FINITE ORDER AUTOMORPHISMS OF HIGGS BUNDLES: THEORY AND APPLICATION

BY

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DISSERTATION

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Abstract

We study the $\mathbb{C}^*$ action on the moduli space of G-Higgs bundles. Focus is especially put on Higgs bundle which are not fixed points of the whole action but are fixed by a roots of unity subgroup of $\mathbb{C}^*$. When G is a complex simple Lie group, we classify these "cyclic Higgs bundles". One main property of cyclic Higgs bundles is that the corresponding equivariant harmonic map to the symmetric space of G admits a canonical harmonic (in fact minimal) lift to a homogeneous space fibering over the symmetric space. In terms of the hermitian metric solving the Hitchin equations, such a lift implies extra symmetries of the solution metric. Such properties were first studied by Baraglia [Bar10] for Higgs bundles in the Hitchin component which are fixed by an n^{th} roots of unity action. The extra symmetries of the metric are used to study the asymptotics of Higgs bundles in the Hitchin component along certain rays. This analysis allows us to partially understand the asymptotic holonomy of certain families of Hitchin representations.

For G a complex simple Lie group and m_ℓ the length of the longest root of the Lie algebra $\mathfrak{g}$ of G, the lifted harmonic maps associated to a fixed point of the $(m_\ell+1)$-roots of unity are study in detail. When such fixed points which arise from a G_0-Higgs bundle, where G_0 is the split real form of G, these lifted maps satisfy an additional "reality" symmetry. For these equivariant harmonic maps we prove a rigidity result generalizing Labourie's work in [Lab14]. We build on the work of [BGPG12] to parameterize the connected components of maximal $\mathsf{PSp}(4,\mathbb{R}) = \mathsf{SO}_0(2,3)$-Higgs bundles which contain fixed points of the 4^{th}-roots of unity action as the product of a vector bundle over a symmetric product of the surface with the vector space of holomorphic quadratic differentials. Generalizing Labourie's work [Lab14] on the Hitchin component, the rigidity results above yield a unique "preferred" Riemann surface structure to each maximal $\mathsf{SO}_0(2,3)$-representation. As a consequence, we obtain a mapping class group invariant parameterization of the $4g-3$-connected components (which we call Gothen components) of maximal $\mathsf{SO}_0(2,3)$-representations which contain fixed points. Finally, we generalize our parameterization of the Gothen components to provide $n(2g-2)$ connected components of the $\mathsf{SO}_0(n,n+1)$-Higgs bundle moduli space which generalize the $\mathsf{SO}_0(n,n+1)$-Hitchin component. When $n \geq 3$, this is the first example of non-maximal and non-Hitchin connected components which are not labeled by a topological invariant in $\pi_1(\mathsf{G})$.

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Chapter 1

Overview and statement of results

For a closed surface S with genus at least two, the nonabelian Hodge correspondence asserts that, for each Riemann surface structure Σ on S, there is a homeomorphism between the moduli space $\mathcal{M}(\mathsf{G})$ of G-Higgs bundles on Σ and the moduli space $\mathcal{X}(\pi_1(S), \mathsf{G})$ of reductive representations of the fundamental group of S in G. As illuminated by Hitchin [Hit87a, Hit87b], Simpson [Sim88], and others, the rich geometric and algebraic structures on the Higgs bundle moduli space provide a distinctive set of tools for studying the topology of $\mathcal{M}(\mathsf{G})$, and thus, through the nonabelian Hodge correspondence, the topology of $\mathcal{X}(\pi_1(S), \mathsf{G})$. One such geometric aspect is the natural $\mathbb{C}^*$-action on $\mathcal{M}(\mathsf{G})$ defined by scaling the Higgs field. The fixed points of this action correspond to critical points of a Morse function on $\mathcal{M}(\mathsf{G})$, and a proper analysis of these fixed points yields information on the cohomology of $\mathcal{M}(\mathsf{G})$. This thesis is dedicated to the study of Higgs bundles which have nontrivial stabilizer for the $\mathbb{C}^*$ action, i.e. fixed points of roots of unity subgroups $\langle \zeta_k \rangle \subset \mathbb{C}^*$.

Since Higgs bundles were introduced in 1987 [Hit87a], they have found application in parameterizing connected components of surface group representations. In particular, Hitchin gave an explicit parameterization of all but one of the connected components of $\mathcal{X}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))$ (these are the components with nonzero Toledo invariant), as vector bundles over symmetric products of a Riemann surface. After this ground breaking work, Hitchin showed that, for G a split real form such as $\mathsf{PSL}(n, \mathbb{R}), \mathsf{PSp}(2n, \mathbb{R})$, or $\mathsf{SO}_0(n, n+1)$, there is a connected component of $\mathcal{X}(\pi_1, \mathsf{G})$ which directly generalizes Teichmüller space [Hit92]. Moreover, Hitchin gave an explicit parameterization of this connected component, now called the Hitchin or Hitchin-Teichmüller component. More precisely, he showed that for each Riemann surface structure on the topological surface S, the Hitchin component $\mathsf{Hit}(\mathsf{G})$ is parameterized by a vector space of holomorphic differentials which generalizes Wolf's parameterization of Teichmüller space by Hopf differentials of harmonic maps [Wol89].

The effective tools of Higgs bundles however come at a cost: since they require fixing a Riemann surface structure, they break the mapping class group action on $\mathcal{X}(\pi_1(S), \mathsf{G})$. Using the properties we establish for finite order fixed points, we use Higgs bundles to provide a mapping class group invariant parameterization of certain connected components of maximal $\mathsf{PSp}(4,\mathbb{R})$ representations as fiber bundles over Teichmüller space. Generalizing this to higher rank, we also give a Higgs bundle parameterization of $n(2g-2)$ smooth connected components of $\mathcal{X}(\pi_1(S),\mathsf{SO}_0(n,n+1))$, one of which is $\mathsf{Hit}(\mathsf{SO}_0(n,n+1))$. For $n=1$, we recover Hitchin's parameterization for $\mathsf{SO}_0(1,2)=\mathsf{PSL}(2,\mathbb{R})$, for $n=2$ these are the maximal $\mathsf{PSp}(4,\mathbb{R})=\mathsf{SO}_0(2,3)$ components discussed above, and for $n\geq 3$, these provides $n(2g-2)-1$ new connected components.

Since the techniques and results that follow concern the interplay between Lie theory, Higgs bundles, harmonic map theory, and surface group representations, we have devoted Chapter 2 and 3 to a lengthy introduction to the relevant background. We hope this is beneficial to anyone interested in learning the subject. For a complex Lie group G a Higgs bundle consists of a pair $(\mathcal{E},\varphi)$ where $\mathcal{E}\to\Sigma$ is a holomorphic principal G-bundle and φ is a holomorphic section of the adjoint bundle twisted by the canonical bundle K of Σ, i.e. $\varphi\in H^0((\mathcal{E}\times_{\mathsf{G}}\mathfrak{g})\otimes K)$. The natural $\mathbb{C}^*$-action on the space of Higgs bundles is defined $(\mathcal{E},\varphi)\to(\mathcal{E},\lambda\varphi)$. In chapter 4, for complex group simple groups G, we classify the Higgs bundles which are fixed by a root of unity subgroup of $\mathbb{C}^*$ in terms of labellings of the extended Dynkin diagram of the Lie algebra $\mathfrak{g}$ (Theorem 4.2.2). For the group $\mathsf{SL}(n,\mathbb{C})$ this was done by Simpson in [Sim09], however we will see that generalizing this work to groups that are not of type A is subtle and requires some care. One key observation which lead to this classification was a reinterpretation of the well known classification of fixed points of the whole $\mathbb{C}^*$-action as coming from $\mathbb{Z}$-gradings on the Lie algebra $\mathfrak{g}$. As is the case of $\mathbb{C}^*$-fixed points, a Higgs bundle $(\mathcal{E},\varphi)$ which is fixed by a roots of unity action admits a holomorphic reduction of $\mathcal{E}$ to a subgroup G_0. However, unlike fixed points of the $\mathbb{C}^*$ action, when $\mathsf{G}\neq\mathsf{SL}(n,\mathbb{C})$, the subgroup G_0 need not be the Levi factor of a parabolic subgroup of G.

Using the standard representation of $\mathsf{SL}(n,\mathbb{C})$ on $\mathbb{C}^n$, an $\mathsf{SL}(n,\mathbb{C})$ Higgs bundle over a Riemann surface Σ is equivalent to a pair $(\mathcal{E},\phi)$ where $\mathcal{E}\to\Sigma$ is a holomorphic rank n vector bundle with $det(\mathcal{E})=\mathcal{O}$ and ϕ is a traceless holomorphic section of $\mathsf{End}(\mathcal{E})\otimes K$. One direction of the nonabelian Hodge correspondence is provided by the relation of stability of Higgs bundles and solutions to the gauge theoretic Higgs bundle equations. For $\mathsf{SL}(n,\mathbb{C})$ this works as follows: if $(\mathcal{E},\phi)$ is a stable Higgs bundle then there exists a unique hermitian metric h (with Chern connection A_h) on $\mathcal{E}$ so that

$$F_{A_h}+[\phi,\phi^{*_h}]=0, \tag{1.0.1}$$

here ϕ^{*_h} denotes the hermitian adjoint of ϕ. Given a solution to (1.0.1), the connection $A_h+\phi+\phi^{*_h}$ is a flat $\mathsf{SL}(n,\mathbb{C})$-connection whose holonomy representation is reductive. The other direction of the correspondence concerns harmonic metrics on flat G bundles, and is provided Corlette's Theorem [Cor88]. Given a representations $\rho : \pi_1(S)\to\mathsf{G}$, any metric on the flat G-bundle $\widetilde{S}\times_\rho \mathsf{G}$ (i.e. reduction of structure group to the maximal compact subgroup $\mathsf{H}\subset\mathsf{G}$) is equivalent to a ρ-equivariant map $\widetilde{S}\to\mathsf{G}/\mathsf{H}$. If the conjugacy class of ρ defines a point in $\mathcal{X}(\pi_1(S),\mathsf{G})$, then Corlette proved that, for each Riemann surface structure Σ on S, there is a metric $h_\rho : \widetilde{\Sigma}\to\mathsf{G}/\mathsf{H}$ which is a *harmonic* map.

The rest of chapter 4 is devoted to studying the consequences of a G-Higgs bundle being a fixed point of a root of unity action. The first such application is that the metric solving the Hitchin equations for stable and simple G-Higgs bundles is compatible the holomorphic reduction to G_0. In other words, we have the following commuting diagram of reductions of structure:

$$\begin{tikzcd}
\mathsf{G}/\mathsf{G}_0 & \mathsf{G}/\mathsf{H}_0 \arrow[l] \arrow[d] \\
\widetilde{\Sigma} \arrow[u] \arrow[ur] \arrow[r, "h_\rho"'] & \mathsf{G}/\mathsf{H}
\end{tikzcd}$$

where $\mathsf{H}_0 = \mathsf{G}_0\cap\mathsf{H}$. In section 4.3, polystable G-Higgs bundles which are fixed by $\langle\zeta_k\rangle$ are interpreted in terms of harmonic maps $f : \widetilde{\Sigma}\to\mathsf{G}/\mathsf{H}_0$ which satisfy certain symmetries (see Theorem 4.3.4). This analysis is used to prove Theorem 4.3.7 which answers a question of Toledo on G-variations of Hodge structures.

For $\mathsf{SL}(n,\mathbb{C})$ the group G_0 is always a Levi factor of a parabolic subgroup of $\mathsf{SL}(n,\mathbb{C})$. If $(\mathcal{E},\phi)$ is an $\mathsf{SL}(n,\mathbb{C})$-Higgs bundle fixed by the k^{th} roots of unity, then, in terms of vector bundles, the rank n holomorphic vector bundle $\mathcal{E}$ splits holomorphically as $\mathcal{E}=\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_k$. Moreover, with respect to the metric solving the Higgs bundle equations, this holomorphic splitting is also orthogonal, i.e. the metric h is a direct sum $h = h_1\oplus\cdots h_k$. If we decompose the Higgs field $\phi:\mathcal{E}\to\mathcal{E}\otimes K$ in terms of the above holomorphic splitting, then, by Theorem 4.1.6, the only components which are nonzero are $\phi_j:\mathcal{E}_j\to\mathcal{E}_{j+1\ mod\ k}$. In this case, the Higgs bundle equations simplify into a fully coupled system of simpler equations:

$$F_{A_{h_j}} + \phi_{j-1}\wedge\phi_{j-1}^* + \phi_j^*\wedge\phi_j = 0. \tag{1.0.2}$$

If G is a real group and $(\mathcal{E},\phi)$ is a G-Higgs bundle which is fixed by a k^{th} roots of unity action and with the propert that the corresponding $\mathsf{G}_\mathbb{C}$ Higgs bundle is stable, then the simplification the Hitchin equations (1.0.2) for the complex group $\mathsf{G}_\mathbb{C}$ has extra symmetries which reflect the real form G. Chapter 5 is based on a joint work with Q. Li [CL14]. In this work, the extra symmetries of the metric for fixed points of the n^{th} and $(n-1)^{st}$ roots of unity actions in the $\mathsf{SL}(n,\mathbb{R})$-Hitchin component are exploited to solve the Higgs bundle equations asymptotically. This analysis also allows us to analyze the asymptotic holonomy of the corresponding family of representations of $\pi_1(S)$.

To describe this, we recall that the $\mathsf{SL}(n,\mathbb{R})$-Hitchin component $\mathsf{Hit}(\mathsf{SL}(n,\mathbb{R}))$ is parameterized by the vector space $\bigoplus_{j=2}^{n} H^0(\Sigma, K^j)$ of holomorphic differentials, and the Higgs bundle $(\mathcal{E},\phi)$ associated to a tuple $(q_2,\dots,q_n)$ of differentials is

$$\mathcal{E} = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$$

and

$$\phi = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_{n-1} & q_n \\ \frac{n-1}{2} & 0 & q_2 & \cdots & q_{n-2} & q_{n-1} \\ & \ddots & & \ddots & & \\ & & & & q_2 & q_3 \\ & & & \frac{n-3}{2} & 0 & q_2 \\ & & & & \frac{n-1}{2} & 0 \end{pmatrix} : \mathcal{E} \longrightarrow \mathcal{E}\otimes K.$$

Such a ϕ will be denoted by $\tilde{e}_1 + q_2e_1 + q_3e_2 + \cdots + q_ne_{n-1}$. Moreover, $(\mathcal{E},\phi)$ is a fixed point of the k^{th} roots of unity if and only if $\phi = \tilde{e}_1 + \sum_{j=0 \ mod \ k} q_je_{j-1}$ (see Proposition4.2.7). Using the $\mathsf{SL}(n,\mathbb{R})$-symmetry the following key corollary can be deduced:

Corollary 4.2.8. *For $k=n$ and $k=n-1$ the harmonic metric splits as $h_1\oplus h_2\oplus\cdots\oplus h_2^{-1}\oplus h_1^{-1}$ on the line bundles $K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$.*

For $k=n$, this was proven by Baraglia [Bar15], and was used to study, amongst other things, the relation between the Hitchin equations and the affine Toda equations. We first obtain estimates for the solution metric h_t of the Hitchin equations as $t\to\infty$ by repeatedly using the maximum principle and a standard "telescope" trick.

Theorem 5.1.1. *For every point $p\in\Sigma$ away from the zeros of q_n or q_{n-1}, as $t\to\infty$*

1. *For $(\Sigma,\ \tilde{e}_1 + tq_ne_{n-1}) \in \mathsf{Hit}(\mathsf{SL}(n,\mathbb{R}))$, the metric $h_j(t)$ on $K^{\frac{n+1-2j}{2}}$ admits the expansion*

$$h_j(t) = (t|q_n|)^{-\frac{n+1-2j}{n}}\left(1+O\left(t^{-\frac{2}{n}}\right)\right) \qquad \textit{for all } j$$

2. *For $(\Sigma,\ \tilde{e}_1 + tq_{n-1}e_{n-2}) \in \mathsf{Hit}(\mathsf{SL}(n,\mathbb{R}))$, the metric $h_j(t)$ on $K^{\frac{n+1-2j}{2}}$ admits the expansion*

$$h_j(t) = \begin{cases} (t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}\left(1+O\left(t^{-\frac{2}{n-1}}\right)\right) & \textit{for } j=1 \ \textit{ and } \ j=n \\ (2t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}\left(1+O\left(t^{-\frac{2}{n-1}}\right)\right) & \textit{for } 1<j<n \end{cases}$$

Using the asymptotic estimates of the solution metric and error estimates, we integrate the ODE defined by the flat connection. This yields an estimate of the parallel transport matrices $T_{P,P'}(t)$ as $t\to\infty$. For $(\Sigma,(0,\dots,0,tq_n))\in \mathsf{Hit}(\mathsf{SL}(n,\mathbb{R}))$, let $P\in\widetilde{\Sigma}$ be a point at which q_n does not vanish. Choose a neighborhood $\mathcal{U}_P$ centered at P, with coordinate z, so that $q_n=dz^n$. Any $P'\in\mathcal{U}_P$ may be written in polar coordinate as $P'=Le^{i\theta}$. Suppose $\gamma(s)$ is a $|q_n|^{\frac{2}{n}}-$geodesic from P to P' parametrized by arc length s. With some work and an extra condition on the path, we obtain the entire set of eigenvalues of the parallel transport operator along the path asymptotically.

Theorem 5.3.2. *Suppose P, P' and the path $\gamma(s)$ are as above. If P' has the property that for every s,*

$$s<d(\gamma(s)):=min\{d(\gamma(s),z_0)|\ \textit{for all zeros}\ z_0\ \textit{of}\ q_n\},$$

then there exists a constant unitary matrix S, not depending on the pair (P,P'), so that as $t\to\infty$,

$$T_{P,P'}(t)=\left(Id+O\left(t^{-\frac{1}{2n}}\right)\right)S\begin{pmatrix}e^{-Lt^{\frac{1}{n}}\mu_1} & & \\ & \ddots & \\ & & e^{-Lt^{\frac{1}{n}}\mu_n}\end{pmatrix}S^{-1}$$

where $\mu_j=2cos\left(\theta+\frac{2\pi(j-1)}{n}\right)$.

Remark 1.0.4. For $(\Sigma,(0,\cdots,0,tq_{n-1},0))\in\mathsf{Hit}(\mathsf{SL}(n,\mathbb{R}))$, we have similar results in Theorem 5.2.8. In particular, in this case, $\mu_1=0$ and for $j>1$, $\mu_j=2cos\left(\theta+\frac{2\pi(j-2)}{n-1}\right)$. When P and P' both project to the same point in Σ, the projected path is a loop. In this case, the above asymptotics correspond to the values of the associated family of representations on the homotopy class of the loop.

By studying the geometry of the family of harmonic equivariant maps $h_t:\widetilde{\Sigma}\to\mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n)$ corresponding to the family of Higgs fields $\phi_t=\tilde{e}_1+tq_ne_{n-1}$ we also obtain a geometric interpretation of the 'boundary point' (see Theorem 5.3.2). In particular, with the proper interpretation, this proves a conjecture of Katzarkov, Noll, Pandit, and Simpson [KNPS15] on the Hitchin-WKB problem in a special case.

For a simple complex Lie group G let m_ℓ be the height of the highest root of the Lie algebra. Stable G-Higgs bundles which are fixed points of the $(m_\ell+1)$-roots of unity action always correspond compatible reduction of structure diagrams given by

$$\begin{tikzcd} \mathsf{G}/\mathsf{C} & \mathsf{G}/\mathsf{T} \arrow[l] \arrow[d] \\ \widetilde{\Sigma} \arrow[u] \arrow[ur] \arrow[r, "h_\rho"'] & \mathsf{G}/\mathsf{H} \end{tikzcd}$$

where C is a maximal complex torus and T is a maximal compact torus. In Chapter 6 we define a special class of equivariant harmonic maps to the space G/T. For the split real form $\mathsf{G}_0 \subset \mathsf{G}$, the G_0-Higgs bundles fixed by $(m_\ell+1)$-roots of unity give rise to such maps which satisfy additional symmetries. Specifying a bit more, we introduce the notion of a *cyclic surface* which generalize those defined by Labourie [Lab14]. For this special class of maps we prove a rigidity result (Theorem 6.1.37). This analysis allows us to use Higgs bundles to understand the mapping class group action on certain connected components of $\mathcal{X}(\pi_1, \mathsf{PSp}(4,\mathbb{R}))$, which we now describe.

For a group of hermitian type such as $\mathsf{PSp}(2n,\mathbb{R})$, the set of maximal representations and maximal Higgs bundles are an especially interesting class of objects. In particular, these representations are all discrete and faithful and the mapping class group $\mathsf{Mod}(S)$ acts properly discontinuously on maximal representations [BILW05]. Later in chapter 5, we analyze fixed points of 4^{th} roots of unity on the space of maximal $\mathsf{PSp}(4,\mathbb{R})$ Higgs bundles. Generalizing the work of Bradlow, Garcia-Prada, and Gothen [BGPG12] for maximal $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundles, we describe all maximal $\mathsf{PSp}(4,\mathbb{R})$-Higgs bundles by exploiting the low dimensional isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}_0(2,3)$. Denote the set of maximal $\mathsf{SO}_0(2,3)$-Higgs bundles by $\mathcal{M}^{2g-2}(\mathsf{SO}_0(2,3))$, we prove:

Theorem 6.2.16. *For each $d \in (0, 4g-4]$, there is a smooth connected component $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ of $\mathcal{M}^{2g-2}(\mathsf{SO}_0(2,3))$ and a diffeomorphism*

$$\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3)) \cong \mathcal{F}^d_\Sigma \times H^0(K^2)$$

where $\mathcal{F}^d_\Sigma$ is a rank $d+3g-3$ vector bundle over the symmetric product $Sym^{-d+4g-4}(\Sigma)$.

Corollary 6.2.18. *$\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ deformation retracts onto $Sym^{-d+4g-4}(\Sigma)$. In particular, there is an isomorphism of cohomology rings*

$$H^*(\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))) \cong H^*(Sym^{-d+4g-4}(\Sigma)).$$

Remark 1.0.7. Since the connected components $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ generalize those discovered for $\mathsf{Sp}(4,\mathbb{R})$ by Gothen [Got01], we will call these *Gothen components*. For $d = 4g-4$, the space $\mathcal{F}^{4g-4}_\Sigma = H^0(K^4)$ and we recover the $\mathsf{SO}_0(2,3)$-Hitchin component. For $0 < d < 4g-4$, it clear from the above theorem that $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ is noncontractible. Furthermore, we show that, if $\mathsf{Goth}_d(\mathsf{SO}_0(2,3))$ is the connected component of $\mathcal{X}(\pi_1, \mathsf{SO}_0(2,3))$ corresponding to $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$, then all representations $\rho \in \mathsf{Goth}_d(\mathsf{SO}_0(2,3))$ are Zariski dense, again generalizing what is known for $\mathsf{Sp}(4,\mathbb{R})$ [BGPG12].

Applying the cyclic surface analysis mentioned above to the Gothen components we prove:

Theorem 6.3.5. *For each $\rho \in \mathsf{Goth}_d(\mathsf{SO}_0(2,3))$ there exists a unique Riemann surface structure Σ_ρ in which the harmonic map $h_\rho : \widetilde{\Sigma}_\rho \to \mathsf{SO}_0(2,3)/(\mathsf{SO}(2)\times\mathsf{SO}(3))$ is a minimal immersion.*

It is not hard to show that, in Theorem 6.2.16, the quadratic differential in the parameterization is a constant multiple of the Hopf differential of the corresponding equivariant harmonic map $h : \widetilde{\Sigma} \to \mathsf{SO}_0(2,3)/(\mathsf{SO}(2)\times\mathsf{SO}(3))$. As a result, we obtain a mapping class group invariant parameterization of $\mathsf{Goth}_d(\mathsf{SO}_0(2,3))$ as a fiber bundle over Teichmüller space.

Theorem 6.3.6. *There is a $\mathsf{Mod}(S)$-invariant diffeomorphism between $\mathsf{Goth}_d(\mathsf{SO}_0(2,3))$ and the fiber bundle $\mathcal{F}_d \to \mathsf{Teich}(S)$ with fiber $\mathcal{F}_d^\Sigma$ from Theorem 6.2.16 over a Riemann surface $\Sigma \in \mathsf{Teich}(S)$.*

Remark 1.0.10. When $d = 4g-4$, Theorems 6.3.5 and 6.3.6 recover results of Labourie which describes the $\mathsf{PSp}(4,\mathbb{R})$-Hitchin component as the vector bundle of holomorphic quartic differentials over Teichmüller space [Lab14].

In the final section of the thesis we discuss generalizations of the Gothen components to higher rank groups. For $n \geq 3$, the space of maximal $\mathsf{PSp}(2n,\mathbb{R})$ representations is not as rich as the space of maximal $\mathsf{PSp}(4,\mathbb{R})$ representations. For instance, for $n \geq 3$ there are $3 \cdot 2^{2g}$ connected components of maximal $\mathsf{Sp}(2n,\mathbb{R})$-Higgs bundles, and since every connected component of maximal $\mathsf{Sp}(2n,\mathbb{R})$ representations can be deformed to either a Hitchin representations or a 'twisted product representation' [GW10], none of these components behave like the Gothen components. For these reasons, it was believed that the diversity of $\mathsf{Sp}(4,\mathbb{R})$-maximal representations and Higgs bundles was an anomaly of low dimensions. However, motivated by the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}_0(2,3)$, we show that the Gothen components should be thought of as an $\mathsf{SO}_0(n,n+1)$ phenomenon.

Theorem 6.4.1. *For each $0 < d \leq n(2g-2)$ there is a smooth connected component $\mathcal{M}_d(\mathsf{SO}_0(n,n+1)) \subset \mathcal{M}(\mathsf{SO}_0(n,n+1))$ which is smooth and parameterized by $\mathcal{F}_\Sigma^d \times \bigoplus_{j=1}^{n-1} H^0(\Sigma, K^{2j})$ where $\mathcal{F}_\Sigma^d \to Sym^{-d+n(2n-2)}(\Sigma)$ is a vector bundle of rank $d+(2n-1)(g-1)$. Moreover, $\mathcal{M}_{n(2g-2)}(\mathsf{SO}_0(n,n+1)) = \mathsf{Hit}(\mathsf{SO}_0(n,n+1))$.*

In particular, using $\mathsf{SO}_0(1,2) \cong \mathsf{PSL}(2,\mathbb{R})$, we recover Hitchin's [Hit87a] parameterization of all connected components of $\mathcal{M}(\mathsf{PSL}(2,\mathbb{R}))$ with positive Toledo invariant mentioned earlier.

Corollary 6.4.2. *For $0 < d \leq 2g-2$, there is a connected component of $\mathcal{M}(\mathsf{PSL}(2,\mathbb{R}))$ which is parameterized by a rank $d+2g-2$ vector bundle over $Sym^{-d+2g-2}(\Sigma)$.*

Remark 1.0.13. The group $\mathsf{SO}_0(n,n+1)$ is not a group of hermitian type for $n > 2$, thus there is no notion of maximality. As a result, the geometry of the corresponding representations is completely unexplored.

Corollary 6.4.3. *Each of the spaces* $\mathcal{M}_d(\mathsf{SO}_0(n, n+1))$ *is homotopy equivalent to* $Sym^{-d+n(2g-2)}(\Sigma)$*. In particular, there is an isomorphism* $H^*(\mathcal{M}_d(\mathsf{SO}_0(n, n+1))) \cong H^*(Sym^{-d+n(2g-2)}(\Sigma))$.

Corollary 6.4.4. *The moduli space* $\mathcal{M}(\mathsf{SO}_0(n, n+1))$*, and hence* $\mathcal{X}(\pi_1, \mathsf{SO}_0(n, n+1))$*, has at least* $n(2g-2)+4$ *connected components.*

Chapter 2

Lie theory and homogeneous space background

2.1 Lie Theory for real reductive Lie groups and Lie algebras

For most Lie theory facts we follow [Kna02], [Oni04] and [Vin94]. Let $\mathfrak{g}$ be a Lie algebra, and let $\mathsf{Aut}(\mathfrak{g})$ denote the group of Lie algebra automorphisms. The adjoint representation is given by

$$\begin{aligned} ad : \mathfrak{g} &\longrightarrow \mathsf{End}(\mathfrak{g}) \\ X &\longmapsto ad_X = [X, -] \end{aligned}$$

The *group of inner automorphisms* $\mathsf{Inn}(\mathfrak{g}) \subset \mathsf{Aut}(\mathfrak{g})$ is defined to be the subgroup generated by $exp(ad_X)$ for all $X \in \mathfrak{g}$. The group $\mathsf{Inn}(\mathfrak{g})$ is a normal subgroup and the quotient $\mathsf{Out}(\mathfrak{g})$ is the group of outer automorphism:

$$1 \longrightarrow \mathsf{Inn}(\mathfrak{g}) \longrightarrow \mathsf{Aut}(\mathfrak{g}) \longrightarrow \mathsf{Out}(\mathfrak{g}) \longrightarrow 1 \ .$$

The Killing form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is defined by

$$B_{\mathfrak{g}}(X, Y) = Tr(ad(X) \circ ad(Y));$$

it is symmetric and $\mathsf{Inn}(\mathfrak{g})$-invariant, i.e. $B_{\mathfrak{g}}([X, Z], Y) = B_{\mathfrak{g}}(X, [Y, Z])$.

Definition 2.1.1. A Lie algebra $\mathfrak{g}$ is called *simple* if it has no nontrivial ideals, and *semisimple* if it is a direct sum of simple Lie algebras. A Lie algebra $\mathfrak{g}$ is is called *reductive* if $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}^{ss}$ where $\mathfrak{z}(\mathfrak{g})$ is an abelian subalgebra and $\mathfrak{g}^{ss}$ is semisimple.

We will mostly deal with semisimple Lie algebras, but on occasion we will need to work with a reductive Lie algebra. Cartan showed that the Killing form $B_{\mathfrak{g}}$ is nondegenerate if and only if $B_{\mathfrak{g}}$ is semisimple. In particular, semisimple Lie algebras have trivial center:

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid ad_X \equiv 0\} = \{0\}.$$

If H is a Lie group with Lie algebra $\mathfrak{h}$, then H is compact if and only if the Killing form $B_{\mathfrak{k}}$ is negative semidefinite (Corollary 4.36 [Kna02]). Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an involutary Lie algebra isomorphism with ± 1 eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \ , \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \ , \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \ . \tag{2.1.1}$$

Hence, $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, and the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is $ad_{\mathfrak{h}}$-invariant. If G is a Lie group with Lie algebra $\mathfrak{g}$ and $\mathsf{H} \subset \mathsf{G}$ is a Lie subgroup with Lie algebra $\mathfrak{h}$, then the splitting $\mathfrak{h} \oplus \mathfrak{m}$ is Ad_{H}-invariant.

Definition 2.1.2. Let $\mathfrak{g}$ be a real semisimple Lie algebra, an involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ so that

$$B_\sigma(X, Y) = -B_{\mathfrak{g}}(X, \sigma(Y))$$

is a symmetric positive definite bilinear form is called a *Cartan involution.*

For a Cartan involution, it follows that the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is orthogonal and that $B_{\mathfrak{g}}$ is positive definite on $\mathfrak{m}$ and negative definite on $\mathfrak{h}$. Thus, $\mathfrak{h}$ is the Lie algebra of a maximal compact subgroup $\mathsf{H} \subset \mathsf{G}$. If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $\mathfrak{g}$, then an involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ extends to a complex linear involution of $\mathfrak{g}_{\mathbb{C}}$, and the splitting $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$ is $Ad_{\mathsf{H}_{\mathbb{C}}}$-invariant.

Cartan involutions exist and are unique up to conjugation. Furthermore, under the conjugation action, the stabilizer of a Cartan involution θ is the group H_θ. Thus we obtain:

Proposition 2.1.3. *Let G be a real simple Lie group with maximal compact H then*

$$\mathsf{G}/\mathsf{H} \cong \{\theta : \mathfrak{g} \to \mathfrak{g} \mid \theta \text{ a Cartan involution}\}.$$

Example 2.1.4. Let $\mathsf{SL}(n, \mathbb{C})$ be the Lie group of determinant 1 complex valued $n \times n$ matrices, its Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ consists of all $n \times n$ traceless matrices. The involution $X \longmapsto -\overline{X}^T$ is a Cartan involution with Cartan decomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n) = \mathfrak{su}(n) \oplus herm_0(n),$$

where $\mathfrak{su}(n)$ consists of all skew adjoint matrices and $herm_0(n)$ consists of all traceless hermitian matrices.

Example 2.1.5. Let $\mathsf{SL}(n, \mathbb{R})$ be the Lie group of determinant 1 real valued $n \times n$ matrices, its Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ consists of all $n \times n$ traceless matrices. The involution $X \longmapsto -X^T$ is a Cartan involution with

Cartan decomposition

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}(n) \oplus sym_0(n)$$

where $\mathfrak{so}(n)$ consists of all skew symmetric matrices and $sym_0(n)$ consists of all traceless symmetric matrices.

Definition 2.1.6. Let $\mathfrak{g}$ be a *complex* Lie algebra, a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is a *real form* of $\mathfrak{g}$ if $\mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{g}$. Equivalently, $\mathfrak{g}_0$ is a real form if it is the $+1$-eigenspace of a *conjugate linear* involution $\lambda : \mathfrak{g} \to \mathfrak{g}$. The ± 1-eigenspace decomposition of $\mathfrak{g}$ with respect to λ is given by $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$.

A complex Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ for a semisimple compact Lie algebra $\mathfrak{k}$ (Theorem 6.11 [Kna02]). In particular, complex semisimple Lie algebras always have compact real forms $\mathfrak{k} \subset \mathfrak{g}$. We will always denote the involution associated to a compact real form by θ.

Proposition 2.1.7. *(Ch 2, Prop 1 [Oni04]) Let λ_0 and λ_1 be two real forms of a complex Lie algebra $\mathfrak{g}$. If $\mathfrak{g}^{\lambda_0}$ and $\mathfrak{g}^{\lambda_1}$ are the corresponding $+1$-eigenspaces then $\mathfrak{g}^{\lambda_0} \cong \mathfrak{g}^{\lambda_1}$ if and only if there is an automorphism $g \in \mathsf{Aut}(\mathfrak{g})$ so that $\lambda_0 = g\lambda_1 g^{-1}$.*

Fix a real form λ, given any other real form τ, the composition $\lambda \circ \tau : \mathfrak{g} \to \mathfrak{g}$ is a complex linear automorphism. The map $\lambda\tau$ is an involution if and only if $\lambda\tau = \tau\lambda$. In this case, the real form τ is invariant under the involutions λ and $\lambda\tau$. If θ is a fixed compact real form, then Cartan proved that for any other real form λ there exists an inner automorphism g so that $g\lambda g^{-1}$ commutes with θ. In this way, one can study real forms in terms of *complex linear* involutions that commute with a fixed compact real form.

2.1.1 Roots and parabolics for complex Lie algebras

The root theory reviewed here will be used throughout the thesis. In particular, the $\mathbb{Z}$-gradings play a vital role in describing fixed points of the $\mathbb{C}^*$ action on the Higgs bundle moduli space and the $\mathbb{Z}/k\mathbb{Z}$-gradings are an essential part of the classification theorem (Theorem 4.2.2) of fixed points of roots of unity actions.

A maximal abelian subalgebra $\mathfrak{c} \subset \mathfrak{g}$ consisting of semisimple elements is called a be a *Cartan subalgebra*. Cartan subalgebras exist and are unique up to conjugation. The dimension of $\mathfrak{c}$ is called the *rank* of $\mathfrak{g}$ and the restriction of the Killing form $B_{\mathfrak{g}}|_{\mathfrak{c}\times\mathfrak{c}}$ is nondegenerate. An element $\alpha \in \mathfrak{c}^*$ is called a *root* if $\alpha \neq 0$ and

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{c}\} \neq \{0\}.$$

Denote the set of roots by $\Delta(\mathfrak{g}, \mathfrak{c}) \subset \mathfrak{c}^*$. If α is a root, the space $\mathfrak{g}_\alpha$ is called the root space of α; the dimension of a root space $\mathfrak{g}_\alpha$ is always 1. Given two roots $\alpha, \beta \in \Delta(\mathfrak{g}, \mathfrak{c})$, a simple calculation shows $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Note that if α is a root, then $-\alpha$ is also a root. This allows us to choose a subset $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{c})$ of *positive roots* such that $\alpha \in \Delta^+$ if and only if $-\alpha \notin \Delta^+$. A choice of positivity defines a set of *simple roots*

$$\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subset \Delta(\mathfrak{g}, \mathfrak{c}) \subset \mathfrak{c}^*,$$

where $\alpha \in \Delta^+$ implies $\alpha = \sum_{j=1}^{\ell} n_j \alpha_j$ with $n_j \in \mathbb{N}$ and $\alpha_j \in \Pi$. The integer $l(\alpha) = \sum_j n_j$ is called the *height* or length of the root α. Let m_ℓ be the maximum height, then, since $\mathfrak{g}$ is simple, there is a unique root μ with $l(\mu) = m_\ell$ called the *highest root*.

Let $\mathsf{C} \subset \mathsf{Inn}(\mathfrak{g})$ be the maximal torus (with Lie algebra $\mathfrak{c}$). Any inner automorphism h is conjugate to an element on C and moreover, two elements of C are conjugate if and only if they are equivalent under the action of the *Weyl group* $N(\mathsf{C})/\mathsf{C}$ where $N(\mathsf{C})$ is the normalizer of C. Although the Weyl group action preserves a Cartan subalgebra, if one fixes a notion of positivity we have the following:

Proposition 2.1.8. *The group* G *acts transitively on the space of a Cartan subalgebra with a choice of simple roots, and the stabilizer of a point is the corresponding Lie group* $\mathsf{C} \subset \mathsf{G}$ *with Lie algebra* $\mathfrak{c}$; *thus*

$$\mathsf{G}/\mathsf{C} \cong \{(\mathfrak{c}, \Delta^+) \mid \mathfrak{c} \subset \mathfrak{g} \text{ a Cartan subalgebra }, \Delta^+ \subset \mathfrak{c}^* \text{a positive root system}\}. \tag{2.1.2}$$

If $\mathfrak{c}(\mathbb{R}) = \{H \in \mathfrak{c} \mid \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{c})\}$, then $\mathfrak{c}(\mathbb{R})$ is a real form of $\mathfrak{c}$. The Killing form $B_\mathfrak{g}$ is real and positive definite on $\mathfrak{c}(\mathbb{R})$ and $\mathfrak{c}(\mathbb{R})^* = Span\{\Delta(\mathfrak{g}, \mathfrak{c})\}$. Furthermore, the Killing form satisfies

$$B_\mathfrak{g}(X, Y) = 0 \quad \text{for} \quad X \in \mathfrak{g}_\alpha, \quad Y \in \mathfrak{g}_\beta \quad \text{with} \quad \alpha + \beta \neq 0. \tag{2.1.3}$$

Thus $\mathfrak{c}$ and the vector subspaces $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ of $\mathfrak{g}$ are pairwise orthogonal.

Since the Killing form restricted to $\mathfrak{c}$ is nondegenerate, define the *coroot* $H_\alpha \in \mathfrak{c}$ of a root α by duality

$$\beta(H_\alpha) = \frac{2B_{\mathfrak{g}^*}(\beta, \alpha)}{B_{\mathfrak{g}^*}(\alpha, \alpha)}.$$

By construction, $\alpha(H_\alpha) = 2$, $H_\alpha \in \mathfrak{c}(\mathbb{R})$ and $\{H_{\alpha_i}\}_{i=1}^{\ell}$ is a basis for $\mathfrak{c}(\mathbb{R})$. A collection $\{X_\alpha\}_{\alpha \in \Delta}$ satisfying

- $[X_\alpha, X_{-\alpha}] = H_\alpha$
- $[X_\alpha, X_\beta] = N_{\alpha,\beta}\ X_{\alpha+\beta}$ with $N_{\alpha,\beta} = -N_{-\alpha,-\beta} \in \mathbb{N}$ and $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root.

is called a *Chevalley basis*; Chevalley bases exist (Theorem 6.6 [Kna02]).

Definition 2.1.9. A Cartan involution which globally preserves a Cartan subalgebra $\mathfrak{c}$ is called a $\mathfrak{c}$-*Cartan* *involution.*

Lemma 2.1.10. *A $\mathfrak{c}$-Cartan involution θ takes a root space $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ and $\theta(H_\alpha) = -H_\alpha$.*

Proof. Since θ is an isomorphism and $\alpha(H_\beta)$ real, for all $X \in \mathfrak{g}_\alpha$, we have $\theta([H_\beta, X]) = \alpha(H_\beta)\theta(X)$. So θ takes root spaces to roots spaces. Recall that for $\alpha + \beta \neq 0$, the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal. By definitinion of a Cartan involution, $-B_\mathfrak{g}(\cdot, \theta\cdot)$ is positive definite. Thus θ takes $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$. Let $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ with $[X_\alpha, X_{-\alpha}] = H_\alpha$, then

$$\theta(H_\alpha) = [\theta(X_\alpha), \theta(X_{-\alpha})] = [\lambda_1 X_{-\alpha}, \lambda_2 X_\alpha] = -\lambda_1\lambda_2 H_\alpha.$$

Since, θ is an involution, $B_\mathfrak{g}(H_\alpha, H_\alpha) > 0$ and $-B_\mathfrak{g}(H_\alpha, \theta(H_\alpha) > 0$ we conclude $\theta(H_\alpha) = -H_\alpha$. $\square$

The existence of a Chevalley basis gives the existence of two real forms, the split real form and the compact real form. The Lie subalgebra

$$\mathfrak{g}' = \bigoplus_{i=1}^{\ell} \mathbb{R}H_{\alpha_i} \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}X_\alpha \tag{2.1.4}$$

is a split real form (Corollary 6.10 [Kna02]). In terms of the Chevalley basis, $\mathfrak{g}'$ is the fixed point set of the *conjugate linear involution* λ defined by $\lambda(X_\alpha) = X_{-\alpha}$ and $\lambda(H_{\alpha_i}) = H_{\alpha_i}$. The subalgebra

$$\mathfrak{k} = \bigoplus_{i=1}^{\ell} \mathbb{R}iH_{\alpha_i} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{c})} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \mathbb{R}i(X_\alpha + X_{-\alpha})$$

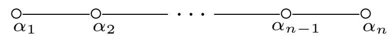
is a compact real form of $\mathfrak{g}$ (Theorem 6.11 [Kna02]). In terms of the Chevalley basis, $\mathfrak{k}$ is the fixed point set the conjugate linear Cartan involution defined by

$$\theta(X_\alpha) = -X_{-\alpha} \qquad \text{and} \qquad \theta(H_\alpha) = -H_\alpha\ . \tag{2.1.5}$$

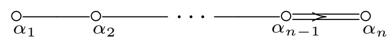
Complex simple Lie algebras are classified by a diagram associated to a set of simple roots $\Pi \subset \Delta$ called its *Dynkin diagram.* Fix a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{g}$ and a set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subset \Delta$. With this choice, the Dynkin diagram has exactly one vertex for each simple root α_i and an edge is drawn between each nonorthogonal pair of vectors. The edge is undirected and single if the root vectors make an angle of $\frac{2\pi}{3}$, it is a directed double edge the root vectors make an angle of $\frac{3\pi}{4}$, and a directed triple edge the root vectors make an angle of $\frac{5\pi}{6}$. If directed, the edge points towards the shorter root.

Example 2.1.11. The classical Lie algebras have the following Dynkin Diagrams

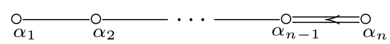
- For $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ there are n simple roots $\alpha_1 \cdots, \alpha_n$ with α_i orthogonal to α_j if and only if $j \neq i \pm 1$. Furthermore, the angle between all nonorthogonal roots is $\frac{2\pi}{3}$, thus the Dynkin diagram is



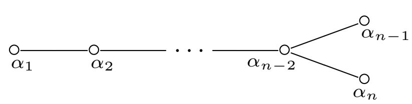
- For $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ there are n simple roots $\alpha_1 \cdots, \alpha_n$ with α_i orthogonal to α_j if and only if $j \neq i \pm 1$. Furthermore, for $i < n-1$, the angle between α_i and α_{i+1} is $\frac{2\pi}{3}$ and the angle between α_{n-1} and α_n is $\frac{3\pi}{4}$ with α_n being the shorter root, thus the Dynkin diagram is



- For $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ there are again n simple roots which satisfy the same orthogonality conditions and angle conditions as $\mathfrak{so}(2n+1, \mathbb{C})$ but with α_{n-1} shorter than α_n, thus the Dynkin Diagram is



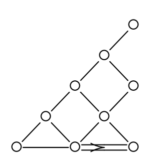
- For $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ there are n simple roots $\alpha_1, \cdots, \alpha_n$ with α_i orthogonal to α_j if $i < n-1$ and $j \neq i \pm 1$ and α_n is orthogonal to all roots except α_{n-2}. Furthermore all angles are $\frac{2\pi}{3}$, hence the Dynkin diagram is given by



Example 2.1.12. For $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$, the rank of $\mathfrak{g}$ is 3 and if $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ is a set of simple roots, the positive roots is given by

$$\Delta^+ = \{\alpha_1, \alpha_2,\ \alpha_3,\ \alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + 2\alpha_3,\ \alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_1 + 2\alpha_2 + 2\alpha_3\}$$

The highest root $\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3$ has height 5. This can be depicted in the root poset of $\mathfrak{so}(7, \mathbb{C})$:



Parabolics and $\mathbb{Z}$-gradings on $\mathfrak{g}$

There are many equivalent definitions of parabolic subgroups $\mathsf{P} \subset \mathsf{G}$, for instance, a subgroup P with G/P compact. If we fix a root system with a notion of positivity, then the standard classification says parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ are in one-to-one correspondence with subsets of simple roots.

Definition 2.1.13. Let $\Pi \subset \Delta(\mathfrak{g}, \mathfrak{c})$ be a set of simple roots. For a positive root system and for a subset $A \subset \Pi$, define the set $R_A = \{\beta \sum_{\alpha_j \in \Pi} m_j \alpha_j \in \Delta(\mathfrak{g}, \mathfrak{c}) | m_j \geq 0 \text{ for } \alpha_j \in A\}$. The parabolic subalgebra associated to $A \subset \Pi$ is defined by

$$\mathfrak{p}_A = \mathfrak{c} \oplus \bigoplus_{\alpha \in R_A} \mathfrak{g}_\alpha.$$

Note that for the two extremes $A = \Pi$ and $A = \emptyset$ we have $\mathfrak{p}_\Pi = \mathfrak{c} \oplus \bigoplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{c})} \mathfrak{g}_\alpha$ and $\mathfrak{p}_\emptyset = \mathfrak{g}$. Given a parabolic $\mathfrak{p}_A$ denote the connected Lie subgroup with Lie algebra $\mathfrak{p}_A$ by $\mathsf{P}_A \subset \mathsf{G}$.

Definition 2.1.14. Given a subset $A \subset \Pi$ define $R_A^0 = \{\beta = \sum_{\alpha_j \in \Pi} m_j \alpha_j \in \Delta(\mathfrak{g}, \mathfrak{c}) | m_j = 0 \text{ for all } \alpha_j \in A\}$. The subalgebras of the parabolic $\mathfrak{p}_A$ given by

$$\mathfrak{l}_A = \mathfrak{c} \oplus \bigoplus_{\alpha \in R_A^0} \mathfrak{g}_\alpha \qquad\qquad \mathfrak{u}_A = \bigoplus_{\alpha \in R_A \setminus R_A^0} \mathfrak{g}_\alpha$$

are respectively called the Levi and unipotent radical subalgebra, and $\mathfrak{p}_A = \mathfrak{l}_A \oplus \mathfrak{u}_A$.

Recall that if $\alpha \in \mathfrak{c}^*$ is a root, then the coroot $H_\alpha \in \mathfrak{c}$ was defined by duality with respect to the Killing form. Denote the root lattice of $\mathfrak{g}$ by $Q \subset \mathfrak{c}(\mathbb{R})^*$. Consider the map $\mathfrak{c} \rightarrow \mathsf{C}$ defined by $x \mapsto exp(2\pi i x)$, the kernel of this map is the lattice $\check{P}$ dual to Q. The lattice $\check{P}$ is generated by the fundamental weights $\{\pi_j\}$ defined previously, and if $\check{Q} \subset \mathfrak{c}$ is the coroot lattice, then $\check{Q} \subset \check{P}$. The fundamental group $\pi_1(\mathsf{Inn}(\mathfrak{g}))$ is canonically isomorphic to $\check{P}/\check{Q}$.

The set of simple coroots generate define the set of *fundamental weights* $\{\pi_j\} \in \mathfrak{c}^*$ as the dual basis:

$$\pi_j(H_{\alpha_i}) = \delta_{ij}.$$

It is not hard to show that center of $\mathfrak{p}_A$ is the same as the center of $\mathfrak{l}_A$ and that

$$\mathfrak{z}(\mathfrak{p}_A) = \mathfrak{z}(\mathfrak{l}_A) = \bigcap_{\alpha_j \in \Pi \setminus A} Ker(\pi_{\alpha_j}). \tag{2.1.6}$$

The set of characters of an arbitrary complex Lie algebra consists of the the set of maps $\mathfrak{g} \rightarrow \mathbb{C}$ which factor through $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, i.e. they are given by elements of the dual of the center $\mathfrak{z}^*(\mathfrak{g})$. Any character χ of a parabolic $\mathfrak{p}_A$ can be written as $\chi = \sum_{\alpha_j \in A} n_j \pi_j$ for $n_j \in \mathbb{R}$. For a semisimple Lie algebra the only character is the constant map. Characters of parabolic subalgebras will play an important role in the definition of stability for Higgs bundles.

Definition 2.1.15. An *(anti)dominant character* of a parabolic $\mathfrak{p}_A$ is an element of $\mathfrak{z}(\mathfrak{l}_A)$ of the form $\chi = \sum_{\alpha_j \in A} n_j \pi_j$ with $n_j \geq 0$, $(n_j \leq 0)$, if the inequality is strict for all $\alpha_j \in A$ the character is called *strictly (anti)dominant.*

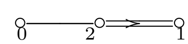
Given a parabolic $\mathfrak{p}_A$ the strictly dominant character $\chi = \sum \pi_j$ defines a $\mathbb{Z}$ grading on $\mathfrak{g}$ called the *height* grading of the parabolic $\mathfrak{p}_A$

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \tag{2.1.7}$$

If $i \neq 0$, $\mathfrak{g}_i$ consists of the root spaces $\mathfrak{g}_\alpha$ where $\alpha = \sum_{\alpha_j \in \Pi} n_j \alpha_j$ with $\sum_{\alpha_j \in \Pi \setminus A} n_j = i$ and $\mathfrak{g}_0 = \mathfrak{l}_A$. Since $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, the decomposition (2.1.7) is a splitting as $\mathfrak{l}_A$-modules with respect to which $\mathfrak{g}_i$ is the dual representation of $\mathfrak{g}_{-i}$.

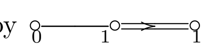
A $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is equivalent to a homomorphism $\gamma : \mathbb{C}^* \to \mathsf{Aut}(\mathfrak{g})$ and such an object is defined by any element $h \in \mathfrak{g}$ so that $d\gamma(1) = ad(h)$, i.e. $exp(2\pi i ad(h)) = id$. For a fixed Cartan subalgebra $\mathfrak{c}$ with simple roots Π, we can act by inner automorphisms to arrange $h \in \mathfrak{c}$ and $\alpha_j(h) \geq 0$ for all $\alpha_j \in \Pi$. If we set $p_j = \alpha_j(h)$, then the vector subspace $\mathfrak{g}_n$ consists of root spaces $\mathfrak{g}_\alpha$ with $\alpha = \sum_{\alpha_j \in \Pi} n_j \alpha_j$ and $\sum n_j p_j = n$. Given a $\mathbb{Z}$-grading, let $A = \{\alpha_j \in \Pi \mid p_j \neq 0\}$ then the Lie algebra $\mathfrak{p}_A = \bigoplus_{j \geq 0} \mathfrak{g}_j$ is a parabolic subalgebra, furthermore, γ is a dominant character of $\mathfrak{p}_A$.

Remark 2.1.16. From the above discussion, we see that $\mathbb{Z}$-gradings are in one-to-one correspondence with labelings of the Dynkin diagram with integers. For any $\mathbb{Z}$-grading, the Lie subalgebra $\mathfrak{g}_0$ is always a Levi factor of a parabolic. Moreover, the height grading corresponding to a parabolic $\mathfrak{p}_A \subset \mathfrak{g}$ is determined by a $\mathbb{Z}$-grading with only 1's and 0's.

Example 2.1.17. Recall from example 2.1.12, $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$ has simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ and the highest root is given by $\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3$. The labeling of the $\mathfrak{so}(7, \mathbb{C})$ Dynkin diagram 

defines the parabolic $\mathfrak{p}_A$ with $A = \{\alpha_2, \alpha_3\}$ and gives the follow $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-6}^{6} \mathfrak{g}_j$ where

$$\begin{array}{llll} \mathfrak{g}_0 = \mathfrak{c} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} & \mathfrak{g}_1 = \mathfrak{g}_{\alpha_3} & \mathfrak{g}_2 = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} & \mathfrak{g}_3 = \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \\ \mathfrak{g}_4 = \mathfrak{g}_{\alpha_2+2\alpha_3} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+2\alpha_3} & \mathfrak{g}_5 = \{0\} & \mathfrak{g}_6 = \mathfrak{g}_{\alpha_1+2\alpha_2+2\alpha_3} & \end{array}$$

Moreover, the height grading of the parabolic $\mathfrak{p}_A$ is defined by labeling the Dynkin diagram by 

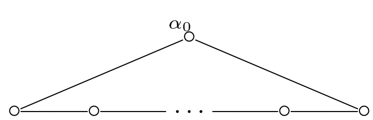
$\mathbb{Z}/k\mathbb{Z}$-gradings

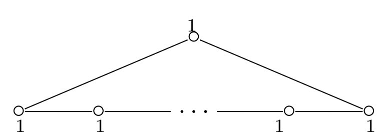
A $\mathbb{Z}/k\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ is equivalent to defining a homomorphism $\gamma : \mathbb{Z}/k\mathbb{Z} \to \mathsf{Aut}(\mathfrak{g})$. Unlike $\mathbb{Z}$-gradings, the image of $\gamma : \mathbb{Z}/k\mathbb{Z} \to \mathsf{Aut}(\mathfrak{g})$ does not necessarily land in $\mathsf{Inn}(\mathfrak{g})$ since $\mathbb{Z}/k\mathbb{Z}$ is not connected. However, we will only discuss $\mathbb{Z}/k/Z$-gradings which arise from homomorphisms $\mathbb{Z}/k\mathbb{Z} \to \mathsf{Inn}(\mathfrak{g})$. Such an object arises from an element $\sigma \in \mathsf{Inn}(\mathfrak{g})$ with $\sigma^m = id$. A good reference for this subsection is Chapter 3 of [Vin94]. Just as $\mathbb{Z}$-gradings correspond to integer labelings of the Dynkin diagram, $\mathbb{Z}/k\mathbb{Z}$-gradings with $\sigma \in \mathsf{Inn}(\mathfrak{g})$ correspond to certain labelings of the extended Dynkin diagram of $\mathfrak{g}$.

Let $\mathfrak{c} \subset \mathfrak{g}$ be a Cartan subalgebra with $\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subset \mathfrak{c}^*$ a set of simple roots. Denote the unique highest root by $\mu = \sum_{j=1}^{\ell} n_j \alpha_j$ and set $\alpha_0 = -\mu$. The extended system of simple roots $\widetilde{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_\ell\}$ is an admissible system of vectors. Its metric properties are described by the so-called *extended Dynkin diagram* (see Chapter 1 [Vin94]). If $n_0 = 1$ then the elements of the system $\widetilde{\Pi}$ satisfy the linear relation that $\sum_{j=0}^{\ell} n_j \alpha_j = 0$. Any automorphism of the system Π can be extended to an automorphism of $\widetilde{\Pi}$ is such a way so that $\mathsf{Aut}(\Pi)$ is a subgroup of $\mathsf{Aut}(\widetilde{\Pi})$, and the fundamental group of the adjoint group $\pi_1(\mathsf{Inn}(\mathsf{G})) = \check{P}/\check{Q}$ is naturally identified with a subgroup of $\mathsf{Aut}(\widetilde{\Pi})$ which acts simply transitively on the set of roots $\alpha_j \in \widetilde{\Pi}$ with $n_j = 1$.

Example 2.1.18. For the classical Lie algebras the extended Dynkin diagrams are given by:

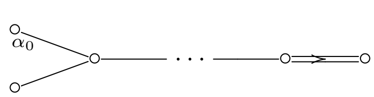
- For $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ if $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots then $\alpha_0 = \sum_{j=1}^{n} -\alpha_j$ and the extended Dynkin diagram is given by

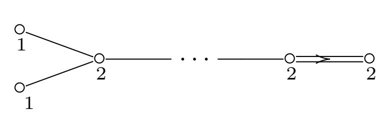




  where the labels on second diagram correspond to $\{n_j\}$.

- For $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ if $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots then $\alpha_0 = \sum_{j=1}^{n} -n_j \alpha_j$ where $n_j = 2$ for $j \geq 2$ and $n_1 = 1$. The extended Dynkin Diagram is given by





  where the labels on second diagram correspond to $\{n_j\}$.

- For $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ if $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots then $\alpha_0 = \sum_{j=1}^{n} -n_j \alpha_j$ where $n_j = 2$ for $j \leq n-1$ and $n_n = 1$. The extended Dynkin Diagram is given by

where the labels on second diagram correspond to $\{n_j\}$.

- For $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ if $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots then $\alpha_0 = \sum_{j=1}^{n} -n_j \alpha_j$ where $n_j = 1$ for $j = 1, n-1, n$ and $n_j = 2$ otherwise. The extended Dynkin Diagram is given by

where the labels on second diagram correspond to $\{n_j\}$.

Let $\mathfrak{c}$ be a Cartan subalgebra with simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ and highest root μ. For any element $x \in \mathfrak{c}$ define the coordinates $(x_0, \cdots, x_\ell)$ by

$$x_0 = 1 - \mu(x) \qquad x_1 = \alpha_1(x) \qquad x_2 = \alpha_2(x) \qquad \cdots \qquad x_\ell = \alpha_\ell(x)\ .$$

We will make extensive use of the following theorem.

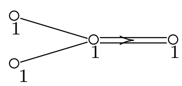
Theorem 2.1.19. *(Theorem 3.11 [Vin94]) Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{c}$ be a Cartan subalgebra with simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ and highest root μ. Any inner semisimple automorphism of $\mathfrak{g}$ is conjugate to an automorphism of the form $exp(2\pi i x)$, where $x \in \mathfrak{c}$ and $Re(x_j) \geq 0$ for all j and if $Re(x_j) = 0$ then $Im(x_j) \geq 0$. Moreover, two automorphisms $exp(2\pi i x)$ and $exp(2\pi i x')$ of such a form are conjugate if and only if the the coordinates $(x_0, \cdots, x_\ell)$ can be taken to $(x'_0, \cdots, x'_\ell)$ by an element of $\pi_1(\mathsf{Inn}(\mathfrak{g}))$.*

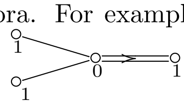
Note, that by ignoring the group $\pi_1(\mathsf{Inn}(\mathfrak{g}))$ we obtain a classification of semisimple elements of the simply connected Lie group G with Lie algebra G. If $\pi_1(\mathsf{Inn}(\mathfrak{g}))$ is replaced by $\mathsf{Aut}(\widetilde{\Pi})$ then we obtain an classification of inner semisimple automorphisms up to conjugacy in $\mathsf{Aut}(\mathfrak{g})$.

Using this theorem, for any inner automorphism σ we may assume that $\sigma = exp(2\pi i x)$ where $x \in \mathfrak{c}$ has coordinates $(x_0, \cdots, x_\ell)$ satisfying the hypotheses of Theorem 2.1.19. If $\sigma^k = Id$ then we have $x_j = \frac{p_j}{k}$ for $p_j \in \mathbb{N}$ such that $\sum_{j=0}^{\ell} n_j p_j = k$. Thus, a $\mathbb{Z}/k\mathbb{Z}$-grading arising from an inner automorphism can be defined by a labeling of the extended Dynkin diagram by non negative integers p_j satisfying $\sum_{j=0}^{\ell} n_j p_j = k$. Two such $\mathbb{Z}/k\mathbb{Z}$-gradings can be taken to one another via an automorphism if and only if the labeled extended Dynkin diagrams can be taken to each other by a diagram automorphism.

Example 2.1.20.

- Given a $\mathbb{Z}$ grading $\mathfrak{g} = \bigoplus_{j=-n}^{n} \mathfrak{g}_j$ on can obtain a $\mathbb{Z}/k\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ by setting $\mathfrak{g}_j = \bigoplus_{i=j \ mod\ k} \mathfrak{g}_i$. One particular instance of such a grading which will be important later is the following: Let $\mu = \sum_{j=1}^{\ell} n_j \alpha_j$ be the highest root and set $k = 1 + \sum_{j=1}^{\ell} n_j$. The $\mathbb{Z}$-grading given by labeling the Dynkin diagram with a 1 on each simple root is given by $\mathfrak{g} = \bigoplus_{j=-k+1}^{k-1} \mathfrak{g}_j$, in particular $\mathfrak{g}_0 = \mathfrak{c}$. The associated $\mathbb{Z}/k\mathbb{Z}$ grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ has $\widehat{\mathfrak{g}}_0 = \mathfrak{c}$ and $\widehat{\mathfrak{g}}_j = \mathfrak{g}_j \oplus \mathfrak{g}_{-k+j}$. The labeling of the extended Dynkin diagram has a 1 on each vertex. For example, for $\mathfrak{so}(7,\mathbb{C})$, $k = 8$ and the labeling of the extended Dynkin diagram is



- Unlike $\mathbb{Z}$-gradings, the identity eigenspace $\widehat{\mathfrak{g}}_0$ does not need to be the Levi factor of a parabolic subalgebra. For example, the $\mathbb{Z}/4\mathbb{Z}$-grading of $\mathfrak{so}(7,\mathbb{C})$ associated to the extended Dynkin diagram labeling  is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$ which is not the Levi factor any parabolic of $\mathfrak{so}(7,\mathbb{C})$.

Remark 2.1.21. Note that is $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ then for any $\mathbb{Z}/k\mathbb{Z}$-grading the Lie subalgebra $\widehat{\mathfrak{g}}_0$ is the Levi factor of a parabolic subalgebra.

The identity eigenspace $\widehat{\mathfrak{g}}_0$ of a finite order automorphism $\sigma \in \mathsf{Inn}(\mathfrak{g})$ is a reductive Lie subgroup of $\mathfrak{g}$ and its type can be determined by the labeling of the extended Dynkin diagram. The simple roots labeled with a 0 define a root system for $\widehat{\mathfrak{g}}_0$, thus one simply removes all vertices without a 0 label to obtain the Dynkin diagram of $\widehat{\mathfrak{g}}_0$.

Remark 2.1.22. Let $\widehat{\mathfrak{g}}_1 = \bigoplus \widehat{\mathfrak{g}}_1^{\nu}$ be the decomposition of $\widehat{\mathfrak{g}}_1$ into irreducible $\widehat{\mathfrak{g}}_0$ representations. If there are roots labeled with a 0, then the irreducible representations which appear are in one-to-one correspondence with the connected components which contain a root labeled with a 1 of the Dynkin diagram which results from removing the roots labeled with a 0 from the extended Dynkin diagram. If there are no roots labeled with a 0 then $\widehat{\mathfrak{g}}_0 = \mathfrak{c}$ and the root space of each root labeled with a 1 defines an irreducible representations.

2.1.2 Maximally compact Cartan subalgebras and roots for real Lie algebras

The interaction between the root theory of a complex simple Lie algebra and its real forms will be important for the definition of Hitchin triples and the notion of a cyclic surface introduced in Chapter 6. The main reference for this subsection is Chapter 6 sections 6-10 of [Kna02]. For this subsection fix $\mathfrak{g}_0$ a real form of a complex simple Lie algebra $\mathfrak{g}$ with Cartan involution $\theta : \mathfrak{g}_0 \to \mathfrak{g}_0$ and corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{m}$.

Definition 2.1.23. A θ-stable *Cartan subalgebra of* $\mathfrak{g}_0$ is a maximal abelian subalgebra $\mathfrak{c}_0 \subset \mathfrak{g}_0$ such that $\mathfrak{c}_0 \otimes \mathbb{C} \subset \mathfrak{g}$ is a Cartan subalgebra and $\theta(\mathfrak{c}_0) = \mathfrak{c}_0$. This gives a decomposition of $\mathfrak{c}_0$ into compact and noncompact parts: $\mathfrak{c}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \subset \mathfrak{h} \oplus \mathfrak{m}$.

Definition 2.1.24. Let $\mathfrak{c}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \subset \mathfrak{h} \oplus \mathfrak{m}$ be a Cartan subalgebra. If $dim(\mathfrak{t}_0)$ maximal amongst all Cartan subalgebras then $\mathfrak{c}_0$ is called *maximally compact* and if $dim(\mathfrak{a}_0)$ is maximal then $\mathfrak{c}_0$ is called *maximally noncompact*.

For a fixed real simple Lie algebra $\mathfrak{g}_0$ with Cartan involution θ, a Cartan subalgebras $\mathfrak{c}_0 \subset \mathfrak{g}_0$ compatible with θ is not unique up to conjugation. If $\mathfrak{c}_0$ and $\mathfrak{c}_0'$ are two such Cartan subalgebras with different compact dimensions then they are clearly not conjugate; even if they have the same compact dimension, they are not necessarily conjugate. However, for maximally compact and maximally noncompact Cartan subalgebras we have:

Proposition 2.1.25. *If* $\mathfrak{c}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \subset \mathfrak{h} \oplus \mathfrak{m}$ *is a maximally compact or maximally noncompact Cartan subalgebra then* $\mathfrak{c}_0$ *is unique up to conjugation.*

Definition 2.1.26. If $\mathfrak{c}_0 \subset \mathfrak{g}$ is a maximally noncompact Cartan with $\mathfrak{t}_0 = \{0\}$ the real Lie algebra $\mathfrak{g}_0$ is the *split real form* of $\mathfrak{g}$. The split real forms of $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{sp}(2n, \mathbb{C})$ and $\mathfrak{so}(2n, \mathbb{C})$ are $\mathfrak{sl}(n, \mathbb{R}), \mathfrak{so}(n, n+1), \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{so}(n, n)$ respectively.

Remark 2.1.27. We will give an equivalent definition of split real forms in terms of maximally compact Cartan subalgebras in Proposition 2.1.43. The Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is a compact real form if and only if $\mathfrak{a}_0 = \{0\}$ for any Cartan subalgebra.

Let $\mathfrak{c}_0 \subset \mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{m}$ be a Cartan subalgebra. If $\mathfrak{c} = \mathfrak{c}_0 \otimes \mathbb{C}$ then we have a decomposition $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{t}_0 \otimes \mathbb{C}$ and $\mathfrak{a} = \mathfrak{a}_0 \otimes \mathbb{C}$. Denote the complex linear extension of the Cartan ivolution to $\mathfrak{g}^*$ by θ also, the set of roots $\Delta \subset \mathfrak{c}^*$ is preserved by the involution θ. Furthermore, since the Killing form is positive definite on the set roots, Δ lives in the noncompact part of $\mathfrak{c}^*$, i.e.

$$\Delta \subset i\mathfrak{t}_0^* \oplus \mathfrak{a}_0^*.$$

This leads to the notion of real, imaginary and complex roots:

Definition 2.1.28. A root $\alpha \in (i\mathfrak{t}^* \oplus \mathfrak{a}^*)$ is called *real* if $\alpha|_{i\mathfrak{t}} = 0$, *imaginary* if $\alpha|_{\mathfrak{a}} = 0$ and *complex* otherwise. By construction, if α is real then $\theta(\alpha) = -\alpha$, if α is imaginary then $\theta(\alpha) = \alpha$ and if α is complex then $\theta(\alpha)$ is a root different than α.

For a root α, the Cartan involution satisfies $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\theta(\alpha)}$. Thus, if α is real then $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and if α is imaginary then $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$. Since the root space $\mathfrak{g}_\alpha$ is one dimensional for an imaginary root α either $\mathfrak{g}_\alpha \subset \mathfrak{h}_\mathbb{C}$ or $\mathfrak{g}_\alpha \subset \mathfrak{m}_\mathbb{C}$.

Definition 2.1.29. An imaginary root α is called *compact* if $\mathfrak{g}_\alpha \subset \mathfrak{h}_\mathbb{C}$, and *noncompact* if $\mathfrak{g}_\alpha \subset \mathfrak{m}_\mathbb{C}$.

The number of each type of roots depends on the dimension of the compact part of a Cartan subalgebra, for maximal compact and maximal noncompact Cartan subalgebras we have the following classification.

Proposition 2.1.30. *A Cartan subalgebra $\mathfrak{c}_0 \subset \mathfrak{g}_0$ is maximally compact if and only if there are* no real roots *and a maximally noncompact if and only if there are* no noncompact imaginary roots.

For any choice of positive roots, if a complex root α is simple then $\theta(\alpha)$ is also simple and $\theta(\alpha)$ is the image of α under a nontrivial automorphism of the Dynkin diagram. In particular, when the Dynkin diagram has no nontrivial automorphism, there are never complex roots.

One way to classify real forms of a complex simple Lie algebra $\mathfrak{g}_\mathbb{C}$ is the notion of Vogan diagrams. This is a done by decorating the Dynkin diagram of $\mathfrak{g}_\mathbb{C}$ to encode the data of the real form.

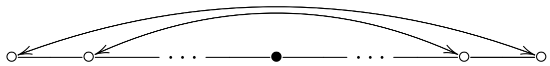
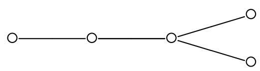
Definition 2.1.31. (Vogan Diagram see Chapter 6.8 [Kna02]) Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a real form with Cartan involution θ and choose a maximally compact θ-stable Cartan subalgebra $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ and choose a notion of positivity Δ^+ for the corresponding roots. The *Vogan diagram* of the triple $(\mathfrak{g}_0, \mathfrak{c}, \Delta^+)$ is Dynkin diagram of Δ^+ where the order 2 orbits of θ have been labeled and the order 1 orbits of θ are painted if and only if they correspond to noncompact imaginary roots.

Theorem 2.1.32. *(Theorem 6.74 in [Kna02]) Let $(\mathfrak{g}_0, \mathfrak{c}, \Delta^+) \subset \mathfrak{g}$ and $(\mathfrak{g}_0', \mathfrak{c}', (\Delta^+)') \subset \mathfrak{g}$ be two real forms with fixed maximally compact Cartan subalgebra and choice of positivity. If the correspond Vogan diagrams are the same, then $\mathfrak{g}_0 \cong \mathfrak{g}_0'$.*

Thus, non-isomorphic real forms always give different Vogan diagrams. An abstract Vogan diagram is defined as a choice of painting of the Dynkin diagram.

Theorem 2.1.33. *(Theorem 6.88 in [Kna02]) For any abstract Vogan diagram, there is a real form $\mathfrak{g}_0 \subset \mathfrak{g}$ with Cartan involution θ, maximally compact Cartan subalgebra $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ and notion of positivity Δ^+ with this Vogan diagram.*

Example 2.1.34. Here are some examples of Vogan diagrams

- $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$: 
- $\mathfrak{g} = \mathfrak{so}(n, n+1)$:
- $\mathfrak{g} = \mathfrak{sp}(4, 8)$:
- $\mathfrak{g} = \mathfrak{so}(10)$: 

Given a real form $\mathfrak{g}_0 \subset \mathfrak{g}$, the form of the corresponding Vogan diagram depends on the choice of positive roots. Due to this redundancy, Vogan diagrams do not classify real forms of $\mathfrak{g}$. However, we have the following classification.

Theorem 2.1.35. *(Theorem 6.96 in [Kna02]) For every real form $\mathfrak{g}_0 \subset \mathfrak{g}$ with Cartan involution θ and maximally compact θ-stable Cartan subalgebra $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$, there exists a unique choice of simple roots so that the Vogan diagram has exactly one painted root.*

Putting this all together, there is a one-to-one correspondence between real forms of a complex simple Lie algebra $\mathfrak{g}$ and Vogan diagrams with exactly one painted root. With respect to such a choice of positivity, all but at most one simple root is complex or compact. This is not the notion of positivity we want to use for the Higgs bundles considered later.

2.1.3 Principal three dimensional subalgebras

The definition and properties of the principal three dimensional subalgebra developed below play a crucial role throughout the thesis. In particular, it is needed to define the Hitchin component, it is necessary for the notion of the cyclic surfaces of Chapter 6 and is important for the generalizations of Hitchin representations of Theorem 6.4.1.

Following Kostant [Kos59], we define the principal three dimensional subalgebra (PTDS) with respect to the Chevalley basis $\{H_{\alpha_i}, X_{\pm\alpha}\}$. If $\{\pi_1, \cdots, \pi_\ell\}$ is the set of fundamental weights (i.e. the basis of $\mathfrak{c}$ dual to the simple roots), set

$$x = \sum_{i=1}^{\ell} \pi_i \;=\; \frac{1}{2} \sum_{\alpha \in \Delta^+} H_\alpha \;=\; \frac{1}{2} \sum_{\imath=1}^{\ell} r_{\alpha_i} H_{\alpha_i} \;. \tag{2.1.8}$$

By construction of x, if $X \in \mathfrak{g}_\alpha$ and $l(\alpha)$ is the height of the root α, then $[x, X] = l(\alpha)X$. The eigenspace decomposition of $\mathfrak{g}$ with respect to ad_x gives a $\mathbb{Z}$-grading on $\mathfrak{g}$ called the height decomposition:

$$\mathfrak{g} = \mathfrak{g}_{-m_\ell} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m_\ell} \tag{2.1.9}$$

where $\mathfrak{g}_j = \bigoplus_{l(\alpha)=j} \mathfrak{g}_\alpha$. Define

$$e_1 = \sum_{i=1}^{\ell} \sqrt{r_{\alpha_i}} X_{\alpha_i} \qquad \text{and} \qquad \tilde{e}_1 = \sum_{i=1}^{\ell} \sqrt{r_{\alpha_i}} X_{-\alpha_i} \ .$$

By construction $\mathfrak{s} = \langle \tilde{e}_1, x, e_1 \rangle$ satisfies the bracket relations

$$[x, e_1] = e_1 \ , \qquad [x, \tilde{e}_1] = -\tilde{e}_1 \ , \qquad [e_1, \tilde{e}_1] = x \ ,$$

and thus $\mathfrak{s} \cong \mathfrak{sl}(2, \mathbb{C})$.

The adjoint action of $\mathfrak{s}$ on $\mathfrak{g}$ decomposes into a direct sum of irreducible $\mathfrak{sl}(2, \mathbb{C})$-representations $\mathfrak{g} = \bigoplus V_j$. Kostant [Kos59] showed that there are exactly $\ell = rank(\mathfrak{g})$ irreducible summands

$$\mathfrak{g} = \bigoplus_{j=1}^{\ell} V_j. \tag{2.1.10}$$

Furthermore, $dim(V_j) = 2m_j + 1$ and the integers $\{m_j\}$ are independent of all the choices. The numbers $\{m_1, \cdots, m_\ell\}$ are called the exponents of $\mathfrak{g}$ and always satisfy $m_1 = 1$ and $m_\ell = l(\mu)$ where μ is the highest root. A three dimensional subalgebra with this property is unique up to conjugation [Kos59].

Definition 2.1.36. Any subalgebra $\mathfrak{s}'$ conjugate to $\mathfrak{s}$ is called a *principal three dimensional subalgebra (PTDS)*, if $\mathfrak{s}' \cap \mathfrak{c} \neq \{0\}$ the PTDS is called a $\mathfrak{c}$-PTDS.

Theorem 2.1.37. *(Theorem 4.2 [Kos59]) Let $\mathfrak{s} \subset \mathfrak{g}$ be PTDS and $x \in \mathfrak{s}$ be a semisimple element with centeralizer $\mathfrak{g}_x$. Then any other PTDS $\mathfrak{s}' \subset \mathfrak{g}$ containing x is conjugate to $\mathfrak{s}$ by an element in Lie group G_x with Lie algebra $\mathfrak{g}_x = Ker(ad_x)$.*

Let $e_j \in V_j$ be the highest weight vector, by definition, $[e_1, e_j] = 0$. Since $[x, e_\ell] = m_\ell e_\ell$, one can always take $e_\ell = X_\mu$, where μ is the highest root. The decomposition (2.1.10) allows us to define an involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ by

$$\sigma(e_j) = -e_j \qquad \sigma(\tilde{e}_1) = -\tilde{e}_1 \tag{2.1.11}$$

and extended by the bracket relations.

Proposition 2.1.38. *([Kos59]) The involution σ commutes with the $\mathfrak{c}$-Cartan involution θ defined by $\theta(X_\alpha) = -X_{-\alpha}$. Furthermore, the resulting real form $\lambda = \theta \circ \sigma$ is a split real form.*

Remark 2.1.39. The involution σ can be represented pictorial using the theory of irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. For instance, when $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$ the exponents $(m_1, m_2, m_3) = (1, 2, 3)$. The irreducible representations of equation (2.1.10) have dimensions $(3,5,7)$ and the involution σ is defined by:

$$
\begin{array}{cccccccc}
 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
V_1 & & & \overset{\tilde{e}_1}{\underset{-1}{\bullet}} & \overset{x}{\underset{1}{\bullet}} & \overset{e_1}{\underset{-1}{\bullet}} & & \\
V_2 & & \overset{ad_{\tilde{e}_1}^4(e_2)}{\underset{-1}{\bullet}} & \underset{1}{\bullet} & \underset{-1}{\bullet} & \underset{1}{\bullet} & \overset{e_2}{\underset{-1}{\bullet}} & \\
V_3 & \overset{ad_{\tilde{e}_1}^6(e_3)}{\underset{-1}{\bullet}} & \underset{1}{\bullet} & \underset{-1}{\bullet} & \underset{1}{\bullet} & \underset{-1}{\bullet} & \underset{1}{\bullet} & \overset{e_3}{\underset{-1}{\bullet}}
\end{array}
\tag{2.1.12}
$$

where the ± 1 below each bullet is the value of the involution σ, and the top row represents the height grading of (2.1.9). By construction, the involution σ is complex linear and preserves the height grading of (2.1.9). In particular, it preserves the middle column which is the Cartan subalgebra $\mathfrak{c}$. Thus $\{ad_{\tilde{e}_1}^{m_j} e_j\}$ generate the $\mathfrak{c}$, and whenever m_j is odd, $\sigma(ad_{\tilde{e}_1}^{m_j} e_j) = 1$.

Recall that we may take $e_{m_\ell} = X_\mu$ where μ is the highest root. Since σ commutes with θ,

$$\sigma(X_{-\mu}) = \theta(\sigma(\theta(X_{-\mu}))) = -X_{-\mu},$$

and thus, $\sigma(H_\mu) = \sigma([X_\mu, X_{-\mu}]) = H_\mu$. Following Labourie [Lab14], we note that the involution σ is unique.

Proposition 2.1.40. *(Proposition 2.5.6 [Lab14]) Let $\mathfrak{c}$ be a Cartan subalgebra with a positive root system and μ the highest root. If σ is an involution which preserves globally preserves $\mathfrak{c}$ and a $\mathfrak{c}$-PTDS $\mathfrak{s}$ with $\sigma(H_\mu) = H_\mu$, then σ is unique.*

The involutions θ and σ give eigenspace decompositions

$$\mathfrak{g} = \mathfrak{g}^\theta \oplus i\mathfrak{g}^\theta \qquad \mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{g}^{-\sigma}.$$

Since the compact form θ and the involution σ commute, the restriction of σ to the split real form $\mathfrak{g}_0 = \mathfrak{g}^\lambda$ is a Cartan involution for $\mathfrak{g}_0$:

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{m} = (\mathfrak{g}^\theta \cap \mathfrak{g}^\sigma) \oplus (i\mathfrak{g}^\theta \cap \mathfrak{g}^{-\sigma}).$$

Since both θ and σ globally preserve $\mathfrak{c}$, we may write $\mathfrak{c}^\lambda = \mathfrak{c}_0 = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} \subset \mathfrak{h}$ and $\mathfrak{a} \subset \mathfrak{m}$,

$$\mathfrak{c} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C}.$$

Recall that the coroots $\{H_\alpha\} \subset \mathfrak{c}$ are in the (-1)-eigenspace of the compact real form θ. In terms of the decomposition $\mathfrak{c} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C}$, the (-1)-eigenspace of θ is $i\mathfrak{t} \oplus \mathfrak{a}$.

By definition, the Cartan involution $\sigma|_{\mathfrak{g}_0}$ preserves the set of positive roots, so there are no real roots. Thus, the Cartan subalgebra $\mathfrak{c}_0$ is a maximally compact Cartan subalgebra. Furthermore, since $\sigma(e_1) = -e_1$, by definition of e_1, it follows that there are no imaginary compact simple roots. Thus, we have proven:

Proposition 2.1.41. *The Cartan subalgebra $\mathfrak{c}_0 \subset \mathfrak{g}_0$ is a maximally compact Cartan subalgebra and, with respect to the Cartan involution σ on $\mathfrak{g}_0$, all simple roots are noncompact imaginary or complex. Furthermore, the subgroup $\mathsf{T} \subset \mathsf{G}$ with Lie algebra $\mathfrak{t}$ is a maximal compact torus of G_0.*

Remark 2.1.42. It is important to note that the split real form $\mathfrak{g}_0 = \mathfrak{g}^{\sigma\circ\theta}$ is very different than the split real form $\mathfrak{g}'$ of equation (2.1.4). For $\mathfrak{g}_0$, the Cartan subalgebra $\mathfrak{c}$ is maximally compact, and for $\mathfrak{g}'$, the Cartan subalgebra $\mathfrak{c}$ is maximally noncompact. Thus

$$\mathfrak{c} \cap \mathfrak{k} \cap \mathfrak{g}_0 = \mathfrak{t} \neq \emptyset \qquad \text{and} \qquad \mathfrak{c} \cap \mathfrak{k} \cap \mathfrak{g}' = \emptyset\ .$$

This gives the following formulation of a split real form, although this must certainly be know by Lie theory experts, to my knowledge, it does not appear in the literature.

Proposition 2.1.43. *Let $\mathfrak{g}$ be a split real simple Lie algebra fix a Cartan involution σ with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and let $\mathfrak{c} \subset \mathfrak{g}$ be a maximally compact Cartan subalgebra. There exists a set of simple roots*

$$\{\alpha_1, \cdots, \alpha_\ell\} \subset \mathfrak{c}^*$$

such that, for all i, α_i is either complex or compact imaginary.

Remark 2.1.44. The converse of Proposition 2.1.43 holds for complex simple Lie algebras $\mathfrak{g}$ whose Dynkin diagram has no automorphisms, or equivalently, when there are no complex roots. Namely, for a real form $\mathfrak{g}_0$ of such a $\mathfrak{g}$, there is a choice of simple roots such that no simple root is compact imaginary (equivalently all simple roots are noncompact imaginary) if and only if $\mathfrak{g}_0$ is a split real form. When $\mathfrak{g}$ has outer automorphisms, then there is a choice of simple roots such that no simple root is compact imaginary if and only if $\mathfrak{g}_0$ is a *quasi-split* real form. For the classical Lie algebras $\mathfrak{sl}(2n,\mathbb{C})$, $\mathfrak{sl}(2n+1,\mathbb{C})$ and $\mathfrak{so}(2n,\mathbb{C})$ the quasi split real forms which are not split are $\mathfrak{su}(n,n)$, $\mathfrak{su}(n,n+1)$ and $\mathfrak{so}(n,n+2)$ respectively.

Remark 2.1.45. Complex roots only appear in types A and E, thus, for the split real forms of types B, C, D, F and G, all simple roots are noncompact imaginary. For types B, C, D, G the involution σ preserves the all roots, and is defined on the Chevalley basis $\{e_\alpha\}_{\alpha\in\pm\Pi}$ by

$$\sigma(e_\alpha) = -e_\alpha.$$

For type A_{2n+1} all simple roots are complex, thus the simple roots come in pairs $\{\alpha, \sigma(\alpha)\}$ the involution σ is defined on the Chevalley bases $\{e_\alpha\}_{\alpha\in\pm\Pi}$ by

$$\sigma(e_\alpha) = -e_{\sigma(\alpha)}.$$

For type A_{2n} there is one noncompact imaginary simple root $\hat{\alpha}$ and the rest are complex. Thus σ is defined on the Chevalley bases $\{e_\alpha\}_{\alpha\in\pm\Pi}$ by

$$\begin{cases} \sigma(e_\alpha) = -e_{\sigma(\alpha)} & \alpha \neq \hat{\alpha} \\ \sigma(e_{\hat{\alpha}}) = -e_{\hat{\alpha}} \end{cases}$$

2.2 Homogeneous spaces and reductions of structure

The geometry of reductive homogeneous spaces will also be essential for the rest of the thesis, a good reference for this is [BR90]. Let X be a manifold with a smooth transitive action of G. If we fix a base point $x_0 \in X$ and define $\mathsf{H} = Stab_{\mathsf{G}}(x_0)$, then, since the action is transitive, we have a principal H-bundle

$$\begin{array}{rcl} \mathsf{H} — \mathsf{G} & \xrightarrow{\pi} & X \\ g & \longmapsto & g \cdot x_0 \end{array}$$

Thus, the tangent bundle is given by $TX = \mathsf{G} \times_H \mathfrak{g}/\mathfrak{h}$.

2.2.1 The geometry of reductive homogeneous spaces

We will mostly be interested in reductive homogeneous spaces.

Definition 2.2.1. A homogeneous space X is called *reductive* if the Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as Ad_{H}-modules.

If W is a linear representation of H we will denote the associated bundle by $\mathsf{G} \times_{\mathsf{H}} W = [W]$. Thus, $[\mathfrak{m}] \cong TX$. Since $\mathfrak{m}$ is an Ad_{H}-invariant subspace of $\mathfrak{g}$, we have $[\mathfrak{m}] \subset [\mathfrak{g}]$. The action of H on $\mathfrak{g}$ is the restriction of the G action, hence $[\mathfrak{g}]$ is trivializable

$$\begin{array}{rcl} \mathsf{G}[\mathfrak{g}] & \xleftrightarrow{\cong} & X \times \mathfrak{g} \\ [g, \xi] & \longmapsto & (\pi(g), Ad_g \xi) \end{array}$$

Example 2.2.2. When G is a complex simple Lie group with maximal compact K, the symmetric space

G/K is a reductive homogeneous space. Furthermore, since $\mathfrak{k}\otimes\mathbb{C}=\mathfrak{g}$, we have

$$T(\mathsf{G}/\mathsf{K})\otimes\mathbb{C}\cong[\mathfrak{k}]\oplus[i\mathfrak{k}]=[\mathfrak{g}]\cong\mathsf{G}/\mathsf{H}\times\mathfrak{g}.$$

Using $[\mathfrak{m}]\cong TX$, we have $TX\subset[\mathfrak{g}]\cong X\times\mathfrak{g}$. This inclusions can be interpreted as an H-equivariant 1-form on X valued in $\mathfrak{g}$.

Definition 2.2.3. The equivariant $\mathfrak{g}$-valued 1-form $\omega\in\Omega^1(X,\mathfrak{g})$ is called the *Maurer Cartan form* of the homogeneous space X.

We will view a reductive homogeneous space as coming equipped with a fixed summand $\mathfrak{m}\subset\mathfrak{g}$. Let $\omega_{\mathsf{G}}\in\Omega^1(\mathsf{G},\mathfrak{g})^{\mathsf{G}}$ be the left Maurer-Cartan form of G, it is G-equivariant. Since $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, we may split ω_G in terms of its projections onto $\mathfrak{h}$ and $\mathfrak{m}$

$$\omega_{\mathsf{G}}=\omega_{\mathsf{G}}^{\mathfrak{h}}\oplus\omega_{\mathsf{G}}^{\mathfrak{m}}.$$

This an Ad_{H}-invariant splitting since $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ is Ad_{H}-invariant, thus

$$\omega_{\mathsf{G}}^{\mathfrak{h}}\in\Omega^1(\mathsf{G},\mathfrak{h})^{\mathsf{H}}\qquad\text{and}\qquad\omega_{\mathsf{G}}^{\mathfrak{m}}\in\Omega^1(\mathsf{G},\mathfrak{m})^{\mathsf{H}}.$$

The form $\omega_{\mathsf{G}}^{\mathfrak{h}}$ is a connection on the principal H-bundle $\mathsf{G}\to X$ which we call the *canonical connection.* For any H-representation V, the canonical connection induces a covariant derivative ∇^c on any associated bundle $[V]$. By construction, if $s\in C^\infty(X,[V])$ is G-equivariant, then $\nabla^c s=0$. The form $\omega_{\mathsf{G}}^{\mathfrak{m}}$ is an equivariant horizontal 1-form, i.e. it vanishes along vector fields induced by the action. Thus, $\omega_{\mathsf{G}}^{\mathfrak{m}}$ descends to a 1-form on X valued in $[\mathfrak{m}]$ which is the Maurer Cartan form ω.

When V is the restriction of a representation of G, $[V]$ is trivializable, in which case, there is a simple relationship between flat differentiation on $X\times V$ and covariant differentiation by the canonical connection. This will be important for our later considerations of cyclic surfaces and the Hitchin equations.

Lemma 2.2.4. *(see chapter 1 [BR90]) Let $f:X\to X\times V$ be a smooth section, then $df=\nabla^c f+\omega\cdot f$. If $V=\mathfrak{g}$ is the adjoint representation, then $\nabla^c=d-ad_\omega$ and the torsion of the canonical connection on $TX=[\mathfrak{m}]$ is given by $T_{\nabla^c}=-\frac{1}{2}[\omega,\omega]^{\mathfrak{m}}$.*

Remark 2.2.5. If $[\ ,\ ]^{\mathfrak{m}}$ and $[\ ,\ ]^{\mathfrak{h}}$ denote the projections onto $[\mathfrak{m}]$ and $[\mathfrak{h}]$, then the flatness of d can be written in terms of ∇^c and ω as

$$\begin{cases} F_{\nabla^c}+\dfrac{1}{2}[\omega,\omega]^{\mathfrak{h}}=0 & \mathfrak{h}-part\\ d^{\nabla^c}\omega+\dfrac{1}{2}[\omega,\omega]^{\mathfrak{m}}=0 & \mathfrak{m}-part\end{cases}$$

Moreover, if we decompose $\mathfrak{m} = \bigoplus_j \mathfrak{m}_j$ into irreducible H-representations then

$$TX \cong \bigoplus_j \mathsf{G} \times_{\mathsf{H}} \mathfrak{m}_j = \bigoplus_j [\mathfrak{m}_j].$$

This gives a decomposition of the trivial bundle $[\mathfrak{g}] \to X$ as a direct sum of ∇^c-parallel vector bundles

$$[\mathfrak{g}] = [\mathfrak{h}] \oplus \bigoplus_j [\mathfrak{m}_j].$$

Furthermore, the Maurer Cartan form decomposes $\omega = \sum_j \omega_j$, and the zero curvature equations are

$$\begin{aligned} F_{\nabla^c} + \frac{1}{2}\sum_{j,k}[\omega_j, \omega_k]^{\mathfrak{h}} &= 0 \qquad \mathfrak{h} - part \\ d^{\nabla^c}\omega_j + \frac{1}{2}\sum_{k,\ell}[\omega_k, \omega_\ell]^{\mathfrak{m}_j} &= 0 \qquad \mathfrak{m}_j - part \end{aligned} \tag{2.2.1}$$

Example 2.2.6. A homogeneous space X is called a *symmetric space* if there is an involution $\sigma : \mathsf{G} \to \mathsf{G}$ with $(\mathsf{G}^\sigma)_0 \subset \mathsf{H} \subset \mathsf{G}^\sigma$. In this case, $\mathfrak{h} = \mathfrak{g}^\sigma$, $\mathfrak{m} = \mathfrak{g}^{-\sigma}$ and by equation (2.1.1) $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. When G is a semisimple Lie group, with $\mathsf{K} \subset \mathsf{G}$ a maximal compact, any G-invariant metric on G/K is a G-equivariant section of an associated bundle. Thus, the canonical connection ∇^c is a metric connection. Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, by Lemma 2.2.4, the torsion of the canonical connection vanishes. Hence, for a symmetric space G/K, the canonical connection is the Levi Civita connection of any G-invariant metric. Furthermore, the flatness equations decompose as

$$\begin{cases} F_{\nabla^c} + \frac{1}{2}[\omega, \omega] = 0 & \mathfrak{k} - part \\ d^{\nabla^c}\omega = 0 & \mathfrak{m} - part \end{cases} \tag{2.2.2}$$

2.2.2 Reductions of structure group

Let M be a manifold, G be a semisimple Lie group and $i : \mathsf{H} \to \mathsf{G}$ be a subgroup so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is Ad_{H} invariant. An important example of this is when M is a closed surface, and G is a noncompact semisimple Lie group (for example $\mathsf{SL}(n, \mathbb{R})$ or $\mathsf{SL}(n, \mathbb{C})$) and H is the maximal compact subgroup of G (for example $\mathsf{SO}(n) \subset \mathsf{SL}(n, \mathbb{R})$ or $\mathsf{SU}(n) \subset \mathsf{SL}(n, \mathbb{C})$). Given a be a principal G-bundle $E_{\mathsf{G}} \to M$, and a manifold X with a G-action, denote the associated fiber bundle by $E_{\mathsf{G}}[X] = E_{\mathsf{G}} \times_{\mathsf{G}} X$.

Definition 2.2.7. A reduction of structure of E from G to H is a principal H-subbundle $E_{\mathsf{H}} \hookrightarrow E_{\mathsf{G}}$ so that $E_{\mathsf{H}}[\mathsf{G}] \cong E_{\mathsf{G}}$.

Reductions of structure are in one to one correspondence with sections (we will always work in the smooth category) of the associated G/H fiber bundle $E_{\mathsf{G}}[\mathsf{G}/\mathsf{H}]\to M$. To see this, the following diagram is very helpful:

$$\begin{tikzcd}
G \arrow[d, dash] & H \arrow[d, dash] & & \\
E_{\mathsf{G}} \arrow[d] & E_{\mathsf{G}} \arrow[d] & & \\
M & E_{\mathsf{G}}/\mathsf{H} \arrow[l, "\pi"'] \arrow[r, leftrightarrow, "\cong"] & E_{\mathsf{G}}[\mathsf{G}/\mathsf{H}] \arrow[r, dash] & \mathsf{G}/\mathsf{H}
\end{tikzcd}$$

Given a section $\sigma \in C^{\infty}(M; E_{\mathsf{G}}[\mathsf{G}/\mathsf{H}])$, we can pullback $E_{\mathsf{G}}\to E_{\mathsf{G}}/\mathsf{H}$ to a H bundle $\sigma^* E_{\mathsf{G}}$ over M which naturally includes, H-equivariantly, in $E_{\mathsf{G}}\to M$

$$\begin{tikzcd}
\mathsf{H} \arrow[dr, dash] & & G \arrow[d, dash] & \mathsf{H} \arrow[d, dash] & \\
& \sigma^* E_{\mathsf{G}} \arrow[r, "\tilde{\sigma}"] \arrow[dr] & E_{\mathsf{G}} \arrow[d] & E_{\mathsf{G}} \arrow[d] & \\
& & M \arrow[r, bend right, "\sigma"'] & E_{\mathsf{G}}[\mathsf{G}/\mathsf{H}] \arrow[l, "\pi"'] \arrow[r, dash] & \mathsf{G}/\mathsf{H}
\end{tikzcd}$$

Remark 2.2.8. Sections of the associated bundle $E_G[\mathsf{G}/\mathsf{H}]$ are equivalent to G-equivariant maps $P_{\mathsf{G}}\to\mathsf{G}/\mathsf{H}$. It will be useful to sometimes think of reductions as G-equivariant maps.

Reductions and connections

Given a principal bundle $\mathsf{G} — E_{\mathsf{G}} \xrightarrow{\pi} M$ we get a exact sequence of tangent bundles

$$0 \longrightarrow ker(d\pi) \longrightarrow TE_{\mathsf{G}} \xrightarrow{d\pi} TM \longrightarrow 0$$

the bundle $ker(d\pi)$ is the vertical bundle $\mathcal{V}_{\mathsf{G}}\to E_{\mathsf{G}}$.

Recall that a connection on a principal bundle is given by a 1-form $B \in \Omega^1(E_{\mathsf{G}}, \mathfrak{g})$ satisfying:

1. (Vertical) For all $X \in \mathfrak{g}$ let X_P be the vector field determined by the G action, then $B(X_P) = X$.
2. (Equivariance) If $R_g : E_{\mathsf{G}}\to E_{\mathsf{G}}$ is the diffeomorphism of E_{G} given by the right action of G then we require $(R_g^* B)(Y) = Ad_{g^{-1}} B(Y)$ for all $g \in \mathsf{G}$ and $Y \in C^{\infty}(E_{\mathsf{G}}; TE_{\mathsf{G}})$.

Such a B defines an equivariant projection $TE_{\mathsf{G}}\to\mathcal{V}_{\mathsf{G}}$ and thus gives an equivariant splitting $TE_{\mathsf{G}} \cong \mathcal{V}_{\mathsf{G}} \oplus ker(B)$, with $ker(B) \cong TM$. The subbundle $ker(B)$ is called the horizontal distribution associated to B and will be denoted $\mathcal{H}_B$.

Recall that a section of $\Omega^*(E_{\mathsf{G}}, \mathfrak{g})$ is called horizontal if it vanishes on vertical vector fields. Equivariant horizontal sections of a principal bundle are called *basic* and are in one-to-one correspondence with sections of $E_{\mathsf{G}}[\mathfrak{g}] \to M$, i.e. $\Omega^*(E_{\mathsf{G}}, \mathfrak{g})_{basic} \cong \Omega^*(M, E_{\mathsf{G}}[\mathfrak{g}])$.

Now fix a connection $B \in \Omega^1(E_{\mathsf{G}}, \mathfrak{g})$, and consider the following diagram

$$\begin{tikzcd} (E_{\mathsf{G}}, B) \arrow[d] \arrow[dr] & \\ M & E_{\mathsf{G}}/\mathsf{H} \arrow[l, "\pi"'] \end{tikzcd}$$

If $pr_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ denotes the projection then set $B^{\mathfrak{h}} = pr_{\mathfrak{h}} \circ B$. Since $B^{\mathfrak{h}} \in \Omega^1(E_{\mathsf{G}}, \mathfrak{h})$ is H equivariant and $B^{\mathfrak{h}}(X_P) = X$ for all $X \in \mathfrak{h}$, the 1-form $B^{\mathfrak{h}}$ defines connection on $E_{\mathsf{G}} \to E_{\mathsf{G}}/\mathsf{H}$.

We will write

$$B = B^{\mathfrak{h}} + \mu$$

where $\mu \in \Omega^1(E_{\mathsf{G}}, \mathfrak{m})^{\mathsf{H}}$. It is straight forward to check that μ is a basic form and hence descends to

$$\hat{\mu} \in \Omega^1(E_{\mathsf{G}}/\mathsf{H}, E_{\mathsf{G}} \times_H \mathfrak{m}).$$

Note that the vertical bundle $ker(d\pi) \subset TE_{\mathsf{G}}/\mathsf{H}$ is isomorphic to $E_{\mathsf{G}} \times_{\mathsf{H}} \mathfrak{m}$ so $\hat{\mu}$ defines a projection

$$\hat{\mu} : \; TE_{\mathsf{G}}/\mathsf{H} \to \mathcal{V}_{\mathsf{G}/\mathsf{H}}.$$

Given a reduction of structure $\sigma : M \to E_{\mathsf{G}}/\mathsf{H}$ we get a principal H bundle

$$\begin{tikzcd} (\sigma^* E_{\mathsf{G}}, \tilde{\sigma}^* B) \arrow[r, "\tilde{\sigma}"] \arrow[dr] & (E_{\mathsf{G}}, B) \arrow[d] \arrow[dr] & \\ & M \arrow[r, bend right, "\sigma"'] & E_{\mathsf{G}}/\mathsf{H} \arrow[l] \end{tikzcd}$$

Definition 2.2.9. Given a reduction of structure $\sigma : M \to E_{\mathsf{G}}/\mathsf{H}$ we define the *vertical derivative* of σ, with respect to the connection B, to be $D_B \sigma = \hat{\mu} \circ d\sigma$. That is

$$\begin{tikzcd} TM \arrow[r, "d\sigma"] \arrow[rr, bend right, "D_B \sigma"'] & TE_{\mathsf{G}}/\mathsf{H} \arrow[r, "\hat{\mu}"] & \mathcal{V}_{\mathsf{G}/\mathsf{H}} \cong E_{\mathsf{G}} \times_{\mathsf{H}} \mathfrak{m} \end{tikzcd}$$

Here $D_B \sigma \in C^\infty(M; T^*M \otimes \sigma^* \mathcal{V}_{\mathsf{G}/\mathsf{H}}) = \Omega^1(M, \sigma^* E_{\mathsf{G}} \times_{\mathsf{H}} \mathfrak{m})$.

We can pull back B by $\tilde{\sigma}$ to $\tilde{\sigma}^* B \in \Omega^1(\sigma^* E_{\mathsf{G}}, \mathfrak{g})$, since the map $\tilde{\sigma}$ is H equivariant, we get the decomposition

$$\tilde{\sigma}^* B = \tilde{\sigma}^* B^{\mathfrak{h}} + \tilde{\sigma}^* \mu.$$

We will use the following notation

$$A_{\sigma} = \tilde{\sigma}^* B^{\mathfrak{h}} \in \Omega^1(\tilde{\sigma}^* E_{\mathsf{G}}, \mathfrak{h}) \qquad \text{and} \qquad \psi_{\sigma} = \tilde{\sigma}^* \mu \in \Omega^1(\tilde{\sigma}^* E_{\mathsf{G}}, \mathfrak{m}) \ .$$

As before, A_{σ} defines a connection on $\sigma^* E_{\mathsf{G}}$ and ψ_{σ} is a basic 1-form valued in $\mathfrak{m}$ which we identify with a section $\hat{\psi}_{\sigma} \in \Omega^1(M, \sigma^* E_{\mathsf{G}} \times_{\mathsf{H}} \mathfrak{m})$.

Proposition 2.2.10. *With the set up above, $\hat{\psi}_{\sigma}$ can be identified with vertical derivative $D_{B}\sigma$.*

Proof. With the above set up, the proof is straight forward. We defined ψ_{σ} by $\psi_{\sigma} = \tilde{\sigma}^* \mu$ and saw that both μ and ψ were basic forms so descend to sections $\hat{\mu}$ and $\hat{\psi}$ of the appropriate bundles. We have that $\hat{\psi}_{\sigma} = \sigma^* \hat{\mu}$, thus for a vector field $X \in C^{\infty}(M; TM)$

$$\hat{\psi}_{\sigma}(X) = (\sigma^* \hat{\mu})(X) = \hat{\mu}(d\sigma(X)) = D_{B}(\sigma)(X)$$

by definition of the vertical derivative. $\square$

Chapter 3

Background on nonabelian Hodge correspondence

3.1 Character varieties

A good reference for the basics of character varieties discussed below is [Gol84]. Let G be a real reductive Lie group and M be a smooth manifold with fundamental group π_1 acting on the universal cover $\widetilde{M}$ by deck transformations.

Definition 3.1.1. The G-representation variety of π_1 is the space of group homomorphisms $\mathsf{Hom}(\pi_1, \mathsf{G})$.

The space $\mathsf{Hom}(\pi_1, \mathsf{G})$ is a real analytic variety which is algebraic if G is algebraic. The groups $\mathsf{Aut}(\pi_1)$ and G both act on $\mathsf{Hom}(\pi_1, \mathsf{G})$ by pre and post composition: if $(f, g) \in \mathsf{Aut}(\pi_1) \times \mathsf{G}$ and $\rho \in \mathsf{Hom}(\pi_1, \mathsf{G})$ then for all $\gamma \in \pi_1$

$$((f, g) \cdot \rho)(\gamma) = Ad_g(\rho(f(\gamma))).$$

Denote the composition $Ad \circ \rho : \pi_1 \to \mathsf{G} \to \mathsf{Aut}(\mathfrak{g})$ by Ad_ρ. The tangent space to a representation $\rho \in \mathsf{Hom}(\pi_1, \mathsf{G})$ is the set of Ad_ρ-twisted group homomorphisms $\pi_1 \to \mathsf{Aut}(\mathfrak{g})$ or equivalently the space $Z^1_\rho(\pi_1, \mathfrak{g})$ of twisted 1-cocycles valued in $\mathfrak{g}$,

$$T_\rho \mathsf{Hom}(\pi_1, \mathsf{G}) = \{u : \pi_1 \to \mathsf{Aut}(\mathfrak{g}) | u(\gamma\delta) = u(\gamma) + Ad_\rho(\gamma) \circ u(\delta)\} = Z^1_\rho(\pi_1, \mathfrak{g}).$$

This can be seen by differentiating at $t = 0$ any family $\rho_t = exp(tu + O(t^2)) \cdot \rho$, the fact that ρ is group homomorphism implies the cocycle condition. When M is a closed surface of genus g we have the following result.

Proposition 3.1.2. *([Gol84]) Let S be a closed surface of genus g and $\rho \in \mathsf{Hom}(\pi_1, \mathsf{G})$. If $\mathcal{Z}(\rho)$ is the centralizer of $\rho(\pi_1)$ then*

$$dim(Z^1_\rho(\pi_1, \mathfrak{g})) = (2g - 1)dim(\mathsf{G}) + dim(\mathcal{Z}(\rho)). \tag{3.1.1}$$

In particular, $T_\rho \mathsf{Hom}(\pi_1, \mathsf{G})$ is of minimal dimension if and only if $dim(\mathcal{Z}(\rho)/\mathcal{Z}(\mathsf{G})) = 0$.

One is usually only interested in representations up to the conjugation action of G. To calculate the tangent space of the orbit G_ρ through ρ, note that any family ρ_t in the or bit G_ρ is defined by $\rho_t = g_t^{-1}\rho g_t$ for some family g_t in G. Writing $g_t = exp(tu_0 + O(t^2))$ and differentiating implies the cocycle u corresponding to ρ_t satisfies $u(\gamma) = Ad_\rho(\gamma u_0 - u_0)$. In other words, u is the coboundary ∂u_0, and the space $B^1_\rho(\pi_1, \mathfrak{g})$ of coboundaries is isomorphic to the vector space $\mathfrak{g}/\mathfrak{z}(\rho)$ where $\mathfrak{z}(\rho)$ is the Lie algebra of $\mathcal{Z}(\rho)$. Thus,

$$dim(\mathsf{G}_\rho) = dim(B^1_\rho(\pi, \mathfrak{g})) = dim(G) - dim(\mathcal{Z}(\rho)). \tag{3.1.2}$$

Definition 3.1.3. A representation $\rho \in \mathsf{Hom}(\pi_1, \mathsf{G})$ is completely reducible if the composition $Ad_G \circ \rho : \pi_1 \to \mathsf{GL}(\mathfrak{g})$ is completely reducible.

Definition 3.1.4. Denote the set of completely reducible representations by $\mathsf{Hom}^+(\pi_1, \mathsf{G})$. The G-character variety $\mathcal{X}(\pi_1, \mathsf{G})$ is defined by $\mathsf{Hom}^+(\pi_1, \mathsf{G})/\mathsf{G}$. This is equivalent to taking the GIT-quotient: $\mathcal{X}(\pi_1, \mathsf{G}) = \mathsf{Hom}(\pi_1, \mathsf{G})//\mathsf{G}$ when G is a reductive complex algebraic group.

To simplify notation, we will usually denote a conjugacy class $[\rho] \in \mathcal{X}(\pi_1, \mathsf{G})$ by ρ. By the above discussion, the tangent space to $\rho \in \mathcal{X}(\pi_1, \mathsf{G})$ is defined by the twisted cohomology group $H^1_\rho(\pi_1, \mathfrak{g})$:

$$T_\rho \mathcal{X}(\pi_1, \mathfrak{g}) = Z^1_\rho(\pi_1, \mathfrak{g})/B^1_\rho(\pi_1, \mathfrak{g}) = H^1_\rho(\pi_1, \mathfrak{g}). \tag{3.1.3}$$

So the dimension of the tangent space $T_\rho \mathcal{X}(\pi_1, \mathfrak{g})$ is $|\chi(S)| \cdot dim(\mathsf{G}) + 2dim(\mathcal{Z}(\rho))$.

Definition 3.1.5. A point $\mathcal{X}(\pi_1, \mathsf{G})$ is called infinitesimally simple if the tangent space $T_\rho \mathcal{X}(\pi_1, \mathsf{G})$ has minimal dimension. By equations (3.1.2) and (3.1.1) this is equivalent to $dim(\mathcal{Z}(\rho)/\mathcal{Z}(\mathsf{G})) = 0$.

Proposition 3.1.6. *A point $\rho \in \mathcal{X}(\pi_1, \mathsf{G})$ is smooth if and only if $\mathcal{Z}(\rho) = \mathcal{Z}(\mathsf{G})$, in particular, such a ρ is irreducible and infinitesimally simple.*

The group $\mathsf{Aut}(\pi_1)$ acts on the character variety as above. An inner automorphisms is defined by conjugating by a fixed element $\delta \in \pi_1$. If $\rho \in \mathcal{X}(\pi_1, \mathsf{G})$ then for all $\gamma \in \pi_1$:

$$\delta \cdot \rho(\gamma) = \rho(\delta \cdot \gamma \cdot \delta^{-1}) = \rho(\delta)\rho(\gamma)\rho(\delta)^{-1}.$$

Since ρ is conjugate to $\delta \cdot \rho$, they define the same point in $\mathcal{X}(\pi_1, \mathsf{G})$. This gives rise to a well defined action of the outer automorphisms $\mathsf{Out}(\pi_1) = \mathsf{Aut}(\pi_1)/\mathsf{Inn}(\pi_1)$ on the character variety $\mathcal{X}(\pi_1, \mathsf{G})$.

Remark 3.1.7. The *mapping class group* $\mathsf{Mod}(M)$ of M is the group of isotopy classes of orientation-preserving homeomorphisms of M. There is always a map $\mathsf{Mod}(M) \to \mathsf{Out}(\pi_1)$, so the mapping class group acts on the character variety. We will mostly be interested in the case where the manifold M is a closed surface of genus $g \geq 2$ which we will denote by S. In this case the group $\mathsf{Mod}(S)$ are isomorphic to an index two subgroup of $\mathsf{Out}(\pi)1)$ by the Dehn-Nielsen-Baer Theorem (see Chapter 8 of [FM12]).

Example 3.1.8. For a closed surface S define the set of *Fuchsian* representations by

$$\mathsf{Fuch}(S) = \{\rho \in \mathcal{X}(\pi_1, \mathsf{PSL}(2,\mathbb{R})) \mid \rho \text{ is discrete and faithful}\}.$$

Using the isomorphism of $\mathsf{PSL}(2,\mathbb{R})$ with the orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$, if $\rho \in \mathsf{Fuch}(S)$ then $\widetilde{S}/\rho = \mathbb{H}^2/\rho(\pi_1) = S$ and the hyperbolic metric descends to the surface S. In fact, $\mathsf{Fuch}(S)$ defines two connected component of $\mathcal{X}(\pi_1, \mathsf{PSL}(2,\mathbb{R}))$ [Gol88] and is homeomorphic to two copies of Teichmüller space $\mathsf{Teich}(S)$ of isotopy classes of marked hyperbolic structures on S. The two components come from a choice of orientation on S. In particular, by a classical result of Fricke, the mapping class group $\mathsf{Mod}(S)$ acts properly discontinuously on $\mathsf{Fuch}(S)$.

Recall from Proposition 2.1.38, Kostant's principal three dimensional subalgebra defines an irreducible representations of $\mathsf{PSL}(2,\mathbb{R})$ into any split real form G which we denote by $i : \mathsf{PSL}(2,\mathbb{R}) \hookrightarrow \mathsf{G}$.

Definition 3.1.9. Let G be a split real form. The *Hitchin component* $\mathsf{Hit}(\mathsf{G}) \subset \mathcal{X}(\pi_1, \mathsf{G})$ is the connected component containing $i(\mathsf{Fuch}(S))$.

The Hitchin component is a natural object to consider since it is the deformation space of $\mathsf{Fuch}(S)$. However, to understand this component, we will need Higgs bundles.

3.1.1 Flat connections and the Riemann-Hilbert correspondence

The universal cover $\widetilde{M}$ of a manifold defines a principal π_1-bundle over M.

Definition 3.1.10. A principal G-bundle $E \to M$ is *flat* if the transition functions can be chosen to be locally constant.

In particular, $\widetilde{M} \to M$ is a flat bundle since π_1 is discrete. Given a representation $\rho \in \mathsf{Hom}(\pi_1, \mathsf{G})$ the associated bundle $\widetilde{M} \times_\rho \mathsf{G} \longrightarrow M$ inherits a flat structure from $\widetilde{M} \to M$. Furthermore, if $\rho' = g\rho g^{-1}$ is conjugate to ρ then the flat bundles associated to ρ and ρ' are isomorphic. Thus, there is a map from the G-character variety to the set of flat structures on G-bundles over M

$$\begin{array}{ccc} \mathcal{X}(\pi_1, \mathsf{G}) & \longrightarrow & \{\text{flat } \mathsf{G}\text{-structures on } M\}/Iso\ . \\ [\rho] & \longmapsto & [\widetilde{M} \times_\rho \mathsf{G}] \end{array}$$

The Riemann-Hilbert correspondence asserts that the above map is an isomorphism onto the space of reductive flat connections. Establishing this correspondence involves the standard exercise of showing that a principal G-bundle E is flat if and only if there exists a flat connection, i.e. a connection 1-form $B \in \Omega^1(E, \mathfrak{g})$ with curvature $F_B = 0$.

Any connection B on a principal G-bundle defines a parallel transport operator form the path groupoid to the category of G torsors $Trans_B : \Pi(M) \to \mathsf{G}\text{-}\mathsf{tors}$. The map $Trans_B$ descends to the fundamental groupoid $\Pi_1(M)$ if and only if the connection B is flat. Restricting to the space of smooth based loops $\Omega_*(M)$ defines a holonomy map

$$\mathsf{Hol}_B : \Omega_*(M) \to \mathsf{G}.$$

Denote the space of flat connections on G bundles over M by $\mathcal{B}(M)$. Holonomy defines a map:

$$\mathsf{Hol} : \mathcal{B}(M) \longrightarrow \mathsf{Hom}(\pi_1, \mathsf{G}).$$

However, the space of connections is an infinite dimensional affine space modeled on the vector space $\Omega^1(E, \mathfrak{g})^{\mathsf{G}}$ of equivariant 1-forms. Fortunately, the natural group of isomorphism is also infinite dimensional.

Definition 3.1.11. Let $E \to M$ be a principal G-bundle, then the gauge group $\mathcal{G}_{\mathsf{G}}$ is the group of smooth bundle isomorphisms.

As with the character variety, the action of $\mathcal{G}_{\mathsf{G}}$ on the space of flat connections does not in general admit a Hausdorff quotient. However, if we restrict to a subset of flat connections a Hausdorff quotient can be defined.

Definition 3.1.12. A flat connection B is called reductive if the holonomy map $\mathsf{Hol}_B : \pi_1 \to \mathsf{G}$ is a completely reducible representation in the sense of Definition 3.1.3.

Theorem 3.1.13. *(Riemann-Hilbert correspondence) The space of isomorphism classes of reductive flat* G*-connections* $\mathcal{B}^{red}(M)/\mathcal{G}_{\mathsf{G}}$ *is homeomorphic to the character variety* $\mathcal{X}(\pi_1, \mathsf{G})$*. Furthermore the homeomorphism is analytic*

Remark 3.1.14. When G is algebraic, both $\mathcal{B}^{red}(M)/\mathcal{G}_{\mathsf{G}}$ and $\mathcal{X}(\pi_1, \mathsf{G})$ are algebraic varieties. However, since the holonomy map involves exponentiating, the homeomorphism in the Riemann-Hilbert correspondence is *not* algebraic.

3.2 Harmonic maps and Corlette's Theorem

3.2.1 Harmonic map basics

A good source for this subsection is Chapter 8 of [Jos08]. Let (M, g) and (N, h) be Riemannian manifolds with M compact, denote their Levi-Civita connections by ∇^g and ∇^h. They are given by Kozul's formula

$$2g(\nabla^g_X Y, Z) = X(g(Y, Z) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y).$$

With respect to coordinates $\{x_1, \ldots, x_m\}$ on M and $\{y_1, \ldots, y_n\}$ on N, we have

$$\nabla^g_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \Gamma^k_{ij} \frac{\partial}{\partial x_k} \qquad \text{and} \qquad \nabla^h_{\frac{\partial}{\partial y_\beta}} \frac{\partial}{\partial y_\alpha} = \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial y_\gamma}$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij}\right) \qquad \text{and} \qquad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta}\left(\frac{\partial}{\partial y_\alpha} h_{\beta\delta} + \frac{\partial}{\partial y_\beta} h_{\alpha\delta} - \frac{\partial}{\partial y_\delta} h_{\alpha\beta}\right)$$

Let $f : M \to N$ be a smooth map, the differential df is a section of the bundle $T^*M \otimes f^*TN$. The bundle $T^*M \otimes f^*TN$ has metric $g^* \otimes f^*h$ and connection $\nabla^{g^*} \otimes f^*\nabla^h$. The Christoffel symbols for ∇^{g^*} are given by

$$\nabla^{g^*}_{\frac{\partial}{\partial x_j}} dx_i = \tilde{\Gamma}^k_{ij} dx_k = -\Gamma^j_{ik} dx_k$$

We can view df in a slightly manor, namely $df \in \Omega^1(M, f^*TN)$. The covariant derivative $f^*\nabla^h$ induces an exterior differential operator

$$d^{f^*\nabla^h} : \Omega^*(M, f^*TN) \to \Omega^{*+1}(M, f^*TN)$$

here $d^{f^*\nabla^h}$ is the skew symmetrization of $\nabla^{\Lambda^\bullet g^*} \otimes f^*\nabla^h$.

Definition 3.2.1. Let (M, g) and (N, h) be Riemannian manifolds with M compact, the *energy* of a C^1 map $f : (M, g) \to (N, h)$ is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_M \langle df, df \rangle_{T^*M \otimes f^*TN} dVol_M$$

Locally we have $|df(x)|^2 = \frac{1}{2} g^{ij} h_{\alpha\beta}(f(x)) \frac{\partial f^\alpha(x)}{\partial x^i} \frac{\partial f^\beta(x)}{\partial x^j}$.

Definition 3.2.2. A C^1 map $f : (M, g) \to (N, h)$ is called *harmonic* if it is a critical point of the energy function $\mathcal{E}$.

Euler Lagrange equations for the energy

We want to find the Euler Lagrange Equations for harmonic maps. To do this, we start with a map $f : M \to N$ and consider a variation ξ of f. By a variation we mean a vector field along f, that is, a section $\xi \in \Gamma(M, f^*TN)$. Given $\xi \in \Gamma(M, f^*TN)$ consider the one parameter family

$$F_\xi(x,t) = exp_{f(x)}(\xi(x)t) : M \times [-\epsilon, \epsilon] \to TN$$

The critical points of $\mathcal{E}$ are the maps f so that, for all ξ, $\frac{\partial}{\partial t}(\mathcal{E}(F_\xi)) = 0$.

Theorem 3.2.3. *The Euler Lagrange Equations for the energy functional are* $(d^{f^*\nabla^h})^*(df) = 0$ *or equivalently,* $Tr_g(\nabla^{g^*} \otimes f^*\nabla^h)(df) = 0$.

Locally the Euler Lagrange equations for a map $f : (M, g) \to (N, h)$ are given by

$$trace(\nabla df) = g^{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} - g^{ij}\Gamma^i_{jk}\frac{\partial f^\alpha}{\partial x_k} + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x_i}\frac{\partial f^\gamma}{\partial x_j}$$

The first part is the Laplace Beltrami operator

$$\Delta_M f = \frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial x_i}(\sqrt{det(g)}g^{ij}\frac{\partial f^\alpha}{\partial x_j}) = g^{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} - g^{ij}\Gamma^i_{jk}\frac{\partial f^\alpha}{\partial x_k}$$

So the Euler Lagrange equations become $\Delta_M f + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x_i}\frac{\partial f^\gamma}{\partial x_j} = 0$, or equivalently

$$\frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial x_i}(\sqrt{det(g)}g^{ij}\frac{\partial f^\alpha}{\partial x_j}) + g^{ij}\Gamma^\alpha_{\beta\gamma}(f(x))\frac{\partial f^\beta}{\partial x_i}\frac{\partial f^\gamma}{x_j} = 0$$

Remark 3.2.4. Since the Cristoffel symbols for $\mathbb{R}$ with the Euclidean metric are zero, we see that a function $f : (M, g) \to \mathbb{R}$ is harmonic if and only if $\Delta_M f = 0$; recovering the standard notion.

Harmonic maps from a Riemann Surface

For maps with a closed surface domain, the harmonic map equations simplify and only depend on the conformal class of the domain metric.

Definition 3.2.5. Let Σ be a Riemann surface, a Riemannian metric g on Σ is called *conformal* if, in local coordinates, it can be written as $\rho^2(z)dz \otimes d\bar{z}$ for ρ a positive real valued function:

$$g(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) = g(\tfrac{\partial}{\partial \bar{z}}, \tfrac{\partial}{\partial \bar{z}}) = 0 \qquad \text{and} \qquad g(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial \bar{z}}) = \rho^2(z)$$

In real coordinates $z = x + iy$ this means

$$g(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = g(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = \rho^2(z) \qquad \text{and} \qquad g(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}) = 0 \ .$$

Form now on, (Σ, g) will be a Riemann surface with a conformal metric.

Definition 3.2.6. A C^1 map $f : (\Sigma, g) \to (N, h)$ is called *conformal* if $h(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}) = 0$.

Proposition 3.2.7. *The energy of a map $f : \Sigma \to (N, h)$ is conformally invariant.*

Proof. The energy of f is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_\Sigma \langle df, df \rangle_{T^*\Sigma \otimes f^*TN} dVol_\Sigma$$

writing the integrand locally and remembering the metric on $T^*\Sigma$ has conformal factor $\frac{1}{\rho}$ we have

$$\langle df, df \rangle_{T^*M \otimes f^*TN} dVol_M = \frac{4}{\rho^2(x)} h_{\alpha\beta} (\frac{\partial f^\alpha}{\partial z} \frac{\partial f^\beta}{\partial \bar{z}}) \frac{\sqrt{-1}}{2} \rho^2(z) dz \wedge d\bar{z}$$

Thus the energy is given by the conformally invariant expression:

$$\mathcal{E}(f) = \sqrt{-1} \int_\Sigma h_{\alpha\beta} \frac{\partial f^\alpha}{\partial z} \frac{\partial f^\beta}{\partial \bar{z}} dz \wedge d\bar{z}$$

□

Lemma 3.2.8. *The Laplace-Beltrami operator for (Σ, g) is given by $\Delta = \dfrac{4}{\rho^2} \dfrac{\partial^2}{\partial z \partial \bar{z}}$.*

Proof. We compute, the metric is given by $\begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix}$

$$\Delta = \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x_i} (\sqrt{det(g)} g^{ij} \frac{\partial}{\partial x_j}) = \frac{1}{\rho^2} (\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} (\rho^2 \begin{pmatrix} \rho^{-2} & 0 \\ 0 & \rho^{-2} \end{pmatrix}) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix})$$

$$= \frac{1}{\rho^2} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) = \frac{4}{\rho^2} \frac{\partial^2}{\partial z \partial \bar{z}}$$

□

With this we have the following form of the harmonic map equations

Lemma 3.2.9. *The harmonic map equations for* $f : (\Sigma, g) \to (N, h)$ *are* $(d^{\nabla^{f^*h}})^{0,1} d^{1,0} f = 0$ *locally the harmonic map equations are*

$$(d^{\nabla^{f^*h}}(\frac{\partial f^\alpha}{\partial z}dz))(\frac{\partial}{\partial \bar{z}}) = -\frac{\partial^2 f^\alpha}{\partial z \partial \bar{z}} dz \wedge d\bar{z} - \Gamma^\alpha_{\beta\gamma} \frac{\partial f^\beta}{\partial z} \frac{\partial f^\gamma}{\partial \bar{z}} dz \wedge d\bar{z} = 0$$

Proof. The harmonic map equations are $\Delta f + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x_i}\frac{\partial f^\gamma}{\partial x_j} = 0$. By the previous lemma and after converting everything to complex coordinates we may rewrite them as:

$$\frac{4}{\rho^2}(\frac{\partial^2 f^\alpha}{\partial z \partial \bar{z}} + \Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial z}\frac{\partial f^\gamma}{\partial \bar{z}}) = 0$$

□

Corollary 3.2.10. *The harmonic map equations for maps* $(\Sigma, g) \to (N, h)$ *only depend on the conformal of* Σ *and not on the actual metric* g.

The Hopf differential and Minimal surfaces

Definition 3.2.11. The Hopf differential q_f of a map $f : (\Sigma, g) \to (N, h)$ is the quadratic differential $q_f = (f^*h)^{2,0} \in \Omega^0(\Sigma, K^2)$, in real coordinates $z = x + iy$ the Hopf differential is given by

$$q_f = h\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}\right) dz^2 - h\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}\right) dz^2 - 2ih\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) dz^2 \tag{3.2.1}$$

Lemma 3.2.12. *If* $f : (\Sigma, g) \to (N, h)$ *is harmonic then the Hopf differential* q_f *is holomorphic.*

Proof. We calculate $\bar{\partial} q_f$. By Lemma 3.2.9, locally $d^{1,0} f = \frac{\partial f}{\partial z}$

$$\bar{\partial} q_f = \bar{\partial} h\left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}\right) dz^2 = 2h\left((\nabla^{f^*TN})^{0,1}\frac{\partial f}{\partial z}, \frac{\partial f}{\partial z}\right) dz^2 = 0.$$

□

Remark 3.2.13. Since any conformal metric on Σ has tensor type $dz \otimes d\bar{z}$, the Hopf differential measures the failure of a harmonic map to be conformal. A harmonic map f is *weakly conformal* (i.e. conformal away from singularities) if and only if the Hopf differential q_f vanishes. This is equivalent to f being a branched minimal immersion [SU82, SY79]. From equation (3.2.1) it is clear that the differential df of a weakly conformal map is always of rank 2 or 0. Thus, if f is a weakly conformal map with nowhere vanishing differential, then f is a conformal immersion, or equivalently, a *minimal immersion*.

Example 3.2.14. Fix a Riemann surface structure Σ on S. Recall from example 3.1.8 that the Teichmüller space $\mathsf{Teich}(S)$ of hyperbolic metrics on S is equivalent to the set of Fuchsian representations. By classical results of Eels-Sampson [ES64], for each hyperbolic surface $(S, g) \in \mathsf{Teich}(S)$, there is a unique harmonic map $f_g : \Sigma \to (S, g)$ which is isotopic to the identity. The Hopf differential of f_g gives a map

$$\begin{array}{rcl} \mathsf{Teich}(S) & \longrightarrow & H^0(K^2) \\ (S,g) & \longmapsto & (f_g^* g)^{(2,0)} \end{array}$$

In [Wol89], Wolf showed that, for each Riemann surface structure Σ, this map gives a diffeomorphism of $\mathsf{Teich}(S) \cong_\Sigma H^0(K^2)$ using harmonic map techniques.

3.2.2 Corlette's Theorem

Given a flat G-bundle, a metric is defined by a reduction of structure group from G to the maximal compact subgroup $\mathsf{H} \subset \mathsf{G}$. Since there is a homotopy equivalence between G and its maximal compact subgroup H, such reductions of structure always exist. Unless the corresponding representation has Zariski closure in H, this reduction will not be a flat bundle.

As discussed above, if $E \to M$ is a principal G-bundle then a reduction of structure group to H is defined by an G-equivariant map $E \to \mathsf{G}/\mathsf{H}$. If E is flat than it arises via a representation $\rho \in \mathsf{Hom}(\pi_1, \mathsf{G})$ and extension of structure group from the principal π_1-bundle $\widetilde{M} \to M$. Thus, for a flat bundle, a reduction of structure is equivalent to a ρ-equivariant map from the universal cover to symmetric space

$$\sigma_\rho : \widetilde{M} \longrightarrow \mathsf{G}/\mathsf{H}.$$

Recall from 2.2.4 and Example 2.2.6, in terms of the canonical connection and the Maurer-Cartan form, flat differentiation on the trivial $[\mathfrak{g}] \to \mathsf{G}/\mathsf{H}$ is given by $d = \nabla^c + \omega$. Pulling back the H bundle $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ and the trivial bundle $[\mathfrak{g}] \to \mathsf{G}/\mathsf{H}$ by σ_ρ gives a principal H-bundle $\sigma_\rho^* \mathsf{G}$ and an $\mathfrak{m}$-bundle $\sigma_\rho^*[\mathfrak{m}]$ over $\widetilde{M}$. Moreover, pulling back the canonical connection and the Maurer-Cartan form defines a connection $\widetilde{\nabla}_A = \sigma_\rho^* \nabla^c$ on $\sigma_\rho^* \mathsf{G}$ and a form $\widetilde{\psi} \in \Omega^1(\widetilde{M}, \sigma_\rho^*[\mathfrak{m}])$ which descend to an H-bundle $E_\mathsf{H} \to M$ with connection ∇_A and $\psi \in \Omega^1(M, E_\mathsf{H}[\mathfrak{m}])$ on M. These objects satisfy the flatness equations

$$F_A + \tfrac{1}{2}[\psi, \psi] = 0 \qquad \text{and} \qquad d^{\nabla_A} \psi = 0. \tag{3.2.2}$$

on the Lie algebra bundle $E_\mathsf{H}[\mathfrak{g}] \to M$.

Remark 3.2.15. By definition, $\widetilde{\psi} = \sigma_\rho^*\omega : T\widetilde{\Sigma} \to [\mathfrak{m}]$ is defined by $T\widetilde{\Sigma} \xrightarrow{d\sigma_\rho} T\mathsf{G}/\mathsf{H} \xrightarrow{\omega} [\mathfrak{m}]$. Thus, the tensor ψ is identified with the derivative of σ_ρ.

Recall that the canonical connection on $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ induces the Levi-Civita connection ∇^c on $[\mathfrak{m}] = T\mathsf{G}/\mathsf{H}$. The equivariant map σ_ρ is a harmonic map if and only if $(d^{\nabla_A})^*\psi = 0$. For a harmonic equivariant map σ_ρ the flatness equations (3.2.2) satisfy an extra equation

$$F_A + \tfrac{1}{2}[\psi, \psi] = 0 \qquad , \qquad d^{\nabla_A}\psi \qquad \text{and} \qquad (d^{\nabla_A})^*\psi = 0 \tag{3.2.3}$$

Now let M be a closed surface of genus at least 2 and fix a Riemann surface structure Σ on M. If we complexify everything we obtain a metric connection ∇_A on $\mathsf{H}_\mathbb{C}$-bundle $E_{\mathsf{H}_\mathbb{C}}$ (which is holomorphic with respect to $\nabla_A^{(0,1)}$) and $\psi = \psi^{(1,0)} \oplus \psi^{(0,1)} \in \Omega^{(1,0)}(\Sigma, E_{\mathsf{H}_\mathbb{C}}[\mathfrak{m}_\mathbb{C}]) \oplus \Omega^{(0,1)}(\Sigma, E_{\mathsf{H}_\mathbb{C}}[\mathfrak{m}_\mathbb{C}])$ which satisfy:

$$F_A + [\psi^{(1,0)}, \psi^{(0,1)}] = 0 \qquad , \qquad d^{\nabla_A,(0,1)}\psi^{(1,0)} = 0 \qquad \text{and} \qquad d^{\nabla_A,(1,0)}\psi^{(0,1)} = 0. \tag{3.2.4}$$

This leads to Corlette's Theorem:

Theorem 3.2.16. *(Corlette [Cor88]) Let M be compact and $\rho \in \mathcal{X}(\pi_1(M), \mathsf{G})$ then for each Riemannian metric g on M, there exists a ρ-equivariant map $h_\rho : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}$ which is harmonic. Moreover, h_ρ is unique up to the centralizer of ρ.*

Remark 3.2.17. We will call such an equivariant harmonic map a harmonic metric.

If we restrict $M = S$ a closed surface of genus at least 2, then we have seen that harmonicity depends only on a conformal class of a metric. Thus, fix a Riemann surface structure Σ on S, then a ρ-equivariant map σ_ρ is a harmonic map if and only if $d\sigma_\rho^{(1,0)}$ is holomorphic, that is $d^{\nabla^c,(0,1)}(d^{(1,0)}\sigma_\rho) = 0$. By Remark 3.2.15, σ_ρ is harmonic if and only if $(d^{\nabla_A})^{(0,1)}\psi^{(1,0)} = 0$. Thus equations (3.2.4) are given by

$$F_A + [\psi^{(1,0)}, \psi^{(0,1)}] = 0 \qquad \text{and} \qquad (d^{\nabla_A})^{(0,1)}\psi^{(1,0)} = 0 \tag{3.2.5}$$

Remark 3.2.18. If $\sigma_\rho : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}$ is harmonic, then, since the metric on G/H is induced from the Killing form on $\mathfrak{g}$, the Hopf differential of σ_ρ is a constant multiple of $Tr(ad(\psi^{1,0}) \otimes ad(\psi^{1,0}))$.

3.3 Higgs bundles

Higgs bundles over a Riemann surface were introduced by Hitchin in [Hit87a] and studied in detail for the groups $\mathsf{SL}(2,\mathbb{C})$ and $\mathsf{SL}(2,\mathbb{R})$. Simpson [Sim92] studied Higgs bundles for general G over compact Kähler manifolds. For our purposes, we will focus on G-Higgs bundles over compact Riemann surfaces.

Let Σ be a closed Riemann surface of genus $g \geq 2$ and $K = T^{*,(1,0)}\Sigma$ be the canonical bundle. Let G be a real reductive Lie group with maximal compact H and fix a Cartan involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Definition 3.3.1. A G-Higgs bundle over Σ is a pair $(\mathcal{E}, \varphi)$ where

- $\mathcal{E} \to \Sigma$ is a holomorphic principal $\mathsf{H}_\mathbb{C}$-bundle
- $\varphi \in H^0(\Sigma, \mathcal{E}[\mathfrak{m}_\mathbb{C}] \otimes K)$ (the Higgs field)

Remark 3.3.2. By Corlette's Theorem, for every representation $\rho \in \mathcal{X}(\pi_1(S), \mathsf{G})$ there is a corresponding Higgs bundle. The construction works as follows: Let $h_\rho : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}$ be a harmonic metric, the Higgs bundle associated to ρ is given by $(h_\rho^* \mathsf{G}[\mathsf{H}_\mathbb{C}], (h_\rho^* \omega_{MC})^{1,0})$. Here we are pulling back the H bundle (with total space G) and extending the structure group to obtain a holomorphic $\mathsf{H}_\mathbb{C}$-bundle, and pulling back the complexification of the Maurer-Cartan form and taking its $(1,0)$ part to obtain a holomorphic section of $(h_\rho^* \mathsf{G}[\mathfrak{m}_\mathbb{C}]) \otimes K$.

Remark 3.3.3. Under the correspondence between harmonic metrics and Higgs bundles, the Higgs field ϕ is identified with $d^{(1,0)} h_\rho$. In particular, the Hopf differential of h_ρ is a constant multiple of the holomorphic quadratic differential $Tr(\phi^2)$.

We start with some examples:

Example 3.3.4. G-compact: When the group is compact, then $\mathfrak{h}_\mathbb{C} = \mathfrak{g}_\mathbb{C}$ and $\mathfrak{m}_\mathbb{C} = \{0\}$. In this case, a G-Higgs bundle is just a holomorphic $\mathsf{G}_\mathbb{C}$ bundle over Σ.

Example 3.3.5. G-complex: If G is a complex Lie group, then $\mathfrak{h}_\mathbb{C} = \mathfrak{g}$ and $\mathfrak{m}_\mathbb{C} = \mathfrak{g}$. Thus, a G-Higgs bundle is given by a holomorphic G-bundle $\mathcal{E}$ over Σ together with a holomorphic section of the adjoint bundle twisted by K, $\varphi \in H^0(\Sigma, \mathcal{E}[\mathfrak{g}] \otimes K)$. When $\mathsf{G} = \mathsf{SL}(n,\mathbb{C})$ this data is equivalent to a holomorphic vector bundle $\mathcal{E}$ of rank n with fixed determinant together with a traceless holomorphic section of $\mathsf{End}(\mathcal{E}) \otimes K$.

Example 3.3.6. $\mathsf{G} = \mathsf{SL}(n,\mathbb{R})$: For the group $\mathsf{SL}(n,\mathbb{R})$, the maximal compact subgroup is $\mathsf{SO}(n) \subset \mathsf{SL}(n,\mathbb{R})$. The Cartan decomposition of $\mathfrak{sl}(n,\mathbb{R})$ is given by $\mathfrak{so}(n) \oplus sym_0(\mathbb{R}^n)$, thus $\mathfrak{m}_\mathbb{C}$ is the space of complex traceless symmetric $n \times n$ matrices $sym_0(\mathbb{C}^n)$. An $\mathsf{SL}(n,\mathbb{R})$-Higgs bundle consists of a holomorphic $\mathsf{SO}(n,\mathbb{C})$-bundle $\mathcal{E}$

over Σ together with a holomorphic section of $\varphi \in H^0(\mathcal{E}[sym_0(\mathbb{C}^n)] \otimes K)$. Using the standard representation of $\mathsf{SO}(n,\mathbb{C})$ on $\mathbb{C}^n$, this data is equivalent to a triple $(\mathcal{E}, Q, \phi)$ where:

- $(\mathcal{E}, Q)$ is a holomorphic bundle rank n with $det(\mathcal{E}) = \mathcal{O}$ and an orthogonal structure, which we will think of as a symmetric holomorphic isomorphism $Q : \mathcal{E} \to \mathcal{E}^*$.
- ϕ is a traceless holomorphic section ϕ of $\mathsf{End}(\mathcal{E})$ that is symmetric, i.e. $Q\phi^T Q = \phi$.

The associated $\mathsf{SL}(n,\mathbb{C})$ Higgs bundle is given by forgetting the orthogonal structure $(\mathcal{E}, \phi)$.

Example 3.3.7. $\mathsf{G} = \mathsf{Sp}(2n,\mathbb{R})$: The maximal compact subgroup of $\mathsf{Sp}(2n,\mathbb{R})$ is $\mathsf{H} \cong \mathsf{U}(n)$, and, if $Sym^2(V)$ is the second symmetric tensor product of the standard representation of $\mathsf{GL}(n,\mathbb{C})$, the complexification of the Cartan decomposition is given by

$$\mathfrak{sp}(2n,\mathbb{C}) \cong \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C}) \oplus (Sym^2(V) \oplus Sym^2(V^*)).$$

Thus a $\mathsf{Sp}(2n,\mathbb{R})$ Higgs bundle is given a holomorphic $\mathsf{GL}(n,\mathbb{C})$-bundle $\mathcal{E}$ together with a holomorphic section $\varphi \in H^0(\Sigma, \mathcal{E}[Sym^2(V) \oplus Sym^2(V^*]) \otimes K)$. This data is equivalent to a triple $(\mathcal{V}, \beta, \gamma)$ where:

- $\mathcal{V}$ is a holomorphic rank n vector bundle
- $\beta \in H^0(\Sigma, Sym^2(\mathcal{V}^*) \otimes K)$ and $\gamma \in H^0(\Sigma, Sym^2(\mathcal{V}) \otimes K)$

The associated $\mathsf{SL}(2n,\mathbb{C})$ bundle is given by

$$(\mathcal{E}, \phi) = \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right).$$

Example 3.3.8. $\mathsf{G} = \mathsf{SO}_0(p,q)$: The maximal compact of $\mathsf{SO}_0(p,q)$ is $\mathsf{H} = \mathsf{SO}(p) \times \mathsf{SO}(q)$. If V and W denote the standard representations of $\mathsf{SO}(p,\mathbb{C})$ and $\mathsf{SO}(q,\mathbb{C})$ respectively then complexified Cartan decomposition of $\mathfrak{so}(p,q)$ is given by

$$\mathfrak{so}(p+q,\mathbb{C}) \cong \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \cong (\mathfrak{so}(p,\mathbb{C}) \oplus \mathfrak{so}(q,\mathbb{C})) \oplus (V^* \otimes W).$$

Thus a $\mathsf{SO}(p,q)$ Higgs bundle is given a holomorphic $\mathsf{SO}(p,\mathbb{C}) \times \mathsf{SO}(q,\mathbb{C})$-bundle $\mathcal{E}$ together with a holomorphic section $\varphi \in H^0(\Sigma, \mathcal{E}[(V^* \otimes W)] \otimes K)$. This data is equivalent to the data $(\mathcal{V}, Q_{\mathcal{V}}, \mathcal{W}, Q_{\mathcal{W}}, \eta)$ where:

- $(\mathcal{V}, Q_{\mathcal{V}})$ and $(\mathcal{W}, Q_{\mathcal{W}})$ are holomorphic orthogonal bundles of rank p and q respectively.
- $\eta \in H^0(\Sigma, \mathcal{V}^* \otimes \mathcal{W} \otimes K)$.

If $\eta^* : \mathcal{W}^* \to \mathcal{V}^* \otimes K$ is the induced on the duals, then define $\eta^T = (Q_{\mathcal{V}} \otimes Id_K) \circ \eta^* \circ Q_{\mathcal{W}} : \mathcal{W} \to \mathcal{V} \otimes K$:

$$\mathcal{W} \xrightarrow{Q_{\mathcal{W}}} \mathcal{W}^* \xrightarrow{\eta^*} \mathcal{V}^* \otimes K \xrightarrow{Q_{\mathcal{V}} \otimes Id_K} \mathcal{V} \otimes K \ .$$

The $\mathsf{SL}(p+q,\mathbb{C})$-Higgs bundle associated to the data $(\mathcal{V}, Q_{\mathcal{V}}, \mathcal{W}, Q_{\mathcal{W}}, \eta)$ is given by

$$(\mathcal{E}, \phi) = \left(\mathcal{V} \oplus \mathcal{W}, \begin{pmatrix} 0 & \eta^T \\ \eta & 0 \end{pmatrix} \right).$$

We will sometimes need the more general notion of a L-twisted Higgs pair.

Definition 3.3.9. Let $\mathsf{H}_{\mathbb{C}}$ be a complex reductive Lie group and $\rho : \mathsf{H}_{\mathbb{C}} \to \mathsf{GL}(V)$ be a linear representation. If L is holomorphic line bundle then an *L-twisted Higgs pair* over Σ is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E} \to \Sigma$ is a holomorphic principal $\mathsf{H}_{\mathbb{C}}$ bundle and φ is a holomorphic section of $(\mathcal{E} \times_\rho V) \otimes L$.

Remark 3.3.10. We will only consider K-twisted Higgs pairs. Note that if $V = \mathfrak{m}_{\mathbb{C}}$ and the representation $\rho : \mathsf{H}_{\mathbb{C}} \to \mathsf{GL}(\mathfrak{g}_{\mathbb{C}})$ is the restriction of the adjoint action of $\mathsf{G}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ then we recover the definition of a Higgs bundle.

3.3.1 Stability and moduli spaces

The moduli space of G-Higgs bundles consists of isomorphism classes of semistable G-Higgs bundles. The notion of stability for $\mathsf{SL}(n,\mathbb{C})$-Higgs bundles is a straight forward slope condition on invariant subbundles, however, for general G it is significantly more subtle. We start with the definition of the degree.

Definition 3.3.11. Let P be a complex Lie group, $\mathcal{E} \to \Sigma$ be a holomorphic P bundle, and let χ be a character of P. Define the degree $deg(\mathcal{E}, \chi)$ to be the degree of the associated $\mathbb{C}^*$ bundle $\mathcal{E} \times_\chi \mathbb{C}^*$

$$deg(\mathcal{E}, \chi) = deg(\mathcal{E} \times_\chi \mathbb{C}^*).$$

Remark 3.3.12. If we have a character of $\chi : \mathfrak{p} \to \mathbb{C}$ of the Lie algebra such that $exp(n\chi) : \mathsf{P} \to \mathbb{C}^*$ defines a character of the group, then we can define the degree of $\mathcal{E}$ by normalizing the degree of the associated line bundle.

$$deg(\mathcal{E}, exp(n\chi)) = \frac{1}{n} deg(\mathcal{E} \times_\chi \mathbb{C}).$$

For $\mathsf{P} \subset \mathsf{H}_{\mathbb{C}}$ a parabolic subgroup and χ an antidominant character of $\mathfrak{p}$, let $s_\chi \in \mathfrak{p}$ be the element corresponding to $\chi \in \mathfrak{p}^*$. Define the subset $(\mathfrak{m}_{\mathbb{C}}^-)_\chi \subset \mathfrak{m}_{\mathbb{C}}$ to be the set of $v \in \mathfrak{m}_{\mathbb{C}}$ so that $Ad_{e^{ts_\chi}} v$ is bounded as $t \to \infty$ and define $(\mathfrak{m}_{\mathbb{C}}^0)_\chi \subset \mathfrak{m}_{\mathbb{C}}$ to be the set of $v \in \mathfrak{m}_{\mathbb{C}}$ so that $Ad_{e^{ts_\chi}} v = v$.

Given a P bundle E_{P} and an anti dominant character χ, denote the associated $(\mathfrak{m}^-_{\mathbb{C}})_\chi$-bundle by $E_\chi[\mathfrak{m}^-_{\mathbb{C}}]$. If $\mathsf{L} \subset \mathsf{P}$ is the Levi factor of P and F is an L-bundle, denote the corresponding $(\mathfrak{m}^0_{\mathbb{C}})_\chi$ bundle by by $F_\chi[\mathfrak{m}^0_{\mathbb{C}}]$.

To form a moduli space, we need to discuss the appropriate notion of stability, for this general set up, the reference is [GGMiR09].

Definition 3.3.13. A G-Higgs bundle $(\mathcal{E}, \varphi)$ is:

- *semistable* if, for any parabolic subgroup $\mathsf{P} \subset \mathsf{H}_{\mathbb{C}}$, any strictly antidominant character χ of $\mathfrak{p}$ and any holomorphic reduction $\sigma \in H^0(\mathcal{E}(\mathsf{H}_{\mathbb{C}}/\mathsf{P}))$ such that $\varphi \in H^0(\sigma^*\mathcal{E}_\chi[\mathfrak{m}^-_{\mathbb{C}}] \otimes K)$, we have $deg(E)(\sigma, \chi) \leq 0$.
- *stable* if the inequality is always strict, $deg(E)(\sigma, \chi) < 0$.
- *polystable* if it is semistable and for any parabolic subgroup $\mathsf{P} \subset \mathsf{H}_{\mathbb{C}}$, any *strictly* antidominant character χ of $\mathfrak{p}$ and any holomorphic reduction $\sigma \in H^0(\mathcal{E}(\mathsf{H}_{\mathbb{C}}/\mathsf{P}))$ such that $\varphi \in H^0(\sigma^*\mathcal{E}_\chi[\mathfrak{m}^-_{\mathbb{C}}] \otimes K)$ with $deg(E)(\sigma, \chi) = 0$, there is a further *holomorphic* reduction of structure group σ_{L} of the P-bundle $\sigma^*\mathcal{E}_{\mathsf{H}_{\mathbb{C}}}$ to L. Furthermore, with respect to this reduction, $\varphi \in H^0(\sigma_L^*\mathcal{E}_\chi[\mathfrak{m}^0_{\mathbb{C}}] \otimes K)$.

Recall that the $\mathsf{H}_{\mathbb{C}}$-gauge group $\mathcal{G}_{\mathsf{H}_{\mathbb{C}}}$ is the group of *smooth* bundle automorphisms of a $\mathsf{H}_{\mathbb{C}}$ bundle $E_{\mathsf{H}_{\mathbb{C}}}$. A gauge transformation $g \in \mathcal{G}_{\mathsf{H}_{\mathbb{C}}}$ acts on a Higgs field ϕ by $g \cdot \phi = Ad_g \phi$ where Ad_g denotes the restriction of the adjoint action of g on $\mathcal{E}[\mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}] \otimes K$.

Definition 3.3.14. The moduli space of G-Higgs bundles $\mathcal{M}(\mathsf{G})$ is defined to the set of isomorphism classes of polystable G-Higgs bundles.

Remark 3.3.15. The set $\mathcal{M}(\mathsf{G})$ described above can be given the structure of a quasi-projective complex variety as in [Hit87a, Sim92]. When the group G is complex semisimple the moduli space $\mathcal{M}(\mathsf{G})$ has a hyperKahler structure [Hit87a, Sim92].

For most cases we will consider, $\mathsf{SL}(n, \mathbb{C})$-stability will be sufficient.

Definition 3.3.16. ($\mathsf{SL}(n, \mathbb{C})$-stability) An $\mathsf{SL}(n, \mathbb{C})$-Higgs bundle (E, ϕ) is *semistable* if all ϕ-invariant subbundles $F \subset E$ satisfy $deg(F) \leq 0$ and *stable* all ϕ invariant subbundles F satisfy $deg(F) < 0$. A semistable Higgs bundle (E, ϕ) is *polystable* if it decomposes as a direct sum of stable Higgs bundles $(E, \phi) = \bigoplus_j (E_j, \phi_j)$.

Definition 3.3.17. Given a Higgs bundle $(\mathcal{E}, \phi)$ define the automorphism group by

$$\mathsf{Aut}(\mathcal{E}, \phi) = \{g \in \mathcal{G}_{\mathsf{H}_{\mathbb{C}}} | g \cdot (\mathcal{E}, \phi) = (\mathcal{E}, \phi)\}$$

and the *infinitesimal* automorphism group by

$$\mathsf{aut}(\mathcal{E},\phi)=\{s\in H^0(E[\mathfrak{h}_{\mathbb{C}}])|ad_s\phi=0\}$$

Define a Higgs bundle $(\mathcal{E},\phi)$ to be *infinitesimally simple* if

$$\mathsf{aut}(\mathcal{E},\phi)=H^0(E[\mathfrak{z}(\mathfrak{h}_{\mathbb{C}})\cap Ker(Ad:\mathsf{H}_{\mathbb{C}}\to\mathfrak{m}_{\mathbb{C}})]).$$

Being infinitesimally simple is equivalent to the dimension of $\mathsf{Aut}(\mathcal{E},\phi)$ being the same as $Z(\mathsf{H}_{\mathbb{C}})\cap Ker(Ad:\mathsf{H}_{\mathbb{C}}\to\mathfrak{m}_{\mathbb{C}})$.

Definition 3.3.18. A Higgs bundle $(\mathcal{E},\phi)$ is *simple* if $\mathcal{Z}(\mathcal{E},\phi)=Z(\mathsf{H}_{\mathbb{C}})\cap Ker(Ad:\mathsf{H}_{\mathbb{C}}\to\mathfrak{m}_{\mathbb{C}})$.

Proposition 3.3.19. *(See section 3 of [GGMiR09]) If $(\mathcal{E},\phi)$ be a stable and simple G-Higgs bundle that is stable as a $\mathsf{G}_{\mathbb{C}}$ Higgs bundle, then the isomorphism class of $(\mathcal{E},\phi)$ in $\mathcal{M}(\mathsf{G})$ is a smooth point.*

3.3.2 Hitchin fibration and Hitchin component

For G a complex semisimple Lie group of rank ℓ, let $p_1,\cdots,p_\ell$ be a basis of the G-invariant polynomials $\mathbb{C}[\mathfrak{g}]^{\mathsf{G}}$. If $\{m_j\}$ are the exponents of $\mathfrak{g}$ then $deg(p_j)=m_j+1$. Since the polynomials are Ad_{G}-invariant, they can be evaluated on the Higgs field of a G-Higgs bundle $(\mathcal{E},\varphi)$, and $p_j(\varphi)\in H^0(K^{m_j+1})$.

Definition 3.3.20. The map $H:\mathcal{M}(\mathsf{G})\longrightarrow\bigoplus_{j=1}^{\ell}H^0(K^{m_j+1})$ obtain by applying $(p_1,\cdots,p_\ell)$ is called the Hitchin fibration. The space $\bigoplus_{j=1}^{\ell}H^0(K^{m_j+1})$ is called the Hitchin base.

In [Hit87b], Hitchin proved that the map H is proper and has abelian varieties as generic fibers. Moreover, Hitchin proved that $H:\mathcal{M}(\mathsf{G})\longrightarrow\bigoplus_{j=1}^{\ell}H^0(K^{m_j+1})$ defines a completely integrable system. While the integrable system aspects of the Hitchin fibration are extremely important, they will not play a role in the rest of the thesis.

In [Hit92], Hitchin showed that there is a section of the above fibration whose image naturally generalizes the Teichmüller component of Example 3.3.21. The definition of this section relies on Kostant's principal three dimensional subalgebra and works as follows.

Example 3.3.21. Consider $\mathsf{SL}(2,\mathbb{R})$-Higgs bundles given by $(K^{\frac{1}{2}}\oplus K^{-\frac{1}{2}},\begin{pmatrix}0&q_2\\1&0\end{pmatrix})$ where $K^{\frac{1}{2}}$ is a fixed square root of K and $q_2\in H^0(K^2)$ is a holomorphic quadratic differential. Up to scaling, there is only one invariant polynomial for $\mathsf{SL}(2,\mathbb{C})$ given by $p_1(X)=\frac{1}{2}Tr(X^2)$. Applying the invariant polynomial to ϕ above gives $p_1(\phi)=q_2$. This gives the Hitchin section for $\mathsf{SL}(2,\mathbb{C})$.

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{s} \subset \mathfrak{g}$ be the PTDS. Recall from (2.1.8) that the grading element x of the PTDS gives the height $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-m_\ell}^{m_\ell} \mathfrak{g}_j$ of $\mathfrak{g}$, here m_ℓ is the height of the highest root of $\mathfrak{g}$. The inclusion $x \to \mathfrak{g}$ defines an inclusion of $\mathbb{C} = \mathfrak{so}(2,\mathbb{C}) \subset \mathfrak{h}_\mathbb{C} \subset \mathfrak{g}$, where $\mathfrak{h}_\mathbb{C}$ is the complexification of the maximal compact Lie algebra of the split real form of $\mathfrak{g}$. If G is the adjoint group of $\mathfrak{g}$ and $\mathsf{H}_\mathbb{C}$ is the complexification of the maximal compact of the split real form of G then we have an inclusion $i : \mathsf{SO}(2,\mathbb{C}) \to \mathsf{H}_\mathbb{C} \subset \mathsf{G}$. Denote the principal $\mathsf{SO}(2,\mathbb{C})$-bundle from Example 3.3.21 by $\mathcal{E}$, extending the structure group gives a holomorphic $\mathsf{H}_\mathbb{C}$ bundle $P_{\mathsf{H}_\mathbb{C}} = \mathcal{E} \times_i \mathsf{H}_\mathbb{C}$. Moreover, the $\mathfrak{g}$-bundle $\mathcal{E} \times_i \mathfrak{g}$ decomposes in terms of the $\mathbb{Z}$-grading defined by x:

$$\mathcal{E} \times_i \mathfrak{g} = \bigoplus_{j=-m_\ell}^{m_\ell} \mathfrak{g}_j \otimes K^j \tag{3.3.1}$$

It also decomposes into line bundles in terms of the irreducible representation $\bigoplus_{j=1}^{\ell} V_j$ of $ad : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{g})$ from (2.1.10). Recall from section 2.1.3 that the irreducible representations V_j have dimension $2m_j + 1$ where $\{m_j\}$ are the exponents of $\mathfrak{g}$. The highest weight vector e_j of each V_j has height m_j. If $\mathfrak{g} = \mathfrak{h}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C}$ is the complexified Cartan decomposition of the split real form, then $e_j \in \mathfrak{m}_\mathbb{C}$ by (2.1.11).

For $(q_{m_1+1}, q_{m_2+1}, \cdots, q_{m_\ell+1}) \in \bigoplus_{j=1}^{m_\ell} H^0(K^{m_j+1})$, define the Higgs field $\varphi \subset H^0(P \times_i \mathfrak{m}_\mathbb{C} \otimes K)$ by

$$\varphi = \tilde{e}_1 + \sum_{j=1}^{m_\ell} e_j \otimes q_{m_j+1} \tag{3.3.2}$$

Recall that Kostant [Kos59] showed that there is a basis $(p_1, \cdots, p_\ell)$ of the invariant polynomials $\mathbb{C}[\mathfrak{g}]^{Ad_\mathsf{G}}$ with the property that for all elements in $\mathfrak{g}$ of the form $\tilde{e}_1 + \sum_{j=1}^{m_\ell} y_j e_j$

$$p_j(\tilde{e}_1 + \sum_{j=1}^{m_\ell} y_j e_j) = y_j.$$

Thus we obtain:

Proposition 3.3.22. *The map* $s_H : \bigoplus_{j=1}^{m_\ell} H^0(K^{m_j+1}) \to \mathcal{M}(\mathsf{G})$ *given by*

$$s_H(q_{m_1+1}, \cdots, q_{m_\ell+1}) = (\mathcal{E} \times_i \mathsf{G}, \tilde{e}_1 + \sum_{j=1}^{m_\ell} e_j \otimes q_{m_j+1})$$

is a section of the Hitchin fibration. Moreover, if $\mathsf{G}_0 \subset \mathsf{G}$ *is the split real form then all Higgs bundles in the image of* s_H *are actually* G_0*-Higgs bundles.*

Using the above set up, Hitchin proved that the section s_H is onto a connected component of G_0 Higgs bundles.

Theorem 3.3.23. *(Hitchin [Hit92]) Let G_0 be a split real form of a simple complex Lie group. There exists a connected component of $\mathcal{M}(\mathsf{G}_0)$ which is diffeomorphic to the Hitchin base $\bigoplus_{j=1}^{m_\ell} H^0(K^{m_j+1})$.*

The Hitchin component will be the central focus of Chapter 5 and in Chapter 6 we will focus on certain generalizations of the Hitchin component.

3.3.3 Hitchin equations

To go from an isomorphism class of polystable Higgs bundles to a representations, one must produce a harmonic metric out of a Higgs bundle. This is the role of the Hitchin equations. Let $(\mathcal{E}, \varphi)$ be a G-Higgs bundle. A reduction of structure group $\sigma : \Sigma \to \mathcal{E}/\mathsf{H}$ gives an H-bundle $\sigma^*\mathcal{E} \subset \mathcal{E}$ and also a splitting of the adjoint bundle $\mathcal{E} \times_{\mathsf{H}_\mathbb{C}} (\mathfrak{h}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C}) = \sigma^* Ee \times_{\mathsf{H}} (\mathfrak{h} \oplus i\mathfrak{h} \oplus \mathfrak{m} \oplus i\mathfrak{m})$. Moreover, if $\varphi \in H^0(\Sigma, [\mathfrak{m}_\mathbb{C}] \otimes K)$ and a compact real form τ is fixed on $\mathfrak{g}_\mathbb{C}$ with $\sigma\tau$ giving the real form $\mathfrak{g}$, then the 1-form $\varphi - \tau(\varphi)$ satisfies:

$$\tau(\varphi - \tau(\varphi)) = -(\varphi - \tau(\varphi)). \tag{3.3.3}$$

Thus $\varphi - \tau\varphi$ takes values in $\sigma^*\mathcal{E} \times_{\mathsf{H}} \mathfrak{m}$. Given a metric connection A, the connection $A + \varphi - \tau\varphi$ is a G-connection on $\sigma^*\mathcal{E} \times_{\mathsf{H}} \mathsf{G}$. Moreover, $-\tau(\varphi)$ is the Hermitian adjoint of φ with respect to the metric induced by the Killing form.

Given a holomorphic $\mathsf{H}_\mathbb{C}$-bundle $\mathcal{E}_{\mathsf{H}_\mathbb{C}}$ and a reduction of structure $\mathcal{E}_{\mathsf{H}} \subset \mathcal{E}_{\mathsf{H}_\mathbb{C}}$ (i.e. a metric), there is a unique connection A (called the Chern connection) that is compatible with both the holomorphic structure and the metric reduction. In other words, there is a unique connection A on $\mathcal{E}_H$ such that the $(0,1)$ part of A induces the holomorphic structure on $\mathcal{E}_{\mathsf{H}_\mathbb{C}}$. For holomorphic vector bundles, this is classical, for instance see [Kob87]. For the general set up see [MiR00].

Theorem 3.3.24. *Let $(\mathcal{E}, \varphi)$ be a polystable G-Higgs bundle and fix a Cartan involution τ on $\mathfrak{g}_\mathbb{C}$, then there exists a reduction of structure of $\mathcal{E}$ from $\mathsf{H}_\mathbb{C}$ to H which solves the following equations*

$$F_A + [\varphi, -\tau(\varphi)] = 0 \qquad \text{and} \qquad \nabla_A^{(0,1)}\varphi = 0 \tag{3.3.4}$$

where A is the Chern connection of the reduction. Moreover, if $(\mathcal{E}, \varphi)$ is stable, then the metric reduction σ is unique.

Note that, by definition of the Chern connection, the equation $\nabla_A^{(0,1)}\varphi = 0$ just says φ is holomorphic.

This theorem was originally proven by Hitchin [Hit87a] for $\mathsf{G} = \mathsf{SL}(2, \mathbb{C})$ and extended to all complex groups by Simpson [Sim92]. The form stated above can be found in [BGPMiR03, GGMiR09].

Remark 3.3.25. From a solution to Hitchin's equations we obtain a flat G-connection $D = A + \phi - \tau(\phi)$, thus giving a map from the Higgs bundle moduli space to the G-character variety. Since the Hitchin equations are the same as the interpretation of the harmonic metric equations in (3.2.5) with $\phi = \psi^{(1,0)}$ and $-\tau(\phi) = \psi^{(0,1)}$, the induced reduction of structure of the flat G-bundle given by σ can be interpreted as an equivariant *harmonic map* $h_D : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}$. This gives an equivalence between harmonic metrics on flat G-bundles and polystable G-Higgs bundles.

Theorem 3.3.24 completes the correspondence between the moduli space of G-Higgs bundles and the G-character variety. In fact, the bijection $\mathcal{M}(\mathsf{G}) \longleftrightarrow \mathcal{X}(\pi_1, \mathsf{G})$ defines a homeomorphism [Sim92].

Remark 3.3.26. For the group $\mathsf{SL}(n, \mathbb{C})$, Theorem 3.3.24 says that given a stable Higgs bundle (E, ϕ), there exists a unique metric hermitian metric H (with Chern connection A) on E which solves the equation $F_A + [\phi, \phi^{*_H}] = 0$.

Example 3.3.27. Recall that Higgs bundles in the $\mathsf{SL}(2, \mathbb{R})$-Hitchin component are given by a choice of square root of K and a holomorphic quadratic differential $q_2 \in H^0(K^2)$. The corresponding $\mathsf{SL}(2, \mathbb{C})$-Higgs bundle is defined by

$$(\mathcal{E}, \phi) = (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}).$$

A metric H on $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ which solves $F_A + [\phi, \phi^*] = 0$ splits as $H = h \oplus h^{-1}$ since it comes from a metric on a $\mathsf{SO}(2, \mathbb{C})$-bundle. The adjoint of ϕ is given by

$$\phi^* = \begin{pmatrix} h^{-1} & \\ & h \end{pmatrix} \begin{pmatrix} & 1 \\ \bar{q}_2 & \end{pmatrix} \begin{pmatrix} h & \\ & h^{-1} \end{pmatrix} = \begin{pmatrix} & h^{-2} \\ h^2 \bar{q}_2 & \end{pmatrix}$$

and the Hitchin equations are given by

$$F_{A_h} + h^2 q_2 \wedge \bar{q}_2 - h^{-2} = 0. \tag{3.3.5}$$

In [Hit87a], Hitchin showed that solving for the metric h on $K^{\frac{1}{2}}$ is equivalent to finding a metric on Σ with constant negative curvature. Moreover, if $q_2 = 0$ then the corresponding hyperbolic metric is the which uniformizes the surface Σ. In this way, for each Riemann surface structure Σ on S, Hitchin parameterized $\mathsf{Teich}(S)$ by $H^0(K^2)$. Since $Tr(\phi^2) = 2q_2$, this is equivalent to the Hopf differential parameterization of $\mathsf{Teich}(S)$ by Wolf [Wol89].

Remark 3.3.28. Note that if $\mathsf{G} \subset \mathsf{SL}(N, \mathbb{C})$ is a real form a subgroup of $\mathsf{SL}(N, \mathbb{C})$ then the inclusion induces a map gives a map between the moduli spaces $\mathcal{M}(\mathsf{G}) \to \mathcal{M}(\mathsf{SL}(N, \mathbb{C}))$. In particular, a G-Higgs bundle is polystable if and only if the corresponding $\mathsf{SL}(N, \mathbb{C})$ Higgs bundle is polystable. Equivalently, if the corresponding $\mathsf{SL}(N, \mathbb{C})$-Higgs bundle is unstable, then the G-Higgs bundle is also unstable. Thus, when determining whether or not the isomorphism class of a G-Higgs bundle defines a point in the moduli space, we can use the simpler version of stability for $\mathsf{SL}(N, \mathbb{C})$-Higgs bundles.

We will also need a slightly more general theorem concerning K-twisted Higgs pairs (see Definition 3.3.9). These objects are an instance of the more general notion of an augmented bundle. Through the work of many authors, including Bradlow, Garcia-Prada, King, and Mundet, the notions of stability have appropriate generalizations to the setting of augmented bundles. For this more general set up, the analog of Theorem 3.3.24 also holds, see [BGPMiR03, GGMiR09].

Theorem 3.3.29. *Let $(\mathcal{E}, \varphi)$ be a polystable K-twisted Higgs pair, then there exists a reduction of structure of $\mathcal{E}$ from $\mathsf{H}_{\mathbb{C}}$ to H which solves the following equations*

$$F_A + [\varphi, \varphi^*] = 0 \qquad \text{and} \qquad \nabla_A^{(0,1)} \varphi = 0 \tag{3.3.6}$$

where A is the Chern connection on $\mathcal{E}(V)$ induced by the reduction and φ^* is the hermitian adjoint with respect to the metric reduction.

Chapter 4

Fixed points

The Higgs bundle moduli space $\mathcal{M}(\mathsf{G})$ has a natural action of $\mathbb{C}^*$ defined by scaling the Higgs field. For $\lambda \in \mathbb{C}^*$, the action is given by $\lambda \cdot (\mathcal{E}, \varphi) = (\mathcal{E}, \lambda\varphi)$. The fixed points of this action are the critical points of the function on $\mathcal{M}(\mathsf{G})$ defined by taking the L^2-norm of the Higgs field with respect to the harmonic metric:

$$H(\mathcal{E}, \varphi) = \int_\Sigma ||\varphi||^2.$$

This function is a Morse-Bott function and is usually called the Hitchin function [Hit87a, Sim92]. Thus, studying the fixed points of the $\mathbb{C}^*$-action gives information on the topology of the Higgs bundles moduli space. This has been successfully carried out by many authors, for instance [Hit92, Sim92, Got01, GPGMiR13, BGPG03].

In this chapter we study the Higgs bundles which are fixed by a root of unity subgroup $\langle \zeta_k \rangle \subset \mathbb{C}^*$. For complex simple Lie groups, we classify the Higgs bundles fixed by this action. We start by recalling the work of Simpson [Sim09] for $\mathsf{SL}(n, \mathbb{C})$ and discuss how the Hitchin equations for these fixed points simplify to a version of K-twisted quiver bundle equations considered in [ÁCGP03]. This relation will be important for the asymptotics considered in Chapter 5. After relating Higgs bundles fixed by the $\mathbb{C}^*$ action to $\mathbb{Z}$-gradings gradings on Lie algebras we classify the fixed points of $\langle \zeta_k \rangle \subset \mathbb{C}^*$ in $\mathcal{M}(\mathsf{G})$ for a complex simple Lie group G. Finally, we discuss the relation between these fixed points and the equivariant harmonic map from Corlette's Theorem, this analysis will be crucial for Chapter 6.

4.1 $\mathsf{SL}(n, \mathbb{C})$ and relation with quiver bundles

Recall that an $\mathsf{SL}(n, \mathbb{C})$-Higgs bundle is given by a pair $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a holomorphic vector bundle with trivial determinant and $\phi \in H^0(\Sigma, \mathsf{End}_0(\mathcal{E}) \otimes K)$ is a traceless twisted endomorphism. The fixed points we will study are special types of twisted quiver bundles developed by [ÁCGP03].

Definition 4.1.1. A K-twisted quiver bundle is a collection of holomorphic vector bundles $\{\mathcal{E}_j\}_{j=1}^k$ together with a collection of holomorphic K-twisted bundle maps $\Phi_{ij} : \mathcal{E}_i \to \mathcal{E}_j \otimes K$.

For quiver bundles, there is a stability condition which has a parameter developed in [ÁCGP03]. When the stability parameter is 0, there is a close relation between quiver bundles and Higgs bundles; we will discuss this case below. This stability condition is used to prove the following theorem.

Theorem 4.1.2. *Given a 0-stable K-twisted quiver bundle $(\{\mathcal{E}_j\}_{j=1}^k, \{\Phi_{ij}\})$ with $det(\bigoplus \mathcal{E}_j) = \mathcal{O}$, there is a unique collection of metrics $\{h_j\}_{j=1}^k$ on the bundles $\{\mathcal{E}_j\}_{j=1}^k$ which solve the quiver bundle equations:*

$$F_{h_j} + \sum_{i,k} \Phi_{kj}^* \wedge \Phi_{jk} + \Phi_{ji} \wedge \Phi_{ij}^* = 0.$$

Here $\Phi_{ij}^* : \mathcal{E}_j \to \mathcal{E}_i \otimes \overline{K}$ is the adjoint defined with respect to the metrics h_i and h_j.

Remark 4.1.3. Given a K-twisted quiver bundle $(\{\mathcal{E}_j\}_{j=1}^k, \{\Phi_{ij}\})$ the holomorphic bundle $\mathcal{E} = \bigoplus\limits_{j=1}^{k} \mathcal{E}_j$ together with $\{\Phi_{ij}\}$ define an $\mathsf{GL}(\sum_j rank(\mathcal{E}_j), \mathbb{C})$-Higgs bundle. Moreover, the stability condition of [ÁCGP03] has the property that the quiver bundle $(\{\mathcal{E}_j\}_{j=1}^k, \{\Phi_{ij}\})$ is stable if and only if the corresponding Higgs bundle is stable. In this case there are two special metrics on $\mathcal{E}$, the quiver bundle metric and the Higgs bundle metric. In general, if the holomorphic bundle $\mathcal{E}$ admits a holomorphic decomposition, such a splitting is not orthogonal with respect to the Higgs bundle metric. We will show that, for fixed points, the holomorphic splitting is indeed orthogonal, and hence the quiver bundle metric and the Higgs bundle metric agree.

The fixed points of the $\mathbb{C}^*$ correspond to special K-twisted quiver bundles called holomorphic chains.

Definition 4.1.4. A K-twisted holomorphic chain is a K-twisted quiver bundle $(\{\mathcal{E}_j\}_{j=1}^k, \{\Phi_{ij}\})$ with $\Phi_{ij} = 0$ if $i+1 \neq j$. Set $\Phi_{j,j+1} = \phi_j$, we will represent K-twisted holomorphic chains by

$$\mathcal{E}_1 \xrightarrow[\phi_1]{} \mathcal{E}_2 \xrightarrow[\phi_2]{} \cdots \xrightarrow[\phi_{k-2}]{} \mathcal{E}_k \xrightarrow[\phi_{k-1}]{} \mathcal{E}_k \tag{4.1.1}$$

where the twisting has been suppressed from he notation.

The relation between fixed points of $\mathbb{C}^*$ action and holomorphic chains is straight forward.

Proposition 4.1.5. *Let $(\mathcal{E}, \phi)$ be a polystable $\mathsf{SL}(n, \mathbb{C})$-Higgs bundle with $(\mathcal{E}, \phi) \cong (\mathcal{E}, \lambda\phi)$ for all $\lambda \in \mathbb{C}^*$ then $(\mathcal{E}, \phi)$ is a K-twisted holomorphic chain with $k > 1$. If $(\mathcal{E}, \phi)$ is stable then each $\Phi_{j,j+1} \neq 0$.*

The proof of this proposition is very similar to classifying fixed points of roots of unity actions given below. The fact that $(\mathcal{E}, \phi)$ stable implies that each Higgs field components $\phi_j \neq 0$ follows from the fact that if some ϕ_j is zero then $(\mathcal{E}, \phi)$ has an invariant subbundle with an invariant compliment, and it is strictly polystable. Given a K-twisted holomorphic chain as in (4.1.2) the quiver bundle equations simplify to:

$$\begin{aligned} F_{h_1} + \phi_1^* \wedge \phi_1 &= 0 \\ F_{h_j} + \phi_j^* \wedge \phi_j + \phi_{j-1} \wedge \phi_{j-1}^* &= 0 \qquad\qquad 2 \leq j \leq k-1 \\ F_{h_k} + \phi_{k-1} \wedge \phi_{k-1}^* &= 0 \end{aligned} \tag{4.1.2}$$

For $\mathsf{SL}(n, \mathbb{C})$, Higgs bundles which are fixed by a root of unity subgroup $\left\langle e^{\frac{2\pi i}{k}} \right\rangle = \langle \zeta_k \rangle \subset \mathbb{C}^*$ but not necessarily all of $\mathbb{C}^*$ were first studied and classified by Simpson in [Sim09]. We will give a very explicit proof of this classification.

Theorem 4.1.6. *Let $(\mathcal{E}, \phi)$ be a stable $\mathsf{SL}(n, \mathbb{C})$-Higgs bundle, $(\mathcal{E}, \phi)$ is a fixed point of $\langle \zeta_k \rangle$ if and only if either $(\mathcal{E}, \phi)$ is fixed by all of $\mathbb{C}^*$ or $(\mathcal{E}, \phi)$ is a K-twisted quiver bundle $(\{\mathcal{E}_j\}_{j=1}^k, \{\Phi_{ij}\})$ with $\Phi_{ij} \neq 0$ if and only if $i + 1 = j$ mod k. Setting $\Phi_{j,j+1} = \phi_j$, such fixed points are given by:*

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r, "\phi_1"'] & \mathcal{E}_2 \arrow[r, "\phi_2"'] & \cdots \arrow[r, "\phi_{k-2}"'] & \mathcal{E}_k \arrow[r, "\phi_{k-1}"'] & \mathcal{E}_k \arrow[llll, bend right, "\phi_k"'] \end{tikzcd} \tag{4.1.3}$$

Proof. Clearly if $(\mathcal{E}, \phi)$ is a fixed point of the $\mathbb{C}^*$ action then it is a fixed point of the k^{th}-roots of unity action, so let $(\mathcal{E}, \phi)$ be of the second type above. To see that a Higgs bundle of the form (4.1.3) is fixed by $\langle \zeta_k \rangle$, consider the following gauge transformation of $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$:

$$g = \begin{pmatrix} Id_{\mathcal{E}_1} \zeta_{nk}^j & & & \\ & Id_{\mathcal{E}_2} \zeta_{nk}^j \zeta_k^1 & & \\ & & \ddots & \\ & & & Id_{\mathcal{E}_k} \zeta_{nk}^j \zeta_k^{k-1} \end{pmatrix}$$

It is straight forward to check that $Ad_g \phi = \zeta_k \phi$, furthermore, j can be chosen so that $det(g) = 1$.

Now assume that $(\mathcal{E}, \phi) = (\mathcal{E}, \zeta_k \phi)$ and $(\mathcal{E}, \phi)$ is stable, then there is a $\mathsf{SL}(n, \mathbb{C})$ holomorphic gauge transformation $g : \mathcal{E} \rightarrow \mathcal{E}$ so that $Ad_g \phi = \zeta_k \phi$ and $g^k = \zeta_n^j Id_{\mathcal{E}}$ for some integer j. Thus the eigenbundles of g can have eigenvalues $\zeta_{nk}^j \zeta_k^i$ for $0 \leq i \leq k-1$. Let $\{\zeta_{nk}^j \zeta_k^{a_i}\}_{i=1}^{l}$ be the distinct eigenvalues of g, and $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$ be the g-eigenbundle decomposition of $\mathcal{E}$. In this splitting g is given by

$$g = \begin{pmatrix} Id_{\mathcal{E}_1} \zeta_{nk}^j \zeta_k^{a_1} & & \\ & \ddots & \\ & & Id_{\mathcal{E}_l} \zeta_{nk}^j \zeta_k^{a_l} \end{pmatrix}$$

Write $\phi = \{\phi_{ij}\}$ in terms of the eigenbundle decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$. The action of Ad_g on ϕ_{ij} is given by $Ad_g(\phi_{ij}) = \zeta_k^{a_i - a_j}\phi_{ij}$. By assumption, $Ad_g\phi = \zeta_k^1\phi$, so

$$a_i - a_j \neq 1 \ mod \ k \implies \phi_{ij} = 0.$$

Thus, there are at most l nonzero ϕ_{ij}'s (at most 1 nonzero ϕ_{ij} per row and at most 1 nonzero ϕ_{ij} per column). Stability of $(\mathcal{E}, \phi)$ implies there must be at least $l-1$ nonzero ϕ_{ij}'s, otherwise there would be a ϕ invariant destabilizing bundle. If there are exactly $l-1$ nonzero ϕ_{ij}'s then $(\mathcal{E}, \phi)$ is a holomorphic chain, and thus a fixed point of the $\mathbb{C}^*$ action. If $(\mathcal{E}, \phi)$ is not a fixed point of the $\mathbb{C}^*$ action then there are exactly l nonzero ϕ_{ij}'s. Finally, if there are exactly l nonzero ϕ_{ij}'s then we have a collection of l distinct numbers $\{a_1, \ldots, a_l\}$ from the set $\{0, \ldots, k-1\}$ with exactly l pairwise differences equal to 1 *mod* k. This implies $l = k$, and proves $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_k$. If $\mathcal{E}_i$ is the eigenbundle with eigenvalue $\zeta_{nk}^j \zeta_k^i$, then the Higgs bundle is of the form of equation (4.1.3). ☐

Remark 4.1.7. The two cases in Theorem 4.1.6 are not disjoint. For instance, if $a = \begin{pmatrix} \phi_1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix}$, the holomorphic chain $E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} E_3$ can also be written as $\begin{tikzcd} E_1 \oplus E_3 \arrow[r, "a"'] & E_2 \arrow[l, bend right, "b"'] \end{tikzcd}$.

As a corollary, we have the following description of polystable fixed points.

Corollary 4.1.8. *Let $(\mathcal{E}, \phi) = \bigoplus_{j=1}^{l} (\mathcal{E}_j, \phi_j)$ be strictly polystable with each $(\mathcal{E}_j, \phi_j)$ stable. Then $(\mathcal{E}, \phi)$ is a fixed point of $\langle \zeta_k \rangle$ if and only if each $(\mathcal{E}_j, \phi_j)$ is fixed by $\langle \zeta_d \rangle$ for some d which divides k.*

The quiver bundle equations associated to Higgs bundles fixed by $\langle \zeta_k \rangle$ and not $\mathbb{C}^*$ are:

$$\begin{aligned} F_{h_1} + \phi_1^* \wedge \phi_1 + \phi_k \wedge \phi_k^* &= 0 \\ F_{h_j} + \phi_j^* \wedge \phi_j + \phi_{j-1} \wedge \phi_{j-1}^* &= 0 \qquad 2 \leq j \leq k-1 \\ F_{h_k} + \phi_k^* \wedge \phi_k + \phi_{k-1} \wedge \phi_{k-1}^* &= 0 \end{aligned} \tag{4.1.4}$$

For fixed points, the holomorphic splitting is orthogonal with respect to the Higgs bundle metric. Thus, the Higgs bundle metric is the same as the quiver bundle metric for fixed points.

Theorem 4.1.9. *Let $(\mathcal{E}, \phi)$ be a polystable Higgs bundle that is fixed by $\langle \zeta_k \rangle$, then the holomorphic decomposition of $\mathcal{E}$ in Theorem 4.1.6 is orthogonal with respect to the Higgs bundle metric. Moreover, the Higgs bundle equations simplify to the quiver bundle equations.*

This extra symmetry condition on the Higgs bundle metric is the starting point for all applications considered later. We will provide two proofs of Theorem 4.1.9, one which is direct and another which uses the quiver bundle results. Denote the Dolbeault operator associated to the holomorphic structure on $\mathcal{E}$ by $\bar{\partial}_{\mathcal{E}}$. Given a stable Higgs bundle $(\bar{\partial}_E, \phi)$ there is a unique metric H solving the Higgs bundle equations. For any $\mathsf{SL}(n,\mathbb{C})$-gauge transformation g, the pair $(g^{-1}\bar{\partial}_E g, g^{-1}\phi g)$ also has a unique metric H' solving the Higgs bundle equations. The metrics H and H' are related by $H' = Hg^{*_h}g$. This follows from general gauge theoretic arguments, for example see section 3 of [Bra90].

Proof. Let $(\mathcal{E}, \phi)$ be a stable Higgs bundle that is a fixed point of the $\langle\zeta_k\rangle$-action that is not fixed by $\mathbb{C}^*$, and let H be the metric on $\mathcal{E}$ solves the Higgs bundle equations. To see that the metric H splits, we will show the holomorphic gauge transformation $g : \mathcal{E} \to \mathcal{E}$ which acts as $g^{-1}\phi g = \zeta_k\phi$ is unitary, that is $g^{*_H}g = Id$. Since the triple $(\bar{\partial}_{\mathcal{E}}, \phi, H)$ solves the Higgs bundle equations, the triple $(g^{-1}\bar{\partial}_{\mathcal{E}}g, g^{-1}\phi g, Hg^{H}g)$ also solves the Higgs bundle equations. Since g is holomorphic $(g^{-1}\bar{\partial}_{\mathcal{E}}g, g^{-1}\phi g) = (\bar{\partial}_{\mathcal{E}}, \zeta_k\phi)$, and thus $(\bar{\partial}_{\mathcal{E}}, \zeta_k\phi, Hg^{*_H}g)$ solves the Higgs bundle equations as well. Now, using the fact that the $\mathsf{U}(1)$-action preserves the metric, the triple $(\bar{\partial}_{\mathcal{E}}, \phi, Hg^{*_H}g)$ solves the equations. By uniqueness of the metric, we conclude $g^{*_H}g = Id$. Recall that the splitting $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ is an eigenbundle splitting for g, since g is both unitary and preserves the eigenbundle splitting $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$, the metric H splits as $H = h_1 \oplus \cdots \oplus h_k$. ☐

Proof. (Quiver bundle proof) The proof for fixed points of $\mathbb{C}^*$ and $\langle\zeta_k\rangle$ are very similar, assume $(\mathcal{E}, \phi)$ is a stable fixed point of $\langle\zeta_k\rangle$ which is not fixed by $\mathbb{C}^*$. The holomorphic bundle decomposes as $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ with ϕ as in Theorem 4.1.6. By [ÁCGP03], there is a collection of metrics on $\{h_j\}$ on $\{\mathcal{E}_j\}$ which solve the quiver bundle equations (4.1.4). With respect the metric $H = h_1 \oplus \cdots \oplus h_l$ on $\mathcal{E}$ the adjoint of the Higgs is $\phi^{*_H} = H^{-1}\overline{\phi}^T H$

$$\phi^{*_H} = \begin{pmatrix} & h_1^{-1}\overline{\phi_1}^T h_2 & & \\ & & \ddots & \\ & & & h_{k-1}^{-1}\overline{\phi_{k-1}}^T h_k \\ h_k^{-1}\overline{\phi_k}^T h_1 & & & \end{pmatrix}.$$

Since $h_j^{-1}\phi_j h_{j+1} = \phi_j^*$, the bracket $[\phi, \phi^{*_H}]$ is given by

$$[\phi, \phi^{*_H}] = \begin{pmatrix} \phi_k \wedge \phi_k^* + \phi_1^* \wedge \phi_1 & & \\ & \ddots & \\ & & \phi_{k-1} \wedge \phi_{k-1}^* + \phi_k^* \wedge \phi_k \end{pmatrix}.$$

Thus, the quiver bundle metric H solves the Higgs bundle equations $F_H + [\phi, \phi^{*_H}] = 0$, and we conclude the Higgs bundle metric on $\mathcal{E}$ is diagonal with respect to the holomorphic splitting $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$. ☐

4.2 Fixed points of the $\langle \zeta_k \rangle \subset \mathbb{C}^*$-action on $\mathcal{M}(\mathsf{G})$

4.2.1 G-complex

Recall that a $\mathbb{Z}$-grading associated to the height grading of a parabolic subalgebra corresponds to a labeling of the Dynkin diagram with only 1's and 0's (see Chapter 2.1.1, in particular Example 2.1.17). Let G be a complex simple Lie group, we first phrase the classification of fixed points of the $\mathbb{C}^*$-action on $\mathcal{M}_G$ in terms of $\mathbb{Z}$-gradings.

Let $\mathsf{P} \subset \mathsf{G}$ be a parabolic subgroup with Levi factor L and denote the corresponding $\mathbb{Z}$-grading of $\mathfrak{g}$ by $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$. Let $\mathfrak{c}$ be a Cartan subalgebra, $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ a set of simple roots and $A \subset \Pi$ the subset which gives the parabolic P_A conjugate to P. Recall that the corresponding $\mathbb{Z}$-grading arises from a one parameter family of elements $g_t \in \mathsf{Inn}(\mathfrak{g})$ where $g_t = exp(2\pi i t x)$ for an element $x \in \mathfrak{z}(\mathfrak{l}_A)$; this implies g_t is in the center of L_A.

Recall from Definition 3.3.9 that an K-twisted $(\mathsf{L}_A, \mathfrak{g}_1)$ Higgs pair consists of a holomorphic L_A-bundle $\mathcal{E}_L \to \Sigma$ and a holomorphic section of the $\mathfrak{g}_1$-associated bundle $\varphi \in H^0(\mathcal{E}_L[\mathfrak{g}_1] \otimes K)$. Denote the moduli space of such objects by $\mathcal{M}(\mathsf{L}_A, \mathfrak{g}_1)$. We have

$$\coprod_{A \subset \Pi} \mathcal{M}(\mathsf{L}_A, \mathfrak{g}_1) = \coprod_{\text{Dynkin Diagram labelings w/ 1's and 0's}} \mathcal{M}(\mathsf{G}_0, \mathfrak{g}_1)$$

Theorem 4.2.1. *Let G be a complex simple Lie group and denote the subvariety consisting of fixed points of the $\mathbb{C}^*$ action by $\mathcal{F}(\mathsf{G})$. Then there is a map $\mathcal{F}(\mathsf{G}) \longrightarrow \coprod_{A \subset \Pi} \mathcal{M}(\mathsf{L}_A, \mathfrak{g}_1)$ and extension of structure group gives a surjective map $\coprod_{A \subset \Pi} \mathcal{M}(\mathsf{L}_A, \mathfrak{g}_1) \twoheadrightarrow \mathcal{F}(\mathsf{G})$. Moreover, if $\mathfrak{g}_1 = \bigoplus \mathfrak{g}_1^\nu$ is the decomposition of $\mathfrak{g}_1$ into irreducible representations of L_A then the objects which map to smooth fixed points consist of $(\mathsf{L}_A, \mathfrak{g}_1)$ K-twisted Higgs pairs $(\mathcal{E}, \phi)$ with $\phi^\nu \neq 0$ for all ν.*

Proof. Given a polystable $(\mathsf{L}_A, \mathfrak{g}_1)$ K-twisted Higgs bundle $(\mathcal{E}_\mathsf{L}, \phi)$ extending the structure group to G defines a G-Higgs bundles $(\mathcal{E}_\mathsf{G}, \phi)$. This will be a polystable Higgs bundle since the reduction of structure from Theorem 3.3.29 which solves the $(L_A, \mathfrak{g}_1)$-Higgs bundle equations also solves the G-Higgs bundle equations. To see that this extended object is a fixed point of the $\mathbb{C}^*$-action, note that there is an element $x \in \mathfrak{z}(\mathfrak{l}_A)$ so that ad_x defines the $\mathbb{Z}$-grading. In particular, the $ad_x(y) = y$ for all $y \in \mathfrak{g}_1$. Exponentiating gives a 1-parameter family of $g_t = exp(2\pi i t x)$ in the center of L_A. This family defines a family of holomorphic gauge transformations of the bundle $\mathcal{E}_\mathsf{L}$ which acts of the Higgs field ϕ by $g_t \cdot \phi = e^{2\pi i t}\phi$. This makes the extended object a fixed point of the $\mathbb{C}^*$-action.

Given a polystable G-Higgs bundle $(\mathcal{E}, \phi)$ which is a point of the $\mathbb{C}^*$ action, there is a 1-parameter family of holomorphic gauge transformations g_t with $Ad_{g_t}\phi = e^{2\pi i t}\phi$. This family gives a $\mathbb{Z}$-grading on the fibers of the adjoint bundle $\mathcal{E}[\mathfrak{g}]$. Moreover, for each t, the coefficients of the characteristic polynomial of Ad_{g_t} are holomorphic functions on Σ, thus the eigenvalues of Ad_{g_t} are constant. Hence the family of g_t of gauge transformations gives a $\mathbb{Z}$-grading of the adjoint bundle $\mathcal{E}([\mathfrak{g}]) = \bigoplus\limits_{j\in\mathbb{Z}} \underline{\mathfrak{g}}_j$ with $\phi \in \underline{\mathfrak{g}}_1$. The Lie algebra $\mathfrak{g}_0$ is a Levi factor of a parabolic, let $\mathsf{G}_0 \subset \mathsf{G}$ be the corresponding connected subgroup. The Lie subalgebra bundle $\underline{\mathfrak{g}}_0 \subset \mathcal{E}(\mathfrak{g})$ defines a reduction of structure group $\mathcal{E}_{\mathsf{G}_0}$of $\mathcal{E}$ from G to G_0. This defines the map $\mathcal{F}(\mathsf{G})\to \coprod\limits_{A\subset\Pi} \mathcal{M}(\mathsf{L}_A, \mathfrak{g}_1)$ proves surjectivity of the map $\coprod\limits_{A\subset\Pi} \mathcal{M}(\mathsf{L}_A, \mathfrak{g}_1)\to\mathcal{F}(\mathsf{G})$.

The bundles $\underline{\mathfrak{g}}_j$ are given by associated bundles $\mathcal{E}_{\mathsf{G}_0}[\mathfrak{g}_j]$. Thus each $\underline{\mathfrak{g}}_j$ decomposes into a direct sum of irreducible G_0 representations. In particular, $\mathcal{E}_{\mathsf{G}_0}[\mathfrak{g}_1] = \bigoplus \mathfrak{g}_1^\nu$, where each $\mathfrak{g}_1^\nu$ is a generalized root space of the parabolic with Levi factor $\mathfrak{g}_0$. The Higgs field $\phi \in \mathcal{E}_{\mathsf{G}_0}[\mathfrak{g}_1]$ also decomposes as $\phi = \bigoplus \phi^\nu$. If $\phi_\nu = 0$ for some ν, then there is an extension of structure group of $\mathcal{E}_{\mathsf{G}_0}$ to a Levi factor of a larger parabolic for which ϕ is in the Levi subalgebra of the larger parabolic. This implies that the Higgs bundle $(\mathcal{E}, \phi)$ is strictly polystable. $\square$

Fixed points of the $\mathbb{C}^*$-action have been understood for awhile, however they are not usually phrased in terms of $\mathbb{Z}$-gradings. With this set up, generalizing from fixed points of the $\mathbb{C}^*$-action to k^{th}-roots of unity $\langle\zeta_k\rangle \subset \mathsf{U}(1)$ is more straight forward. For each $\mathbb{Z}/k\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus\limits_{j\in\mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ arising from a labeling of the extended Dynkin diagram, let $\mathsf{G}_0 \subset \mathsf{G}$ denote the connected Lie group with Lie algebra $\widehat{\mathfrak{g}}_0$ and denote the moduli space of K-twisted $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$-Higgs pairs by $\mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$. Define the sets:

- B the set of all $\mathbb{Z}/k\mathbb{Z}$-gradings on the Lie algebra $\mathfrak{g}$ which arise from labeling the extended Dynkin diagram of $\mathfrak{g}$.
- $B' \subset B$ the set of all $\mathbb{Z}/k\mathbb{Z}$-gradings on the Lie algebra $\mathfrak{g}$ which arise from labeling the extended Dynkin diagram with only 1s and 0s on simple roots α_j with $n_j \neq 1$. Here the longest root μ is defined by $\sum n_j\alpha_j$ (see section 2.1.1).
- $B'' \subset B'$ the set of all $\mathbb{Z}/k\mathbb{Z}$-gradings on the Lie algebra $\mathfrak{g}$ which arise from labeling the extended Dynkin diagram with only 1s and 0s.

Denote the subvariety of fixed points of the $\mathbb{C}^*$-action and the $\langle\zeta_k\rangle$-action on $\mathcal{M}(\mathsf{G})$ by $\mathcal{F}$ and $\mathcal{F}_k$ respectively.

Theorem 4.2.2. *Let G be a complex simple Lie group, extension of structure group gives a map*

$$\coprod_B \mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1) \longrightarrow \mathcal{F}_k\ .$$

If $\mathcal{F}^{sm}$ and $\mathcal{F}_k^{sm}$ denote the smooth fixed points and B' and B'' are as above, then there are maps

$$\mathcal{F}_k^{sm} \xrightarrow{f} \coprod_{B'} \mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1) \qquad\qquad \mathcal{F}_k^{sm} \setminus \mathcal{F}^{sm} \xrightarrow{f|_{\mathcal{F}_k^{sm}\setminus\mathcal{F}^{sm}}} \coprod_{B''} \mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$$

Furthermore, if $\widehat{\mathfrak{g}}_1 = \bigoplus \widehat{\mathfrak{g}}_1^\nu$ is the decomposition of $\widehat{\mathfrak{g}}_1$ into irreducible representations of G_0, then the fixed points of $\langle \zeta_k \rangle$ which are not fixed by all of $\mathbb{C}^*$ satisfy the extra condition that $\phi^\nu \neq 0$ for all ν.

Remark 4.2.3. This gives a one-to-one correspondence between polystable $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ K-twisted Higgs pairs $(\mathcal{E}, \phi)$ which arise from $\mathbb{Z}/k\mathbb{Z}$-gradings corresponding to labeling the extended Dynkin diagram with only 1's and 0's and which satisfy $\phi^\nu \neq 0$ for all ν and stable simple fixed points of $\langle \zeta_k \rangle$ in $\mathcal{M}(\mathsf{G})$ which are not fixed by all of $\mathbb{C}^*$.

Proof. Let $\mathfrak{g} = \bigoplus\limits_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ be a $\mathbb{Z}/k\mathbb{Z}$-grading which arises from a labeling of the extended Dynkin diagram with 1's and 0's. Recall that this grading arises from an inner automorphism $g \in \mathsf{Inn}(\mathfrak{g})$ with $g^k = Id$. Moreover, if $\mathsf{G}_0' \subset \mathsf{Inn}(\mathfrak{g})$ is the connected subgroup of $\mathsf{Inn}(\mathfrak{g})$ with Lie algebra $\widehat{\mathfrak{g}}_0$, then g lies in the center of G_0'. Let $\mathsf{G}_0 \subset \mathsf{G}$ be the connected Lie group with Lie algebra $\widehat{\mathfrak{g}}_0$. To get an element of G we must choose a lift of $g \in \mathsf{Inn}(\mathfrak{g})$, we will denote this lift by g also. Note that g is a central element of G_0 and g^k is a central element of G.

Given a polystable $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ K-twisted Higgs bundle $(\mathcal{E}_{\mathsf{G}_0}, \phi)$ extending the structure group to G defines a G-Higgs bundles $(\mathcal{E}_{\mathsf{G}}, \phi)$. This will be a polystable Higgs bundle since the reduction of structure which solves the $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$-Higgs bundle equations will also solve the G-Higgs bundle equations. To see that this extended object is a fixed point of the roots of unity-action $\langle \zeta_k \rangle \subset \mathsf{U}(1)$, note that the central element $g \in \mathcal{Z}(\mathsf{G}_0)$ which defines the $\mathbb{Z}/k\mathbb{Z}$-grading gives a well defined holomorphic gauge transformation $gId_{\mathcal{E}_{\mathsf{G}_0}}$ of $\mathcal{E}_{\mathsf{G}_0}$ which acts on ϕ by multiplication by ζ_k.

Now assume $(\mathcal{E}, \phi)$ is a stable and simple G-Higgs bundle with $(\mathcal{E}, \phi) \cong (\mathcal{E}, \zeta_k \phi)$. Let $g \in \mathcal{G}_{\mathsf{G}}$ be a holomorphic gauge transformation which acts as $Ad_g \phi = \zeta_k \phi$. Thus $Ad_{g^k} \phi = \phi$ and g^k is in the center of G since. The Ad_g-eigenbundle decomposition of the adjoint bundle $\mathcal{E}[\mathfrak{g}]$ defines a $\mathbb{Z}/k\mathbb{Z}$-grading $\mathcal{E}[\mathfrak{g}] = \bigoplus\limits_{j \in \mathbb{Z}/k\mathbb{Z}} \underline{\widehat{\mathfrak{g}}}_j$ with $\phi \in H^0(\Sigma, \underline{\widehat{\mathfrak{g}}}_1 \otimes K)$. The Lie algebra subbundle $\underline{\widehat{\mathfrak{g}}}_0 \subset \mathcal{E}[\mathfrak{g}]$ defines a reduction of structure group $\mathcal{E}_{\mathsf{G}_0}$ of $\mathcal{E}$ from G to G_0, and the bundle $\underline{\widehat{\mathfrak{g}}_j}$ are associated bundles $\mathcal{E}_{\mathsf{G}_0}[\widehat{\mathfrak{g}}_j]$. Thus, a stable and simple fixed point determines a $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ K-twisted Higgs pair.

Recall that G_0 is not necessarily a Levi factor of a parabolic of G, thus polystability of the $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ K-twisted Higgs pair does not follow automatically from polystability of $(\mathcal{E}, \phi)$. However, since $(\mathcal{E}, \phi)$ is stable and simple, the metric which solves the G-Higgs bundle equations is unique. Denote the Chern connection of the metric solving the Higgs bundle equations by ∇_A, since it solves the equations for both ϕ and $\zeta_k \phi$, the gauge transformation g is covariantly constant, $\nabla_A g = 0$. Therefore, ∇_A preserves the eigenbundles of Ad_g which implies the connection 1-form A takes values in the bundle identity eigenbundle $\underline{\widehat{\mathfrak{g}}}_0$. But the bundle $\underline{\widehat{\mathfrak{g}}}_0$ is the adjoint bundle of $\mathcal{E}_{\mathsf{G}_0}$ thus the metric connection A solves the K-twisted $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$-Higgs bundle equations. This proves polystability of the K-twisted $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ Higgs pair associated to a stable and simple fixed point of $\langle \zeta_k \rangle$.

The $\mathbb{Z}/k\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus \widehat{\mathfrak{g}}_j$ came from an inner automorphism, it corresponds to a labeling of the extended Dynkin diagram. Thus, we get a map $f : \mathcal{F}_k^{sm} \to \bigsqcup_B \mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$. Let $\widetilde{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_\ell\}$ be the extended simple roots, recall that $\alpha_0 = \sum_{j=1}^{\ell} -n_j \alpha_j$ is the lowest root. Suppose the labeling of the extended Dynkin diagram has a nonzero label on the root α_j. If $n_j \neq 1$ then consider the $\mathbb{Z}/n_j\mathbb{Z}$ grading on $\mathfrak{g}$ corresponding to labeling the extended Dynkin Diagram with a 1 on α_j and 0's on all other roots. Denote the this grading by $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/n_j\mathbb{Z}} \widehat{\mathfrak{g}}_j'$, then $\widehat{\mathfrak{g}}_0 \subset \widehat{\mathfrak{g}}_0'$ and $\widehat{\mathfrak{g}}_1 \subset \widehat{\mathfrak{g}}_0'$. The element g' which gives this second grading acts trivially on the Higgs bundle $(\mathcal{E}, \phi)$ but is not in the center of G. This contradicts the simplicity of $(\mathcal{E}, \phi)$, and proves that the image of the map $f : \mathcal{F}_k \to \bigsqcup_B \mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ lies in $\bigsqcup_{B'} \mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$. If $n_j = 1$, then, after acting by an automorphism of the extended Dynkin diagram, we may assume $\alpha_j = \alpha_0$. In this case, we obtain a $\mathbb{Z}$-grading on $\mathfrak{g}$ with ϕ at height 1, thus $(\mathcal{E}, \phi)$ is fixed by all of $\mathbb{C}^*$. Thus, the image of the restriction of the map f to the space $\mathcal{F}_k^{sm} \setminus \mathcal{F}^{sm}$ lies in $\bigsqcup_{B''} \mathcal{M}(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$.

Let $\widehat{\mathfrak{g}}_1 = \bigoplus \mathfrak{g}^\nu$ be the decomposition of $\widehat{\mathfrak{g}}_1$ into irreducible representations of G_0. The Higgs field $\phi \in H^0(\Sigma, \mathcal{E}[\widehat{\mathfrak{g}}_1] \otimes K)$ decomposes as $\phi = \sum \phi^\nu$. To prove the last part of the theorem, recall from Remark 2.1.22 there are two case to consider. First, assume that the $\mathbb{Z}/k\mathbb{Z}$-grading under consideration has no roots labeled with a 0, then all roots have are labeled with a 1. Assume that $\phi^\nu = 0$ for a root $\{\alpha_j\}$ in the Dynkin diagram. As above, if $n_j \neq 1$, then consider the $\mathbb{Z}/n_j\mathbb{Z}$-grading associated to labeling the root α_j with a 1 and labeling all other roots in $\widetilde{\Pi}$ with a 0. Such a Higgs bundle is not simple since, by assumption, the Higgs field is in the identity eigenspace of this grading and there is a gauge transformation acting trivially on the Higgs bundle that is not in the center of G. If $n_j = 1$ then we can assume $\alpha_j = \alpha_0$ and $(\mathcal{E}, \phi)$ is a fixed point of the $\mathbb{C}^*$-action. If there is a root with 0 label, then let $\widehat{\mathfrak{g}}_1^\mu$ be the irreducible representations with $\phi^\mu = 0$. There are again two cases, $\widehat{\mathfrak{g}}_1^\mu$ is one dimensional and corresponds to a root space $\mathfrak{g}_{\alpha_j}$ with $n_j = 1$ or not. In the first case, as before, we can assume $\alpha_j = \alpha_0$ and the Higgs bundle $(\mathcal{E}, \phi)$ will be fixed by all of $\mathbb{C}^*$. For the second case, in the finite order grading corresponding to labeling all roots in the irreducible representations $\widehat{\mathfrak{g}}_1^\mu$ with a 1 and all other roots 0, the Higgs field will lie in the identity eigenspace, and the Higgs bundle $(\mathcal{E}, \phi)$ will not be simple. $\square$

4.2.2 G-real

For real groups G the classification of stable simple fixed points is more subtle. For instance one needs to understand how a finite a finite order element of $\mathsf{Inn}(\mathfrak{h}_\mathbb{C})$ acts on the isotropy subspace $\mathfrak{m}_\mathbb{C}$. Fixed points of $\langle\zeta_k\rangle$ in $\mathcal{M}(\mathsf{G})$ will correspond to polystable $(\mathsf{H}_0, \widehat{\mathfrak{m}_\mathbb{C}}^1)$-Higgs pairs where $\mathfrak{g}_\mathbb{C} = \bigoplus\limits_{j\in\mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}^j_\mathbb{C}$ and each $\widehat{\mathfrak{g}}^j_\mathbb{C}$ decomposes as $\widehat{\mathfrak{g}}^j = \widehat{\mathfrak{h}_\mathbb{C}}^j \oplus \widehat{\mathfrak{m}_\mathbb{C}}^j$.

For some groups however, there is a way around this subtlety by using results on simplifications of stability. In [GPGMiR13], it is shown that if (V,β,γ) is a stable and simple $\mathsf{Sp}(2n,\mathbb{R})$-Higgs bundle with $V \ncong V^*$, then the associated $\mathsf{SL}(n,\mathbb{C})$-Higgs bundle $\left(V\oplus V^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0\end{pmatrix}\right)$ is stable. Also, a stable simple $\mathsf{SL}(n,\mathbb{R})$-Higgs bundle $(\mathcal{E},Q,\phi)$ the corresponding $\mathsf{SL}(n,\mathbb{C})$-Higgs bundle is also stable and simple. Using Simpson's classification of fixed points of $\langle\zeta_k\rangle$ for $\mathsf{SL}(n,\mathbb{C})$-Higgs bundles (Theorem 4.1.6), we have the following classification of stable simple fixed points for $\mathsf{Sp}(2n,\mathbb{R})$.

Theorem 4.2.4. *Let (V,β,γ) be a stable simple $\mathsf{Sp}(2n,\mathbb{R})$-Higgs bundle with $V \ncong V^*$. If $(V,\beta,\gamma) \cong (V,\zeta_k\gamma,\zeta_k\beta)$ and is not fixed by all of $\mathbb{C}^*$, then k is even and $V = V_1\oplus V_2\oplus\cdots\oplus V_{\frac{k}{2}}$ with $V_j = V_{\frac{k}{2}+1-j}$. Morever, if $\beta_{ij} : V_j^*\to V_i\otimes K$ and $\gamma_{ij} : V_j \to V_i^*\otimes K$ then $\beta_{ij}^T = \beta_{ji}$, $\gamma_{ij}^T = \gamma_{ji}$ and $\beta_{ij}\neq 0$ if and only if $(i+j) = 2 \ mod\ \frac{k}{2}$ and $\gamma_{ij} = 0$ if and only if $(i+j) = 1\ mod\ \frac{k}{2}$.*

Proof. The $\mathsf{SL}(2n,\mathbb{C})$ Higgs bundle $\left(V\oplus V^*, \phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0\end{pmatrix}\right)$ is stable. Since $(V,\beta,\gamma)\cong(V,\zeta_k\beta,\zeta_k\gamma)$, there is a gauge transformation $g: V\to V$ with the properties $g\gamma g = \zeta_k\gamma$ and $g^*\beta g^* = \zeta_k\beta$, thus $g^k = \pm Id_V$. Let $\tilde{g} = g\oplus g^*$ be the corresponding gauge transformation of $V\oplus V^*$, note that $Ad_{\tilde{g}}\phi = \zeta_k\phi$ and $(V\oplus V^*,\phi)$ is not fixed by all of $\mathbb{C}^*$. Thus, by theorem 4.1.6, $V\oplus V^*$ decomposes as $\mathcal{E}_1\oplus\mathcal{E}_2\oplus\cdots\oplus\mathcal{E}_k$ with $\phi_{ij}:\mathcal{E}_j\to\mathcal{E}_i\otimes K$ equal 0 if and only if $j-i = -1\ mod\ k$. Since $\phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0\end{pmatrix}$ we must have k even and $V = \bigoplus\limits_{j\ odd}\mathcal{E}_j = V_1\oplus V_2\oplus\cdots\oplus V_{\frac{k}{2}}$ and $V^* = \bigoplus\limits_{j\ even}\mathcal{E}_j = V^*_{\frac{k}{2}}\oplus V_{\frac{k}{2}-1}\oplus\cdots\oplus V_1^*$. The form of β and γ follow from rearranging the splitting $\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_k$ as $V_1\oplus V_2\oplus\cdots\oplus V_{\frac{k}{2}}\oplus V_1^*\oplus\cdots\oplus V^*_{\frac{k}{2}}$.

□

For stable and simple $\mathsf{SL}(n,\mathbb{R})$-Higgs bundles $(\mathcal{E},Q,\phi)$ we have the following classification theorem.

Theorem 4.2.5. *Let $(\mathcal{E},Q,\phi)$ be a stable and simple $\mathsf{SL}(n,\mathbb{R})$-Higgs bundle. If $(\mathcal{E},Q,\phi)\cong(\mathcal{E},Q,\zeta_k\phi)$ then $\mathcal{E} = \mathcal{E}_1\oplus\mathcal{E}_2\oplus\cdots\mathcal{E}_k$, and either $Q_{ij}:\mathcal{E}_j\to\mathcal{E}_i$ is 0 if $(i+j)\neq 1\ mod\ k$ and an isomorphism otherwise, or n is even and $Q_{ij} = 0$ if $(i+j)\neq 2\ mod\ k$ and an isomorphism otherwise. Furthermore, if $\phi_{ij}:\mathcal{E}_j\to\mathcal{E}_i\otimes K$ then $\phi_{ij} = 0$ if and only if $(i+j) = 1\ mod\ k$ and $\phi^TQ\phi = Q$.*

Proof. Recall that the orthogonal structure Q is a symmetric isomorphism $\mathcal{E}\to\mathcal{E}^*$. Since $(\mathcal{E},Q,\phi)\cong(\mathcal{E},Q,\zeta_k\phi)$ and $((\mathcal{E},Q,\phi)$ is stable and simple, there is a $\mathsf{SO}(n,\mathbb{C})$ gauge transformation g for which $Ad_g\phi = \zeta_k$ with g^k

a central element of $\mathsf{SO}(n,\mathbb{C})$. There are 2 cases to consider, $g^k = Id_{(\mathcal{E},Q)}$ and $g^k = -Id_{(\mathcal{E},Q)}$ (the second case is only possible if n is even).

Suppose $g^k = Id$ and let $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ be the eigenbundle decomposition of $\mathcal{E}$ with $g|_{\mathcal{E}_j} = Id_{\mathcal{E}_j}\zeta_k^{j-1}$. Denote the decomposition of the orthogonal structure $Q : \mathcal{E} \to \mathcal{E}^*$ by Q_{ij}. The gauge transformation g acts on Q_{ij} by $\zeta_k^{j-1}\zeta_k^{i-1}Q_{ij}$. Since $g^*Qg = Q$, we must have $Q_{ij} = 0$ for $(j+i) \neq 2 \ mod \ k$ and $Q_{ij} : \mathcal{E}_j \to \mathcal{E}_i^*$ an isomorphism for $(j+i) = 2 \ mod \ k$. Now suppose n is even and $g^k = -Id_{(\mathcal{E},Q)}$, let $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ be the corresponding decomposition with $g|_{\mathcal{E}_j} = \zeta_{2k}^{2j-1}$. The gauge transformation acts trivially on the orthogonal structure Q and as $\zeta_{2k}^{2j-1}\zeta_{2k}^{2i-1}Q_{ij} = \zeta_k^{i+j-1}Q_{ij}$, thus $Q_{ij} = 0$ if $i+j \neq 1 \ mod \ k$ and is an isomorphism otherwise. In both cases, the properties of the Higgs field follow from the definition of $\mathsf{SL}(n,\mathbb{R})$ Higgs bundles and the fact that $(\mathcal{E},\phi)$ is a stable $\mathsf{SL}(n,\mathbb{C})$ Higgs bundle fixed by $\langle \zeta_k \rangle$. $\square$

Remark 4.2.6. When k is even and the gauge transformation g satisfies $g^k = Id_{(\mathcal{E},Q)}$ then there are two self dual bundles $\mathcal{E}_1$ and $\mathcal{E}_{\frac{k}{2}+1}$, and if $g^k = -Id$ then there are *no* self dual bundles. When k-odd is odd, there is always only one self dual bundle $\mathcal{E}_1$. Also, when n is odd and k is odd, after rearranging the eigenbundles of g, the orthogonal structure can be made to be of the form $Q_{ij} = 0$ if and only if $i + j = 1 \ mod \ k$.

4.2.3 Fixed points in the Hitchin component for all simple split real forms

For the Hitchin component the fixed points of $\langle \zeta_k \rangle$ are easy to classify. For classical groups, we will use the extra symmetries of $\mathsf{SL}(n,\mathbb{R})$, $\mathsf{SO}(n,n+1)$, $\mathsf{SO}(n,n)$, $\mathsf{Sp}(2n,\mathbb{R})$-Higgs bundles, we will deduce stronger metric splitting properties or equivalently, more symmetries in the quiver bundle equations.

Proposition 4.2.7. *Let $(\mathcal{E},\phi)$ be a Higgs bundle in the G-Hitchin component with $\phi = \tilde{e}_1 + \sum\limits_{j=1}^{\ell} e_j \otimes q_{m_j}$, then $(\mathcal{E},\phi) \cong (\mathcal{E},\zeta_k\phi)$ if and only if*

$$\phi = \tilde{e}_1 + \sum_{\substack{m_j+1 \equiv 0 \\ mod \ k}} e_j \otimes q_{m_j+1}.$$

Proof. The Hitchin component is the image of a section s_h of the Hitchin fibration

$$\begin{tikzcd}
\mathcal{M}_{\mathsf{G}} \arrow[r, "p_1,\cdots,p_\ell"] & \bigoplus\limits_{j=1}^{\ell} H^0(\Sigma, K^{m_j+1}) \arrow[dl, bend left, "s_h"] \\
\mathsf{Hit}(\mathsf{G}) \arrow[u, hook] &
\end{tikzcd}$$

and $p_j(\mathcal{E},\lambda\phi) = \lambda^{m_j+1}p_j(\mathcal{E},\phi)$ for all $\lambda \in \mathbb{C}^*$. Thus if $(\mathcal{E},\phi) \cong (\mathcal{E},\zeta_k\phi)$ then $p_j(\mathcal{E},\phi) = p_j(\mathcal{E},\zeta_k\phi)$, for $j = 1,\cdots,\ell$. But by definition of the Higgs fields in the Hitchin component if $\phi = \tilde{e}_1 + \sum\limits_{j=1}^{\ell} e_j \otimes q_{m_j+1}$ then

$p_j(\phi) = q_{m_j+1}$, and $p_j(\zeta_k\phi) = \zeta_k^{m_j+1} q_{m_j+1}$. Thus if $(\mathcal{E}, \phi)$ is a fixed point of $\langle \zeta_k \rangle$ then

$$p_j(\phi) = q_{m_j+1} = 0 \quad \text{for} \qquad m_j + 1 \neq 0 \bmod k.$$

Conversely if $\phi = \tilde{e}_1 + \sum\limits_{\substack{m_j+1\equiv 0 \\ \bmod k}} e_j \otimes q_{m_j+1}$ and $x \otimes Id \in \underline{\mathfrak{t}_{\mathbb{C}}} \otimes \mathcal{O} \subset \mathcal{E} \times_{H_{\mathbb{C}}} \mathfrak{h}_{\mathbb{C}}$ is the grading element of the principal three dimensional subalgebra from which the Higgs bundle in the Hitchin component are derived, then

$$ad_x(\phi) = -\tilde{e}_1 + \sum_{\substack{m_j+1\equiv 0 \\ \bmod k}} m_j e_j \otimes q_{m_j+1}.$$

Exponentiating, we have $g_k = exp\left(\dfrac{2\pi i x}{k}\right) \in \mathcal{G}_{H_{\mathbb{C}}}(\mathcal{E})$ and $Ad_{g_k}\phi = \zeta_k^{-1}\tilde{e}_1 + \sum\limits_{\substack{m_j+1\equiv 0 \\ \bmod k}} \zeta_k^{m_j} e_j \otimes q_{m_j+1} = \zeta_k^{-1}\phi$. Thus $Ad_{g_k^{-1}}\phi = \zeta_k\phi$ as desired. The subbundle $V \subset \mathcal{E} \times_{H_{\mathbb{C}}} (\mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}})$ fixed by Ad_{g_k} is $\bigoplus\limits_{j=0 \bmod k} \mathfrak{g}_j \otimes K^j$. The Lie algebra subbundle $W \subset \mathcal{E} \times_{H_{\mathbb{C}}} \mathfrak{h}_{\mathbb{C}}$ given by $W = \bigoplus\limits_{j=0 \bmod k} \mathfrak{h}_j \otimes K^j$ yields a corresponding reduction of structure of $\mathcal{E}$ compatible with the metric solving the Higgs bundle equations. $\square$

The following corollaries of Theorem 4.2.7 was the starting point to prove the asymptotic results of Chapter 5.

Corollary 4.2.8. *For $(\mathcal{E}, \phi)$ a fixed point of $\langle \zeta_k \rangle$ in the $\mathsf{SL}(n, \mathbb{R})$-Hitchin component, the splitting*

$$\mathcal{E} = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$$

with $\mathcal{E}_j = K^{\frac{n-1}{2}-j+1} \oplus K^{\frac{n-1}{2}-j+1-k} \oplus K^{\frac{n-1}{2}-j+1-2k} \oplus \cdots$, is unitary with respect to the metric solving the Higgs bundle equations.

Corollary 4.2.9. *For $(\mathcal{E}, \phi)$ a fixed point of $\left\langle \zeta_{n-1} \right\rangle$ in the $\mathsf{SL}(n, \mathbb{R})$-Hitchin component, the line bundle splitting*

$$\mathcal{E} = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$$

is unitary with respect to the metric H solving the Higgs bundle equations. Moreover, the metric is given by $H = h_1 \oplus h_2 \oplus \cdots h_2^{-1} \oplus h_1^{-1}$.

Proof. By the previous corollary the splitting $(K^{\frac{n-1}{2}} \oplus K^{-\frac{n-1}{2}}) \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}}$ is unitary with respect to the metric solving the Higgs bundle equations. But $K^{\frac{n-1}{2}} \oplus K^{-\frac{n-1}{2}}$ is an $\mathsf{SO}(2, \mathbb{C})$ bundle thus the metric is splits as $h_1 \oplus h_1^{-1}$ on $K^{\frac{n-1}{2}} \oplus K^{-\frac{n-1}{2}}$. The form of the metric follows from the compatibility of the metric with the orthogonal structure. $\square$

4.3 Harmonic maps, fixed points and G-variations of Hodge structure

We now discuss the harmonic maps of fixed points of roots of unity actions. Let G is a complex simple Lie group. By Theorem 4.2.2, if $(\mathcal{E}, \phi)$ is a stable, simple G-Higgs bundle which is a fixed point of $\langle \zeta_k \rangle \subset \mathbb{C}^*$, then it arises from a K-twisted $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ Higgs pair where $\mathfrak{g} = \bigoplus\limits_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ is a $\mathbb{Z}/k\mathbb{Z}$-grading and G_0 is the connected Lie subgroup with Lie algebra $\widehat{\mathfrak{g}}_0$. This gives rise to a commuting diagram of compatible reductions of structure group, equivalently equivariant maps

$$\begin{tikzcd}
\mathsf{G}/\mathsf{G}_0 & \mathsf{G}/\mathsf{H}_0 \arrow[l] \arrow[d] \\
\widetilde{\Sigma} \arrow[u] \arrow[ur, "f"] \arrow[r, "h"'] & \mathsf{G}/\mathsf{H}
\end{tikzcd} \tag{4.3.1}$$

where the map h is the harmonic metric and $\mathsf{H}_0 = \mathsf{G}_0 \cap \mathsf{H}$ is the maximal compact subgroup of G_0.

We want to rephrase the condition of being fixed by $\langle \zeta_k \rangle$ in terms of harmonic maps. Before doing this, we need to develop a little geometry of homogeneous spaces. Recall from Theorem 3.2.3 that a map $f : M \to N$ between Riemannian manifolds is harmonic if and only if $(d^{f^*\nabla_{LC}})^*(df) = 0$ where ∇_{LC} denotes the Levi Civita connection on N. Recall also (see section 2.2.1) that every reductive homogeneous space has a canonical connection, but the canonical connection is the Levi Civita connection for an G-invariant metric if and only if the homogeneous space is a symmetric space. In general translating the harmonic map equations into equations with respect to the canonical connection is a little complicated. Fortunately, for some special homogeneous spaces this is not complicated.

Definition 4.3.1. A homogeneous space G/Q is a *naturally reductive homogeneous space* there exists a G-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ and an Ad_{Q} invariant splitting $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{m}$ so that

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0$$

for all $X, Y, Z \in \mathfrak{m}$ where $[\ ,\ ]_{\mathfrak{m}}$ denotes the projection onto $\mathfrak{m}$.

We will use the following proposition, see [Woo03].

Proposition 4.3.2. *Let G/Q be a naturally reductive homogeneous space with canonical connection ∇_c. Let M be a Riemannian manifold, a smooth map $f : M \to \mathsf{G}/\mathsf{Q}$ is harmonic if and only if*

$$(d^{f^*\nabla_c})^*(df) = 0.$$

If M is a Riemann surface then harmonicity is equivalent to $d^{(1,0)}f$ being holomorphic with respect $f^*\nabla_c$, that is $(d^{f^*\nabla_c})^{(0,1)}(d^{(1,0)}f) = 0$.

The space we are interested in is a naturally reductive homogeneous space.

Lemma 4.3.3. *The reductive homogeneous space G/H_0 is a naturally reductive homogeneous space with the metric induced by the Riemannian metric $\langle X, Y\rangle = -B_{\mathfrak{g}}(X, \theta(Y))$.*

Proof. Recall that if $\mu = \sum_{j=1}^{\ell} n_j\alpha_j$ is the highest root of $\mathfrak{g}$, then a $\mathbb{Z}/k\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ is equivalent to a labeling of the extended Dynkin diagram with integers $\{a_0, \cdots, a_\ell\}$ so that $a_0 + \sum_{j=1}^{\ell} n_j a_j = k$. Every root α can be written as $\alpha = -m_0\mu + \sum_{j=1}^{\ell} m_j\alpha_j$ with $m_0 = 0$ if α is a positive root and $m_0 = -1$ for α a negative root. Each summand $\widehat{\mathfrak{g}}_j$ is a direct sum of root spaces $\mathfrak{g}_\alpha$ with $\sum_{j=0}^{\ell} a_j m_j = j \mod k$. In particular, $\alpha \in \widehat{\mathfrak{g}}_j$ if and only if $-\alpha \in \widehat{\mathfrak{g}}_{-j}$. Thus, by (2.1.3) the splitting $(\widehat{\mathfrak{g}}_0 \cap \mathfrak{h}) \oplus (\widehat{\mathfrak{g}}_0 \cap i\mathfrak{h}) \oplus \bigoplus_{j=1}^{k-1} \widehat{\mathfrak{g}}_j = (\widehat{\mathfrak{h}_0}) \oplus \mathfrak{m}$ is orthogonal with respect to the inner product $\langle X, Y\rangle = -B_{\mathfrak{g}}(X, \theta(Y))$. Hence, for all $X, Y, Z \in \mathfrak{m}$

$$0 = \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = \langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle + \langle [X,Y]_{\mathfrak{h}_0}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{h}_0}\rangle$$

$$= \langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle$$

□

Recall that we have the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and complexifying gives $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$ (since $\mathfrak{g}$ is complex both $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{m}_{\mathbb{C}}$ are isomorphic to $\mathfrak{g}$). The complexified tangent bundle of G/H is $T_{\mathbb{C}}\mathsf{G}/\mathsf{H} = \mathsf{G} \times_{\mathsf{H}} \mathfrak{m}_{\mathbb{C}}$. Given a $\mathbb{Z}/k\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$, recall that the Cartan involution θ acts on this splitting as

$$\theta(\widehat{\mathfrak{g}}_j) = \widehat{\mathfrak{g}}_{-j}.$$

This gives the splitting $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{m}_0 \oplus \bigoplus_{j\neq 0} \widehat{\mathfrak{g}}_j$. Thus the tangent bundle of G/H_0 is $\mathsf{G} \times_{\mathsf{H}_0} (\mathfrak{m}_0 \oplus \bigoplus_{j\neq 0} \widehat{\mathfrak{g}}_j)$. The complexified tangent bundle is

$$T_{\mathbb{C}}\mathsf{G}/\mathsf{H}_0 = \mathsf{G} \times_{\mathsf{H}_0} (\mathfrak{m}_0^{\mathbb{C}} \oplus \bigoplus_{j\neq 0} \widehat{\mathfrak{h}}_j^{\mathbb{C}} \oplus \widehat{\mathfrak{m}}_j^{\mathbb{C}}) = \bigoplus_{j\in\mathbb{Z}/k\mathbb{Z}} \mathsf{G} \times_{\mathsf{H}_0} \widehat{\mathfrak{m}}_j^{\mathbb{C}} \oplus \bigoplus_{j\neq 0} \mathsf{G} \times_{\mathsf{H}_0} \widehat{\mathfrak{h}}_j^{\mathbb{C}}.$$

With respect to this splitting, the map $d\pi : T_{\mathbb{C}}\mathsf{G}/\mathsf{H}_0 \to T_{\mathbb{C}}\mathsf{G}/\mathsf{H}$ is given by

$$d\pi = (Id, 0) : \bigoplus_{j\in\mathbb{Z}/k\mathbb{Z}} \mathsf{G} \times_{\mathsf{H}_0} \widehat{\mathfrak{m}}_j^{\mathbb{C}} \oplus \bigoplus_{j\neq 0} \mathsf{G} \times_{\mathsf{H}_0} \widehat{\mathfrak{h}}_j^{\mathbb{C}} \longrightarrow \mathsf{G} \times_H \mathfrak{m}_{\mathbb{C}}.$$

Theorem 4.3.4. *Let $(\mathcal{E}, \varphi)$ be a stable and simple G-Higgs bundle. If $(\mathcal{E}, \varphi)$ is a fixed point of the $\langle \zeta_k \rangle$ then the map equivariant map $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}_0$ is harmonic and $df(T^{1,0}\widetilde{\Sigma}) \subset \mathsf{G} \times_{\mathsf{G}_0} \widehat{\mathfrak{m}}_1^{\mathbb{C}}$. Conversely, let $\rho \in \mathcal{X}(\pi_1, \mathsf{G})$ and $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ be a $\mathbb{Z}/k\mathbb{Z}$-grading with G_0 and H_0 as before, if $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}_0$ is an equivariant harmonic map with $df(T^{1,0}\widetilde{\Sigma}) \subset \mathsf{G} \times_{\mathsf{G}_0} \widehat{\mathfrak{m}}_1^{\mathbb{C}}$ then the Higgs bundle $(f^*\mathsf{G} \times_{\mathsf{H}_0} \mathsf{G}, d^{(1,0)}f)$ is a polystable Higgs bundle which is a fixed point of $\langle \zeta_k \rangle$.*

Proof. If $(\mathcal{E}, \varphi)$ is a stable and simple fixed point of $\langle \zeta_k \rangle \subset \mathbb{C}^*$ then, by Theorem 4.2.2, there is a $\mathbb{Z}/k\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ with $\mathsf{G}_0 \subset \mathsf{G}$ the connected Lie group with Lie algebra $\widehat{\mathfrak{g}}_0$ so that $(\mathcal{E}, \varphi)$ arises from a polystable K-twisted $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ Higgs pair $(\mathcal{E}_0, \widehat{\varphi})$ via extension of structure group. This gives the following commuting diagram of reductions of structures (equivalently equivariant maps)

$$\begin{tikzcd}
\mathsf{G}/\mathsf{G}_0 & \mathsf{G}/\mathsf{H}_0 \arrow[l] \arrow[d] \\
\widetilde{\Sigma} \arrow[u] \arrow[ur, "f"] \arrow[r, "h"'] & \mathsf{G}/\mathsf{H}
\end{tikzcd}$$

where $\mathsf{H}_0 = \mathsf{G}_0 \cap \mathsf{H}$ be the maximal compact subgroup of G_0. Since the Higgs field $\widehat{\varphi} \in H^0(\mathcal{E}_0 \times_{\mathsf{H}_0} \widehat{\mathfrak{g}}_1 \otimes K)$ is identified with the $(1,0)$ part of the derivative of the map f and $\widehat{\mathfrak{g}}_1 = \widehat{\mathfrak{m}}_1^{\mathbb{C}}$, we have $df(T^{1,0}\widetilde{\Sigma}) \subset \mathsf{G} \times_{\mathsf{G}_0} \widehat{\mathfrak{m}}_1^{\mathbb{C}}$. Moreover, since G/H_0 is a naturally reductive homogeneous space, holomorphicity of $\widehat{\varphi}$ implies the map f is harmonic.

Now suppose $\rho \in \mathcal{X}(\pi_1, \mathsf{G})$ and let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} \widehat{\mathfrak{g}}_j$ be a $\mathbb{Z}/k\mathbb{Z}$-grading with G_0 and H_0 as before. Let $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}_0$ be a ρ-equivariant harmonic map with $df(T^{1,0}\widetilde{\Sigma}) \subset \mathsf{G} \times_{\mathsf{G}_0} \widehat{\mathfrak{m}}_1^{\mathbb{C}}$. Pulling back the H_0 bundle $\mathsf{G} \to \mathsf{G}/\mathsf{H}_0$ and extending the structure group to G_0 gives a holomorphic G_0 bundle $\mathcal{E}_{\mathsf{G}_0} \to \Sigma$. Since the f is harmonic, $df(T^{1,0}\widetilde{\Sigma}) \subset \mathsf{G} \times_{\mathsf{G}_0} \widehat{\mathfrak{m}}_1^{\mathbb{C}}$ and G/H_0 is a naturally reductive homogeneous space, $d^{(1,0)}f = \widehat{\varphi} \in H^0(\mathcal{E}_{\mathsf{G}_0} \times_{\mathsf{H}_0} \widehat{\mathfrak{m}}_{\mathbb{C}}^1 \otimes K)$. Thus $(f^*\mathsf{G} \times_{\mathsf{H}_0} \mathsf{G}_0, d^{(1,0)}f) = (\mathcal{E}_{\mathsf{G}_0}, \widehat{\varphi})$ defines a $(\mathsf{G}_0, \widehat{\mathfrak{g}}_1)$ K-twisted Higgs Higgs pair, and extension of structure group to G defines a polystable G-Higgs bundle which is a fixed point of $\langle \zeta_k \rangle$. $\square$

4.3.1 G-variations of Hodge structure and harmonic maps

Let G be a real reductive Lie group with the property that G has a maximal torus T which is compact, such a G is called a group of *Hodge type*. This is equivalent to the Lie algebra $\mathfrak{g}$ having no complex roots with respect to a maximally compact Cartan subalgebra (see 2.1.2). The classical nonexamples are any complex reductive Lie group (thought of as real), $\mathsf{SL}(n, \mathbb{R})$, and $\mathsf{SO}(p, q)$ with p, q both odd; there are also two real forms of $E6$ which are not of Hodge type. If $\mathsf{H} \subset \mathsf{G}$ is a maximal compact subgroup then the condition that a Lie group be of Hodge type is equivalent to $rank(\mathsf{G}_{\mathbb{C}}) = rank(\mathsf{H}_{\mathbb{C}})$. For example, for $\mathsf{G} = \mathsf{Sp}(2n, \mathbb{R})$, we have $\mathsf{G}_{\mathbb{C}} = \mathsf{Sp}(2n, \mathbb{C})$, $\mathsf{H}_{\mathbb{C}} = \mathsf{GL}(n, \mathbb{C})$ and $rank(\mathsf{Sp}(2n, \mathbb{C})) = n = rank(\mathsf{GL}(n, \mathbb{C}))$ while for $\mathsf{G} = \mathsf{SL}(2n, \mathbb{R})$, we have $\mathsf{G}_{\mathbb{C}} = \mathsf{SL}(2n, \mathbb{C})$, $\mathsf{H}_{\mathbb{C}} = \mathsf{SO}(2n, \mathbb{C})$ and

$$rank(\mathsf{SL}(2n, \mathbb{C})) = 2n - 1 \qquad \text{and} \qquad rank(\mathsf{SO}(2n, \mathbb{C})) = n.$$

The following comes from [Sim88], although we will follow the set up of [GRT13]. Let G be simple and of Hodge type, fix a maximal compact $\mathsf{H} \subset \mathsf{G}$ and corresponding Cartan involution $\Theta : \mathsf{G} \to \mathsf{G}$. Let $\mathsf{T} \subset \mathsf{H}$ be the maximal torus, $\mathsf{T}' \subset \mathsf{T}$ a subtorus and let $\mathsf{V} = Z_{\mathsf{G}}(\mathsf{T}')$ be the centralizer of T' in G. The inclusions $\mathsf{T} \subset \mathsf{V} \subset \mathsf{H} \subset \mathsf{G}$ give a fibration

$$\mathsf{H}/\mathsf{V} - \mathsf{G}/\mathsf{V} \to \mathsf{G}/\mathsf{H}$$

over the symmetric space; G/V is called a *flag domain*. On the Lie algebra level we have $\mathfrak{t}' \subset \mathfrak{t} \subset \mathfrak{v} \subset \mathfrak{h} \subset \mathfrak{g}$ with Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ giving $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. This splitting to be orthogonal with respect to the Killing form. Note that the complexification $\mathsf{V}_{\mathbb{C}} \subset \mathsf{G}_{\mathbb{C}}$ is the Levi factor of a parabolic subgroup.

Note that the roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ satisfy $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \subset i\mathfrak{t}^*$. Since $\theta|_{\mathfrak{t}_{\mathbb{C}}} = +1$, for all $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ we have $\theta(\alpha) = \alpha$. Recall from 2.1.2 that a root α is called *compact* if the root space $\mathfrak{g}_\alpha \subset \mathfrak{h}_{\mathbb{C}}$ and *noncompact* if $\mathfrak{g}_\alpha \subset \mathfrak{m}_{\mathbb{C}}$, and that

- if α, β compact then $\alpha + \beta$ compact
- if α, β noncompact then $\alpha + \beta$ compact
- if α compact and β noncompact then $\alpha + \beta$ noncompact.

For $\mathfrak{s} \subset \mathfrak{g}_{\mathbb{C}}$ define $\Delta(\mathfrak{s}) = \{\alpha \in \Delta | \mathfrak{g}_\alpha \subset \mathfrak{s}\}$. If $[t_{\mathbb{C}}, \mathfrak{s}] \subset \mathfrak{s}$ we have $\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{s})} \mathfrak{g}_\alpha$. In particular,

$$\mathfrak{v}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{v}_{\mathbb{C}})} \mathfrak{g}_\alpha \qquad \mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{h}_{\mathbb{C}})} \mathfrak{g}_\alpha \qquad \mathfrak{m}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta(\mathfrak{m}_{\mathbb{C}})} \mathfrak{g}_\alpha$$

Picking a positive root system (or equivalently, a Borel subalgebra), gives us simple roots $\Pi \subset \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Fix a set of positive simple roots $\Pi \subset \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, and let $\{\epsilon_i\}$ be the basis for $\mathfrak{t}_{\mathbb{C}}$ dual to $\{\alpha_i\}$. Define

$$\psi = \sum_{\pi_i \in \Delta(\mathfrak{m}_{\mathbb{C}})} \pi_i \; + \sum_{\pi_i \in \Delta(\mathfrak{h}_{\mathbb{C}}) \setminus \Delta(\mathfrak{v}_{\mathbb{C}})} 2\pi_i$$

where $\{\pi_i\}$ are the fundamental weights (i.e. $\alpha_j(\pi_i) = \delta_{ij}$). Since $\psi \in \mathfrak{t}_{\mathbb{C}}$ is semisimple, the Lie algebra $\mathfrak{g}_{\mathbb{C}}$

decomposes as a direct sum of eigenspaces of ψ,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_k \oplus \mathfrak{g}_{k-1} \oplus \cdots \oplus \mathfrak{g}_{-k+1} \oplus \mathfrak{g}_{-k} \tag{4.3.2}$$

where $\mathfrak{g}_m = \{\xi \in \mathfrak{g}_{\mathbb{C}} | ad_\psi(\xi) = m\xi\}$, since all roots are integer combinations of simple roots, the eigenvalues of ad_ψ are integers.

Remark 4.3.5. This is the $\mathbb{Z}$-grading which arises from labeling all noncompact imaginary roots in the Vogan diagram with a 1 and all compact imaginary roots in the Vogan diagram with a 2.

The space $\mathfrak{g}_1$ consists of all noncompact root spaces of height 1, since G is assumed to be noncompact, we have $dim(\mathfrak{g}_1) \geq 1$. Moreover, since the roots are purely imaginary, we have

$$\overline{\mathfrak{g}_m} = \mathfrak{g}_{-m}$$

for the compact conjugation. Setting $\mathfrak{g}_{\mathbb{C}}^{m,-m} = \mathfrak{g}_m$, we recover a real, weight zero Hodge structure on $\mathfrak{g}$; since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the Hodge structure is polarized by the Killing form (see [Sim88]).

The element ψ is the grading element for the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}_{\mathbb{C}}$. Since the decomposition of a compact root into a linear combination of simple roots must have an even number of noncompact contributions, we have

$$\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{v}_{\mathbb{C}} = \mathfrak{g}_0 \qquad \mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{even} = \bigoplus_m \mathfrak{g}_{2m} \qquad \mathfrak{m}_{\mathbb{C}} = \mathfrak{g}_{odd} = \bigoplus_m \mathfrak{g}_{2m+1}$$

Decompose $\mathfrak{g}_{\mathbb{C}}$ into positive and negative eigenspaces

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ = \mathfrak{h}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{m}_- \oplus \mathfrak{m}_+$$

and let $\mathfrak{q}_m = \mathfrak{g} \cap (\mathfrak{g}_m \oplus \mathfrak{g}_{-m})$, then $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_k$.

The real tangent space of G/V is given by $T(\mathsf{G}/\mathsf{V}) = \bigoplus_{j=1}^{k} \mathsf{G} \times_{\mathsf{V}} \mathfrak{q}_j$. Complex structures on G/V are given by specifying $T_{\mathbb{C}}(\mathsf{G}/\mathsf{V}) = T^{1,0}\mathsf{G}/\mathsf{V} \oplus T^{01}\mathsf{G}/\mathsf{V}$ with

$$T^{1,0}\mathsf{G}/\mathsf{V} = \mathsf{G} \times_{\mathsf{V}} \mathfrak{g}_- \quad \text{and} \quad T^{0,1}\mathsf{G}/\mathsf{V} = \mathsf{G} \times_{\mathsf{V}} \mathfrak{g}_+$$

Thus each choice of positive roots gives a complex structure. Since the splitting (4.3.2) is V-invariant the complexified tangent bundle decomposes as

$$T_{\mathbb{C}}\mathsf{G}/\mathsf{V}=\bigoplus_{0<|j|\leq k}\mathsf{G}\times_V\mathfrak{g}_j$$

with $T^{1,0}\mathsf{G}/\mathsf{V}=\bigoplus_{j=-1}^{-k}\mathsf{G}\times_{\mathsf{V}}\mathfrak{g}_j$. We are now ready to define a G-variation of Hodge structure.

Definition 4.3.6. Let Σ be a compact Riemann surface, a G-variation of Hodge structure is a triple $(\rho,\mathsf{G}/\mathsf{V},F)$ where $\rho:\pi_1(\Sigma)\to\mathsf{G}$ is a representation, G/V is a flag domain for G, and $F:\widetilde{\Sigma}\to\mathsf{G}/\mathsf{V}$ is an ρ-equivariant holomorphic map with $dF(T^{1,0}\widetilde{\Sigma})\subset\mathsf{G}\times_{\mathsf{V}}\mathfrak{g}_{-1}\subset T^{1,0}\mathsf{G}/\mathsf{V}$.

A G-variation of Hodge structure $(\rho,\mathsf{G}/\mathsf{V},F)$ gives rise to a G-Higgs bundle $(\mathcal{E},\varphi)=(F^*\mathsf{G}\times_V\mathsf{H}_{\mathbb{C}},d^{(1,0)}F)$; here we are pulling back the V-bundle with total space G and extending the structure group to $\mathsf{H}_{\mathbb{C}}$ and pulling back the Maurer-Cartan form (identified with dF) and taking its $(1,0)$-part. Moreover, since the grading element ψ is in the center of $\mathfrak{v}$, exponentiating $exp(\lambda\psi)$ gives a one parameter family of holomorphic gauge transformation which acts on the Higgs field by $e^{-\lambda}$. Thus the Higgs bundle $(\mathcal{E},\varphi)$ associated to a G-variation of Hodge structure is a fixed point of the $\mathbb{C}^*$-action. In [Sim88], Simpson proved that if $(\mathcal{E},\phi)$ is a G-Higgs bundle that is a fixed point of the $\mathbb{C}^*$ action, then it gives rise to a G-variation of Hodge structure.

The correspondence between G-variations of Hodge structure and fixed points of the $\mathbb{C}^*$ action in $\mathcal{M}(\mathsf{G})$ relies on the holomorphicity of the map F in Definition 4.3.6. In [Tol13], Toledo asked whether this condition is equivalent to a harmonic condition on F. We now show that this is not the case, and if one only requires the map F in Definition 4.3.6 to be harmonic, then one is naturally lead to finite order fixed points.

Theorem 4.3.7. *Let G/V be a flag domain with $\mathfrak{g}_{2k-1}$ the highest nonzero summand in the corresponding $\mathbb{Z}$-grading. Let Σ be a closed Riemann surface and $\rho:\pi_1(\Sigma)\to\mathsf{G}$ be a representations. If $F:\widetilde{\Sigma}\to\mathsf{G}/\mathsf{V}$ is a ρ-equivariant harmonic map with $dF(T^{(1,0)}\widetilde{\Sigma})\subset\mathsf{G}\times_V(\mathfrak{g}_{-1}\oplus\mathfrak{g}_{2k-1})$ that is not holomorphic then $(F^*\mathsf{G}\times_{\mathsf{V}}\mathsf{H}_{\mathbb{C}},dF^{(1,0)})$ is a polystable G-Higgs bundle fixed by $\langle\zeta_{2k}\rangle$ and not by all of $\mathbb{C}^*$.*

Proof. We first show that G/V is a naturally reductive homogeneous space (see Definition 4.3.1). Recall that the Killing form satisfies the identity $B_{\mathfrak{g}}([X,Y],Z)+B_{\mathfrak{g}}(Y,[X,Z])=0$ for all $X,Y,Z\in\mathfrak{g}$. The Ad_{V} splitting $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{m}$ is orthogonal with respect to the Riemannian metric $\langle X,Y\rangle=-B_{\mathfrak{g}}(X,\theta(Y))$ by (2.1.3). Thus for all $X,Y,Z\in\mathfrak{m}$ we have

$$0=\langle[X,Y],Z\rangle+\langle Y,[X,Z]\rangle=\langle[X,Y]_{\mathfrak{m}},Z\rangle+\langle Y,[X,Z]_{\mathfrak{m}}\rangle+\langle[X,Y]_{\mathfrak{l}},Z\rangle+\langle Y,[X,Z]_{\mathfrak{l}}\rangle$$

$$=\langle[X,Y]_{\mathfrak{m}},Z\rangle+\langle Y,[X,Z]_{\mathfrak{m}}\rangle$$

Thus, G/V is a naturally reductive homogeneous space and harmonicity of map F is equivalent to holomorphicity of $d^{(1,0)}F$ with respect to the canonical connection.

The bundle $\mathcal{E}_{\mathsf{H}_\mathbb{C}} = F^*\mathsf{G}\times_\mathsf{V}\mathsf{H}_\mathbb{C}$ is a holomorphic $\mathsf{H}_\mathbb{C}$ bundle and the Higgs field is $d^{(1,0)}F \in \Omega^{1,0}(\mathcal{E}_{\mathsf{H}_\mathbb{C}}\times_{\mathsf{H}_\mathbb{C}}\mathfrak{m}_\mathbb{C})$. Moreover, since F is harmonic, by the above discussion, $\varphi = d^{(1,0)}F$ is holomorphic with respect the pullback of the canonical connection on G/V. The Higgs bundle $(\mathcal{E}_{\mathsf{H}_\mathbb{C}}, \varphi)$ is polystable since the pullback of the flatness equations (2.2.1) by F and the holomorphicity of $d^{(1,0)}F$ with respect to the pull back of the canonical connection solve the Higgs bundle equations.

Finally, recall that the grading element $\psi \in \mathfrak{z}(\mathfrak{v})$ defines a $\mathbb{Z}$ grading $\mathfrak{g}_\mathbb{C} = \bigoplus\limits_{j\in\mathbb{Z}} \mathfrak{g}_j$ with $\mathfrak{g}_{2i+1} \subset \mathfrak{m}_\mathbb{C}$ and $\mathfrak{g}_{2i} \subset \mathfrak{h}_\mathbb{C}$ with $\mathfrak{g}_{2k-1}$ the highest nonzero summand in $\mathfrak{m}_\mathbb{C}$. By assumption, the Higgs field is given by $d^{(1,0)}F \in F^*\mathsf{G} \times_V (\mathfrak{g}_{-1} \oplus \mathfrak{g}_{2k-1}) \otimes K$. Thus, $g_{2k} = exp(\frac{-2\pi i\psi}{2k})$ defines a holomorphic gauge transformation of $\mathcal{E}_{\mathsf{H}_\mathbb{C}}$ which acts as $Ad_{g_{2k}}\varphi = \zeta_{2k}\varphi$. Moreover, F is holomorphic if and only if $d^{(1,0)}F \subset F^*\mathsf{G} \times_\mathsf{V} \bigoplus\limits_{j<0} \mathfrak{g}_j \otimes K$. Hence, the harmonic map F is not holomorphic if and only if the component of $d^{(1,0)}F$ in $\mathfrak{g}_{2k-1}$ is nonzero, this is equivalent to the Higgs bundle $(\mathcal{E}_{\mathsf{H}_\mathbb{C}}, \varphi)$ being fixed by $\langle\zeta_{2k}\rangle$ but not all of $\mathbb{C}^*$. $\square$

Remark 4.3.8. Note that for a variation of Hodge structure, the condition $dF(T^{(1,0)}\widetilde{\Sigma}) \subset \mathsf{G} \times_\mathsf{V} \mathfrak{g}_{-1}$ is an extra condition on the holomorphic map. That is, any map which satisfies $dF(T^{(1,0)}\widetilde{\Sigma}) \subset \mathsf{G} \times_\mathsf{V} \mathfrak{g}_{-1}$ is automatically holomorphic. This is not the case for the harmonic maps considered in Theorem 4.3.7. Namely, maps which satisfy $dF(T^{(1,0)}\widetilde{\Sigma}) \subset \mathsf{G} \times_\mathsf{V} (\mathfrak{g}_{-1} \oplus \mathfrak{g}_{2k-1})$ are not automatically harmonic. There are however, many examples of such maps (see Chapter 6).

Chapter 5

Asymptotics of fixed points in the Hitchin component

In this chapter we summarize the joint work with Q. Li in [CL14]. We include this work here since it is an application of finite order fixed points on $\mathcal{M}(\mathsf{G})$. Most details and proofs have been omitted, we direct the interested reader to [CL14]. For fixed points of $\langle\zeta_n\rangle$ and $\langle\zeta_{n-1}\rangle$ in the $\mathsf{SL}(n,\mathbb{R})$-Hitchin components, we investigate the asymptotics of the nonabelian Hodge correspondence. More precisely, by Proposition 4.2.7, in terms of the holomorphic differential paramenterization of the Hitchin component, these fixed points are given by $(0,\cdots,q_{n-1},0)$ and $(0,\cdots,0,q_n)$, and along the rays $(0,\cdots,tq_{n-1},0)$ and $(0,\cdots,0,tq_n)$ in the Hitchin component, we study the asymptotics of the metric h_t solving the Higgs bundle equations, the harmonic maps $f_t:\widetilde{\Sigma}\to\mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n)$ and the parallel transport holonomy. This analysis leads to a proof of a conjecture by Katzarkov, Pandit, Noll, and Simpson [KNPS15] on the Hitchin WKB-problem.

5.1 Equations, flat connections and metric asymptotics

In this section, the metric splitting property of fixed points will be used to write the Hitchin equations as a system of $\lfloor\frac{n}{2}\rfloor$ fully coupled nonlinear elliptic equations, and to give an explicit description of the corresponding flat connections. There are slight differences when n is even compared to when n is odd. We will always work in the even case and mention what the differences are for the odd case. One obvious difference in the odd case is the middle line bundle of E is a trivial bundle; for both $\phi=\tilde{e}_1+q_ne_{n-1}$ and $\phi=\tilde{e}_1+q_{n-1}e_{n-2}$, the metric on the trivial line bundle is the standard one on $\mathbb{C}$.

5.1.1 Equations

Since the metric splits as $h = h_1 \oplus h_2 \oplus \cdots \oplus h_2^{-1} \oplus h_1^{-1}$, the adjoints of the Higgs fields $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$ are respectively

$$\phi^* = \begin{pmatrix} 0 & h_1^{-1}h_2 & 0 & & \\ & & h_2^{-1}h_3 & & \\ & & & \ddots & \\ 0 & & & 0 & h_1^{-1}h_2 \\ h_1^2\bar{q}_n & 0 & & & 0 \end{pmatrix} \qquad \phi^* = \begin{pmatrix} 0 & h_1^{-1}h_2 & 0 & & \\ & & h_2^{-1}h_3 & & \\ & & & \ddots & \\ h_1 h_2 \bar{q}_{n-1} & & & 0 & h_1^{-1}h_2 \\ 0 & h_1 h_2 \bar{q}_{n-1} & & & 0 \end{pmatrix}$$

We are interested in the corresponding family of flat connections as the differentials q_n and q_{n-1} are scaled by a real parameter t. Using the simplification of the Hitchin equations for fixed points, the Hitchin equations for n-cyclic Higgs field $\phi = \tilde{e}_1 + tq_n e_{n-1}$ become:

$$\begin{cases} F_{A_1} + t^2 h_1^2 q_n \wedge \bar{q}_n - h_1^{-1}h_2 = 0 \\ F_{A_j} + h_{j-1}^{-1}h_j - h_j^{-1}h_{j+1} = 0 \qquad 1 < j < \dfrac{n}{2} \\ F_{A_{\frac{n}{2}}} + h_{\frac{n}{2}-1}^{-1} h_{\frac{n}{2}} - h_{\frac{n}{2}}^{-2} = 0 \end{cases} \tag{5.1.1}$$

Here all the metrics, and hence, all the curvature forms depend on t. We will suppress the t dependence from the notation. When n is odd, the last equation is changed to $F_{A_{\frac{n-1}{2}}} + h_{\frac{n-1}{2}-1}^{-1} h_{\frac{n-1}{2}} - h_{\frac{n-1}{2}}^{-1} = 0$.

To understand the flat connection we choose a local coordinate z on Σ. Such a choice gives a local holomorphic frame $(s_1, s_2, \ldots, s_2^*, s_1^*)$ for E, where $s_j = dz^{\frac{n+1-2j}{2}}$ is the local frame of $K^{\frac{n+1-2j}{2}}$ induced by the coordinate z. With respect to this choice of coordinates, the Higgs field is locally given by

$$\phi = \begin{pmatrix} 0 & & & f_n \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix}$$

where $q_n = f_n dz^n$, for some function f_n.

With respect to this frame, locally represent the metric h_j by $e^{-\lambda^j}$, here the j is a superscript and **not** an exponent. Recall that in a holomorphic frame, the Chern connection has connection 1-form $A = H^{-1}\partial H$ and curvature 2-form given by $F_A = \bar{\partial}(H^{-1}\partial H)$. Since h_j is a metric on a line bundle, the expressions simplify to

$$A_j = -\lambda^j_z dz \qquad \text{and} \qquad F_{A_j} = \lambda^j_{z\bar{z}} dz \wedge d\bar{z}.$$

The equations may be rewritten as:

$$\begin{cases} \lambda^1_{z\bar{z}} + t^2 e^{-2\lambda^1} |q_n|^2 - e^{\lambda^1 - \lambda^2} = 0 \\ \lambda^j_{z\bar{z}} + e^{\lambda^{j-1} - \lambda^j} - e^{\lambda^j - \lambda^{j+1}} = 0 \qquad 1 < j < \dfrac{n}{2} \\ \lambda^{\frac{n}{2}}_{z\bar{z}} + e^{\lambda^{\frac{n}{2}-1} - \lambda^{\frac{n}{2}}} - e^{2\lambda^{\frac{n}{2}}} = 0 \end{cases} \tag{5.1.2}$$

Similarly for $(n-1)$-cylcic Higgs field $\phi = \tilde{e}_1 + t q_{n-1} e_{n-2}$, we may rewrite the Hitchin equations as

$$\begin{cases} \lambda^1_{z\bar{z}} + t^2 e^{-\lambda^1 - \lambda^2} |q_{n-1}|^2 - e^{\lambda^1 - \lambda^2} = 0 \\ \lambda^2_{z\bar{z}} + t^2 e^{-\lambda^1 - \lambda^2} |q_{n-1}|^2 + e^{\lambda^1 - \lambda^2} - e^{\lambda^2 - \lambda^3} = 0 \\ \lambda^j_{z\bar{z}} + e^{\lambda^{j-1} - \lambda^j} - e^{\lambda^j - \lambda^{j+1}} = 0 \qquad\qquad 2 < j < \dfrac{n}{2} \\ \lambda^{\frac{n}{2}}_{z\bar{z}} + e^{\lambda^{\frac{n}{2}-1} - \lambda^{\frac{n}{2}}} - e^{2\lambda^{\frac{n}{2}}} = 0 \end{cases} \tag{5.1.3}$$

Again, in the odd case, the last equation is changed to $\lambda^{\frac{n-1}{2}}_{z\bar{z}} + e^{\lambda^{\frac{n-1}{2}-1} - \lambda^{\frac{n-1}{2}}} - e^{\lambda^{\frac{n-1}{2}}} = 0$.

5.1.2 Flat connections

The flat connection is given by $D = A_h + \phi + \phi^*$. If, in the holomorphic frame $(s_1, \ldots, s_{\frac{n}{2}}, s^*_{\frac{n}{2}}, \ldots, s^*_1)$, we have

$$q_n = f_n dz^n \quad \text{and} \quad q_{n-1} = f_{n-1} dz^{n-1}, \tag{5.1.4}$$

then the flat connection for the n-cyclic $\phi = \tilde{e}_1 + t q_n e_{n-1}$ is given by

$$D = \begin{pmatrix} -\lambda^1_z dz & & & & t f_n \\ 1 & -\lambda^2_z dz & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda^2_z dz & 0 \\ & & & 1 & \lambda^1_z dz \end{pmatrix} + \begin{pmatrix} 0 & e^{\lambda^1 - \lambda^2} & & & \\ 0 & 0 & e^{\lambda^2 - \lambda^3} & & \\ & & & \ddots & \\ & & & & e^{\lambda^1 - \lambda^2} \\ t e^{-2\lambda^1} \bar{f}_n & & & & 0 \end{pmatrix}, \tag{5.1.5}$$

and the flat connection for the $(n-1)$-cyclic $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$, is

$$D = \begin{pmatrix} -\lambda^1_z dz & & & tf_{n-1} & 0 \\ 1 & -\lambda^2_z dz & & & tf_{n-1} \\ 0 & 1 & -\lambda^3_z dz & & \\ & & & \ddots & \\ & & & 1 & \lambda^1_z dz \end{pmatrix} + \begin{pmatrix} 0 & e^{\lambda^1-\lambda^2} & & & \\ 0 & 0 & e^{\lambda^2-\lambda^3} & & \\ & & & \ddots & \\ te^{-\lambda^1-\lambda^2}\bar{f}_{n-1} & & & & e^{\lambda^1-\lambda^2} \\ 0 & te^{-\lambda^1-\lambda^2}\bar{f}_{n-1} & & & 0 \end{pmatrix}. \tag{5.1.6}$$

We want to calculate the behavior of the flat connection in the limit $t\to\infty$. To do so, we need to understand the asymptotics of the λ^j's and the asymptotics of their first derivatives λ^j_z. In order to use the maximum principle, we will make a change of variables. Let $\Omega_n \subset \Sigma$ be a compact set away from the zeros of q_n and fix a background metric g_n on Σ with the following properties:

$$\begin{cases} g_n = |q_n|^{\frac{2}{n}} & \text{on } \Omega_n \\ \dfrac{|q_n|^2}{(g_n)^n} \leq 1 & \text{on } \Sigma \end{cases} \tag{5.1.7}$$

Using this metric, we make the following change of variables:

$$u^j = \lambda^j - \frac{n+1-2j}{2}\ln(g_n).$$

For $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$, we define the analogous compact set Ω_{n-1} and background metric g_{n-1} with the property

$$\begin{cases} g_{n-1} = |q_{n-1}|^{\frac{2}{n-1}} & \text{on } \Omega_{n-1} \\ \dfrac{|q_{n-1}|^2}{(g_{n-1})^{n-1}} \leq 1 & \text{on } \Sigma \end{cases} \tag{5.1.8}$$

Using g_{n-1}, we make the change of variables

$$v^j = \lambda^j - \frac{n+1-2j}{2}\ln(g_{n-1}).$$

Recall that the Laplace-Beltrami operator of a conformal metric g on a Riemann surface is given by $\Delta_g = \frac{4}{g}\partial_{z\bar{z}}$ and the scalar curvature is

$$K_g = -\frac{1}{2}\Delta_g \ln(g) = -\frac{2}{g}\partial_{z\bar{z}}\ln(g).$$

Because q_n and q_{n-1} are holomorphic, $K_{g_n} = 0 = K_{g_{n-1}}$ on Ω_n and Ω_{n-1}.

With respect to u^j, the equations for $\phi = \tilde{e}_1 + tq_n e_{n-1}$ become

$$\begin{cases} (u^1 + \dfrac{n-1}{2}\ln(g_n))_{z\bar{z}} + t^2 e^{-2u^1-(n-1)\ln(g_n)}|q_n|^2 - e^{u^1-u^2+\ln(g_n)} = 0 \\ (u^j + \dfrac{n+1-2j}{2}\ln(g_n))_{z\bar{z}} + e^{u^{j-1}-u^j+\ln(g_n)} - e^{u^j-u^{j+1}+\ln(g_n)} = 0 \qquad 1 < j < \dfrac{n}{2} \\ (u^{\frac{n}{2}} + \dfrac{1}{2}\ln(g_n))_{z\bar{z}} + e^{u^{\frac{n}{2}-1}-u^{\frac{n}{2}}+\ln(g_n)} - e^{2u^{\frac{n}{2}}+\ln(g_n)} = 0 \end{cases} \tag{5.1.9}$$

Using our knowledge of K_{g_n} and Δ_{g_n}, we rewrite the equations as

$$\begin{cases} -\dfrac{1}{4}\Delta_{g_n}u^1 = -\dfrac{n-1}{4}K_{g_n} + \dfrac{t^2|q_n|^2}{g_n^n}e^{-2u^1} - e^{u^1-u^2} \\ -\dfrac{1}{4}\Delta_{g_n}u^j = -\dfrac{n+1-2j}{4}K_{g_n} + e^{u^{j-1}-u^j} - e^{u^j-u^{j+1}} \qquad 1 < j < \dfrac{n}{2} \\ -\dfrac{1}{4}\Delta_{g_n}u^{\frac{n}{2}} = -\dfrac{1}{4}K_{g_n} + e^{u^{\frac{n}{2}-1}-u^{\frac{n}{2}}} - e^{2u^{\frac{n}{2}}} \end{cases} \tag{5.1.10}$$

We will show

$$\lim_{t\to\infty} e^{u^j} = t^{\frac{n+1-2j}{n}} \qquad 1 \le j \le \frac{n}{2}.$$

Similarly, in terms of the v^j's, the equations for $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$ become

$$\begin{cases} -\dfrac{1}{4}\Delta_{g_{n-1}}v^1 = -\dfrac{n-1}{4}K_{g_{n-1}} + \dfrac{t^2|q_{n-1}|^2}{g_{n-1}^{n-1}}e^{-v^1-v^2} - e^{v^1-v^2} \\ -\dfrac{1}{4}\Delta_{g_{n-1}}v^2 = -\dfrac{n-1}{4}K_{g_{n-1}} + \dfrac{t^2|q_{n-1}|^2}{g_{n-1}^{n-1}}e^{-v^1-v^2} + e^{v^1-v^2} - e^{v^2-v^3} \\ -\dfrac{1}{4}\Delta_{g_{n-1}}v^j = -\dfrac{n+1-2j}{4}K_{g_{n-1}} + e^{v^{j-1}-v^j} - e^{v^j-v^{j+1}} \qquad 2 < j < \dfrac{n}{2} \\ -\dfrac{1}{4}\Delta_{g_{n-1}}v^{\frac{n}{2}} = -\dfrac{1}{4}K_{g_{n-1}} + e^{v^{\frac{n}{2}-1}-v^{\frac{n}{2}}} - e^{2v^{\frac{n}{2}}} \end{cases} \tag{5.1.11}$$

In this case, it will be shown that

$$\lim_{t\to\infty} e^{v^1} = t$$

$$\lim_{t\to\infty} e^{v^j} = (2t)^{\frac{n+1-2j}{n-1}} \qquad 1 < j \le \frac{n}{2}$$

5.1.3 Estimates on asymptotics of λ^j and λ^j_z

In order to understand the asymptotics of the family of flat connections above, we need to understand the asymptotics of the metric and its first derivative. For the metric, we have the following theorem.

Theorem 5.1.1. *For every point $p \in \Sigma$ away from the zeros of q_n or q_{n-1}, as $t\to\infty$*

1. For $(\Sigma,\ \tilde{e}_1 + tq_n e_{n-1}) \in Hit_n(S)$, the metric $h_j(t)$ on $K^{\frac{n+1-2j}{2}}$ admits the expansion

$$h_j(t) = (t|q_n|)^{-\frac{n+1-2j}{n}}\left(1 + O\left(t^{-\frac{2}{n}}\right)\right) \qquad \textit{for all } j$$

2. For $(\Sigma,\ \tilde{e}_1 + tq_{n-1} e_{n-2}) \in Hit_n(S)$, the metric $h_j(t)$ on $K^{\frac{n+1-2j}{2}}$ admits the expansion

$$h_j(t) = \begin{cases} (t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}\left(1 + O\left(t^{-\frac{2}{n-1}}\right)\right) & \textit{for } j = 1 \quad \textit{and} \quad j = n \\ (2t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}\left(1 + O\left(t^{-\frac{2}{n-1}}\right)\right) & \textit{for } 1 < j < n \end{cases}$$

In terms of the u^j's and v^j's, Theorem 5.1.1 says the asymptotics of the metric solving the Hitchin equations on Ω_n are

$$e^{u^j} = t^{\frac{n+1-2j}{n}}\left(1 + O\left(t^{-\frac{2}{n}}\right)\right) \qquad 1 \le j \le \frac{n}{2},$$

and for $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$, the asymptotics of the metric solving the Hitchin equations on Ω_{n-1} are

$$e^{v^j} = (2t)^{\frac{n+1-2j}{n-1}}\left(1 + O\left(t^{-\frac{2}{n-1}}\right)\right) \qquad 1 < j \le \frac{n}{2}$$

$$e^{v^1} = t\left(1 + O\left(t^{-\frac{2}{n-1}}\right)\right).$$

Using our understanding of the u^j's, the v^j's, and their Laplacians, we gain control of their first derivatives.

Proposition 5.1.2. *Let z be a local coordinate so that $q_n = dz^n$, then there is a constant $C_n = C_n(\Sigma, q_n, \Omega_n)$ so that*

$$|u^j_z| \le C_n t^{-\frac{1}{n}} \ .$$

Similarly, let z be a local coordinate so that $q_{n-1} = dz^{n-1}$, then there is a constant $C_{n-1} = C_{n-1}(\Sigma, q_{n-1}, \Omega_{n-1})$ so that

$$|v^j_z| \le C_{n-1} t^{-\frac{1}{n-1}} \ .$$

5.2 Parallel transport asymptotics

In this section, the parallel transport ODE we wish to integrate is setup. To avoid some redundancy, we will sometimes use a subscript or superscript b will be used to denote objects corresponding to the b-cyclic Higgs field $\phi_b = \tilde{e}_1 + tq_b e_{b-1}$ for $b = n, n-1$. We will also work in the universal cover $\widetilde{\Sigma}$ of Σ, all objects should be pulled back to the universal cover.

Let $P \in \widetilde{\Sigma}$ be away from the zeros of the differential q_b, and choose a neighborhood $\mathcal{U}_P$ centered at P, with coordinate z, so that $q_b = dz^b$. Note that for this to make sense, $\mathcal{U}_P$ must be disjoint from the zero set of q_b. In this neighborhood, $u^j = \lambda^j$ for $b = n$ and $v^j = \lambda^j$ for $b = n-1$.

As before, the choice of local coordinate z defines a local holomorphic frame $(s_1, \ldots, s_{\frac{n}{2}}, s^*_{\frac{n}{2}}, \ldots, s^*_1)$ for

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-1}{2}},$$

where $s_j = dz^{\frac{n+1-2j}{2}}$. In this frame, the connection 1-form of the corresponding flat connection is given by (5.1.5) and (5.1.6). By our choice of coordinates, the f_b in (5.1.4) is identically 1.

Using our estimates from Theorem 5.1.1 and Proposition 5.1.2, we will solve for the transport matrix $T_{P,P'}(t)$ along paths starting at P and ending at a point P' in the neighborhood $\mathcal{U}_P$. In fact, $T_{P,P'}(t)$ will be calculated along geodesics of the background metric $g_b = |dz|^{\frac{2}{b}}$ which start at P and end at P'. Since the connection is flat, the value of $T_{P,P'}(t)$ is path independent in $\mathcal{U}_P$.

We rescale the holomorphic frame $(s_1, \cdots, s^*_1)$ so that it stays bounded away from 0 and ∞ as $t \to \infty$. For $\phi = \tilde{e}_1 + tq_n e_{n-1}$, the rescaled frame is given by $F_n = (\sigma_1, \ldots, \sigma^*_1)$ where

$$\sigma_j = t^{\frac{n+1-2j}{2n}} s_j \qquad \sigma^*_j = t^{-\frac{n+1-2j}{2n}} s^*_j. \tag{5.2.1}$$

Remark 5.2.1. By Theorem 5.1.1, in the rescaled frame, the metric $h = Id\left(1 + O\left(t^{-\frac{2}{n}}\right)\right)$. To see this, consider

$$h(s_i, s_j) = \delta_{ij} t^{\frac{i+j-n-1}{n}} \left(1 + O\left(t^{-\frac{2}{n}}\right)\right)$$

thus

$$h(\sigma_i, \sigma_j) = h(t^{\frac{n+1-2i}{2n}} s_i, t^{\frac{n+1-2j}{2n}} s_j) = t^{\frac{n+1-(i+j)}{n}} h(s_i, s_j) = \delta_{ij} \left(1 + O\left(t^{-\frac{2}{n}}\right)\right).$$

For $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$, the rescaled frame is denote by $F_{n-1} = (\sigma_1, \ldots, \sigma^*_1)$, it is given by

$$\sigma_1 = t^{\frac{1}{2}} s_1 \qquad \sigma^*_1 = t^{-\frac{1}{2}} s^*_1$$

$$\sigma_j = (2t)^{\frac{n+1-2j}{2(n-1)}} s_j \qquad \sigma^*_j = (2t)^{-\frac{n+1-2j}{2(n-1)}} s^*_j \qquad j = 2, \ldots, \frac{n}{2}.$$

As in the previous case, the harmonic metric in this frame is $h = Id\left(1 + O\left(t^{-\frac{2}{n-1}}\right)\right)$.

If we denote the flat connection by $D_b = U_b dz + V_b d\bar{z}$, then, by the estimates from Theorem 5.1.1 and Proposition 5.1.2, the matrices in the connection 1-form are given by:

1. For $\phi = \tilde{e}_1 + q_n e_{n-1}$,

$$U_n = \begin{pmatrix} -\overset{1}{u}_z & & & t^{\frac{1}{n}} \\ t^{\frac{1}{n}} & -\overset{2}{u}_z & & \\ & \ddots & \ddots & \\ & & t^{\frac{1}{n}} & \overset{1}{u}_z \end{pmatrix} = t^{\frac{1}{n}} \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} + O\left(t^{-\frac{1}{n}}\right)$$

$$V_n = \begin{pmatrix} & t^{-\frac{1}{n}} e^{\overset{1}{u} - \overset{2}{u}} & & \\ & & \ddots & \\ & & & t^{-\frac{1}{n}} e^{\overset{1}{u} - \overset{2}{u}} \\ t^{-\frac{2n-1}{n}} e^{-2\overset{1}{u}} & & & \end{pmatrix} = t^{\frac{1}{n}} \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix} + O\left(t^{-\frac{1}{n}}\right)$$

where $O\left(t^{-\frac{1}{n}}\right)$ is uniform as $t \to \infty$ for all points in Ω_n.

2. For $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$,

$$U_{n-1} = \begin{pmatrix} -\overset{1}{v}_z & & & 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} & \\ 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} & -\overset{2}{v}_z & & & 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} \\ & 2^{\frac{1}{n-1}} t^{\frac{1}{n-1}} & & & \\ & & \ddots & & \\ & & & 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} & \overset{1}{v}_z \end{pmatrix}$$

$$= (2t)^{\frac{1}{n-1}} \begin{pmatrix} & & & \frac{1}{\sqrt{2}} & \\ \frac{1}{\sqrt{2}} & & & & \frac{1}{\sqrt{2}} \\ & 1 & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{2}} & \end{pmatrix} + O\left(t^{-\frac{1}{n-1}}\right)$$

$$V_{n-1} = \begin{pmatrix} & e^{\overset{1}{v} - \overset{2}{v}} 2^{\frac{n-3}{2(n-1)}} t^{-\frac{1}{n-1}} & & \\ & & \ddots & \\ e^{-\overset{1}{v} - \overset{2}{v}} 2^{\frac{n-3}{2(n-1)}} t^{\frac{2n-3}{n-1}} & & & e^{\overset{1}{v} - \overset{2}{v}} 2^{\frac{n-3}{2(n-1)}} t^{-\frac{1}{n-1}} \\ & e^{-\overset{1}{v} - \overset{2}{v}} 2^{\frac{n-3}{2(n-1)}} t^{\frac{2n-3}{n-1}} & & \end{pmatrix}$$

$$= (2t)^{\frac{1}{n-1}} \begin{pmatrix} & \frac{1}{\sqrt{2}} & & & \\ & & 1 & & \\ & & & \ddots & \\ \frac{1}{\sqrt{2}} & & & & \frac{1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & & & \end{pmatrix} + O\left(t^{-\frac{1}{n-1}}\right)$$

where $O\left(t^{-\frac{1}{n-1}}\right)$ is uniform as $t \to \infty$ for all points in Ω_{n-1}.

As noted above, we will integrate the initial value problem along geodesics of the metric $|q_b|^{\frac{2}{b}}$ which avoid the zeros of q_b. Any $P' \in \mathcal{U}_P$, can be expressed in polar coordinates $P' = Le^{i\theta}$; the geodesic γ of the metric $|q_b|^{\frac{2}{b}}$ which starts at P and ends at P' is the straight line

$$\gamma(s) = se^{i\theta} \qquad \text{for } s \in [0, L].$$

To avoid an overload of notation, when there is no confusion, the b will be dropped from the notation. We start at P with the initial rescaled holomorphic frame $F(P)$. For a fixed t, parallel transportation along the geodesic $\gamma(s) : [0, L] \to \widetilde{\Sigma}$ with respect to the flat connection yields a family of frames $G(\gamma(s))(t)$ along γ given by

$$G(\gamma(s))(t) = T_{P,\gamma(s)}(t)(F(P)) \qquad \text{with} \qquad T_{P,\gamma(0)}(t) = Id.$$

For each t, consider the family of matrices $\Psi_t(s)$ satisfying

$$\Psi_t(0) = Id \quad \text{ and } \quad \Psi_t(s)G(\gamma(s))(t) = F(\gamma(s)).$$

Since $G(\gamma(s))(t)$ is parallel along γ, rewriting $\nabla_{\frac{\partial}{\partial s}} F(\gamma(s))$ in terms of $G(\gamma(s))(t)$ yields

$$\nabla_{\frac{\partial}{\partial s}} F(\gamma(s)) = \frac{d\Psi_t}{ds} G(\gamma(s))(t).$$

Also,

$$\nabla_{\frac{\partial}{\partial s}} F(\gamma(s)) = (e^{i\theta}U + e^{-i\theta}V)F(\gamma(s)) = (e^{i\theta}U + e^{-i\theta}V)\Psi_t G(\gamma(s))(t),$$

hence,

$$\frac{d\Psi_t}{ds} = \left(e^{i\theta}U + e^{-i\theta}V\right)\Psi_t.$$

Rewriting $T_{P,\gamma(s)}(t)$ in terms of Ψ_t gives

$$T_{P,\gamma(s)}(t)(F(P)) = G(\gamma(s))(t) = \Psi_t(\gamma(s))^{-1}F(\gamma(s)). \tag{5.2.2}$$

Thus $T_{P,\gamma(s)}(t) = \Psi_t(\gamma(s))^{-1}$, and we obtain the following proposition.

Proposition 5.2.2. *With respect to the frame $(\sigma_1, \ldots, \sigma_{\frac{n}{2}}, \sigma^*_{\frac{n}{2}}, \ldots, \sigma^*_1)$, parallel transport along the geodesic from P to P' for the flat connection is given by $\Psi_t(L)^{-1}\left(1 + O\left(t^{-\frac{2}{b}}\right)\right)$, where Ψ_t solves the initial value* *problem*

$$\Psi_t(0) = I \qquad\qquad \frac{d\Psi_t}{ds} = \left(e^{i\theta}U + e^{-i\theta}V\right)\Psi_t$$

Explicitly, we have

1. For $\phi = \tilde{e}_1 + tq_n e_{n-1}$,

$$\frac{d\Psi_t}{ds} = \left[t^{\frac{1}{n}} \begin{pmatrix} 0 & e^{-i\theta} & & & 0 & e^{i\theta} \\ e^{i\theta} & 0 & e^{-i\theta} & & & \\ & \ddots & & \ddots & & \\ 0 & & & e^{i\theta} & 0 & e^{-i\theta} \\ e^{-i\theta} & & & & e^{i\theta} & 0 \end{pmatrix} + O\left(t^{-\frac{1}{n}}\right) \right] \Psi_t$$

2. For $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$,

$$\frac{d\Psi_t}{ds} = \left[(2t)^{\frac{1}{n-1}} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}e^{-i\theta} & & & \frac{1}{\sqrt{2}}e^{i\theta} & 0 \\ \frac{1}{\sqrt{2}}e^{i\theta} & 0 & e^{-i\theta} & & & \frac{1}{\sqrt{2}}e^{i\theta} \\ & \ddots & & \ddots & & \\ \frac{1}{\sqrt{2}}e^{-i\theta} & & & e^{i\theta} & 0 & \frac{1}{\sqrt{2}}e^{-i\theta} \\ 0 & \frac{1}{\sqrt{2}}e^{-i\theta} & & & \frac{1}{\sqrt{2}}e^{i\theta} & 0 \end{pmatrix} + O\left(t^{-\frac{1}{n-1}}\right) \right] \Psi_t$$

In the above expressions, the matrix inside the bracket may be diagonalized by a constant unitary matrix S, and thus can be written as

$$S \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} S^{-1}$$

where the set $\{\mu_j\}$ is the set of roots of the characteristic polynomial $det(\mu I - (e^{i\theta}U + e^{-i\theta}V))$. More precisely,

1. For the case $\phi = \tilde{e}_1 + q_n e_{n-1}$, $\mu_j = 2\cos(\theta + \frac{2\pi j}{n})$.

2. For the case $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$, $\mu_1 = 0$, and for $j \geq 2$, $\mu_j = 2\cos(\theta + \frac{2\pi j}{n-1})$.

To compute $\Psi(L)$, we compute $\Phi = S^{-1}\Psi S$

$$\Phi(0) = I, \qquad\qquad \frac{d\Phi}{ds} = \left[t^{\frac{1}{b}}M(\theta) + R\right]\Phi \tag{5.2.3}$$

where $M(\theta) = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}$, and $S^{-1}RS$ is the error term in Proposition 5.2.2.

To integrate this initial value problem, we employ the following strategy:

Consider the solution Φ_0 to the initial value problem

$$\Phi_0(0) = I, \qquad \frac{d\Phi_0}{ds} = t^{\frac{1}{n}} M(\theta)\Phi_0.$$

Hence $\Phi_0(s) = \begin{pmatrix} e^{st^{\frac{1}{b}}\mu_1} & & & \\ & e^{st^{\frac{1}{b}}\mu_2} & & \\ & & \ddots & \\ & & & e^{st^{\frac{1}{b}}\mu_n} \end{pmatrix}$. Instead of solving for Φ asymptotically, we solve for $(\Phi_0)^{-1}\Phi$. Note that $(\Phi_0)^{-1}\Phi$ solves the initial value problem

$$(\Phi_0)^{-1}\Phi(0) = I, \qquad \frac{d((\Phi_0)^{-1}\Phi)}{ds} = (\Phi_0)^{-1}R\Phi_0 \cdot (\Phi_0)^{-1}\Phi. \tag{5.2.4}$$

This can be seen by using the product rule

$$\begin{aligned} \frac{d((\Phi_0)^{-1}\Phi)}{ds} &= \frac{d\Phi_0}{ds}\Phi + (\Phi_0)^{-1}\frac{d\Phi}{ds} \\ &= -(\Phi_0)^{-1}\frac{d\Phi_0}{ds}(\Phi_0)^{-1}\Phi + (\Phi_0)^{-1}\frac{d\Phi}{ds} \\ &= -(\Phi_0)^{-1}t^{\frac{1}{b}}M(\theta)\Phi + (\Phi_0)^{-1}(t^{\frac{1}{b}}M(\theta) + R)\Phi \\ &= (\Phi_0)^{-1}R\Phi \\ &= (\Phi_0)^{-1}R\Phi_0 \cdot (\Phi_0)^{-1}\Phi. \end{aligned}$$

For the initial value problem (5.2.4), we will show $(\Phi_0)^{-1}R\Phi_0$ is $o(1)$, and that $(\Phi_0)^{-1}\Phi$ is $Id + o(1)$; hence

$$\Phi = \Phi_0(Id + o(1)).$$

Before doing this, we need a more in-depth understanding of the error term.

The estimate of the error term for the ODE relies mainly on the error estimate of the u^{j}'s and v^{j}'s. For the n-cyclic case, we introduce the following notation for the error term for u^{j} coming from Theorem 5.1.1

$$\tilde{u}^{j} = u^{j} - ln\,|tq_n|^{\frac{n+1-2j}{n}}.$$

Similarly for the $(n-1)$-cyclic case set

$$\tilde{v}^{j} = \begin{cases} v^{j} - ln\,|tq_{n-1}| & j = 1 \\ v^{j} - ln\,|2tq_{n-1}|^{\frac{n+1-2j}{n-1}} & otherwise \end{cases}$$

For the n-cyclic case, writing the error term R for the ODE (5.2.3) in terms of $\tilde{u}^j$ gives

$$S^{-1}\left(e^{i\theta}\begin{pmatrix}\tilde{u}^1_z & & & \\ & \tilde{u}^2_z & & \\ & & \ddots & \\ & & & -\tilde{u}^1_z\end{pmatrix}+t^{\frac{1}{n}}e^{-i\theta}\begin{pmatrix}0 & e^{\tilde{u}^1-\tilde{u}^2}-1 & & \\ & \ddots & \ddots & \\ & & 0 & e^{\tilde{u}^1-\tilde{u}^2}-1 \\ e^{-2\tilde{u}^1}-1 & & & 0\end{pmatrix}\right)S \tag{5.2.5}$$

which we will write as $R = B^1_n + t^{\frac{1}{n}} B^2_n$. In a similar fashion, the error term for the $(n-1)$-cyclic case is

$$S^{-1}\left(e^{i\theta}\begin{pmatrix}\tilde{v}^1_z & & & & \\ & \tilde{v}^2_z & & & \\ & & \ddots & & \\ & & & -\tilde{v}^2_z & \\ & & & & -\tilde{v}^1_z\end{pmatrix}\right.$$

$$\left.+(2t)^{\frac{1}{n-1}}e^{-i\theta}\begin{pmatrix} & \frac{1}{\sqrt{2}}(e^{\tilde{v}^1-\tilde{v}^2}-1) & & & & \\ & & e^{\tilde{v}^2-\tilde{v}^3}-1 & & & \\ & & & \ddots & & \\ & & & & e^{\tilde{v}^2-\tilde{v}^3}-1 & \\ \frac{1}{\sqrt{2}}(e^{-\tilde{v}^1-\tilde{v}^2}-1) & & & & & \frac{1}{\sqrt{2}}(e^{\tilde{v}^1-\tilde{v}^2}-1) \\ & \frac{1}{\sqrt{2}}(e^{-\tilde{v}^1-\tilde{v}^2}-1) & & & & \end{pmatrix}\right)S \tag{5.2.6}$$

5.2.1 The n-cyclic case

The following theorem concerning estimates of the errors will be crucial.

Theorem 5.2.3. *Let $d(p)$ be the minimum distance from a point p to the zeros of q_n. Then for any $d < d(p)$, as $t \to +\infty$, the (k,l)-entry of R satisfies*

$$R_{kl}(p) = O\left(t^{-\frac{1}{2n}}e^{-2|1-\zeta_n^{k-l}|t^{\frac{1}{n}}d}\right).$$

Assuming Theorem 5.2.3, we can now prove the main theorem concerning the asymptotics of the parallel transport operator with an extra condition on the path.

Theorem 5.2.4. *Suppose P, P' and the path $\gamma(s)$ are as above. If P' has the property that for every s,*

$$s < d(\gamma(s)) := min\{d(\gamma(s), z_0)| \text{ for all zeros } z_0 \text{ of } q_n\},$$

then there exists a constant unitary matrix S, not depending on the pair (P, P'), so that as $t \to \infty$,

$$T_{P,P'}(t) = \left(Id + O\left(t^{-\frac{1}{2n}}\right)\right) S \begin{pmatrix} e^{-Lt^{\frac{1}{n}}\mu_1} & & & \\ & e^{-Lt^{\frac{1}{n}}\mu_2} & & \\ & & \ddots & \\ & & & e^{-Lt^{\frac{1}{n}}\mu_n} \end{pmatrix} S^{-1}$$

where $\mu_j = 2cos\left(\theta + \frac{2\pi(j-1)}{n}\right)$.

Remark 5.2.5. The extra condition on the path is necessary for our method of proof, as the distance from the zeros of the holomorphic differential q_n controls the decay rate of the error terms. However, for sufficiently short paths, the extra condition is automatically satisfied. Thus, for each point z away from the zeros of q_n, there is a neighborhood U for which all $|q_n|^{\frac{2}{n}}$-geodesics in U satisfy the extra condition. Furthermore, if, for all zeros z_0 of q_n, the angle $<_{z_0} (P, P')$ is less than $\pi/3$, then the $|q_n|^{\frac{2}{n}}$-geodesic from P to P' satisfies the condition.

When P and P' both project to the same point in Σ, the projected path is a loop. In this case, the above asymptotics correspond to the values of the associated family of representations on the homotopy class of the loop.

Proof. By Theorem 5.2.3, the (k, l)-entry of the error term $(\Phi_0^b)^{-1} R \Phi_0^b$ is

$$R_{k,l}(\gamma(s)) e^{(\mu_k - \mu_l)st^{\frac{1}{n}}} = O\left(t^{-\frac{1}{2n}} e^{-2|1-\zeta_n^{k-l}|t^{\frac{1}{n}} d(\gamma(s))} e^{(\mu_k-\mu_l)st^{\frac{1}{n}}}\right).$$

Observe that

$$\begin{aligned} |\mu_k - \mu_l| &= \left|2\cos\left(\theta + \frac{2\pi(k-1)}{n}\right) - 2\cos\left(\theta + \frac{2\pi(l-1)}{n}\right)\right| \\ &= \left|4\sin\left(\frac{\pi(k-l)}{n}\right)\sin\left(\theta + \frac{\pi(k+l-2)}{n}\right)\right| \\ &\leq \left|4\sin\left(\frac{\pi(k-l)}{n}\right)\right| \\ &= 2|1 - \zeta_n^{k-l}|. \end{aligned}$$

Hence, the (k, l)-entry of $(\Phi_0^b)^{-1} R \Phi_0^b$ is $O\left(t^{-\frac{1}{2n}} e^{2|1-\zeta_n^{k-l}|t^{\frac{1}{n}}(s-d(\gamma(s)))}\right)$. Since $\gamma(s)$ satisfies the condition that for every s, $s < d(\gamma(s))$, we obtain $(\Phi_0^b)^{-1} R \Phi_0^b = O\left(t^{-\frac{1}{2n}}\right)$.

We make use of the following classical theorem in ODE theory, for a nice proof, see appendix B of [DW14].

Lemma 5.2.6. *Let $A : [a,b] \to \mathfrak{gl}_n(\mathbb{R})$ be a continuous function. For the equation $F'(s) = F(s)A(s)$ on an interval $[a,b] \subset \mathbb{R}$, there exists $C, \delta_0 > 0$ such that if $\|A(t)\| < \delta < \delta_0$ for all $s \in [a,b]$, then the solution F with $F(a) = I$ satisfies $|F(s) - I| < C\delta$ for all $s \in [a,b]$.*

Applying Lemma 5.2.6 and $(\Phi_0^b)^{-1} R \Phi_0^b = O\left(t^{-\frac{1}{2n}}\right)$ to the ODE

$$(\Phi_0^b)^{-1}\Phi^b(0) = I, \qquad \frac{d((\Phi_0^b)^{-1}\Phi^b)}{ds} = (\Phi_0^b)^{-1} R \Phi_0^b \cdot (\Phi_0^b)^{-1}\Phi^b,$$

we obtain

$$(\Phi_0^b)^{-1}\Phi^b = Id + O\left(t^{-\frac{1}{2n}}\right).$$

Therefore $\Phi^b = \Phi_0^b\left(Id + O\left(t^{-\frac{1}{2n}}\right)\right)$. $\square$

5.2.2 The $(n-1)$-cyclic case

For the $(n-1)$-cyclic case, the crucial error estimate theorem is the following.

Theorem 5.2.7. *Let $d(p)$ be the minimum distance from a point p to the zeros of q_{n-1}. Then for any $d < d(p)$, as $t \to +\infty$, the (k,l)-entry of R satisfies*

$$R_{kl}(p) = \begin{cases} O\left(t^{-\frac{1}{2(n-1)}} e^{-2|1-\zeta_{n-1}^{k-l}|(2t)^{\frac{1}{n-1}} d}\right) & k, l \geq 2 \\ 0 & k = l = 1 \\ O\left(t^{-\frac{1}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}} d}\right) & otherwise \end{cases}$$

As with the n-cyclic case, we will assume Theorem 5.2.7 for now and prove the main theorem concerning the asymptotic of the parallel transport operator with an extra condition on the path.

Theorem 5.2.8. *Suppose P, P' and the path $\gamma(s)$ are as above. If P' has the property that for every s*

$$s < d(\gamma(s)) := min\{d(\gamma(s), z_0)| \text{ for all zeros } z_0 \text{ of } q_{n-1}\},$$

then there exists a constant unitary matrix S, not depending on the pair P and P', so that as $t \to \infty$,

$$T_{P,P'}(t) = \left(Id + O\left(t^{-\frac{1}{2(n-1)}}\right)\right) S \begin{pmatrix} e^{-Lt^{\frac{1}{n-1}}\mu_1} & & & \\ & e^{-Lt^{\frac{1}{n-1}}\mu_2} & & \\ & & \ddots & \\ & & & e^{-Lt^{\frac{1}{n-1}}\mu_n} \end{pmatrix} S^{-1}$$

where $\mu_1 = 0$, *for* $j \geq 2, \mu_j = 2cos\left(\theta + \frac{2\pi(j-2)}{n-1}\right)$.

Remark 5.2.9. The extra condition on the path is necessary for our method of proof, as the distance from the zeros of the holomorphic differential q_n controls the decay rate of error terms. However, for sufficiently short paths, the extra condition is automatically satisfied. Thus, for each point z away from the zeros of q_n, there is a neighborhood U for which all $|q_n|^{\frac{2}{n}}$-geodesics in U satisfy the extra condition. Furthermore, if the angle $<_{z_0}(P, P')$ satisfies $<_{z_0}(P, P') < \pi/3$ for all zeros z_0 of q_{n-1}, then the $|q_{n-1}|^{\frac{2}{n-1}}$-geodesic from P to P' satisfies the extra condition in Theorem 5.2.8.

When P and P' both project to the same point in Σ, the projected path is a loop. In this case, the above asymptotics correspond to the values of the associated family of representations on the homotopy class of the loop.

Proof. By Theorem 5.2.7, we have the (k, l)-entry of the error term $(\Phi_0^b)^{-1}R\Phi_0^b$ is

$$R_{k,l}(\gamma(s))e^{(\mu_k-\mu_l)st^{\frac{1}{n}}}.$$

For $k, l \geq 2$, similar to the proof of Theorem 5.2.4, $|\mu_k - \mu_l| \leq 2|1 - \zeta_{n-1}^{k-l}|$. Hence for $k, l \geq 2$, the (k, l)-entry of $(\Phi_0^b)^{-1}R\Phi_0^b$ is $O\left(t^{-\frac{1}{2(n-1)}}e^{2|1-\zeta_n^{k-l}|(2t)^{\frac{1}{n-1}}(s-d(\gamma(s))}\right)$.

For $k = l = 1$, we have $\mu_1 = 0$, hence the $(1, 1)$-entry of $(\Phi_0^b)^{-1}R\Phi_0^b$ is $O\left(t^{-\frac{1}{2(n-1)}}\right)$. If $k = 1$ and $l \neq 1$, then

$$|\mu_k - \mu_l| = |0 - 2\cos(\theta + \frac{2\pi(l-1)}{n-1})| \leq 2.$$

Also, if $l = 1$ and $k \neq 1$, we have $|\mu_k - \mu_l| = |2\cos(\theta + \frac{2\pi(k-1)}{n-1}) - 0| \leq 2$. Thus for $k = 1, l \neq 1$ or $l = 1, k \neq 1$, the (k, l)-entry of $(\Phi_0^b)^{-1}R\Phi_0^b$ is

$$O\left(t^{-\frac{1}{2(n-1)}}e^{2(2t)^{\frac{1}{n-1}}(s-d(\gamma(s)))}\right).$$

Since $\gamma(s)$ satisfies the condition that for every s, $s < d(\gamma(s))$, we obtain that $(\Phi_0^b)^{-1}R\Phi_0^b = O\left(t^{-\frac{1}{2(n-1)}}\right)$. As in the n-cyclic case, we apply Lemma 5.2.6 and obtain $(\Phi_0^b)^{-1}\Phi_0^b = Id + O\left(t^{-\frac{1}{2(n-1)}}\right)$, and thus

$$\Phi^b = \Phi_0^b\left(Id + O\left(t^{-\frac{1}{2(n-1)}}\right)\right).$$

□

5.3 Harmonic maps into symmetric spaces

We continue to work in the universal cover $\widetilde{\Sigma}$ of Σ, all objects should be pulled back from the surface. As in previous sections, we will use a subscript b to work with the two cases $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$ simultaneously.

A Hermitian metric h on a flat bundle E gives rise to an equivariant map to the symmetric space $\mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n)$. To see this, fix a positively oriented unitary frame $\{x_i(P)\}$ over a base point $P \in \widetilde{\Sigma}$. With respect to the flat connection, parallel transport of the frame $\{x_i(P)\}$ gives a global frame $\{x_i\}$. Define a $\pi_1(\Sigma)$-equivariant map by,

$$\begin{aligned} f : \widetilde{\Sigma} \quad &\longrightarrow \quad \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n) \\ P' \quad &\longmapsto \quad \{h(x_i(P'), x_j(P'))\}. \end{aligned}$$

By Corlette's Theorem [Cor88], the family of harmonic metrics h_t considered above, gives a family of ρ_t-equivariant harmonic maps

$$f_t : \widetilde{\Sigma} \to \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n).$$

Remark 5.3.1. The image of the family f_t lies in a copy of the real symmetric space

$$\mathsf{SL}(n, \mathbb{R})/\mathsf{SO}(n, \mathbb{R}) \subset \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n).$$

This is because the family of representations ρ_t has image in the real group $\mathsf{SL}(n, \mathbb{R})$.

Pick a base point $P \in \widetilde{\Sigma}$ away from zeros of the differential q_b. Recall that $\mathcal{U}_P$ is a local coordinate such that $q_b = dz^b$, and $F_b = F = (\sigma_1, \ldots, \sigma_{\frac{n}{2}}, \sigma^*_{\frac{n}{2}}, \ldots, \sigma^*_1)$ is a rescaled holomorphic frame (5.2.1). By Remark 5.2.1, we can choose a unitary and orthogonal (with respect to the orthogonal structure Q) basis $N(P)$ at P so that

$$F(P) = N(P)(1 + O(t^{-\frac{2}{b}})).$$

Using the flat connection, parallel transport the unitary basis $N(P)$ to obtain a frame N. Note that N is not a unitary frame since the flat connection does not have holonomy in $\mathsf{SU}(n)$; however, it retains its $\mathsf{SL}(n, \mathbb{R})$ symmetry. As a result, the image of f_t is contained in a copy of $\mathsf{SL}(n, \mathbb{R})/\mathsf{SO}(n, \mathbb{R}) \hookrightarrow \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n)$. The inclusion is determined by the inclusion of $\mathsf{SO}(n, \mathbb{R}) \subset \mathsf{SU}(n)$ given by Q-orthogonal unitary matrices, and the intersection of Q-symmetric matrices with determinant 1 Hermitian matrices.

At a point P', denote the j^{th} column of N by $N^j(P')$. Recall from equation (5.2.2), that the parallel transport of the rescaled holomorphic frame F at P has been denoted by G, and

$$G(P') = N(P')(1 + O(t^{-\frac{2}{b}})).$$

If we denote the j^{th} column of $G(P')$ by $G^j(P')$, we have

$$f_t(P') = \{h_t(P')(N^i(P'), N^j(P'))\}$$

$$= \{h_t(P')\left(G^i(P'), G^j(P')\right)(1 + O(t^{-\frac{2}{b}}))\}.$$

By Proposition 5.2.2, we understand $h_t(P')$ in the frame F, thus, we change coordinates $\Psi_t(P')G(P') = F(P')$. In terms of columns, we have

$$G^i(P') = \Psi_t^{-1}(P')_{ik}F^k(P').$$

Thus $f_t(P')$ is given by

$$f_t(P') = \{\ h_t(P')\left(\Psi_t^{-1}(P')_{ik}F^k(P')\ ,\ \Psi_t^{-1}(P')_{jl}F^l(P')\right)(1 + O(t^{-\frac{2}{b}}))\ \}.$$

In the frame F, the metric h_t is diagonal, thus

$$f_t(P') = \Psi_t^{-1}(P')^T\ h_t^F\ \overline{\Psi_t^{-1}}(P')\ (1 + O(t^{-\frac{2}{b}}))$$

where h_t^F denotes the metric in the rescaled holomorphic frame F. Now, using Theorem 5.1.1 and Remark 5.2.1, we have

$$f_t(P') =\ \Psi_t^{-1}(P')^T \cdot (1 + O(t^{-\frac{2}{b}})) \cdot \overline{\Psi_t^{-1}}(P')\ .$$

Therefore, by applying estimates for $\Psi(L)$ in Theorems 5.2.4 and 5.2.8, for any $P' = Le^{i\theta} = \gamma(L)$ with the property that, for all s, $s < d(\gamma(s)) := min\{d(\gamma(s), z_0)|$ for all zeros z_0 of $q_n\}$, as $t \to \infty$

$$f_t(P') = \left(Id + O\left(t^{-\frac{1}{2b}}\right)\right)\overline{S}\begin{pmatrix} e^{-2Lt^{\frac{1}{b}}\mu_1} & & & \\ & e^{-2Lt^{\frac{1}{b}}\mu_2} & & \\ & & \ddots & \\ & & & e^{-2Lt^{\frac{1}{b}}\mu_n}\end{pmatrix} S^T\left(Id + O\left(t^{-\frac{1}{2b}}\right)\right). \tag{5.3.1}$$

Here S and the $\{\mu_j\}$'s satisfy the same conditions as in Theorems 5.2.4 and 5.2.8. By Remark 5.2.5, the above equation can be interpreted as saying that for all such P, there exists a neighborhood $\mathcal{U}_P$, so that the ρ_t-equivariant maps $f_t : \tilde{\Sigma} \to \mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n,\mathbb{R})$ send $\mathcal{U}_P$ asymptotically into a *flat* of the symmetric space.

Given two points P, P' in the symmetric space $\mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n,\mathbb{R})$, the vector distance between them is defined by $\vec{d}(P,P') = P - P'$, where the difference is taken in a positive Weyl chamber of a *flat* (isometric to $\mathbb{A}^{n-1}$) containing both points. One can show $\vec{d}(P,P')$ is independent of the choice of flat. For example, in the standard flat of $\mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n,\mathbb{R})$ consisting of all diagonal matrices of determinant 1, the vector distance is defined by

$$\vec{d}\left(\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}\right) = (\lambda_{i_1}, \cdots, \lambda_{i_n})$$

where $\lambda_{i_1} \geq \lambda_{i_2} \geq \cdots \geq \lambda_{i_n}$. Since all flats in $\mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n,\mathbb{R})$ are conjugate to the standard flat, the vector distance can be defined in a similar way.

The asymptotic expression (5.3.1) for f_t, together with the definition of vector distance, gives the following theorem.

Theorem 5.3.2. *With the same assumptions as the parallel transport asymptotitcs Theorem 5.2.4, for a path γ satisfying*

$$s < d(\gamma(s)) := min\{d(\gamma(s), z_0)| \text{ for all zeros } z_0 \text{ of } q_n\} \tag{5.3.2}$$

we have $\lim_{t\to\infty} \frac{1}{t^{\frac{1}{n}}} \vec{d}(f_t(\gamma(0)), f_t(\gamma(1))) = (\lambda_1, \cdots, \lambda_n)$ *where* $\lambda_1 \geq \cdots \geq \lambda_n$ *is a reordering of*

$$\left(-2L\cos(\theta), -2L\cos\left(\theta + \frac{2\pi}{n}\right), \ldots, -2L\cos\left(\theta + \frac{2\pi(n-1)}{n}\right)\right) \qquad \textit{for} \qquad \phi = \tilde{e}_1 + q_n e_{n-1},$$

$$\left(0, -2L\cos(\theta), -2L\cos\left(\theta + \frac{2\pi}{n-1}\right), \ldots, -2L\cos\left(\theta + \frac{2\pi(n-2)}{n-1}\right)\right) \qquad \textit{for} \qquad \phi = \tilde{e}_1 + q_{n-1}e_{n-2}.$$

With algebraic techniques generalizing methods of Morgan-Shalen [MS84], Parreau [Par12] provided a compactification of the Hitchin component. In this paper we pursue a more geometric approach to the compactification of the Hitchin component. Our main motivation is Wolf's [Wol89] harmonic map interpretation of Thurston's compactification [FLP12] of Teichmüller space with measured foliations.

Roughly, Thurston's compactification works as follows: Let $\mathcal{S}$ denote the space of isotopy classes of simple closed curves and denote the projectivation of the space of nonnegative functions on $\mathcal{S}$ by $\mathbb{P}\mathbb{R}_+^{\mathcal{S}}$. The map which assigns the projectived length spectrum of each hyperbolic metric is an embedding of Teichmüller space inside $\mathbb{P}\mathbb{R}_+^{\mathcal{S}}$. The image is homeomorphic to an open ball of dimension $6g-6$, and the boundary corresponds to projective classes of measured foliations. Furthermore, the action of the mapping class group extends to the boundary. This compactification was further extended to the character variety for $\mathsf{SL}(2,\mathbb{C})$

(see [Bes88] and [DDW00]).

Fix a Riemann surface structure Σ on S. To each hyperbolic metric h on S, the Hopf differential of the unique harmonic map $f : \Sigma \to (S, h)$ isotopic to the identity associates a holomophic quadratic differential to h. Wolf showed that this procedure provides a homeomorphism between Teichmüller space and the vector space of holomorphic quadratic differentials $H^0(K^2)$. Adjoing points at ∞ to rays in $H^0(K^2)$ provides a compactification of Teichmüller space. Let q_2 be a holomorphic quadratic differential, away from the zeros of q_2 choose a coordinate z such that $q_2 = dz^2$. In such coordinates we have local measured foliations $(\mathcal{F}, \mu) = (\{Re(z) = const\}, |dRe(z)|)$, which piece together to form the vertical measured foliation $\mathcal{F}(q_2)$ of q_2. For $t > 0$, consider the ray $tq_2 \in H^0(K^2)$, and let h_t be the corresponding family of hyperbolic metrics and $f_t : \Sigma \to (S, h_t)$ be the corresponding family of harmonic maps. The key step in showing the harmonic map compactification agrees with Thurston's measured foliations compactification is to show the length spectrum of h_t is asymptotically the same as the length spectrum of the vertical measured foliation of tq_2. That is, for any closed curve γ on Σ, as $t \to \infty$,

$$l(f_t(\gamma)) = l_\gamma(h_t) \sim i(\mathcal{F}(tq_2), \gamma). \tag{5.3.3}$$

Here, $i(\mathcal{F}(tq_2), \gamma)$ is the intersection number of γ with the measured foliation $\mathcal{F}(tq_2)$.

Hitchin's parameterization generalizes the parameterization of Teichmüller space by holomorphic quadratic differentials. By Corlette's Theorem [Cor88], for each representation $\rho \in \mathsf{Hit}(\mathsf{PSL}(n, \mathbb{R}))$, there is a unique ρ-equivariant harmonic map from the universal cover $\widetilde{\Sigma}$ to the symmetric space $\mathsf{SL}(n, \mathbb{R})/\mathsf{SO}(n, \mathbb{R})$. In this paper, we study the families of Hitchin representations parameterized by rays $(0, \cdots, tq_n)$ and $(0, \cdots, tq_{n-1}, 0)$, and relate the asymptotics of the corresponding harmonic maps with the geometry of the holomorphic differentials q_n and q_{n-1}. This is formulated in terms of the following generalizations of measured foliations and length spectrum:

- For $q_n \in H^0(K^n)$ choose a local coordinate (away from the zeros of q_n) so that $q_n = dz^n$. In this coordinate, we have n foliations $\mathcal{F}_1(q_n), \cdots, \mathcal{F}_n(q_n)$ with signed measure defined by

$$\mathcal{F}_k(q_n) = \left(\{Re(e^{\frac{2(k-1)\pi i}{n}} z = const\}, dRe(e^{\frac{2(k-1)\pi i}{n}} z) \right).$$

  Unlike the rank 2 case, these local foliations do not piece together to define global foliations.

- Any two points P and P' in the symmetric space $\mathsf{SL}(n, \mathbb{R})/\mathsf{SO}(n, \mathbb{R})$, are contained in a *flat* isometric to the Euclidean space $\mathbb{R}^{n-1}$. The vector distance $\vec{d}(P, P')$ is then defined as the vector from P to P'

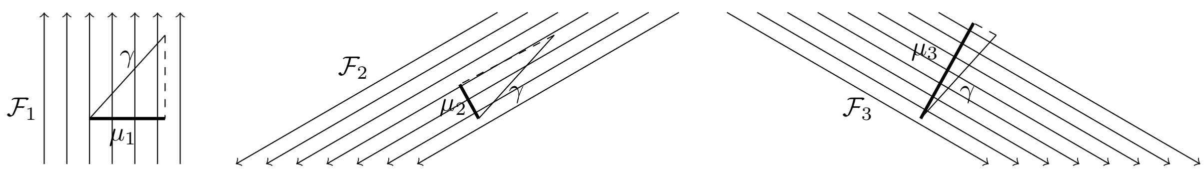
in the flat. This is independent of the choice of flat.

For $q_n \in H^0(K^n)$, let ρ_t and f_t be the family of representations and ρ_t-equvariant harmonic maps associated to the ray $(0, \cdots, tq_n)$ in the Hitchin component. As a generalization of the asymptotic length formula (5.3.3), from Theorem 5.3.2 we deduce:

Theorem 5.3.3. *Let $\widetilde{\Sigma}$ be the universal cover of Σ. For a path $\gamma \subset \widetilde{\Sigma}$ which does not pass through the zeros of q_n, choose a local coordinate so that $q_n = dz^n$, and denote the local foliations associated to tq_n by $\mathcal{F}_1, \cdots, \mathcal{F}_n$. If γ satisfies the regularity condition (5.3.2), then as $t \to \infty$,*

$$\overrightarrow{d}(f_t(\gamma(0)), f_t(\gamma(1))) \sim (i(\mathcal{F}_1, \gamma), i(\mathcal{F}_2, \gamma), \cdots, i(\mathcal{F}_n, \gamma)).$$

For $\mathsf{SL}(3, \mathbb{R})$ the picture is



In [KNPS15] the following asymptotic question (called the complex WKB problem by the authors) is studied: Fix a representation $\rho \in \mathcal{X}(\pi_1, \mathsf{SL}(n, \mathbb{C}))$ and let $(\mathcal{E}, \nabla)$ be the corresponding flat holomorphic vector bundle. If θ is a holomorphic (with respect to the flat connection ∇) section of $End(\mathcal{E}) \otimes K$ then

$$\nabla_t = \nabla + t\theta \tag{5.3.4}$$

is a family of flat holomorphic connections. The asymptotics of the family ∇_t is called the complex WKB problem. The asymptotic problem studied in this paper (called the Hitchin WKB problem in [KNPS15]) is significantly different than the complex WKB problem. In particular, in the complex WKB problem there is no PDE to solve. Also, the $(0, 1)$ part of the family of flat connections in (5.3.4) is constant while the $(0, 1)$ part of the family of flat connections

$$\nabla_t = \nabla_{h_t} + t\phi + t\phi^{*_{h_t}} \tag{5.3.5}$$

is $\nabla^{0,1}_{h_t} + t\phi^{*_{h_t}}$ and thus varies with the solution metric h_t. Despite these differences, in [KNPS15] it is conjectured that the asymptotics of these two families are similar.

More precisely, given a family of representations δ_t, any fixed metric h defines a family of $\nabla + t\theta$ equivariant maps

$$f_t : \widetilde{\Sigma} \longrightarrow \mathsf{SL}(n, \mathbb{C})/\mathsf{SU}(n).$$

For the family of flat connections $\nabla_t = \nabla + t\theta$ in the complex WKB problem, Katzarkov et al prove that, for any fixed metric h and any 'noncritical path' $\gamma : [0, L] \to \Sigma$ the family of equivariant maps satisfies

$$\frac{1}{t}\vec{d}\left(f_t(\gamma(0)), f_t(\gamma(L))\right) \sim \left(\int_0^L Re\gamma^*\xi_1, \cdots, \int_0^L Re\gamma^*\xi_n\right). \tag{5.3.6}$$

Here $\xi_1, \cdots, \xi_n$ are the local eigenvalues of the spectral curve associated the Higgs bundle $(\mathcal{E}, \theta)$ such that the entries of the vector are decreasing. They conjecture that the family of equivariant *harmonic* maps associated to the family of flat connections $\nabla_t = \nabla_{h_t} + t\theta + t\theta^{*h_t}$ satisfies the same asymptotics.

In a local coordinate z with $q_n = dz^n$, the spectral curve associated to the Higgs field $\phi = \tilde{e}_1 + e_{n-1}q_n$ has local eigenvalues

$$\{\xi_1, \cdots, \xi_n\} = \{1dz, e^{\frac{2\pi i}{n}}dz, e^{\frac{2\cdot 2\pi i}{n}}dz, \cdots, e^{\frac{2(n-1)\pi i}{n}}dz\}.$$

Similarly in a local coordinate z with $q_{n-1} = dz^{n-1}$, the spectral curve associated to the Higgs field $\phi = \tilde{e}_1 + e_{n-2}q_{n-1}$ has local eigenvalues

$$\{\xi_1, \cdots, \xi_n\} = \{0, 1dz, e^{\frac{2\pi i}{n-1}}dz, e^{\frac{2\cdot 2\pi i}{n-1}}dz, \cdots, e^{\frac{2(n-2)\pi i}{n-1}}dz\}.$$

Thus, for a path $\gamma : [0, L] \to \Sigma$ the expression $\left(\int_0^L Re\gamma^*\xi_1, \cdots, \int_0^L Re\gamma^*\xi_n\right)$ is given by

$$\left(-2L\cos(\theta), -2L\cos\left(\theta + \frac{2\pi}{n}\right), \ldots, -2L\cos\left(\theta + \frac{2\pi(n-1)}{n}\right)\right) \quad \text{for} \quad \phi = \tilde{e}_1 + q_n e_{n-1},$$

$$\left(0, -2L\cos(\theta), -2L\cos\left(\theta + \frac{2\pi}{n-1}\right), \ldots, -2L\cos\left(\theta + \frac{2\pi(n-2)}{n-1}\right)\right) \quad \text{for} \quad \phi = \tilde{e}_1 + q_{n-1}e_{n-2}.$$

Hence, Theorem 5.3.2 proves the conjecture that the asymptotics of (5.3.6) for the Hitchin WKB problem and the complex WKB problem are the same for the Higgs bundles in the Hitchin component with $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$. After this joint work with Q. Li, the conjecture was proven in general by Mochizuki [Moc15].

To close, we briefly discuss the behavior of the 'limit map' f_∞ associated to the family f_t, studied extensively in [KNPS15]. To obtain better information about the behavior of the maps f_t as $t \to \infty$, we rescale the metric on the symmetric space and consider the family of maps

$$f_t : \widetilde{\Sigma} \to \left(\mathsf{SL}(n, \mathbb{R}) / \mathsf{SO}(n, \mathbb{R}), \frac{1}{t^{\frac{1}{b}}} d \right).$$

The limit of $\left(\mathsf{SL}(n, \mathbb{R}) / \mathsf{SO}(n, \mathbb{R}), \frac{1}{t^{\frac{1}{b}}} d \right)$ as $t \to \infty$ is not well defined, however, by the work of Kleiner-Leeb [KL97] and Parreau [Par12], a Gromov limit of $\left(\mathsf{SL}(n, \mathbb{R}) / \mathsf{SO}(n, \mathbb{R}), \frac{1}{t^{\frac{1}{b}}} d \right)$ is an affine building modeled on $\mathbb{A}^{n-1}$. The limit construction depends on the choice of ultrafilter ω on $\mathbb{R}$ with countable support; with this choice, the limit is called the asymptotic cone and is denoted $Cone_\omega$. In [Par12], Parreau showed that, given a diverging family of representations ρ_t, the limit of the vector length spectra of ρ_t arises from the length spectrum of a limit action ρ_ω on $Cone_\omega$. This gives a harmonic map

$$f_\omega : \widetilde{\Sigma} \to Cone_\omega,$$

which is equivariant for the limiting action ρ_ω of $\pi_1(S)$ on $Cone_\omega$.

In this language, the asymptotic expression (5.3.1) of f_t implies that for the families of rays

$$(\Sigma, 0, \cdots, 0, tq_n), \ (\Sigma, 0, \cdots, tq_{n-1}, 0) \in \mathsf{Hit}(\mathsf{SL}(n, \mathbb{R}))$$

and for any P away from the zeros of q_n and q_{n-1}, there exists a neighborhood $\mathcal{U}_P$ so that the ρ_ω-equivariant map

$$f_\omega : \widetilde{\Sigma} \to Cone_\omega$$

sends $\mathcal{U}_P$ into a single apartment of the building $Cone_\omega$.

Chapter 6

Cyclic surfaces and maximal $\mathsf{PSp}(4,\mathbb{R}) = \mathsf{SO}_0(2,3)$-Higgs bundles

For a simple split real group G, recall that, for each Riemann surface structure Σ on S, the Hitchin component $\mathsf{Hit}(\mathsf{G}) \subset \mathcal{X}(\mathsf{G})$ is parameterized by the following vector space of holomorphic differentials $\mathsf{Hit}(\mathsf{G}) \cong \bigoplus_{j=1}^{rk(\mathsf{G})} H^0(\Sigma, K^{m_j+1})$ where $\{m_j\}$ are the exponents of G [Hit92]. Furthermore, for each $\rho \in \mathsf{Hit}(\mathsf{G})$ the quadratic differential in this parameterization corresponding to ρ is a constant multiple of the Hopf differential of the unique ρ-equivariant harmonic map $h_\rho : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}$. Representations in the Hitchin component are deformations discrete and faithful representations into $\mathsf{PSL}(2,\mathbb{R})$. Furthermore, Labourie has shown [Lab06] that Hitchin representations are examples of Anosov representations. As a result, every Hitchin representation is discrete and faithful and the mapping class group $\mathsf{Mod}(S)$ acts properly discontinuously on $\mathsf{Hit}(\mathsf{G})$. Since Hitchin's parameterization by holomorphic differentials depends on fixing a conformal structure, it breaks the $\mathsf{Mod}(S)$-symmetry. Labourie conjectured that for each $\rho \in \mathsf{Hit}(\mathsf{G})$, there exists a unique preferred conformal structure:

Conjecture 6.0.1. *(Labourie [Lab06]) For each $\rho \in \mathsf{Hit}(\mathsf{G})$ there exists a unique conformal structure $(S, J_\rho) = \Sigma_\rho$ in which $h_\rho : \widetilde{\Sigma}_\rho \to \mathsf{G}/\mathsf{H}$ is a branched minimal immersion.*

Since the quadratic differential in Hitchin's parameterization is a constant multiple of the Hopf differential of h_ρ and the Hopf differential of h_ρ vanishes if and only if h_ρ is a branched minimal immersion, a positive answer to this conjecture together with Hitchin's parameterization would provide a $\mathsf{Mod}(S)$-invariant parameterization of $\mathsf{Hit}(\mathsf{G})$ as a vector bundle over Teichmüller space $\pi : E \to \mathsf{Teich}(S)$, wher the fiber over a Riemann surface $\Sigma \in \mathsf{Teich}(S)$ is given by $\pi^{-1}(\Sigma) = \bigoplus_{j=2}^{rk(\mathsf{G})} H^0(\Sigma, K^{m_j+1})$, note the sum starts at $j = 2$.

In general, Conjecture 6.0.1 is an important open question in higher Teichmüller theory. It has however been established for some low rank groups. For $\mathsf{G} = \mathsf{PSL}(3,\mathbb{R})$ Loftin [Lof07] and Labourie [Lab07] independently proved the conjecture using the geometry of convex foliated $\mathbb{RP}^2$-structures and affine spheres. In [Lab14] Labourie proved the conjecture for all G of rank 2, that is $\mathsf{G} = \mathsf{PSL}(3,\mathbb{R}), \mathsf{PSp}(4,\mathbb{R}), \mathsf{G}_2$.

Fix a representation $\rho \in \mathcal{X}(\pi_1, \mathsf{G})$, and for each conformal structure denote the corresponding harmonic metric by h_ρ. Consider the following energy function $\mathcal{E}_\rho$ on the Teichmüller space $\mathsf{Teich}(S)$

$$\mathcal{E}_\rho(J)=\mathcal{E}_J(h_\rho)=\frac{1}{2}\int\limits_S |dh_\rho|^2 dvol \ : \ \mathsf{Teich}(S)\longrightarrow\mathbb{R} \tag{6.0.1}$$

Remark 6.0.2. By [SU82, SY79], critical points of $\mathcal{E}_J(h_\rho)$ are a branched minimal immersions, or equivalently, weakly conformal maps. Note that the harmonic map h_ρ, the norm $|dh_\rho|^2$ and the volume element all depend of J.

In [Lab08], Labourie proved the following theorem:

Theorem 6.0.3. *([Lab08]) If ρ is an Anosov representation then the energy function $\mathcal{E}_\rho : \mathsf{Teich}(S)\to\mathbb{R}$ is smooth and proper.*

Since $\mathcal{E}_\rho$ is proper and bounded below by zero, $\mathcal{E}_\rho$ attains a minimum. This gives

Corollary 6.0.4. *For all Anosov representations ρ there exists a conformal structure in which h_ρ is a branched minimal immersion. In particular, the existence part of Conjecture 6.0.1 holds.*

For the groups $\mathsf{SO}_0(2,3)\cong\mathsf{PSp}(4,\mathbb{R})$ and $\mathsf{Sp}(4,\mathbb{R})$ we will study the class of Anosov representations called maximal representations. There are $4g-3$ special connected components of maximal $\mathsf{PSp}(4,\mathbb{R})$ representations, which we call the *Gothen components*. This class of representations include $\mathsf{PSp}(4,\mathbb{R})$-Hitchin representations. Using existence of a conformal structure in which h_ρ is a branched minimal immersion, we will show that this conformal structure is unique for all Gothen representations. Using a Higgs bundle parameterization of the Gothen components, we obtain a mapping class group invariant parameterization of all Gothen components. In the final section, we show that the Gothen components are not an $\mathsf{PSp}(2n,\mathbb{R})$ phenomenon but rather an $\mathsf{SO}_0(n,n+1)$ phenomenon. In particular we prove:

Theorem 6.4.1. *For each $0<d\leq n(2g-2)$ there is a connected component $\mathcal{M}_d(\mathsf{SO}_0(n,n+1))\subset\mathcal{M}(\mathsf{SO}_0(n,n+1))$ which is smooth and parameterized by $\mathcal{F}^d_\Sigma\times\bigoplus\limits_{j=1}^{n-1}H^0(\Sigma,K^{2j})$ where $\mathcal{F}^d_\Sigma\to Sym^{-d+n(2n-2)}(\Sigma)$ is a vector bundle of rank $d+(2n-1)(g-1)$. Moreover, $\mathcal{M}_{n(2g-2)}(\mathsf{SO}_0(n,n+1))=\mathsf{Hit}(\mathsf{SO}_0(n,n+1))$.*

Other than Theorem 6.4.1, most of the contents of this chapter have been published in [Col15]. However, for the results on maximal representations and minimal surfaces, the results in [Col15] only concern $\mathsf{Sp}(4,\mathbb{R})$ and not $\mathsf{PSp}(4,\mathbb{R})$. Also, the parameterizations given in Theorem 6.3.6 and Theorem 6.2.21 are simpler than those in [Col15].

6.1 G-Cyclic surfaces

The surfaces we will be interested are solutions to certain Pfaffian systems in the spaces of Cartan triples and Hitchin triples. The cyclic surfaces defined below are more general than [Lab14], yet, we show that deformations of the below cyclic surfaces have many similarities with deformations of Labourie's cyclic surfaces.

6.1.1 Cartan triples and Hitchin triples

We now define the main reductive homogeneous spaces we will study. The spaces we will be interested in are G/T and G/T_0 where G is a complex simple Lie group and T is a maximal compact torus of G and T_0 is the maximal compact torus of a split real form of $\mathsf{G}_0 \subset \mathsf{G}$. We start by considering a more geometric set of objects.

Definition 6.1.1. A *Cartan triple is a triple* $(\mathfrak{c}, \Delta^+, \theta)$ where

- $\mathfrak{c} \subset \mathfrak{g}$ is a Cartan subalgebra
- $\Delta^+ \subset \mathfrak{c}^*$ is a choice of positive roots
- θ is a $\mathfrak{c}$-Cartan involution

Let $\mathsf{T} \subset \mathsf{G}$ be a maximal compact torus, Proposition 2.1.2 and Lemma 2.1.10 imply the following proposition.

Proposition 6.1.2. *The space of Cartan triples is isomorphic to* G/T

Note that we could equivalently define $(\mathfrak{c}, \Delta^+, \theta)$ to be a Cartan triple where θ is Cartan involution, and $(\mathfrak{c}, \Delta^+)$ a Cartan subalgebra with positive root system and $\mathfrak{c}$ is preserved by θ. There are natural projection maps

$$\begin{tikzcd} \mathsf{G}/\mathsf{C} & \mathsf{G}/\mathsf{T} \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] \\ & \mathsf{G}/\mathsf{K} \end{tikzcd}$$

where $\pi_1(\mathfrak{c}, \Delta^+, \theta) = (\mathfrak{c}, \Delta^+)$ and $\pi_2(\mathfrak{c}, \Delta^+, \theta) = \theta$.

Definition 6.1.3. A *Hitchin triple* is a triple $(\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda)$ where

- $\mathfrak{c}$ is a Cartan subalgebra
- $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{c}) \subset \mathfrak{c}^*$ is a choice of positive roots

- θ is a $\mathfrak{c}$-Cartan involution which globally preserves a PTDS $\mathfrak{s}$ which contains $x = \frac{1}{2}\sum_{\alpha\in\Delta^+} H_\alpha$.
- λ is a split real form which commutes with θ, globally preserves $\mathfrak{c}$, globally preserves a PTDS $\mathfrak{s}$ which contains $x = \frac{1}{2}\sum_{\alpha\in\Delta^+} H_\alpha$ and satisfies $\lambda(H_\mu) = -H_\mu$.

Proposition 6.1.4. *Let G be a complex simple Lie group, and G_0 be a split real form of G. The space of Hitchin triple is diffeomorphic to G/T_0 where T_0 is the maximal compact torus of G_0.*

Proof. We first show that the G acts transitively on the space of Hitchin triple. Let $(\Delta_1^+ \subset \mathfrak{c}_1^*, \theta_1, \lambda_1)$ and $(\Delta_2^+ \subset \mathfrak{c}_2^*, \theta_2, \lambda_2)$ be two such Hitchin triples. By Remark 2.1.8, we can conjugate $(\Delta_2^+ \subset \mathfrak{c}_2^*)$ to $(\Delta_1^+ \subset \mathfrak{c}_1^*)$. Thus we may assume $(\Delta_1^+ \subset \mathfrak{c}_1^*) = (\Delta_2^+ \subset \mathfrak{c}_2^*)$. Let $x = \frac{1}{2}\sum_{\alpha\in\Delta_1^+} H_\alpha$, and suppose θ_1 stabilizes an $\mathfrak{c}$-PTDS $\mathfrak{s}_1$ and θ_2 stabilizes an $\mathfrak{c}$-PTDS $\mathfrak{s}_2$ with $x \in \mathfrak{s}_1$ and $x \in \mathfrak{s}_2$. By Theorem 4.2 of [Kos59] (q.f. Theorem 2.1.37), the PTDSs $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are conjugate via an element of C. Thus we may assume $\mathfrak{s}_1 = \mathfrak{s}_2$. Since θ_1 and θ_2 are both $\mathfrak{c}$-Cartan involutions, $\theta_1|_\mathfrak{c} = \theta_2|_\mathfrak{c}$. Furthermore, θ_1 and θ_2 are both $\mathfrak{c}\cap\mathfrak{s}$-Cartan involutions of $\mathfrak{s}$, by Proposition 6.1.2, $\theta_2|_\mathfrak{s}$ can be conjugated to $\theta_1|_\mathfrak{s}$ by an element of the subgroup $\mathsf{C}' \subset \mathsf{C}$ with Lie algebra $\mathfrak{c}\cap\mathfrak{s}$. Observe that conjugating by C' preserves $(\Delta_1^+ \subset \mathfrak{c}_1^*, \mathfrak{s}_1)$. Furthermore, $\mathfrak{g}$ is generated by $\mathfrak{c}+\mathfrak{s}$, thus after conjugating by such an element of C', we obtain $\theta_1 = \theta_2$. Since $\theta_1 = \theta_2$ and $\mathfrak{s}_1 = \mathfrak{s}_2$, by uniqueness of the involution σ, the splits real forms λ_1 and λ_2 are equal.

The stabilizer of $(\Delta^+ \subset \mathfrak{c},)$ is a maximal torus C, and the stabilizer of a $\mathfrak{c}$-Cartan involution is $\mathsf{C}\cap\mathsf{K}$. The stabilizer of the split real form λ is the corresponding split real group $\mathsf{G}_0 \subset \mathsf{G}$. Thus the stabilizer of a Hitchin triple $(\Delta^+ \subset \mathfrak{c}, \theta, \lambda)$ is $\mathsf{T}_0 = \mathsf{G}_0 \cap \mathsf{K} \cap \mathsf{C}$. ☐

Remark 6.1.5. A real form G_0 is called a *group of Hodge type* if the maximal compact torus $T_0 \subset \mathsf{G}_0$ is a maximal compact torus of the complex group G. For split real forms, only $\mathsf{SL}(n,\mathbb{R})$, $\mathsf{SO}_0(2n+1,2n+1)$, and the split real form of E_6 are *not* of Hodge type. When a split real form G_0 is of Hodge type, the space of Cartan triples and the space of Hitchin triples are the same. In this case, the involution σ determined by a $\mathfrak{c}$-PTDS containing $x = \frac{1}{2}\sum_{\alpha\in\Delta^+} H_\alpha$ acts as $+Id$ on $\mathfrak{c}$, and $\mathfrak{c} = \mathfrak{t}\oplus i\mathfrak{t}$.

Let M be the space of Cartan triples of Definition 6.1.1, then $M \cong \mathsf{G}/\mathsf{T}$ where T is the maximal compact torus of G. If $(\mathfrak{c}, \Delta^+, \theta)$ is a Cartan triple, let $\mathfrak{t} = \mathfrak{c}^\theta$, then $\mathfrak{t}$ is the Lie algebra of T. We have the following Ad_T invariant decompositions

$$\mathfrak{g} = \mathfrak{t}\oplus i\mathfrak{t}\oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{c})} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g} = \mathfrak{k}\oplus i\mathfrak{k}.$$

Thus, the Lie algebra bundle $[\mathfrak{g}] = \underline{\mathfrak{g}} \to \mathsf{G}/\mathsf{T}$ has corresponding compatible ∇^c-parallel decompositions

$$\underline{\mathfrak{g}} = [\mathfrak{t}] \oplus [i\mathfrak{t}] \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{c})} [\mathfrak{g}_\alpha] \qquad \text{and} \qquad \underline{\mathfrak{g}} = [\mathfrak{k}] \oplus [i\mathfrak{k}].$$

Recall that $T\mathsf{G}/\mathsf{T} \cong [i\mathfrak{t}] \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{c})} [\mathfrak{g}_\alpha]$, thus the Maurer-Cartan form vanishes of $[\mathfrak{t}]$, i.e. $\omega|_{[\mathfrak{t}]} \equiv 0$.

If $\ell = rank(\mathfrak{g})$, then a set simple roots gives a $\mathbb{Z}^\ell$-grading of $\mathfrak{g}$ called the root space decomposition

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{c})} \mathfrak{g}_\alpha.$$

Since this decomposition is Ad_{T}-invariant and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, the zero curvature equations decomposes as

$$\begin{cases} F_{\nabla^c} + \displaystyle\sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{c})} [\omega_\alpha, \omega_{-\alpha}]^{\mathfrak{t}} = 0 & \mathfrak{t} - part \\ d^{\nabla^c} \omega_{i\mathfrak{t}} + \displaystyle\sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{c})} [\omega_\alpha, \omega_{-\alpha}]^{i\mathfrak{t}} = 0 & i\mathfrak{t} - part \\ d^{\nabla^c} \omega_\alpha + [\omega_{i\mathfrak{t}}, \omega_\alpha] + \displaystyle\sum_{\substack{\beta,\gamma \in \Delta(\mathfrak{g},\mathfrak{c}) \\ \alpha = \beta+\gamma}} [\omega_\beta, \omega_\gamma] = 0 & \mathfrak{m}_\alpha - part \end{cases} \tag{6.1.1}$$

Recall that if $\{\alpha_i\}$ is the collection of simple roots, then every root α can be written uniquely as $\alpha = \sum n_i \alpha_i$, and the integer $\ell(\alpha) = \sum n_i$ is called the height of α. From equation (2.1.9), the grading element x from the PTDS $\mathfrak{s}$ gives a $\mathbb{Z}$-grading on $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_{-m_\ell} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m_\ell}$$

where $\mathfrak{g}_j = \bigoplus_{l(\alpha)=j} \mathfrak{g}_\alpha$. Since, $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$, in terms of the height decomposition, the flatness equations decompose as

$$\begin{cases} F_{\nabla^c} + d^{\nabla^c} \omega_0 + \displaystyle\sum_{j>0} [\omega_j, \omega_{-j}] = 0 & \mathfrak{c} - part \\ d^{\nabla^c} \omega_j + \dfrac{1}{2} \displaystyle\sum_k [\omega_k, \omega_{j-k}] = 0 & \mathfrak{g}_j - part \end{cases} \tag{6.1.2}$$

Set $g_+ = exp(\frac{2\pi i \cdot x}{m_\ell+1})$, and consider the autormorphism $Ad_{g_+} : \mathfrak{g} \to \mathfrak{g}$. Since $ad(x)$ acts on $\mathfrak{g}_j$ with eigenvalue j, the automorphism Ad_{g_+} acts on $\mathfrak{g}_j$ with eigenvalue $\zeta^j_{m_\ell+1} = e^{\frac{2\pi i \cdot j}{m_\ell+1}}$. Note that, by construction, $Ad_{g_+}(X) = X$ if and only if $X \in \mathfrak{c}$. An eigenspace decomposition of Ad_g gives a $\mathbb{Z}/(m_\ell+1)\mathbb{Z}$-grading on $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/(m_\ell+1)\mathbb{Z}} \widehat{\mathfrak{g}}_j$$

where $\widehat{\mathfrak{g}}_j = \bigoplus_{k=j \ mod \ m_\ell+1} \mathfrak{g}_k$. The Maurer Cartan form decomposes as

$$\omega = \sum_{j\in\mathbb{Z}/(m_\ell+1)\mathbb{Z}} \widehat{\omega}_j \tag{6.1.3}$$

and the flatness equations decompose as

$$\begin{cases} F_{\nabla^c} + d^{\nabla^c}\widehat{\omega}_0 + \sum_{j>0}[\widehat{\omega}_j, \widehat{\omega}_{-j}] = 0 & \widehat{\mathfrak{g}}_0 = \mathfrak{c} - part \\ d^{\nabla^c}\widehat{\omega}_j + \frac{1}{2}\sum_k[\widehat{\omega}_k, \widehat{\omega}_{j-k}] = 0 & \widehat{\mathfrak{g}}_j - part \end{cases} \tag{6.1.4}$$

Remark 6.1.6. This grading will be essential for our definition of cyclic surfaces. The automorphism Ad_{g_+} makes the space G/T into a $(m_\ell+1)$-symmetric space. It will be important that the subspaces $\widehat{\mathfrak{g}}_{\pm1}$ are

$$\widehat{\mathfrak{g}}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_{-m_\ell} = \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_\ell} \oplus \mathfrak{g}_{-\mu} \quad \text{and} \quad \widehat{\mathfrak{g}}_{-1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{m_\ell} = \mathfrak{g}_{-\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{-\alpha_\ell} \oplus \mathfrak{g}_{\mu} \tag{6.1.5}$$

where $\{\alpha_i\}$ is the set of simple roots and μ is the highest root. Furthermore, the compact involution θ maps $\widehat{\mathfrak{g}}_1$ to $\widehat{\mathfrak{g}}_{-1}$.

For the space of Hitchin triples G/T_0, the Cartan subalgebra decomposes as $\mathfrak{c} = \mathfrak{t}_0 \oplus i\mathfrak{t} \oplus \mathfrak{a} \oplus i\mathfrak{a}$. The tangent bundle is given by

$$T\mathsf{G}/\mathsf{T}_0 = [i\mathfrak{t}_0] \oplus [\mathfrak{a}] \oplus [i\mathfrak{a}] \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{c})} [\mathfrak{g}_\alpha]$$

and Maurer Cartan form vanishes on $[\mathfrak{t}_0]$. The decompositions (6.1.1), (6.1.2), and (6.1.4) of the flatness equations still hold.

Lemma 6.1.7. *Let $\mathfrak{t}_0 \oplus \mathfrak{m} = \mathfrak{g}$ be the reductive decomposition corresponding to a Hitchin triple. The trivial Lie algebra bundle $\underline{\mathfrak{g}}\to\mathsf{G}/\mathsf{T}$ has the following data*

- *$\omega \in \Omega^1(\mathsf{G}/\mathsf{T}, [\mathfrak{m}] \subset \underline{\mathfrak{g}})$ the Maurer Cartan form*
- *the canonical connection ∇^c with flat differentiation given by $d = \nabla^c + ad_\omega$*
- *$[\mathfrak{c}] \subset \underline{\mathfrak{g}}$ which decomposes as $[\mathfrak{c}] = [\mathfrak{t}_0] \oplus [i\mathfrak{t}_0] \oplus [\mathfrak{a}]$*
- *∇^c-parallel subbundles $[\mathfrak{n}^+] \subset \underline{\mathfrak{g}}$ and $[\mathfrak{n}^-] \subset \underline{\mathfrak{g}}$ with $[\mathfrak{n}^-] \oplus [\mathfrak{c}] \oplus [\mathfrak{n}^+] = \underline{\mathfrak{g}}$.*
- *∇^c-parallel conjugate linear involution $\Theta : \underline{\mathfrak{g}}\to\underline{\mathfrak{g}}$ and $\Lambda : \underline{\mathfrak{g}}\to\underline{\mathfrak{g}}$ with fixed point set $[\mathfrak{k}]$ and $[\mathfrak{g}_0]$.*
- *A ∇^c-parallel complex linear involution $\sigma = \Theta \circ \Lambda$ with eigenbundle decomposition $\underline{\mathfrak{g}} = [\mathfrak{h}_\mathbb{C}] \oplus [\mathfrak{m}_\mathbb{C}]$, where $[\mathfrak{h}] \subset [\mathfrak{g}_0]$ is the fixed point set of $\Theta|_{[\mathfrak{g}_0]}$.*

- *A ∇^c-parallel order $(m_\ell+1)$ automorphism $\mathcal{X}_+ : \underline{\mathfrak{g}} \to \underline{\mathfrak{g}}$ with eigenbundles $[\widehat{\mathfrak{g}}_j]$ and $[\mathfrak{c}]$ the identity eigenbundle.*

Proof. The splitting of $\mathfrak{g}$ into root space is Ad_{T_0} invariant, thus we have ∇^c-parallel subbundles

$$[\mathfrak{n}^+] = \bigoplus_{\alpha\in\Delta^+} [\mathfrak{g}_\alpha] \qquad \text{and} \qquad [\mathfrak{n}^-] = \bigoplus_{\alpha\in\Delta^-} [\mathfrak{g}_\alpha]\ .$$

The fiber of $[\mathfrak{n}^+]$ over a Hitchin triple $(\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda)$ is $\bigoplus\limits_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. For $X \in \mathfrak{g}$, the conjugate linear involutions Θ and Λ are defined by

$$\Theta((\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda), X)) = ((\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda), \theta(X)) \qquad \text{and} \qquad \Lambda((\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda), X)) = ((\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda), \lambda(X))\ .$$

The subbundle $[\mathfrak{t}_0]$ is defined by

$$[\mathfrak{t}_0] = \{X \in [\mathfrak{c}] \mid \Lambda(X) = X = \Theta(X)\}.$$

By definition, the conjugate linear involutions Θ and Λ commute. Thus, we also obtain a complex linear involution σ which is the complex linear extension of a Cartan involution of the split real form $\mathfrak{g}_0$. If $\mathfrak{g}_0 = \mathfrak{h}\oplus\mathfrak{m}$ is the corresponding Cartan decomposition, then the eigenbundle splitting of $\underline{\mathfrak{g}}$ is given by

$$\sigma = \Theta \circ \Lambda : [\mathfrak{h}_{\mathbb{C}}] \oplus [\mathfrak{m}_{\mathbb{C}}].$$

Recall that for $x = \frac{1}{2}\sum\limits_{\alpha\in\Delta^+} H_\alpha$, and if the highest root has height m_ℓ then we defined $g_+ = exp(\frac{2\pi i x}{m_\ell+1})$. The ∇^c-parallel automorphism $\mathcal{X}^+$ is defined by

$$\mathcal{X}^+((\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda), X)) = ((\Delta^+ \subset \mathfrak{c}^*, \theta, \lambda), g_+(X)).$$

$\square$

Recall from Proposition 2.1.3, the symmetric space G/K is the space of Cartan involutions. The following lemma will be important for our defintion of cyclic surfaces.

Lemma 6.1.8. *Let $\underline{\mathfrak{g}} = [\mathfrak{k}] \oplus [\mathfrak{m}]$ denote the trivializable Lie algebra bundle over $M = \mathsf{G}/\mathsf{K}$ the symmetric space of Cartan involutions of $\mathfrak{g}$. There is a canonical automorphism $\Theta : \underline{\mathfrak{g}} \longrightarrow \underline{\mathfrak{g}}$ given by*

$$\Theta(\theta, X) = (\theta, \theta(X)).$$

Furthermore, the invariant metric on $\underline{\mathfrak{g}}$ *induced by the Killing form is given by* $B_\Theta(X,Y) = -B_{\mathfrak{g}}(X,\Theta(Y))$, *and is parallel with respect to the canonical connection,* $\nabla^c B_\Theta = 0$.

Remark 6.1.9. The automorphism Θ has a natural extension to complex forms valued in $\underline{\mathfrak{g}}$. If $\alpha \in \Omega^*(\mathsf{G}/\mathsf{K}, \underline{\mathfrak{g}})$ is of the form $\alpha = A\cdot a$ where $A \in \Omega^*(\mathsf{G}/\mathsf{K})$ and a is a section of $\underline{\mathfrak{g}}$, then $\Theta(\alpha) = \overline{A}\cdot\Theta(a)$.

Proposition 6.1.10. *Let* N *be a simply connected manifold and* $(\widetilde{\mathfrak{g}}, \widetilde{D})$ *be a flat* $\mathfrak{g}$*-bundle. Suppose*

- $\widetilde{\Theta} : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ *be a smoothly varying Cartan involution with* $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{m}}$ *the corresponding eigenbundle decomposition.*
- $\widetilde{\nabla}$ *a connection with* $\widetilde{\nabla}\widetilde{\Theta} = 0$
- $\widetilde{\omega} \in \Omega^1(N, \widetilde{\mathfrak{m}})$ *with* $\widetilde{D} = \widetilde{\nabla} + ad_{\widetilde{\omega}}$.

Then there exists a map $f : N \to \mathsf{G}/\mathsf{K}$, *unique up to postcomposition by an element of* G *so that*

$$f^*(\ \underline{\mathfrak{g}}\ ,\ \Theta\ ,\ \nabla^c\ ,\ \omega\ )\ =\ (\ \widetilde{\mathfrak{g}}\ ,\ \widetilde{\Theta}\ ,\ \widetilde{\nabla}\ ,\ \widetilde{\omega}\ ).$$

Proof. Since N is simply connected, choose a trivialization $(\widetilde{\mathfrak{g}}, \widetilde{D}) = (N \times \mathfrak{g}, d)$. In this trivialization, the gauge transformation $\widetilde{\Theta}$ defines the map $f : N \to \mathsf{G}/\mathsf{K}$ with $(f^*\underline{\mathfrak{g}}, f^*\Theta) = (\widetilde{\mathfrak{g}}, \widetilde{\Theta})$. Another trivialization produces a map which differs from f by postcomposition by an element of G.

Thus, $\widetilde{\Theta}$ is parallel with respect to $f^*\nabla^c$ and $\widetilde{\nabla}$. Since the stabilizer of a Cartan involution is K, we have $f^*\nabla^c - \widetilde{\nabla} \in \Omega(N, \widetilde{\mathfrak{k}})$, and thus $f^*\omega - \widetilde{\omega} \in \Omega^1(N, \widetilde{\mathfrak{k}})$. But $f^*\omega - \widetilde{\omega} \in \Omega^1(N, \widetilde{\mathfrak{m}})$, thus $f^*\nabla^c = \widetilde{\nabla}$ and $f^*\omega = \widetilde{\omega}$. ∎

The following proposition and corollary are proven in section 4 of [Lab14], the proofs are analogous to Proposition 6.1.10.

Proposition 6.1.11. *Let* N *be a smooth simply connected manifold and* $\mathfrak{g}$ *be a complex simple Lie algebra. Let* $(\widetilde{\mathfrak{g}}, \widetilde{D}) \to N$ *be a flat* $\mathfrak{g}$*-Lie algebra bundle with the following*

- *A smoothly varying Hitchin triple* $(\widetilde{\mathfrak{c}}, \widetilde{\mathfrak{n}^+}, \widetilde{\Theta}, \widetilde{\Lambda})$ *with corresponding decompositions*

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{t}}_0 \oplus i\widetilde{\mathfrak{t}} \oplus \widetilde{\mathfrak{a}} \oplus \widetilde{\mathfrak{n}}^+ \oplus \widetilde{\mathfrak{n}}^-\ =\ \widetilde{\mathfrak{t}}_0 \oplus \widetilde{\mathfrak{m}}$$

- $\widetilde{\nabla}$ *a connection so that* $(\widetilde{\mathfrak{c}}, \widetilde{\mathfrak{n}^+}, \widetilde{\Theta}, \widetilde{\Lambda})$ *is parallel.*
- $\widetilde{\omega} \in \Omega^1(N, \widetilde{\mathfrak{m}})$ *with* $\widetilde{\nabla} + ad_{\widetilde{\omega}} = \widetilde{D}$.

Then there is a map $f : N \to \mathsf{G}/\mathsf{T}_0$, unique up to post composition by an element of G, so that

$$(\widetilde{\mathfrak{g}},\ \widetilde{\nabla},\ \widetilde{\omega},\ \widetilde{\mathfrak{c}},\ \widetilde{\mathfrak{n}}^+,\ \widetilde{\Theta},\ \widetilde{\Lambda}) = f^*(\underline{\mathfrak{g}},\ \nabla^c,\ \omega,\ [\mathfrak{c}],\ [\mathfrak{n}^+],\ \Theta,\ \Lambda).$$

Corollary 6.1.12. *Let N be a smooth manifold and $\widetilde{\mathfrak{g}} \to N$ be a flat $\mathfrak{g}$-Lie algebra bundle equipped with the structure of Proposition 6.1.11, then there exists*

1. *A representation $\rho : \pi_1(N) \to \mathsf{G}$ unique up to conjugation*
2. *A ρ-equivariant map f from the universal cover $\widetilde{N}$ of N to the space of Hitchin triples G/T_0 satisfying the conclusion of Proposition 6.1.11.*

Remark 6.1.13. Lemma 6.1.7, Proposition 6.1.11 and Corollary 6.1.12 all have analogous versions for the space of Cartan triples.

6.1.2 Cyclic Pfaffian systems and cyclic surfaces

The general Pfaffian system definitions in this section come from section 7 of [Lab14].

Definition 6.1.14. Let $E \to N$ be a vector bundle over a smooth manifold N, and $(\eta_1, \cdots, \eta_n)$ be a collection of differential forms on N valued in E. A submanifold $L \subset N$ is called a solution to the *Pfaffian system defined by* $(\eta_1, \cdots, \eta_n)$ if $\eta_j|_L \equiv 0$ for all j.

The Pfaffian systems we will be interested are defined as follows:

Definition 6.1.15. Let $\omega \in \Omega^1(\mathsf{G}/\mathsf{T}, \underline{\mathfrak{g}})$ be the Maurer Cartan form of the space of Cartan triples G/T. A G-*cyclic Pfaffian system* is defined by the vanishing of the following $\underline{\mathfrak{g}}$-valued forms

$$((\widehat{\omega}_0, \widehat{\omega}_2, \cdots, \widehat{\omega}_{m_\ell - 1})\ ,\ [\widehat{\omega}_{-1}, \widehat{\omega}_{-1}]\ ,\ \widehat{\omega}_{-1} + \Theta(\widehat{\omega}_1))$$

where $\omega = \sum \widehat{\omega}_j$ is the decomposition of (6.1.3).

For the space of Hitchin triples, we define a G_0-cyclic Pfaffian system as follows.

Definition 6.1.16. Let $\omega \in \Omega^1(\mathsf{G}/\mathsf{T}_0, \underline{\mathfrak{g}})$ be the Maurer-Cartan form of the space of Hitchin triples G/T_0. The G_0-*cyclic Pfaffian system* is defined by the vanishing of the following $\underline{\mathfrak{g}}$-valued forms

$$((\widehat{\omega}_0, \widehat{\omega}_2, \cdots, \widehat{\omega}_{m_\ell - 1})\ ,\ [\widehat{\omega}_{-1}, \widehat{\omega}_{-1}]\ ,\ \widehat{\omega} + \Theta(\widehat{\omega})\ ,\ \Lambda(\omega) - \omega)$$

where $\omega = \sum \widehat{\omega}_j$ is the decomposition of (6.1.3).

The above definition are related to the τ-primitive maps consider for compact groups G in [BPW95]. In the context of representations of surface groups, we are interested in maps from a Riemann surface Σ to the spaces of Cartan triples and Hitchin triples.

Definition 6.1.17. Let Σ be a Riemann surface (not necessarily compact), a map $f : \Sigma \to \mathsf{G}/\mathsf{T}$ is a G-*cyclic surface* if it is a G-cyclic Pfaffian system and $f^*\widehat{\omega}_{-1}$ is a $(1,0)$-form. Similarly, a map $f : \Sigma \to \mathsf{G}/\mathsf{T}_0$ is a G_0-*cyclic surface* if it is a G_0-cyclic Pfaffian system and $f^*\widehat{\omega}_{-1}$ is a $(1,0)$-form.

Remark 6.1.18. The reality condition $f^*(\Lambda(\omega)) = f^*(\omega)$ for a G_0-cyclic surface implies $f(\Sigma)$ lies in a G_0 orbit. If G_0 is a split real form of Hodge type, then $T_0 = T$, and the G_0-cyclic condition is just an extra symmetry a G-cyclic map must satisfy.

The following theorem relates equivariant cyclic surfaces and Higgs bundles that are fixed points of $\left\langle \zeta_{m_\ell} \right\rangle \subset \mathsf{U}(1)$.

Theorem 6.1.19. *Let G be a complex simple Lie group of rank at least 2, and $\rho \in \mathcal{X}(\mathsf{G})$. If $\underline{\mathfrak{g}} \to \mathsf{G}/\mathsf{T}$ is the associated Lie algebra bundle and $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{T}$ be a ρ-equivariant G-cyclic surface, then $(f^*\underline{\mathfrak{g}}, (f^*\nabla^c)^{01}, f^*\widehat{\omega}_{-1})$ is a G-Higgs bundle that is a fixed point of the $\left\langle \zeta_{m_\ell+1} \right\rangle$-action. Furthermore, f^*B_Θ solves the Hitchin equations which simplify to*

$$F_{f^*\nabla^c} + \sum_{i=1}^{\ell} [f^*\omega_{\alpha_i}, f^*\omega_{-\alpha_i}] + [f^*\omega_\mu, f^*\omega_{-\mu}] = 0.$$

Proof. To prove that $(f^*\underline{\mathfrak{g}}, (f^*\nabla^c)^{01}, f^*\widehat{\omega}_{-1})$ is a G-Higgs bundle we just need to show $f^*\widehat{\omega}_{-1}$ is holomorphic. By equations (6.1.3), the flatness equations for $\nabla^c + \omega$ we have

$$d^\nabla \widehat{\omega}_{-1} + \sum_{j=0}^{m_\ell} [\widehat{\omega}_j, \widehat{\omega}_{-j-1}] = 0.$$

By the cyclic assumption, $f^*\widehat{\omega}_j = 0$ for $j \neq \pm 1$, thus, pulling back the flatness equations, we have

$$d^{f^*\nabla}(f^*\widehat{\omega}_{-1}) = 0.$$

Since $f^*\widehat{\omega}_{-1}$ is a $(1,0)$-form, we conclude that $(d^{f^*\nabla})^{01} f^*\omega_{-1} = 0$.

To see that it is a fixed point of $\left\langle \zeta_{m_\ell+1} \right\rangle$, recall from Lemma 6.1.7 that there is an automorphism $\mathcal{X}_+ : \underline{\mathfrak{g}} \to \underline{\mathfrak{g}}$, of order $(m_\ell+1)$, which acts as $\zeta_{m_\ell+1}^{-1}$ on $[\widehat{\mathfrak{g}}_{-1}]$. Thus $f^*(\mathcal{X}_+)^{-1}$ is a gauge transformation of $f^*\underline{\mathfrak{g}}$ which acts as $\zeta_{m_\ell+1}$ on the Higgs field $f^*\widehat{\omega}_{-1}$.

Recall that by definition of a G-cyclic surface, we have $f^*(-\Theta(\widehat{\omega}_{-1})) = f^*\widehat{\omega}_1$, thus the adjoint of the Higgs field $f^*\widehat{\omega}_{-1}$ is given by $f^*(-\Theta(\widehat{\omega}_{-1})) = f^*\widehat{\omega}_1$. Using the decompositions of (6.1.5) we have

$$\widehat{\omega}_1 = \sum_{i=1}^{\ell} \omega_{\alpha_i} + \omega_{-\mu} \qquad\qquad \widehat{\omega}_{-1} = \sum_{i=1}^{\ell} \omega_{-\alpha_i} + \omega_{\mu}\ .$$

The assumption that $f^*\omega_0 = 0$, and the flatness equations of (6.1.3) imply

$$F_{f^*\nabla^c} + \sum_{i=1}^{\ell}[f^*\omega_{\alpha_i}, f^*\omega_{-\alpha_i}] + [f^*\omega_\mu, f^*\omega_{-\mu}] = 0.$$

Since $f^*\nabla^c$ is a metric connection for the hermitian metric f^*B_Θ, and the holomorphic structure is on $f^*\mathfrak{g}$ is defined to be $(f^*\nabla^c)^{01}$, we conclude that f^*B_Θ solves the Hitchin equations. ☐

Corollary 6.1.20. *Let G be a complex simple Lie group with rank at least 2, $\rho \in \mathcal{X}(\mathsf{G})$, and $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{T}$ be an ρ-equivariant G-cyclic surface, then the associated equivariant harmonic map $h_{\rho,J} = f \circ \pi : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{H}$ is a minimal surface.*

Proof. Since the Higgs bundle admits a solution to the Hitchin equations, it is polystable. Since it is a fixed point of $\langle \zeta_{m_\ell+1} \rangle$ and $rank(\mathfrak{g}) \geq 2$, the quadratic differential is the image of the Hitchin fibration vanishes, thus the Hopf differential of the harmonic map is zero and we conclude the harmonic map is a branched minimal immersion. ☐

Similarly, for G_0-cyclic surfaces we have the following theorem.

Theorem 6.1.21. *Let G be a complex simple Lie group of rank at least 2, and $\rho \in \mathcal{X}(\mathsf{G})$. If $\underline{\mathfrak{g}} \to \mathsf{G}/\mathsf{T}_0$ is the associated Lie algebra bundle and $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{T}_0$ be a ρ-equivariant G_0-cyclic surface, then $(f^*[\mathfrak{h}_\mathbb{C}], (f^*\nabla^c)^{01}, f^*\widehat{\omega}_{-1})$ is a G_0-Higgs bundle that is a fixed point of the $\left\langle \zeta_{m_\ell+1} \right\rangle$-action. Furthermore, f^*B_Θ solves the Hitchin equations which simplify to*

$$F_{f^*\nabla^c} + \sum_{i=1}^{\ell}[f^*\omega_{-\alpha_i}, f^*\omega_{\alpha_i}] + [f^*\omega_\mu, f^*\omega_{-\mu}] = 0.$$

Remark 6.1.22. In this case, the representation $\rho \in \mathcal{X}(\mathsf{G})$ is actually in $\mathcal{X}(\mathsf{G}_0)$.

Proof. Recall from Lemma 6.1.7, that the Lie algebra bundle $\underline{\mathfrak{g}} \to \mathsf{G}/\mathsf{T}_0$ has a complex linear involution $\sigma = \Theta \circ \Lambda$ which has eigenbundle decomposition $\underline{\mathfrak{g}} = [\mathfrak{h}_\mathbb{C}] \oplus [\mathfrak{m}_\mathbb{C}]$ where $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{m}$ is the corresponding Cartan decomposition. To show that $(f^*[\mathfrak{h}_\mathbb{C}], (f^*\nabla^c)^{01}, f^*\widehat{\omega}_{-1})$ is a G_0-Higgs bundle, we must show that $f^*\widehat{\omega}_{-1} \in \Omega^{10}(\Sigma, f^*[\mathfrak{m}_\mathbb{C}])$. Recall from Remark 2.1.39, the involution σ preserves the height grading

$$\mathfrak{g}_{-m_\ell} \oplus \cdots \mathfrak{g}_{-1} \oplus \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m_\ell},$$

and thus, σ preserves both $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_{-1}$. By the definition of a G_0-cyclic surface 6.1.17, $f^*\omega = f^*\widehat{\omega}_{-1} + f^*\widehat{\omega}_1$ and

$$f^*\Theta(f^*\widehat{\omega}_{-1} + f^*\widehat{\omega}_1) = -f^*\widehat{\omega}_1 - f^*\widehat{\omega}_{-1} \qquad f^*\Lambda(f^*\widehat{\omega}_{-1} + f^*\widehat{\omega}_1) = f^*\widehat{\omega}_{-1} + f^*\widehat{\omega}_1 \ .$$

Hence, $\sigma(f^*\widehat{\omega}_{-1} + f^*\widehat{\omega}_1) = -(f^*\widehat{\omega}_{-1} + f^*\widehat{\omega}_1)$, and, furthermore, since σ preserves $[\widehat{\mathfrak{g}}_{-1}]$,

$$\sigma(\widehat{\omega}_{-1}) = -\widehat{\omega}_{-1}.$$

This proves that $f^*\widehat{\omega}_{-1} \in \Omega^{10}(\Sigma, f^*[\mathfrak{m}_{\mathbb{C}}])$.

We also need to check that the gauge transformation $f^*\mathcal{X}_+$ is an $f^*[\mathfrak{h}_{\mathbb{C}}]$-gauge transformation. Recall that the grading element x of the PTDS is in the $+1$-eigenspace of σ (see Remark 2.1.39). Since $\mathcal{X}_+$ is obtained from exponentiating x, it follows that $f^*\mathcal{X}_+$ is an $[\mathfrak{h}_{\mathbb{C}}]$-gauge transformation. The proof of the rest of the theorem is identical to the proof of Theorem 6.1.19. $\square$

6.1.3 Deformations of cyclic Pfaffian systems and cyclic surfaces

Definition 6.1.23. Let $F = (f_t) : L \to N$ be a one parameter family with f_0 being the inclusion and set

$$\xi = \left.\frac{d}{dt}\right|_{t=0} f_t.$$

Then $\xi \in \Omega^0(L, f_0^*TN)$ is a vector field along L in N called the *tangent vector field to the family F*. A family $F = (f_t)$ is a *first order deformation of the Pfaffian system L defined by* $(\eta_1, \cdots, \eta_n)$ if, for all j,

$$\left.\frac{d}{dt}\right|_{t=0} f_t^*\eta_i = 0.$$

In the above definition, we have chosen a connection to identify f_t^*E and f_0^*E, this choice does not effect the definition.

Definition 6.1.24. A vector field ξ along a solution L of a Pfaffian system given by $\eta = (\eta_1, \cdots, \eta_n)$ is an *infinitesimal variation of the Pfaffian system* if, for any connection ∇, and all j,

$$\iota_\xi d^\nabla \eta_j\big|_L = -d^\nabla(\iota_\xi \eta_j)\big|_L \, .$$

The relation between first order deformations and variations is given by Proposition 7.1.4 of [Lab14]:

Proposition 6.1.25. *Let ξ be a tangent vector to a family of first order deformations of a Pfaffian system*

$\eta = (\eta_1, \cdots, \eta_n)$*, then* ξ *is an infinitesimal variation of the Pfaffian system.*

Proof. See Proposition 7.1.4 of [Lab14]. $\square$

Definition 6.1.26. An *infinitesimal variation of a* G*-cyclic surface* is an infinitesimal variation of a G-cyclic Pfaffian system. An *infinitesimal variation of a* G_0*-cyclic surface* is an infinitesimal variation ξ of a G_0-cyclic Pfaffian system such that $\Lambda(\xi) = \xi$.

Definition 6.1.27. Let $\rho : \pi_1(S) \to \mathsf{G}$ be a representation and $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{T}$ be a ρ-equivariant G-cyclic surface. If ξ is an infinitesimal variation of a G-cyclic surface, then ξ is an *infinitesimal variation of the equivariant* G*-cyclic surface* if it is ρ-equivariant. Similarly for an equivariant G_0-cyclic surface.

The signs in the following lemma will be crucial.

Lemma 6.1.28. *Let* Σ *be a compact Riemann surface and* f *a* G*-cyclic surface or a* G/T_0*-cyclic surface. Let* $\alpha \in \Omega^{10}(\Sigma, f^*\underline{\mathfrak{g}})$ *and* $\beta \in \Omega^{01}(\Sigma, f^*\underline{\mathfrak{g}})$ *then*

$$-i\int_\Sigma B_{\mathfrak{g}}(\alpha, \Theta\alpha) \geq 0 \qquad \textit{and} \qquad i\int_\Sigma B_{\mathfrak{g}}(\beta, \Theta\beta) \geq 0\ .$$

Also, if $\alpha, \beta \in \Omega^1(\Sigma, f^*\underline{\mathfrak{g}})$ *and* $\gamma \in \Omega^0(\Sigma, f^*\underline{\mathfrak{g}})$*, then*

$$B_{\mathfrak{g}}(\gamma, [\beta, \alpha]) = B_{\mathfrak{g}}([\gamma, \alpha], \beta). \tag{6.1.6}$$

Proof. It is suffices to check the sign on a form $\alpha = A \cdot a$ where a is a section of $f^*\underline{\mathfrak{g}}$ and $A \in \Omega^{10}(\Sigma)$. By Remark 6.1.9, $\Theta(\alpha) = \overline{A} \cdot \Theta(a)$ and hence, since $-B_{\mathfrak{g}}(\cdot, \Theta\cdot)$ is positive definite,

$$-i\int_\Sigma B_{\mathfrak{g}}(\alpha, \Theta\alpha) = -i\int_\Sigma A \wedge \overline{A} \cdot B_{\mathfrak{g}}(a, \Theta a) \geq 0.$$

Equation (6.1.6) follows from a calculation using invariance of the Killing form. $\square$

Let $f : \Sigma \to \mathsf{G}/\mathsf{T}$ be a G-cyclic surface, we will use the following notation

$$\widehat{\omega}_{-1}\big|_{f(\Sigma)} = \Phi = \Phi_{-1} + \Phi_{m_\ell} \qquad \text{and} \qquad \widehat{\omega}_1\big|_{f(\Sigma)} = \Phi^* = \Phi_1^* + \Phi_{-m_\ell}^*.$$

Let ξ is an infinitesimal variation of f, and denote the contraction with ω by

$$\zeta = \iota_\xi(\omega).$$

Using the various decompositions of the Maurer Cartan form ω, we have the following decompositions

$$\zeta = \zeta_0 + \sum_{\alpha \in \Delta(\mathfrak{c},\mathfrak{g})} \zeta_\alpha \ , \qquad \zeta = \sum_{j=-m_\ell}^{m_\ell} \zeta_j \ , \qquad \zeta = \sum_{j \in \mathbb{Z}/(m_\ell+1)\mathbb{Z}} \widehat{\zeta}_j \ . \tag{6.1.7}$$

The following notation will also be useful

$$\zeta = \widehat{\zeta}_{-1} + \widehat{\zeta}_0 + \widehat{\zeta}_1 + \widehat{\zeta}_Y, \tag{6.1.8}$$

where $\widehat{\zeta}_Y = \sum_{j \neq 0,1,-1} \widehat{\zeta}_j$.

Using the decomposition of the flatness equations (6.1.2) we have

$$d^{\nabla^c}\omega_j + \sum_{k=-m_\ell}^{m_\ell} [\omega_k, \omega_{j-k}] = 0.$$

By Definition 6.1.24, on the surface $f(\Sigma)$, we have $\iota_\xi(d^{\nabla^c}\omega_j) = -d^{\nabla^c}(\zeta_j)$ for $j \neq -m_\ell, -1, 1, m_\ell$. Contracting the wedge product is given by

$$\iota_\xi[\omega_j, \omega_{j-k}] \ = \ [\zeta_j, \omega_{j-k}] - [\omega_j, \zeta_{j-k}] \ = \ [\zeta_j, \omega_{j-k}] + [\zeta_{j-k}, \omega_j].$$

Thus, contracting the flatness equations with ξ yields

$$d^{\nabla^c}(\zeta_j) = \sum_{k=-m_\ell}^{m_\ell} ([\zeta_k, \omega_{j-k}] + [\zeta_{j-k}, \omega_k]) \qquad j \neq -m_\ell, -1, 1, m_\ell \tag{6.1.9}$$

The assumption on a cyclic surface that $f^*\widehat{\omega}_j = 0$ for $j \neq \pm 1$ and the fact that Φ is a $(1,0)$-form and Φ^* is a $(0,1)$-form allows us to simplify the equations. For $1 < j < m_\ell$ we have

$$\partial^{\nabla^c}(\zeta_j) = 2([\zeta_{j+1}, \Phi_{-1}] + [\zeta_{j-m_\ell}, \Phi_{m_\ell})] \qquad \text{and} \qquad \bar{\partial}^{\nabla^c}(\zeta_j) = 2[\zeta_{j-1}, \Phi_1^*] \tag{6.1.10}$$

and for $-m_\ell < j < -1$ we have

$$\partial^{\nabla^c}(\zeta_j) = 2[\zeta_{j+1}, \Phi_{-1}] \qquad \text{and} \qquad \bar{\partial}^{\nabla^c}(\zeta_j) = 2([\zeta_{j-1}, \Phi_1^*] + [\zeta_{j+m_\ell}, \Phi^*_{-m_\ell}]) \tag{6.1.11}$$

Let π_Y denote the projection onto the Y component of equation (6.1.8), then equations (6.1.10) and (6.1.11) can be written compactly as:

$$\partial^{\nabla^c}\widehat{\zeta}_Y = 2\pi_Y\left([\Phi,(\widehat{\zeta}_Y+\widehat{\zeta}_{-1})]\right) \qquad \text{and} \qquad \bar{\partial}^{\nabla^c}\widehat{\zeta}_Y = 2\pi_Y\left([\Phi^*,(\widehat{\zeta}_Y+\widehat{\zeta}_1)]\right) \tag{6.1.12}$$

For $j=-1,1$ even though $\iota_\xi\left(d^{\nabla^c}\widehat{\omega}_j\right)\big|_\Sigma \neq -d^{\nabla^c}\widehat{\zeta}_j$, by equations (6.1.4) we have

$$\iota_\xi(\partial^{\nabla^c}\widehat{\omega}_{-1})\big|_\Sigma = -2[\zeta_0,\Phi] \qquad \text{and} \qquad \iota_\xi(\bar{\partial}^{\nabla^c}\widehat{\omega}_1)\big|_\Sigma = -2[\zeta_0,\Phi^*] \tag{6.1.13}$$

Similarly,

$$\begin{aligned} &\iota_\xi(\bar{\partial}^{\nabla^c}\omega_{-1})\big|_\Sigma = -2([\zeta_{-2},\Phi_1^*]+[\zeta_{-1+m_\ell},\Phi^*_{-m_\ell}])\ , \qquad &&\iota_\xi(\partial^{\nabla^c}\omega_1)\big|_\Sigma = -2([\zeta_2,\Phi_{-1}]+[\zeta_{-1-m_\ell},\Phi_{m_\ell}])\ , \\ &\iota_\xi(\bar{\partial}^{\nabla^c}\omega_{m_\ell})\big|_\Sigma = -2[\zeta_{m_\ell-1},\Phi_1^*]\ , &&\iota_\xi(\partial^{\nabla^c}\omega_{-m_\ell})\big|_\Sigma = -2[\zeta_{-m_\ell+1},\Phi_{-1}]\ . \end{aligned} \tag{6.1.14}$$

Proposition 6.1.29. *The second derivatives are given by*

$$\bar{\partial}^{\nabla^c}(\partial^{\nabla^c}\widehat{\zeta}_Y) = 4\pi_Y\left(\left[\left[\widehat{\zeta}_Y,\Phi^*\right],\Phi\right]\right)\ , \qquad \partial^{\nabla^c}(\bar{\partial}^{\nabla^c}\widehat{\zeta}_Y) = 4\pi_Y\left(\left[\left[\widehat{\zeta}_Y,\Phi\right],\Phi^*\right]\right)\ , \tag{6.1.15}$$

$$\bar{\partial}^{\nabla^c}(\partial^{\nabla^c}\widehat{\zeta}_0) = 4\pi_{it}\left(\left[\left[\widehat{\zeta}_0,\Phi^*\right],\Phi\right]\right)\ , \qquad \partial^{\nabla^c}(\bar{\partial}^{\nabla^c}\widehat{\zeta}_0) = 4\pi_{it}\left(\left[\left[\widehat{\zeta}_0,\Phi\right],\Phi^*\right]\right)\ . \tag{6.1.16}$$

Proof. Recall that on a G-cyclic surface we have $\partial^{\nabla^c}\Phi = \bar{\partial}^{\nabla^c}\Phi = \partial^{\nabla^c}\Phi^* = \bar{\partial}^{\nabla^c}\Phi^* = 0$. We will first show equation (6.1.15). Using equations (6.1.10) and (6.1.11), a direct computation shows

$$\begin{cases} \bar{\partial}^{\nabla^c}(\partial^{\nabla^c}\zeta_j) = 4\left([[\zeta_j,\Phi_1^*],\Phi_{-1}]+[[\zeta_{j-m_\ell-1},\Phi_1^*],\Phi_{m_\ell}]+\left[\left[\zeta_j,\Phi^*_{-m_\ell}\right],\Phi_{m_\ell}\right]\right) & 1<j<m_\ell-1 \\ \bar{\partial}^{\nabla^c}(\partial^{\nabla^c}\zeta_j) = 4\left([[\zeta_j,\Phi_1^*],\Phi_{-1}]+\left[\left[\zeta_{j+1+m_\ell},\Phi^*_{-m_\ell}\right],\Phi_{-1}\right]\right) & -m_\ell<j<-2 \\ \partial^{\nabla^c}(\bar{\partial}^{\nabla^c}\zeta_j) = 4\left([[\zeta_j,\Phi_{-1}],\Phi_1^*]+[[\zeta_{j-1-m_\ell},\Phi_{m_\ell}],\Phi_1^*]\right) & 2<j<m_\ell \\ \partial^{\nabla^c}(\bar{\partial}^{\nabla^c}\zeta_j) = 4\left([[\zeta_j,\Phi_{-1}],\Phi_1^*]+\left[[\zeta_{j+m_\ell+1},\Phi_{-1}],\Phi^*_{-m_\ell}\right]+\left[[\zeta_j,\Phi m_\ell],\Phi^*_{-m_\ell}\right]\right) & -m_\ell+1<j<-1 \end{cases}$$

The remaining cases are given by

$$\begin{aligned} &\bar{\partial}^{\nabla^c}(\partial^{\nabla^c}\zeta_{m_\ell-1}) = 2\left(\left[\bar{\partial}^{\nabla^c}\zeta_{m_\ell},\Phi_{-1}\right]+\left[\bar{\partial}^{\nabla^c}\zeta_{-1},\Phi_{m_\ell}\right]\right)\ , \qquad &&\bar{\partial}^{\nabla^c}(\partial^{\nabla^c}\zeta_{-2}) = 2[\bar{\partial}^{\nabla^c}\zeta_{-1},\Phi_{-1}]\ , \\ &\partial^{\nabla^c}(\bar{\partial}^{\nabla^c}\zeta_{-m_\ell+1}) = 2\left(\left[\partial^{\nabla^c}\zeta_{-m_\ell},\Phi_1^*\right]+\left[\partial^{\nabla^c}\zeta_1,\Phi^*_{-m_\ell}\right]\right)\ , &&\partial^{\nabla^c}(\bar{\partial}^{\nabla^c}\zeta_2) = 2[\partial^{\nabla^c}\zeta_1,\Phi_1^*]\ . \end{aligned}$$

We will compute the first two cases, the remaining two cases follow by a symmetric argument. Recall that

$\partial^{\nabla^c}\Phi = \bar\partial^{\nabla^c}\Phi = \partial^{\nabla^c}\Phi^* = \bar\partial^{\nabla^c}\Phi^* = 0$ on a cyclic surface, thus

$$\bar\partial^{\nabla^c}\left(\iota_\xi[\omega_{m_\ell},\omega_{-1}]\right) = [\bar\partial^{\nabla^c}\zeta_{m_\ell},\Phi_{-1}] + [\bar\partial^{\nabla^c}\zeta_{-1},\Phi_{m_\ell}]\,, \qquad \bar\partial^{\nabla^c}\left(\iota_\xi[\omega_{-1},\omega_{-1}]\right) = 2[\bar\partial^{\nabla^c}\zeta_{-1},\Phi_{-1}] \tag{6.1.17}$$

However, since $[\omega_1,\omega_1] = [\omega_1,\omega_{-m_\ell}] = [\omega_{-1},\omega_{-1}] = [\omega_{-1},\omega_{m_\ell}] = 0$ on a cyclic surface, we have

$$\begin{cases} \bar\partial^{\nabla^c}\left(\iota_\xi\left[\omega_{m_\ell},\omega_{-1}\right]\right) = -\left[\iota_\xi(\bar\partial^{\nabla^c}\omega_{m_\ell})\Big|_\Sigma,\Phi_{-1}\right] - \left[\iota_\xi(\bar\partial^{\nabla^c}\omega_{-1})\Big|_\Sigma,\Phi_{m_\ell}\right] \\ \bar\partial^{\nabla^c}\left(\iota_\xi\left[\omega_{-1},\omega_{-1}\right]\right) = -2\left[\iota_\xi(\bar\partial^{\nabla^c}\omega_{-1})\Big|_\Sigma,\Phi_{-1}\right] \end{cases} \tag{6.1.18}$$

Using equations (6.1.14) and (6.1.17), we have the desired result:

$$2([\bar\partial^{\nabla^c}\zeta_{m_\ell},\Phi_{-1}] + [\bar\partial^{\nabla^c}\zeta_{-1},\Phi_{m_\ell}]) = 4\left([[\zeta_{m_\ell-1},\Phi_1^*],\Phi_{-1}] + [[\zeta_{-2},\Phi_2^*],\Phi_{m_\ell}] + \left[\left[\zeta_{-1+m_\ell},\Phi^*_{-m_\ell}\right],\Phi_{m_\ell}\right]\right)$$

and

$$2\left[\bar\partial^{\nabla^c}\zeta_{-1},\Phi_{-1}\right] = 4\left([[\zeta_{-2},\Phi_1^*],\Phi_{-1}] + \left[\left[\zeta_{-1+m_\ell},\Phi^*_{-m_\ell}\right],\Phi_{-1}\right]\right).$$

Thus, we obtain the desired formula:

$$\bar\partial^{\nabla^c}(\partial^{\nabla^c}\widehat\zeta_Y) = 4\pi_Y\left(\left[\left[\widehat\zeta_Y,\Phi^*\right],\Phi\right]\right) \qquad \text{and} \qquad \partial^{\nabla^c}(\bar\partial^{\nabla^c}\widehat\zeta_Y) = 4\pi_Y\left(\left[\left[\widehat\zeta_Y,\Phi\right],\Phi^*\right]\right)$$

We now prove formula (6.1.16), for $\bar\partial^{\nabla^c}\partial^{\nabla^c}\widehat\zeta_0$ and $\partial^{\nabla^c}\bar\partial^{\nabla^c}\widehat\zeta_0$. Since $\widehat\omega_0$ vanishes along a G-cyclic surface, by the flatness equations (6.1.9), we have

$$\partial^{\nabla^c}\widehat\zeta_0 = 2[\widehat\zeta_1,\Phi] \qquad \text{and} \qquad \bar\partial^{\nabla^c}\widehat\zeta_0 = 2[\widehat\zeta_{-1},\Phi^*].$$

Recall that $\widehat\zeta_0$ vanishes along the subbundle $[\mathfrak{t}]$, that is, $\pi_{it}\widehat\zeta_0 = \widehat\zeta_0$. Thus, the second derivatives are

$$\bar\partial^{\nabla^c}\partial^{\nabla^c}\widehat\zeta_0 = 2[\bar\partial^{\nabla^c}\widehat\zeta_1,\Phi] = 2\pi_{it}\left([\bar\partial^{\nabla^c}\widehat\zeta_1,\Phi]\right)\,, \qquad \partial^{\nabla^c}\bar\partial^{\nabla^c}\widehat\zeta_0 = 2[\partial^{\nabla^c}\widehat\zeta_{-1},\Phi^*] = 2\pi_{it}\left([\partial^{\nabla^c}\widehat\zeta_{-1},\Phi^*]\right). \tag{6.1.19}$$

Since $\widehat\omega_1 = -\Theta(\widehat\omega_{-1})$ on a G-cyclic surface, it follows that $\pi_{it}([\widehat\omega_1,\widehat\omega_{-1}]) = 0$ along a G-cyclic surface. Thus,

$$\iota_\xi d^{\nabla^c}\pi_{it}([\widehat\omega_1,\widehat\omega_{-1}])\Big|_\Sigma = -d^{\nabla^c}\left(\iota_\xi\pi_{it}([\widehat\omega_1,\widehat\omega_{-1}])|_\Sigma\right).$$

The subbundle $[it]$ is parallel with respect to ∇^c, thus

$$\pi_{it}\left(\iota_\xi d^{\nabla^c}([\widehat{\omega}_1,\widehat{\omega}_{-1}])\Big|_\Sigma\right) = -\pi_{it}\left(d^{\nabla^c}\ (\iota_\xi([\widehat{\omega}_1,\widehat{\omega}_{-1}])|_\Sigma)\right).$$

For the (1,0) part, we have

$$\pi_{it}\left(\iota_\xi \partial^{\nabla^c}([\widehat{\omega}_1,\widehat{\omega}_{-1}])\Big|_\Sigma\right) = \pi_{it}\left([\,(\partial^{\nabla^c}\widehat{\omega}_1)\Big|_\Sigma,\Phi] + [\,(\partial^{\nabla^c}\widehat{\omega}_{-1})\Big|_\Sigma,\Phi^*]\right) = \pi_{it}\left((\partial^{\nabla^c}\widehat{\omega}_{-1})\Big|_\Sigma,\Phi^*])\right).$$

The term $[\partial^{\nabla^c}\widehat{\zeta}_1,\Phi]$ vanishes since it is a $(2,0)$-form. A similar calculations for the (0,1) part gives

$$\pi_{it}\left(\iota_\xi\left(\bar{\partial}^{\nabla^c}([\widehat{\omega}_1,\widehat{\omega}_{-1}])\right)\big|_\Sigma\right) = \pi_{it}\left([\bar{\partial}^{\nabla^c}\widehat{\zeta}_1,\Phi]\right).$$

Thus, by equations (6.1.19),

$$\bar{\partial}^{\nabla^c}\partial^{\nabla^c}\widehat{\zeta}_0 = 2\pi_{it}\left(\iota_\xi\left(\bar{\partial}^{\nabla^c}([\widehat{\omega}_1,\widehat{\omega}_{-1}])\right)\big|_\Sigma\right) \qquad \text{and} \qquad \partial^{\nabla^c}\bar{\partial}^{\nabla^c}\widehat{\zeta}_0 \ = 2\pi_{it}\left(\iota_\xi\partial^{\nabla^c}([\widehat{\omega}_1,\widehat{\omega}_{-1}])\Big|_\Sigma\right)$$

The term $-\pi_{it}\left(d^{\nabla^c}\ (\iota_\xi([\widehat{\omega}_1,\widehat{\omega}_{-1}])|_\Sigma)\right)$ is computed using equation (6.1.13):

$$-\pi_{it}\left(\partial^{\nabla^c}\ (\iota_\xi([\widehat{\omega}_1,\widehat{\omega}_{-1}])|_\Sigma)\right) = -\pi_{it}\left([\left(\partial^{\nabla^c}\iota_\xi\widehat{\omega}_1\right)\Big|_\Sigma,\Phi] + [\left(\partial^{\nabla^c}\iota_\xi\widehat{\omega}_{-1}\right)\Big|_\Sigma,\Phi^*]\right) = 2\pi_{it}\left([[\widehat{\zeta}_0,\Phi],\Phi^*]\right).$$

A similar computation shows

$$-\pi_{it}\left(\bar{\partial}^{\nabla^c}\ (\iota_\xi([\widehat{\omega}_1,\widehat{\omega}_{-1}])|_\Sigma)\right) = 2\pi_{it}\left([[\widehat{\zeta}_0,\Phi^*],\Phi]\right).$$

Thus, on a G-cyclic surface,

$$\partial^{\nabla^c}\bar{\partial}^{\nabla^c}\widehat{\zeta}_0 = 4\pi_{it}\left([[\widehat{\zeta}_0,\Phi],\Phi^*]\right) \qquad \text{and} \qquad \bar{\partial}^{\nabla^c}\partial^{\nabla^c}\widehat{\zeta}_0 \ = 4\pi_{it}\left([[\widehat{\zeta}_0,\Phi^*],\Phi]\right).$$

$\square$

Proposition 6.1.30. *Let $\rho:\pi_1(S)\to\mathsf{G}$ and $f:\widetilde{\Sigma}\to\mathsf{G}/\mathsf{T}$ be a ρ-equivariant G-cyclic surface. Let $\widehat{\zeta}_Y$, $\widehat{\zeta}_0$, Φ and Φ^* be as above, then*

$$\partial^{\nabla^c}\widehat{\zeta}_Y = 0\ , \qquad \bar{\partial}^{\nabla^c}\widehat{\zeta}_Y = 0\ , \qquad [\Phi,\widehat{\zeta}_Y] = 0\ , \qquad [\Phi^*,\widehat{\zeta}_Y] = 0$$

and

$$\partial^{\nabla^c}\widehat{\zeta}_0 = 0\ , \qquad \bar{\partial}^{\nabla^c}\widehat{\zeta}_0 = 0\ , \qquad [\Phi, \widehat{\zeta}_0] = 0\ , \qquad [\Phi^*, \widehat{\zeta}_0] = 0\ .$$

Proof. Recall from Lemma 6.1.28 that

$$0 \leq -i \int_\Sigma B_{\mathfrak{g}}\left(\partial^{\nabla^c}\widehat{\zeta}_Y, \Theta\left(\partial^{\nabla^c}\widehat{\zeta}_Y\right)\right).$$

Since the canonical connection is a metric connection, we have

$$d\left(-B_{\mathfrak{g}}\left(\widehat{\zeta}_Y, \Theta\left(\partial^{\nabla^c}\widehat{\zeta}_Y\right)\right)\right) = -B_{\mathfrak{g}}\left(\partial^{\nabla^c}\widehat{\zeta}_Y, \Theta\left(\partial^{\nabla^c}\widehat{\zeta}_Y\right)\right) - B_{\mathfrak{g}}\left(\widehat{\zeta}_Y, \Theta\left(\bar{\partial}^{\nabla^c}\partial^{\nabla^c}\widehat{\zeta}_Y\right)\right).$$

Integrating over Σ gives

$$0 \leq -i \int_\Sigma B_{\mathfrak{g}}\left(\partial^{\nabla^c}\widehat{\zeta}_Y, \Theta\left(\partial^{\nabla^c}\widehat{\zeta}_Y\right)\right) = i \int_\Sigma B_{\mathfrak{g}}\left(\widehat{\zeta}_Y, \Theta\left(\bar{\partial}^{\nabla^c}\partial^{\nabla^c}\widehat{\zeta}_Y\right)\right).$$

Recall that $[\widehat{\mathfrak{g}}_Y] = \bigoplus_{j\neq -1,0,1} [\widehat{\mathfrak{g}}_j]$ and $\Theta\left([\widehat{\mathfrak{g}}_j]\right) \subset [\widehat{\mathfrak{g}}_{-j}]$. Also, if $i + j \neq 0\ mod\ \ (m_\ell + 1)$ then $\widehat{\mathfrak{g}}_j$ and $\widehat{\mathfrak{g}}_i$ are orthogonal with respect to B_Θ. Thus, the bundles $[\widehat{\mathfrak{g}}_j]$ and $[\widehat{\mathfrak{g}}_i]$ are orthogonal. Thus, using equations (6.1.15) we have

$$0 \leq 4i \int_\Sigma B_{\mathfrak{g}}\left(\widehat{\zeta}_Y, \Theta\left(\pi_Y\left(\left[\left[\widehat{\zeta}_Y, \Phi^*\right], \Phi\right]\right)\right)\right) = 4i \int_\Sigma B_{\mathfrak{g}}\left(\widehat{\zeta}_Y, \Theta\left(\left[\left[\widehat{\zeta}_Y, \Phi^*\right], \Phi\right]\right)\right).$$

Lemma 6.1.28 and the cyclic surface assumption $\Phi = -\Theta\left(\Phi^*\right)$ yield

$$0 \leq -i \int_\Sigma B_{\mathfrak{g}}\left(\partial^{\nabla^c}\widehat{\zeta}_Y, \Theta\left(\partial^{\nabla^c}\widehat{\zeta}_Y\right)\right) = -4i \int_\Sigma B_{\mathfrak{g}}\left(\left[\widehat{\zeta}_Y, \Phi^*\right], \Theta\left(\left[\widehat{\zeta}_Y, \Phi^*\right]\right)\right) \leq 0.$$

Thus

$$\partial^{\nabla^c}\widehat{\zeta}_Y = 0 \qquad \text{and} \qquad [\Phi^*, \widehat{\zeta}_Y] = 0 \tag{6.1.20}$$

By a symmetric argument, we obtain

$$\bar{\partial}^{\nabla^c}\widehat{\zeta}_Y = 0 \qquad \text{and} \qquad [\Phi, \widehat{\zeta}_Y] = 0 \tag{6.1.21}$$

For $\widehat{\zeta}_0$, consider the following integral:

$$0 \leq -i\int_\Sigma B_{\mathfrak{g}}\left(\partial^{\nabla^c}\widehat{\zeta}_0, \Theta(\partial^{\nabla^c}\widehat{\zeta}_0)\right) = i\int_\Sigma B_{\mathfrak{g}}\left(\widehat{\zeta}_0, \Theta(\bar{\partial}^{\nabla^c}\partial^{\nabla^c}\widehat{\zeta}_0)\right)$$

Using equations (6.1.16), the fact that $i\mathfrak{t}\oplus\mathfrak{t}$ is is orthogonal, Lemma 6.1.28 and $\Theta(\Phi^*) = -\Phi$ we have

$$0 \leq 4i\int_\Sigma B_{\mathfrak{g}}\left(\widehat{\zeta}_0, \Theta([[\widehat{\zeta}_0, \Phi^*], \Phi])\right) = -4i\int_\Sigma B_{\mathfrak{g}}\left([\widehat{\zeta}_0, \Phi^*], \Theta\left([\widehat{\zeta}_0, \Phi^*]\right)\right) \leq 0.$$

Thus,

$$\partial^{\nabla^c}\widehat{\zeta}_0 = 0 \qquad \text{and} \qquad [\Phi^*, \widehat{\zeta}_0] = 0\ .$$

A symmetric argument shows

$$\bar{\partial}^{\nabla^c}\widehat{\zeta}_0 = 0 \qquad \text{and} \qquad [\Phi, \widehat{\zeta}_0] = 0\ .$$

□

The same calculations show that the analogous proposition for equivariant G_0-cyclic surfaces is also true.

Corollary 6.1.31. *Let $\rho : \pi_1(S)\to\mathsf{G}$ and $f : \widetilde{\Sigma}\to\mathsf{G}/\mathsf{T}_0$ be a ρ-equivariant G_0-cyclic surface. Let $\widehat{\zeta}_Y$, Φ and Φ^* be as above, then*

$$\partial^{\nabla^c}\widehat{\zeta}_Y = 0\ , \qquad \bar{\partial}^{\nabla^c}\widehat{\zeta}_Y = 0\ , \qquad [\Phi, \widehat{\zeta}_Y] = 0\ , \qquad [\Phi^*, \widehat{\zeta}_Y] = 0\ ,$$

$$\partial^{\nabla^c}\widehat{\zeta}_0 = 0\ , \qquad \bar{\partial}^{\nabla^c}\widehat{\zeta}_0 = 0\ , \qquad [\Phi, \widehat{\zeta}_0] = 0\ , \qquad [\Phi^*, \widehat{\zeta}_0] = 0\ .$$

Furthermore, if $\mathfrak{c} = \mathfrak{t}\oplus i\mathfrak{t}\oplus\mathfrak{a}\oplus i\mathfrak{a}$ is the decomposition of the Cartan subalgebra, then $\widehat{\zeta}_0$ vanishes along $i\mathfrak{t}\oplus i\mathfrak{a}$. In particular, if G_0 is of Hodge type then $\widehat{\zeta}_0 = 0$

Proof. The first part is an immediate corollary of the proof of Proposition 6.1.30. The variation $\widehat{\zeta}_0$ is along $i\mathfrak{t}\oplus\mathfrak{a}\oplus i\mathfrak{a}$, where Λ acts as $+1$ on $\mathfrak{a}$ and -1 on $i\mathfrak{t}\oplus i\mathfrak{a}$. But, by the reality condition of variations of G_0-cyclic surfaces, $\Lambda(\widehat{\zeta}_0) = \widehat{\zeta}_0$; thus, $\widehat{\zeta}_0$ vanishes along $i\mathfrak{t}\oplus i\mathfrak{a}$. Recall that a G_0 is of Hodge type then the $\mathfrak{a} = \{0\}$, thus, in this case $\widehat{\zeta}_0 = 0$. □

If $\rho : \pi_1(S)\to\mathsf{G}$ is representation and $f : \widetilde{\Sigma}\to\mathsf{G}/\mathsf{T}$ is a G-cyclic surface, then Proposition 6.1.30 says that $\widehat{\zeta}_0$ and $\widehat{\zeta}_Y$ are covariantly constant with respect to the *flat connection* $f^*\nabla^c + \Phi + \Phi^*$. Thus, if either $\widehat{\zeta}_0$ or $\widehat{\zeta}_Y$ is non zero, then they are in the centralizer of the representation ρ. However, if ρ is a smooth point, then the centralizing subalgebra is zero by Proposition 3.1.6; thus we have the following proposition.

Proposition 6.1.32. *Let G be a complex simple Lie group, and $\rho : \pi_1(S)\to\mathsf{G}$ be an irreducible representation. If $f : \widetilde{\Sigma}\to\mathsf{G}/\mathsf{T}$ be a ρ-equivariant cyclic surface, then for any variation ξ, we have*

$$\iota_\xi(f^*\widehat{\omega}_0) = \widehat{\zeta}_0 = 0 \qquad \textit{and} \qquad \iota_\xi(f^*\widehat{\omega}_Y) = \widehat{\zeta}_Y = 0\ .$$

6.1.4 Special cyclic surfaces

In this subsection we consider equivariant cyclic surfaces with extra conditions on $f^*\omega_{-1}$ and show that for these special equivariant cyclic surfaces are rigid.

Proposition 6.1.33. *Let $(S, J) = \Sigma$ be a compact Riemann surface, G be a complex simple Lie group of rank at least 2 and not $\mathsf{SL}(3,\mathbb{C})$. Let $\rho : \pi_1(S)\to\mathsf{G}$ be an irreducible representation and $f : \widetilde{\Sigma}\to\mathsf{G}/\mathsf{T}_0$ be a ρ-equivariant G_0-cyclic surface so that $f^*\omega_{-\alpha_i} \neq 0$ for all simple roots α_i. If ξ is an infinitesimal variation with the property that there exists a simple root α with $\iota_\xi\omega_{-\alpha} \equiv 0$, then*

$$\iota_\xi\omega \equiv 0.$$

Remark 6.1.34. The analogous statement follows for G-cyclic surfaces if one assumes that there are simple roots α and β so that $\iota_\xi\omega_{-\alpha} \equiv 0 \equiv \iota_\xi\omega_{+\beta}$. For G_0-cyclic surfaces, if $\iota_\xi\omega_{-\alpha} \equiv 0$, the reality condition $\Lambda\xi = \xi$ on an infinitesimal variation implies that $\iota_\xi\omega_{\Lambda(-\alpha)} \equiv 0$. Furthermore, since Θ flips positive simple roots and negative simple roots, σ preserves the set of positive simple roots and $\Lambda = \Theta\circ\sigma$, it follows that $\Lambda(-\alpha)$ is a positive simple root. If G_0 is of Hodge type, then $\Lambda(-\alpha) = \alpha$.

Proof. Let ξ be a variation of the ρ-equivariant G_0-cyclic surface $f : \widetilde{\Sigma}\to\mathsf{G}/\mathsf{T}_0$, and $\zeta = \iota_\xi\omega$. Using the decompositions of (6.1.7) and (6.1.8), by Corollary 6.1.31,

$$\widehat{\zeta}_0 = 0 \qquad \text{and} \qquad \widehat{\zeta}_Y = 0\ .$$

It remains to show $\widehat{\zeta}_1 = 0 = \widehat{\zeta}_{-1}$. Recall that $\mathsf{G} \neq \mathsf{SL}(3,\mathbb{C})$, thus, $\widehat{\mathfrak{g}}_Y \neq \{0\}$, in particular $\mathfrak{g}_{\pm 2} \neq \{0\}$. A infinitesimal variation ξ of a G_0-cyclic surface satisfies the reality condition $\Lambda\xi = \xi$. By Lemma 2.1.10, $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for all roots, and by Proposition 2.1.41, the involution σ sends roots simple root spaces to simple root spaces. Since there is a simple root α so that $\zeta_{-\alpha} \equiv 0$ and $\Lambda\zeta = \zeta$, it follows that there is a simple root $-\Lambda(\alpha)$ so that $\zeta_{-\Lambda\alpha} \equiv 0$.

By equation (6.1.12), we have

$$0 = \partial^{\nabla^c}(\zeta_{-2}) = 2[\zeta_{-1}, \Phi_{-1}] \qquad \text{and} \qquad 0 = \bar\partial^{\nabla^c}(\zeta_2) = 2[\zeta_1, \Phi_1^*] \ .$$

Thus for each pair of simple roots α_i, α_j so that $\alpha_i + \alpha_j$ is a root, we have

$$[\zeta_{-\alpha_i}, \Phi_{-\alpha_j}] + [\zeta_{-\alpha_j}, \Phi_{-\alpha_i}] = 0 \qquad \text{and} \qquad [\zeta_{\alpha_i}, \Phi^*_{\alpha_j}] + [\zeta_{\alpha_j}, \Phi^*_{\alpha_i}] = 0 \ .$$

Since $\Phi_{-\alpha_i} = f^*\omega_{-\alpha_i}$, by assumption $\Phi_{-\alpha_i}$ is a nonzero holomorphic section. By the definition of a G_0-cyclic surface,

$$f^*\Theta(\Phi_{\alpha_i}) = f^*(\Theta\omega_{-\alpha_i}) = -f^*(\omega_{\alpha_i}) = -\Phi^*_{\alpha_i}.$$

Thus, $\Phi^*_{\alpha_i}$ is also nonzero for all simple roots.

The group G is simple, thus the Dynkin diagram is connected and we conclude that $\zeta_{\pm\alpha_i} = 0$ for all simple roots. It remains to show that for the highest root μ, we have $\zeta_{\pm m_\ell} = \zeta_{\pm\mu} = 0$. By equations (6.1.12), we have

$$0 = \bar\partial^{\nabla^c}(\zeta_{m_\ell - 1}) = 2[\zeta_{m_\ell}, \Phi_1^*] \qquad \text{and} \qquad 0 = \partial^{\nabla^c}(\zeta_{-m_\ell+1}) = 2[\zeta_{m_\ell}, \Phi_{-1}] \ .$$

Since $\mathsf{G} \neq \mathsf{SL}(3,\mathbb{C})$, we have $\mathfrak{g}_{\pm 1} \neq \mathfrak{g}_{\pm(m_\ell - 1)} \neq \{0\}$. Thus, for each roots $\gamma = \mu - \alpha_i \in \mathfrak{g}_{m_\ell - 1}$ we have $0 = [\zeta_\mu, \Phi_{-\alpha_i}]$. Hence $\zeta_\mu = 0$, and similarly, $\zeta_{-\mu} = 0$. $\square$

Remark 6.1.35. Proposition 6.1.33 is also true when $\mathsf{G} = \mathsf{SL}(3,\mathbb{C})$, see Proposition 7.7.4 of [Lab14].

In [Lab14], Labourie considers maps $f : S \to \mathsf{G}/\mathsf{T}_0$ from the surface S, without a conformal structure, to the space of Hitchin Triples G/T_0 that satisfy $f^*\widehat\omega_j = 0$ for $j \neq \pm 1$, $f^*(\Theta(\widehat\omega_{-1})) = -f^*(\widehat\omega_1)$, $f^*([\widehat\omega_{-1}, \widehat\omega_{-1}]) = 0$, $f^*(\Lambda\omega) = f^*\omega$ and satisfy the extra assumption that *for all* simple roots α_i,

$$f^*\omega_{-\alpha_i} \quad \text{is nowhere vanishing.}$$

It is then proven that there is a unique conformal structure on S so that $f^*\widehat\omega_{-1}$ is a $(1,0)$-form.

Proposition 6.1.36. *Let rank$(\mathsf{G}) \geq 2$, a map $f : S \to \mathsf{G}/\mathsf{T}$ satisfies: $f^*\widehat\omega_j = 0$ for $j \neq \pm 1$ and*

$$f^*(\Theta(\widehat\omega_{-1})) = -f^*(\widehat\omega_1) \ , \qquad f^*([\widehat\omega_{-1}, \widehat\omega_{-1}]) = 0 \ , \qquad f^*(\Lambda\omega) = f^*\omega.$$

Suppose that $f^*\omega_{\alpha_i}$ has discrete zeros for all simple roots α_i and that there exists a simple root β so that

$$f^*\omega_{-\beta} \quad \textit{is nowhere vanishing.}$$

Then there exists a unique conformal structure $(S, J) = \Sigma$, so that $f : \Sigma \to \mathsf{G}/\mathsf{T}$ is a cyclic surface.

Thus, Definition 6.1.17 and the cyclic surfaces in Proposition 6.1.33 are generalizations of the cyclic surfaces in [Lab14]. The cyclic surfaces related to maximal $\mathsf{Sp}(4, \mathbb{R})$ representations are more special than those considered in Proposition 6.1.33 and more general than Labourie's. Namely, we only require that there exists a simple root α_i so that $f^*\omega_{-\alpha_i}$ is nowhere vanishing.

Proof. Let $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{c})$ be a simple roots for which $f^*\omega_{-\beta}$ is nowhere vanishing. Since $f^*\omega$ is nowhere vanishing, $df : TS \to [\mathfrak{g}_{-\beta}]$ is an isomorphism. Thus, there is a unique complex structure $(S, J) = \Sigma$ so that $f^*\omega_{-\beta}$ is a $(1, 0)$-form.

Since $f^*([\widehat{\omega}_{-1}, \widehat{\omega}_{-1}]) = 0$, decomposing this in terms of root spaces we have for all simple roots α and γ

$$[f^*\omega_{-\alpha}, f^*\omega_{-\gamma}] = 0 \qquad \text{and} \qquad [f^*\omega_{-\alpha}, f^*\omega_{\mu}] = 0\ .$$

Recall that $\mathfrak{g}$ is simple, so there is a simple root α so that $-\beta - \alpha$ is a root, in particular,

$$[[\mathfrak{g}_{-\alpha}], [\mathfrak{g}_{-\beta}]] \neq 0.$$

By $[f^*\omega_{-\alpha}, f^*\omega_{-\beta}] = 0$, it follows that $f^*\omega_{-\alpha}$ is a $(1, 0)$-form. Using the fact that $\mathfrak{g}$ is simple and that $f^*\omega_{-\alpha}$ has discrete zeros, we conclude that for all simple roots α, the form $f^*\omega_{-\alpha}$ is a $(1, 0)$-form. Similarly, there is there is a simple root α so that $\mu - \alpha$ is a root. We again conclude that $f^*\omega_{\mu}$ is a $(1, 0)$-form, proving $f^*\widehat{\omega}_{-1}$ is a $(1, 0)$-form. ∎

Putting everything together, we obtain the following theorem which is the analogue to the transversality of the Hitchin map in [Lab14].

Theorem 6.1.37. *Let G be a complex simple Lie group of rank at least 2, $\rho : \pi_1(S) \to \mathsf{G}$ an irreducible representation, and $(S, J) = \Sigma$ be a conformal structure. Suppose $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{T}_0$ a ρ-equivariant G_0-cyclic surface such that there exists a simple root α so that $f^*\omega_{-\alpha}$ is nowhere vanishing and, for all simple roots α_i, the form $f^*\omega_{-\alpha_i}$ is nonzero. Let (ρ_t, J_t) is a one parameter family with $(\rho_0, J_0) = (\rho, J)$ and $f_t : \widetilde{(S, J_t)} \to \mathsf{G}/\mathsf{T}_0$ be a family of ρ_t-equivariant G_0-cyclic surfaces with $f_0 = f$. If $[\frac{d}{dt}\big|_{t=0} \rho_t] = 0$, then $\frac{d}{dt}\big|_{t=0} J_t = 0$.*

Proof. Let $\rho \in \mathcal{X}(\mathsf{G})$ be an irreducible representation and let $(S, J) = \Sigma$ be a conformal structure. Let $f : \widetilde{\Sigma} \to \mathsf{G}/\mathsf{T}_0$ be a ρ-equivariant G_0-cyclic surface so that there is a simple root α with $f^*\omega_{-\alpha}$ nowhere vanishing and, for all simple roots α_i, the form $f^*\omega_{-\alpha_i} \neq 0$. Suppose (ρ_t, J_t) is a one parameter family and $f_t : \widetilde{(S, J_t)} \to \mathsf{G}/\mathsf{T}_0$ is a family of ρ_t-equivariant G_0-cyclic surfaces with $f_0 = f$, that is for all $\gamma \in \pi_1(S)$,

$$f_t(\gamma(s)) = \rho_t(\gamma) \cdot f_t(s).$$

If $[\frac{d}{dt}\big|_{t=0} \rho_t] = [\rho]$, then the tangent space at ρ is given by $T_\rho\mathcal{X}(\mathsf{G}) = H^1_\rho(S, \mathfrak{g})$ since ρ is irreducible (see (3.1.3)). Thus, after conjugating the family ρ_t by a family of elements of G, and preforming a similar transformation for f_t, for all $\gamma \in \pi_1(S)$ we have

$$\frac{d}{dt}\bigg|_{t=0} f_t(\gamma(s)) = \rho(\gamma) \cdot \frac{d}{dt}\bigg|_{t=0} f_t(s).$$

In particular, $\xi(s) = \frac{d}{dt}\big|_{t=0} f_t(s)$ is an ρ-equivariant infinitesimal deformation of f. Since $f^*\omega_{-\alpha}$ is nowhere vanishing, $f^*\omega_{-\alpha} : T\Sigma \to [\mathfrak{g}_{-\alpha}]$ is a bijection. Let X be the vector field along Σ so that $\iota_{\omega_{-\alpha}}\xi = f^*\omega_{-\alpha(X)}$, then $df(X)$ is an infinitesimal variation of f. By construction, $\xi - df(X)$ is an equivariant infinitesimal variation of f which vanishes along the simple root space $[\mathfrak{g}_\alpha]$. Thus, by Proposition 6.1.33, $\xi - df(X) = 0$.

To see that $\frac{d}{dt}\big|_{t=0} J_t = 0$, we employ an argument of Marco Spinaci [Spi]. We have

$$\xi = \frac{\partial f_t}{\partial t}\bigg|_{t=0} = df(X),$$

thus

$$\zeta = \omega(df_0(X)) = \Phi(X) + \Phi^*(X).$$

In particular, ζ is self adjoint and hence lives in the subbundle $[i\mathfrak{k}]$. Also, $\widehat{\zeta}_{-1} = \Phi(X)$ is holomorphic and $\widehat{\zeta}_1 = \Phi^*(X)$ is antiholomorphic.

Let $\Psi_t = f_t^*\omega = \Phi_t + \Phi_t$, by definition, for all tangent vectors, we have

$$\Psi_t(J_t v) = i\Phi_t(v) - i\Phi_t^*(v) = \left(i\pi_{\widehat{\mathfrak{g}}_{-1}} - i\pi_{\widehat{\mathfrak{g}}_1}\right)\Psi_t(v). \tag{6.1.22}$$

Recall that, for vector fields Y and Y' on G/T_0, we have

$$\omega(d^\nabla_Y Y') = \omega(d^\nabla_{Y'} Y) + \omega([Y, Y']) + \omega(T(Y, Y')) \tag{6.1.23}$$

where $T(Y, Y')$ is the torsion tensor given by Lemma 2.2.4. Differentiating equation (6.1.22) yields

$$d^{f_t^*\nabla^c}_{\frac{\partial}{\partial t}}\left(f_t^*\omega(J_t v)\right)\Big|_{t=0} = \left(i\pi_{\widehat{\mathfrak{g}}_{-1}} - i\pi_{\widehat{\mathfrak{g}}_1}\right) d^{f_t^*\nabla^c}_{\frac{\partial}{\partial t}}\left(f_t^*\omega(v)\right)\Big|_{t=0}.$$

Using the pullback of equation (6.1.23) by f_t, the left hand side of the above equations is given by

$$\left(d^{f_t^*\nabla^c}_{J_t v}\left(f_t^*\omega\left(\frac{\partial}{\partial t}\right)\right) + f_t^*\omega\left[\frac{\partial}{\partial t}, J_t v\right] + f_t^*\omega\left(T\left(\frac{\partial}{\partial t}, J_t v\right)\right)\right)\Big|_{t=0}.$$

The expression for the torsion in Lemma 2.2.4 and the decomposition $\mathfrak{g} = \mathfrak{t}_0 \oplus \mathfrak{m}$ imply the above expression can be rewritten as

$$\left(d^{f_t^*\nabla^c}_{J_t v}\left(\omega\left(\frac{\partial f_t}{\partial t}\right)\right) + \omega\left(df_t\left(\frac{\partial J_t}{\partial t}v\right)\right) + \pi_{[\mathfrak{m}]}\left(\left[f_t^*\omega\left(\frac{\partial}{\partial t}\right), f_t^*\omega(J_t v)\right]\right)\right)\Big|_{t=0}.$$

Using $\Psi_0 = f_0^*\omega$ and $\zeta = \frac{\partial f_t}{\partial t}\Big|_0$, evaluating at $t = 0$ yields

$$d^{f_0^*\nabla^c}_{J_0 v}(\zeta) + \Psi_0\left(\frac{\partial J_t}{\partial t}\Big|_0 v\right) + \pi_{[\mathfrak{m}]}\left(\Psi_0(J_0 v), \zeta\right).$$

Since $\left[\frac{\partial}{\partial t}, v\right] = 0$, a similar computation shows that the left hand side of equation (6.1.22) is

$$d^{f_0^*\nabla^c}_{v}(\zeta) + \pi_{[\mathfrak{m}]}\left(\Psi_0(v), \zeta\right).$$

Thus, we have

$$d^{f_0^*\nabla^c}_{J_0 v}(\zeta) + \Psi_0\left(\frac{\partial J_t}{\partial t}\Big|_0 v\right) + \pi_{[\mathfrak{m}]}\left(\Psi_0(J_0 v), \zeta\right) = \left(i\pi_{\widehat{\mathfrak{g}}_{-1}} - i\pi_{\widehat{\mathfrak{g}}_1}\right)\left(d^{f_0^*\nabla^c}_{v}(\zeta) + \pi_{[\mathfrak{m}]}\left(\Psi_0(v), \zeta\right)\right) \tag{6.1.24}$$

Recall $\zeta = \omega\left(\frac{\partial f_t}{\partial t}\Big|_0\right)$ is in $\omega(df(T\Sigma)) \subset [i\mathfrak{k}]$. Also, $\pi_{[\mathfrak{m}]}$ and $\left(i\pi_{\widehat{\mathfrak{g}}_{-1}} - i\pi_{\widehat{\mathfrak{g}}_1}\right)$ commute with the Cartan involution Θ. Thus, we can consider the $[i\mathfrak{k}]$ part of equation (6.1.24). This yields

$$d^{f_0^*\nabla^c}_{J_0 v}(\zeta) + \Psi_0\left(\frac{\partial J_t}{\partial t}\Big|_0 v\right) = \left(i\pi_{\widehat{\mathfrak{g}}_{-1}} - i\pi_{\widehat{\mathfrak{g}}_1}\right)\left(d^{f_0^*\nabla^c}_{v}(\zeta)\right). \tag{6.1.25}$$

Rearranging the equations and using the fact that $\widehat{\zeta}_{-1}$ is holomorphic and $\widehat{\zeta}_1$ is antiholomorphic gives

$$\Psi_0\left(\frac{\partial J_t}{\partial t}\Big|_0 v\right) = 2i(\bar{\partial}\widehat{\zeta}_{-1} - \partial\widehat{\zeta}_1) = 0.$$

Since Ψ is injective, it follows that $\left.\frac{\partial J_t}{\partial t}\right|_{t=0} = 0$, as desired. $\square$

6.2 Maximal $\mathsf{Sp}(4,\mathbb{R})$ & $\mathsf{SO}_0(2,3)$ representations and Higgs bundles

We will now apply the theory and results of cyclic surfaces to special components of maximal $\mathsf{Sp}(4,\mathbb{R})$ and $\mathsf{SO}_0(2,3) \cong \mathsf{PSp}(4,\mathbb{R})$-Higgs bundles which we will call Gothen representations. For the $\mathsf{Sp}(4,\mathbb{R})$ Gothen components (or equivalently the $\mathsf{PSp}(4,\mathbb{R})$-Gothen representations which lift to $\mathsf{Sp}(4,\mathbb{R})$) the below results were published in [Col15].

For $\mathsf{G} = \mathsf{Sp}(4,\mathbb{R})$, the complexification of the maximal compact subgroup is $\mathsf{H}_{\mathbb{C}} = \mathsf{GL}(2,\mathbb{C})$. For a $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ), $\tau = deg(V) \in \mathbb{Z}$ defines an integer invariant called the Toledo invariant. Given two $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundles (V,β,γ) and (V',β',γ'), if $deg(V) \neq deg(V')$ then (V,β,γ) and (V',β',γ') are in different connected components. This gives a decomposition

$$\mathcal{M}_J(\mathsf{Sp}(4,\mathbb{R})) = \bigsqcup_{\tau\in\mathbb{Z}} \mathcal{M}_{J,\tau}(\mathsf{Sp}(4,\mathbb{R})).$$

The map sending (V,β,γ) to (V^*,γ,β) gives an isomorphism $\mathcal{M}_{J,\tau}(\mathsf{Sp}(4,\mathbb{R})) \cong \mathcal{M}_{J,-\tau}(\mathsf{Sp}(4,\mathbb{R}))$. The invariant τ satisfies a Milnor-Wood inequality $|\tau| \leq 2g-2$ and

$$\mathcal{M}_J(\mathsf{Sp}(4,\mathbb{R})) = \bigsqcup_{-2g+2\leq\tau\leq 2g-2} \mathcal{M}_{J,\tau}(\mathsf{Sp}(4,\mathbb{R})).$$

We will show that Milnor-Wood inequality for $\mathsf{PSp}(4,\mathbb{R}) = \mathsf{SO}_0(2,3)$-Higgs bundles, below. Gothen [Got01] showed that, for $\tau = 0$, $\mathcal{M}_{J,\tau}(\mathsf{Sp}(4,\mathbb{R}))$ is connected, and, for $|\tau| = 2g-2$, the moduli space $\mathcal{M}_{J,\tau}(\mathsf{Sp}(4,\mathbb{R}))$ has $3^{2g}+2g-4$ connected components. In [GPMiR04], it is shown that $\mathcal{M}_{J,\tau}(\mathsf{Sp}(4,\mathbb{R}))$ is connected for all other values of the Toledo invariant. This gives $1+2(2g-1)+2(3^{2g}+2g-4)$ total connected components for $\mathcal{M}_J(\mathsf{Sp}(4,\mathbb{R}))$.

$\mathsf{SO}_0(2,3)$-Higgs bundles

To use vector bundles for the group $\mathsf{PSp}(4,\mathbb{R})$ we will make use of the low dimensional isomorphism $\mathsf{SO}_0(2,3) \cong \mathsf{PSp}(4,\mathbb{R})$. This works as follows, let W be a 4 dimensional vector space with a symplectic form $\Omega \in \Lambda^2 W$. The 6 dimensional vector space $\Lambda^2 W$ has a natural orthogonal structure of signature $(3,3)$. Since $\Omega \in \Lambda^2 W$, the orthogonal complement of the 1-dimensional subspace spanned by Ω defines a

5 dimensional vector space with an orthogonal structure of signature $(2,3)$. This defines a surjective map $\mathsf{Sp}(4,\mathbb{R})\to\mathsf{SO}_0(2,3)$. Furthermore, since we are taking the second exterior product, the kernel of this map is $\pm Id$, giving an isomorphism

$$\mathsf{Sp}(4,\mathbb{R})/\{\pm Id\} = \mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}_0(2,3).$$

An $\mathsf{SO}_0(2,3)$-Higgs bundle is determined by a holomorphic $\mathsf{SO}(2,\mathbb{C})$ bundle $(L\oplus L^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, a holomorphic $\mathsf{SO}(3,\mathbb{C})$-bundle (W, Q_W) and two holomorphic bundle maps β and γ where

$$\beta : L^{-1}\to W\otimes K \qquad\qquad \gamma : L\to W\otimes K.$$

There are two types of topological invariants, the degree of L and the second Stiefel-Whitney class of (W, Q_W) which we will denote by

$$\tau = deg(L) \in \mathbb{Z} \qquad\qquad w_2 \in \mathbb{Z}/2\mathbb{Z}.$$

The associated $\mathsf{SL}(5,\mathbb{C})$-Higgs bundle associated to a quadruple (L, W, β, γ) is $(L\oplus W\oplus L^{-1}, \phi)$ where ϕ consists of the maps γ and β along with the induced maps $\gamma^T : W\to L^{-1}K$ and $\beta^T : W\to LK$ which are defined by $\gamma^T = \gamma^*\circ Q_W$ and $\beta^T = \beta^*\circ Q_W$:

$$W \xrightarrow{Q_W} W^* \xrightarrow{\gamma^*} L^{-1}K \qquad\qquad W \xrightarrow{Q_W} W^* \xrightarrow{\beta^*} LK\ .$$

We will represent these pictorially by

$$\begin{tikzcd} L \arrow[r, bend right, "\gamma"'] & W \arrow[l, bend right, "\beta^T"'] \arrow[r, bend right, "\gamma^T"'] & L^{-1} \arrow[l, bend right, "\beta"'] \end{tikzcd}$$

where we have suppressed the twisting by K from the notation. The bound $|\tau| \leq 2g-2$ on the Toledo invariant can be see by considering the following compositions

$$LK^{-1} \xrightarrow[\gamma]{} W \xrightarrow[Q_W]{} W^* \xrightarrow[\gamma^*]{} L^{-1}K\ .$$

$$L^{-1}K^{-1} \xrightarrow[\beta]{} W \xrightarrow[Q_W]{} W^* \xrightarrow[\beta^*]{} LK$$

If $deg(L) > 0$ then by Remark 6.2.5, $\gamma \neq 0$ and thus $deg(L) \leq 2g-2$. Similarly, if $deg(L) < 0$ then $\beta \neq 0$, and thus $deg(L) \geq -2g+2$.

If $\mathcal{M}^{\tau,w_2}(\mathsf{SO}_0(2,3))$ denotes the moduli space with invariants τ and w_2, then, since the $deg(L)$ is bounded, we obtain the following decomposition of the moduli space of Higgs bundles and the character variety

$$\mathcal{M}(\mathsf{SO}_0(2,3)) = \coprod_{\substack{|\tau|\leq 2g-2\\ w_2\in\mathbb{Z}/2\mathbb{Z}}} \mathcal{M}^{\tau,w_2}(\mathsf{SO}_0(2,3)) \cong \coprod_{\substack{|\tau|\leq 2g-2\\ w_2\in\mathbb{Z}/2\mathbb{Z}}} \mathcal{X}^{\tau,w_2}(\mathsf{SO}_0(2,3)) = \mathcal{X}(\mathsf{SO}_0(2,3)).$$

The Higgs bundles and representation in the components with maximal Toledo invariant are called maximal.

Proposition 6.2.1. *Given an $\mathsf{Sp}(4,\mathbb{R})$-Higgs (V,β,γ) the associated $\mathsf{SO}_0(2,3)$-Higgs bundle is*

$$(L,W,\beta,\gamma) = (\Lambda^2 V, S^2V^*\otimes\Lambda^2 V,\beta,\gamma)$$

Proof. Given an $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ), the corresponding $\mathsf{SO}_0(2,3)$-Higgs bundle is determined by the map $\mathsf{Sp}(4,\mathbb{C})\to\mathsf{SO}(5,\mathbb{C})$. For the bundle, one takes the second exterior product

$$\Lambda^2(V\oplus V^*)\cong\Lambda^2(V)\oplus V\otimes V^*\oplus\Lambda^2(V^*) = \Lambda^2V\oplus\Lambda^2(V^*)\oplus Hom(V,V) \tag{6.2.1}$$

The orthogonal structure on this bundle is given by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ on $\Lambda^2(V)\oplus\Lambda^2(V^*)$ and the Killing form on $Hom(V,V)$ (i.e. $\langle A,B\rangle = Tr(AB)$). The symplectic structure $\Omega = \begin{pmatrix}0 & Id\\ -Id & 0\end{pmatrix}\in\Lambda^2(V^*\oplus V)$ corresponds to $Id\in Hom(V,V)$. If $Hom_0(V,V)$ is the space of traceless homomorphisms, then

$$\langle\Omega\rangle^\perp\subset\Lambda^2V\oplus Hom(V,V)\oplus\Lambda^2V^* = \Lambda^2V\oplus Hom_0(V,V)\oplus\Lambda^2V^*.$$

If $\mathcal{V}$ is the standard representation of $\mathsf{GL}(2,\mathbb{C})$ then it is straight forward to check that $Hom_0(\mathcal{V},\mathcal{V})$ is the representation $S^2\mathcal{V}\otimes\Lambda^2\mathcal{V}^*\cong S^2\mathcal{V}^*\otimes\Lambda^2\mathcal{V}$. Thus, $Hom_0(V,V) = S^2V\otimes\Lambda^2V^*\cong S^2(V^*)\otimes\Lambda^2V$. This gives $L=\Lambda^2V$ and $W = S^2V\otimes\Lambda^2V^*\cong S^2(V^*)\otimes\Lambda^2V$. Finally, note that $\gamma\in H^0(S^2V^*\otimes K) = H^0(L^{-1}\otimes W\otimes K)$ and $\beta\in H^0(S^2V\otimes K) = H^0(L\otimes W\otimes K)$. □

Using this correspondence and the Milnor wood in equality for $\mathsf{SO}_0(2,3)$-Higgs bundles one obtains the Milnor inequality for $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundles.

Proposition 6.2.2. *Let $\rho\in\mathcal{X}^{\tau,w_2}(\mathsf{SO}_0(2,3))$ then ρ lifts to $\tilde{\rho}\in\mathcal{X}^\tau(\mathsf{Sp}(4,\mathbb{R}))$ if and only if $w_2+\tau = 0$ mod 2.*

Proof. By a similar procedure as above, the groups complex groups $\mathsf{SO}(5,\mathbb{C})$ and $\mathsf{PSp}(4,\mathbb{C})$ are isomorphic. Since the cover $\mathsf{Sp}(4,\mathbb{C})\to\mathsf{SO}(5,\mathbb{C})$ is $2:1$, we have $\mathsf{Sp}(4,\mathbb{C})\cong\mathsf{Spin}(5,\mathbb{C})$. Thus, the representations $\rho:\pi_1\to\mathsf{SO}_0(2,3)$ will lift if and only if the second Stiefel class of the $\mathsf{SO}(5,\mathbb{C})$ bundle $(L\oplus L^{-1}\oplus W)$ is zero.

Since the first Stiefel-Whitney class of the bundle $L\oplus L^{-1}$ and (W,Q_W) is zero, if w is the total Stiefel-Whitney class then we have:

$$w(L\oplus L^{-1}\oplus W)=(1+w_2(L\oplus L^{-1}))\smile(1+w_2(W,Q_W))=1+w_2(L\oplus L^{-1}+w_2(W,Q_W).$$

Thus $w_2(L\oplus L^{-1}\oplus W)=degL\ mod\ 2+w_2(W,Q_W)$, which is 0 if and only if $w_2(W,Q_W)+\tau=0\ mod\ 2$. $\square$

Remark 6.2.3. For maximal $\mathsf{SO}_0(2,3)$-Higgs bundles, the corresponding representations will lift to $\mathsf{Sp}(4,\mathbb{R})$ if and only if $w_2=0$.

6.2.1 Maximal components for $\mathsf{Sp}(4,\mathbb{R})$ & $\mathsf{SO}_0(2,3)$

Higgs bundles (V,β,γ) with $|deg(V)|=|\tau|=2g-2$ are called maximal. When $\tau=2g-2$, polystablility forces the holomorphic map $\gamma:V\to V^*\otimes K$ to be an isomorphism [Got01]. Using this fact, to a maximal $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ) one associates a $\mathsf{GL}(2,\mathbb{R})$ K^2-twisted Higgs bundle (W,φ) (i.e. a $\mathsf{GL}(2,\mathbb{R})$-Higgs bundle where the Higgs field is twisted by K^2 instead of K), called its *Cayley partner*. The Cartan decomposition of $\mathfrak{gl}(2,\mathbb{R})$ is $\mathfrak{o}(2,\mathbb{R})\oplus sym(\mathbb{R}^2)$, and, complexifying, we have

$$\mathfrak{gl}(2,\mathbb{C})=\mathfrak{o}(2,\mathbb{C})\oplus sym(\mathbb{C}^2).$$

Thus, a K^2-twisted $\mathsf{GL}(2,\mathbb{R})$-Higgs bundle is a triple (W,Q_W,φ) where (W,Q_W) is a $\mathsf{O}(2,\mathbb{C})$ bundle and $\varphi\in H^0(End(W)\otimes K^2)$ satisfying $\varphi^TQ_W=Q_W\varphi$.

The characteristic classes of the Cayley partner help to distinguish connected components of $\mathcal{M}_J^{2g-2}(\mathsf{Sp}(4,\mathbb{R}))$. We will recall how this works for $\mathsf{Sp}(4,\mathbb{R})$ [Got01, BGPG12], for a general development of the theory of Cayley partners see [RN12]. Fix a square root of the canonical bundle $K^{\frac{1}{2}}$ and set $W=V^*\otimes K^{\frac{1}{2}}$. Using the fact that $\gamma:V\to V^*\otimes K$ is an isomorphism, define an orthogonal structure $Q_W:W^*\to W$ by

$$Q_W=\gamma\otimes Id_{K^{-\frac{1}{2}}}:V\otimes K^{-\frac{1}{2}}\longrightarrow V^*\otimes K\otimes K^{-\frac{1}{2}}. \tag{6.2.2}$$

For the Cayley partner, the Higgs field $\varphi:W\to W\otimes K^2$ is given by $\varphi=(\gamma\otimes Id_{K\otimes K^{\frac{1}{2}}})\circ(\beta\otimes K^{\frac{1}{2}})$, i.e.

$$W=\ V^*\otimes K^{\frac{1}{2}}\xrightarrow{\beta\otimes Id}V\otimes K\otimes K^{\frac{1}{2}}\xrightarrow{\gamma\otimes Id}V^*\otimes K\otimes K\otimes K^{\frac{1}{2}}\ =W\otimes K^2.$$

The map φ is Q_W-symmetric, thus (W,Q_W,φ) defines a K^2-twisted $\mathsf{GL}(2,\mathbb{R})$-Higgs bundle.

The $\mathsf{O}(2,\mathbb{C})$ bundle (W,Q_W) has a first and second Stiefel-Whitney class

$$w_1(W,Q_W)\in H^1(\Sigma,\mathbb{Z}/2\mathbb{Z})=(\mathbb{Z}/2\mathbb{Z})^{2g} \qquad \text{and} \qquad w_2(W,Q_W)\in H^2(\Sigma,\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}\ .$$

There are $2\cdot 2^{2g}-2$ possible values for $(w_1(W,Q_W),w_2(W,Q_W))$ with $w_1(W,Q_W)\neq 0$. When $w_1(W,Q_W)=0$, the structure group of the $\mathsf{O}(2,\mathbb{C})$-bundle lifts to $\mathsf{SO}(2,\mathbb{C})$, in this case, we have a *Chern class,* and *Proposition 3.20* of [BGPG12].

Proposition 6.2.4. *Let* (V,β,γ) *be a maximal* $\mathsf{Sp}(4,\mathbb{R})$*-Higgs bundle with Cayley partner* (W,Q_W) *and* $w_1(W,Q_W)=0$, *then there is a line bundle* $N\to\Sigma$ *so that* $V=N\oplus N^{-1}K$. *With respect to this decomposition,*

$$\beta=\begin{pmatrix}\nu & q_2\\ q_2 & \mu\end{pmatrix}:N^{-1}\oplus NK^{-1}\to NK\oplus N^{-1}K^2 \qquad \textit{and} \qquad \gamma=\begin{pmatrix}0&1\\1&0\end{pmatrix}:N\oplus N^{-1}K\to N^{-1}K\oplus N.$$

The line bundle N *satisfies a degree bound,* $g-1\leq deg(N)\leq 3g-3$; *for* $g-1<deg(N)$, *the line bundle* N *is* unique *and when* $deg(N)=g-1$, *the line bundle* N *is unique up to a multiple of a square root of* $\mathcal{O}$.

The proof of this proposition makes extensive use of Mumford's classification of rank 2 holomorphic orthogonal bundles [Mum71]. The degree of N provides $2g-1$ extra invariants; set $d=deg(N)$, and denote the corresponding moduli space by $\mathcal{M}^d_J(\mathsf{Sp}(4,\mathbb{R}))$. For $deg(N)=3g-3$, stability forces $N^2=K^3$, and there are at least 2^{2g} connected components corresponding to choices of square roots of K. Thus, there are

$$2\cdot 2^{2g}-2+2g-2+2^g \ = \ 3\cdot 2^{2g}+2g-4$$

invariants for $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundles with $deg(V)=2g-2$, and we have

$$\mathcal{M}^{2g-2}(\mathsf{Sp}(4,\mathbb{R}))=\coprod_{w_1\neq 0}\mathcal{M}^{2g-2}_{w_1,w_2}(\mathsf{Sp}(4,\mathbb{R}))\coprod_{g-1\leq d<3g-3}\mathcal{M}^{2g-2}_d(\mathsf{Sp}(4,\mathbb{R}))\coprod_{L^2=K}\mathcal{M}^{2g-2}_L(\mathsf{Sp}(4,\mathbb{R})). \tag{6.2.3}$$

In [Got01], it is shown that each of the above moduli space is nonempty and connected. For $deg(N)=3g-3$, the 2^{2g} connected components are the Hitchin components. When $g-1\leq d<3g-3$, we call the components $\mathcal{M}^d_J(\mathsf{Sp}(4,\mathbb{R}))$ the *Gothen components.*

Remark 6.2.5. By Proposition 6.2.4, the $\mathsf{SL}(4,\mathbb{C})$-Higgs bundle associated to a Higgs bundle $(V,\beta,\gamma)\in\mathcal{M}^d_J(\mathsf{Sp}(4,\mathbb{R}))$ is of the form

$$(E,\phi)=\left(N\oplus N^{-1}K\oplus N^{-1}\oplus NK^1,\begin{pmatrix} & & \nu & q_2\\ & & q_2 & \mu\\ 0&1& & \\ 1&0& & \end{pmatrix}\right) \tag{6.2.4}$$

If $\mu = 0$, then $N \oplus NK^{-1}$ is an invariant subbundle of E with of degree $2d - 2g + 2$. Thus, for $g - 1 < d$ stability forces $\mu \neq 0$. Furthermore, by Proposition 3.24 of [BGPG12], for $g - 1 < d$, all isomorphism classes in $\mathcal{M}_J^d(\mathsf{Sp}(4, \mathbb{R}))$ are stable and simple. When $d = g - 1$, the Higgs bundle is stable if and only if $\mu \neq 0$. It follows that $\mathcal{M}_J^d(\mathsf{Sp}(4, \mathbb{R}))$, and hence $\mathcal{X}_d^{2g-2}(\mathsf{Sp}(4, \mathbb{R}))$, is *smooth* if and only if $g - 1 < d \leq 3g - 3$.

Let $\mathcal{X}_d^{2g-2}(\mathsf{Sp}(4, \mathbb{R}))$ be the component of Gothen representations which corresponds to $\mathcal{M}_d^{2g-2}(\mathsf{Sp}(4, \mathbb{R}))$. Using the description of the possible Zariski closures of maximal $\mathsf{Sp}(4, \mathbb{R})$ representations of [BIW10], Bradlow, Garcia-Prada, and Gothen showed [BGPG12], if $g - 1 < d < 3g - 3$ and $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{Sp}(4, \mathbb{R}))$, then ρ is Zariski dense. Furthermore, by Remark 6.2.5, the Gothen components $\mathcal{X}_d^{2g-2}(\mathsf{Sp}(4, \mathbb{R}))$ for $g - 1 < d < 3g - 3$ are smooth.

Maximal $\mathsf{SO}_0(2, 3)$-Higgs bundles

Maximal $\mathsf{SO}_0(2, 3)$-Higgs bundles satisfy the following important extra symmetry.

Proposition 6.2.6. *If (L, W, β, γ) be a maximal $\mathsf{SO}_0(2, 3)$-Higgs bundle with $\tau = 2g - 2$ then the map $\gamma : L \to W \otimes K$ is nowhere vanishing.*

Proof. Note that γ cannot be zero by stability. Consider the composition

$$LK^{-1} \xrightarrow{\gamma} W \xrightarrow{Q} W^* \xrightarrow{\gamma^*} L^{-1}K$$

Then $\gamma \circ Q \circ \gamma^* \in H^0(L^{-2}K^2)$ is a nonzero section. But since $deg(L) = 2g - 2$ we must have $L^{-2}K^2 = \mathcal{O}$ and $\gamma \circ Q \circ \gamma^*$ nowhere vanishing, proving γ is injective. ☐

Proposition 6.2.7. *If (L, W, β, γ) be a maximal $\mathsf{SO}_0(2, 3)$-Higgs bundle with $\tau = 2g - 2$ then $(LK^{-1})^2 = \mathcal{O}$ is an $\mathsf{O}(1, \mathbb{C})$ bundle and W decomposes holomorphically and orthogonally as an $W = LK^{-1} \oplus (V, Q_V)$ where (V, Q_V) is a $\mathsf{O}(2, \mathbb{C})$-bundle.*

Proof. By the previous proposition, $(LK^{-1})^2 = \mathcal{O}$, thus LK^{-1} is an $\mathsf{O}(1, \mathbb{C})$-bundle and W decomposes as the image of $\gamma : LK^{-1} \to W$ and its orthogonal complement which is an $\mathsf{O}(2, \mathbb{C})$-bundle. ☐

For maximal Higgs bundles, the decomposition $(W, Q_W) = (LK^{-1}, q) \oplus (V, Q_V)$ gives finner topological invariants. Namely the first Stiefel-Whitney class of (LK^{-1}, q) and (V, Q_V) and the second Stiefel-Whitney class of (V, Q_V). However we have

$$w(W, Q_W) = (1 + w_2(W, Q_W) = (1 + w_1(LK^{-1}) \smile (1 + w_1(V, Q_V) + w_2(W, Q_W)$$

$$= 1 + w_1(LK^{-1}, q) + w_1(V, Q_V) + w_1(LK^{-1}, q) \smile w_1(V, Q_V) + w_2(W, Q_W)$$

Thus, $w_1(LK^{-1}, q) = w_1(V, Q_V)$ and thus $w_2(V, Q_V) = w_2(W, Q_W)$. Let $\mathcal{M}^{2g-2,w_2}_{w_1}(\mathsf{SO}_0(2,3)$ be the space of maximal polystable $\mathsf{SO}_0(2,3)$-Higgs bundles with $w_1(V, Q_V) = w_1(L^{-1}K) = w_1$ and $w_2(V, Q_V) = w_2(W, Q_W) = w_2$.

Proposition 6.2.8. *An $\mathsf{SO}_0(2,3)$-Higgs bundle in $\mathcal{M}^{2g-2,w_2}_{w_1}(\mathsf{SO}_0(2,3))$ is determined by a triple $((V, Q_V), \beta', q_2)$ where*

- *(V, Q_V) is an orthogonal bundle with first and second Steifel-Whitney classes w_1 and w_2.*
- *$q_2 \in H^0(K^2)$ and $\beta_V \in H^0(V \otimes LK)$ where $L^{-1}K$ is the $\mathsf{O}(1,\mathbb{C})$-bundle with Stiefel-Whitney class w_1.*

Proof. Let (L, W, β, γ) be a polystable $\mathsf{SO}_0(2,3)$-Higgs bundle. By Proposition 6.2.7, if the isomorphism class of (L, W, β, γ) is in $\mathcal{M}^{2g-2,w_2}_{w_1}(\mathsf{SO}_0(2,3))$ then $(W, Q_W) = LK^{-1} \oplus (V, Q_V)$. Here (V, Q_V) is a holomorphic $\mathsf{O}(2,\mathbb{C})$ bundle with $w_1(V, Q_V) = w_1$ and $w_2(V, Q_V) = w_2$ and $L^{-1}K$ is a holomorphic $\mathsf{O}(1,\mathbb{C})$-bundle with $w_1(L^{-1}K) = w_1$. Since the holomorphic splitting of W was determined by the image of γ, we have

$$(\gamma, \beta) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} q_2 \\ \beta_V \end{pmatrix}\right) : L \oplus L^{-1} \longrightarrow (LK^{-1} \oplus V) \otimes K$$

where $q_2 \in H^0(K^2)$ and $\beta_V \in H^0(V \otimes LK)$. □

Proposition 6.2.9. *An $\mathsf{SO}_0(2,3)$-Higgs bundle in $\mathcal{M}^{2g-2,w_2}_{0}(\mathsf{SO}_0(2,3))$ is determined by a quadruple (M, μ, ν, q_2) where M is a holomorphic line bundle with $0 \leq deg(M) \leq 4g-4$, $\mu \in H^0(M^{-1}K^2)$, $\nu \in H^0(MK^2)$ and $q_2 \in H^0(K^2)$. Furthermore, if $d > 0$ then $\mu \neq 0$ and if $d = 0$ then $\mu = 0 = \nu$ or $\mu \neq 0 \neq \nu$.*

Proof. Let (L, W, β, γ) be a polystable $\mathsf{SO}_0(2,3)$-Higgs bundle. By Proposition 6.2.8, if the isomorphism class of (L, W, β, γ) is in $\mathcal{M}^{2g-2,w_2}_{0}(\mathsf{SO}_0(2,3))$ then (L, W, β, γ) is determined by a triple $((V, Q_V), \beta_V, q_2)$ where $w_1(V) = 0$ and $L^{-1}K = \mathcal{O}$. Thus (V, Q_V) reduces to an $\mathsf{SO}(2,\mathbb{C})$ bundle and there is an holomorphic line bundle M so that $(V, Q_V) = \left(M \oplus M^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$. The map β_V is given by

$$\beta_V := \begin{pmatrix} \nu \\ \mu \end{pmatrix} : K^{-1} \to M \oplus M^{-1}$$

where $\nu \in H^0(MK^2)$ and $\mu \in H^0(M^{-1}K^2)$. The associated $\mathsf{SL}(5,\mathbb{C})$ Higgs bundle is

$$\begin{tikzcd} M \arrow[r, "\mu"'] & K \arrow[r, "1"'] & \mathcal{O} \arrow[r, "1"'] \arrow[l, bend right, "q_2"'] & K^{-1} \arrow[r, "\mu"'] \arrow[l, bend right, "q_2"'] \arrow[lll, bend right, "\nu"'] & M^{-1} \arrow[lll, bend right, "\nu"'] \end{tikzcd} . \tag{6.2.5}$$

Furthermore, if $deg(M) > 0$ then the holomorphic section $\mu \in H^0(M^{-1}K^2)$ must be nonzero or else M would be a positive invariant subbundle. If $deg(M) > 4g-4$ then the holomorphic section μ must be 0, thus $deg(M) \leq 4g-4$. A similar analysis shows that if $deg(M) < 0$ then $-4g+4 \leq deg(M)$ and $\nu \in H^0 = (MK^2) \setminus \{0\}$. Note that the determinant 1 orthogonal gauge transformations

$$g_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} : K \oplus K^{-1} \to K \oplus K^{-1} \quad \text{and} \quad g_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} : \mathcal{O} \oplus M \oplus M^{-1}$$

send the Higgs bundle determined by tuple (M, μ, ν, q_2) to the one determined by $(M^{-1}, -\nu, -\mu, q_2)$. So the Higgs bundles associated to (M, μ, ν, q_2) and $(M^{-1}, -\nu, -\mu, q_2)$ define the same isomorphism class and we may assume $deg(M) \geq 0$. If $d = 0$ and $\mu = 0$ then the corresponding $\mathsf{SL}(5,\mathbb{C})$-Higgs bundles will be polystable if and only if $\nu = 0$. Similarly, when $\nu = 0$, polystability forces $\mu = 0$. $\square$

The extra invariants for maximal $\mathsf{SO}_0(2,3)$-Higgs bundles given the following decomposition:

$$\mathcal{M}^{2g-2}(\mathsf{SO}_0(2,3)) = \coprod_{w_1 \neq 0} \mathcal{M}^{2g-2}_{w_1,w_2}(\mathsf{SO}_0(2,3)) \coprod_{0 \leq d \leq 4g-4} \mathcal{M}^{2g-2}_{d}(\mathsf{SO}_0(2,3)). \tag{6.2.6}$$

Remark 6.2.10. The second Stiefel-Whitney class of bundles in $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ is $d \ mod \ 2$, thus when d is even, the corresponding representations will lift to $\mathsf{Sp}(4,\mathbb{R})$. The component $\mathcal{M}^{2g-2}_{4g-4}(\mathsf{SO}_0(2,3))$ is the $\mathsf{SO}_0(2,3)$-Hitchin component $\mathsf{Hit}(\mathsf{SO}_0(2,3))$ (this will be explained below).

The relation between the invariants of maximal $\mathsf{SO}_0(2,3)$-Higgs bundles which lift to $\mathsf{Sp}(4,\mathbb{R})$ is as follows.

Proposition 6.2.11. *If $(L, W, \beta, \gamma) \in \mathcal{M}^{2g-2}(\mathsf{SO}_0(2,3))$ with $w_2(W, Q_W) = 0$, then the first Stiefel-Whitney class of LK^{-1} is the same as the first Stiefel-Whitney class invariant associated to the corresponding maximal $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle $(V, \tilde{\beta}, \tilde{\gamma})$. Moreover, if $w_1 = 0$ and $(L, W, \beta, \gamma) \in \mathcal{M}^{2g-2}_{2j}(\mathsf{SO}_0(2,3))$ then $(V, \beta, \gamma) \in \mathcal{M}^{2g-2}_{j+g-1}(\mathsf{Sp}(4,\mathbb{R}))$.*

Proof. Let (L, W, β, γ) is a maximal $\mathsf{SO}_0(2,3)$-Higgs bundle with $w_2(W) = 0$. If $(V, \tilde{\beta}, \tilde{\gamma})$ is a maximal $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle lifting (L, W, β, γ) then $L = \Lambda^2 V$. Recall that from the maximal $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle $(V, \tilde{\beta}, \tilde{\gamma})$, one constructs a $\mathsf{O}(2,\mathbb{C})$ bundle $V \otimes K^{-\frac{1}{2}}$ (see equation (6.2.2)). The first Stiefel-Whitney class of $V \otimes K^{-\frac{1}{2}}$ is the same as the first Stiefel-Whitney class of the determinant bundle $\Lambda^2(V \otimes K^{-\frac{1}{2}}) = (\Lambda^2 V) \otimes K^{-1}$. But the first Stiefel-Whitney class invariant of the maximal $\mathsf{SO}_0(2,3)$ bundle is the $w_1(LK^{-1})$, and $\Lambda^2 V = L$. Thus, if the maximal $\mathsf{SO}_0(2,3)$-Higgs bundle (L, W, β, γ) lies in $\mathcal{M}^{2g-2}_{w_1, w_2=0}(\mathsf{SO}_0(2,3))$, then the lifts $(V, \tilde{\beta}, \tilde{\gamma})$ lie in $\mathcal{M}^{2g-2}_{w_1,w_2=0}(\mathsf{Sp}(4,\mathbb{R})) \sqcup \mathcal{M}^{2g-2}_{w_1,w_2=1}(\mathsf{Sp}(4,\mathbb{R}))$.

If $w_1(LK^{-1}) = 0$ then $L = K$ and $W = \mathcal{O} \oplus M \oplus M^{-1}$ with $deg(M) = 2j$ since $w_2(W) = 0$. The lift of (L, W, β, γ) is a maximal $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle $(V, \tilde{\beta}, \tilde{\gamma})$ with first Stiefel-Whitney class invariant vanishing. Thus, it is given by a $V \oplus V^* = N \oplus N^{-1}K \oplus N^{-1} \oplus NK^{-1}$ with symplectic form $\Omega = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ and $deg(N) = d$ for some $d \in [g-1, 3g-3]$. To see that the degree of N is j, we need to calculate the $\mathsf{SO}(3,\mathbb{C})$ bundle W which arises from this $\mathsf{Sp}(4,R)$-Higgs bundle. The corresponding $\mathsf{SO}(3,\mathbb{C})$ bundle is given by taking the orthogonal complement of the symplectic form Ω inside $V \otimes V^*$. Using the decomposition $V = N \oplus N^{-1}K$ we have

$$V \otimes V^* = N^2K^{-1} \oplus \mathcal{O} \oplus \mathcal{O} \oplus N^{-2}K.$$

Moreover, the symplectic form defines a trivial subbundle of the $\mathcal{O} \oplus \mathcal{O}$. Thus $M = N^2K^{-1}$ and $deg(M) = 2deg(N) - 2g + 2$, proving the result. □

To determine the smooth points of $\mathcal{M}^{2g-2,sw_2}(\mathsf{SO}_0(2,3))$ we need a more refined notion of stability for $\mathsf{SO}_0(2,3)$-Higgs bundles. This is made precise in [Arr09] and [ACGP+16].

Definition 6.2.12. An $\mathsf{SO}_0(2,3)$-Higgs bundle $(L, (W, Q_W), \beta, \gamma)$ with $deg(L) \neq 0$ is *stable if, whenever $N \subset W$ is an isotropic subbundle, one of the following holds*

- *if* $\gamma(L) \subset N$ *and* $\beta^T(N) = \{0\}$ *then* $deg(L) + deg(N) < 0$,
- *if* $\beta(L^{-1}) \subset N$ *and* $\gamma^T(N) = \{0\}$ *then* $-deg(L) + deg(N) < 0$,
- *if* $N \neq \{0\}$, $\gamma^T(N) = \{0\}$ *and* $\beta^T(N) = \{0\}$ *then* $deg(N) < 0$.

In [Arr09] it is shown that the isomorphism class of an $\mathsf{SO}_0(2,3)$ Higgs bundle defines a smooth point of $\mathcal{M}(\mathsf{SO}_0(2,3))$ if and only if it is stable an simple.

Proposition 6.2.13. *For* $d \in (0, 4g-4]$, *the space* $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ *is smooth and an isomorphism class* $[(M, \mu, \nu, q_2)] \in \mathcal{M}^{2g-2}_0(\mathsf{SO}_0(2,3))$ *is smooth if and only if*

- $M \ncong M^{-1}$ *and* $\mu \neq 0 \neq \nu$ *or*
- $M \cong M^{-1}$, $\mu \neq 0 \neq \nu$ *and* $\mu \neq \lambda\nu$ *for all* $\lambda \in \mathbb{C}^*$.

Proof. By Proposition 6.2.9, a polystable $\mathsf{SO}_0(2,3)$-Higgs bundle whose isomorphism defines a point in $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ is determined by a tuple (M,μ,ν,q_2), the associated $\mathsf{SL}(5,\mathbb{C})$ Higgs bundle $(\mathcal{E},\phi)$ is given by (6.2.5). Consider holomorphic orthogonal gauge transformations

$$g_1=\begin{pmatrix} e^{\lambda} & 0\\ 0 & e^{-\lambda}\end{pmatrix}: K\oplus K^{-1}\to K\oplus K^{-1}\quad\text{ and }\quad g_2=\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & j\end{pmatrix}: M\oplus M^{-1}\oplus\mathcal{O}\to M\oplus M^{-1}\oplus\mathcal{O}.$$

The action of (g_1,g_2) on the Higgs field (β,γ) is by

$$(g_1,g_2)\cdot(\beta,\gamma)=(g_2\beta e^{\lambda},g_2\gamma e^{-\lambda}).$$

A straight forward calculation shows

$$g_2\gamma e^{\lambda}=g_2\begin{pmatrix}0\\0\\1\end{pmatrix}e^{\lambda}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

if and only if $g=h=0$ and $e^{\lambda}j=1$. Using $g=h=0$, another calculation shows $g_2^*Q_Wg_2=Q_W$ implies

$$c=0\qquad f=0\qquad ad=0\qquad be=0\qquad j^2=1\qquad ae+bd=1\ .$$

Since $det(g_2)=1$, we must have

$$(g_1,g_2)=\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}e^{\lambda}&0&0\\0&e^{-\lambda}&0\\0&0&1\end{pmatrix}\right)\quad\text{ or }\quad(g_1,g_2)=\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&e^{\lambda}&0\\e^{-\lambda}&0&0\\0&0&-1\end{pmatrix}\right).\tag{6.2.7}$$

The first type of gauge transformation acts on $\beta^T=(\mu,\nu,q_2)$ by $(e^{\lambda}\mu,e^{-\lambda}\nu,q_2)$. If (M,μ,ν,q_2) is a stable Higgs bundle then $\mu\neq 0$ and the first type of gauge transformation acts trivially if and only if it is the identity.

The second type of gauge transformation in (6.2.7) will be holomorphic if and only if $M\cong M^{-1}$ and can only occur when $d=0$. It acts on $\beta^T=(\mu,\nu,q_2)$ by $(-e^{\lambda}\nu,-e^{-\lambda}\mu,q_2)$. Thus, by Proposition 6.2.9, when $M\cong M^{-1}$ and $\mu=-e^{\lambda}\nu\neq 0$ the Higgs bundle associated to (M,μ,ν,q_2) is stable and not simple. $\square$

Since the moduli spaces $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ generalize the $\mathsf{Sp}(4,\mathbb{R})$-Gothen components, we will also call them Gothen representations.

Definition 6.2.14. The components $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ with $d>0$ are called the *Gothen components* and maximal representations in the corresponding components $\mathcal{X}_d^{2g-2}(\pi,\mathsf{SO}_0(2,3))$ are called *Gothen representations.*

Proposition 6.2.15. *For $0<d\leq 4g-4$ the Gothen components $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ are connected and parameterized by tuples (M,μ,ν,q_2) where M is a holomorphic line bundle of degree d so that $M^{-1}K^2$ has a nonzero holomorphic section, $\mu\in H^0(M^{-1}K^2)\setminus\{0\}$, $\nu\in H^0(MK^2)$ and $q_2\in H^0(K^2)$. Up to the $\mathbb{C}^*$ action $(M,\mu,\nu,q_2)\longrightarrow(M,\lambda\mu,\lambda^{-1}\nu,q_2)$ a tuples (M,μ,ν,q_2) parameterize points in $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$.*

Proof. By Proposition 6.2.11 a Higgs bundle in a Gothen component $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ is determined by a tuple $(M,\mu,\nu,q_2)\in Pic^d(\Sigma)\times(H^0(M^{-1}K)\setminus\{0\})\times H^0(MK^2)\times H^0(K^2)$. The corresponding $\mathsf{SO}_0(2,3)$-Higgs bundle is given by

$$(L,W,\beta,\gamma)=(K,\mathcal{O}\oplus M\oplus M^{-1},\begin{pmatrix}q_2\\ \nu\\ \mu\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix})$$

We will show that there is only a 1-parameter gauge symmetry of (L,W,β,γ) which sends (M,μ,ν,q_2) to another point in $Pic^d(\Sigma)\times(H^0(M^{-1}K)\setminus\{0\})\times H^0(MK^2)\times H^0(K^2)$. Denote the orthogonal structures on $K\oplus K^{-1}$ and $\mathcal{O}\oplus M\oplus M^{-1}$, recall that if $g_1\in\mathcal{G}_{\mathsf{SO}(2,\mathbb{C})}(K\oplus K^{-1},Q_1)$ and $g_2\in\mathcal{G}_{\mathsf{SO}(3,\mathbb{C})}(\mathcal{O}\oplus M\oplus M^{-1},Q_2)$ then

$$(g_1,g_2)\cdot\begin{pmatrix}\beta^T\\ \gamma^T\end{pmatrix})=g_1^{-1}\begin{pmatrix}\beta^T\\ \gamma^T\end{pmatrix}g_2$$

where $\begin{pmatrix}\beta^T\\ \gamma^T\end{pmatrix}=\eta^T:\mathcal{O}\oplus M\oplus M^{-1}\to K^2\oplus\mathcal{O}$. If the bundles are of the form $K\oplus K^{-1}$ and $\mathcal{O}\oplus M\oplus M^{-1}$ then the gauge transformations must preserve the orthogonal structure, a computation shows

$$g_1=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\qquad\text{and}\qquad g_2=\begin{pmatrix}1&0&-\lambda^{-1}b\\ b&\lambda&0\\ 0&0&\lambda^{-1}\end{pmatrix}$$

where $a,\lambda\in\mathbb{C}^*$, and $b\in H^0(M)$. By definition of γ^T and β^T we have

$$g_1^{-1}\begin{pmatrix}\beta^T\\ \gamma^T\end{pmatrix}g_2=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\begin{pmatrix}q_2&\mu&\nu\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&-\lambda^{-1}b\\ b&\lambda&0\\ 0&0&\lambda^{-1}\end{pmatrix}=\begin{pmatrix}\mu b+aq_2&a\lambda\mu&\nu a\lambda^{-1}-\lambda^{-1}baq_2\\ a^{-1}&0&-b\lambda^{-1}a^{-1}\end{pmatrix}$$

Thus $a=1$ and $b=0$ and there is a 1-parameter gauge symmetry g_λ acting as

$$g_\lambda\cdot(M,\mu,\nu,q_2)=(M,\lambda\mu,\lambda^{-1}\nu,q_2)$$

Up to this symmetry, every point in $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ is determined uniquely by a tuple (M, μ, ν, q_2). $\square$

Putting everything discussed above together, it is not hard to prove the following.

Theorem 6.2.16. *The moduli space $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ is diffeomorphic to $\mathcal{F}^d_\Sigma \times H^0(K^2)$ where $\mathcal{F}^d_\Sigma$ is a rank $d+3g-3$ vector bundle over the symmetric product $Sym^{-d+4g-4}(\Sigma)$.*

Proof. The set of divisors of degree $-d+4g-4$ on a Σ is parameterized by $Sym^{-d+4g-4}(\Sigma)$. A projective classes of $\mu \in H^0(M^{-1}K^2) \setminus \{0\}$ is in one-to-one correspondence with divisors of degree $-d+4g-4$. Since the line bundle M can be recovered from such a divisor by inverting and tensoring with K^2, the data $(M, [\mu])$ is one one-to-one correspondence with divisors D of degree $-d+4g-4$. Since a point in $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ is determined by a tuple (M, μ, ν, q_2) with $\mu \neq 0$ and $(M, \mu, \nu, q_2) \cong (M, \lambda\mu, \lambda^{-1}\nu, q_2)$ for all $\lambda \in \mathbb{C}^*$, the moduli space is given by a vector bundle of rank $h^0(\Sigma, MK^2) = d+3g-3$ over $Sym^{-d+4g-4}(\Sigma)$ times the space $H^0(K^2)$. $\square$

When $d = 4g-4$, we recover Hitchin's parameterization of the Hitchin component.

Corollary 6.2.17. *The Hitchin component $\mathsf{Hit}(\mathsf{SO}_0(2,3)) = \mathcal{M}^{2g-2}_{4g-4}(\mathsf{SO}_0(2,3))$ is diffeomorphic to a vector space $H^0(K^2) \oplus H^0(K^4)$. Also, for $d = 4g-3$ the space $\mathcal{F}^d_\Sigma$ is a vector bundle over the surface Σ.*

Corollary 6.2.18. *Since the $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ deformation retracts onto $Sym^{-d+4g-4}(\Sigma)$, there is a homotopy equivalence between $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ and $Sym^{-d+4g-4}(\Sigma)$. In particular, there is a cohomology isomorphism $H^*(\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))) \cong H^*(Sym^{-d+4g-4}(\Sigma))$; the cohomology ring $H^*(Sym^{-d+4g-4}(\Sigma))$ was computed by McDonald [Mac62].*

Remark 6.2.19. The parameterization of $\mathsf{Sp}(4,\mathbb{R})$ Gothen components in [BGPG12] and [Col15] is equivalent to Theorem 6.2.16 but more complicated and less explicit. In particular, it is not clear what the cohomology of these spaces is.

Corollary 6.2.20. *For $0 < d < 4g-4$, the spaces $\mathcal{M}_d(\mathsf{SO}_0(2,3))$ are smooth and contain only Zariski dense representations.*

Proof. By Proposition 6.2.13, the components are all smooth. Moreover, generalizations of the proof that $\mathcal{M}^{2g-2}_d(\mathsf{Sp}(4,\mathbb{R}))$ contains only Zariski dense representations for $g-1 < d < 3g-3$ imply that $\mathcal{M}^{2g-2}_d(\mathsf{SO}_0(2,3))$ contains only Zariski dense representations for $0 < d < 4g-4$. $\square$

For Gothen components of $\mathsf{Sp}(4,\mathbb{R})$, one obtains a similar parameterization. Recall that a $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundle in the Gothen component $\mathcal{M}^{2g-2}_d(\mathsf{Sp}(4,\mathbb{R}))$ is determined by a tuple (N, μ, ν, q_2) where $N \in Pic^d(\Sigma)$ and $g-1 < d \leq 3g-3$. Moreover, up to a 1-parameter family of isomorphisms acting by

$(N,\mu,\nu,q_2)\longrightarrow(N,\lambda\mu,\lambda^{-1}\nu,q_2)$ such a tuple uniquely determines a point in $\mathcal{M}_d^{2g-2}(\mathsf{Sp}(4,\mathbb{R}))$. By Proposition 6.2.11 the corresponding $\mathsf{SO}_0(2,3)$-Gothen Higgs bundle is given by (M,μ,ν,q_2) with $M=N^2K^{-1}$. Define the space

$$\widetilde{\mathcal{F}}_\Sigma^d=\{(N,\mu,\nu)|N\in Pic(\Sigma)\text{ with }h^0(N^{-2}K^3)\neq 0,\ \mu\in H^0(N^{-2}K^3)\setminus\{0\}\ \nu\in H^0(N^2K)\}/\mathbb{C}^*$$

where the $\mathbb{C}^*$ action is given as above. Using the results above for $\mathsf{SO}_0(2,3)\cong\mathsf{PSp}(4,\mathbb{R})$ we have the following parameterizations of $\mathcal{M}_d^{2g-2}(\mathsf{Sp}(4,\mathbb{R}))$.

Theorem 6.2.21. *Let $g-1<d\leq 3g-3$ then the Gothen component $\mathcal{M}_d^{2g-2}(\mathsf{Sp}(4,\mathbb{R}))$ is diffeomorphic the space $\widetilde{\mathcal{F}}_\Sigma^d\times H^0(K^2)$. Moreover, the $2:1$ map $\pi:\mathsf{Sp}(4,\mathbb{R})\to\mathsf{SO}_0(2,3)$ gives rise to a $2^{2g}:1$ map*

$$\tilde{\pi}:\widetilde{\mathcal{F}}_\Sigma^d\times H^0(K^2)\longrightarrow\mathcal{F}_\Sigma^{2d-2g+2}\times H^0(K^2)\cong\mathcal{M}_{2d-2g+2}^{2g-2}(\mathsf{SO}_0(2,3))$$

which has fiber $\tilde{\pi}^{-1}(M,\mu,\nu,q_2)$ corresponding to the 2^{2g} square roots of $\mathcal{O}$.

Proof. Given an Higgs bundle $(N,\mu,nu,q_2)\in\mathcal{M}_d^{2g-2}(\mathsf{Sp}(4,\mathbb{R}))$, the corresponding $\mathsf{SO}_0(2,3)$-Higgs bundle is given by $(N^2K^{-1},\mu,\nu,q_2)\in\mathcal{M}_{2d-2g+2}^{2g-2}(\mathsf{SO}_0(2,3))$. Thus, for each of the 2^{2g} line bundles L satisfying $L^2=\mathcal{O}$ the $\mathsf{Sp}(4,\mathbb{R})$-Higgs bundles associated to (N,μ,ν,q_2) and $(N\otimes L,\mu,\nu,q_2)$ map to the same point in $\mathcal{M}_{2d-2g+2}^{2g-2}(\mathsf{SO}_0(2,3))$. Thus the space $\widetilde{\mathcal{F}}_\Sigma^2$ is a vector bundle over the symmetric $Sym^{2d-2g+2}(\Sigma)/(\mathbb{Z}/2^{2g}\mathbb{Z})$ where two divisors (N^2K^{-1},μ) is equivalent to $(LN)^2K^{-1}$ for each of the 2^{2g} square roots of the trivial bundle. $\square$

6.3 Gothen components and unique minimal immersions

In this section we will do everything for the groups $\mathsf{SO}_0(2,3)\cong\mathsf{PSp}(4,\mathbb{R})$, when one restricts to the maximal $\mathsf{SO}_0(2,3)$-Higgs bundles which have vanishing w_2 then all the statements hold for maximal $\mathsf{Sp}(4,\mathbb{R})$. Recall that $0\leq deg(M)\leq 4g-4$, the $\mathsf{SL}(5,\mathbb{C})$-Higgs bundle (6.2.4) is determined by the quadruple (M,μ,ν,q_2). The bundle $\mathcal{E}=K\oplus K^{-1}\oplus\mathcal{O}\oplus M\oplus M^{-1}$ has orthogonal structure Q and Higgs field ϕ given by:

$$Q=\begin{pmatrix}0&1&&&\\1&0&&&\\&&1&&\\&&&0&1\\&&&1&0\end{pmatrix}:\mathcal{E}\longrightarrow\mathcal{E}^*\qquad\phi=\begin{pmatrix}&&q_2&\mu&\nu\\&&1&0&0\\1&q_2&&&\\0&\nu&&&\\0&\mu&&&\end{pmatrix}:\mathcal{E}\longrightarrow\mathcal{E}\otimes K.\tag{6.3.1}$$

Note that the Higgs field satisfies which $\phi^TQ+Q\phi=0$, thus, (6.2.4)is in fact an $\mathsf{SO}(5,\mathbb{C})$-Higgs bundle. For the group $\mathsf{SO}(5,\mathbb{C})$, the polynomial ring $\mathbb{C}[\mathfrak{so}(5,\mathbb{C})]^{\mathsf{SO}(5,\mathbb{C})}$ has two homogeneous generators (p_1,p_2) of degree two and four. One choice of generators is

$$p_1 = Tr(X^2) \qquad \text{and} \qquad p_2 = Tr(X^4)\ .$$

For any other basis (p_1', p_2'), there are constants A, B, C so that

$$p_1' = Ap_1 \qquad \text{and} \qquad p_2' = Bp_1^2 + Cp_2\ .$$

Thus, for any choice of basis of the invariant polynomials, the holomorphic quadratic and quartic differentials associate the (E, ϕ) via the Hitchin fibration are

$$ATr(\phi^2) = 4Aq_2 \qquad \text{and} \qquad BTr(\phi^2) + CTr(\phi^4) = 16Bq_2 \otimes q_2 + C4\mu \otimes \nu\ .$$

Lemma 6.3.1. *Let $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ and fix a conformal structure $\Sigma = (S, J)$. If the harmonic ρ-equivariant map h_ρ is a branched minimal immersion, then the corresponding Higgs bundle is given by*

$$(M, \mu, \nu, 0) \tag{6.3.2}$$

Furthermore, up to a constant, the associated holomorphic quartic differential in the Hitchin base is given by $q_4 = \mu \otimes \nu$.

Proof. Let $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ and fix a conformal structure $(S, J) = \Sigma$. By Proposition 6.2.9, the $\mathsf{SO}_0(2,3)$-Higgs bundle corresponding to ρ is given by (6.3.1). By Remark 3.3.3, h_ρ is a branched minimal immersion if and only if

$$Tr(\phi^2) = 4q_2 = 0.$$

In this case, any choice of basis for $\mathbb{C}[\mathfrak{sp}(4,\mathbb{C})]^{\mathsf{Sp}(4,\mathbb{C})}$ gives $q_4 = p_2(\phi) = CTr(\phi^4) = 4C\mu \otimes \nu$. ☐

Lemma 6.3.2. *Let $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ and choose a conformal structure J so that the corresponding ρ-equivariant harmonic map h_ρ is a branched minimal immersion, then h_ρ is a* minimal immersion.

Proof. By Lemma 6.3.1, in the conformal structure J, the $\mathsf{SO}_0(2,3)$-Higgs bundle (L, W, β, γ) associated to ρ is given by (6.3.2). By Remark 3.3.3 Higgs field represents the $(1,0)$ part of dh_ρ. Since γ is injective, the Higgs field is nowhere vanishing, by Remark 3.2.13, the branched minimal immersion h_ρ is branch point free. ☐

For maximal representations $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ with $0 \leq d \leq 4g-4$ we obtain local uniqueness of the conformal structures J_ρ in which the ρ-equivariant harmonic map is minimal.

Theorem 6.3.3. *Let $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ for $0 < d \leq 4g-4$ or $\rho \in \mathcal{X}_0(\mathsf{SO}_0(2,3))$ and ρ irreducible. Then the collection of conformal structures $\{J_\rho\}$ so that the ρ-equivariant harmonic mapping $\widetilde{\Sigma} \to \mathsf{SO}_0(2,3)/(\mathsf{SO}(2) \times \mathsf{SO}(3))$ is a minimal immersion is nonempty and discrete.*

Proof. Fix a representation $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ and let (S, J) be a conformal structure in which the harmonic map is minimal. By Lemma 6.3.1, the Higgs bundle corresponding to ρ in this conformal structure is given by a tuple $(M, \mu, \nu, 0)$ with $\mu \neq 0$. We will show that this defines a $\mathsf{SO}_0(2,3)$-cyclic surface satisfying the hypothesis of Theorem 6.1.37. Consider the $\mathsf{SO}(5,\mathbb{C})$-Higgs bundle $(\mathcal{E}, Q, \phi)$ from (6.3.1) associated to $(M, \mu, \nu, 0)$, if we rearrange the holomorphic bundle $\mathcal{E}$ to be $M \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus M^{-1}$ then the Higgs field is given by

$$\phi = \begin{pmatrix} 0 & 0 & 0 & \nu & 0 \\ \mu & 0 & 0 & 0 & \nu \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix} : M \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus M^{-1} \longrightarrow MK \oplus K^2 \oplus K \oplus \mathcal{O} \oplus M^{-1}K \tag{6.3.3}$$

A computation shows that the gauge transformation

$$g = diag(-1, i, 1, -i, -1) : M \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus M^{-1} \to M \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus M^{-1}$$

acts as $g^{-1}\phi g = i\phi$. Furthermore, the gauge transformation g is in the gauge group fro $\mathsf{SO}(2,\mathbb{C}) \times \mathsf{SO}(3,\mathbb{C})$. Thus, such a Higgs bundle is a fixed point of 4^{th}-roots of unity action in $\mathcal{M}(\mathsf{SO}_0(2,3))$. The unique equivariant harmonic metric $h_\rho : \widetilde{\Sigma} \to \mathsf{SO}(5,\mathbb{C})/\mathsf{SO}(5)$ lifts to the space of Hitchin triples $\mathsf{SO}_0(5,\mathbb{C})/(\mathsf{SO}(2) \times \mathsf{SO}(2))$. Recall for $\mathfrak{so}(5,\mathbb{C})$, there are two simple roots α_1, α_2 and the set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_1\alpha_2, \alpha_1 + 2\alpha_2\}$. The $\mathsf{SO}(5,\mathbb{C})$-adjoint bundle of $M \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus M^{-1}$ is given by

$$(M^{-1}K^{-1} \otimes \mathfrak{g}_{-\alpha_1-2\alpha_2}) \oplus (M \otimes \mathfrak{g}_{-\alpha_1-\alpha_2}) \oplus (K^{-1} \otimes \mathfrak{g}_{-\alpha_2}) \oplus M^{-1}K \otimes \mathfrak{g}_{-\alpha_1}) \oplus (\mathcal{O} \otimes \mathfrak{c})$$

$$\oplus (MK^{-1} \otimes \mathfrak{g}_{\alpha_1}) \oplus (K \otimes (\mathfrak{g}_{\alpha_2}) \oplus (M^{-1} \otimes \mathfrak{g}_{\alpha_1+\alpha_2}) \oplus (MK \otimes \mathfrak{g}_{\alpha_1+2\alpha_2})$$

and the Higgs field is given by $\phi = 1 \otimes \mathfrak{g}_{-\alpha_2} + \mu \otimes \mathfrak{g}_{-\alpha_1} \oplus \nu \otimes \mathfrak{g}_{\alpha_1+2\alpha_2}$. Since the Higgs field ϕ is no where vanishing along the simple root space $\mathfrak{g}_{-\alpha_2} \otimes K$ and is not identically zero along the simple root space $\mathfrak{g}_{-\alpha_1} \otimes K$, the Higgs bundle defines a $\mathsf{SO}_0(2,3)$-cyclic surface satisfying the hypothesis of Theorem 6.1.37, proving local uniqueness. $\square$

To go from local uniqueness to global uniqueness we will follow Labourie's general differential geometric arguments in section 8 of [Lab14].

Theorem 6.3.4. *(Theorem 8.1.1 [Lab14])Let $\pi : P \to M$ be a smooth fiber bundle with connected fibers and $F : P \to \mathbb{R}$ be a positive smooth function. Define*

$$N = \{x \in P \mid \; d_x(F|_{P_{\pi(x)}}) = 0\}.$$

and assume for all $m \in M$ the function $F|_{P_m}$ is proper and that N is connected and everywhere transverse to the fibers. Then π is a diffeomeorphism from N onto M and $F|_{P_m}$ has a unique critical point which is an absolute minimum.

Theorem 6.3.5. *If S be a closed surface of genus at least 2 and $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ for $0 < d \leq 4g-4$, then there exists a unique conformal structure $(S, J_\rho) = \Sigma$ so that the ρ-equivariant harmonic map*

$$h_\rho : \widetilde{\Sigma} \to \mathsf{SO}_0(2,3)/(\mathsf{SO}(2) \times \mathsf{SO}(3))$$

is a minimal immersion.

Proof. Existence is covered by Corollary 6.0.4 of Labourie's theorem and Lemma 6.3.2, and local uniqueness is covered by Theorem 6.3.3. By Proposition 6.2.13, the space $\mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ is smooth if and only if $0 < d \leq 4g-4$. Consider the fiber bundle $\pi : \mathsf{Teich}(S) \times \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3)) \to \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$. Define a positive function F by

$$F((J,\rho)) = \mathcal{E}_\rho(J) = \mathcal{E}_J(h_\rho) = \frac{1}{2}\int_S |dh_\rho|^2 dvol_J.$$

By [Lab08], the map $F|_\rho$ is proper and smooth, furthermore, the critical points of $F|_{P_\rho}$ are minimal surfaces. Set

$$N = \{(J,\rho) \in P \mid \; d_x(F|_{P_\rho}) = 0\}.$$

By Theorem 6.3.3, N is everywhere transverse to the fibers. Applying Theorem 6.3.4, when $0 < d \leq 4g-4$, for each $\rho \in \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ there is a unique conformal structure $(S, J_\rho) = \Sigma$ in which the ρ-equivariant harmonic map

$$h_\rho : \widetilde{\Sigma} \to \mathsf{SO}_0(2,3)/(\mathsf{SO}(2) \times \mathsf{SO}(3))$$

is a minimal immersion.

$\square$

6.3.1 Parameterizations of $\mathsf{SO}_0(2,3)$ and $\mathsf{Sp}(4,\mathbb{R})$ Gothen components

Recall that Higgs bundles in the Gothen component $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ are given by tuples (M,μ,ν,q_2) with $deg(M)=d$ and $\mu\neq 0$. This only describes representatives of the isomorphism classes of Higgs bundles, and by Theorem 6.2.15, there is only a 1-parameter family of gauge symmetries to account for which acts by:

$$(M,\mu,q_2,\nu)\xrightarrow{g_\lambda}(M,\lambda^2\mu,q_2,\lambda^{-2}\nu)\ . \tag{6.3.4}$$

Since the $\mathbb{C}^*$-action of (6.3.4) acts trivially on the holomorphic quadratic differential, the space $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ is parameterized by a rank $d+3g-3$ vector bundle $\mathcal{F}_d^{\Sigma}\to Sym^{-d+4g-4}(\Sigma)$ times the space of holomorphic quadratic differentials as in Theorem 6.2.16.

Theorem 6.3.6. *For $0<d\leq 4g-4$, let $\mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ be the component of the maximal $\mathsf{SO}_0(2,3)$ representation variety corresponding to the Higgs bundle component $\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$. If $\pi:\mathcal{F}^d\to\mathsf{Teich}(S)$ is the fiber bundle over Teichmüller space with $\pi^{-1}([\Sigma])=\mathcal{F}_\Sigma^d$ is the vector bundle over $Sym^{-d+4g-4}(\Sigma)$ from Theorem 6.2.16, then there is a mapping class group equivariant diffeomorphism*

$$\Psi:\mathcal{F}^d\longrightarrow\mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3)).$$

Proof. Let $\rho_{\Sigma,M,\mu,\nu}\in\mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))$ be the representation associated to the $\mathsf{SO}_0(2,3)$)-Higgs bundle $(M,\mu,\nu,0)\in\mathcal{M}_d^{2g-2}(\mathsf{SO}_0(2,3))$ over the Riemann surface Σ. The map Ψ is defined by

$$\begin{array}{ccc}\mathcal{F}^d & \xrightarrow{\quad\Psi\quad} & \mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))\\ (\Sigma,[M,\mu,\nu]) & \longmapsto & \rho_{\Sigma,M,\mu,\nu}\end{array}$$

The inverse of Ψ is defined by Theorem 6.3.5,

$$\begin{array}{ccc}\mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3)) & \xrightarrow{\quad\Psi^{-1}\quad} & \mathcal{F}^d\\ \rho & \longmapsto & (\Sigma_\rho,[M,\mu,\nu])\end{array}$$

Moreover, this bijection is an immersion by Theorems 6.1.37 and 6.3.5. □

Since the mapping class group $\mathsf{Mod}(S)$ acts properly discontinuously on $\mathcal{X}^{2g-2}(\mathsf{SO}_0(2,3))$ we can take the quotient.

Corollary 6.3.7. *The space $\mathcal{X}^{2g-2}(\mathsf{SO}_0(2,3))/\mathsf{Mod}(S)$ has at least $4g-4$ connected components and the spaces $\mathcal{X}_d^{2g-2}(\mathsf{SO}_0(2,3))/\mathsf{Mod}(S)$ are fiber bundles over the moduli of curves $\mathcal{M}_g$.*

Remark 6.3.8. For the Hitchin component, $\mathcal{F}_\Sigma^{3g-3} = H^0(K^4)$ and we recover Labourie's mapping class group invariant parameterization of the Hitchin component as a vector bundle over Teichmüller space.

For the group $\mathsf{Sp}(4,\mathbb{R})$, using Theorem 6.2.21, we have

Theorem 6.3.9. *There is a mapping class group invariant diffeomorphism $\mathcal{X}_d^{2g-2}$ with a bundle $\pi : \widetilde{\mathcal{F}}_d \to \mathsf{Teich}(S)$ with fiber $\pi^{-1}(\Sigma) = \widetilde{\mathcal{F}}_\Sigma^d$. The $2:1$ map $\pi : \mathsf{Sp}(4,\mathbb{R}) \to \mathsf{SO}_0(2,3)$ gives rise to a $2^{2g} : 1$ map $\varpi : \mathcal{X}_d^{2g-2}(\mathsf{Sp}(4,\mathbb{R})) \longrightarrow \mathcal{X}_{2d-2g+2}^{2g-2}(\mathsf{SO}_0(2,3))$. In terms of the parameterizations $\mathcal{X}_d^{2g-2}(\mathsf{Sp}(4,\mathbb{R})) \cong \widetilde{\mathcal{F}}_d \xrightarrow{\tilde{\pi}} \mathsf{Teich}(S)$ and $\mathcal{X}_{2d-2g+2}^{2g-2}(\mathsf{SO}_0(2,3)) \cong \mathcal{F}_{2d-2g+2} \xrightarrow{\pi} \mathsf{Teich}(S)$, $\widetilde{\mathcal{F}}_d$ is a 2^{2g} cover:*

$$\begin{tikzcd} \widetilde{\mathcal{F}}_d \arrow[r, "\varpi"] \arrow[rr, bend right, "\tilde{\pi}"'] & \mathcal{F}_{2d-2g+2} \arrow[r, "\pi"] & \mathsf{Teich}(S) \end{tikzcd}$$

6.4 Connected components of $\mathcal{M}(\mathsf{SO}_0(n,n+1))$ and higher rank Gothen representations

Since the group $\mathsf{PSp}(2n,\mathbb{R})$ is a group of Hermitian type, one can talk about maximal representations. As we have seen, the collection of maximal $\mathsf{PSp}(4,\mathbb{R})$-representations is especially rich. In particular, there are $2(2^{2g}-1)+4g-3$ connected components of maximal representations and the $4g-4$ Gothen components have a very nice description as bundles over $\mathsf{Teich}(S)$. For the group $\mathsf{Sp}(2n,\mathbb{R})$ however there are only $3\cdot 2^{2g}$ connected components [GPGMiR13], for $\mathsf{PSp}(2n,\mathbb{R})$ there are 3 connected components when $n\geq 3$ is odd and at least $2^{2g}+2$ connected components when $n\geq 3$ and even [GW10]. Furthermore, none of these components behave like the Gothen components for $\mathsf{PSp}(4,\mathbb{R})$ [GW10].

In this section we show that Higgs bundle parameterization of the $\mathsf{PSp}(4,\mathbb{R}) = \mathsf{SO}_0(2,3)$-Gothen components generalizes to the split real group $\mathsf{SO}_0(n,n+1)$. Unlike $\mathsf{PSp}(2n,\mathbb{R})$, the group $\mathsf{SO}_0(n,n+1)$ is not a group of Hermitian type for $n\geq 3$, so there is no notion of maximality. We will prove the following theorem and corollaries.

Theorem 6.4.1. *For each $0<d\leq n(2g-2)$ there is a smooth connected component $\mathcal{X}_d(\mathsf{SO}_0(n,n+1))$ of $\mathcal{X}(\mathsf{SO}_0(n,n+1))$ and for each Riemann surface structure*

$$\mathcal{X}_d(\mathsf{SO}_0(n,n+1)) \;\cong\; \mathcal{F}_\Sigma^d \times \bigoplus_{j=1}^{n-1} H^0(\Sigma, K^{2j})$$

where $\mathcal{F}_\Sigma^d \to Sym^{-d+n(2n-2)}(\Sigma)$ is a vector bundle of rank $d+(2n-1)(g-1)$. Moreover, $\mathcal{X}_{n(2g-2)}(\mathsf{SO}_0(n,n+1)) = \mathsf{Hit}(\mathsf{SO}_0(n,n+1))$.

In particular, using the isomorphism $\mathsf{SO}_0(1,2) \cong \mathsf{PSL}(2,\mathbb{R})$, we recover Hitchin's [Hit87a] parameterization of all connected components of $\mathcal{X}(\mathsf{PSL}(2,\mathbb{R}))$ with positive Toledo invariant.

Corollary 6.4.2. *For each integer $0 < d \leq 2g-2$, there is a connected component of $\mathcal{X}(\mathsf{PSL}(2,\mathbb{R}))$ which is parameterized by a rank $d+2g-2$ vector bundle over $Sym^{-d+2g-2}(\Sigma)$.*

This result is proven by showing the existence of components $\mathcal{M}_d(\mathsf{SO}_0(n,n+1))$ in the Higgs bundle moduli space $\mathcal{M}(\mathsf{SO}_0(n,n+1))$.

Corollary 6.4.3. *Each of the spaces $\mathcal{M}_d(\mathsf{SO}_0(n,n+1))$ deformation retracts onto the $Sym^{-d+n(2g-2)}(\Sigma)$. Thus there is an isomorphism $H^*(\mathcal{M}_d(\mathsf{SO}_0(n,n+1))) \cong H^*(Sym^{-d+n(2g-2)}(\Sigma))$.*

The topological invariants of an $\mathsf{SO}_0(n,n+1)$-Higgs bundle (V,W,η) are the second Stiefel Whitney classes of the orthogonal bundles V and W. Since none of these Higgs field in the components $\mathcal{M}_d(\mathsf{SO}_0(n,n+1))$ can be deformed to zero, we obtain a lower bound on the number of connected components.

Corollary 6.4.4. *The moduli space $\mathcal{M}(\mathsf{SO}_0(n,n+1))$ and hence $\mathcal{X}(\pi_1,\mathsf{SO}_0(n,n+1))$ has at least $n(2g-2)+4$ connected components.*

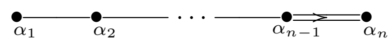
The starting point is the fixed points of the $\mathbb{C}^*$ action on $\mathcal{M}(\mathsf{SO}_0(n,n+1))$ discovered in [Arr09]. Namely, it is proven that, for each $0 < d \leq n(2g-2)$ there exists smooth minima of the Hitchin function in $\mathcal{M}(\mathsf{SO}_0(n,n+1))$ parameterized by:

$$M \xrightarrow{\mu} K^{n-1} \xrightarrow{1} K^{n-2} \xrightarrow{1} \cdots \xrightarrow{1} K \xrightarrow{1} \mathcal{O} \xrightarrow{1} K^{-1} \xrightarrow{1} \cdots \xrightarrow{1} K^{1-n} \xrightarrow{\mu} M^{-1} \tag{6.4.1}$$

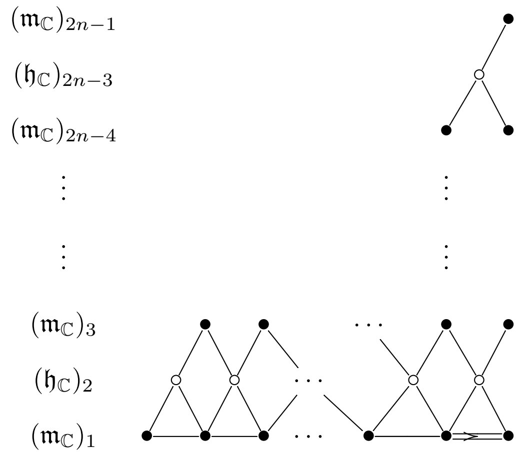
In terms of an $\mathsf{SO}_0(n,n+1)$-Higgs bundle $(V,Q_V,W,Q_W,\eta : V \to W \otimes K)$ this is given by

$$\left(K^{n-1} \oplus K^{n-3} \oplus \cdots \oplus K^{1-n}, \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}, M \oplus K^{n-2} \oplus \cdots \oplus K^{2-n} \oplus M^{-1}, \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \\ & & & \mu \end{pmatrix} \right).$$

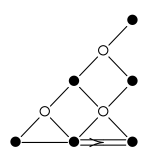
Since the bundle reduces to a direct sum of line bundles, we can write the Higgs field as a section of adjoint bundle and a collection of roots. Fix a Cartan involution θ giving $\mathfrak{so}(n,n+1) = \mathfrak{h} \oplus \mathfrak{m}$ and a maximally compact Cartan subalgebra $\mathfrak{t}$. Recall that the real form $\mathsf{SO}_0(n,n+1)$ of $\mathsf{SO}(2n+1,\mathbb{C})$ is both of Hodge type and split. Thus, the Cartan subalgebra has no noncompact part, $\mathfrak{t} \subset \mathfrak{h}$, and we can choose a set of simple roots $\{\alpha_1, \cdots, \alpha_n\}$ so that they are all noncompact imaginary, i.e. $(\mathfrak{g}_{\mathbb{C}})_{\alpha_j} \in \mathfrak{m}_{\mathbb{C}}$ for all j. With respect to these choices, the Vogan diagram of $\mathfrak{so}(n,n+1)$ has all roots painted (see Proposition 2.1.43):



With these choices, the height grading $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j=-2n}^{2n} (\mathfrak{g}_{\mathbb{C}})_j$ with $(\mathfrak{g}_{\mathbb{C}})_j \subset \mathfrak{h}_{\mathbb{C}}$ if and only if j is even. Using the Vogan diagram above, we can see the decomposition $\mathfrak{so}(2n+1,\mathbb{C}) = \mathfrak{h}_C \oplus \mathfrak{m}_{\mathbb{C}}$ in terms of root spaces, namely the root poset is given by



For $\mathfrak{so}(3,4)$ the root poset is given by



A positive roots $\alpha = \sum n_i\alpha_i$ has $n_1 \neq 0$ if and only if it is farthest to the left in one of the rows of the root poset. Note also that if we remove the simple root α_1 then the simple roots $\alpha_2, \cdots, \alpha_n$ span an embedded $\mathfrak{so}(2n-1,\mathbb{C}) \subset \mathfrak{so}(2n+1,\mathbb{C})$ with all simple roots noncompact imaginary, thus giving an embedding of $\mathfrak{so}(n-1,n) \subset \mathfrak{so}(n,n+1)$.

Denote the $\mathsf{SO}(2n+1,\mathbb{C})$ bundle associated to $(V \oplus W, Q_V \oplus Q_W)$ above by $\mathcal{E}$. Since $\mathcal{E}$ reduces holomorphically to the maximal torus, the adjoint bundle $\mathcal{E}(\mathfrak{so}(2n,\mathbb{C}))$ decomposes holomorphically as a direct of sum of root spaces $\mathcal{E}(\mathfrak{so}(2n,\mathbb{C})) = \mathcal{O} \otimes \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} (L_\alpha \otimes \mathfrak{g}_\alpha)$ for some line bundles L_α. For bundle $M \oplus K^{n-1} \oplus \cdots \oplus K^{1-n} \oplus M^{-1}$ above the line bundles for the negative simple root spaces are given by:

$$L_{-\alpha_1} = M^{-1}K^{n-1} \qquad L_{-\alpha_2} = K^{-1} \qquad L_{-\alpha_3} = K^{-1} \qquad \cdots \qquad L_{-\alpha_n} = K^{-1} \,.$$

The Higgs field $\phi = \begin{pmatrix} & \eta^T \\ \eta & \end{pmatrix} \in H^0(\Sigma, E(\mathfrak{so}(2n+1)\otimes K)$ from (6.4.1) is defined by

$$\phi = \mu \otimes \mathfrak{g}_{-\alpha_1} + \sum_{j=2}^{n} 1 \otimes \mathfrak{g}_{-\alpha_j}.$$

Let $(V' \oplus W', \phi')$ denote a Higgs bundle in the $\mathsf{SO}_0(n-1,n)$-Hitchin component, recall that the Higgs field is defined by $\phi' = \tilde{e}_1 + \sum_{j=1}^{n-1} q_{2j} \otimes e_j$. Let (M, μ, ν) be a tuple of a line bundle M of degree $0 < d \le n(2g-2)$, $\mu \in H^0(M^{-1}K^n) \setminus \{0\}$ and $\nu \in H^0(MK^n)$. Denote the embedding of $\mathfrak{so}(n-1,n)$ in $\mathfrak{so}(n,n+1)$ given by taking the span of $\{\alpha_2, \cdots, \alpha_n\}$ by $\iota : \mathfrak{so}(n-1,n) \longrightarrow \mathfrak{so}(n,n+1)$. For each such tuple (M, μ, ν) define the $\mathsf{SO}_0(n, n+1)$-Higgs bundle $(V \oplus W, \phi)$ by

$$V \oplus W = W' \oplus (M \oplus V' \oplus M^{-1}) \qquad \text{and} \qquad \phi = \mu \otimes \mathfrak{g}_{-\alpha_1} + i(\phi') + \nu \otimes \mathfrak{g}_{\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n}$$

The space of Higgs bundles obtained this way is determined by the tuples $(M, \mu, \nu, q_2, q_4, \cdots, q_{2n-2})$, and setting the parameters $(\nu, q_2, \cdots, q_{2n-2})$ all equal to zero gives rise to the minima of the Hitchin function described in (6.4.1).

Remark 6.4.5. For the case $\mathsf{SO}_0(2,3)$ the above construction gives the $\mathsf{SO}_0(2,3)$-Gothen representations. This can be seen in terms of vector bundles as follows: start with a $\mathsf{SO}_0(1,2)$-Hitchin component Higgs bundle (i.e. a Fuchsian one) $(V', W') = (\mathcal{O}, K \oplus K^{-1})$ with Higgs field $\phi = \tilde{e}_1 + q_2 \otimes e_1 = 1 \otimes \mathfrak{g}_{-\beta} + q_2 \otimes \mathfrak{g}_{\beta}$, here β is a choice of positive root in $\mathsf{SO}(3, \mathbb{C})$. We can represent such an object as the twisted endomorphism

$$\phi = \begin{pmatrix} 0 & q_2 & 0 \\ 1 & 0 & q_2 \\ 0 & 1 & 0 \end{pmatrix} : K \oplus \mathcal{O} \oplus K^{-1} \longrightarrow (K \oplus \mathcal{O} \oplus K^{-1}) \otimes K.$$

To obtain the $\mathsf{SO}_0(2,3)$-Gothen component Higgs bundles one adds $M \oplus M^{-1}$ to the $\mathsf{SO}(1,\mathbb{C})$ bundle $\mathcal{O}$ in $\mathsf{SO}_0(1,2)$-Hitchin component Higgs bundle. That is, the $\mathsf{SO}_0(2,3)$-Gothen component Higgs bundles are given by

$$(V, W, \phi) = (K \oplus K^{-1}, M \oplus \mathcal{O} \oplus M^{-1}, \mu \otimes \mathfrak{g}_{-\alpha_1} + 1 \otimes \mathfrak{g}_{\alpha_2} + q_2 \otimes \mathfrak{g}_{\alpha_2} + \nu \otimes \mathfrak{g}_{\alpha_1 + 2\alpha_2})$$

which we can represent as the twisted endomorphism

$$\phi = \begin{pmatrix} 0 & 0 & 0 & \nu & 0 \\ \mu & 0 & q_2 & 0 & \nu \\ 0 & 1 & 0 & q_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \end{pmatrix} : M \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus M^{-1} \longrightarrow (M \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus M^{-1}) \otimes K$$

If we start with a point in $\mathsf{SO}_0(2,3)$-Hitchin component given by

$$\phi' = \begin{pmatrix} 0 & 3q_2 & 0 & q_4 & 0 \\ 1 & 0 & q_2 & 0 & q_4 \\ 0 & 1 & 0 & q_2 & 0 \\ 0 & 0 & 1 & 0 & 3q_2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} : K^2 \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus K^{-2} \longrightarrow (K^2 \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus K^{-2}) \otimes K$$

Then given a tuple (M,μ,ν), the corresponding $\mathsf{SO}_0(3,4)$-Higgs bundle obtained from is given by:

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \nu & 0 \\ \mu & 0 & 3q_2 & 0 & q_4 & 0 & \nu \\ 0 & 1 & 0 & q_2 & 0 & q_4 & 0 \\ 0 & 0 & 1 & 0 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3q_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 \end{pmatrix} : M \oplus K^2 \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus K^{-2} \oplus M^{-1} \longrightarrow (M \oplus K^2 \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus K^{-2} \oplus M^{-1}) \otimes K \tag{6.4.2}$$

Proposition 6.4.6. *The parameters $(M,\mu,\nu,q_2,\cdots,q_{2n-2})$ define a unique isomorphism class of Higgs bundles up to the symmetry $(M,\mu,\nu,q_2,\cdots,q_{2n-2}) \cong (M,\lambda\mu,\lambda^{-1}\nu,q_2,\cdots,q_{2n-2})$ for all $\lambda \in \mathbb{C}^*$.*

Proof. Let $\xi = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$ be the highest root of $\mathfrak{so}(2n+1,\mathbb{C})$ and

$$\phi = \mu \otimes \mathfrak{g}_{\alpha_1} + \iota(\phi') + \nu \otimes \mathfrak{g}_\xi = \mu \otimes \mathfrak{g}_{\alpha_1} + \sum_{j=1}^{n-1} q_{2j} \otimes \iota(e_j) + \nu \otimes \mathfrak{g}_\xi$$

be a Higgs field as above. In particular, for any root $\alpha = \sum_{j=1}^{n} n_j\alpha_j$ with $n_1 = 1$, we have $\phi_\alpha = 0$, where ϕ_α be the component of ϕ along a root α. To prove the proposition, we need to show that for any $\mathsf{SO}(n,\mathbb{C}) \times \mathsf{SO}(n+1,\mathbb{C})$ gauge transformation g with $Ad_g\phi = \mu' + \sum_{j=1}^{n-1} q'_{2j} \otimes \iota(e_j) + \nu' \otimes \mathfrak{g}_\xi$ must act by $\mu' = \lambda\mu$, $q'_{2j} = q_{2j}$ and $\nu' = \lambda^{-1}\nu$ for $\lambda \in \mathbb{C}^*$. It is sufficient to show any holomorphic orthogonal bundle automorphism g of

$$M \oplus K^{n-1} \oplus \cdots \oplus K \oplus \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{1-n} \oplus M^{-1} \tag{6.4.3}$$

which doesn't act by $(\lambda\mu, \lambda^{-1}\nu, q_2, \cdots, q_{2n-2})$ is necessarily the identity. Thinking of the holomorphic gauge transformation g as a $(2n+1)\times(2n+1)$ matrix (g_{ij}) with respect to the splitting (6.4.3), it has the property:

$$\begin{gathered} g_{ij}=0 \text{ if } i+j \text{ is odd or } i=1 \text{ and } j=2n+1 \\ g_{ij}=0 \text{ if } j-i<-1 \text{ for } i\neq 1 \text{ or } j\neq 2n+1 \end{gathered} \tag{6.4.4}$$

The first two properties follow from the splitting $\mathfrak{so}(2n+1,\mathbb{C})=\mathfrak{h}_{\mathbb{C}}\oplus\mathfrak{m}_{\mathbb{C}}$ and the last property follows from the holomorphicity of g. In this splitting, the Higgs field ϕ is expressed as a matrix with

$$\begin{gathered} \phi_{ij}=0 \text{ for } j-i<-1 \text{ and } i+j= \text{ even} \\ \phi_{ij}=0 \text{ for } i=1 \text{ and } j\neq 2n \\ \phi_{ij}=0 \text{ for } j=2n+1 \text{ and } i\neq 2 \\ \phi_{2,1}=\phi_{2n+1,2n}=\mu \text{ and } \phi_{i,i+1}=1 \text{ for } 2<i<2n+1 \end{gathered} \tag{6.4.5}$$

(see (6.4.2) above for the case $\mathsf{SO}_0(3,4)$).

Claim 6.4.7. *Suppose $(Ad_g\phi)_{i,i-1}=1$ for $2<i<2n+1$ and $(Ad_g\phi)_{1,2j}=0$ and $(Ad_g\phi)_{2n+1,2j}=0$ for all $j<n$ then*

- *for $\lambda\in\mathbb{C}^*$, $g_{1,1}=\lambda$, $g_{2n+1,2n+1}=\lambda^{-1}$, and $g_{i,i}=1$ for $1<i<2n+1$*
- *$g_{1,2j+1}=0$, $g_{2j+1,2n+1}=0$, $g_{2j+1,1}=0$ and $g_{2n+1,2j+1}=0$ for all $j<n$.*

Proof. (of Claim) Since g preserves the orthogonal structure Q on $M\oplus K^{n-1}\oplus\cdots\oplus K^{1-n}\oplus M^{-1}$, we have $g^TQg=Q$. Thus $g^{-1}=Qg^TQ$, and $(g^{-1})_{ij}=g_{2n+2-j,2n+2-i}$. Using (6.4.4), a calculation shows $(Ad_g\phi)_{i,i-1}=g_{i,i}\phi_{i,i-1}(g^{-1})_{i-1,i-1}$. When $2<i<2n$, using (6.4.5), the condition $Ad_g\phi_{i,i-1}=1$ implies $g_{i,i}=g_{i-1,i-1}$ for all $2<i<2n$. Since $g_{2n+2-i,2n+2-i}=\frac{1}{g_{i,i}}$, we have $g_{i,i}=1$ for all $2\leq i\leq 2n$. This proves the first statement.

The second statement will be proven by induction. First we show that $g_{1,3}=g_{2n+1,3}=0$. Since $(g^{-1})_{ij}=0$ for $j-i<-1$ with $i\neq 1$ or $j\neq 2n+1$, a simple calculation shows $(Ad_g\phi)_{1,2}=g_{1,3}(g^{-1})_{2,2}$. Since $(g)^{-1}_{2,2}=1$ and we are assuming $(Ad_g\phi)_{1,2}=0$ it follows that $g_{1,3}=0$. Another calculation shows $(Ad_g\phi)_{2n,1}=g_{2n,2n}(g^{-1})_{2n-1,1}$, now orthogonality of g implies $g_{2n+1,3}=0$.

For the induction step, assume $g_{1,2j+1}=0$ for all $1\leq j<k<n$. We are assuming $(Ad_g\phi)_{1,2k}=0$. By the induction hypothesis, a computation shows $(Ad_g\phi)_{1,2k}=g_{1,2k+1}(g^{-1})_{2k,2k}$. Thus $g_{1,2j+1}=0$ for all j. Similarly, assume $g_{2n+1,2j+1}=0$ for $1\leq j<k<n$, by orthogonality, $(g^{-1})_{2(n-j)+3,1}=0$ for $1\leq j<k\leq n$. Since $(Ad_g\phi)_{2(n-k+1),1}=0$, another computation shows $(Ad_g\phi)_{2(n-k+1),1}=g_{2(n-k),2(n-k)}(g^{-1})_{2(n-k+1),1}$. Hence $(g^{-1})_{2(n-j)+3,1}=g_{2n+1,2j+1}=0$ for all j. To complete the proof note that, $g_{1,2j+1}=0=g_{2n+1,2j-1}=0$ for all $1\leq j\leq n$ and $g^TQg=Q$ imply that $g_{2j+1,1}=0=g_{2j-1,2n+1}$ for all $1\leq j\leq n$.

□

By the claim we can write

$$g_\lambda = \begin{pmatrix} \lambda & & \\ & g' & \\ & & \lambda^{-1} \end{pmatrix} : M \oplus (K^{n-1} \oplus \cdots K^{1-n}) \oplus M^{-1} \tag{6.4.6}$$

where g' is an $\mathsf{SO}_0(n-1,n)$ gauge transformation which acts on the $\mathsf{SO}_0(n-1,n)$-Hitchin component. But by Hitchin's parameterization [Hit92] of the Hitchin component we have $g' = Id$. Furthermore, the gauge transformation g_λ in (6.4.6) with $g' = Id$ acts on the data $(M, \mu, \nu, q_2, q_4, \cdots, q_{2n-2})$ as

$$(M, \mu, \nu, q_2, q_4, \cdots, q_{2n-2}) \xrightarrow{\quad g_\lambda \quad} (M, \lambda\mu, \lambda^{-1}\nu, q_2, q_4, \cdots, q_{2n-2}) .$$

Thus up to this $\mathbb{C}^*$ action, the tuple $(M, \mu, \nu, q_2, q_4, \cdots, q_{2n-2})$ determines a point in $\mathcal{M}(\mathsf{SO}_0(n, n+1))$. □

Proposition 6.4.8. *The dimension of the space of Higgs bundles parameterized by $(M, \mu, \nu, q_2, \cdots, q_{2n-2})$ is maximal.*

Proof. Like the $\mathsf{SO}_0(2,3)$ Gothen components, the space of (M, μ, ν) (where $(M, \mu, \nu) \cong (M, \lambda\mu, \lambda^{-1}\nu)$) is parameterized by a rank $d + n(2g-2)$ vector bundle over over $Sym^{-d+n(2g-2)}(\Sigma)$. This space has complex dimension $g - 1 - d + n(2g-2) + d + n(2g-2) = (2n+1)(3g-3)$. Thus the space parameterized by tuples $(M, \mu, \nu, q_2, \cdots, q_{2n-2})$ has complex dimension

$$(3g-3) + (7g-7) + \cdots (2n+1)(g-1) = maxdim(\mathcal{M}(\mathsf{SO}_0(n, n+1)).$$

□

Remark 6.4.9. So far we have described an open set around the minima which is the same dimension as the moduli space. To prove that this defines a connected component, we will show that it is also closed.

Lemma 6.4.10. *The open set parameterized by $(M, \mu, \nu, q_2, \cdots, q_{2n-2})$ is closed in $\mathcal{M}(\mathsf{SO}_0(n, n+1))$.*

Proof. Since for all $\lambda \in \mathbb{C}^*$ the tuples $(M, \mu, \nu, q_2, \cdots, q_{2n-2})$ and $(M, \lambda\mu, \lambda^{-1}\nu, q_2, \cdots, q_{2n-2})$ are isomorphic, we can normalize the norm of the nonzero section μ. Let $(M, \mu^t, \nu^t, q_2^t, \cdots, q_{2n-2}^t)$ be a diverging family in the parameters with the norm of μ^t normalized to 1. We claim that the family of Higgs bundles $(\mathcal{E}, \phi_t)$ corresponding to this family diverges in the moduli space $\mathcal{M}(\mathsf{SO}_0(n, n+1))$. Indeed, since the Hitchin fibration is proper, if the norms of any of the parameters $\nu^t, q_2^t, \cdots, q_{2n-2}^t$ go to infinity then the corresponding point in the Hitchin base associated to ϕ_t also goes to infinity. Thus the family $(\mathcal{E}, \phi_t)$ diverges. □

Putting together Proposition 6.4.6, Proposition 6.4.8 and Lemma 6.4.10 we obtain Theorem 6.4.1. Namely, there is a for each $0 < d \leq n(2g-2)$ there is a smooth connected component $\mathcal{M}_d(\mathsf{SO}_0(n,n+1))$ of $\mathcal{M}(\mathsf{SO}_0(n,n+1))$ which is smooth and parameterized by $\mathcal{F}^d_\Sigma \times \bigoplus\limits_{j=1}^{n-1} H^0(\Sigma, K^{2j})$ where $\mathcal{F}^d_\Sigma \to Sym^{-d+n(2n-2)}(\Sigma)$ is a vector bundle of rank $d+(2n-1)(g-1)$. Moreover, $\mathcal{M}_{n(2g-2)}(\mathsf{SO}_0(n,n+1)) = \mathsf{Hit}(\mathsf{SO}_0(n,n+1))$.

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