\bar\eta\right)=D_{U^\alpha}K_{U^\alpha_\eta}\left(h(z,w,\eta),z';\bar\zeta,\bar\zeta'\right). \tag{3.5}$$

Theorem 3.2. *For $(z,z',w;\zeta,\zeta',\eta)\in V^\gamma\times V^\gamma$, let $l(z,w,\eta)$ and D_{V^γ} be as above. Then*

$$K_{V^\gamma}\left(z,z',w;\bar\zeta,\bar\zeta',\bar\eta\right)=D_{V^\gamma}K_{V^\gamma_\eta}\left(l(z,w,\eta),z';\bar\zeta,\bar\zeta'\right). \tag{3.6}$$

Proof of Theorem 3.1. Let $K_1(z,z',w;\bar\zeta,\bar\zeta',\bar\eta)$ denote the right-hand side of (3.5). By the argument in Section 3.1, $K_1(z,z',w;\bar\zeta,\bar\zeta',\bar\eta)$ is defined on $U^\alpha\times U^\alpha$ and it has the expression:

$$\sum_{\mathbf{a},\mathbf{b},\mathbf{c}}c_{\mathbf{a},\mathbf{b},\mathbf{c}}(z\bar\zeta)^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}(w\bar\eta)^{\mathbf{c}}.$$

To show K_1 is the Bergman kernel, it suffices to verify that K_1 reproduces every element in $A^2(U^\alpha)$. For arbitrary $z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^{\mathbf{c}} \in A^2(U^\alpha)$,

$$\begin{aligned}
&\int_{U^\alpha} K_1(z, z', w; \bar{\zeta}, \bar{\zeta}', \bar{\eta})\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta')\eta^{\mathbf{c}} dV \\
=&\int_{\mathbb{B}^k} \eta^{\mathbf{c}} \int_{U^\alpha_\eta} D_{U^\alpha} K_{U^\alpha_\eta} \left(h(z, w, \eta), z'; \bar{\zeta}; \bar{\zeta}'\right) \zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta') dV(\zeta, \zeta') dV(\eta). \quad (3.7)
\end{aligned}$$

Using the reproducing property of $K_{U^\alpha_\eta}$ on U^α_η and the fact that

$$\prod_{i=1}^{k} \left(iI + \sum_{j=1}^{n} \alpha_j \left(I + z_j \frac{\partial}{\partial z_j} \right) \right) (z^{\mathbf{a}}) = \prod_{i=1}^{k} \left(i + \alpha \cdot (\mathbf{a} + \mathbf{1})\right) z^{\mathbf{a}},$$

we have

$$\begin{aligned}
&\int_{U^\alpha_\eta} D_{U^\alpha} K_{U^\alpha_\eta} \left(h(z, w, \eta), z'; \bar{\zeta}; \bar{\zeta}'\right) \zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta') dV(\zeta, \zeta') \\
=&\prod_{i=1}^{k} \left(i + \alpha \cdot (\mathbf{a} + \mathbf{1})\right) \frac{(1 - \|\eta\|^2)^{\alpha\cdot\mathbf{1}}}{\pi(1 - \langle w, \eta\rangle)^{2+\alpha\cdot\mathbf{1}}} h(z, w, \eta)^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z'). \quad (3.8)
\end{aligned}$$

Let $C_{\alpha,\mathbf{a}} = \prod_{i=1}^{k} \left(i + \alpha \cdot (\mathbf{a} + \mathbf{1})\right)$. Then the integral in the second line of (3.7) becomes

$$C_{\alpha,\mathbf{a}} \int_{\mathbb{B}^k} \frac{(1 - \|\eta\|^2)^{\alpha\cdot\mathbf{1}}\eta^{\mathbf{c}} h(z, w, \eta)^{\mathbf{a}}}{\pi^k(1 - \langle w, \eta\rangle)^{1+k+\alpha\cdot\mathbf{1}}} dV(\eta). \quad (3.9)$$

Since $h(z, w, \eta) = (z_1(\frac{1-\|\eta\|^2}{1-\langle w,\eta\rangle})^{\alpha_1}, \ldots, z_n(\frac{1-\|\eta\|^2}{1-\langle w,\eta\rangle})^{\alpha_n})$, (3.9) is equal to

$$\frac{C_{\alpha,\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')}{\pi^k} \int_{\mathbb{B}^k} \frac{(1 - \|\eta\|^2)^{\alpha\cdot(\mathbf{a}+\mathbf{1})}\eta^{\mathbf{c}}}{(1 - \langle w, \eta\rangle)^{1+k+\alpha\cdot(\mathbf{a}+\mathbf{1})}} dV(\eta). \quad (3.10)$$

Expanding the denominator in (3.10), we have

$$\begin{aligned}
&z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z') \int_{\mathbb{B}^k} \sum_{\mathbf{p}} \frac{(1 + \alpha \cdot (\mathbf{a} + \mathbf{1}))_{(\mathbf{j}\cdot\mathbf{1})+k} \left(1 - \|\eta\|^2\right)^{\alpha\cdot(\mathbf{a}+\mathbf{1})} (w\bar{\eta})^{\mathbf{p}}}{\pi^k \prod_{p=1}^{k}(p_j)!} \eta^{\mathbf{c}} dV \\
=&z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^{\mathbf{c}} \int_{\mathbb{B}^k} \frac{(1 + \alpha \cdot (\mathbf{a} + \mathbf{1}))_{(\mathbf{c}\cdot\mathbf{1})+k} \left(1 - \|\eta\|^2\right)^{\alpha\cdot(\mathbf{a}+\mathbf{1})} |\eta|^{2\mathbf{c}}}{\pi^k \prod_{j=1}^{k}(c_j)!} dV. \quad (3.11)
\end{aligned}$$

By letting $r_j = |\eta_j|^2$, we have

$$\int_{\mathbb{B}^k} (1 - \|\eta\|^2)^{\alpha\cdot(\mathbf{a}+\mathbf{1})}|\eta|^{2\mathbf{c}} dV = \pi^k \int_{\mathbf{B}^k_+} (1 - \sum_{j=1}^{k} r_j)^{\alpha\cdot(\mathbf{a}+\mathbf{1})} r^{\mathbf{c}} dV \quad (3.12)$$

where $\mathbf{B}_+^k = \{(r_1, \ldots, r_k) \in \mathbb{R}_+^k; \sum_{j=1}^k r_j < 1\}$. We claim that

$$\pi^k \int_{\mathbf{B}_+^k} (1 - \sum_{j=1}^k r_j)^{\alpha \cdot (\mathbf{a}+\mathbf{1})} r^{\mathbf{c}} dV = \frac{\pi^k \prod_{j=1}^k (c_j)!}{(1 + \alpha \cdot (\mathbf{a}+\mathbf{1}))_{(\mathbf{c}\cdot\mathbf{1})+k}}. \tag{3.13}$$

Then the term in the second line 3.11 becomes

$$z^{\mathbf{a}} \phi_{\mathbf{a},\mathbf{b}}(z') w^{\mathbf{c}},$$

which completes the proof.

To prove (3.13), we do induction on k. When $k = 1$, we have

$$\int_0^1 (1-r)^{\alpha \cdot (\mathbf{a}+\mathbf{1})} r^c dV = \frac{\Gamma(1 + \alpha \cdot (\mathbf{a}+\mathbf{1}))\Gamma(c+1)}{\Gamma(2 + c + \alpha \cdot (\mathbf{a}+\mathbf{1}))}.$$

Thus (3.13) holds for $k = 1$. Suppose (3.13) holds for $k < N$. When $k = N$,

$$\begin{aligned} &\int_{\mathbb{B}_+^N} (1 - \sum_{j=1}^N r_j)^{\alpha \cdot (\mathbf{a}+\mathbf{1})} r^{\mathbf{c}} dV \\ =&\int_0^1 {r_N}^{c_N} \int_{W_{r_N}} (1 - \sum_{j=1}^N r_j)^{\alpha \cdot (\mathbf{a}+\mathbf{1})} \prod_{j=1}^{N-1} r_j^{c_j} dr_1 \ldots dr_{N-1} dr_N, \end{aligned} \tag{3.14}$$

where $W_{r_N} = \{(r_1, \ldots, r_{N-1}) \in \mathbb{R}_+^{N-1} : \sum_{j=1}^{N-1} r_j < 1 - r_N\}$. By substituting $t_j = \frac{r_j}{1-r_N}$ in the second line of (3.14) for $1 \le j \le N-1$, we obtain

$$\begin{aligned} &\left(\int_0^1 {r_N}^{c_N} (1-r_N)^{\alpha \cdot (\mathbf{a}+\mathbf{1}) + \sum_{j=1}^{N-1}(c_j+1)} dr_N \right) \\ &\times \left(\int_{\mathbb{B}_+^{N-1}} (1 - \sum_{j=1}^{N-1} t_j)^{\alpha \cdot (\mathbf{a}+\mathbf{1})} \prod_{j=1}^{N-1} t_j^{c_j} dt_1 \ldots dt_{N-1} \right). \end{aligned} \tag{3.15}$$

Applying the definition of the beta function and the induction hypothesis, (3.15) becomes

$$\begin{aligned} &\frac{\Gamma(c_N+1)\Gamma(\alpha \cdot (\mathbf{a}+\mathbf{1}) + \sum_{j=1}^{N-1}(c_j+1) + 1)}{\Gamma(\alpha \cdot (\mathbf{a}+\mathbf{1}) + \sum_{j=1}^{N}(c_j+1) + 1)} \times \frac{\prod_{j=1}^{N-1}(c_j)!}{(1 + \alpha \cdot (\mathbf{a}+\mathbf{1}))_{\sum_{j=1}^{N-1}(c_j+1)}} \\ =&\frac{\prod_{j=1}^{N}(c_j)!}{(\alpha \cdot (\mathbf{a}+\mathbf{1}) + \sum_{j=1}^{N-1} c_j + N)_{c_N+1} (1 + \alpha \cdot (\mathbf{a}+\mathbf{1}))_{\sum_{j=1}^{N-1}(c_j+1)}} \\ =&\frac{\prod_{j=1}^{N}(c_j)!}{(1 + \alpha \cdot (\mathbf{a}+\mathbf{1}))_{\mathbf{c}\cdot\mathbf{1}+N}}. \end{aligned}$$

Therefore (3.13) holds for all k. $\square$

Proof of Theorem 3.2. Let K_2 denote the right-hand side of (3.6). Then K_2 is defined on $V^\gamma \times V^\gamma$ and can be written as:

$$\sum_{\mathbf{a},\mathbf{b}} c_{\mathbf{a},\mathbf{b},\mathbf{c}}(z\bar{\zeta})^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\overline{\phi_{\mathbf{a},\mathbf{b}}(\zeta')}(w\bar{\eta})^{\mathbf{c}}.$$

We verify that K_2 reproduces every element in $A^2(V^\gamma)$. For arbitrary $z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^c \in A^2(V^\gamma)$,

$$\begin{aligned}&\int_{V^\gamma} K_2(z,z',w;\bar{\zeta},\bar{\zeta}',\bar{\eta})\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta')\eta^{\mathbf{c}}dV \\ =&\int_{\mathbb{C}^k}\eta^{\mathbf{c}}\int_{V^\gamma_\eta} D_{V^\alpha}K_{V^\gamma_\eta}(l(z,w,\eta),z';\bar{\zeta},\bar{\zeta}')\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(\zeta')dV(\zeta,\zeta')dV(\eta). \end{aligned} \tag{3.16}$$

Applying the reproducing property of K_{V^γ} and the fact that

$$\left(\sum_{j=1}^{n}\gamma_j\left(I+z_j\frac{\partial}{\partial z_j}\right)\right)^k (z^{\mathbf{a}}) = (\gamma\cdot(\mathbf{a}+\mathbf{1}))^k z^{\mathbf{a}}$$

to the inner integral in the last line of (3.16) yield

$$\begin{aligned}&\int_{V^\alpha_\eta} D_{V^\alpha}K_{V^\gamma_\eta}\left(l\left(z,w,\eta\right),z';\bar{\zeta},\bar{\zeta}'\right)\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}\left(\zeta'\right)dV\left(\zeta,\zeta'\right) \\ =&\pi^{-k}\left(\gamma\cdot(\mathbf{a}+\mathbf{1})\right)^k\phi_{\mathbf{a},\mathbf{b}}\left(z'\right)e^{(\alpha\cdot(\mathbf{a}+\mathbf{1}))(w\bar{\eta}-|\eta|^2)}z^{\mathbf{a}}. \end{aligned} \tag{3.17}$$

Therefore the integral in the last line of (3.16) becomes

$$\pi^{-k}(\alpha\cdot(\mathbf{a}+\mathbf{1}))^k z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\int_{\mathbb{C}^k}\frac{e^{\gamma\cdot(\mathbf{a}+\mathbf{1})\langle w,\eta\rangle}\eta^{\mathbf{c}}}{e^{(\gamma\cdot(\mathbf{a}+\mathbf{1}))\|\eta\|^2}}dV(\eta). \tag{3.18}$$

Expanding $e^{\gamma\cdot(\mathbf{a}+\mathbf{1})\langle w,\eta\rangle}$ in (3.18), we have

$$\begin{aligned}(3.18) =&z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\prod_{i=1}^{k}\int_{\mathbb{C}^1}\sum_{j=0}^{\infty}\frac{(\gamma\cdot(\mathbf{a}+\mathbf{1}))^{j+1}(w_i\bar{\eta}_i)^j}{\pi^k j! e^{\gamma\cdot(\mathbf{a}+\mathbf{1})|\eta_i|^2}}\eta_i^{c_i}dV(\eta_i) \\ =&z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^{c_i}\prod_{i=1}^{k}\int_{\mathbb{C}^1}\frac{(\gamma\cdot(\mathbf{a}+\mathbf{1}))^{c_i+1}|\eta_i|^{2c_i}}{\pi^k c_i! e^{\gamma\cdot(\mathbf{a}+\mathbf{1})|\eta_i|^2}}dV(\eta_i). \end{aligned} \tag{3.19}$$

Letting $t=\gamma\cdot(\mathbf{a}+\mathbf{1})|\eta|^2$ and using polar coordinates, the last line of (3.19) becomes

$$z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^c\prod_{i=1}^{k}\int_0^\infty \frac{t^{c_i}}{\mathbf{c}!}e^{-t}dt,$$

which equals $z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')w^{\mathbf{c}}$. Therefore K_2 is the Bergman kernel on V^γ. $\square$

3.3 Examples

Theorems 3.1 and 3.2 enable us to explicitly compute the Bergman kernel in new situations. We use Theorem 3.1 to give a new proof of the explicit formula in [D'A94]. Then we compute the kernel function in some new cases.

Example 3.1. *Let the "base" domain Ω be the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. For $p > 0$, put $\alpha = (\frac{1}{p}, \dots, \frac{1}{p})$ and let $w \in \mathbb{C}$. We have*

$$U^\alpha = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} :, \|z\|^{2p} + |w|^2 < 1\}.$$

By Theorem 3.1, the Bergman kernel function K_{U^α} equals:

$$\frac{n!}{\pi^{n+1}p}\frac{(n+p)(1-w\overline{\eta})^{\frac{1}{p}} + (1-p)\langle z,\zeta\rangle}{(1-w\overline{\eta})^{2-\frac{1}{p}}((1-w\overline{\eta})^{\frac{1}{p}} - \langle z,\zeta\rangle)^{n+2}}.$$

We consider domain $U^{\alpha\prime} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m :, \|z\|^{2p} + \|w\|^2 < 1\}$. Applying the inflation method in [BFS99] to K_{U^α} yields the Bergman kernel function on $U^{\alpha\prime}$:

$$\frac{n!}{\pi^{m+n}p}\Big(\frac{\partial}{\partial t}\Big)^{m-1}\frac{(n+p)(1-\langle w,\eta\rangle)^{\frac{1}{p}} + (1-p)\langle z,\zeta\rangle}{(1-\langle w,\eta\rangle)^{2-\frac{1}{p}}\Big((1-\langle w,\eta\rangle)^{\frac{1}{p}} - \langle z,\zeta\rangle\Big)^{n+2}},$$

where $t = \langle w,\eta\rangle$. We may also apply Theorem 3.1 to $U^{\alpha\prime}$ for $w \in \mathbb{C}^k$ to obtain $K_{U^{\alpha\prime}}$:

$$\frac{1}{(1-\langle w,\eta\rangle)^{1+m-\frac{1}{p}}}\prod_{i=1}^{m}\left(iI + \sum_{j=1}^{n}\frac{1}{p}\left(I + z_j\frac{\partial}{\partial z_j}\right)\right)\frac{n!}{\pi^{n+m}((1-\langle w,\eta\rangle)^{\frac{1}{p}} - \langle z,\zeta\rangle)^{n+1}}.$$

Note that if we let the above p tend to ∞, then U^α becomes $\mathbb{B}^n \times \mathbb{B}^m$ and K_{U^α} equals $K_{\mathbb{B}^n} \cdot K_{\mathbb{B}^m}$.

Example 3.2. *Suppose $\Omega = \{(z,z') \in \mathbb{C}^n \times \mathbb{C}^m : \|z\|^2 + \|z'\|^2 < 1\}$ and $\alpha = (1, \cdots, 1)$, then the domain*

$$U^\alpha = \{(z,z',w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : |w| < 1, \|z\|^2 + \|z'\|^2 + |w|^2 < 1 + |w|^2\|z'\|^2\}$$

has the Bergman kernel function:

$$K_{U^\alpha} = \frac{(m+n)!}{\pi^{m+n+1}} \frac{(1-w\bar{\eta})^m(n+1-(n+1)\langle z', \zeta'\rangle + m\frac{\langle z,\zeta\rangle}{1-w\bar{\eta}})}{(1-w\bar{\eta}-\langle z,\zeta\rangle - \langle z',\zeta'\rangle + w\bar{\eta}\langle z',\zeta'\rangle)^{m+n+2}}. \tag{3.20}$$

When $m = 0$, we have

$$\frac{n+1}{\pi}\frac{n!}{\pi^n}\frac{1}{(1-w\bar{\eta}-\langle z,\zeta\rangle)^{n+2}},$$

the Bergman kernel function on the unit ball $\mathbb{B}^{n+1}$. When $n = m = 1$,

$$U^\alpha = \{(z, z', w) \in \mathbb{C}^3 : |w| < 1, |z|^2 + |z'|^2 + |w|^2 < 1 + |w|^2|z'|^2\}.$$

By (3.20), we obtain the kernel function on U^α:

$$K_{U^\alpha} = \frac{2}{\pi^3}\frac{(1-w\bar{\eta})(2-2z'\bar{\zeta}' + \frac{z\bar{\zeta}}{1-w\bar{\eta}})}{(1-w\bar{\eta}-z\bar{\zeta}-z'\bar{\zeta}'+w\bar{\eta}z'\bar{\zeta}')^4}.$$

Example 3.3. *Let Ω be as above and $\gamma = (\gamma_1, \ldots, \gamma_n)$, then*

$$V^\gamma = \left\{ \Big(z, z', w\Big) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : \sum_{j=1}^{n} e^{\gamma_j|w|^2}|z_j|^2 + \|z'\|^2 < 1 \right\}.$$

Put $\rho(z, z', w; \bar{\zeta}, \bar{\zeta}', \bar{\eta}) = 1 - \sum_{j=1}^n e^{\gamma_j w\bar{\eta}} z_j\bar{\zeta}_j - \langle z', \zeta'\rangle$. Then the Bergman kernel function satisfies

$$K_{V^\gamma} = \frac{(m+n)!e^{(\gamma\cdot\mathbf{1})w\bar{\eta}}}{\pi^{m+n+1}}\left(\frac{\gamma\cdot\mathbf{1}}{\rho^{m+n+1}} + \frac{(m+n+1)\sum_{j=1}^n \gamma_j e^{\gamma_j w\bar{\eta}} z_j\bar{\zeta}_j}{\rho^{m+n+2}}\right). \tag{3.21}$$

When γ tends to $\mathbf{0}$, the domain V^γ becomes $\mathbb{B}^{n+m} \times \mathbb{C}^1$, and the kernel function K_V^γ goes to 0, which is the Bergman kernel on $\mathbb{B}^{n+m} \times \mathbb{C}^1$. When $\gamma = \mathbf{1}$ and $n = m = 1$,

$$V^\gamma = \Big\{(z, z', w) \in \mathbb{C}^3 : e^{|w|^2}|z|^2 + |z'|^2 < 1\Big\} \tag{3.22}$$

and the kernel function on it is:

$$K_{V^\gamma} = \frac{2}{\pi^3}\frac{e^{w\bar{\eta}}(1-z'\bar{\zeta}' + 2e^{w\bar{\eta}}z\bar{\zeta})}{(1-e^{w\bar{\eta}}z\bar{\zeta}-z'\bar{\zeta}')^4}. \tag{3.23}$$

In the next section, we will use (3.20) and (3.21) to obtain the boundary behavior of the Bergman kernel on the domains in Example 3.2 and 3.3.

In Theorems 3.1 and 3.2, we only need our "base" domain to be n-star-shaped Hartogs. Note that the "target" domains U^α and V^γ are also $(n+k)$-star-shaped Hartogs in z and w. Therefore we can repeat using Theorems 3.1 and 3.2 to obtain the Bergman kernel on more complicated domains. In fact, repeated use k times of Theorem 3.2 for $w \in \mathbb{C}$ yields Theorem 3.2 for $w \in \mathbb{C}^k$.

Example 3.4 (Repeated use of Theorems 3.1 and 3.2)**.** *The diagram below indicates how to obtain the kernel function explicitly on increasingly complicated domains.*

$$\left\{z \in \mathbb{C}^1 : |z|^2 < 1\right\}$$
$$\Downarrow$$
$$\left\{z \in \mathbb{C}^2 : |z_1|^{2p} + |z_2|^2 < 1\right\}$$
$$\Downarrow$$
$$\left\{z \in \mathbb{C}^3 : |z_1|^{2p} + \exp\left\{|z_3|^2\right\}|z_2|^2 < 1\right\}$$
$$\Downarrow$$
$$\left\{z \in \mathbb{C}^4 : |z_1|^{2p_1} + \exp\left\{\frac{|z_3|^2}{(1-|z_4|^2)^{p_2}}\right\}|z_2|^2 < 1, |z_4| < 1\right\}$$
$$\Downarrow$$
$$\left\{z \in \mathbb{C}^5 : \frac{|z_1|^{2p_1}}{(1-|z_5|^2)^{p_3}} + \exp\left\{\frac{|z_3|^2}{(1-|z_4|^2)^{p_2}}\right\}|z_2|^2 < 1, |z_4| < 1, |z_5| < 1\right\}$$
$$\Downarrow$$
$$\left\{z \in \mathbb{C}^6 : \frac{|z_1|^{2p_1}}{(1-e^{|z_6|^2}|z_5|^2)^{p_3}} + \exp\left\{\frac{|z_3|^2}{(1-|z_4|^2)^{p_2}}\right\}|z_2|^2 < 1, |z_4| < 1, e^{|z_6|^2}|z_5|^2 < 1\right\}$$
$$\Downarrow$$
$$\vdots$$

The Bergman kernels in the first two cases are known. The kernel in the third case equals

$$\begin{aligned}
\frac{e^{z_3\bar{\zeta}_3}}{\pi^3 p}\Bigg(& \frac{(1+p)(1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{\frac{1}{p}} + (1-p)z_1\bar{\zeta}_1}{(1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{2-\frac{1}{p}}((1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{\frac{1}{p}} - z_1\bar{\zeta}_1)^3} \\
&+ \frac{(p-1)e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2\big((2+\frac{1}{p})(1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{\frac{1}{p}} + (2-\frac{1}{p})z_1\bar{\zeta}_1\big)}{(1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{3-\frac{1}{p}}((1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{\frac{1}{p}} - z_1\bar{\zeta}_1)^3} \\
&+ \frac{2e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2\big((2+\frac{1}{p})(1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{\frac{1}{p}} + (2-\frac{2}{p})z_1\bar{\zeta}_1\big)}{(1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{3-\frac{2}{p}}((1-e^{z_3\bar{\zeta}_3}z_2\bar{\zeta}_2)^{\frac{1}{p}} - z_1\bar{\zeta}_1)^4}\Bigg).
\end{aligned}$$

For the last three domains, the kernel functions are explicit but rather complicated. Hence we omit the formulas.

3.4 A case other than U^{α} and V^{γ}

Let $\Omega \subseteq \mathbb{C}^{n+m}$ be an n-star-shaped Hartogs domain. Recall the conjugate invariant domain $\mathcal{D} \subseteq \mathbb{C}^k$, the Hermitian symmetric functions ψ_j's on $\mathcal{D}\times\mathcal{D}$, and the domain $\mathcal{U}$ constructed using Ω and ψ_j's:

$$\{(z,\zeta,w) \in \mathbb{C}^{n+m+k} : (\psi_1(w,\bar{w})z_1, \ldots, \psi_n(w,\bar{w})z_n, \zeta) \in \Omega \text{ and } w \in \mathcal{D}\}.$$

Theorems 3.1 and 3.2 relate the Bergman kernel on $\mathcal{U}$ to the kernel functions on the lower dimensional slices $\mathcal{U}_w$ when ψ_j is chosen in the following two ways:

(i) $\psi_j(w,\bar{\eta}) = (1-\langle w,\eta\rangle)^{-\frac{\alpha_j}{2}}$ where $\|w\| < 1$ and $\alpha_j \geq 0$.

(ii) $\psi_j(w,\bar{w}) = \exp\{\frac{\gamma_j}{2}\langle w,\eta\rangle\}$ where $w \in \mathbb{C}^k$ and $\gamma_j > 0$.

It is natural to ask whether the same method works for other cases. In this section, we consider the case where $\Omega = \mathbb{B}$ and compute the Bergman kernel on $\mathcal{U}$ when $\mathcal{D}$ is the punctured disk $\mathbb{B}^*$ in $\mathbb{C}$ and $\psi_j(w,\bar{w})$ equals $|w|^{-\alpha_j}$ with $\alpha_j \in \mathbb{Q}^+$. We will see that, for such a $\mathcal{U}$, the kernel function $K_{\mathcal{U}}$ can be obtained by applying a similar technique not just to the Bergman kernel but also to some weighted kernel functions on the slice $\mathcal{U}_w$.

Let Ω be the unit disk in $\mathbb{C}$. Let $\psi(w,\bar{w}) = |w|^{-\frac{p}{q}}$ with p, q positive integers. Then $\mathcal{U} = \{(z,w) \in \mathbb{C}^2 : |z|^q < |w|^p < 1\}$. When $p = q$, the domain $\mathcal{U}$ becomes the Hartogs triangle and is biholomorphic to the product domain $\mathbb{B} \times \mathbb{B}^*$. When $p \neq q$, we call $\mathcal{U}$ the generalized Hartogs triangle. Edholm, in [Edh15], computed the Bergman kernel $K_{\mathcal{U}}$ when either p or q equals 1 and obtained the explicit formula for $K_{\mathcal{U}}$ in these cases:

1. For $\frac{p}{q} = \frac{1}{k}$ with $k \in \mathbb{N}^+$, the Bergman kernel on $\mathcal{U}$ is given by

$$K_{\mathcal{U}}\left(z,w;\bar{\zeta},\bar{\eta}\right) = \frac{p_k(z\bar{\zeta})\left((w\bar{\eta})^2 + \left(z\bar{\zeta}\right)^k\right) + q_k\left(z\bar{\zeta}\right)w\bar{\eta}}{k\pi^2\left(1-w\bar{\eta}\right)^2\left(w\bar{\eta}-(z\bar{\zeta})^k\right)^2}, \tag{3.24}$$

where p_k and q_k are polynomials

$$p_k(s) = \sum_{n=1}^{k-1} n(k-n)s^{n-1}, \quad q_k(s) = \sum_{n=1}^{k}(n^2+(k-n)^2s^k)s^{n-1}.$$

2. For $\frac{p}{q} = k$ with $k \in \mathbb{N}^+$, the Bergman kernel on $\mathcal{U}$ is given by

$$K_{\mathcal{U}}\left(z, w; \bar{\zeta}, \bar{\eta}\right) = \frac{(w\bar{\eta})^k}{\pi^2 (1 - w\bar{\eta})^2 \left((w\bar{\eta})^k - z\bar{\zeta}\right)^2}. \tag{3.25}$$

Using our technique, we rediscover Edholm's result.

For positive integers p and q, let $\mathcal{U}^{\frac{p}{q}}$ denote the set

$$\left\{(z, w) \in \mathbb{C}^2 : |z|^q < |w|^p < 1\right\}.$$

Since $\mathcal{U}^{\frac{p}{q}}$ is Reinhardt, every holomorphic function on $\mathcal{U}^{\frac{p}{q}}$ has a Laurent series expansion. Noting that $\mathcal{U}^{\frac{p}{q}}$ is 1-star-shaped Hartogs in z, we have the following lemma:

Lemma 3.2. *Let $f \in A\left(\mathcal{U}^{\frac{p}{q}}\right)$. Then f has a unique expansion:*

$$f\left(z, z', w\right) = \sum_{c\in\mathbb{Z}} \sum_{a\in\mathbb{N}} z^a w^c,$$

where the series converges normally in $\mathcal{U}^{\frac{p}{q}}$.

Lemma 3.3. *For $a \in \mathbb{N}$, let $\mathfrak{I}_a$ denote the set $\{c \in \mathbb{Z} : c+1+\frac{p}{q}(a+1) > 0\}$. Then the holomorphic function z^a is square-integrable on Ω if and only if $z^a w^c$ is square-integrable on $\mathcal{U}^{\frac{p}{q}}$ for all $c \in \mathfrak{I}_a$. Moreover, $z^a w^c$ is not square-integrable on $\mathcal{U}^{\frac{p}{q}}$ for $z^a \phi(z') \in A^2(\Omega)$ if $c \notin \mathfrak{I}_a$.*

Proof. Suppose $z^a w^c \in A^2(\mathcal{U}^{\frac{p}{q}})$. Then

$$\int_{\mathcal{U}^{\frac{p}{q}}} |z^a|^2 |w|^{2c} dV(z, w) = \|z^a w^c\|^2_{L^2(U^\alpha)} < \infty. \tag{3.26}$$

Substituting $t = z|w|^{-\frac{p}{q}}$ for $1 \le j \le n$ and applying Fubini's theorem to the integral in (3.26) yield:

$$\begin{aligned}
&\int_{\mathcal{U}^{\frac{p}{q}}} |z|^{2a} |w|^{2c} dV(z, z', w) \\
=&\int_{\mathbb{B}^*} |w|^{2c+2\frac{p}{q}(a+1)} dV(w) \int_{\mathbb{B}} |t|^{2a} dV(t) \\
=&2\pi \int_0^1 r^{2c+2\frac{p}{q}(a+1)+1} dr \left\|z^a\right\|^2_{L^2(\mathbb{B})}.
\end{aligned} \tag{3.27}$$

Since $\int_0^1 r^{2c+2\frac{p}{q}(a+1)+1} dr < \infty$ when $c \in \mathfrak{I}_a$, we have $z^a \in A^2(\Omega)$ for $c \in \mathfrak{I}_a$. By (3.27), the converse also holds for $c \in \mathfrak{I}_a$. When $c \notin \mathfrak{I}_a$, the first integral in the last line of (3.27) blows up. Therefore $z^a w^c$ is not square-integrable on $\mathcal{U}^{\frac{p}{q}}$ for $c \notin \mathfrak{I}_a$. $\square$

Combining Lemmas 3.2 and 3.3 yields the following:

Corollary 3.1. *Let $K_{\mathcal{U}^{\frac{p}{q}}}$ be the Bergman kernel on $\mathcal{U}^{\frac{p}{q}}$. Let $\mathfrak{I}_a$ be as in Lemma 3.3. Then $K_{\mathcal{U}^{\frac{p}{q}}}$ has the following expansion:*

$$K_{\mathcal{U}^{\frac{p}{q}}}(z, w; \bar{\zeta}, \bar{\eta}) = \sum_{a\in\mathbb{N}} \sum_{c\in\mathfrak{I}_a} \frac{(z\bar{\zeta})^a (w\bar{\eta})^c}{\|z^a w^c\|^2_{L^2(\mathcal{U}^{\frac{p}{q}})}}. \tag{3.28}$$

For $\eta \in \mathbb{B}^*$, let $\mathcal{U}_\eta^{\frac{p}{q}}$ denote the slice of $\mathcal{U}^{\frac{p}{q}}$:

$$\{z \in \mathbb{C} : |z|^q < |\eta|^p < 1\}.$$

For $k \in \mathbb{R}$, let $K^k_{\mathcal{U}_\eta^{\frac{p}{q}}}$ denote the reproducing kernel for the space $A^2(\mathcal{U}_\eta^{\frac{p}{q}}, |z|^k)$, the weighted Bergman kernel on $\mathcal{U}_\eta^{\frac{p}{q}}$ with weighted measure $|z|^k dV$.

Let $\lfloor\cdot\rfloor$ denote the floor function. Let $m_k = \lfloor -\frac{p}{q}(k+1)\rfloor$. Set g equal to $\frac{(w\bar{\eta})^{m_k}}{(1-w\bar{\eta})}$, and let M_g be the multiplication operator with multiplier g. Let I denote the identity operator. Let D_k denote the differential operator:

$$D_k = \frac{q(z\bar{\zeta})^k |\eta|^{2\frac{k+1}{q}\frac{p}{q}}}{\pi} \left(CI + \frac{p}{q} z \frac{\partial}{\partial z_j} + w\frac{\partial}{\partial w} \right) M_g, \tag{3.29}$$

where $C = \frac{(k+1)p}{q} + 1$.

Let $\mathcal{G}_1(z, w, \eta)$ and $\mathcal{G}_2(z, w, \eta)$ denote the modifying functions:

$$\mathcal{G}_1(z, w, \eta) = \frac{z^q}{(w\bar{\eta})^p} |\eta|^{\frac{p}{q}+p} \quad \text{and} \quad \mathcal{G}_2(z, w) = z^q |\eta|^{\frac{p}{q}-p}.$$

Then we have our result for $K_{\mathcal{U}^{\frac{p}{q}}}$:

Theorem 3.3. *For $(z, w), (\zeta, \eta) \in \mathcal{U}^{\frac{p}{q}}$, let $\mathcal{G}_1$, $\mathcal{G}_2$ and D_k be as above. Then the Bergman kernel $K_{\mathcal{U}^{\frac{p}{q}}}(z, w; \bar{\zeta}, \bar{\eta})$ equals*

$$\sum_{k=0}^{q-1} D_k K^{c(k,q)}_{\mathcal{U}_\eta^{\frac{p}{q}}} \left(\mathcal{G}_1(z, w, \eta); \overline{\mathcal{G}_2(\zeta, \eta)} \right), \tag{3.30}$$

where $c\,(k, q) = \frac{2(k+1-q)}{q}$.

Proof. By its definition, $\mathcal{U}_\eta^{\frac{p}{q}}$ contains both $\mathcal{G}_1(z, w, \eta)$ and $\mathcal{G}_2(\zeta, \eta)$. Therefore (3.30) is defined on $\mathcal{U}^{\frac{p}{q}} \times \mathcal{U}^{\frac{p}{q}}$. To show (3.30) is the Bergman kernel, it suffices to prove the term with the index k in (3.30) equals

$$\sum_{(a,c)\in\mathfrak{K}_k} \frac{(z\bar{\zeta})^a(w\bar{\eta})^c}{\|z^a w^c\|^2_{L^2(\mathcal{U}^{\frac{p}{q}})}}, \tag{3.31}$$

where $\mathfrak{K}_k = \{(a,c) : a = k \mod q \text{ and } c \in \mathfrak{I}_a\}$. Let K_k denote the term with the index k in (3.30). We first show that K_k can be expanded as follows:

$$\sum_{(a,c)\in\mathfrak{K}_k} C_{a,c}(z\bar{\zeta})^a(w\bar{\eta})^c. \tag{3.32}$$

Consider the biholomorphic mapping f from $\mathcal{U}_\eta^{\frac{p}{q}}$ to Ω:

$$f(z) = z|\eta|^{-\frac{p}{q}}.$$

Then a similar formula for biholomorphic transformation relates the weighted Bergman kernel $K^k_{\mathbb{B}}$ to $K^k_{\mathcal{U}_\eta^{\frac{p}{q}}}$:

$$K^k_{\mathcal{U}_\eta^{\frac{p}{q}}}(z;\bar{\zeta}) = |\eta|^{-(k+2)\frac{p}{q}} K^k_{\mathbb{B}}(f(z);\overline{f(\zeta)}). \tag{3.33}$$

By formula (3.33), the function $|\eta|^{2\frac{k+1}{q}\frac{p}{q}} K^{c(k,q)}_{\mathcal{U}_\eta^{\frac{p}{q}}}(\mathcal{G}_1(z,w,\eta);\bar{\mathcal{G}}_2(\zeta,\eta))$ becomes

$$\frac{(z\bar{\zeta})^k(w\bar{\eta})^{m_k}}{\pi(1-w\bar{\eta})} K^{c(k,q)}_{\mathbb{B}}(\frac{z^q}{w^p};\frac{\bar{\zeta}^q}{\bar{\eta}^p}). \tag{3.34}$$

Since $c(k,q) > -2$, all monomials z^a are in the weighted space $A^2\left(\mathbb{B}, |z|^{c(k,q)}\right)$. Therefore we have

$$K^{c(k,q)}_{\mathbb{B}}\left(z;\bar{\zeta}\right) = \sum_{a\in\mathbb{N}} \frac{(z\bar{\zeta})^a}{\|z^a\|^2_{L^2(\mathbb{B},|z|^{c(k,q)})}}. \tag{3.35}$$

Applying (3.35) to (3.34) yields a sum of form

$$\sum_{s\in\mathbb{N}} \frac{C_s(w\bar{\eta})^{m_k}}{(1-w\bar{\eta})} \frac{(z\bar{\zeta})^{sq+k}}{(w\bar{\eta})^{sp}}. \tag{3.36}$$

Expanding $\frac{1}{(1-w\bar{\eta})}$ in (3.36), we obtain

$$\sum_{c,s\in\mathbb{N}} C_s(w\bar{\eta})^{m_k-sp+c}(z\bar{\zeta})^{sq+k}. \tag{3.37}$$

Since

$$
\begin{aligned}
&(m_k - sp + c) + 1 + \frac{p}{q}(sq + k + 1) \\
=&\lfloor -\frac{p}{q}(k+1) \rfloor + c + 1 + \frac{p}{q}(k+1) > c \geq 0,
\end{aligned} \tag{3.38}
$$

w's exponent $m_k - sp + c$ is in $\mathfrak{I}_{sq+k}$ if and only if $c \in \mathbb{N}$. Thus, (3.32) and (3.37) are equivalent and K_k can be expanded as a series like (3.32).

To prove that K_k equals (3.32), we show that K_k reproduces each $z^a w^c$ for $(a, c) \in \mathfrak{K}_k$. Consider the integral

$$
\int_{\mathcal{U}^{\frac{p}{q}}} K_k(z, w; \bar{\zeta}, \bar{\eta}) \zeta^a \eta^c dV(\zeta, \eta). \tag{3.39}
$$

For $a = sq \in \mathbb{N}$ and $c \in \mathbb{Z}$,

$$
\left(CI + \frac{p}{q} z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w} \right) z^a w^c = (C + ps + c)\, z^a w^c.
$$

Let $C_{p,c,s}$ denote the constant $C + ps + c$. When $(a, c) \in \mathfrak{K}_k$, there exists a unique $s \in \mathbb{N}$ and $0 \leq k \leq q - 1$ such that $a = sq + k$. Therefore (3.39) equals

$$
C_{p,c,s} \int_{\mathbb{B}^*} \frac{q z^k (w\bar{\eta})^{m_k} |\eta|^{2\frac{(k+1)p}{q^2}} \eta^c}{\pi(1 - w\bar{\eta})} \int_{\mathcal{U}_\eta^{\frac{p}{q}}} K^{c(k,q)}_{\mathcal{U}_\eta^{\frac{p}{q}}}(\mathcal{G}_1; \bar{\mathcal{G}}_2) \zeta^a \bar{\zeta}^k dV(\zeta) dV(\eta). \tag{3.40}
$$

Note that $\mathcal{G}_2(\cdot, \eta)$ is a proper map from $\mathcal{U}_\eta^{\frac{p}{q}}$ onto itself. By substituting $t = G_2(\zeta, \eta)$ and dividing the domain $\mathcal{U}_\eta^{\frac{p}{q}}$ into q branches, (3.40) becomes

$$
\begin{aligned}
&\int_{\mathcal{U}_\eta^{\frac{p}{q}}} K^{c(k,q)}_{\mathcal{U}_\eta^{\frac{p}{q}}}(\mathcal{G}_1; \bar{\mathcal{G}}_2) \zeta^a \bar{\zeta}^k dV(\zeta) \\
=&\frac{|\eta|^{(p-\frac{p}{q})(\frac{2+k+a}{q})}}{q} \int_{\mathcal{U}_\eta^{\frac{p}{q}}} K^{c(k,q)}_{\mathcal{U}_\eta^{\frac{p}{q}}}(\mathcal{G}_1; \bar{t}) t^s |t|^{c(k,q)} dV(t).
\end{aligned} \tag{3.41}
$$

Applying the reproducing property of the weighted Bergman kernel yields

$$
\begin{aligned}
&\frac{|\eta|^{(p-\frac{p}{q})\frac{2+k+a}{q}}}{q^1} \int_{\mathcal{U}_\eta^{\frac{p}{q}}} K^{c(k,q)}_{\mathcal{U}_\eta^{\frac{p}{q}}}(\mathcal{G}_1, z'; \bar{t}, \bar{\zeta}') t^s |t|^{c(k,q)} dV(t) \\
=&\frac{|\eta|^{(p-\frac{p}{q})\frac{2+k+a}{q}}}{q^1} \frac{z^{sq} |\eta|^{(p+\frac{1}{q})s}}{(w\bar{\eta})^{ps}} \\
=&\frac{z^{sq} |\eta|^{(p-\frac{p}{q})\frac{2+2k}{q} + 2sp}}{q^1 (w\bar{\eta})^{ps}}.
\end{aligned} \tag{3.42}
$$

Substituting (3.42) to (3.40), the integral (3.39) becomes

$$C_{p,c,s}z^a \int_{\mathbb{B}^*} \frac{|\eta|^{p\frac{2+2k}{q}+2sp}\eta^c}{\pi(1-w\bar{\eta})(w\bar{\eta})^{m_k+ps}} dV(\eta). \tag{3.43}$$

Expanding $\frac{1}{(1-w\bar{\eta})}$ in (3.43) and using polar coordinates, we have

$$\begin{aligned}(3.43) =& \frac{C_{p,c,s}z^a w^c}{\pi} \int_{\mathbb{B}^*} |\eta|^{p\frac{2+2k}{q}+2sp+2c} dV(\eta) \\ =& 2C_{p,c,s}z^a w^c \int_0^1 r^{p\frac{2+2k}{q}+2sp+2c+1} dr \\ =& \frac{C_{p,c,s}z^a w^c}{\frac{p(1+k)}{q}+sp+c+1} = z^a w^c. \end{aligned} \tag{3.44}$$

Hence K_k equals (3.32) and the proof is complete. □

By explicitly computing the series, the weighted kernel function $K_{\mathbb{B}}^{c(k,q)}$ is given by:

$$K_{\mathbb{B}}^{c(k,q)}(z;\bar{\zeta}) = \frac{1}{\pi(1-z\bar{\zeta})^2} + \frac{c(k,q)}{2\pi(1-z\bar{\zeta})}. \tag{3.45}$$

Applying (3.45) and (3.33) to (3.30) yields the explicit formula for $K_{\mathcal{U}^{\frac{p}{q}}}$:

Corollary 3.2. *Let $s = z\bar{\zeta}$ and $t = w\bar{\eta}$. Then the Bergman kernel on the generalized Hartogs triangle $\{(z,w) \in \mathbb{C}^2 : |z|^q < |w|^p < 1\}$ is given by:*

$$\sum_{k=0}^{q-1} \frac{t^{p+m_k}s^k\left((1-m_k-C)\,t+m_k+C\right)\left((t^p-s^q)\,(k+1-q)+qt^p\right)}{\pi^2(t^p-s^q)^2(1-t)^2}, \tag{3.46}$$

where $C = \frac{(k+1)p}{q} + 1$.

When $p = 1$, $m_k = \lfloor -\frac{1}{q}(k+1) \rfloor = -1$ and $C = \frac{k+1+q}{q}$. Then (3.46) becomes

$$\sum_{k=0}^{q-1} \frac{s^k\left((q-k-1)\,t+k+1\right)\left((t-s^q)\,(k+1-q)+qt\right)}{q\pi^2\,(t-s^q)^2\,(1-t)^2}. \tag{3.47}$$

When $q = 1$, $m_k = \lfloor -p(k+1) \rfloor = -p$ and $C = p+1$. In this case, (3.46) becomes

$$\frac{t^p}{\pi^2(t^p-s)^2(1-t)^2}. \tag{3.48}$$

Both (3.47) and (3.48) are consistent with (3.24) and (3.25).

For more general "base" domains Ω (such as n-star-shaped Hartogs domains), our computation does not work for the the kernel function on the “target” domain

$$\left\{\big(z,z',w\big)\in\mathbb{C}^{n+m+1}:\left(\frac{z_1}{|w|^{\frac{p_1}{q_1}}},\ldots,\frac{z_n}{|w|^{\frac{p_n}{q_n}}},z'\right)\in\Omega\right\}.$$

However, as we will see in Theorem 5.8, the Bergman kernel on the domain

$$\Omega^{\mathbf{q}}=\left\{\big(z,z'\big)\in\mathbb{C}^{n+m}:\big(z_1^{q_1},\ldots,z_n^{q_n},z'\big)\in\Omega\right\}.$$

can be obtained as a finite sum of weighted Bergman kernels on Ω.

Since $\Omega\times\mathbb{B}^*$ is biholomorphic to the domain

$$\mathcal{U}^{\mathbf{p}}=\left\{\big(z,z',w\big)\in\mathbb{C}^{n+m}\times\mathbb{B}^*:\left(\frac{z_1}{|w|^{p_1}},\ldots,\frac{z_n}{|w|^{p_n}},z'\right)\in\Omega\right\},$$

the Bergman kernel on $\mathcal{U}^{\frac{\mathbf{p}}{\mathbf{q}}}$ can be written as a sum of weighted kernel functions on $\Omega\times\mathbb{B}^*$.

Chapter 4

Boundary behavior

Because of its close connection to the boundary regularity of holomorphic mappings, the behavior of the Bergman kernel near the boundary has been studied for many decades. The boundary behavior of the Bergman kernel in the strongly pseudoconvex case is well understood. C. Fefferman [Fef74], L. Boutet de Monvel and J. Sjöstrand [BS76] gave an asymptotic expansion of the kernel function when the domain is bounded smooth and strongly pseudoconvex. In the non-strongly pseudoconvex case, the boundary behavior is difficult to analyze. Near a weakly pseudoconvex point of finite type, certain estimates on the Bergman kernel were obtained by McNeal [McN89,McN94]. Less is known near non-smooth boundary points.

In this chapter, we analyze the boundary behavior of $K_{\mathcal{U}}$ when the "base" domain Ω is pseudoconvex with smooth boundary. We begin by clarifying the relation between the pseudoconvexity of "base" domain Ω and the "target" domain $\mathcal{U}$.

4.1 Pseudoconvexity of $\mathcal{U}$

Let's recall the definitions of pseudoconvex boundary point and pseudoconvex domain.

Definition 4.1. *A domain $\Omega \subseteq \mathbb{C}^n$ is said to have C^k boundary $\mathbf{b}\Omega$ at boundary point p if there exist a neighborhood U of p and a real-valued C^k function r defined in U such that the following properties hold:*

1. $\Omega \bigcap U = \{z \in U : r(z) < 0\}$.

2. $\mathbf{b}\Omega \bigcap U = \{z \in U : r(z) = 0\}$.

3. *The gradient* $\nabla r(z) \neq 0$ *on* $\mathbf{b}\Omega \cap U$.

The function r is called a local defining function for Ω near p. If $\overline{\Omega} \subseteq U$, then r is called a defining function of Ω.

Definition 4.2. *Given a domain $\Omega \subseteq \mathbb{C}^n$, a point $p \in \mathbf{b}\Omega$, and a local defining function r of Ω at p, the holomorphic tangent space of $\mathbf{b}\Omega$ at p is defined by:*

$$T_p^{1,0}(\mathbf{b}\Omega) := \left\{ v \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p) v_j = 0 \right\}.$$

Definition 4.3. *Let Ω be a domain in $\mathbb{C}^n$ with C^2 boundary at $p \in \mathbf{b}\Omega$, and let r be a local defining function for Ω at p. The domain Ω is called pseudoconvex at $p \in \mathbf{b}\Omega$ if for all $v \in T_p^{1,0}(\mathbf{b}\Omega)$, the Levi form*

$$\lambda(p)(v, \bar{v}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) v_j \bar{v}_k \geq 0.$$

The domain is said to be strongly pseudoconvex at p if the Levi form at p is strictly positive for all $0 \neq v \in T_p^{1,0}(\mathbf{b}\Omega)$. Ω is called a pseudoconvex domain if Ω is pseudoconvex at every boundary point. Ω is called a strongly pseudoconvex domain if Ω is strongly pseudoconvex at every boundary point.

Recall the n-star-shaped Hartogs "base" domain $\Omega \subseteq \mathbb{C}^{n+m}$. Suppose Ω is also bounded, and has smooth boundary. Consider polar coordinates $z_j = t_j e^{i\theta_j}$ for $1 \leq j \leq n$. The definition of the n-star-shaped Hartogs domain implies that a defining function of Ω:

$$r\left(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n; z', \bar{z}'\right) \in C^\infty\left(\bar{\Omega}\right)$$

can be chosen so that in a tubular neighborhood of $\mathbf{b}\Omega$, the function r is independent of θ_j and is non-decreasing in t_j. In this chapter, we always use such an r as the defining function for Ω.

Lemma 4.1. *Let r_{z_j} and $r_{\bar{z}_j}$ denote the partial derivative of r with respect to z_j and $\bar{z}_j$. Then on $\mathbf{b}\Omega$, the following holds:*

1. $z_j r_{z_j} = \bar{z}_j r_{\bar{z}_j} \geq 0$.
2. $z_j \bar{r}_{\bar{z}_j \bar{z}_l} = z_j r_{z_j z_l} = \bar{z}_j r_{\bar{z}_j z_l} = \bar{z}_j \bar{r}_{z_j \bar{z}_l}$.
3. *If* $z_j = 0$, *then* $r_{z_j z_k} = 0$ *for all* $1 \leq k \leq n$.
4. *If* $p = (z, z') \in \mathbf{b}\Omega$ *is a strongly pseudoconvex point, then* $r_{z_j}(p) = 0$ *if and only if* $z_j = 0$.

Proof. Since r does not change with respect to θ_j and is non-decreasing in t_j in a tubular neighborhood of $\mathbf{b}\Omega$, we have

$$\frac{\partial}{\partial \theta_j} r\left(t_1 e^{i\theta_1}, \ldots, t_n e^{i\theta_n}, t_1 e^{-i\theta_1}, \ldots, t_n e^{-i\theta_n}; z', \bar{z}'\right) = 0,$$

$$\frac{\partial}{\partial t_j} r\left(t_1 e^{i\theta_1}, \ldots, t_n e^{i\theta_n}, t_1 e^{-i\theta_1}, \ldots, t_n e^{-i\theta_n}; z', \bar{z}'\right) \geq 0.$$

Hence on $\mathbf{b}\Omega$, $z_j r_{z_j} + \bar{z}_j r_{\bar{z}_j} \geq 0$ and $z_j r_{z_j} - \bar{z}_j r_{\bar{z}_j} = 0$. Summing them yields $z_j r_{z_j} = \bar{z}_j r_{\bar{z}_j} \geq 0$. Taking the derivative on both sides of $z_j r_{z_j} = \bar{z}_j r_{\bar{z}_j}$ and applying $\bar{r}_{z_j} = r_{\bar{z}_j}$, we obtain $z_j \bar{r}_{\bar{z}_j \bar{z}_l} = z_j r_{z_j z_l} = \bar{z}_j r_{\bar{z}_j z_l} = \bar{z}_j \bar{r}_{z_j \bar{z}_l}$. For $z_j \neq 0$, we have $e^{2i\theta_j} r_{z_j} = r_{\bar{z}_j}$. Then by continuity, $e^{2i\theta_j} r_{z_j} = r_{\bar{z}_j}$ at $z_j = 0$ for arbitrary $\theta_j \in \mathbb{R}$. Therefore $r_{z_j} = r_{\bar{z}_j} = 0$ when $z_j = 0$. Conversely, when $r_{z_j}(p) = 0$, the vector $v = (0, \ldots, v_j, \ldots, 0) \in T_p^{1,0}(\mathbf{b}\Omega)$. Since p is a strongly pseudoconvex point, $|v_j|^2 r_{z_j \bar{z}_j}(p) > 0$ and hence $r_{z_j \bar{z}_j}(p) > 0$. Note that for $z_j = t_j e^{i\theta_j}$,

$$\frac{\partial^2}{\partial t_j^2} r(p) = 4 r_{z_j \bar{z}_j}(p) > 0.$$

Suppose z_j does not equal 0. Then $\frac{\partial r}{\partial t_j} < 0$ at points near p with a slightly smaller t_j. It contradicts the fact that $\frac{\partial r}{\partial t_j} \geq 0$ in a neighborhood of p and hence z_j must be 0 for each j. $\square$

Recall ψ_j's and the induced "target" domain $\mathcal{U}$ in Section 2.3. Suppose the "base" domain Ω is pseudoconvex. It is natural to ask for what kind of ψ_j's the $\mathcal{U}$ is a pseudoconvex domain. The following theorem gives a partial answer.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{C}^{n+m}$ be an n-star-shaped Hartogs pseudoconex domain with smooth boundary. Let $\mathcal{D} \subseteq \mathbb{C}^k$ be a smooth and pseudoconvex domain. Let Ψ denote a Hermitian symmetric function on $\mathcal{D} \times \mathcal{D}$. Suppose $\Psi(w, \bar{w}) > 0$ and $\log \Psi(w, \bar{w})$ is C^2-plurisubharmonic. Then for $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_j > 0$, the domain*

$$\mathcal{U} = \{(z, z', w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathcal{D} : (z_1 \Psi^{\alpha_1}(w, \bar{w}), \ldots, z_n \Psi^{\alpha_n}(w, \bar{w}), z') \in \Omega\},$$

is also pseudoconvex.

Proof. On the boundary of a domain, the set of non-pseudoconvex points, if not empty, has non-empty interior. To prove $\mathcal{U}$ is pseudoconvex, it suffices to show the set of pseudoconvex boundary points is dense in $\mathbf{b}\mathcal{U}$. Let r be the defining function of Ω. Let $r_{\mathcal{U}}$ denote the function:

$$r_{\mathcal{U}}\left(z, z', w\right) = r\left(z_1 \Psi^{\alpha_1}(w, \bar{w}), \ldots, z_n \Psi^{\alpha_n}(w, \bar{w}), z'\right).$$

Let $\mathbf{b}\mathcal{U}_1$ denote the set

$$\left\{\left(z, z', w\right) \in \mathbf{b}\mathcal{U} : w \notin \mathbf{b}\mathcal{D} \text{ and } r_{\mathcal{U}}(z, z', w) = 0\right\}.$$

Let $\mathbf{b}\mathcal{U}_2$ denote the set

$$\{(z,z',w)\in \mathbf{b}\mathcal{U} : r_{\mathcal{U}}(z,z',w)\neq 0 \ \text{ and } \ w\in \mathbf{b}\mathcal{D}\}.$$

Then $\mathbf{b}\mathcal{U}_1\cup \mathbf{b}\mathcal{U}_2$ is dense in $\mathbf{b}\mathcal{U}$. If $\mathbf{b}\mathcal{U}_2\neq\emptyset$, then the point in $\mathbf{b}\mathcal{U}_2$ is pseudoconvex. It remains to prove the pseudoconvexity of points in $\mathbf{b}\mathcal{U}_1$. Choose $r(z,z')$ as in Lemma 4.1. For $1\le j\le n+m$ and $1\le k\le n+m$, set $r_{j,l}$ to be:

$$r_{j,l}=\begin{cases} r_{z_j\bar z_l} & 1\le j,l\le n\\ r_{z_j\bar z'_{l-n}} & 1\le j\le n \text{ and } n<l\le n+m\\ r_{z'_{j-n}\bar z_l} & 1\le l\le n \text{ and } n<j\le n+m\\ r_{z'_{j-n}\bar z'_{l-n}} & n<j,l\le n+m\end{cases}$$

For $p\in \mathbf{b}\Omega$, let $H_\Omega(p)$ denote the complex Hessian matrix of r at p:

$$H_\Omega(p)=(r_{jl})|_{(z,z')=p}.$$

We set $X(z,w)=(\psi_1(w,\bar w)z_1,\ldots,\psi_n(w,\bar w)z_n)$ and $r_{\mathcal{U}}(z,z',w)=r(X,z')$. For the boundary point $p=(z,z',w)\in \mathbf{b}\mathcal{U}_1$, the gradient $dr_{\mathcal{U}}\neq 0$. Hence $r_{\mathcal{U}}$ is a local defining function of $\mathcal{U}_1$ at p. Let $\Psi_j(w,\bar w)=\frac{\partial\Psi}{\partial w_j}(w,\bar w)$. Set

$$\lambda(z)=\begin{cases} z/\bar z & z\neq 0\\ 0 & z=0\end{cases} \quad\text{and}\quad \mu(z)=\begin{cases} 1/\bar z & z\neq 0\\ 0 & z=0\end{cases}.$$

Taking the derivatives of $r_{\mathcal{U}}$ and applying Lemma 4.1, we have:

$$\begin{aligned}
\frac{\partial^2 r_{\mathcal{U}}}{\partial z_j\partial\bar z_l} =&\Psi^{\alpha_j+\alpha_l}r_{j,l},\quad \frac{\partial^2 r_{\mathcal{U}}}{\partial z'_j\partial\bar z'_l}=r_{j+n,l+n},\\
\frac{\partial^2 r_{\mathcal{U}}}{\partial z_j\partial\bar z'_l} =&\Psi^{\alpha_j}r_{j,l+n},\quad \frac{\partial^2 r_{\mathcal{U}}}{\partial z'_j\partial\bar z_l}=\Psi^{\alpha_l}r_{j+n,l},\\
\frac{\partial^2 r_{\mathcal{U}}}{\partial z_j\partial\bar w_l} =&\Psi^{\alpha_j}\sum_{s=1}^n(\alpha_s\Psi^{\alpha_s-1}\bar\Psi_l r_{j,s}(1+\bar\lambda(z_s)))+\alpha_j\Psi^{\alpha_j-1}\bar\Psi_l r_{z_j}(1-\Psi^{\alpha_j}\bar\mu(z_j)),\\
\frac{\partial^2 r_{\mathcal{U}}}{\partial w_j\partial\bar z_l} =&\Psi^{\alpha_l}\sum_{s=1}^n(\alpha_s\Psi^{\alpha_s-1}\Psi_j r_{s,l}(1+\lambda(z_s)))+\alpha_l\Psi^{\alpha_l-1}\Psi_j r_{\bar z_l}(1-\Psi^{\alpha_l}\mu(z_l)),\\
\frac{\partial^2 r_{\mathcal{U}}}{\partial z'_j\partial\bar w_l} =&\sum_{s=1}^n\alpha_s\Psi^{\alpha_s-1}\bar\Psi_l r_{j+n,s}(1+\bar\lambda(z_s)),\\
\frac{\partial^2 r_{\mathcal{U}}}{\partial w_j\partial\bar z'_l} =&\sum_{s=1}^n\alpha_s\Psi^{\alpha_s-1}\Psi_j r_{s,l+n}(1+\lambda(z_s)),
\end{aligned}$$

$$\frac{\partial^2 r_{\mathcal{U}}}{\partial w_j \partial \bar{w}_l} = \sum_{s=1}^{n}\sum_{t=1}^{n} \alpha_s\alpha_t \Psi^{\alpha_s+\alpha_t-2}\Psi_j\bar{\Psi}_l(r_{s,t}(1+\lambda(z_s))(1+\bar{\lambda}(z_t)) - r_{z_s}\bar{\mu}(z_s) -$$
$$r_{\bar{z}_s}\mu(z_s)) + \sum_{s=1}^{n}\alpha_s\Psi^{\alpha_s-1}r_{z_s}(1+\lambda(z_s))\left(\frac{\partial^2\Psi}{\partial w_j\partial\bar{w}_l} + (\alpha_s-1)\Psi^{-1}\Psi_j\bar{\Psi}_l\right).$$

Let $H_{\mathcal{U}}$ denote the complex Hessian matrix of $r_{\mathcal{U}}$. Since $\mathcal{D} \subseteq \mathbb{C}^k$, $H_{\mathcal{U}}$ is a $(n+m+k)\times(n+m+k)$ matrix. Let $H_\Psi(w)$ denote the complex Hessian of $\log\Psi(w,\bar{w})$. Set $c = \sum_{s=1}^{n}\alpha_s\Psi^{\alpha_s-1}r_{z_s}(1+\lambda(z_s))$. For $p=(z,z',w) \in \mathbf{b}\mathcal{U}_1$, consider $(n+m+k)\times(n+m+k)$ matrix $A(p)$,

$$A(p) = \begin{bmatrix} H_\Omega(X,z') & \bar{M}^T \\ M & cH_\Psi(w) \end{bmatrix}.$$

Here $M = (M_{j,l})$ is a $k\times(n+m)$ matrix with entry

$$M_{j,l} = \begin{cases} \alpha_l\Psi_j\Psi^{-1}r_{\bar{z}_l}(1-\Psi^{\alpha_l}\mu(z_l)) & 1\le l\le n \\ 0 & n<l\le n+m \end{cases}.$$

Set ${}_1B = ({}_1B_{j,l})$ to be an $(n+m)\times(n+m)$ diagonal matrix with entry:

$${}_1B_{j,j} = \begin{cases} \Psi^{\alpha_j} & 1\le j\le n \\ 1 & n<j\le n+m \end{cases}.$$

Set ${}_2B = ({}_2B_{j,l})$ to be a k by $(n+m)$ matrix with entry

$${}_2B_{j,l} = \alpha_l\Psi_j\Psi^{\alpha_l-1}\left(1+\lambda\left(z_l\right)\right).$$

Let I denote the $k\times k$ identity matrix and set $B(p)$ to be the $(n+m+k)\times(n+m+k)$ matrix:

$$B(p) = \begin{bmatrix} {}_1B & \mathbf{0} \\ {}_2B & I \end{bmatrix}.$$

Then $H_{\mathcal{U}}(p) = B(p)A(p)\bar{B}(p)^T$. By its definition, $B(p)$ is invertible. Hence $t \in T^{1,0}(\mathbf{b}\mathcal{U})(p)$ implies $\mathbf{0} \neq tB(p) \in T^{1,0}(\mathbf{b}\mathcal{U}_w)(z,z')\times T^{1,0}(\mathcal{D})(w)$. Since $cH_\Psi(w)$ is positive semidefinite on $T^{1,0}(\mathcal{D})(w)$, the matrix $A(p)$ is positive semidefinite on $T^{1,0}(\mathbf{b}\mathcal{U}_w)(z,z')\times T^{1,0}(\mathcal{D})(w)$. Therefore $H_{\mathcal{U}}(p)$ is positive semidefinite on $T^{1,0}(\mathbf{b}\mathcal{U})(p)$. $\square$

Since both $\log\left(\frac{1}{1-\|w\|^2}\right)$ and $\log e^{\|w\|^2}$ are plurisubharmonic, Theorem 4.1 implies that U^α and V^γ are pseudoconvex when Ω is pseudoconvex with smooth boundary.

4.2 Further analysis of some examples

In this section, we use the explicit formulas of K_{U^α} and K_{V^γ} from Examples 4.2 and 4.3 and some admissible approach regions to analyze their boundary behavior. Before going into these examples, we first look at a simple case. Let Ω be the polydisk $\mathbb{B} \times \mathbb{B}$ in $\mathbb{C}^2$. Since the kernel function on a product domain equals the product of the kernel function on each factor, we have

$$K_\Omega\left(z_1, z_2; \bar{\zeta_1}, \bar{\zeta_2}\right) = \frac{1}{\pi(1 - z_1\bar{\zeta_1})^2} \cdot \frac{1}{\pi(1 - z_2\bar{\zeta_2})^2}.$$

If we approach the boundary point $p = (w_1, w_2)$ along the diagonal, then the boundary behavior of K_Ω depends on w_1 and w_2:

1. If $|w_1| = 1$ and $|w_2| \neq 1$, then in Ω we have:
$$\lim_{z \to p} K_\Omega(z; \bar{z})(1 - |z_1|^2)^2 = \frac{1}{\pi^2(1 - |w_2|^2)^2} \neq 0.$$
2. If $|w_1| \neq 1$ and $|w_2| = 1$, then in Ω we have:
$$\lim_{z \to p} K_\Omega(z; \bar{z})(1 - |z_2|^2)^2 = \frac{1}{\pi^2(1 - |w_1|^2)^2} \neq 0.$$
3. If $|w_1| = |w_2| = 1$, then in Ω we have:
$$\lim_{z \to p} K_\Omega(z; \bar{z})(1 - |z_1|^2)^2(1 - |z_2|^2)^2 = \frac{1}{\pi^2} \neq 0.$$

Note that $\mathbf{b}\Omega$ is not smooth at boundary points in the 3rd case and the boundary behavior of the Bergman kernel depends on the rate at which $|z_1|$ and $|z_2|$ tend to 1. We will see similar phenomena when we analyze the boundary behavior of K_{U^α}.

Example (Exampe 3.2 revisited)**.** *The boundary of U^α is not smooth at $(0, z', w)$ where $\|z'\| = |w| = 1$. We let $\mathcal{S}_1$ denote the set of these non-smooth points. By calculating the Levi form of U^α on the smooth boundary points, one obtains that (z, z', w) is strongly pseudoconvex if both $\|z'\|$ and $|w|$ are not equal to 1. We let $\mathcal{S}_2$ denote the set of these strongly pseudoconvex points. We denote by $\mathcal{S}_3$ the set*

$$\{(0, z', w) \in \mathbf{b}U^\alpha : \|z'\| = 1, |w| \neq 1\}$$

and denote by $\mathcal{S}_4$ *the set*

$$\{(0, z', w) \in \mathbf{b}U^\alpha : \|z'\| \neq 1, |w| = 1\}.$$

Then $\mathbf{b}U^\alpha = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$. *The boundary behavior of the kernel function near the strongly pseudoconvex points in* $\mathcal{S}_2$ *is known. To obtain the result near the points in the other sets, we need an admissible approach region. For* $0 < s < 1$, *let* $\mathcal{W}_s$ *denote the set*

$$\{(z, z', w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : |w| < 1, \|z\|^{2s} + \|z'\|^2 + |w|^2 < 1 + |w|^2\|z'\|^2\}.$$

These sets exhaust U^α *when* s *tends to 1. Moreover,* $\mathcal{S}_1$, $\mathcal{S}_3$ *and* $\mathcal{S}_4$ *are contained in* $b\mathcal{W}_s$. *We will choose* $\mathcal{W}_s$ *as the admissible approach region. Let* $r(z, z', w)$ *denote the function:*

$$1 - \|z'\|^2 - \frac{\|z\|^2}{(1 - |w|^2)}.$$

Then U^α *can also be expressed as the set*

$$\{(z, z', w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C}; |w| < 1, -r(z, z', w) < 0\}.$$

Note that the function

$$\frac{\|z\|^{2s}}{(1 - |w|^2)}$$

is bounded in $\mathcal{W}_s$. *For* $p = (0, z'_0, w_0) \in \mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_4$, *when approaching* p *in* $\mathcal{W}_s$,

$$\frac{\|z\|^2}{(1 - |w|^2)} \to 0. \tag{4.1}$$

Therefore $r(z, z', w)$ *is continuous in the closure of* $\mathcal{W}_s$. *Combining (4.1) and (3.20) yields the following results on boundary behavior:*

1. *For* $p_0 = (0, z'_0, w_0) \in \mathcal{S}_3$, *the admissible limit satisfies:*

$$\lim_{\mathcal{W}_s \ni p \to p_0} K_{U^\alpha}(p; \bar{p}) r^{n+m+1}(p) = \frac{(m+n)!(n+1)}{\pi^{m+n+1}(1 - |w_0|^2)^{n+2}} \neq 0.$$

2. *For* $p_0 = (0, z'_0, w_0) \in \mathcal{S}_4$, *the admissible limit satisfies:*

$$\lim_{\mathcal{W}_s \ni p \to p_0} K_{U^\alpha}(p; \bar{p})(1 - |w|^2)^{n+2} = \frac{(m+n)!(n+1)}{\pi^{m+n+1} r^{n+m+1}(p_0)} \neq 0.$$

3. For $p_0 = (0, z_0', w_0) \in \mathcal{S}_1$, the admissible limit satisfies:

$$\lim_{\mathcal{W}_s \ni p \to p_0} K_{U^\alpha}(p; \bar{p}) r^{n+m+1}(p)(1 - |w|^2)^{n+2} = \frac{(m+n)!(n+1)}{\pi^{m+n+1}} \neq 0.$$

Example (Example 3.3 revisited)**.** *Calculating the Levi form shows that V^γ is a pseudoconvex domain. For any $w_0 \in \mathbb{C}$ and $z_0' \in \mathbb{C}^m$ on the unit sphere, $(0, z_0', w_0)$ is a weakly pseudoconvex point on $\mathbf{b}V^\gamma$. With (3.21), we can obtain the boundary behavior of the Bergman kernel function in an admissible approach region of $(0, z_0', w_0)$.*

Let $0 < s_j < 1$ for $1 \leq j \leq n$. Let $W_{\mathbf{s}}$ denote the domain

$$\left\{ (z, z', w) \in \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} : \sum_{j=1}^n e^{\gamma_j |w|^2} |z_j|^{2s_j} + \|z'\|^2 < 1 \right\}. \tag{4.2}$$

For each $\mathbf{s}$, $W_{\mathbf{s}}$ is contained in V^γ and it exhausts V^γ as each p_j approaches 1. Moreover, $\mathbf{b}W_{\mathbf{s}}$ intersects $\mathbf{b}V^\gamma$ at those weakly pseudoconvex points on $\mathbf{b}V^\gamma$. Let ρ denote a defining function of V^γ:

$$\rho(z, z', w) = 1 - e^{|w|^2} \|z\|^2 - \|z'\|^2.$$

When approaching $p_0 = (0, z_0', w_0)$ in the approach region $W_{\mathbf{s}}$, the admissible limit

$$\lim_{W_{\mathbf{s}} \ni p \to p_0} \frac{\sum_{j=1}^n e^{\gamma_j |w|^2} |z_j|^{2s_j}}{1 - \|z'\|^2} = 0. \tag{4.3}$$

Therefore,

$$\lim_{W_{\mathbf{s}} \ni p \to p_0} \frac{\sum_{j=1}^n e^{\gamma_j |w|^2} |z_j|^{2s_j}}{\rho} = 0. \tag{4.4}$$

Applying (4.4) to (3.21), we have in $W_{\mathbf{s}}$:

$$\lim_{W_{\mathbf{s}} \ni p \to p_0} K_{V^\gamma}(p; \bar{p}) \rho^{m+n+1}(p) = \frac{(m+n)! e^{n|w_0|^2} \sum_{j=1}^n \gamma_j}{\pi^{m+n+1}} \neq 0.$$

4.3 General results for boundary behavior

In the previous section, we used the explicit formula of the Bergman kernel to study its boundary behavior at weakly pseudoconvex boundary points. In general, we do not require an explicit formula for the kernel function on the "base" domain. If enough information on the boundary behavior of the kernel function of the "base" domain is known, we can obtain the boundary behavior of the Bergman kernel on the "target" domain. Here we'll discuss the boundary behavior for U^α and V^γ when the "base" domain Ω is smooth and strongly pseudoconvex and w is a single variable.

For positive numbers α_j's and γ_j's, recall

$$f_\alpha(z,w) = \left(\frac{z_1}{(1-|w|^2)^{\frac{\alpha_1}{2}}}, \ldots, \frac{z_n}{(1-|w|^2)^{\frac{\alpha_n}{2}}} \right)$$

and

$$g_\gamma(z,w) = \left(e^{\frac{\gamma_1|w|^2}{2}} z_1, \ldots, e^{\frac{\gamma_n|w|^2}{2}} z_n \right).$$

Then U^α denotes

$$\left\{ (z,z',w) \in \mathbb{C}^{n+m+1} : |w| < 1 \text{ and } r\left(f_\alpha(z,w), \overline{f_\alpha(z,w)}; z', \bar{z}' \right) < 0 \right\}$$

and V^γ denotes

$$\left\{ (z,z',w) \in \mathbb{C}^{n+m+1} : r\left(g_\gamma(z,w), \overline{g_\gamma(z,w)}; z', \bar{z}' \right) < 0 \right\}.$$

To simplify the notation, we let $K_\Omega(z,z') = K_\Omega(z,z';\bar{z},\bar{z}')$. We let

$$r_{U^\alpha}(z,z',w) = r\left(f_\alpha(z,w), \overline{f_\alpha(z,w)}; z', \bar{z}' \right),$$

$$r_{V^\gamma}(z,z',w) = r\left(g_\gamma(z,w), \overline{g_\gamma(z,w)}; z', \bar{z}' \right).$$

We let ∇_z denote the partial gradient $\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right)$.

We start with U^α. The boundary behavior of the Bergman kernel on U^α is more complicated than on V^α for two reasons:

1. The possible non-smooth boundary points created by the two inequalities of U^α.
2. The singularity of r_{U^α} at points where $|w| = 1$.

On the boundary of U^α, we consider the following subsets:

$$\begin{aligned}
\mathcal{S}_1 &= \left\{ (z,z',w) \in \mathbf{b}U^\alpha : z \neq 0 \text{ and } |w| \neq 1 \right\}, \\
\mathcal{S}_2 &= \left\{ (z,z',w) \in \mathbf{b}U^\alpha : z = 0 \text{ and } |w| \neq 1 \right\}, \\
\mathcal{S}_3 &= \left\{ (z,z',w) \in \mathbf{b}U^\alpha : z = 0,\ |w| = 1 \text{ and } (0,z') \notin \mathbf{b}\Omega \right\}, \\
\mathcal{S}_4 &= \left\{ (z,z',w) \in \mathbf{b}U^\alpha : z = 0,\ |w| = 1 \text{ and } (0,z') \in \mathbf{b}\Omega \right\}.
\end{aligned}$$

$\mathcal{S}_j$'s are distinct subsets of $\mathbf{b}U^\alpha$. By the boundedness of Ω, we have

$$\{(z,z',w) \in \mathbf{b}U^\alpha : z \neq 0, |w| = 1\} = \emptyset.$$

Therefore $\mathbf{b}U^\alpha = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$. As we'll see soon, the points on $\mathcal{S}_1$ are strongly pseudoconvex and the boundary behavior of the Bergman kernel near the boundary points of $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ can be obtained by a suitable choice of approach regions.

Let L_Ω denote the Bergman kernel on the diagonal,

$$L_\Omega(z, z') = K_\Omega(z, z'; \bar{z}, \bar{z}').$$

We recall the result of C. Fefferman [Fef74] for bounded strongly pseudoconvex domain Ω. There exist $\Psi, \Phi \in C^\infty(\bar{\Omega})$, such that

$$L_\Omega(z, z') = \frac{\Psi(z, \bar{z}; z', \bar{z}')}{(-r)^{n+m+1}(z, \bar{z}; z', \bar{z}')} + \Phi(z, \bar{z}; z', \bar{z}') \log(-r(z, \bar{z}; z', \bar{z}')). \quad (4.5)$$

In our situation, when the kernel function on the base domain has no log terms (that is, $\Phi = 0$ in (4.5)), then the kernel function on the target domain also has no log terms.

Applying formula (4.5) and Theorem 3.1, we obtain the following result on the "target" domain U^α.

Theorem 4.2. *Let Ω and U^α be as above. Suppose Ω is strongly pseudoconvex. Then U^α is pseudoconvex. The point $p \in \mathbf{b}U^\alpha$ is a strongly pseudoconvex point if $p \in \mathcal{S}_1$. Near the points of $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$, the kernel function behaves in three different ways:*

1. *For $(z_0, z_0', w_0) \in \mathcal{S}_2$, there exists an admissible approach region W_2 of (z_0, z_0', w_0) such that when approaching (z_0, z_0', w_0) in W_2,*

$$K_{U^\alpha}(z, z', w)(-r_{U^\alpha})^{m+n+1}(z, z', w) \quad (4.6)$$

has a nonzero limit.

2. *For $(z_0, z_0', w_0) \in \mathcal{S}_3$, there exists an admissible approach region W_3 of (z_0, z_0', w_0) such that when approaching (z_0, z_0', w_0) in W_3,*

$$K_{U^\alpha}(z, z', w)(1 - |w|^2)^{2+\alpha\cdot\mathbf{1}} \quad (4.7)$$

has a nonzero limit.

3. *For $(z_0, z_0', w_0) \in \mathcal{S}_4$, there exists an admissible approach region W_4 of (z_0, z_0', w_0) such that when approaching (z_0, z_0', w_0) in W_4,*

$$K_{U^\alpha}(z, z', w)(1 - |w|^2)^{2+\alpha\cdot\mathbf{1}}(-r_{U^\alpha})^{m+n+1}(z, z', w) \quad (4.8)$$

has a nonzero limit.

Proof. Recall

$$f_\alpha(z,w) = \left(\frac{z_1}{(1-|w|^2)^{\frac{\alpha_1}{2}}}, \dots, \frac{z_n}{(1-|w|^2)^{\frac{\alpha_n}{2}}} \right).$$

For simplicity, let X denote the $f_\alpha(z,w)$. Since the range of X on U^α is the same as the range of z on Ω, we can replace z in (4.5) by X and have

$$L_\Omega(X,z') = \frac{\Psi(X,\bar{X};z',\bar{z}')}{(-r_{U^\alpha})^{n+m+1}(z,z',w)} + \Phi(X,\bar{X};z',\bar{z}')\log(-r_{U^\alpha}(z,z',w)) \tag{4.9}$$

where $\Psi(X,\bar{X};z',\bar{z}'), \Phi(X,\bar{X};z',\bar{z}') \in C^\infty(U^\alpha)$. Using change of variables formula,

$$L_\Omega(X,z') = (1-|w|^2)^{\alpha\cdot\mathbf{1}} K_{U_w^\alpha}(z,z').$$

Therefore by Theorem 3.1, we have

$$K_{U^\alpha}(z,z',w) = (c_\alpha I + D)\frac{L_\Omega(X,z')}{\pi(1-|w|^2)^{2+\alpha\cdot\mathbf{1}}} \tag{4.10}$$

where $c_\alpha = (1+\sum_{j=1}^n \alpha_j)$ and $D = \sum_{j=1}^n \alpha_j z_j \frac{\partial}{\partial z_j}$.

Note that $r(X,\bar{X};z',\bar{z}')$ is equal to $r_{U^\alpha}(z,z',w)$. Multiplying both sides of (4.10) by $(1-|w|^2)^{2+\alpha\cdot\mathbf{1}}(-r_{U^\alpha})^{m+n+1}(z,z',w)$, (4.6) becomes

$$\pi^{-1}(-r)^{n+m+1}(X,\bar{X};z',\bar{z}')(c_\alpha I + D)L_\Omega(X,z') = I_1 + I_2 \tag{4.11}$$

where

$$I_1 = \pi^{-1}(-r)^{n+m+1}(X,\bar{X};z',\bar{z}')c_\alpha L_\Omega(X,z')$$

and

$$I_2 = \pi^{-1}(-r)^{n+m+1}(X,\bar{X};z',\bar{z}')D\left(L_\Omega\left(X,z'\right)\right).$$

Applying (4.9) to I_1, we have

$$\pi I_1\left(X,\bar{X};z',\bar{z}'\right) = c_\alpha\left(\Psi + \Phi\left(-r\right)^{n+m+1}\log\left(-r\right)\right). \tag{4.12}$$

Using the product rule,

$$\begin{aligned} &\pi I_2\left(X,\bar{X};z',\bar{z}'\right) \\ =&D\left(\left(-r\right)^{n+m+1}L_\Omega\right) - L_\Omega D\left(-r\right)^{n+m+1} \\ =&J_1(X,\bar{X};z',\bar{z}') - J_2(X,\bar{X};z',\bar{z}'). \end{aligned} \tag{4.13}$$

Substituting (4.9) to J_1 and J_2,

$$J_1\left(X,\bar{X};z',\bar{z}'\right) = D\Psi + (-r)^{n+m+1}\log(-r)Dt + (1+(n+m+1)\log(-r))(-r)^{n+m}D(-r). \quad (4.14)$$

and

$$\begin{aligned} J_2\left(X,\bar{X};z',\bar{z}'\right) &= (n+m+1)\left((-r)^{n+m+1}L_\Omega\right)\frac{D(-r)}{-r} \\ &= (n+m+1)\left(\Psi+\Phi(-r)^{n+m+1}\log(-r)\right)\frac{D(-r)}{-r}. \quad (4.15)\end{aligned}$$

Let $p = (z_0, z_0', w_0)$ be a boundary point U^α. When $|w_0| \neq 1$, we let X_0 denote the corresponding vector X at point p.

Case 1) For $(z_0, z_0', w_0) \in \mathcal{S}_2$, we have $z_0 = 0$, $(X_0, \bar{X}_0, z_0') \in \mathbf{b}\Omega$, and $|w_0| \neq 1$. Then the existence of a nonzero limit of (4.8) is equivalent to the existence of a nonzero limit of (4.6). Since $|w_0| \neq 1$, X is smooth near p. Thus $r_{U^\alpha}(z,z',w)$ is smooth in a neighborhood of p and has limit $r(X,\bar{X}_0;z_0',\bar{z}_0') = 0$. Note that $(-r)\log(-r)$ also has limit 0 at point p. Therefore the limit of I_1 and J_1 exist. To achieve the limit existence of J_2 at p, we need an admissible approach region such that in this region, the limit of $\frac{D(-r)}{-r}$ is 0. We consider the following approach region

$$W_2 = \left\{\left(z,z',w\right) \in U^\alpha : \sum_{j=1}^n |z_j|^q < -r_{U^\alpha}\left(z,z',w\right)\right\}$$

where $0 < q < 1$. First we need to show W_2 is not empty. By Lemma 4.1, $r_{z_j}(0,0;z_0',\bar{z}_0') = r_{\bar{z}_j}(0,0;z_0',\bar{z}_0') = 0$ for all j. When approaching p from inside of U^α in the normal direction, z_j's are identically zero. Hence the region W_2 is not empty and $p \in \mathbf{b}W_2$. By perhaps shrink W_2, we may also consider W_2 as a connected set. Note that

$$\left|\frac{D(-r)(X,\bar{X};z',\bar{z}')}{-r_{U^\alpha}(z,z',w)}\right| = \frac{\sum_{j=1}^n \alpha_j \frac{|z_j|}{(1-|w|^2)^{\alpha_j/2}}}{-r_{U^\alpha}(z,z',w)} < \frac{c\sum_{j=1}^n |z_j|}{-r_{U^\alpha}(z,z',w)}$$

for some constant $c > 0$. In W_2,

$$\frac{\sum_{j=1}^n |z_j|^q}{-r_{U^\alpha}(z,z',w)} < 1.$$

When approaching the boundary point p inside W_2, we have

$$\frac{\sum_{j=1}^{n}|z_j|}{-r_{U^\alpha}(z,z',w)} \leq \frac{(\sum_{k=1}^{n}|z_j|^{1-q})(\sum_{j=1}^{n}|z_j|^q)}{-r_{U^\alpha}(z,z',w)} < \sum_{k=1}^{n}|z_j|^{1-q} \to 0.$$

Hence (4.14) and (4.15) have admissible limit 0 at point p. The strong pseudoconvexity of Ω implies that the left hand side of (4.12) has nonzero limit. Therefore in W_2, the limit of (4.6) at p exists and is unequal to zero.

Case 2) For $(z_0, z_0', w_0) \in \mathcal{S}_3$, we have $z_0 = 0$, $(0, z_0') \notin \mathbf{b}\Omega$, and $|w_0| = 1$. Consider the region

$$W_3 = \left\{ \big(z, z', w\big) \in U^\alpha : \frac{|z_j|^2}{(1-|w|^2)^{p_j}} < 1, \forall\, 1 \leq j \leq n \right\}$$

where $p_j > \alpha_j$ for all j. Similar reasoning as above shows that W_3 is nonempty and connected. When we approaching the bounadary point p in W_3,

$$\frac{|z_j|^2}{(1-|w|^2)^{\alpha_j}} = \frac{|z_j|^2(1-|w|^2)^{p_j-\alpha_j}}{(1-|w|^2)^{p_j}} < (1-|w|^2)^{p_j-\alpha_j} \to 0.$$

Thus X, $D\Psi(X, \bar{X}; z', \bar{z}')$, $D\Phi(X, \bar{X}; z', \bar{z}')$ and $D(-r_{U^\alpha}(z, z', w))$ all tends to 0 at p. Since $(0, z_0') \notin \mathbf{b}\Omega$, the function $-r_{U^\alpha}(z, z', w)$ has a positive limit at point p. Plugging these results into (4.14) and (4.15), we have both J_1 and J_2 tend to 0. The limit of (4.12) is positive since $I_1 = c_\alpha L_\Omega$ and L_Ω is positive at $(0, z_0', \bar{z}_0')$. Therefore when approaching p in W_3, function (4.8) and r_{U^α} has a nonzero limit. Hence the limit of (4.7) is also not zero.

Case 3) When $(z_0, z_0', w_0) \in \mathcal{S}_4$, we have $z_0 = 0$, $(0, z_0') \in \mathbf{b}\Omega$, and $|w_0| = 1$. Consider the approach region $W_4 = W_2 \bigcap W_3$. Since both W_2 and W_3 contains the set $Z\{z_1, \ldots, z_n\} \bigcap U^\alpha$ and $p \in Z\{z_1, \ldots, z_n\} \bigcap \overline{U^\alpha}$, we can approach p in W_4. The argument in Cases 1 and 2 imply X, $D\Psi(X, \bar{X}; z', \bar{z}')$, $D\Phi(X, \bar{X}; z', \bar{z}')$, $r_{U^\alpha}(z, z', w)$, $D(r_{U^\alpha})$, $\frac{D(-r_{U^\alpha})}{-r_{U^\alpha}}$, and $r_{U^\alpha} \log(-r_{U^\alpha})$ all tends to 0. By (4.12), (4.14) and (4.15), limits of J_1 and J_2 both equal zero and the limit of I_1 is equal to a nonzero constant. Therefore (4.8) has a nonzero admissible limit in W_4.

Case 4) When $p = (z_0, z_0', w_0) \in \mathcal{S}_1$, Lemma 4.1 implies $r_{z_j}(p) \neq 0$ for $1 \leq j \leq n$. Then $cH_\Psi(p)$ in Theorem 4.1 is positive. Since (X_0, z_0') is a strongly pseudoconvex boundary point in $\mathbf{b}\Omega$, $H_\Omega(X, z')$ is positive definite on $T^{1,0}(\mathbf{b}\Omega)$. Therefore $H_\mathcal{U}(p)$ is positive definite on $T_p^{1,0}(\mathbf{b}\mathcal{U})$ and p is a strongly pseudoconvex point. $\square$

Compared to U^α, the boundary behavior of the kernel funcion V^γ is simpler. The argument is similar to the proof of Theorem 4.1. We state the result without proof:

Theorem 4.3. *Let Ω and V^γ be as above. Suppose Ω is bounded, smooth, and strongly pseudoconvex. Then V^γ is pseudoconvex. The boundary point $p = (z_0, z_0', w_0)$ is weakly pseudoconvex if $z_0 = 0$. Moreover, for each weakly pseudoconvex boundary point p, there is an admissible approach region W, such that when approaching p inside W:*

$$K_{V^\gamma}(z, z', w)(-r_{V^\gamma})^{n+m+1}(z, z', w)$$

tends to a nonzero constant.

Remark In both Theorems 4.2 and 4.3 above, we assumed the existence of z' components. Because of our assumption, points in $\mathcal{S}_2$, $\mathcal{S}_3$ of Theorem 4.2 and the weakly pseudoconvex boundary points in $\mathbf{b}V^\gamma$ of Theorem 4.3 are of infinite type in the sense of D'Angelo. If there is no z', i.e. $m = 0$ in the definition of Ω, then the boundary geometry of the target domains is different. In this case, V^γ becomes a strongly pseudoconvex domain. The boundary geometry of U^α, on the other hand, depends on the value of α. One can see this fact immediately from Example 3.1.

Chapter 5

Cancellation of singularities for weighted Bergman kernels

The Bergman kernel tends to infinity when it approaches the boundary of a pseudoconvex domain along the diagonal. The kernel function behaves differently when approaching the boundary off the diagonal. Kerzman [Ker72] first obtained this result in the strongly pseudoconvex case. The formula $P = I - \overline{\partial}^* N \overline{\partial}$ relates the Bergman projection to the $\overline{\partial}$-Neumann operator N. Using the pseudo-local property of N on strongly pseudoconvex domains, Kerzman estimated the derivatives of the Bergman kernel off the diagonal and obtained the following theorem:

Theorem 5.1. *Suppose Ω is a smooth, bounded strongly pseudoconvex domain in $\mathbb{C}^n$. Let K_Ω denote the Bergman kernel on Ω. Let $\triangle\Omega$ denote the closed subset defined by $\triangle\Omega = \{(z, w) \in \mathbf{b}\Omega \times \mathbf{b}\Omega : z = w\}$. Then $K_\Omega \in C^\infty(\overline{\Omega} \times \overline{\Omega} - \triangle\Omega)$.*

Later, using subelliptic estimates for N on domains of finite type, Bell [Bel86] and Boas [Boa87] independently generalized Kerzman's theorem to the following two cases:

Theorem 5.2. *Suppose Ω is a smooth, bounded pseudoconvex domain in $\mathbb{C}^n$. If U and V are disjoint open subsets of $\mathbf{b}\Omega$ consisting of points of finite type, then the Bergman kernel on Ω can be smoothly extended to $U \times V$.*

Theorem 5.3. *Suppose Ω is a smooth, bounded pseudoconvex domain in $\mathbb{C}^n$ satisfying Condition R. If U and V are disjoint open subsets of $\mathbf{b}\Omega$ and U consists of points of finite type, then the Bergman kernel on Ω can be smoothly extended to $U \times V$.*

Theorems 5.1 and 5.2 show the differentiability of the Bergman kernel on the boundary off diagonal. If the kernel function can be written as a finite sum of some functions and each of them is blowing up on certain boundary points off diagonal, then a cancellation of singularities among terms in the sum happens when approaching these boundary points.

In Section 5.1, we illustrate a cancellation of singularities among some generalized hypergeometric functions by applying Theorems 5.1 and 5.2 to the Bergman kernel function on the complex ovals. In Section 5.3, we show a cancellation of singularities among some weighted Bergman kernels on some Hartogs domains.

5.1 Generalized hypergeometric functions

Consider the complex oval Ω in $\mathbb{C}^n$ defined by:

$$\{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2a_j} < 1\}$$

where a_j's are positive integers. Francsics and Hanges [FH96] expressed the Bergman kernel on these domains in terms of Appell's 2nd hypergeometric functions. We introduce the definition of these generalized hypergeometric functions.

Definition 5.1. *For $z \in \mathbb{C}^n$ and $\mathbf{m}$ a multi-index, the Appell's 2nd hypergeometric function $F_2^{(n)}$ in $\mathbb{C}^n$ is given by*

$$F_2^{(n)}(\alpha; \beta_1, \cdots, \beta_n; \gamma_1, \cdots, \gamma_n; z) = \sum_{\mathbf{m}} \frac{(\alpha)_{|\mathbf{m}|} \prod_{i=1}^{n} (\beta_i)_{m_i}}{\mathbf{m}! \prod_{i=1}^{n} (\gamma_i)_{m_i}} z^{\mathbf{m}} \tag{5.1}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$.

The series is convergent in the domain $\{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j| < 1\}$ and is divergent at the boundary where all the z_j's are positive.

When $n = 1$, (5.1) becomes

$$F_2^{(1)}(\alpha, \beta, \gamma; z) = \sum_{m=1}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} z^m, \tag{5.2}$$

which is the standard hypergeometric function.

The result of Francsics and Hanges expresses the Bergman kernel in terms of $F_2^{(n)}$:

Theorem 5.4. *For $\Omega = \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\}$, the Bergman kernel is given by*

$$\begin{aligned} K_\Omega(z, \bar{w}) = & \frac{\prod_{i=1}^{n} p_i}{\pi^n} \sum_{q_1=0}^{p_1-1} \cdots \sum_{q_n=0}^{p_n-1} (z\bar{w})^{\mathbf{q}} \frac{\Gamma(1 + \sum_{j=1}^{n} (q_j+1)/p_j)}{\prod_{j=1}^{n} \Gamma((q_j+1)/p_j)} \\ & \times F_2^{(n)}(1 + \sum_{j=1}^{n} \frac{q_j+1}{p_j}, \mathbf{1}, \frac{\mathbf{q}+\mathbf{1}}{\mathbf{p}}; (z\bar{w})^{\mathbf{p}}). \end{aligned} \tag{5.3}$$

Here **p**, **q** *and* **1** *are multi-indices.*

Note that Ω in Theorem 5.4 is smooth, bounded, pseudoconvex, and every point on its boundary is a point of finite type. By Theorem 5.2, the sum on the right-hand side of (5.3) is smooth at the boundary off the diagonal. However, $F_2^{(n)}$ is not necessary smooth at the boundary of its domain of convergence. Therefore each term in (5.3) can blow up at some points in $\mathbf{b}\Omega$. Let's first recall an example where the kernel function has an explicit formula.

Example 5.1. *Let* Ω *denote the complex oval in* $\mathbb{C}^2$ *defined by*

$$\Omega = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1\}.$$

In Chapter 2, we've already obtained the formula for the Bergman kernel function on Ω

$$K_\Omega(z;\bar{w}) = \frac{3(1 - z_1\bar{w}_1)^{\frac{1}{2}} - z_2\bar{w}_2}{2\pi^2((1 - z_1\bar{w}_1)^{\frac{1}{2}} - z_2\bar{w}_2)^3(1 - z_1\bar{w}_1)^{\frac{3}{2}}} \tag{5.4}$$

Using (5.3), K_Ω can also be expressed in the following way:

$$\begin{aligned} K_\Omega(z;\bar{w}) =& \frac{4z_2\bar{w}_2}{\pi^2(1 - z_1\bar{w}_1 - z_2^2\bar{w}_2^2)^3} \\ &+ \frac{3(1 - z_1\bar{w}_1)^2 + 6(1 - z_1\bar{w}_1)z_2^2\bar{w}_2^2 - z_2^4\bar{w}_2^4}{2\pi^2(1 - z_1\bar{w}_1 - z_2^2\bar{w}_2^2)^3(1 - z_1\bar{w}_1)^{\frac{3}{2}}} \end{aligned} \tag{5.5}$$

In equation (5.5), each of the two terms is an $F_2^{(n)}$ with different parameters. For $p = (0, i)$ and $q = (0, -i)$, either Theorem 5.2 or 5.4 guarantees that (5.5) can be smoothly extended at (p, q). However, $(1-z_1\bar{w}_1-z_2^2\bar{w}_2^2)$ is identically 0 if $z \in \mathbf{b}\Omega$ is of purely imaginary coordinates and $w = \bar{z}$. Hence there is no smooth extension for each term in (5.5) to (p, q). But when we sum them up, the singularity of each $F_2^{(n)}$ cancels each other. The cancellation in (5.5) is not hard to observe since each term is an elementary function. In general, $F_2^{(n)}$ with rational parameters does not have necessarily have an elementary expression while such cancellation of singularities phenomena still happens.

Theorem 5.5. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *be a multi-index. Let* U *denote the set* $\{t \in \mathbb{C}^n : \sum_{j=1}^n |t_j|^{p_j} < 1\}$. *Let* $\mathbf{b}U^+$ *denote the set* $\{t \in \mathbf{b}U : t_j \geq 0\}$. *Let*

$F(t)$ denotes the sum

$$\sum_{q_1=0}^{p_1-1}\cdots\sum_{q_n=0}^{p_n-1}(t)^{\mathbf{q}}\frac{\Gamma(1+\sum_{j=1}^{n}(q_j+1)/p_j)}{\prod_{j=1}^{n}\Gamma((q_j+1)/p_j)}$$
$$\times F_2^{(n)}\left(1+\sum_{j=1}^{n}\frac{q_j+1}{p_j},\mathbf{1},\frac{\mathbf{q}+\mathbf{1}}{\mathbf{p}};t^{\mathbf{p}}\right). \tag{5.6}$$

Then F is defined on U and F can be smoothly extended to $\mathbf{b}U-\mathbf{b}U^+$.

Proof. Replacing t by $z\bar{w}$ and multiplying with a constant factor, (5.3) becomes (5.6). Therefore the convergence of $K_\Omega(z,\bar{w})$ on $\Omega\times\Omega$ implies the convergence of (5.3) on U. For $t\in\mathbf{b}U-\mathbf{b}U^+$, there exist $z,w\in\mathbf{b}\Omega$ such that $(z\bar{w})=t$. Since $t\notin\mathbf{b}U^+$, there exists j such that $t_j=z_j\bar{w}_j\notin\mathbb{R}^+$. Hence $z\neq w$. By Theorem 5.2, K_Ω can be extended smoothly to (z,w). Therefore F can be smoothly extended to t. Since t is arbitrary, the proof is complete. □

Remark Consider $t=(t_1,\ldots,t_n)\in\mathbf{b}U-\mathbf{b}U^+$ where $t_j^{p_j}\geq 0$ for all j. By its definition, $F_2^{(n)}$ is divergent at $t^{\mathbf{p}}$. Therefore when approaching t, each term in (5.6) is blowing up to the direction of $(t)^{\mathbf{q}}$ but the sum is finite. This phenomenon indicates a cancellation of singularities happens between terms of different directions and the sum becomes bounded. As we will see in the next section, each term in (5.6) is a weighted Bergman kernel on the unit ball in $\mathbb{C}^n$ and Theorem 5.5 shows a cancellation of singularities among these weighted kernel functions.

5.2 A transformation formula for proper mapping

In [Bel81], Bell had obtained a transformation rule for the Bergman projections under proper holomorphic mappings:

Theorem 5.6. *For $j=1,2$, let P_j denote the Bergman projection associated to the bounded domain $\mathbb{D}_j$ in $\mathbb{C}^n$. If there is a proper holomorphic mapping f from $\mathbb{D}_1$ onto $\mathbb{D}_2$, then*

$$P_1(J(f)\cdot(\phi\circ f))=J(f)\cdot\big((P_2\phi)\circ f\big), \tag{5.7}$$

where $J(f)$ is the holomorphic Jacobian determinant of f and $\phi\in A^2(\mathbb{D}_2)$.

As a consequence of (5.7), a transformation formula for the Bergman kernels on $\mathbb{D}_1$ and $\mathbb{D}_2$ can also be obtained:

Theorem 5.7. *Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be bounded domains in $\mathbb{C}^n$ and f is a proper holomorphic mapping of $\mathbb{D}_1$ onto $\mathbb{D}_2$ of order m. Let $J(f)$ denote the holomorphic Jacobian determinant of f. Let $F_1, \dots, F_m$ be the m local inverses to f defined locally on $D_2 - V$ where V is the null set of $J(f)$. Let $J(F_j)$ denote the holomorphic Jacobian determinant of F_j.*

$$\sum_{j=1}^{m} K_{\mathbb{D}_1}\left(z; \bar{F}_j(\zeta)\right) \overline{J(F_j)}(\zeta) = K_{\mathbb{D}_2}\left(f(z); \bar{\zeta}\right) J(f)(z) \tag{5.8}$$

for all $z \in \mathbb{D}_1$ and $\zeta \in \mathbb{D}_2 - V$.

Remark 5.1. *By the Removable Singularity Theorem, the left hand side of (5.8) extends to be anti-holomorphic in ζ on $\mathbb{D}_2$.*

Bell's transformation formula is explicit for $K_{\mathbb{D}_2}$ but implicit for $K_{\mathbb{D}_1}$. The Bergman kernel on $\mathbb{D}_2$ can be obtained if the kernel function on $\mathbb{D}_1$ is known; conversely, it is difficult to obtain the Bergman kernel $K_{\mathbb{D}_1}$ for given $K_{\mathbb{D}_2}$ due to the possible cancellation among terms in the left hand side of (5.8). In this section, we provide, for some particular $\mathbb{D}_1$ and $\mathbb{D}_2$, a transformation formula which is explicit for $K_{\mathbb{D}_1}$.

In this section, we let Ω be a Hartogs domain in $\mathbb{C}^{n+m}$ with symmetric planes $\{z_j = 0\}$ for $1 \leq j \leq n$. A similar argument in the proof of Lemmas 2.2 and 2.4 yield the following lemmas:

Lemma 5.1. *Let f be a holomorphic function on Ω. Then f has a series expansion:*

$$f(z, z') = \sum_{\mathbf{a} \in \mathbb{Z}^n} z^{\mathbf{a}} \phi_{\mathbf{a}}(z'), \tag{5.9}$$

where $\phi_{\mathbf{a}}$ is holomorphic in z' and the series converges normally in Ω.

Lemma 5.2. *Let f be a square-integrable holomorphic function on Ω. Then we have the following:*

1. *f has a series expansion:*

$$f(z, \zeta) = \sum_{\mathbf{a} \in \mathbb{Z}^n} z^{\mathbf{a}} \phi_{\mathbf{a}}(\zeta), \tag{5.10}$$

   where, for each multi-index $\mathbf{a}$, $\phi_{\mathbf{a}}$ is a square-integrable holomorphic function on $\pi(\Omega)$ with respect to the the measure $\|z^{\mathbf{a}}\|^2_{L^2(\Omega_\zeta)}$.

2. *If $\{\phi_{\mathbf{a},\mathbf{b}}\}$ is a complete orthogonal system for $A^2(\pi(\Omega), \|z^{\mathbf{a}}\|^2_{L^2(\Omega_\zeta)})$, then $\{z^{\mathbf{a}} \phi_{\mathbf{a},\mathbf{b}}\}$ forms a complete orthogonal system for $A^2(\Omega)$.*

By Lemma 5.2, we can choose a complete orthogonal system of the form $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}\}$ for $A^2(\Omega)$. Hence the Bergman kernel on Ω has a series expansion:

$$K_\Omega(z,z';\bar\zeta,\bar\zeta') = \sum_{\mathbf{a},\mathbf{b}} \frac{z^{\mathbf{a}}\bar\zeta'\phi_{\mathbf{a},\mathbf{b}}(z')\bar\phi_{\mathbf{a},\mathbf{b}}(\zeta')}{\|z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\|^2_{L^2(\Omega)}}. \tag{5.11}$$

Note that the series above is different from the series expansion for the Bergman kernel on an n-star-shaped Hartogs domain: the multi-indices are in $\mathbb{N}^n$ for an n-star-shaped Hartogs domain while in (5.11) they are in $\mathbb{Z}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_j's are positive integers. Set

$$\Omega^\alpha = \{(z,z') \in \mathbb{C}^{n+m} : (z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}, z') \in \Omega\}.$$

Then Ω^α is also a Hartogs domain with symmetric planes $\{z_j = 0\}$ for $1 \le j \le n$. Let f_α denote the proper mapping from $\mathbb{C}^n \times \mathbb{C}^m$ to $\mathbb{C}^n \times \mathbb{C}^m$:

$$f_\alpha : (z_1, \ldots, z_n, z') \mapsto (z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}, z').$$

Since f_α maps Ω^α onto Ω, Bell's transformation rule implies that

$$\begin{aligned} &K_\Omega\left(z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}, z'; \bar\zeta_1^{\alpha_1}, \ldots, \bar\zeta_n^{\alpha_n}, \bar\zeta'\right) \\ &= \sum_{j_1=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} K_{\Omega^\alpha}(z,z';\bar\omega_1^{j_1}\bar\zeta_1, \ldots, \bar\omega_n^{j_n}\bar\zeta_n, \bar\zeta') \frac{\bar\omega_1^{j_1}\cdots\bar\omega_n^{j_n}}{\prod_{j=1}^n (\alpha_j^2 (z_j\bar\zeta_j)^{\alpha_j - 1})}, \end{aligned} \tag{5.12}$$

where ω_j is j_n-th root of unity. K_Ω above is written in terms of K_{Ω^α}. In the next theorem, we provide another transformation formula in which K_{Ω^α} is written in terms of some weighted kernel functions on Ω. For multi-index $\mathbf{c}$, let $K_\Omega^{\mathbf{c}}$ denote the weighted Bergman kernel for the space $A^2(\Omega, |z|^{\mathbf{c}}dV)$.

Theorem 5.8. *Let Ω be a Hartogs domain in $\mathbb{C}^n \times \mathbb{C}^m$ with symmetric planes $\{z_j = 0\}$ for $1 \le j \le n$. For $\alpha \in \mathbb{Z}^n$, let Ω^α and f_α be defined as above. Then for (z,z') and (ζ,ζ') in Ω^α, we have:*

$$K_{\Omega^\alpha}(z,z';\bar\zeta,\bar\zeta') = \sum_{j_1=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} \frac{z^{\mathbf{j}}\bar\zeta^{\mathbf{j}}}{\alpha^{\mathbf{1}}} K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}\left(f_\alpha(z), z'; \bar f_\alpha(\zeta), \bar\zeta'\right), \tag{5.13}$$

where $\mathbf{c}(\mathbf{j},\alpha) = \left(\frac{2(j_1+1-\alpha_1)}{\alpha_1}, \ldots, \frac{2(j_n+1-\alpha_n)}{\alpha_n}\right)$.

Proof. Since Ω^α is a Hartogs domain with symmetric planes $\{z_j = 0\}$ for $1 \le j \le n$, a complete orthogonal system of form $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\}$ can be chosen for $A^2(\Omega)$. Let $\mathfrak{I}$ denote the set

$$\{(\mathbf{a},\mathbf{b}) : z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}} \in A^2(\Omega^\alpha)\}.$$

Then the Bergman kernel on Ω^α has a bi-orthonormal series expansion:

$$K_{\Omega^\alpha}(z,z';\bar\zeta,\bar\zeta') = \sum_{(\mathbf{a},\mathbf{b})\in\mathfrak{I}} \frac{z^{\mathbf{a}}\bar\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\bar\phi_{\mathbf{a},\mathbf{b}}(\zeta')}{\|z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\|^2_{L^2(\Omega^\alpha)}}.$$

For multi-index $\mathbf{j} = (j_1,\dots,j_n)$, let $\mathfrak{I}_{\mathbf{j}}$ denote the set

$$\{(\mathbf{a},\mathbf{b}) \in \mathfrak{I} : a_k = j_k \bmod \alpha_k \text{ for } 1 \le k \le n\}.$$

Then $\mathfrak{I} = \cup_{\mathbf{j}\in\mathbb{Z}^n}\mathfrak{I}_{\mathbf{j}}$. We claim, for each $\mathbf{j}$, that

$$\alpha^{\mathbf{1}} z^{\mathbf{j}}\bar\zeta^{\mathbf{j}} K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}(f_\alpha(z),z';\bar f_\alpha(\zeta),\bar\zeta') = \sum_{(\mathbf{a},\mathbf{b})\in\mathfrak{I}_{\mathbf{j}}} \frac{z^{\mathbf{a}}\bar\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\bar\phi_{\mathbf{a},\mathbf{b}}(\zeta')}{\|z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\|^2_{L^2(\Omega^\alpha)}}. \tag{5.14}$$

Then summing up both sides of (5.14) for each $\mathbf{j}$ yields (5.13) and completes the proof. Set $K_{\mathbf{j}}$ as the left hand side of (5.14). To prove (5.14), it suffices to show that $K_{\mathbf{j}}$ can be expanded as follows:

$$K_{\mathbf{j}}(z,z';\bar\zeta,\bar\zeta') = \sum_{(\mathbf{a},\mathbf{b})\in\mathfrak{I}_{\mathbf{j}}} c_{\mathbf{a},\mathbf{b}} z^{\mathbf{a}}\bar\zeta^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')\bar\phi_{\mathbf{a},\mathbf{b}}(\zeta'), \tag{5.15}$$

and reproduces $z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')$ for each $(\mathbf{a},\mathbf{b}) \in \mathfrak{I}_j$.

Let $\{z^{\mathbf{s}}\psi_{\mathbf{s},\mathbf{b}}\}$ be a complete orthogonal system for $A^2(\Omega, |z|^{\mathbf{c}(\mathbf{j},\alpha)}dV)$. Let $\mathfrak{L}_{\mathbf{j}}$ denote the set $\{(\mathbf{s},\mathbf{b}) : z^{\mathbf{s}}\psi_{\mathbf{s},\mathbf{b}}(z') \in A^2(\Omega, |z|^{\mathbf{c}(\mathbf{j},\alpha)}dV)\}$. By Lemma 5.2, $K_{\mathbf{j}}$ has the following series expansion:

$$K_{\mathbf{j}}(z,z';\bar\zeta,\bar\zeta') = \sum_{(\mathbf{s},\mathbf{b})\in\mathfrak{L}_{\mathbf{j}}} c_{\mathbf{s},\mathbf{b}} z^{\mathbf{s}\alpha+\mathbf{j}}\bar\zeta^{\mathbf{s}\alpha+\mathbf{j}}\psi_{\mathbf{s},\mathbf{b}}(z')\bar\psi_{\mathbf{s},\mathbf{b}}(\zeta'), \tag{5.16}$$

where $\mathbf{s}\alpha$ denotes $(s_1\alpha_1,\dots,s_n\alpha_n)$. Since for $z^{\mathbf{s}}f(z') \in A^2(\Omega)$,

$$\int_{\Omega^\alpha} |z|^{2(\mathbf{s}\alpha+\mathbf{j})}|f(z')|^2 dV = \frac{1}{\alpha^{\mathbf{1}}}\int_\Omega |z|^{2\mathbf{s}}|f(z')|^2|z|^{\mathbf{c}(\mathbf{j},\alpha)}dV,$$

if the integral in either side is bounded, $z^{\mathbf{s}}f(z') \in A^2(\Omega, |z|^{\mathbf{c}(\mathbf{j},\alpha)}dV)$ is equivalent to $z^{\mathbf{s}\alpha+\mathbf{j}}f(z') \in A^2(\Omega^\alpha)$ and $(\mathbf{s},\mathbf{b})$ being in $\mathfrak{L}_j$ is the same as $(\mathbf{s}\alpha+\mathbf{j},\mathbf{b})$ being in $\mathfrak{I}_j$. Also, $\psi_{\mathbf{s},\mathbf{b}}$ can be chosen as $\phi_{\mathbf{s}\alpha+\mathbf{j},\mathbf{b}}$. Therefore (5.15) holds.

For $(\mathbf{s}\alpha+\mathbf{j},\mathbf{b})\in\mathfrak{I}_{\mathbf{j}}$,

$$\begin{aligned}&\int_{\Omega^\alpha} K_{\mathbf{j}}(z,z';\bar\zeta,\bar\zeta')\zeta^{\mathbf{s}\alpha+\mathbf{j}}\phi_{\mathbf{s}\alpha+\mathbf{j},\mathbf{b}}(\zeta')dV\\ =&\int_{\Omega^\alpha}\frac{z^{\mathbf{j}}\bar\zeta^{\mathbf{j}}}{\alpha^{\mathbf{1}}}K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}(f_\alpha(z),z';\bar f_\alpha(\zeta),\bar\zeta')\zeta^{\mathbf{s}\alpha+\mathbf{j}}\psi_{\mathbf{s},\mathbf{b}}(\zeta')dV.\end{aligned}\tag{5.17}$$

Substituting $t_k=\zeta_k^{\alpha_k}$ for $1\le k\le n$ yields

$$\begin{aligned}&\int_{\Omega^\alpha}\alpha^{\mathbf{1}}z^{\mathbf{j}}\bar\zeta^{\mathbf{j}}K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}(f_\alpha(z),z';\bar f_\alpha(\zeta),\bar\zeta')\zeta^{\mathbf{s}\alpha+\mathbf{j}}\psi_{\mathbf{s},\mathbf{b}}(\zeta')dV\\ =&z^{\mathbf{j}}\int_{\Omega}\bar\zeta^{\mathbf{j}}K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}(f_\alpha(z),z';\bar t,\bar\zeta')t^{\mathbf{s}}\psi_{\mathbf{s},\mathbf{b}}(\zeta')|t|^{\mathbf{c}(\mathbf{j},\alpha)}dV\\ =&z^{\mathbf{s}\alpha+\mathbf{j}}\psi_{\mathbf{s},\mathbf{b}}(z').\end{aligned}\tag{5.18}$$

Therefore $K_{\mathbf{j}}$ reproduces $z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}(z')$ for $(\mathbf{a},\mathbf{b})\in\mathfrak{I}_{\mathbf{j}}$. □

Remark 5.2. *Formula (5.13) can be used to obtain the explicit formula of K_{Ω^α} when explicit formulas for $K_\Omega^{c(j,q)}$ are known. For $p,q\in\mathbb{N}^+$ and*

$$\Omega=\{(z,w)\in\mathbb{C}^2:|z|<|w|^p<1\},$$

$K_\Omega^{\mathbf{c}(j,q)}$ can be written in terms of elementary functions. Applying (5.13) to $K_\Omega^{c(j,q)}$ yields the Bergman kernel on generalized Hartogs triangle

$$\Omega^q=\{(z,w)\in\mathbb{C}^2:|z|^q<|w|^p<1\}.$$

In general, $K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}$ might not have an explicit formula. Consider the domains $\Omega=\{(z,w)\in\mathbb{C}^2:|z|^4+|w|^4<1\}$ and $\Omega^2=\{(z,w)\in\mathbb{C}^2:|z|^8+|w|^4<1\}$. Park provided, in [Par08], the explicit formula for K_Ω and showed K_{Ω^2} can not be written in terms of elementary functions. This observation indicates that the weighted kernel function $K_\Omega^{c(0,2)}$ does not have an explicit form.

5.3 Cancellation of singularities

Let Ω^α be as in Theorem 5.8. Suppose Ω^α is smooth, bounded and pseudoconvex, and the boundary points are of finite type in the sense of D'Angelo (See e.g. [D'A93]). Applying Theorem 5.2 to Theorem 5.8 yields that

$$\sum_{j_j=1}^{\alpha_1}\cdots\sum_{j_n=1}^{\alpha_n}\frac{z^{\mathbf{j}}\bar\zeta^{\mathbf{j}}}{\alpha^{\mathbf{1}}}K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}(f_\alpha(z),z';\bar f_\alpha(\zeta),\bar\zeta'),\tag{5.19}$$

can be smoothly extended to be in $C^\infty(\bar{\Omega}^\alpha \times \bar{\Omega}^\alpha - \triangle(\Omega^\alpha))$. Such a statement is not obvious since, for boundary points (z, z') and (ζ, z') of Ω^α with $z \neq \zeta$, it is possible that $f_\alpha(z) = f_\alpha(\zeta)$ and $K^{\mathbf{c}(\mathbf{j},\alpha)}$ blows up.

Theorem 5.9. *Let $\Omega \subseteq \mathbb{C}^{n+m}$ be a Hartogs domain with symmetric planes $\{z_j = 0\}$ for $1 \leq j \leq n$. Suppose Ω is smooth, bounded and pseudoconvex, and the boundary points are of finite type in the sense of D'Angelo. Set $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}^+$. Let ω_j denote the α_j-th root of unity and let ω denote $(\omega_1, \ldots, \omega_n)$. Then the function*

$$\sum_{j_j=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} \frac{\omega^{\frac{\mathbf{j}}{\alpha}} |z|^{\frac{2\mathbf{j}}{\alpha}}}{\alpha^{\mathbf{1}}} K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}(z, z'; \bar{z}, \bar{z}'), \tag{5.20}$$

extended smoothly to the boundary of Ω.

Proof. Set

$$\Omega^\alpha = \left\{(z, z') : (z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}, z') \in \Omega\right\}.$$

Since Ω is smooth, bounded and pseudoconvex, Ω^α is also smooth, bounded and pseudoconvex. Moreover, the boundary points of Ω^α are of finite type since points in $b\Omega$ are of finite type. By Theorems 5.2 and 5.8

$$\sum_{j_j=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} \frac{z^{\mathbf{j}} \bar{\zeta}^{\mathbf{j}}}{\alpha^{\mathbf{1}}} K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}\left(f_\alpha(z), z'; \bar{f}_\alpha(\zeta), \bar{\zeta}'\right), \tag{5.21}$$

can be smoothly extended to be in $C^\infty(\bar{\Omega}^\alpha \times \bar{\Omega}^\alpha - \triangle(\Omega^\alpha))$. Substituting $\zeta_s = z_s \omega_s$ and $t = f_\alpha(z)$ in (5.21) yields that $(t, z') \in \Omega$ and

$$\sum_{j_j=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} \frac{\omega^{\frac{\mathbf{j}}{\alpha}} |t|^{\frac{2\mathbf{j}}{\alpha}}}{\alpha^{\mathbf{1}}} K_\Omega^{\mathbf{c}(\mathbf{j},\alpha)}(t, z'; \bar{t}, \bar{z}') \tag{5.22}$$

can be extended smoothly to the boundary of Ω. $\square$

Chapter 6

Complex harmonic homogeneous polynomials

Our technique for computing the Bergman kernel on U^α (or V^γ) uses the fact that $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}w^{\mathbf{c}}\}$ forms a complete orthogonal system for $A^2(U^\alpha)$ (or $A^2(V^\gamma)$) when $\{z^{\mathbf{a}}\phi_{\mathbf{a},\mathbf{b}}\}$ is a complete orthogonal system for $A^2(\Omega)$. The formulas in Theorems 3.1 and 3.2 relate orthonormal elements of $A^2(\Omega)$ to orthonormal elements of $A^2(U^\alpha)$ and $A^2(V^\gamma)$. In this chapter, we use a similar method to relate complex harmonic homogeneous polynomials in $\mathbb{C}^n$ to complex harmonic homogeneous polynomials in the higher dimensional $\mathbb{C}^{n+m}$.

6.1 Decomposition of $P_{m,n}$

Let $P_{m,n}(\mathbb{C}^k)$ denote the space of polynomials that are homogeneous of degree m in $z_1, \ldots, z_k$ and n in $\bar{z}_1, \ldots, \bar{z}_k$. Let $\triangle$ denote the Laplacian; $\triangle = \sum_{j=1}^{k} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ (The usual Laplacian equals $4\triangle$). A polynomial p is called harmonic if $\triangle p = 0$. Let $H_{m,n}(\mathbb{C}^k)$ denote the space of harmonic polynomials that are homogeneous of degree m in $z_1, \ldots, z_k$ and n in $\bar{z}_1, \ldots, \bar{z}_k$.

We consider surface integrals on the unit sphere $\mathbb{S}^{2k-1}$ in $\mathbb{C}^k$. Let $d\sigma$ denote the Lebesgue $2k-1$ dimensional measure on the sphere. We define an inner product on $L^2\left(\mathbb{S}^{2k-1}\right)$ as follows: For p and q in $L^2\left(\mathbb{S}^{2k-1}\right)$,

$$\langle p, q \rangle := \int_{\mathbb{S}^{2k-1}} p\bar{q}\, d\sigma.$$

Lemma 6.1. *Let $p \in H_{m,n}(\mathbb{C}^k)$, and let $q \in H_{a,b}(\mathbb{C}^k)$. If $(a, b) \neq (m, n)$, then $\langle p, q \rangle = 0$.*

Proof. We consider two cases: the case $m - n \neq a - b$ and the case $m - n = a - b$. When $m + n = a + b$, the assumption $(a, b) \neq (m, n)$ shows that

$m-n \neq a-b$. Since the measure $d\sigma$ is invariant under rotation,

$$\begin{aligned}\int_{\mathbb{S}^{2k-1}} p(z,\bar{z})\overline{q(z,\bar{z})}d\sigma &= \int_{\mathbb{S}^{2k-1}} p(e^{i\theta}z, e^{-i\theta}\bar{z})\overline{q\left(e^{i\theta}z, e^{-i\theta}\bar{z}\right)}d\sigma \\ &= e^{i(m-n-a+b)\theta}\int_{\mathbb{S}^{2k-1}} p\left(z,\bar{z}\right)\overline{q(z,\bar{z})}d\sigma, \qquad (6.1)\end{aligned}$$

where θ is real. Because $m-n \neq a-b$ and θ is arbitrary, the integral in the last line of (6.1) equals 0.

For the case $m+n \neq a+b$, we may regard p and q as harmonic homogeneous polynomials of degree $m+n$ in real variables x and $a+b$ in real variables y. Then we have

$$\frac{d}{dr}p(rz,r\bar{z})\Big|_{r=1} = (m+n)r^{m-1}p(x)\Big|_{r=1} = (m+n)p(x).$$

Similarly,

$$\frac{d}{dr}q(rz,r\bar{z})\Big|_{r=1} = (a+b)q(x).$$

Let $\mathbf{n}$ denote the exterior unit normal vector of $\mathbb{S}^{2k-1}$. By Green's Identity:

$$\begin{aligned}(m+n)\langle p,q\rangle &= \int_{\mathbb{S}^{2k-1}} \frac{\partial}{\partial \mathbf{n}}\left(p(z,\bar{z})\right)\bar{q}(z,\bar{z})d\sigma \\ &= \int_{\mathbb{B}^k} \triangle\left(p(z,\bar{z})\right)\bar{q}(z,\bar{z}) + \bigtriangledown p(z,\bar{z})\cdot\bigtriangledown\bar{q}(z,\bar{z})dV \\ &= \int_{\mathbb{B}^k} p(z,\bar{z})\triangle\bar{q}(z,\bar{z}) + \bigtriangledown p(z,\bar{z})\cdot\bigtriangledown\bar{q}(z,\bar{z})dV \\ &= \int_{\mathbb{S}^{2k-1}} p(z,\bar{z})\frac{\partial}{\partial \mathbf{n}}\bar{q}(z,\bar{z})d\sigma = (a+b)\langle p,q\rangle.\end{aligned}$$

Since $m+n \neq a+b$, we have $\langle p,q\rangle = 0$. $\square$

Let $M_{\|z\|^2}$ denote the multiplication operator by $\|z\|^2$:

$$M_{\|z\|^2} : f \mapsto \|z\|^2 f.$$

Then $M_{\|z\|^2}$ maps $P_{m,n}(\mathbb{C}^k)$ to $P_{m+1,n+1}(\mathbb{C}^k)$. Let $P_{m,n}(\mathbb{S}^{2k-1})$ denote the space of functions in $P_{m,n}(\mathbb{C}^k)$ restricted to $\mathbb{S}^{2k-1}$. Since $\|z\|^2 = 1$ on the unit sphere, we have

$$P_{m-1,n-1}\left(\mathbb{S}^{2k-1}\right) = M_{\|z\|^2}\left(P_{m-1,n-1}\left(\mathbb{S}^{2k-1}\right)\right) \subseteq P_{m,n}\left(S^{2k-1}\right).$$

Our next result provides an orthogonal decomposition for $P_{m,n}(\mathbb{S}^{2k-1})$. The proof uses another useful inner product.

For polynomials $p(z,\bar{z})$ and $q(z,\bar{z})$, we define $\prec p, q \succ$ to be

$$p(\bar{D}, D)\bar{q}(z,\bar{z})|_{z=0},$$

where $D = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_k}\right)$. If $p(z,\bar{z}) = z^{\alpha_1}\bar{z}^{\beta_1}$ and $q(z,\bar{z}) = z^{\alpha_2}\bar{z}^{\beta_2}$, we have

$$\prec p, q \succ = \left(\frac{\partial}{\partial \bar{z}}\right)^{\alpha_1}\left(\frac{\partial}{\partial z}\right)^{\beta_1}\bar{z}^{\alpha_2}z^{\beta_2}|_{z=0} = \begin{cases} 0 & \text{if } (\alpha_1,\beta_1) \neq (\alpha_2,\beta_2) \\ \alpha_2!\beta_2! & \text{if } (\alpha_1,\beta_1) = (\alpha_2,\beta_2) \end{cases}.$$

Therefore the space $P_{a,b}$ is orthogonal to $P_{c,d}$ with respect to $\prec \cdot,\cdot \succ$ if $(a,b) \neq (c,d)$. Similarly, $H_{a,b}$ is orthogonal to $H_{c,d}$ under this inner product. By its definition, the inner product $\prec \cdot,\cdot \succ$ satisfies the following:

For polynomials p, q_1 and q_2,

$$\prec p(z,\bar{z}), q_2(D,\bar{D})q_1(z,\bar{z}) \succ = \prec p(z,\bar{z})q_2(z,\bar{z}), q_1(z,\bar{z}) \succ .$$

In particular, the adjoint operator of multiplication by $\|z\|^2$ is the Laplacian, i.e. for $p, q \in P(\mathbb{C}^k)$, we have

$$\prec \|z\|^2 p, q \succ = \prec p, \triangle q \succ . \tag{6.2}$$

Formula (6.2) implies the $\prec \cdot,\cdot \succ$-orthogonality between elements in $H_{m,n}(\mathbb{C}^k)$ and $M_{\|z\|^2}\left(P_{m-1,n-1}(\mathbb{C}^k)\right)$.

Lemma 6.2. *The space $P_{m,n}(\mathbb{C}^k)$ has the following orthogonal decomposition with respect to the inner product $\prec \cdot,\cdot \succ$:*

$$P_{m,n}\left(\mathbb{C}^k\right) = H_{m,n}\left(\mathbb{C}^k\right) \bigoplus M_{\|z\|^2}\left(P_{m-1,n-1}\left(\mathbb{C}^k\right)\right). \tag{6.3}$$

Proof. By (6.2), the space $H_{m,n}(\mathbb{C}^k)$ is orthogonal to $M_{\|z\|^2}(P_{m-1,n-1}(\mathbb{C}^k))$. It suffices to show that the orthogonal complement of $M_{\|z\|^2}(P_{m-1,n-1}(\mathbb{C}^k))$ is contained in $H_{m,n}(\mathbb{C}^k)$. Let $q \in P_{m,n}(\mathbb{C}^k)$ be a polynomial in the orthogonal complement of $M_{\|z\|^2}(P_{m-1,n-1}(\mathbb{C}^k))$. Since $\triangle q \in P_{m-1,n-1}(\mathbb{C}^k)$, we have

$$0 = \prec M_{\|z\|^2}(\triangle q), q \succ = \prec \triangle q, \triangle q \succ .$$

Hence $\triangle q = 0$ and $q \in H_{m,n}(\mathbb{C}^k)$. $\square$

As a consequence of Lemma 6.2, the dimension of $H_{m,n}(\mathbb{C}^k)$ equals:

$$\dim\left(P_{m,n}\left(\mathbb{C}^k\right)\right) - \dim\left(P_{m-1,n-1}\left(\mathbb{C}^k\right)\right). \tag{6.4}$$

Since $\|z\| = 1$ on the unit sphere $\mathbb{S}^{2k-1}$, the decomposition (6.3) is also true with respect to the inner product $\langle \cdot, \cdot \rangle$. When restricting the domain to the unit sphere, the decomposition (6.3) becomes:

$$P_{m,n}\left(\mathbb{S}^{2k-1}\right) = H_{m,n}\left(\mathbb{S}^{2k-1}\right) \bigoplus P_{m-1,n-1}\left(\mathbb{S}^{2k-1}\right). \tag{6.5}$$

Decomposing $P_{m-1,n-1}(\mathbb{S}^{2k-1})$ in (6.5) and repeating the same process yield the last lemma in this section:

Lemma 6.3. $P_{m,n}(\mathbb{S}^{2k-1})$ *has the following orthogonal decomposition with respect to the inner product* $\langle \cdot, \cdot \rangle$*:*

$$P_{m,n}\left(\mathbb{S}^{2k-1}\right) = \bigoplus_{j=0}^{\min\{m,n\}} H_{m-j,n-j}\left(\mathbb{S}^{2k-1}\right).$$

Proof. Since $\|z\| = 1$ on $\mathbb{S}^{2k-1}$, we have

$$H_{m-j,n-j}\left(\mathbb{S}^{2k-1}\right) = M_{\|z\|^2}\left(H_{m-j,n-j}\left(\mathbb{S}^{2k-1}\right)\right) \subseteq P_{m,n}\left(\mathbb{S}^{2k-1}\right).$$

By Lemma 6.1, the spaces $H_{m-j,n-j}(\mathbb{S}^{2k-1})$ with different indices are orthogonal. It suffices to show that the spaces on both sides have the same dimension. Set $h_{a,b} = \dim(H_{a,b}(\mathbb{S}^{2k-1}))$ and $d_{a,b} = \dim(P_{a,b}(\mathbb{S}^{2k-1}))$. We assume, without loss of generality, that $m \geq n$. Then (6.4) gives

$$\sum_{j=0}^{n} h_{m-j,n-j} = \sum_{j=0}^{n-1}(d_{m-j,n-j} - d_{m-j-1,n-j-1}) + h_{m-n,0}. \tag{6.6}$$

Since $H_{m-n,0}(\mathbb{S}^{2k-1}) = P_{m-n,0}(\mathbb{S}^{2k-1})$, $h_{m-n,0} = d_{m-n,0}$. The right hand side of (6.6) equals the dimension of $P_{m,n}(\mathbb{S}^{2k-1})$. □

6.2 Decomposition of $H_{m,n}$

In Lemma 6.3, we decomposed the space of homogeneous polynomials into spaces of harmonic homogeneous polynomials. The harmonic homogeneous polynomial space could also be decomposed into a direct sum. In this section, we present a multiplication operator $M_{a,b,m,n}$ that maps elements in $H_{a,b}(\mathbb{S}^{2k-3})$ to elements in $H_{m,n}(\mathbb{S}^{2k-1})$ with $m \geq a$ and $n \geq b$. For each $a \leq m$ and $b \leq n$, the image of $H_{a,b}(\mathbb{S}^{2k-3})$ under $M_{a,b,m,n}$ is a subspace of $H_{m,n}(\mathbb{S}^{2k-1})$. We show in this section that all these subspaces together induce a decomposition of $H_{m,n}(\mathbb{S}^{2k-1})$.

Lemma 6.4. *For $z \in \mathbb{C}^{k-1}$ and $w \in \mathbb{C}$, there exists a unique sequence $\{c_i\}$ starting with $c_0 = 1$ such that the polynomial*

$$p(z, \bar{z}) \sum_{i=0}^{\min\{m-a,n-b\}} c_i w^{m-a-i} \bar{w}^{n-b-i} \|z\|^{2i} \tag{6.7}$$

is in $H_{m,n}(\mathbb{S}^{2k-1})$ whenever $p(z, \bar{z}) \in H_{a,b}(\mathbb{S}^{2k-3})$.

Proof. It suffices to prove that there exists c_i's such that $\triangle(q_{a,b,m,n}p) = 0$. Set $\triangle_z = \sum_{j=1}^{k-1} \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}$ and $\triangle_w = \frac{\partial}{\partial w} \frac{\partial}{\partial \bar{w}}$. Then

$$\begin{aligned} \triangle(q_{a,b,m,n}p) &= \triangle_z \left(q_{a,b,m,n}p\right) + \triangle_w(q_{a,b,m,n}p) \\ &= p \triangle q_{a,b,m,n} + \sum_{t=1}^{k-1} \left(\frac{\partial}{\partial z_t} q_{a,b,m,n} \frac{\partial}{\partial \bar{z}_t} p + \frac{\partial}{\partial \bar{z}_t} q_{a,b,m,n} \frac{\partial}{\partial z_t} p \right). \end{aligned} \tag{6.8}$$

Set $\mathrm{I} = p \triangle q_{a,b,m,n}$. Set II equal to the sum in (6.8). We have

$$\begin{aligned} \mathrm{I} =& p\Bigg(\sum_{i=1}^{\min\{m-a,n-b\}} i(i+k-2) c_i w^{m-a-i} \bar{w}^{n-b-i} \|z\|^{2i-2} + \\ & \sum_{i=0}^{\min\{m-a,n-b\}-1} (m-a-i)(n-b-i) c_i w^{m-a-i-1} \bar{w}^{n-b-i-1} \|z\|^{2i} \Bigg) \\ =& p\Bigg(\sum_{i=1}^{\min\{m-a,n-b\}} \left(i\left(i+k-2\right) c_i + \left(m-a-i+1\right)\left(n-b-i+1\right) c_{i-1}\right) \\ & w^{m-a-i} \bar{w}^{n-b-i} \|z\|^{2i-2} \Bigg), \end{aligned}$$

and

$$\begin{aligned} \mathrm{II} =& \sum_{i=0}^{\min\{m-a,n-b\}} \sum_{t=1}^{k-1} c_i w^{m-a-i} \bar{w}^{n-b-i} \left(\frac{\partial}{\partial z_t} \|z\|^{2i} \frac{\partial}{\partial \bar{z}_t} p + \frac{\partial}{\partial \bar{z}_t} \|z\|^{2i} \frac{\partial}{\partial z_t} p \right) \\ =& \sum_{i=0}^{\min\{m-a,n-b\}} c_i w^{m-a-i} \bar{w}^{n-b-i} \sum_{t=1}^{k-1} \left(\frac{\partial}{\partial z_t} \|z\|^{2i} \frac{\partial}{\partial \bar{z}_t} p + \frac{\partial}{\partial \bar{z}_t} \|z\|^{2i} \frac{\partial}{\partial z_t} p \right) \\ =& \sum_{i=1}^{\min\{m-a,n-b\}} i c_i w^{m-a-i} \bar{w}^{n-b-i} \|z\|^{2i-2} \sum_{t=1}^{k-1} \left(\bar{z}_t \frac{\partial}{\partial \bar{z}_t} p + z_t \frac{\partial}{\partial z_t} p \right) \\ =& p \sum_{i=1}^{\min\{m-a,n-b\}} i(a+b) c_i w^{m-a-i} \bar{w}^{n-b-i} \|z\|^{2i-2}. \end{aligned}$$

Then $\mathrm{I} + \mathrm{II} = 0$ if and only if $\{c_i\}_{i=0}^{\min\{m-a,n-b\}}$ satisfies:

$$c_i = -\frac{(m-a-i+1)(n-b-i+1)}{i(i+a+b+k-2)} c_{i-1}, \quad \text{for} \quad i \geq 1. \tag{6.9}$$

Formula (6.9) uniquely determines the $\{c_j\}$ given that $c_0 = 1$. $\square$

For the sequence $\{c_j\}$ in Lemma 6.4 with $c_0 = 1$, let $q_{a,b,m,n}$ denote

$$\sum_{i=0}^{\min\{m-a,n-b\}} c_i w^{m-a-i} \bar{w}^{n-b-i} \|z\|^{2i}. \tag{6.10}$$

Lemma 6.4 shows that for a polynomial $w^a \bar{w}^b p(z, \bar{z})$ with p being a complex harmonic homogeneous polynomial, there exists a unique polynomial $q_{a,b,m,n}$ of the form (6.10) such that $q_{a,b,m,n}(z, \bar{z}, w, \bar{w}) p(z, \bar{z})$ is harmonic. We call the product $q_{a,b,m,n}(z, \bar{z}, w, \bar{w}) p(z, \bar{z})$ the *harmonization* of $w^a \bar{w}^b p$. Let $M_{a,b,m,n}$ denote the multiplication operator induced by $q_{a,b,m,n}$ from $H_{a,b}(\mathbb{S}^{2k-3})$ to $H_{m,n}(\mathbb{S}^{2k-1})$:

$$M_{a,b,m,n} : p \mapsto q_{a,b,m,n} p.$$

Fix m and n. For $a \leq m$ and $n \leq n$, we set $\mathcal{M}_{a,b} = M_{a,b,m,n}(H_{a,b}(\mathbb{S}^{2k-3}))$. Then $\mathcal{M}_{a,b}$ is a subspace of $H_{m,n}(\mathbb{S}^{2k-1})$. The next theorem shows not only that $\mathcal{M}_{a_1,b_1}$ is orthogonal to $\mathcal{M}_{a_2,b_2}$ when $(a_1, b_1) \neq (a_2, b_2)$, but also that these $\mathcal{M}_{a,b}$ determine an orthogonal decomposition of $H_{m,n}(\mathbb{S}^{2k-1})$.

Theorem 6.1. *Let $M_{a,b,m,n}$ be as above. With respect to the inner product $\langle \cdot, \cdot \rangle$, the mapping $M_{a,b,m,n}$ preserves orthogonality. Moreover, $H_{m,n}(\mathbb{C}^k)$ have the following orthogonal decomposition:*

$$H_{m,n}\left(\mathbb{S}^{2k-1}\right) = \bigoplus_{a=0,b=0}^{m,n} M_{a,b,m,n}\left(H_{a,b}\left(\mathbb{S}^{2k-3}\right)\right). \tag{6.11}$$

Here the spaces $M_{a,b,m,n}\left(H_{a,b}\left(\mathbb{S}^{2k-3}\right)\right)$ are orthogonal to each other.

Proof. By Lemma 6.4, we can set

$$\tilde{q}_{a,b,m,n}(\|z\|^2, w, \bar{w}) = q_{a,b,m,n}(z, \bar{z}, w, \bar{w}).$$

On the unit sphere $\mathbb{S}^{2k-1}$, $\|z\|^2 + |w|^2 = 1$. Therefore, we have

$$\tilde{q}_{a,b,m,n}(\|z\|^2, w, \bar{w}) = \tilde{q}_{a,b,m,n}(1 - |w|^2, w, \bar{w}).$$

Set $Q_{a,b,m,n}(w, \bar{w}) = \tilde{q}_{a,b,m,n}(1 - |w|^2, w, \bar{w})$. Let $\{p_j\}$ denote an orthogonal basis for $H_{a,b}(\mathbb{S}^{2k-3})$. When $j \neq l$,

$$\begin{aligned}
&\langle M_{a,b,m,n}(p_j), M_{a,b,m,n}(p_l)\rangle \\
=&\int_{\mathbb{S}^{2k-1}} |q_{a,b,m,n}|^2 p_j \bar{p}_l d\sigma \\
=&\int_{\mathbb{S}^{2k-1}} |Q_{a,b,m,n}(w,\bar{w})|^2 p_j(z,\bar{z})\overline{p_l(z,\bar{z})} d\sigma.
\end{aligned}$$

Substituting $z_j = \sqrt{1-|w|^2} t_j$ yields

$$\begin{aligned}
&\int_{\mathbb{S}^{2k-1}} |Q_{a,b,m,n}(w,\bar{w})|^2 p_j(z,\bar{z})\bar{p}_l(z,\bar{z}) d\sigma \\
=&\int_{\mathbb{B}} |Q_{a,b,m,n}(w,\bar{w})|^2 dV(w) \int_{\mathbb{S}^{2k-3}} p_j(z,\bar{z})\bar{p}_l(z,\bar{z})(1-|w|^2)^{k-2} d\sigma(t) \\
=&\int_{\mathbb{B}} |Q_{a,b,m,n}|^2 dV(w) \int_{\mathbb{S}^{2k-3}} (1-|w|^2)^{k+a+b-2} p_j(t,\bar{t})\bar{p}_l(t,\bar{t}) d\sigma(t) \\
=&\int_{\mathbb{B}} |Q_{a,b,m,n}|^2 (1-|w|^2)^{k+a+b-2} dV(w) \int_{\mathbb{S}^{2k-3}} p_j \bar{p}_l d\sigma(t) \\
=&\int_{\mathbb{B}} |Q_{a,b,m,n}|^2 (1-|w|^2)^{k+a+b-2} dV(w) \langle p_j, p_l\rangle = 0.
\end{aligned}$$

Thus the mapping $M_{a,b,m,n}$ preserves orthogonality and we have

$$\dim\left(M_{a,b,m,n}\left(H_{a,b}\left(\mathbb{S}^{2k-1}\right)\right)\right) = \dim\left(H_{a,b}\left(\mathbb{S}^{2k-1}\right)\right).$$

Lemma 6.1 also implies that spaces $M_{a,b,m,n}(H_{a,b}(\mathbb{S}^{2k-3}))$ with different indices a, b are orthogonal to each other. It remains to check that

$$\dim\left(H_{m,n}\left(\mathbb{S}^{2k-1}\right)\right) = \sum_{a=0,b=0}^{m,n} \dim\left(H_{a,b}\left(\mathbb{S}^{2k-3}\right)\right). \tag{6.12}$$

Let $d_{a,b,k}$ denote the dimension of $P_{a,b}(\mathbb{S}^{2k-1})$, and let $h_{a,b,k}$ denote the dimension of $H_{a,b}(\mathbb{S}^{2k-1})$. Lemma 6.2 implies that

$$h_{a,b,k} = d_{a,b,k} - d_{a-1,b-1,k},$$

for $a \geq 1$ and $b \geq 1$. To prove (6.12), we use generating functions of $d_{a,b,k}$ and $h_{a,b,k}$. It is known that $d_{a,b,k}$ equals $\binom{a+k-1}{k-1}\binom{b+k-1}{k-1}$. Therefore,

$$\begin{aligned}
\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} d_{a,b,k} x^a y^b =& \left(\sum_{a=0}^{\infty}\binom{a+k-1}{k-1}x^a\right)\left(\sum_{b=0}^{\infty}\binom{b+k-1}{k-1}y^b\right) \\
=&\frac{1}{(1-x)^k(1-y)^k}.
\end{aligned}$$

Since $h_{a,b,k} = d_{a,b,k} - d_{a-1,b-1,k}$ for $a \geq 1$ and $b \geq 1$, we have

$$\begin{aligned}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} h_{a,b,k}x^a y^b =& \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}(d_{a,b,k} - d_{a-1,b-1,k})x^a y^b \\ =& \frac{1-xy}{(1-x)^k(1-y)^k}.\end{aligned}$$

Factoring out $\frac{1}{(1-x)(1-y)}$ from $\frac{1-xy}{(1-x)^k(1-y)^k}$ and expanding it as a series yield

$$\begin{aligned}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} h_{a,b,k}x^a y^b =& \frac{1-xy}{(1-x)^{k-1}(1-y)^{k-1}}\frac{1}{(1-x)(1-y)} \\ =& \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h_{r,s,k-1}x^r y^s\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j\right) \\ =& \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left(\sum_{j=0}^{a}\sum_{l=0}^{b} h_{j,l,k-1}\right) x^a y^b.\end{aligned}$$

Hence $h_{a,b,k} = \sum_{j=0}^{a}\sum_{l=0}^{b} h_{j,l,k-1}$ and the proof is complete. □

Remark 6.1. *In the argument above, we expand the factor* $\frac{1}{(1-x)(1-y)}$ *from* $\frac{1-xy}{(1-x)^k(1-y)^k}$ *and obtain the identity*

$$h_{a,b,k} = \sum_{j=0}^{a}\sum_{l=0}^{b} h_{j,l,k-1}d_{a-j,l-b,1},$$

with $d_{a-j,l-b,1} = 1$*. Similarly, for positive integer* r *satisfying* $r < k$*, we can expand the factor* $\frac{1}{(1-x)^r(1-y)^r}$ *from* $\frac{1-xy}{(1-x)^k(1-y)^k}$ *and have*

$$\begin{aligned}\frac{1-xy}{(1-x)^k(1-y)^k} =& \frac{1-xy}{(1-x)^{k-r}(1-y)^{k-r}}\frac{1}{(1-x)^r(1-y)^r} \\ =& \left(\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} h_{a,b,k-r}x^a y^b\right)\left(\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} d_{j,l,r}x^j y^l\right) \\ =& \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left(\sum_{j=0}^{a}\sum_{l=0}^{b} h_{j,l,k-r}d_{a-j,b-l,r}\right) x^m y^n,\end{aligned}$$

which implies that

$$h_{a,b,k} = \sum_{j=0}^{a}\sum_{l=0}^{b} h_{j,l,k-r}d_{a-j,b-l,r}.$$

This observation suggests the existence of mappings sending the orthogonal basis of $H_{a,b}(\mathbb{C}^{k-r})$ to $H_{m,n}(\mathbb{C}^k)$ and spanning the whole space. We'll see a higher dimensional analogue of Lemma 6.3 and Theorem 6.1 in the next section.

The polynomials $q_{a,b,m,n}$ in Lemma 6.3 are special functions and are interesting on their own. Theorems in the rest of this section demonstrate the relation of $q_{a,b,m,n}$ to Jacobi polynomials and zonal harmonics.

Theorem 6.2. *For $z \in \mathbb{C}^{n-1}$ and w, let $Q_{a,b,m,n}(w, \bar{w}) = q_{a,b,m,n}(z, \bar{z}, w, \bar{w})$ on $\mathbb{S}^{2k-1}$. Then for any fixed a and b, $\{Q_{a,b,m,n}\}_{m=a,n=b}^{\infty}$ are orthogonal polynomials for the space $L^2(\mathbb{D}, (1-|w|^2)^{k+a+b-2})$.*

Proof. Since the highest order term in $Q_{a,b,m,n}$ is $w^{m-a}\bar{w}^{n-b}$, $\{Q_{a,b,m,n}\}_{m,n}^{\infty}$ already forms a complete system for $P(\mathbb{C})$. It suffices to show that these polynomials are orthogonal to each other in the space $L^2(\mathbb{D}, (1-|w|^2)^{k+a+b-2})$. For $p \in H_{a,b}(\mathbb{S}^{2k-3})$, we let $M_{m,n}$ denote the mapping from $H_{a,b}(\mathbb{S}^{2k-3})$ to $H_{m,n}(\mathbb{S}^{2k-1})$:

$$M_{m,n}(p) = pq_{a,b,m,n}.$$

For simplicity, we just use the notation $q_{m,n}$ for $q_{a,b,m,n}$ and $Q_{m,n}$ for $Q_{a,b,m,n}$. By Theorem 6.1 and Lemma 6.1, $M_{m,n}(p) \in H_{m,n}(\mathbb{S}^{2k-1})$. Thus when $(m_1, n_1) \neq (m_2, n_2)$, we have:

$$\begin{aligned}
0 =& \langle M_{m_1,n_1}(p), M_{m_2,n_2}(p)\rangle \\
=& \int_{\mathbb{S}^{2k-1}} |p|^2 q_{m_1,n_1}\overline{q_{m_2,n_2}} d\sigma \\
=& \int_{\mathbb{S}^{2k-1}} |p|^2 Q_{m_1,n_1}\overline{Q_{m_2,n_2}} d\sigma.
\end{aligned}$$

Substituting $z_i = \sqrt{1-|w|^2}t_i$ yields

$$\begin{aligned}
&\int_{\mathbb{S}^{2k-1}} |p|^2 Q_{m_1,n_1}\overline{Q_{m_2,n_2}} d\sigma \\
=& \int_{\mathbb{B}} Q_{m_1,n_1}\overline{Q_{m_2,n_2}} \int_{\mathbb{S}^{2k-3}} |p(z,\bar{z})|^2 (1-|w|^2)^{k-2} d\sigma(t) d\sigma(w) \\
=& \int_{\mathbb{B}} Q_{m_1,n_1}\overline{Q_{m_2,n_2}} \int_{\mathbb{S}^{2k-3}} (1-|w|^2)^{k+a+b-2} |p(t,\bar{t})|^2 d\sigma(t) d\sigma(w) \\
=& \int_{\mathbb{B}} Q_{m_1,n_1}\overline{Q_{m_2,n_2}} (1-|w|^2)^{k+a+b-2} d\sigma(w) \int_{\mathbb{S}^{2k-3}} |p|^2 d\sigma(t) \\
=& \int_{\mathbb{B}} Q_{m_1,n_1}\overline{Q_{m_2,n_2}} (1-|w|^2)^{k+a+b-2} d\sigma(w) \langle p, p\rangle. \qquad (6.13)
\end{aligned}$$

Noting that $\langle p, p\rangle \neq 0$, we have

$$\int_{\mathbb{B}} Q_{m_1,n_1}\bar{Q}_{m_2,n_2}(1-|w|^2)^{k+a+b-2}d\sigma(w) = 0.$$

Thus $Q_{m,n}$'s are orthogonal in $L^2(\mathbb{B},(1-|w|^2)^{k+a+b-2})$. $\square$

The polynomials $Q_{m,n}$ form an orthogonal system for the function space $L^2(\mathbb{B},(1-|w|^2)^{k+a+b-2}d\sigma(w))$. The following corollary shows that these polynomials are related to orthogonal polynomials:

Corollary 6.1. *Let $J^n_{a,b}$ be the Jacobi polynomial of degree n with the pair (a,b) such that $J^n_{a,b}(1) = 1$; i.e. $\{J^n_{a,b}\}_{n=0}^{\infty}$ is an orthogonal basis for the real function space $L^2([-1,1],(1-x)^a(1+x)^b dx)$. Then we have:*

$$Q_{m,n} = \begin{cases} w^{m-a-n+b}J^n_{k+a+b-2,m-a-n+b}(2|w|^2-1) & m-a \geq n-b \\ \bar{w}^{n-b-m+a}J^m_{k+a+b-2,n-b-m+a}(2|w|^2-1) & m-a < n-b \end{cases} \tag{6.14}$$

Proof. We provide the proof for $m-a \geq n-b$. The proof for the other case is similar. Let $j = m-a-n+b$ and we consider the integral

$$\int_{\mathbb{B}} Q_{n_1+j,n_1}\overline{Q_{n_2+j,n_2}}(1-|w|^2)^{k+a+b-2}d\sigma(w). \tag{6.15}$$

Applying (6.7) of Lemma 6.3 yields:

$$Q_{n+j,n} = w^j\sum_{i=0}^{n-b} c_i|w|^{2i}(1-|w|^2)^i. \tag{6.16}$$

Set $p_n(|w|^2)$ equal to $\sum_{i=0}^{n-b} c_i|w|^{2i}(1-|w|^2)^i$. By substituting (6.16) to (6.15) and using polar coordinates, the integral (6.15) becomes:

$$\int_0^1\int_0^{2\pi} p_{n_1}(r^2)p_{n_2}(r^2)(1-r^2)^{k+a+b-2}r^{2j+1}d\theta dr. \tag{6.17}$$

Substituting $t = 2r^2-1$ to (6.17), we obtain:

$$2\pi\int_{-1}^1 p_{n_1}\left(\frac{t+1}{2}\right)p_{n_2}\left(\frac{t+1}{2}\right)\left(\frac{t-1}{2}\right)^{k+a+b-2}\left(\frac{t+1}{2}\right)^j dt. \tag{6.18}$$

When $n_1 \neq n_2$, the integral (6.15) vanishes. Hence (6.18) also equals zero. Since $p_n(\frac{t+1}{2})$ is a polynomial of degree n, the set $\{p_n(\frac{t+1}{2})\}$ forms an orthogonal basis for $L^2([-1,1],(1-x)^a(1+x)^b dx)$ and $p_n(\frac{t+1}{2}) = c_nJ^n_{k+a+b-2,j}(t)$.

Note that $p_n(\frac{1+1}{2}) = c_0 = 1$ and $J^n_{k+a+b-2,j}(1) = 1$. We have $c_n = 1$ for all n and therefore (6.14) holds. $\square$

When both $a = 0$ and $b = 0$, the polynomial $q_{0,0,m,n}$ is in $H_{m,n}(\mathbb{S}^{2k-1})$. Set $\tilde{q}_{m,n} = q_{0,0,m,n}$. Lemma 6.1 implies that $\tilde{q}_{m,n}$ only depends on w, $\bar{w}$ and $|z|^2$ and uniquely determined by m and n. On the other hand, w, $\bar{w}$ and $|z|^2$ are the lowest order elements in $P(\mathbb{C}^k)$ that are invariant under the unitary group $U(k-1)(z)$. Hence up to a constant coefficient, $\tilde{q}_{m,n}$ is the unique $U(k-1)(z)$ invariant element in $H_{m,n}(\mathbb{S}^{2k-1})$. The next theorem shows that, up to a constant factor, $q_{0,0,m,n}$ equals the zonal harmonics of $H_{m,n}(\mathbb{S}^{2k-1})$ valued at certain point.

Let $\{s_j\}$ denote an orthonormal basis for $H_{m,n}(\mathbb{S}^{2k-1})$. The zonal harmonics $S_{m,n}(\hat{z}, \zeta)$ is the function satisfying:

$$S_{m,n}(\hat{z}, \zeta) = \sum_{j=1}^{h_{m,n,k}} s_j(\hat{z})\overline{s_j(\zeta)}, \tag{6.19}$$

where $\hat{z}$, $\zeta \in \mathbb{C}^k$ with $\hat{z} = (z, w)$. We show below that $S_{m,n}$ is well-defined and invariant under the unitary group $U(k)(\hat{z})$.

Lemma 6.5. *$S_{m,n}(\hat{z}, \zeta)$ is invariant under the unitary group $U(k)(\hat{z})$.*

Proof. First, we prove that $S_{m,n}(\hat{z}, \zeta)$ is well-defined. For any orthonormal basis chosen, the series in (6.19) corresponds to the projection map from $L^2(\mathbb{S}^{2k-1})$ to $H_{m,n}(\mathbb{S}^{2k-1})$: for any $f \in L^2(\mathbb{S}^{2k-1})$,

$$\int_{\mathbb{S}^{2k-1}} \sum_{j=1}^{h_{m,n,k}} s_j(\hat{z})\overline{s_j(\zeta)} f(\zeta) d\sigma(\zeta) = \sum_{j=1}^{h_{m,n,k}} \langle f, s_j \rangle s_j(\hat{z}). \tag{6.20}$$

Thus $S_{m,n}(\hat{z}, \zeta)$ is independent of the choice of orthonormal basis. It suffices to show that any $T \in U(k)(\hat{z})$ send orthonormal basis to orthonormal basis. Let $\{s_j(\zeta)\}_{j=1}^{h_{m,n,k}}$ be an orthonormal basis for $H_{m,n}(\mathbb{S}^{2k-1})$. $\{s_j(T\zeta)\}_{j=1}^{h_{m,n,k}}$.

$$\begin{aligned}
\langle s_j(T\zeta), s_l(T\zeta) \rangle &= \int_{\mathbb{S}^{2k-1}} s_j(T\zeta)\overline{s_l(T\zeta)} d\sigma(\zeta) \\
&= \int_{\mathbb{S}^{2k-1}} s_j(T\zeta)\overline{s_l(T\zeta)} d\sigma(T\zeta) \\
&= \int_{\mathbb{S}^{2k-1}} s_j(\zeta)\overline{s_l(\zeta)} d\sigma(\zeta) = \langle s_j, s_l \rangle.
\end{aligned}$$

Hence, for any $T \in U(k)(\hat{z})$, functions $s_j(T\zeta)$ still form an orthonormal basis. By the well-definedness of $S_{m,n}$, we have:

$$S_{m,n}(\hat{z},\zeta) = \sum_{j=1}^{h_{m,n,k}} s_j(T\hat{z})\overline{s_j(T\zeta)} = S_{m,n}(T(\hat{z}),T(\zeta)).$$

$\square$

Theorem 6.3. *Let* $\{s_j(z,w)\}_{j=1}^{h_{m,n,k}}$ *be an arbitrary orthonormal basis for* $H_{m,n}(\mathbb{S}^{2k-1})$*, then*

$$\tilde{q}_{m,n} = c_{m,n,k}\sum_{j=1}^{h_{m,n,k}} s_j(z,w)\overline{s_j(0,1)}, \tag{6.21}$$

where $c_{m,n,k} = \frac{\sigma(\mathbb{S}^{2k-1})}{h_{m,n,k}}$.

Proof. We first show that the right hand side of (6.21) is invariant under $U(k-1)(z)$. Since $U(k-1)(z) \subseteq U(k)(\hat{z})$, we have for any $T \in U(k-1)(z)$,

$$\tilde{q}_{m,n}(z,w) = \sum_{j=1}^{h_{m,n,k}} s_j(z,w)\overline{s_j(0,1)} = \sum_{j=1}^{h_{m,n,k}} s_j(Tz,w)\overline{s_j(0,1)} = \tilde{q}_{m,n}(Tz,w).$$

To compute $c_{m,n,k}$, we consider $\tilde{Q}_{m,n} := Q_{0,0,m,n}$. Since $\tilde{Q}_{m,n}$ equals $\tilde{q}_{m,n}$ on $\mathbb{S}^{2k-1}$, we have on $\mathbb{S}^{2k-1}$ that:

$$\tilde{Q}_{m,n}(w) = \tilde{q}_{m,n}(z,w) = c_{m,n,k}\sum_{j=1}^{h_{m,n,k}} s_j(z,w)\overline{s_j(0,1)}. \tag{6.22}$$

For fixed $\zeta \in \mathbb{S}^{2k-1}$, let T be a unitary transformation in $U(k)(\hat{z})$ such that $T\zeta = (0,1)$. Then for any $\hat{z} \in \mathbb{S}^{2k-1}$, there exists a $\theta \in \mathbb{C}^{k-1}$ such that $T^{-1}(\theta, \langle \hat{z},\zeta\rangle) = (z,w) = \hat{z}$. Since $\hat{z} \in \mathbb{S}^{2k-1}$, we have $(\theta,\langle\hat{z},\zeta\rangle) \in \mathbb{S}^{2k-1}$ and $||\theta||^2 = 1 - |\langle\hat{z},\zeta\rangle|^2||\theta||^2$. Therefore on $\mathbb{S}^{2k-1}$,

$$\begin{aligned}
\tilde{Q}_{m,n}(\langle\hat{z},\zeta\rangle) &= c_{m,n,k}\sum_{j=1}^{h_{m,n,k}} s_j\left(\theta,\langle\hat{z},\zeta\rangle\right)\overline{s_j(0,1)} \\
&= c_{m,n,k}\sum_{j=1}^{h_{m,n,k}} s_j\left(T^{-1}(\theta,\langle\hat{z},\zeta\rangle)\right)\overline{s_j\left(T^{-1}(0,1)\right)} \\
&= c_{m,n,k}\sum_{j=1}^{h_{m,n,k}} s_j(z,w)\overline{s_j(\zeta)} = c_{m,n,k}\sum_{j=1}^{h_{m,n,k}} s_j(\hat{z})\overline{s_j(\zeta)}.
\end{aligned} \tag{6.23}$$

Setting $\hat{z} = \zeta$ and integrating both sides of (6.23) on $\mathbb{S}^{2k-1}$ yields

$$\begin{aligned}\sigma\left(\mathbb{S}^{2k-1}\right) =& \int_{\mathbb{S}^{2k-1}} \tilde{Q}_{m,n}(1) d\sigma(\zeta) \\ =& \int_{\mathbb{S}^{2k-1}} c_{m,n,k} \sum_{j=1}^{h_{m,n,k}} |s_j(\zeta)|^2 d\sigma(\zeta) \\ =& c_{m,n,k} h_{m,n,k}.\end{aligned}$$

Therefore $c_{m,n,k} = \frac{\sigma(\mathbb{S}^{2k-1})}{h_{m,n,k}}$. □

6.3 A higher dimensional analogue

Corollary 6.2. *Let r be a positive integer satisfying $r < k$. Let $M_{a,b,r,\alpha,\beta}$ denote a map sending orthogonal basis of each $H_{a,b}(\mathbb{S}^{2k-1-2r})$ to orthogonal set of $H_{m,n}(\mathbb{S}^{2k-1})$. Then we have:*

$$H_{m,n}\left(\mathbb{S}^{2k-1}\right) = \bigoplus_{a=0,b=0}^{m,n} \bigoplus_{||\alpha||=m-a,||\beta||=n-b} M_{a,b,r,\alpha,\beta}\left(H_{a,b}\left(\mathbb{S}^{2k-2r-1}\right)\right) \tag{6.24}$$

Proof. By Remark 6.1, the dimension of spaces on both sides of (6.24) are the same. Let $(z, w) \in \mathbb{C}^{k-r} \times \mathbb{C}^r$. For every $p \in H_{a,b}(\mathbb{S}^{2k-2r-1})$ we let $M_{a,b,r,\alpha,\beta}(p) = p(z, \bar{z})q(z, \bar{z}, w, \bar{w})$ where $p(z, \bar{z})q(z, \bar{z}, w, \bar{w})$ is the harmonization of $w^\alpha \bar{w}^\beta p(z, \bar{z})$. To see this harmonization exists for the multivariable case, we just need to repeat the process of Lemma 1 several times. □

Remark 6.2. *Note that when $r = k - 1$, we have*

$$H_{a,0}(\mathbb{S}^1) = P_{a,0}(\mathbb{S}^1), \quad H_{0.b}\left(\mathbb{S}^1\right) = P_{0,b}\left(\mathbb{S}^1\right),$$

and $H_{a,b}(\mathbb{S}^1) = \emptyset$ when both a and b are not zero. If we start from $H_{a,b}(\mathbb{S}^1)$ and consider the mapping $M_{a,b,k-1,\alpha,\beta}$, then the proof of Corollary 6.2 suggest an explicit basis for $H_{m,n}(\mathbb{S}^{2k-1})$. Such a basis and the decomposition (6.24) was first given by Ikeda and Keyama in [IK67], where they obtained (6.7) in Lemma 6.4 using special coordinates and the method of separation of variables. Later, Koornwinder gave a different proof for (6.24) using special functions and obtained an addition formula for Jacobi polynomials. In both their approaches, the explicit basis they obtained depends on the which order new variables w_j were added.

For example, when $p(z, \bar{z}) = z \in H_{1,0}(\mathbb{S}^1)$ and $w^\alpha \bar{w}^\beta = |w_1|^2|w_2|^2 w_2$, we *have two different mappings that send $p(z,\bar{z})$ to the space $H_{4,2}(\mathbb{S}^5)$:*

$$z \longrightarrow z(|w_1|^2 - \tfrac{1}{2}|z|^2) \longrightarrow zw_2(|w_1|^2 - \tfrac{1}{2}|z|^2)\left(|w_2|^2 - \tfrac{1}{2}(|z|^2 + |w_1|^2)\right).$$

$$z \longrightarrow zw_2(|w_2|^2 - |z|^2) \longrightarrow zw_2(|w_2|^2 - |z|^2)\left(|w_1|^2 - \tfrac{1}{5}(|z|^2 + |w_2|^2)\right).$$

The mapping in upper line sends z to the space $H_{2,1}(\mathbb{S}^3)$ with the new variable w_1 first and then to the space $H_{4,2}(\mathbb{S}^5)$ with the new variable w_2. The mapping in lower line sends z to the space $H_{3,1}(\mathbb{S}^3)$ with the new variable w_2 first and then to the space $H_{4,2}(\mathbb{S}^5)$ with the new variable w_1.

A construction of an "order-invariant" $M_{a,b,r,\alpha,\beta}$ could be done by an analogue of Theorem 6.1 and Lemma 6.3. Such a map directly sends the element of $H_{a,b}(\mathbb{S}^{2k-2r-1})$ into $H_{m,n}(\mathbb{S}^{2k-1})$ without passing through the intermediate spaces. The following lemma is a higher dimensional analogue of Lemma 6.3.

Lemma 6.6. *Let $z \in \mathbb{C}^{k-r}$ and $w \in \mathbb{C}^r$. For $w^\alpha \bar{w}^\beta \in P_{m-a,n-b}(\mathbb{C}^r)$ and $p(z,\bar{z}) \in H_{a,b}(\mathbb{S}^{2k-3})$, there exists a unique sequence $\{c_{\mathbf{a},\mathbf{b}}\}$ starting with $c_{\mathbf{0},\mathbf{0}} = 1$ such that the polynomial*

$$p(z,\bar{z}) \sum_{i=0}^{\min\{m-a,n-b\}} \sum_{|\mathbf{a}|=i,|\mathbf{b}|=i} c_{\mathbf{a},\mathbf{b}} w^{\alpha-\mathbf{a}} \bar{w}^{\beta-\mathbf{b}} \|z\|^{2i} \tag{6.25}$$

is in $H_{m,n}(\mathbb{S}^{2k-1})$ whenever $p(z,\bar{z}) \in H_{a,b}(\mathbb{S}^{2k-3})$.

Proof. It suffices to prove that there exists $c_{\mathbf{a},\mathbf{b}}$'s such that $\triangle(q_{a,b,\alpha,\beta}p) = 0$. Set $\triangle_z = \sum_{j=1}^{k-r} \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}$ and $\triangle_w = \sum_{j=1}^{r} \frac{\partial}{\partial w_j}\frac{\partial}{\partial \bar{w}_j}$. Then

$$\begin{aligned}
&\triangle\left(q_{a,b,\alpha,\beta}p\right)\\
&= \triangle_z\left(q_{a,b,\alpha,\beta}p\right) + \triangle_w\left(q_{a,b,\alpha,\beta}p\right)\\
&= p\triangle_z q_{a,b,\alpha,\beta} + \sum_{t=1}^{k-r}\left(\frac{\partial}{\partial z_t}q_{a,b,\alpha,\beta}\frac{\partial}{\partial \bar{z}_t}p + \frac{\partial}{\partial \bar{z}_t}q_{a,b,\alpha,\beta}\frac{\partial}{\partial z_t}p\right) + p\triangle_w q_{a,b,\alpha,\beta}\\
&= p\triangle q_{a,b,\alpha,\beta} + \sum_{t=1}^{k-r}\left(\frac{\partial}{\partial z_t}q_{a,b,\alpha,\beta}\frac{\partial}{\partial \bar{z}_t}p + \frac{\partial}{\partial \bar{z}_t}q_{a,b,\alpha,\beta}\frac{\partial}{\partial z_t}p\right).
\end{aligned} \tag{6.26}$$

Set (i) $= p \bigtriangleup q_{a,b,\alpha,\beta}$. Set (ii) equal to the sum in (6.26). Then

$$\begin{aligned}\text{(i)} =& p \sum_{i=0}^{\min\{m-a,n-b\}} \sum_{|\mathbf{a}|=i,|\mathbf{b}|=i} \Big(i(i+k-2) c_{\mathbf{a},\mathbf{b}} w^{\alpha-\mathbf{a}} \bar{w}^{\beta-\mathbf{b}} \|z\|^{2i-2} \\ & + \sum_{j=1}^{r} (\alpha_j - a_j)(\beta_j - b_j) c_{\mathbf{a},\mathbf{b}} w^{\alpha-\mathbf{a}-\mathbf{1}_j} \bar{w}^{\beta-\mathbf{b}-\mathbf{1}_j} \|z\|^{2i} \Big) \\ =& p \sum_{i=1}^{\min\{m-a,n-b\}} \sum_{|\mathbf{a}|=i,|\mathbf{b}|=i} C_{\mathbf{a},\mathbf{b}} w^{\alpha-\mathbf{a}} \bar{w}^{\beta-\mathbf{b}} \|z\|^{2i-2},\end{aligned}$$

where $C_{\mathbf{a},\mathbf{b}} = i(i+k-2)c_{\mathbf{a},\mathbf{b}} + \sum_{j=1}^{r} (\alpha_j - a_j + 1)(\beta_j - b_j + 1) c_{\mathbf{a}-\mathbf{1}_j,\mathbf{b}-\mathbf{1}_j}$. For (ii), we have

$$\text{(ii)} = p \sum_{i=1}^{\min\{m-a,n-b\}} \sum_{|\mathbf{a}|=i,|\mathbf{b}|=i} i(a+b) c_{\mathbf{a},\mathbf{b}} w^{\alpha-\mathbf{a}} \bar{w}^{\beta-\mathbf{b}} \|z\|^{2i-2}.$$

Hence $\bigtriangleup(q_{a,b,\alpha,\beta} p) = 0$ if and only if $\{c_{\mathbf{a},\mathbf{b}}\}$ satisfies:

$$c_{\mathbf{a},\mathbf{b}} = -\frac{\sum_{j=1}^{r} (\alpha_j - a_j + 1)(\beta_j - b_j + 1) c_{\mathbf{a}-\mathbf{1}_j,\mathbf{b}-\mathbf{1}_j}}{i(i+a+b+k-2)}. \tag{6.27}$$

Formula (6.27) uniquely determine the $\{c_{\mathbf{a},\mathbf{b}}\}$ given that $c_{\mathbf{0},\mathbf{0}} = 1$. $\square$

Let $q_{a,b,\alpha,\beta}$ denote the polynomial

$$\sum_{i=0}^{\min\{m-a,n-b\}} \sum_{|\mathbf{a}|=i,|\mathbf{b}|=i} c_{\mathbf{a},\mathbf{b}} w^{\alpha-\mathbf{a}} \bar{w}^{\beta-\mathbf{b}} \|z\|^{2i}. \tag{6.28}$$

Then $M_{a,b,\alpha,\beta}(p) := p q_{a,b,\alpha,\beta}$ defines a mapping from the space $H_{a,b}(\mathbb{S}^{2k-2r-1})$ to the space $H_{m,n}(\mathbb{S}^{2k-1})$. The following corollary shows that such mapping induces a decomposition for $H_{m,n}(\mathbb{S}^{2k-1})$.

Corollary 6.3. *Let $M_{a,b,\alpha,\beta}$ as above. Then the mapping $M_{ab,\alpha,\beta}$ preserves the orthogonality. Moreover,*

$$H_{m,n}\left(\mathbb{S}^{2k-1}\right) = \bigoplus_{a=0,b=0}^{m,n} \bigoplus_{|\alpha|=m-a,|\beta|=n-b} M_{a,b,\alpha,\beta}\left(H_{a,b}\left(\mathbb{S}^{2k-2r-1}\right)\right). \tag{6.29}$$

Proof. By the Remark 6.1, we have

$$\dim\left(H_{m,n}\left(\mathbb{S}^{2k-1}\right)\right) = \sum_{a=0,b=0}^{m,n} \sum_{|\alpha|=m-a,|\beta|=n-b} \dim\left(H_{a,b}\left(\mathbb{S}^{2k-2r-1}\right)\right).$$

It suffices to show that elements in spaces $M_{a,b,\alpha,\beta}\big(H_{a,b}(\mathbb{S}^{2k-2r-1})\big)$ are linearly independent.

For $p_j, p_l \in \bigoplus_{a=0,b=0}^{m,n} H_{a,b}(\mathbb{S}^{2k-2r-1})$ such that $\langle p_j, p_l\rangle = 0$ in $L^2(\mathbb{S}^{2k-2r-1})$, similar argument in the proof of Theorem 6.1 shows that in $L^2(\mathbb{S}^{2k-1})$

$$\langle M_{a,b,\alpha,\beta}(p_j), M_{a,b,\alpha,\beta}(p_l)\rangle = 0.$$

Therefore $M_{a,b,\alpha,\beta}$ preserve the orthogonality of the elements in $H_{a,b}(\mathbb{S}^{2k-2r-1})$, and $M_{a,b,\alpha,\beta}\left(H_{a_1,b_1}(\mathbb{S}^{2k-2r-1})\right)$ is orthogonal to $M_{a,b,\alpha,\beta}\left(H_{a_2,b_2}(\mathbb{S}^{2k-2r-1})\right)$ when $(a_1, b_1) \neq (a_2, b_2)$.

The highest order term of $q_{a,b,\alpha,\beta}$ in w and $\bar{w}$ is $w^\alpha \bar{w}^\beta$. Polynomials $q_{a,b,\alpha_1,\alpha_2}$ and $q_{a,b,\alpha_1,\alpha_2}$ are linearly independent for $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$. Thus elements in $M_{a,b,\alpha_1,\beta_1}\left(H_{a_1,b_1}(\mathbb{S}^{2k-2r-1})\right)$ and $M_{a,b,\alpha_2,\beta_2}\left(H_{a_1,b_1}(\mathbb{S}^{2k-2r-1})\right)$ are also linearly independent and (6.29) holds. $\square$

Remark 6.3. *Unlike in Theorem 6.1, the spaces* $M_{a,b,\alpha,\beta}\left(H_{a,b}(\mathbb{S}^{2k-2r-1})\right)$ *with different* α *and* β *are not necessarily orthogonal to each other. For example, when* $r \geq 4$*, the polynomials* q_{a,b,α_1,β_1} *and* q_{a,b,α_2,β_2} *with leading term* $w_1\bar{w}_2$ *and* $w_3\bar{w}_4$ *are not orthogonal in* $L^2(\mathbb{B}^r, (1-\|w\|^2)^{k+a+b-1-r}dV)$*. Therefore* $pq_{a,b,\alpha_1,\beta_1}$ *is not orthogonal to* $pq_{a,b,\alpha_2,\beta_2}$ *in* $L^2(\mathbb{S}^{2k-1})$*.*

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