REPUTATION IN CONTINUOUS-TIME GAMES WITH MULTIPLE COMMITMENT TYPES

BY

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DISSERTATION

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ABSTRACT

We study a continuous-time game with imperfect monitoring in which a large player faces a continuum of infinitely-lived small players. We extend Faingold and Sannikov (2011) to a framework in which the support of the prior belief of the small players contains any finite number of commitment types. In this setting, we show the existence of a unique Markov equilibrium, we characterize a partial differential equation (PDE) for the equilibrium payoff, and we derive an optimality condition for the equilibrium actions. Also, we provide a stochastic representation of the Markov equilibrium payoffs, which is the solution to the PDE. Finally, we show that the equilibrium action of the sufficiently patient large player follows a non-stationary process that is determined by the small players' posterior beliefs.

To my parents

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CHAPTER 1: INTRODUCTION

There is nothing so practical
as a good theory.
- Kurt Lewin

Many real-world situations involve the repeated interaction of a large player and a population of small players. For instance, a monopolist (large player) chooses the quality of a product sold to a large population of customers (small players). By selling a high-quality product, the monopolist may be able to increase its profits in the long-run by building reputation of being a good-quality type. This strategy is specially attractive for a large player that is patient and not only cares about today's payoff, but also about its long-run payoff.[1] The question of whether or not a large-player can build reputation of being of a certain type when facing a large population of small players has attracted attention of economists for several decades. Since the classic work of Kreps and Wilson (1982) and Milgrom and Roberts (1982), the question of reputation building has been explored in different settings such as discrete-time repeated games with perfect monitoring (Fudenberg and Levine, 1989), discrete-time repeated games with imperfect monitoring (Fudenberg and Levine, 1992), and continuous-time games with imperfect monitoring (Faingold and Sannikov, 2011). For most of their analysis, Faingold and Sannikov (2011) assume that the support of the prior belief of the small players contain only two possible types: a *normal* type, a player that is fully strategic, and a *commitment* type, a type who is committed to playing a stationary strategy. The contribution of this paper is to extend the framework of Faingold and Sannikov (2011) by allowing the support of prior belief of the small players to contain a finite number of *multiple* commitment types.

The literature on discrete-time repeated games has emphasized the relevance of the support

[1]Other examples of the role of reputations include government's monetary policies (Barro (1986)) or time-inconsistent government policies (Celentani and Pesendorfer (1996)).

of the prior belief of the small players in the ability of the large player to build reputation. Consider, for example, this quote in Fudenberg and Levine (1992): "The power of reputation effects depends on which reputations are *a priori* feasible and this depends on which types have positive prior probability." In their setting, they do not restrict to a single commitment type and, in fact, some of their results are derived for prior beliefs with full support on all the possible commitment types. Ely and Välimäki (2003) show that when the prior belief of the small players does not include the Stackelberg type in its support, even with rational learning after observing long enough histories, the large player cannot build reputation. In their example, the rational large player does not have an incentive to mimic a *bad* commitment type because building reputation would give a lower long-run payoff. Ely, Fudenberg, and Levine (2008) show that by extending the type space so that the Stackelberg type is included in the support of the prior belief of the small players, the long-run player can build reputation even in the presence of bad types. Thus, apart from adding realism to the model, allowing for multiple commitment types delivers results that diverge from what can be achieved with a single commitment type. To the best of my knowledge, this paper is the first attempt to study reputation dynamics in a continuous-time game with imperfect monitoring with multiple commitment types.

The continuous-time framework, however, permits to fully characterize the equilibrium payoffs by using differential equations. In contrast, discrete-time characterization only provides lower and upper bounds on equilibrium payoffs. Faingold and Sannikov (2011) show the existence of a unique Markov equilibrium, characterize *ordinary* differential equation that must be satisfied by the equilibrium payoff, and show that when the large player is sufficiently patient, the equilibrium action converges to the commitment action. In other words, they show that reputation effects exist in the continuous-time framework when there is a single commitment on the Stackelberg action. With multiple commitment types, Faingold and Sannikov (2011) provide the diffusion dynamics of the equilibrium belief process and a result on transitory reputation effects, which are analogous results to the ones proven in the discrete-time framework by Cripps, Mailath, and Samuelson (2004).

We extend the existence result in Faingold and Sannikov (2011) by allowing small players to hold a prior belief with support on multiple commitment types of the large player. We also show the existence of a unique Markov equilibrium payoff to the normal type large player. Moreover, we fully characterize the *partial* differential equation to which the equilibrium payoff function is the unique solution. In addition, we provide a stochastic representation of the equilibrium payoff, which has not been shown yet in literature on continuous-time with a single commitment type. We also establish that, when the large player is patient enough, the equilibrium action is close to a weighted average of the actions of all commitment types. We also characterize a sufficient condition on public signals under which reputation effects do not hold even when the large player is patient enough.

Our main result (Theorem 1) shows that the Markov equilibrium payoff to the rational large player exists and uniquely determined when there is an arbitrary finite number of commitment types in the support of an initial prior. Proposition 5 derives a *partial* differential optimality equation and an optimality condition that the equilibrium payoff and the equilibrium action should satisfy respectively. The optimality equation and optimality condition reduce to the result of Faingold and Sannikov (2011) when there is a single commitment type.[2] Proposition 6 implies that a problem with multiple commitment types cannot be reduced to a problem of a single commitment type because even Bayesian learning cannot eliminate any commitment type that is in the support of the prior. Therefore, it is meaningful to study the case of multiple types separately from the case of a single type, which contrasts with the discrete-time literature's focus on reputation bounds. Similar to Faingold and Sannikov (2011), we also derive the equilibrium degeneracy in a complete information game where the large player is certainly believed to be either a rational type or a specific commitment type.

[2]In this sense, our Theorem 1 is a generalization of Theorem 4 in Faingold and Sannikov (2011).

A second contribution of the paper is to provide a stochastic representation of equilibrium payoffs. Past literature has concentrated on the lower and upper bounds of equilibrium payoffs through reputation building. Even in continuous-time with a type space, the literature has only characterized the optimality equation and the optimal condition, not deriving expressions for equilibrium payoffs. By using the property that equilibrium beliefs do not arrive at the boundary of the whole belief space (Proposition 6), Theorem 2 expresses equilibrium payoffs as an expectation of a discounted sum of flow payoffs up to some moment when the reputation arrives at where some specific type is believed almost surely to be the large player. Since it is well-known that, using Monte Carlo methods, the values represented in such a way can be numerically calculated, this result broadens the application of reputation games to more realistic situations. Furthermore, when there is a single commitment type, we show the opponents' belief are certainly expected to converge close to the rational type before arriving at being a commitment type, strengthening the result in Faingold and Sannikov (2011).

Finally, Theorem 3 provides an example showing that, at the behavioral level, the sufficiently patient large player cannot raise the equilibrium payoff to the level of the Stackelberg payoff by reputation building. In other words, *reputation effects* do not hold. In the example, the rational large player's equilibrium action always stays away from any commitment type's action by a non-trivial distance even as the large player grows arbitrarily patient. This result is based on some conditions on public signals given by Condition 5 that requires, for each commitment type, a barrier on the belief for public signals not to work properly. Proposition 11 generalizes Theorem 5 in Faingold and Sannikov (2011) with multiple commitment types, proving that the large player's equilibrium action converges to some weighted average of all commitment types' action as the discount rate goes to zero. However, contrary to the case of a single commitment type, this result does not tell us that the limit action should be a specific commitment type's action because other commitment types are also possible. Faingold and Sannikov (2011) mention[3] that, if such a barrier on reputation exists then, when the posterior touches the barrier, there would be no changes in reputation and equilibrium actions from that point on. However, we show that the belief process and the corresponding equilibrium action profiles are almost surely believed not to hit the barrier. Therefore, even there is a barrier on each belief about commitment types, we can also expect rich dynamics of Markov equilibria.

1.1 RELATED LITERATURE

This work builds on a rich literature that studies repeated interactions between agents. Asymmetric patience between agents plays a key role in standard reputation games. The literature on reputation games can be roughly divided based on the monitoring technology of long-run/large player's action and the patience of short-run/small opponents. Some literature like Celentani and Pesendorfer (1996) study a model between a long-run *large* player and a continuum of long-run *small* opponents. Each *small* player in a continuum is strategically myopic because she can not change public signals individually. Contrast to the *small* opponents, they call a *large* player if he can affect public signals. Hence, a model with a long-run *large* player and a continuum of long-run *small* opponents is equivalent to the canonical reputation games with public monitoring in Fudenberg and Levine (1992) between a long-run player and a sequence of short-run players in the sense that opponents, who are either long-run small or short-run, myopically behave.

Following Fudenberg and Levine (1989) and Fudenberg and Levine (1992), several papers have studied reputation effects in a discrete-time setting. Celentani and Pesendorfer (1996) show that reputation effects hold in a sufficiently long finitely-truncated dynamic game. With public monitoring, Cripps, Mailath, and Samuelson (2004) prove that, for any fixed level of a discount factor, reputation eventually disappears when the commitment strategy is not a Nash

[3]See p. 798 in Faingold and Sannikov (2011).

equilibrium. Even when the opponents' beliefs are private, Cripps, Mailath, and Samuelson (2007)[4] also show the impermanent reputation. These results imply that *changing types* is necessary for a permanent reputation.

Other papers relax the standard assumptions of reputation games. One such category is a reputation game with non-myopic opponents. When the opponents are also long-lived, the Stackelberg strategy is no longer the best commitment strategy. Nonetheless, Schmidt (1993) show that the reputation effects also hold when the repeated game features *"conflicting interests"*.[5] Cripps, Schmidt, and Thomas (1996) study the tight lower bounds, which was shown to be generally lower than that for canonical reputation games by generalizing Schmidt (1993) to non-conflicting interests. Celetani, Fudenberg, Levine, and Pesendorfer (1996) also prove reputation effects by introducing imperfect monitoring about an intended action of less patient long-lived opponents and a bounded recall to Cripps, Schmidt, and Thomas (1996).

Another category of relaxed reputation games is a model with *changing types*. When the type is not stationary,[6] Phelan (2006) study cyclic reputation, while Ekmekci, Gossner, and Wilson (2012) study permanent reputation by using the *relative entropy*[7] introduced by Gossner (2011). Using the *relative entropy*, Gossner (2011) easily replicate the results of classic reputation games in Fudenberg and Levine (1992). On the other hand, Watson (1993) and Battigalli and Watson (1997) study the case of heterogeneous opponents' beliefs. When the long-run player could estimate a bounded set where all the heterogeneous beliefs are believed to be included by some finite time, both showed that reputation effects still hold. Ely and Välimäki (2003) and Ely, Fudenberg, and Levine (2008) study a separating reputation by introducing a *"bad"* commitment type. Finally, Mailath and Samuelson (2006) is a well-written introduction

[4] Cripps, Mailath, and Samuelson (2007) study a model with the long-lived opponents.

[5] We call a game with *"conflicting interests"* when the commitment strategy of the long-run/large player always makes the opponents to choose an action that achieves their minimax payoff.

[6] With *changing types*, it is no longer possible to use *"grain of truth"* that plays a main role in literature. See p. 164 in Ekmekci, Gossner, and Wilson (2012).

[7] *Relative entropy* is a kind of measure of the difference between any two different absolutely continuous probability distributions. For an introduction to the concept, see Cover and Thomas (2012).

to reputation games in discrete-time. Mailath and Samuelson (2013) provide a survey of more recent works in this field.

In contrast to the literature, Faingold and Sannikov (2011) study reputation games in continuous time. By using Brownian diffusion in continuous-time, they showed the equilibrium degeneracy in complete information games, which contrasts with the result in Fudenberg and Levine (1994) that the set of equilibrium payoffs includes a value that is not generated by static payoffs. Based on methods in Faingold and Sannikov (2011), Bohren (2016) show that a non-trivial incentive is possible in dynamic games even without introducing any type. Bohren (2016) posit that the reputation plays a role as a state variable that follows a diffusion process and has persistent effects on the game. These kinds of continuous-time games build on the influential works of Sannikov (2007) and Sannikov (2008). By using techniques in stochastic calculus, Sannikov (2007) and Sannikov (2008) open a new way to the study of the dynamic game and the principal-agent problem respectively in a rigorous yet more tractable way. Thereafter, many previous works in discrete-time repeated games have been revisited in continuous-time setting. Both Bernard and Frei (2016) and Staudigl (2015) extend Sannikov (2007) by allowing for *multiple* players. In terms of method, Staudigl (2015) use stochastic viability theory, which we also apply to our setting.

Reputation games are also related to literature on the *Folk Theorem* because, by introducing reputation building in repeated games, the equilibrium payoff in the long-run relationship has a lower bound that generally exceeds the static Nash equilibrium payoff. Fudenberg, Levine, and Maskin (1994) study a standard *Folk Theorem* in long-run relationships between two long-lived players. With myopic opponents, Fudenberg, Kreps, and Maskin (1990) also show that, because of lack of possible punishments on the opponents that is useful to the long-run player, inefficiency exists. Also, Fudenberg and Levine (2007) and Fudenberg and Levine (2009) study efficiency in the continuous-time limit when the time interval shrinks.

Another related literature is on *Bayesian learning*.[8] In the standard reputation games, opponents update beliefs after observing public histories and reputation effects are heavily dependent on this learning process. Kalai and Lehrer (1993) show that, by Bayesian learning, long-run player accurately predict the future play of an infinitely repeated game. Lehrer and Smorodinsky (1997) and Kalai and Lehrer (1993) provide an example that absolutely continuity[9] is not necessary for *weak learning*,[10] which plays a key role in reputation games with myopic players. Lehrer and Smorodinsky (1996) introduce *almost weak learning*[11] that is also sufficient for reputation effects to hold with myopic opponents. Sorin (1999) study the *merging of probabilities*. They provided the reputation bounds shown by Fudenberg and Levine (1992) using the *merging* techniques.

The next subchapter 1.2 introduces a simple example explaining the extension to multiple commitment types is important in reputation games. Subchapter 1.3 describes notations and preliminary results based on the set-up in Faingold and Sannikov (2011). Chapter 2 extends the model to case of the multiple commitment types and establishes the existence of a unique Markov equilibrium. In chapter 3, the stochastic representation of the equilibrium payoff is derived. Chapter 4 presents the reputation effects at the behavioral level and establishes a condition under which reputation effects fail. All the proofs are in the Appendix.

1.2 AN EXAMPLE

Consider a game between a large player and a continuum of small players whose individual action cannot affect the payoffs. Both the large and the small players are long-lived. Before the

[8] We also use *"rational"* learning when players update their belief by Bayesian rule. See Kalai and Lehrer (1993).

[9] This is less strict than the *"grain of truth"* assumption in the merging of probabilities.

[10] *"Weak"* is used in the sense that players accurately predict only the next stage.

[11] *"Almost weak learning"* implies that weak learning holds only except for zero density of periods. Lehrer and Smorodinsky (1996) show a diffused belief from the true belief is sufficient for *"almost weak learning"*.

start of each t-stage game, the small players learn all of the actions taken previously by all the small players, but they only observe noisy public signals of the action taken by the large player. Next, the large player chooses an action $a_t \in A = \{U, M, D\}$ and each small player i chooses an action $b_t^i \in B = \{L, C, R\}$, simultaneously. Then, the stage game payoffs are realized. The payoff matrix of the static game is shown in Figure 1.

	L	C	R
U	(3, 2)	(1, 1)	(2, 1)
M	(2, 2)	(4, 3)	(3, 1)
D	(1, 1)	(5, 2)	(2, 3)

Figure 1: Static Payoff

The small players do not know the *type* of the large player, which is in the type space $T = \{T_0, T_U, T_M, T_D\}$. T_0 is the large player's type who behaves strategically. T_k, $k \in \{U, M, D\}$, is called a *commitment type* and corresponds to the type who is believed to choose action k every period regardless of the small players' action.[12] In this example, if the small players' prior belief about a commitment type T_M is low, they do not have incentives to choose C. However, when they believe that large player is a type T_M with sufficiently high probability through learning from public signals, they will choose C instead of L. Knowing this, the large player can build reputation of being type T_M when he is patient enough by choosing M every period.

For simplicity, suppose that all the small players choose the same action at each time and the aggregate distribution of b_t^i denoted by $\bar{b}_t$ is centered on either L, C, or R. The unique Nash equilibrium strategy in the static game is (U, L). Suppose that the small players are certain that the large player is a type T_0 who behaves fully strategically. With a prior on T_0 only, the equilibrium strategy would be the repetition of (U, L) that yields an equilibrium payoff of 3 at every stage. This implies that the large player cannot build any other reputation. However, if we introduce another fixed type of the large player to the support of small players' prior belief, it

[12] FS(2011) calls is behavioral type.

has been known that this result changes. With a non-trivial prior about types of the large player, small players update their beliefs after observing public signals. Knowing this learning process of small players, the large player exploits the commitment power, which generates non-trivial incentives.

For each large player's action $a \in A$, denote $B(a)$ as a set of best responses chosen by small players corresponding to an *observationally equivalent*[13] action to a. For any large player's static game payoff, $g(a, \bar{b})$, the *generalized Stackelberg payoff* is defined by $\bar{g}^s = \max_{a' \in A} \max_{\bar{b}' \in B(a')} g(a', \bar{b}')$ and the Stackelberg action is defined as the action that achieves $\bar{g}^s$. In this example, the Stackelberg action for the long-run player is M that yields the generalized Stackelberg payoff of 4. In other words, it is the greatest payoff that a large player could get with commitment power. This exceeds the payoff of 3 from Nash equilibrium strategy (U, L). Past literature on reputation games with imperfect monitoring has focused on the fact that, when small players have any positive belief on the large player being a type T_M[14], the large player could raise lower bounds of the equilibrium payoff by playing like the type T_M. Moreover, when he is sufficiently patient, the equilibrium payoff would be close enough to the Stackelberg payoff.

However, these results heavily depend on the assumption that small players have positive beliefs on the Stackelberg type when the game starts. If small players' prior does not include the Stackelberg type in its support, then no matter how long the large type mimics the type, there is no way for small players to raise his belief that the large player would choose the Stackelberg action M. For example, suppose that the small players' prior has a support $T^1 = \{T_0, T_D\}$. In other words, there is only one commitment type T_D on the type space. In this case, the large player cannot build a reputation of type T_M no matter how enough the large player is patient. To make matters worse, the large player get a payoff of 2 using the commitment power on D, which

[13]Because of imperfect monitoring, we focus on all the *observationally equivalent* actions to each action $a \in A$ and the *generalized Stackelberg payoff*. See p. 785 in Faingold and Sannikov (2011) for definition.

[14]We call such a commitment type the *Stackelberg type*.

is the only possible reputation to build, that is even less than the Nash equilibrium stage payoff of 3. Theorem 5 in Faingold and Sannikov (2011) demonstrates that *only*[15] when the type space is $T^2 = \{T_0, T_M\}$, the rational large player's equilibrium action and payoff converge to M and 4, respectively, as he becomes sufficiently patient. However, if the type space is T^1 instead of T^2, by using the same logic in their proof, the rational large player's equilibrium action and payoff converge to D and 2, respectively.

On the other hand, this paper allows a type space of T with multiple commitment types so that the Stackelberg type can be included in the support of the small players' prior. With the type space T, Propositon 11 proves that the rational large player's equilibrium action converges to some point in $co\{U, M, D\}$. Moreover, Theorem 3 in our work shows that the rational large player's limit action is bounded away from either U, M, or D, which is dependent on the small players' belief.

1.3 NOTATIONS AND PRELIMINARY RESULTS

As in Faingold and Sannikov (2011) (Henceforce, FS(2011)), a large player lives in infinitely periods, who faces a continuum of infinitely lived small players. At each time $t \in [0, \infty)$, the large player chooses an action $a_t \in A$ and each small player $i \in [0, 1]$ chooses $b_t^i \in B$ where both A and B are compact subsets of Euclidean space. Let $\bar{b}_t \in \Delta(B)$ be the aggregate distribution of small players' actions $\{b^i\}_{i \in [0,1]}$ where $\Delta(B)$ is the space of distributions on B. Every player observes public signals which are distorted by a Brownian motion.

Definition 1 *(FS(2011)) The Public signal $\{X_t\}_{t \geq 0}$ is represented by the diffusion process*

$$dX_t = \mu(a_t, \bar{b}_t)dt + \sigma(\bar{b}_t)dZ_t \tag{1}$$

[15] Faingold and Sannikov (2011) state Theorem 5 under the assumption that the *single* commitment type is the Stackelberg type. However, their proof of Theorem 5 can be applied to a game with *any single* commitment type.

where $\{Z_t\}_{t\geq 0}$ is d-dimensional Brownian motions and both $\mu(a_t,\bar{b}_t)\in\mathbb{R}^d$ and $\sigma(\bar{b}_t)\in\mathbb{R}^{d\times d}$ are Lipschitz continuous functions on $A\times\Delta(B)$ and $\Delta(B)$, respectively.

Let $\{\mathscr{F}_t\}_{t\geq 0}$ be the augmented filtration generated by public signals $\{X_t\}_{t\geq 0}$. In other words, $\mathscr{F}_t=\sigma\{X_s \mid 0\leq s<t\}\cup\mathscr{F}_\phi$ where $\sigma\{X_s \mid 0\leq s<t\}$ is the smallest $\sigma-$field generated by $\{X_s \mid 0\leq s<t\}$ and $\mathscr{F}_\phi$ is the null-set. Each $\mathscr{F}_t$ contains all the information known from observations of public signals up to time $t\geq 0$. The following assumption implies $\sigma(\bar{b})\neq 0$. Hence, we are considering a repeated game with imperfect monitoring as in FS(2011).

Assumption 1 *For the diffusion term of public signals $\sigma(\bar{b})$, we have $\left|\sigma(\bar{b}_t)y\right|\geq c\left|y\right|$, $\forall t>0$, for any $y\in\mathbb{R}^d$ and some $c>0$.*

Each small player could not affect the signals and the large player is believed to affect only the drift term $\mu(\cdot,\cdot)$ through action choices. In general, when we suppose that the large player could affect the diffusion term, small players could estimate a_t chosen by the large player after observing enough history by calculating the quadratic variation of public signals.

Definition 2 *(FS(2011)) Each small player i has the payoff function*

$$\int_0^\infty re^{-rt}h(a_t,b_t^i,\bar{b}_t)dt, \tag{2}$$

where $h: A\times B\times\Delta B\mapsto\mathbb{R}$ is a continuous function. These information is common knowledge to all players.

A large player's payoffs are given by

$$\int_0^\infty re^{-rt}g(a_t,\bar{b}_t)dt, \tag{3}$$

where $g: A\times\Delta B\mapsto\mathbb{R}$ is positive, Lipschitz continuous and uniformly bounded.

The large and small players share the same discount rate $r > 0$.[16] Small players are uncertain about the payoff function of the large player, believing that the large player is a type $T_i \in \{T_0, T_1, ..., T_K\}$ for some $K \geq 1$. Type T_0 is the *normal* type who chooses an action profile $\{a_t\}_{t \geq 0}$ to maximize (3). Type T_i for each $i \in \{1, ..., K\}$ is a *commitment* type who chooses a fixed action $a_i^* \in A$, independent of t, regardless of the aggregate distribution of actions chosen by small players.

At the begining of the game, small players hold a common prior belief over the large player's types denoted by, $\tilde{\theta}_0 = (\theta_{0,0}, \theta_{1,0}, ..., \theta_{K,0})$ with $\sum_{i=0}^{K} \theta_{i,0} = 1$, where $\theta_{i,0} \in [0,1]$ is the probability that each small player assigns that the large player is of type T_i. Let

$$\Delta^{K-1} = \left\{ (\theta_{1,t}, ..., \theta_{K,t}) \in \mathbb{R}^K \mid \sum_{k=1}^{K} \theta_{i,t} < 1 \text{ and } \theta_{i,t} \in (0,1)\,, i = 1, ..., K, \text{ for every} t \geq 0 \right\}$$

be the set of belief process about the commitment types. Therefore, the belief about the normal type for each time t, $\theta_{0,t}$, is given by $1 - \sum_{i=1}^{K} \theta_{i,0}$. From now on, denote $\theta_t = (\theta_{1,t}, ..., \theta_{K,t}) \in \Delta^{K-1}$ with $0 < \sum_{i=1}^{K} \theta_{i,t} < 1$ and $\theta_{0,t} = 1 - \sum_{i=1}^{K} \theta_{i,t}$. Hence, this $\theta_t \in \Delta^{K-1}$, which is a vector-valued belief about commitment types, gives all the information about the belief of being the normal type, $\theta_{0,t}$. Via Bayesian updating, FS(2011) derive a diffusion process that the posterior should follow after observing public signals $\{X_t\}_{t \geq 0}$, which is given by Proposition 1.

Proposition 1 *(Proposition 5 in FS(2011)) Fix a prior*[17] $p \in \Delta^{K-1}$. *A belief process* $\{\theta_t\}_{t \geq 0} = \{(\theta_{1,t}, ..., \theta_{K,t})\}_{t \geq 0}$ *with* $\sum_{i=1}^{K} \theta_{i,t} = 1 - \theta_0$ *for each t is consistent with a strategy profile* $(a_t, \bar{b}_t)_{t \geq 0}$ *if and only if*

[16] In the standard reputation game, it has been usually assumed that the large player is more patient than the opposite player. For example, Fudenberg and Levine (1989), Fudenberg and Levine (1992), Cripps, Mailath, and Samuelson (2004), and Gossner (2011) deal with the case of a long-run player and a sequence of short-run players.

[17] Faingold and Sannikov (2011) define the prior in $(K+1)$-dimensional space, Δ^K, including the belief about the normal type. Instead, for our partial differential equation model, we change the belief space to $(K-1)$-dimensional space, Δ^{K-1}, where only the beliefs about commitment types live. Since $\theta_{0,t} = 1 - \sum_{i=1}^{K} \theta_{k,t}$ and $d\theta_{0,t} = -\sum_{i=1}^{K} d\theta_{i,t}$, it is enough to consider beliefs only about commitment types, $\{\theta_{i,t}\}_{1 \leq i \leq K}$, for each time $t \geq 0$ on Δ^{K-1}.

(a) $(\theta_{1,0}, ..., \theta_{K,0}) = p$,

(b) for each $k \in \{0, 1, ..., K\}$ *and* $t \in [0, \infty)$

$$d\theta_{k,t} = \gamma_k(a_t, \bar{b}_t, \theta_t) \cdot \sigma^{-1}(\bar{b}_t)\big(dX_t - \mu^{\theta_t}(a_t, \bar{b}_t)dt\big) \tag{4}$$

where $\theta^k = \{\theta_{k,t}\}_{t\geq 0}$ *is the probability assigned by small players on a type* T_k *large player, and*

$$\gamma_0(a_t, \bar{b}_t, \theta_t) \equiv \theta_{0,t}\sigma^{-1}(\bar{b}_t)\big(\mu(a_t, \bar{b}_t) - \mu^{\theta_t}(a_t, \bar{b}_t)\big)$$
$$\gamma_k(a_t, \bar{b}_t, \theta_t) \equiv \theta_{k,t}\sigma^{-1}(\bar{b}_t)\big(\mu(a^*_k, \bar{b}_t) - \mu^{\theta_t}(a_t, \bar{b}_t)\big)$$
$$\mu^{\theta_t}(a_t, \bar{b}_t) = \theta_{0,t}\mu(a_t, \bar{b}_t) + \sum_{k=1}^{K} \theta_{k,t}\mu(a^*_k, \bar{b}_t)$$

We use Δ^{K-1} as our domain on which the Markov equilibrium payoff is defined. Next, we define the expected payoffs to the normal type large player at time $t > 0$ when the large player and small players follow a given strategy profile.

Definition 3 *(FS(2011)) The continuation value of the normal type at time* $t \geq 0$ *is given by:*

$$W_t(S) = \mathbb{E}_t\left\{ \int_t^\infty re^{-r(s-t)}g(a_s, \bar{b}_s)ds \;\middle|\; T_0 \right\} \tag{5}$$

where $S = \{(a_s, \bar{b}_s)\}_{s\geq 0}$ *is a strategy profile.*

Let $\mathscr{L}$ be the space of all progressively measurable processes $\alpha = \{\alpha_t\}_{t\geq 0}$, such that $\mathbb{E}\big[\int_0^T |\alpha_s|^2 ds\big] < \infty$ for every $0 < T < \infty$.[18] FS(2011) characterize a diffusion process of the continuation value, W_t, by using Martingale representation theorem.[19].

[18] A stochastic process $\mathbb{X} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is *progressively measurable* if $\mathbb{X}$ is $(\mathbb{P}, \mathscr{B}_{\mathbb{R}^d})$-measurable where $\mathscr{B}_{\mathbb{R}^d}$ is the Borel set in $\mathbb{R}^d$.

[19] This result was also shown in Sannikov (2007) in the general two long-run players game.

Proposition 2 *(Proposition 2 in FS(2011)) A bounded process* $\{W_t\}_{t\geq 0}$ *is the process of continuation values of the normal type under a public strategy profile* $S = \{(a_s, \bar{b}_s)\}_{s\geq 0}$ *if and only if for some* $\beta = \{\beta_t\}_{t\geq 0} \in \mathscr{L}$ *such that*

$$dW_t = r\big(W_t - g(a_t, \bar{b}_t)\big)dt + r\beta_t \cdot \big(dX_t - \mu(a_t, \bar{b}_t)dt\big) \tag{6}$$

A *public strategy* of the normal type is a stochastic process that is progressively measurable with respect to the augmented filtration $\{\mathscr{F}_t\}_{t\geq 0}$ generated by public signals. FS(2011) also define a public sequential equilibrium in the case of multiple commitment types.

Definition 4 *(FS(2011)) A public sequential equilibrium consists of a public strategy* $\{(a_s, b_s^i)\}_{t\geq 0}$ *for each small player* $i \in [0,1]$*, and a belief process* $\{\theta_t\}_{t\geq 0}$ *such that at every times* $t \geq 0$ *and after all histories,*

(a) The strategy $a = \{a_t\}_{t\geq 0}$ *of the normal type large player maximizes his expected payoff:*

$$\mathbb{E}_t\left[\int_0^\infty re^{-rs} g(a_s, \bar{b}_s) ds \mid T_0\right]$$

(b) For each $i \in [0,1]$*,* b_t^i *maximizes each small player i's expected payoff:*

$$\sum_{j=0}^{K} \theta_{i,t} \mathbb{E}_t\left[\int_0^\infty re^{-rs} h(a_s, b_s^i, \bar{b}_s) ds \mid T_i\right]$$

(c) Given the common prior $(\theta_{1,0}, ..., \theta_{K,0}) = p \in \Delta^{K-1}$*, the belief process* $\{\theta_t\}_{t\geq 0}$ *is determined by the Bayes' rule.*

In the diffusion process of public signals (1), note that each small player can not affect public signals. Hence, even in the dynamic game, each choice variable, b_t^i, is a maximizer of the static expected payoff at each time t to a small player i. In this sense, we call them a *small* player who is strategically myopic even though they are long-lived. The following proposition characterizes a *sequentially rational* public strategy profile.

Proposition 3 *(Proposition 3 in FS(2011)) A public strategy profile* $\{(a_t,\bar{b}_t)\}_{t\geq 0}$ *is sequentially rational with respect to a belief process* $\{\theta_t\}_{t\geq 0}$ *if and only if there exist* $\{\beta_t\}_{t\geq 0}$ *in* $\mathscr{L}$ *and a bounded process* $\{W_t\}_{t\geq 0}$ *satisfying* (6)*, such that for all* $t \geq 0$*, and after all public histories,*

$$a_t \in \arg\max_{a' \in A} g(a',\bar{b}_t) + \beta_t \cdot \mu(a',\bar{b}_t) \tag{7}$$

$$b_t \in \arg\max_{b \in B} \theta_{0,t} h(a_t,b,\bar{b}_t) + \sum_{i=1}^{K} \theta_{i,t} h(a_i^*,b,\bar{b}_t),\ \forall b_t \in supp\ \bar{b}_t \tag{8}$$

Finally, in continuous-time with multiple commitment types, FS(2011) characterize a sequential equilibrium as below:

Proposition 4 *(Theorem 7 in FS(2011)) Fix the prior* $p \in \Delta^{K-1}$*. A public strategy profile* $\{(a_t,\bar{b}_t)\}_{t\geq 0}$ *and a belief process* $\{\theta_t\}_{t\geq 0}$ *form a sequential equilibrium with continuation values* $\{W_t\}_{t\geq 0}$ *for the normal type if and only if there exists a random process* $\{\beta_t\}_{t\geq 0}$ *in* $\mathscr{L}$ *such that the following conditions hold:*

(a) $\{\theta_t\}_{t\geq 0}$ *satisfies equation (4) with initial condition* $\theta_0 = p$.

(b) $\{W_t\}_{t\geq 0}$ *is a bounded process satisfying equation (6), given* $\{\beta_t\}_{t\geq 0}$

(c) $\{(a_t,\bar{b}_t)\}_{t\geq 0}$ *satisfy the incentive constraint (7) and (8), given* $\{\beta_t\}_{t\geq 0}$ *and* $\{\theta_t\}_{t\geq 0}$.

The following condition implies that the drift term in public signals by the normal type large player cannot be represented by any convex combination of all drift terms from commitment types' action.

Condition 1 *(FS(2011)) For each* $\theta_t \in \Delta^K$ *and each static Bayesian Nash equilibrium* $(a_t,\bar{b}_t)$

of the game with prior $p \in \Delta^{K-1}$, we have:

$$\mu(a,\bar{b}) \notin co\left\{\mu(a_i^*,\bar{b}) \;\middle|\; i \in \{1,...,K\}\right\}$$

With multiple commitment types, under Condition 1, FS(2011) show that the belief process is believed to eventually converge to where the large player is a normal type from the perspective of the normal type large player. In other words, in every public sequential equilibrium[20],

$$P\left\{\lim_{t\to\infty} \theta_{0,t} = 1\right\} = 1 \text{ under the normal type} \tag{9}$$

In a discrete-time setting, this *impermanent reputation* result is also shown by Cripps, Mailath, and Samuelson (2004) and Cripps, Mailath, and Samuelson (2007).

[20]This is Theorem 8 in Faingold and Sannikov (2011).

CHAPTER 2: EXTENSION TO MULTIPLE COMMITMENT TYPES

In this chapter, we construct the optimal equation and the optimality condition when there are multiple commitment types. With the optimal equation, we show that there exists a unique equilibrium that is Markovian in the small players' beliefs. The first step is to represent the belief process from the perspective of the normal type large player. The following result is easily derived from Proposition 1. All proofs are in the Appendix.

Corollary 1 *For the equilibrium belief process $\{\theta_t\}_{t\geq 0}$ and each $k \in \{0, 1, ..., K\}$,*

$$d\theta_{k,t} = \frac{\gamma_k \cdot \gamma_0}{\theta_{0,t}} dt + \gamma_k \cdot dZ_t^n \tag{10}$$

where $\gamma_k = \gamma_k(a_t^*, \bar{b}_t, \theta)$ for each $k \in \{1, ..., K\}$, $\gamma_0 = \gamma_0(a_t, \bar{b}_t, \theta)$ and Z_t^n is a Brownian motion from the perspective of the normal type large player given by:

$$dZ_t^n = \sigma^{-1}(\bar{b}_t)\big(dX_t - \mu(a_t, \bar{b}_t)dt\big)$$

In contrast to the set-up in FS(2011) that consider a single commitment type, when there are multiple commitment types, we cannot guarantee that the drift term $\frac{\gamma_k \cdot \gamma_0}{\theta_0}$ of each belief process about type T_k is positive unless $k = 0$. This implies that the only possibility is that $\theta^0 = \{\theta_{0,t}\}_{t\geq 0}$ is a *supermartingale*[21], while $\theta^k = \{\theta_{k,t}\}_{t\geq 0}$ for $k \neq 0$ is a *submartingale* only when an angle between *deviated* drift terms from *the weighted drift*, μ^{θ_t}, of type T_k and the normal type is greater than the acute. When the angle is larger than the acute, every component of the *deviated* drift by type T_k is in the opposite direction to the *deviated* drift by the normal

[21] θ is a submartingale when $\theta_t \leq \mathbb{E}[\theta_s \mid \mathscr{F}_t]$ for $t \leq s$. When $\theta_t \geq \mathbb{E}[\theta_s \mid \mathscr{F}_t]$, θ is called a supermartingale.

type. Hence, the expected increment of belief about a type T_k is negative as time goes.[22]

Moreover, γ_k is the volatility of belief θ^k, which can be understood as a speed of the belief process about type k. The further the deviation from the weighted drift is, the higher the volatility of belief about type T_k is. When small players believe that the drift of T_k is same as the weighted drift, for the type T_k, $\gamma_k = 0$ and θ^k does not change and remains at the level. In this case, the type T_k is removed from the support of small players' belief.

For the value function $U(\cdot) : \Delta^{K-1} \to \mathbb{R}$ which is twice continuously differentiable and Markovian in the small player's belief θ_t about commitment types, let $W_t = U(\theta_t)$ where $\theta_t = (\theta_{1,t}, ..., \theta_{K,t})$, $0 < \theta_{k,t} < 1$, $k \in \{1, ..., K\}$, and $\sum_{k=1}^{K} \theta_{k,t} + \theta_{0,t} = 1$ for each time $t \geq 0$. In other words, the continuation value of the normal type large player following a given public sequential strategy is Markovian in the small players' belief θ_t. By using the Ito's formula, we can easily represent the following diffusion equation of the continuation payoff W to the normal type large player. For $\theta_t = (\theta_{1,t}, \theta_{2,t}, ..., \theta_{K,t}) \in \Delta^{K-1}$,

$$dU(\theta_t) = \left\{ \sum_{k=1}^{K} \frac{\partial}{\partial \theta_{k,t}} U(\theta_t) \frac{\gamma_0 \cdot \gamma_k}{\theta_{0,t}} + \frac{1}{2} \sum_{j,k=1}^{K} \frac{\partial^2}{\partial \theta_{j,t} \theta_{k,t}} U(\theta_t) \gamma_j \cdot \gamma_k \right\} dt + \sum_{j=1}^{K} \frac{\partial}{\partial \theta_{j,t}} U(\theta_t) \gamma_j \cdot dZ_t^n,$$

which is a second-order stochastic differential equation. By using the above equations and (6), we can derive the optimality equation and the optimality condition that equilibrium Markov payoff should satisfy.

Proposition 5 *Let $U(\cdot)$ be a bounded function on Δ^{K-1}, which is the solution of the following second-order partial differential equation: for some $r > 0$,*

[22]For example, suppose $d = 1$. In this case, $\mu \in \mathbb{R}$ is a scalar. If $\mu_0 = \mu(a, \bar{b})$ of the normal type and $\mu_k = \mu(a_k^*, \bar{b})$ of k-the commitment type lie in the same side from μ^θ, that is, either "$\mu_0 < \mu^\theta$ and $\mu_k < \mu^\theta$" or "$\mu_0 > \mu^\theta$ and $\mu_k > \mu^\theta$", then $\frac{\gamma_k \cdot \gamma_0}{\theta_0} > 0$. However, $\frac{\gamma_k \cdot \gamma_0}{\theta_0} < 0$ with μ_0 and μ_0 lying in the different sides from μ^θ, that is, either "$\mu_k < \mu^\theta < \mu_0$" or "$\mu_0 < \mu^\theta < \mu_k$".

$$\frac{1}{2}\sum_{i,j=1}^{K}\gamma_i\cdot\gamma_j U_{\theta_{i,t}\theta_{j,t}}(\theta_t)+\sum_{i=1}^{K}\frac{\gamma_0\cdot\gamma_i}{\theta_0}U_{\theta_{i,t}}(\theta_t)-rU(\theta_t)=-rg\big(a_t(\theta_t),\bar{b}_t(\theta_t)\big) \tag{11}$$

where $\big(a_t(\theta_t),\bar{b}_t(\theta_t)\big)$ *is a public strategy at time* $t\geq 0$ *that is consistent with the equilibrium belief process* θ.

Furthermore, this $U(\cdot)$ *satisfies the following condition:*

$$r\beta_t=\sigma^{-1}(\bar{b}_t)\sum_{j=1}^{K}\frac{\partial}{\partial\theta_{j,t}}U(\theta_t)\gamma_j \tag{12}$$

where β_t *is given by Proposition 2.*

Let's say that the equation (11) is the optimality equation and the equation (12) is the optimality condition. We can easily check that, when there is a single commitment type ($K=1$), the above optimality equation (11) and the optimality condition (12) are defined by:

$$\frac{1}{2}\gamma_1\cdot\gamma_1 U_{\theta_{1,t}\theta_{1,t}}(\theta_t)+\frac{\gamma_0\cdot\gamma_1}{\theta_0}U_{\theta_{1,t}}(\theta_t)-rU(\theta_t)=-rg\big(a_{0,t}(\theta_t),\bar{b}(\theta_t)\big)$$
$$r\beta_t=\sigma^{-1}(\bar{b}_t)\frac{d}{d\theta_t}U(\theta_t)\gamma_1$$

Since $\theta_{0,t}+\theta_{1,t}=1$ with $K=1$, it is clear that $U_{\theta_{1,t}}=-U_{\theta_{0,t}}$, $U_{\theta_{1,t}\theta_{1,t}}=U_{\theta_{0,t}\theta_{0,t}}$, and $\gamma_1-\gamma_0=\theta_{1,t}(1-\theta_{0,t})\sigma^{-1}(\bar{b})(\mu_1-\mu_0)$. Therefore, the above optimality partial differential equations (11) and the optimality condition (12) with $K=1$ is exactly same as the case in FS(2011).

In order to characterize the optimality equation (11), let $\Gamma\equiv(\gamma_i\cdot\gamma_j)_{i,j=1}^{K}$ be the $K\times K$ matrix whose (i,j)-element is $\gamma_i\cdot\gamma_j$ for each $i,j\in\{1,...,K\}$. The following technical condition is weaker than that μ_0 is not included in the linear space generated by $\big\{\mu_i \,\big|\, i\in\{1,...,K\}\big\}$. Therefore, Condition 1 in FS(2011) does not imply Condition 2. Furthermore, we can check that the following Condition 2 also does not imply Condition 1 when every μ_i is different from each other. When there is a single commitment type (K = 1), however, Condition 2 is exactly same as Condition 1 in FS(2011).

Condition 2 *For each belief process $\theta = \left\{\theta_t\right\}_{t\geq 0}$ where $\theta_t \in \Delta^{K-1}$, $\forall t \in [0,\infty)$ and each static Bayesian Nash equilibrium $(a,\bar{b})$ of the game with prior $p = (\theta_{1,0},...,\theta_{K,0}) \in \Delta^{K-1}$,*

$$\sum_{i=1}^{K}\left\{\frac{\theta_{i,t}^2}{\sum_{i=1}^{K}\theta_{i,t}^2} - \theta_{i,t}\right\}\mu_i \neq \theta_{0,t}\mu_0$$

When there is a single commitment type, Condition 2 means that $\mu_1 \neq \mu_0$[23]. Condition 2 is not so restrictive because if μ_0 can not be represented as a linear combination of all μ_i for $i = 1,...,K$, then Condition 2 holds. Since we can generally construct μ_j, $j = 0,1,...,K$ such that μ_0 is not a linear combination of all μ_i for $i = 1,...,K$, Condition 2 is a general condition. First, let $d = K+1$ and μ_0 be a d-dimensional vector with first element of $a > 0$. For $i = 1,...,K$, let μ_i be a d-dimensional vector with first element of 0. Since $d = K+1$, we can pick μ_i, $i = 1,...,K$ that are linearly independent. Since $a > 0$, μ_0 can not be represented as a linear combination of all μ_i for $i = 1,...,K$. For such μ_j, $j = 0,1,...,K$, Condition 2 holds. Therefore, there are uncountably many ways to construct such μ_j for $d = K+1$. Since d is arbitrary, for any $d > K$, we can construct such μ_j, $j = 0,1,...,K$ satisfying Condition 2.

Lemma 1 explains why Condition 2 needs to be imposed on Γ to study the optimality equation (11). Nonetheless, it does not guarantee the uniform ellipticity of Γ on Δ^{K-1} because Γ become degenerate on the boundary of Δ^{K-1}.

Lemma 1 *The $K \times K$ matrix Γ is a real symmetric, positive semi-definite with $\Gamma_{ii} = \left|\gamma_i\right|^2 \geq 0$. Under the Condition 2, Γ is positive definite on Δ^{K-1}.*

This lemma implies that the optimality equation (11) is a second-order *degenerate* partial

[23] It can be easily shown that the set of $\theta \in \mathbb{R}^n$ satisfying both Condition 1 and Condition 2 is not empty.

differential equation. Let $M = \overline{\Delta^{K-1}}$ be the closure of Δ^{K-1} and

$$\partial M = \left\{ \theta_t \in \mathbb{R}^K \mid \theta_{0,t} = 0 \right\} \cup \left\{ \theta_{i,t} = 0 \text{ for some } i \in \{1, ..., K\} \right\}$$

be the boundary of M. Intuitively, the boundary of the belief space is the set of beliefs that small players are certain that the large player is neither the normal type or a specific commitment type. Under Condition 2, by Lemma 1, the optimality equation (11) is *degenerate* only on the boundary.[24]

Denote the second order differential operator $L : C(M) \to C(M)$ which is corresponding to the belief process $\left\{\theta_t\right\}_{t \geq 0}$ defined by Corollary 1 where $C(M)$ is the space of all continuous functions defined on M. For $\theta_t \in \Delta^{K-1}$ that follows the diffusion process (10), it is well-known that: for $f \in C(M)$,

$$Lf(\theta_t) = \frac{1}{2} \sum_{i,j=1}^{K} \gamma_i \cdot \gamma_j f_{\theta_{i,t}\theta_{j,t}} + \sum_{i=1}^{K} \frac{\gamma_0 \cdot \gamma_i}{\theta_{0,t}} f_{\theta_{i,t}}.$$

On a general open domain in $\mathbb{R}^K$, the *elliptic* operator L that is defined above is known as the *infinitesimal generator* that is corresponding to a diffusion process that lives on the domain. Even with Condition 2 on Δ^{K-1}, however, since L might be degenerate on ∂M, we need to classify each point on ∂M based on the criteria that if small players' belief process θ is expected to arrive at that point at some time. Freidlin (1985) define the fisrt category of points on ∂M.

Definition 5 *(ε-regular point) A point $\theta^* \in \partial M$ is said to be ε-regular for the operator L (that is, with corresponding process (θ, P_θ)), in the domain Δ^{K-1}, if for any $\varepsilon > 0$,*

$$\lim_{\theta \in \Delta^{K-1}, \theta \to \theta^*} P_\theta \left\{ \tau_{\Delta^K} > \varepsilon \right\} = 0$$

[24] In other words, $x^T \Gamma(\theta_t) x = 0$ for and $\theta_t \in \partial M$ and $\forall x \neq 0$.

where $\tau_{\Delta^{K-1}} = \inf\left\{t \geq 0 : \theta_t \notin \Delta^{K-1}\right\}$ *and* P_θ *is the solution to the Martingale problem corresponding to the optimality equation (11) for any process* θ*.*

When we say θ^* is a ε-regular point, it means that in any neighborhood of θ^*, small players' belief θ is certainly expected to arrive at θ^* immediately. Let $\partial\Delta^\varepsilon \subset \partial M$ be the set of all ε-regular points included in ∂M. Following Friedman (1974), define a *non-attainable set*, which is denoted by Ψ, as below.

Definition 6 *(Non-attainable set) A set* $\Psi \subset \partial M$ *that is closed subset in* $\mathbb{R}^K$*, is said to be non-attainable for the operator L (that is, with corresponding process* (θ, P_θ)*), in the domain* Δ^{K-1}*, if for any consistent belief process* θ*,*

$$P_\theta\left\{\theta_t \in \Psi \textit{ for some } t > 0\right\} = 0$$

A non-attainable set means that the belief process of small players is certainly not expected to touch any point included in the set at any time. Therefore, from the view of normal type large player, small players' belief is not expected to arrive at the point in the non-attainable set. Hence, we do not need to consider those points in the equilibrium from the perspective of normal type large player. In the next proposition, we show that every point in the boundary ∂M is included in the non-attainable set. In other words, there is no ε-regular point on ∂M.

Proposition 6 *For any* $\varepsilon > 0$ *and prior* $p \in \Delta^{K-1}$*,*

$$\partial M = \Psi, \textit{ and } \partial\Delta^\varepsilon = \emptyset$$

with probability 1 from the perspective of the normal type large player.

This proposition separates the problem with multiple commitment types from the problem

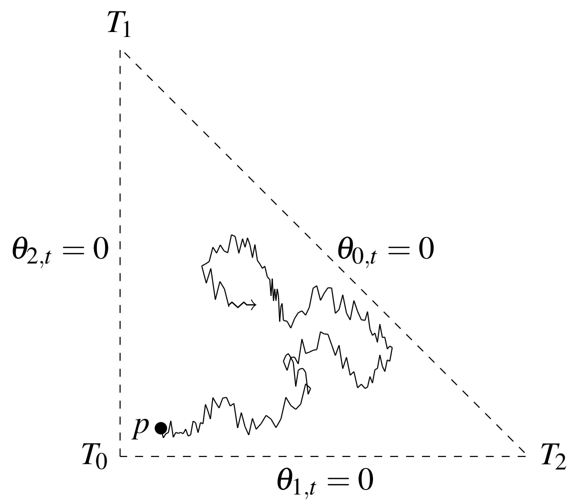

Figure 2: A belief trajectory when $K = 2$

with a single commitment type problem that was studied in FS(2011). It means that, from the perspective of the normal type large player, the K-commitment types problem cannot be even reduced to the $(K-1)$-commitment types problem. Therefore, when there are multiple commitment types, the normal type large player is almost sure that every commitment type and normal type always have positive probability in the small players' belief no matter how long they observe public signals.

Figure 2 shows a belief trajectory of small players in Δ^1 with the initial prior p when there are only two commitment types, $(K = 2)$. The belief process represented by a irregular line lives only in the interior of Δ^1 and does not touch any point on the boundary $\partial\overline{\Delta^1}$ because the boundary is a non-attainable set. This implies that we could not directly deal with the multiple commitment types problem using techniques in FS(2011) that was applied to the case of a single commitment type. Instead, we should consider reputation games separately for each K-number of commitment type(s), which is much different from reputation games in discrete time. Past literature on reputation games has focused on the lower bound of the equilibrium payoff to the normal type. Therefore, even when there are multiple commitment types, the normal type large player could raise the lower bound by sending signals of the Stackelberg type. This result holds regardless of the number of commitment types as long as the Stackelberg type has a positive probability in the small players' prior. However, in continuous time reputation games, we are focusing on the exact characterization of equilibrium payoff to the normal type large player that is Markovian in small players' belief. Since even a sufficiently small belief about some commitment type might affect the equilibrium payoff, we could not ignore any belief when we study the Markovian payoff in continuous time as long as it is positive.

We define the following correspondence that gives out sequential equilibrium strategy profiles for the given level of belief and the discount rate $r > 0$.

Definition 7 *Let $\mathscr{N} : \Delta^{K-1} \times \mathbb{R} \rightrightarrows A \times \Delta(B)$ be a correspondence defined by:*

$$\mathscr{N}(\theta_t, r) = \Bigg\{ (a_t, \bar{b}_t) : a \in argmax_{a' \in A} g(a', \bar{b}) + \left(\sigma(\bar{b}) \cdot \sigma(\bar{b})^T\right)^{-1} \cdot z^T \cdot \begin{pmatrix} \mu_1 - \mu^{\theta_t} \\ \vdots \\ \mu_K - \mu^{\theta_t} \end{pmatrix} \mu(a', \bar{b})$$

$$b \in argmax_{b' \in B} \sum_{i=0}^{K} \theta_i h(a_i, b', \bar{b}) \ \forall b \in supp\, \bar{b} \Bigg\}$$

where $z^T = \frac{1}{r}\left(\theta_{1,t} U_{\theta_{1,t}}, \ldots, \theta_{K,t} U_{\theta_{K,t}}\right)$.

In general, the correspondence $\mathscr{N}(\cdot,\cdot)$ defined above is not continuous because we cannot guarantee that it is lower-hemicontinuous. However, the following assumption makes everything more easy.

Assumption 2 *For each (θ_t, r), the correspondence $\mathscr{N}$ is a singleton. Furthermore, for each belief process $\theta = \{\theta_t\}_{t\geq 0}$ with $\theta_t \in \Delta^{K-1}$, the aggregate distribution $\bar{b}(\theta_t)$ is a mass-distribution.*

If $\mathscr{N}(\theta_t, r)$ is upper-hemicontinuous, Assumption 2 guarantees the $\mathscr{N}(\cdot,\cdot)$ is a continuous function in (θ, r). In other words, the equilibrium action profile $\left(a(\theta), \bar{b}(\theta)\right)$ is continuous in (θ, r). By applying the Maximum theorem, the following proposition is easily shown.

Proposition 7 *The correspondence $\mathcal{N}(\cdot,\cdot)$ is non-empty, compact-valued and upper hemicontinuous. Specifically, under Assumption 2, $\mathcal{N}(\cdot,\cdot)$ is continuous in (θ,r).*

We show the equilibrium degeneracy in the continuous-time dynamic game with complete information regardless of $r > 0$ as is shown in FS(2011) in the case of a single commitment type. Therefore, the result of equilibrium degeneracy is invariant with respect to the number of commitment types. From now on, denote $U_r(\theta_t)$ as the Markov equilibrium payoff at time t for any fixed discount rate $r > 0$.

Proposition 8 *Let $\theta^* = (0,...,0)$ or $(0,...,0,1,0,...,0) \in M$ where small players are certain that the large player is a normal type or some commitment type T_i for any $i \in \{1,...,K\}$. Then, for any $r > 0$,*

$$U_r(\theta^*) \in g\big(\mathcal{N}(z^*,r)\big)$$

where $z^* = 0$ or $\big(0,...,0,\frac{1}{r}\frac{\partial U(\theta_t)}{\partial \theta_{i,t}}\big|_{\theta_{i,t}=1},0,...,0\big)$ for any $i \in \{1,...,K\}$.

This proposition means that when there is no uncertainty about type of large player, the equilibrium payoff to the normal type large player is determined by static payoff at the equilibrium action without any consideration of reputational incentive characterized by the value of z. With $z^* = 0$ or $\big(0,...,0,\frac{1}{r}\frac{\partial U(\theta_t)}{\partial \theta_{i,t}}\big|_{\theta_{i,t}=1},0,...,0\big)$ for any $i \in \{1,...,K\}$, the reputational incentive part, $\big(\sigma(\bar{b})\cdot\sigma(\bar{b})^T\big)^{-1}\cdot z^T \cdot \begin{pmatrix} \mu_1 - \mu^{\theta_t} \\ \vdots \\ \mu_K - \mu^{\theta_t} \end{pmatrix} \mu(a',\bar{b})$ is always zero.

Next, we show that there exists a unique Markov equilibrium payoff function that satisfies the optimality equation (11) and the optimality condition (12). For any domain $D \in \mathbb{R}^K$, let

$\mathscr{W}^2(D)$ be the Sobolev space[25] of all Borel functions $h : D \to \mathbb{R}$ that are square integrable on D. In other words, for any multi-index α and weak-derivatives D^{α}:

$$\mathscr{W}^2(D) = \left\{ h : D \to \mathbb{R} \,\middle|\, \|h\|_{\mathscr{W}^2(D)} < \infty \right\}$$

where $\|h\|^2_{\mathscr{W}^2(D)} = \sum_{|\alpha|\leq 2} \int_D |D^{\alpha} h(x)|^2 ds$. Denote $\mathscr{W}^2_{loc}(D)$ be the space of all Borel functions $h : D \to \mathbb{R}$ that belong to $\mathscr{W}^2(D')$ for any open subset D' such that its closure $\overline{D'}$ is also included in D. The following theorem is our first main result, which is an extension of Theorem 4 in Faingold and Sannikov (2011) to the case of multiple commitment types.

Theorem 1 *Under Condition 2 and Assumption 2, for any given discount rate $r > 0$, there exists a unique Markov equilibrium payoff function $U_r(\cdot)$ defined on the space of belief process, M, satisfying the optimality equation:*

$$\frac{1}{2}\sum_{i,j=1}^{K} \gamma_i \cdot \gamma_j (U_r)_{\theta_{i,t}\theta_{j,t}}(\theta_t) + \sum_{i=1}^{K} \frac{\gamma_0 \cdot \gamma_i}{\theta_{0,t}} (U_r)_{\theta_{i,t}}(\theta_t) - rU_r(\theta_t) = -rg\big(a_t(\theta_t), \bar{b}_t(\theta_t)\big) \tag{13}$$

where $\big(a_t(\theta_t), \bar{b}_t(\theta_t)\big) \in \mathscr{N}(\theta_t, r)$[26] is a public strategy at time $t \geq 0$ and $U_r(\theta_t) = W_t$.

Furthermore, for any static payoff function $g(\cdot,\cdot)$ that is continuous on M[27], the equilibrium payoff $U_r(\cdot)$ that is the solution to the equation (13) is also continuous on M and belongs to $\mathscr{W}^2_{loc}(\Delta^{K-1})$.

The main difference from the case of a single commitment type that is studied in FS(2011) is the characterization of equilibrium payoff, $U_r(\theta_t)$, when θ_t is on the boundary of Δ^{K-1}. In the case of a single commitment type, the equilibrium payoff converges to the static payoff as small players' belief converges to the boundary where the large player is certainly believed as

[25] In the Sobolev space, the derivative is defined in the sense of distributions.

[26] This implies that U_r also satisfies the optimality condition (12).

[27] Since $\mathscr{N}$ is continuous under Assumption 2, this holds with $g(\cdot,\cdot)$ that is continuous in $\big(a, \bar{b}\big)$.

either the normal type or the commitment type. This is because the boundary in the case of a single commitment type is where small players are certain about the large player's type. On the other hand, on the boundary of Δ^{K-1} with multiple commitment types, small players are only sure that the large player is not a specific commitment type. Hence, it is ambiguous to determine what the equilibrium payoff is for such a belief at which other commitment types are still possible. Furthermore, it is generally required to define the equilibrium payoff function on every boundary to determine a solution to a partial differential equation.

Fortunately, by Proposition 6, every boundary point is included in a non-attainable set, which make our problem to show the existence of equilibrium payoff tractable. Although we do not know the value of equilibrium payoff on the whole boundary, we are certain that there is a Markov equilibrium payoff that is characterized as the unique solution to the optimality equation and the optimality condition.

CHAPTER 3: STOCHASTIC REPRESENTATION OF MARKOV EQUILIBRIUM PAYOFF

Although we show that there is a unique Markov equilibrium with multiple commitment types, Theorem 1 does not give any information about a specific form of Markov equilibrium payoff function. It just shows an existence of the equilibrium payoff. Proposition 6 implies that, in small players' belief, every commitment type has positive probability even after observing sufficiently long history, which means that there is no need to consider boundary values of equilibrium payoff. This is the "punch-line" to show a unique existence of Markov equilibrium payoff.

However, according to Proposition 8, we already know that the equilibrium payoff should be determined as the value of static payoff at all vertices of the belief space, M, where small players absolutely believe that the large player is either some specific commitment type or the normal type. By using this degeneracy of equilibrium, we can impose some regularity restrictions on the equilibrium payoff. In this section, we find a stochastic representation[28] of the Markov equilibrium payoff. This kind of representation has not been shown yet in past literature on reputation games even in a continuous-time set-up. Moreover, it is well-known that we can solve numerically the optimality equation when the solution is represented by this kind of expression. Hence, from an applied perspective, Theorem 2 opens the way to calculate values of the Markov equilibrium payoffs in a specific reputation game by using Monte Carlo methods.

First, instead of the original problem, we consider a reduced problem on a restricted belief domain. For each sufficiently small $\delta > 0$, construct $D^{\delta} \subset \Delta^{K-1}$ that is a convex and connected

[28] We adopt the expression of "*stochastic representation*" by following Feehan and Pop (2015)

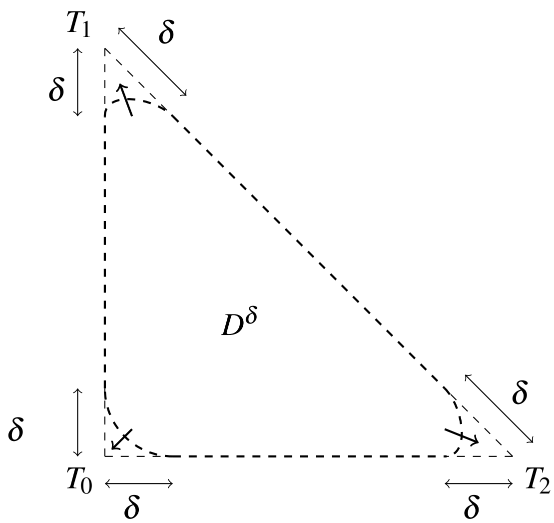

Figure 3: A construction of D^δ when $K = 2$

open subset with boundary[29] $\partial D^\delta = \partial \overline{D}^\delta$[30] such that

(a) $\cup_{\delta>0} D^\delta = \Delta^{K-1}$ and $D^{\delta_1} \subset D^{\delta_2}$ for any $\delta_1 > \delta_2$

(b) $\partial D^\delta\big|_{\Omega_\delta} = \partial M\big|_{\Omega_\delta}$

(c) $\partial D^\delta \backslash \Omega_\delta \subset \cup_{i\in\{0,1,..,K\}} \{\theta_i > 1-\delta\}$

where Ω_δ is a subset of ∂M such that $\delta < \theta_i < 1-\delta$ for some $i \in \{0,1,...,K\}$ and $\theta_j = 0$ for some $j \neq i$. In the construction of D^δ, (a) means that $\{D^\delta\}$ is an increasing sequence of restricted domains converging to Δ^{K-1} as δ goes to 0. (b) implies that D^δ shares Ω_δ with Δ^{K-1} as a part of their boundary. Finally, (c) means that the remaining part of boundary of D^δ except for Ω_δ is where some specific commitment type has sufficiently high probability in small players' belief.

Figure 3 describes a construction of D^δ when there are only 2 commitment types. The dashed line represents Ω_δ which is included in ∂M where belief process are not expected to

[29]Actually, it is enough to find D^δ having some part of C^3-boundary in ∂D^δ.
[30]Here, the boundary of an open set is defined as the boundary of the closure.

touch. The dashed curves correspond to $\partial D^{\delta}\backslash\Omega_{\delta}$. These curves make up the boundary of D^{δ} where some commitment type has sufficiently high probability in small players' belief because they are included in the δ-neighborhood from a vertex of Δ^{K-1}. The following lemma shows that these parts of boundary represented by blue dashed curves are included in a set of ε-regular points where we should impose some boundary condition.

Lemma 2 *Under Condition 2, for any sufficiently small $\delta \in (0,1)$, Ω_{δ} is an non-attainable set and $\partial D^{\delta}\backslash\Omega_{\delta}$ is a set of ε-regular points in ∂D^{δ}.*

By Lemma 2, for any $\delta \in (0,1)$, Ω_{δ} is still a non-attainable set that is included in the boundary of Δ^{K-1}. However, $\partial D^{\delta}\backslash\Omega_{\delta}$ is included in the neighborhoods of each vertex of Δ^{K-1} where belief process is expected to touch immediately. Denote $\partial\Delta_{\delta}^{\varepsilon} = \partial D^{\delta}\backslash\Omega_{\delta}$, which is the ε-regular part of ∂D^{δ}, where, because of the equilibrium degeneracy at the vertices of M shown by Proposition 8, we expect equilibrium payoff is sufficiently close to the static payoff. For each $\delta > 0$ and any given $r > 0$, we consider the following reduced problem on the restricted domain D^{δ}:

$$
\begin{aligned}
LU^{\delta}(\theta_t) - rU^{\delta}(\theta_t) &= -rg\big(a_t(\theta_t), \bar{b}_t(\theta_t)\big) \text{ on } D^{\delta} \\
U^{\delta}(\theta_t) &= g\big(a_t(\theta_t), \bar{b}_t(\theta_t)\big) \text{ on } \partial\Delta_{\delta}^{\varepsilon}
\end{aligned}
$$

Figure 4 depicts a reduced problem on the restricted domain when there are two commitment types. The dashed curves represent $\partial\Delta_{\delta}^{\varepsilon}$ where we should impose boundary conditions. Since those parts are included in δ-neighborhood from some vertex of Δ^{K-1}, it is reasonable to impose the large player's static payoff $g(\cdot,\cdot)$ on the parts for small enough δ. Let $U^{\delta}(\theta)$ be the solution to the above second-order partial differential equation defined on the restricted domain D^{δ}. This is the equilibrium payoff to the normal type large player that is Markovian in small players' belief $\theta \in D^{\delta}$ when we restrict small players' belief space to D^{δ} instead of Δ^{K-1}. Let $\tau^{\delta} =$

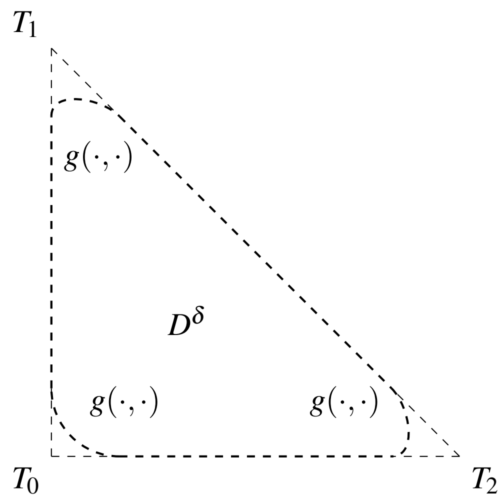

Figure 4: An reduced problem when $K = 2$

$\inf\{t > 0 \mid \theta_t \notin D^\delta\}$ be the first hitting time when small players' belief arrives at some boundary of D^δ. Since Ω_δ is included in a non-attainable set, we certainly expect that, at τ^δ, small players' belief process θ_{τ^δ} would touch some ε-regular point that is included in $\partial\Delta^\varepsilon_\delta$.

Proposition 9 *Under the Condition 2 and Assumption 2, when we restrict on D^δ for some $\delta \in (0,1)$, there exist a unique Markov equilibrium payoff $U^\delta(\theta)$ that satisfies the optimality equation (11), which is bounded, measurable and continuous almost everywhere on D^δ. The payoff $U^\delta(\theta)$ has the following form: for any $\theta \in D^\delta$ and $0 < r < \infty$,*

$$U^\delta(\theta) = \mathbb{E}_\theta\left[g\big(a(\theta_{\tau^\delta}),\bar{b}(\theta_{\tau^\delta})\big)\exp\{-r\tau^\delta\}\right] + r\mathbb{E}_\theta\int_0^{\tau^\delta} g\big(a_0(\theta_s),\bar{b}(\theta_s)\big)\exp\{-rs\}ds \quad (14)$$

Furthermore, the equilibrium payoff satisfies the following boundary conditions near each vertex of Δ^{K-1} : for any $\theta_{\tau^\delta} \in \partial\Delta^\varepsilon_\delta$,

$$U^\delta(\theta_{\tau^\delta}) = g\big(a(\theta_{\tau^\delta}),\bar{b}(\theta_{\tau^\delta})\big)$$

Proposition 9 provides a stochastic representation of the Markov equilibrium payoff when we are restricted to the small belief domain D^{δ} for some sufficiently small $\delta \in (0,1)$. This representation looks similar to the *Feynman-Kac* representation of the solution to a second-order parabolic partial differential equation. In order to derive the same form of representation of the equilibrium payoff on the original belief space, we need to find, for each θ, the limit of $\{U^{\delta}(\theta)\}$, each of which is given by (14), when δ goes to zero. For this purpose, pick a decreasing sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty}\delta_i = 0$. Corresponding to the sequence $\{\delta_i\}_{i\in\mathbb{N}}$, construct an increasing sequence of restricted domains $\{D^{\delta_i}\}_{i\in\mathbb{N}}$ that is convergent to Δ^{K-1} and each D^{δ_i} satisfies above (a), (b), and (c). Let $\tau = \lim_{i\to\infty}\tau^{\delta_i}$ be the limit of a sequence of corresponding hitting times. Since $\lim_{i\to\infty} D^{\delta_i} = \Delta^{K-1}$ and θ_t is a continuous process with probability 1, it is trivial that $\tau = \tau_{\Delta^{K-1}} = \{t \geq 0 \,:\, \theta_t \notin \Delta^{K-1}\}$. Let $\mathscr{V}(A)$ be the set of all vertices of A. The following theorem shows the stochastic representation of Markov equilibrium payoff of the original problem.

Theorem 2 *Under Condition 1, 2, and Assumption 2, there exists a unique Markov equilibrium payoff function $U(\cdot)$ that is an approximate*[31] *solution to the optimality equation (12) on Δ^{K-1}. This is a bounded and measurable function on Δ^{K-1} with the following boundary condition at all the vertices of Δ^{K-1}: for any $\theta^* \in \mathscr{V}(\Delta^{K-1})$,*

$$\lim_{\theta\to\theta^*} U(\theta) = g\big(a(\theta^*), \bar{b}(\theta^*)\big)$$

Furthermore, the Markov equilibrium payoff $U(\theta)$ is given by: for any given $\theta \in \Delta^{K-1}$ and $0 < r < \infty$,

$$U(\theta) = \mathbb{E}_{\theta}\left[g\big(a(\theta_\tau), \bar{b}(\theta_\tau)\big)\exp\{-r\tau\}\right] + r\mathbb{E}_{\theta}\int_0^{\tau} g\big(a_0(\theta_s), \bar{b}(\theta_s)\big)\exp\{-rs\}ds \qquad (15)$$

[31] Actually, this is an *approximate* equilibrium payoff on Δ^{K-1} because it is a pointwise limit of $\{U^{\delta_i}(\theta)\}$ for each θ as $i\to\infty$. Each U^{δ_i} in the sequence is defined for $\theta \in D^{\delta_i}$.

Sketch of Proof. For each given $\delta_i > 0$, consider D^{δ_i} where a belief trajectory lives on. While we could not assign any boundary condition on ∂M, by Lemma 2, each ∂D^{δ_i} has the ε-regular boundaries that is near of each vertex of Δ^{K-1}. Only on those regular boundaries, we could assign boundary values that equilibrium payoff should satisfy. Since $D^{\delta_i} \backslash \Omega_{\delta_i}$ is included in where some of commitment type has sufficiently high probability that is greater than $1 - \delta_i$, it is reasonable to expect that the equilibrium values at those point are included in some neighborhood of the value of static payoff by the degeneracy result shown by Proposition 8. With this boundary condition on $\partial \Delta^{\varepsilon}_{\delta}$, we could apply the Dynkin's formula to the second-order partial differential equation. The result is shown by Proposition 9.[32] Finally, we show that, when the sequence $\{\delta_i\}$ converges to zero, the sequence of equilibrium payoffs $\{U^{\delta_i}(\theta)\}$ on the reduced domain D^{δ_i} has a pointwise limit for each $\theta \in D^{\delta_i}$. Since the belief trajectory $\{\theta_t\}_{t \geq 0}$ is continuous in probability 1, the sequence of stopping times $\{\tau^{\delta_i}\}$ when the belief touch the ε-regular boundary $D^{\delta_i} \backslash \Omega_{\delta_i}$ is an increasing sequence in decreasing $\{\delta_i\}$. Furthermore, Condition 1 guarantees that the belief process is certainly expected to touch this regular boundary for any prior, which means that both term of equilibrium payoff in (14) on the reduced domain D^{δ_i} are expectations of uniformly integrable random variables. Therefore, for each $\theta \in D^{\delta_i}$, the sequence of equilibrium payoffs on reduced domains has a pointwise limit as $\delta_i \to 0$. This pointwise limit approximates the equilibrium payoff on the whole belief space Δ^{K-1}. Hence, this limit is a stochastic representation of Markov equilibrium payoff on Δ^{K-1} with multiple commitment types. ■

In past literature on reputation games, to the best of my knowledge, there has been no result about a representation of equilibrium payoff. Even in FS(2011) with a single commitment type, only the characterization of ordinary differential equation and the optimality condition that Markov equilibrium payoff should satisfy were provided without any mention about forms

[32]This is also guaranteed by Theorem 5.2 in Stroock and Varadhan (1972). It is well-known that, when we deal with a *degenerate* partial differential equation, the boundary condition should be assigned only on parts of boundary that is regular in a strong sense.

of the equilibrium payoff. Therefore, this stochastic representation in Theorem 2, which is derived via Feynman-Kac type formula, sheds light on the application of reputation games into more real world problem. By applying Monte Carlo methods to the representation, it is already well-known that we can calculate the equilibrium payoff function numerically. Therefore, in a specific example, it is possible to evaluate the value of equilibrium utilities of normal type large player with respect to the reputation of small players.

3.1. When $d = 1$ and $K = 1$

In this subchapter, suppose that $d = 1$ and there is a single commitment type ($K = 1$). Hence, the drift term μ_i, the diffusion term γ_i, and $\sigma(\bar{b})$ are scalar-valued. Moreover, the public signals $\{X_t\}_{t\geq 0}$ are one-dimensional diffusion process. When we restrict on $D^\delta = \{\theta_1 \in (\delta, 1-\delta)\}$ for some $\delta > 0$, without loss of generality, we can assume that $\delta < \theta_{0,t} < 1-\delta$ for all $t \geq 0$ because D^δ is away from every vertex of Δ^0 by some distance that is characterized by δ and $\theta_{0,t} + \theta_{1,t} = 1$ for each $t \geq 0$. Denote $I_\delta = (\delta, 1-\delta)$ be the space of beliefs about the normal type large player. Let $T_\delta = \inf\{t > 0 \mid \theta_{0,t} = \delta\}$ and $T_{1-\delta} = \inf\{t > 0 \mid \theta_{0,t} = 1-\delta\}$ be the first hitting times for each boundary point of I_δ. Define $\tau_\delta = T_\delta \wedge T_{1-\delta}$ is the first hitting time when θ_t arrived at any boundary of I_δ. The following proposition show that, for each $\delta \in (0,1)$, the belief about the normal type large player is certainly expected to touch $\theta_{0,\tau} = \delta$ first before it arrives at $1-\delta$.

Proposition 10 *Under Condition 2, for any prior $\theta_0 \in I_\delta$,*

$$\lim_{\delta\to 0} P_{\theta_0}\left\{\inf_{0\leq t<\tau_\delta} \theta_{0,t} > \delta\right\} = \lim_{\delta\to 0} P_{\theta_0}\left\{\theta_{0,\tau_\delta} = 1-\delta\right\} = 1$$

from the perspective of the normal type large player.

This result is similar to the conclusion (9) that is shown by FS(2011) under Condition 1. However, Proposition 10 is a more strong result in the sense that it shows that the first exit time of $\theta_{0,t}$ in I_δ always take place where $\theta_{0,t}$ is sufficiently close to 1 before it arrives at some point where $\theta_{0,t}$ is sufficiently small. With this conclusion, we further characterize the stochastic representation shown by Theorem 2 when $d = 1$ and $K = 1$.

Corollary 2 *Assume that $d = 1$ and there is a single commitment type. Under Condition 1, 2, and Assumption 2, the equilibrium Markov payoff is given by: for any given $\theta \in \Delta^0$,*

$$U(\theta) = g(a^*, \bar{b}^*)\mathbb{E}_\theta\left[\exp\{-r\tau\}\right] + r\mathbb{E}_\theta \int_0^\tau g\big(a_0(\theta_s), \bar{b}(\theta_s)\big)\exp\{-rs\}ds \tag{16}$$

where

$$(a^*, \bar{b}^*) \in \left\{(a, \bar{b}) : a \in argmax_{a' \in A} g(a', \bar{b}),\ and\ b \in argmax_{b' \in B} h(a, b', \bar{b})\ \forall b \in supp\, \bar{b}\right\}$$

CHAPTER 4: A FAIL OF REPUTATION EFFECTS AT THE BEHAVIORAL LEVEL

Past literature on reputation games has shown that the normal type large player's equilibrium payoff converges to the Stackelberg payoff by choosing the corresponding Stackelberg action as the large player becomes sufficiently patient. FS(2011) show a similar result at the behavioral level that the normal type's equilibrium action also converges to the Stackelberg action as the discount rate goes to zero.[33] In this chapter, we show that although large player becomes sufficiently patient, the normal type's equilibrium action need not converge to an action of any commitment type when the prior is given at some specific level. First, we introduce the following condition under which the normal type's equilibrium action converges to some point in the sub-manifold generated by every commitment type's fixed action.

Condition 3 *For any prior $p \in \Delta^{K-1}$, an equilibrium strategy $\left\{a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right\}_{t\geq 0}$, and a consistent belief process $\left\{\theta_t\right\}_{t\geq 0}$ that is corresponding to $\left\{a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right\}_{t\geq 0}$:*

(a) There exists a $C_1 > 0$ such that $\frac{\partial U(\theta_t)}{\partial \theta_{i,t}} > C_1$ for every $i \in \{1, ..., K\}$.

(b) There exists a $C_2 > 0$ such that

$$\left|\mu(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)) - \mu^{\theta_t}(\theta_t)\right| \geq C_2 \left| a_{0,t}(\theta_t) - \sum_{i=1}^{K} \theta_{i,t} a_i^*(\theta_t) - \theta_{0,t} a_{0,t}(\theta_t) \right|$$

Condition 3(a) implies every commitment type is a "good" type. In other words, the normal type large player's equilibrium payoff increases with respect to the belief about any commitment type. Hence, reputation about any commitment type has positive value to the normal type large

[33] In this case, the commitment type should be the Stackelberg type. Because small players are uncertain about payoff-related types about the large player, it is unnatural to assume that small players recognize the exact Stackelberg action of the large player as the only commitment type.

player. If the static payoff function $g(\theta_t) = g\big(a_t(\theta), \bar{b}_t(\theta)\big)$ has this property, then Condition 3(a) also holds. Condition 3(b) means that if the normal type's equilibrium action is different from the weighted average of actions chosen by every types, then the drift term from the normal type and the average of drift terms by every types are also different. When the drift term of public signals, $\mu\left(a_t(\theta_t), \bar{b}_t(\theta_t)\right)$, is quasi-linear in the action profile $a(\theta)$, then Condition 3(b) is also satisfied under the following condition:

Condition 4 [34] *For any prior $p \in \Delta^{K-1}$, an equilibrium strategy $\left\{a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right\}_{t\geq 0}$, and a consistent belief process $\left\{\theta_t\right\}_{t\geq 0}$ corresponding to $\left\{a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right\}_{t\geq 0}$:*

(b) There exists a $C_2 > 0$ such that*

$$\left|\mu\left(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right) - \mu\left(a'(\theta_t), \bar{b}_t(\theta_t)\right)\right| \geq C_2\left|a_{0,t}(\theta_t) - a'(\theta_t)\right|$$

Condition 4 implies that different actions chosen by large player are expected to have different effects on public signals from the view of the normal type. Under Condition 3, we show that the normal type's equilibrium action converges to a convex combination of actions of all commitment type's as discount rate goes to zero. When there is a single commitment type ($K = 1$), this result is consistent with Theorem 5 in FS(2011).

Proposition 11 *Under Condition 2 and 3, and Assumption 2, for any given belief θ_t, the equilibrium action of the normal type large player converges to a convex combination of every commitment type's action as the large player becomes sufficiently patient. In other words, as $r \to 0$,*

$$a_{0,t}(\theta_t) \;\longrightarrow\; \sum_{i=1}^{K} \frac{\theta_{i,t}}{1-\theta_{0,t}} a_i^*$$

[34] This condition is same as Condition 3(a) in Faingold and Sannikov (2011).

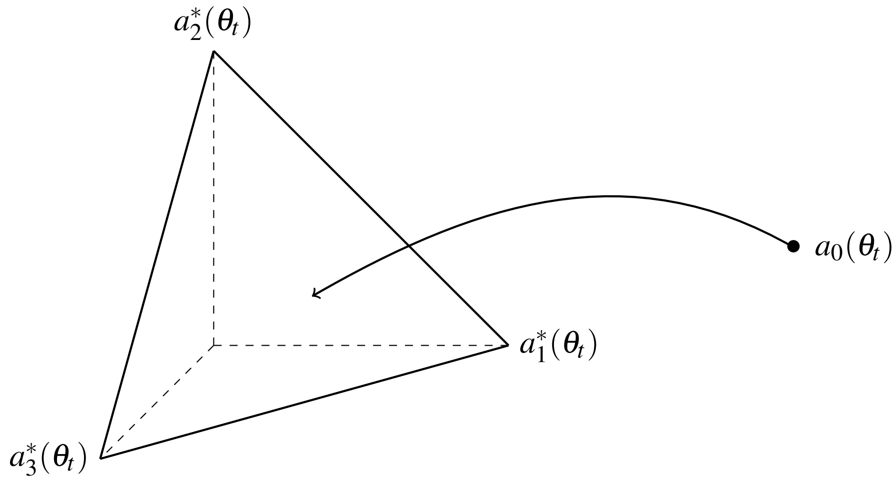

Figure 5: Convergence when $K = 3$

Denote $A^*(\theta_t) = \sum_{i=1}^{K} \frac{\theta_{i,t}}{1-\theta_{0,t}} a_i^*$ be the limit action that is a convex combination of every commitment type's action for any given $\theta_t \in \Delta^{K-1}$. Proposition 11 implies that $a_{0,t}(\theta_t)$ converges to $A^*(\theta_t)$ as r goes to 0. Therefore, the normal type's equilibrium action might converge to any commitment type T_i's action when the belief process converges to where $\theta_{i,t} = 1$ after observing enough history. In FS(2011), because there is a single commitment type, the sub-manifold generated by the commitment type's action is a singleton. Therefore, the normal type large player's equilibrium action always converges to the commitment type's action as the large player becomes patient. Figure 5 describes the result of Proposition 11 when there are 3 commitment types.

However, under the following condition, with multiple commitment types, this limit action stays away from any commitment type action because the equilibrium belief process is trapped in some specific area in the belief space. This is a main difference between the case of a single commitment type and the case of multiple commitment types.

Condition 5 *For each $i \in \{1, ..., K\}$, there exists only one $\beta_i \in (0,1)$ such that:*

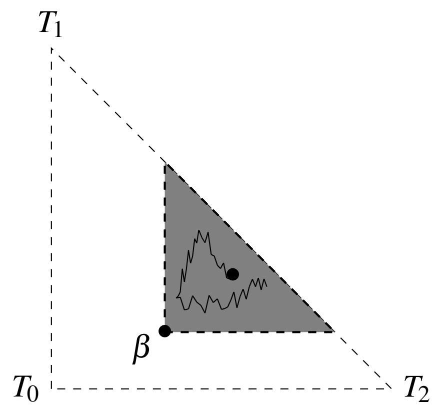

Figure 6: Condition 5 when $K = 2$

(a) $0 < \sum_{i=1}^{K} \beta_i < 1$

(b) $\mu\left(a_i^*(\theta_t^{\beta_i}), \bar{b}_t(\theta_t^{\beta_i})\right) = \sum_{j \neq i}^{K} \frac{\theta_{j,t}}{1-\beta_i} \mu\left(a_j^*(\theta_t^{\beta_i}), \bar{b}_t(\theta_t^{\beta_i})\right) + \frac{\theta_{0,t}}{1-\beta_i} \mu\left(a_{0,t}(\theta_t^{\beta_i}), \bar{b}(\theta_t^{\beta_i})\right)$

for any $\theta_t^{\beta_i} = \left(\theta_{1,t}, ..., \theta_{i-1,t}, \beta_i, \theta_{i+1,t}, ..., \theta_{K,t}\right) \in \Delta^{K-1}$ *with* $\sum_{j \neq i}^{K} \theta_{j,t} + \theta_{0,t} = 1 - \beta_i$.

Condition 5(a) guarantees that the belief β lies in Δ^{K-1}. At the belief $\beta = \left(\beta_1, ..., \beta_K\right) \in \Delta^{K-1}$, Condition 5(b) implies that $\mu\left(a_{0,t}(\beta), \bar{b}_t(\beta)\right) = \mu\left(a_1^*(\beta), \bar{b}_t(\beta)\right) = \cdots = \mu\left(a_K^*(\beta), \bar{b}_t(\beta)\right)$. In other words, all drift terms in public signals by every commitment type and the normal type are same at the belief $\beta \in \Delta^{K-1}$. If all drift terms in public signals are same across all the commitment types and the normal type, then Condition 5(b) also holds and the posterior remains as same as the prior no matter how long small players observe public signals. Condition 5 is not necessary but a sufficient condition for Theorem 3.[35]

Under the following condition, at $\theta_t^{\beta_i} \neq \beta$, Condition 5(b) does not imply that all drift terms in public signals are same. This is possible because each drift term from a commitment type is affected by the aggregate distribution of small players' actions as well as a commitment action, which are also dependent on the level of small players' belief.

[35] We can find other conditions that are sufficient for Theorem 3 to hold by characterizing an invariant set where the belief process lives in different ways.

Condition 6 *Let $\theta_t^{\beta_i} \neq \beta$. For each $i, j \in \{0,1,...,K\}$, there exist $C_3 > 0$ such that:*

$$\left|\mu(a_i^*(\theta_t^{\beta_i}), \bar{b}_t(\theta_t^{\beta_i})) - \mu(a_j^*(\theta_t^{\beta_i}), \bar{b}_t(\theta_t^{\beta_i}))\right| \geq C_3 \left|a_i^*(\theta_t^{\beta_i}) - a_j^*(\theta_t^{\beta_i})\right|$$

where $a_0^*\left(\theta_t^{\beta_i}\right) = a_0\left(\theta_t^{\beta_i}\right)$ is the normal type large player's equilibrium action at the belief $\theta_t^{\beta_i}$.

Figure 6 describes the effects of Condition 5 on the equilibrium belief process when there are two commitment types. Since each β_i plays a role of a barrier on the equilibrium belief about each commitment type T_i, even after observing long enough public signals, the equilibrium belief trajectory, which is represented by blue irregular line, is trapped in the gray area. This gray area is an invariant set. In other words, the posterior of small players with prior in the area keeps staying in the area with probability 1. Note that Condition 5(b) only requires that for each commitment type, there is only one level of belief $0 < \beta_i < 1$. Therefore, we do not require that on the whole belief space, every drift term should be represented by convex combinations of drift terms by other types. Let $\dot{F} = \left\{\theta_t \in \Delta^{K-1} \middle| \beta_i < \theta_{i,t} < 1 \text{ for every } i \in \left\{1,...,K\right\}\right\}$ be an open subset in Δ^{K-1}, which is characterized by $\beta \in \Delta^{K-1}$.

Theorem 3 *Suppose that Condition 2 holds for $\theta_t \in \dot{F}$. Under Condition 3 and 5 in $\dot{F}$, the limit action is uniformly away from each commitment action. In other words, there exists a $\alpha > 0$ such that*

$$\left|A^*(\theta_t) - a_i^*\right| \geq \alpha$$

for any $i \in \{1,...,K\}$,

Sketch of Proof. By Proposition 11, we know that, for any θ_t, the limit action $A^*(\theta_t)$ is in the sub-manifold generated by $\left\{a_1^*,...,a_K^*\right\}$. By Condition 5, we set up a barrier to each $\theta_{i,t}$ that prevents $\theta_{i,t}$ from decreasing below β_i. Since $\dot{F}$ is an invariant set, there is also a unique Markov

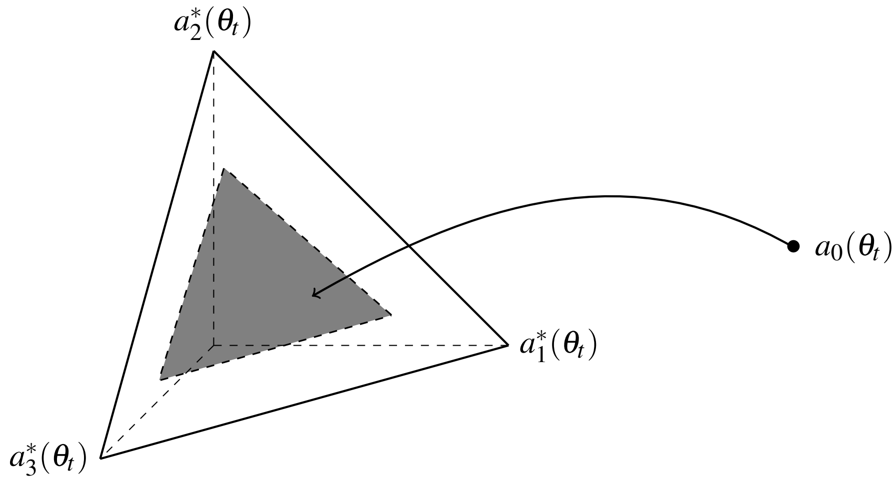

Figure 7: Convergence when $K = 3$

equilibrium payoff, $U_{\dot{F}}(\theta)$, when we restrict the original problem on $\dot{F}$. Therefore, with a prior in $\dot{F}$, the equilibrium belief process is certainly believed to live in $\dot{F}$. With Condition 3 imposed on this $U_{\dot{F}}$, the limit action $A^*(\theta_t)$ for $\theta_t \in \dot{F}$ stays away from every vertex of Δ^{K-1} ■

Theorem 3 means that although the large player becomes patient enough, the normal type's equilibrium action is always different from any commitment action by some non-trivial distance. Therefore, reputation effects do not hold in this case at the behavioral level. In other words, large player could not fix his equilibrium payoff at the level of the Stackelberg payoff in spite that one of the multiple commitment types is the Stackelberg type. Figure 7 describes Theorem 3 when there are 3 commitment types. Each commitment type's action consists of vertex of a tetrahedron. The gray area represents the invariant set F that is characterized by Condition 5. When the prior about commitment types lives in this gray area, then the equilibrium posterior also lives in this area. Therefore, the normal type large player's equilibrium action a_0 converges to some point in the gray area, which is away from every vertex of Δ^{K-1}.

CHAPTER 5: CONCLUSION

We study a continuous-time reputation game when there are multiple commitment types. Theorem 1 shows that there exists a unique equilibrium payoff to the normal type large player that is Markovian in small players' beliefs. In Theorem 2, we find a stochastic representation of the equilibrium payoff. Theorem 3 provides an example that, under some conditions on public signals, the equilibrium action of the normal type does not converge to any commitment type action even though the large player is sufficiently patient. In other words, reputation effects do not hold. Although we focus only on the reputation games, these results we have derived could be easily adjusted to more general continuous-time games that study a Markov equilibrium payoff with respect to belief processes.

Although this is a partial extension of FS(2011) by allowing multiple commitment types, there is still an open question about what if the best response correspondence $\mathscr{N}(\theta,r)$ is not a singleton. When $\mathscr{N}(\theta,r)$ is a singleton as we assume in this paper, it is a continuous function in (θ,r). However, when $\mathscr{N}(\theta,r)$ yields multiple equilibrium action profiles for each given (θ,r), it is no longer guaranteed to be lower-hemicontinuous. Hence, we are not sure that equilibrium action profiles $\big(a(\theta),\bar{b}(\theta)\big) \in \mathscr{N}(\theta,r)$ is continuous in (θ,r). FS(2011) deal with such a case by using the techniques in differential inclusions and the concept of viscosity solutions[36] when there is a single commitment type. With multiple commitment types, it might be more technically difficult to show the existence of the Markov equilibrium payoff function. However, one possible approach to the difficulty is an imposing a condition on a set $\mathscr{N}(\theta,r)$ under which each player choose an action among all best responses. We leave this open question as a future research.

Furthermore, we assume that the large player acts against a continuum of opponents. There-

[36]This is a kind of solution to a differential equation in a more generalized sense because it may be assumed to be non-differentiable. For an introduction to concepts and related properties of viscosity solutions, see Crandall, Ishii, and Lions (1992).

fore, although the opponents are long-lived players with a same discount rate as the large player, each of them do not have any power on the progress of the game. In this sense, we call the opponents small compared to the large player who can affect the public signals by himself. This implies that each small player plays myopically as is living only once. However, when there are finite number of opponents who are against the large player, things are absolutely different. Each of the opponents is no longer a small player because their choice could change the game through affecting public signals directly. We expect that, in this case, the reputation effects are hard to hold relative to the case of small players. Although there has been lots of literature on reputation games with non-myopic opponent players, it is also an interesting future research question to study whether the result of reputation games with finite number of opponents converges to the result of games with a continuum of small players when the number of opponents increases.

REFERENCES

BARRO, R. J. (1986): "Reputation in a Model of Monetary Policy with Incomplete Information," *Journal of Monetary Economics*, 17(1), 3–20.

BATTIGALLI, P., AND J. WATSON (1997): "On "Reputation" Refinements with Heterogeneous Beliefs," *Econometrica*, 65(2), 369–374.

BERNARD, B., AND C. FREI (2016): "The Folk Theorem with Imperfect Public Information in Continuous Time," *Theoretical Economics*, 11(2), 411–453.

BOHREN, A. (2016): "Using Persistence to Generate Incentives in a Dynamic Moral Hazard Problem.," Discussion paper, Working Paper.

CANNARSA, P., G. DA PRATO, AND H. FRANKOWSKA (2010): "Invariant Measures Associated to Degenerate Elliptic Operators," *Indiana University Mathematics Journal*, 59(1), 53–78.

CELENTANI, M., AND W. PESENDORFER (1996): "Reputation in Dynamic Games," *Journal of Economic Theory*, 70(1), 109–132.

CELETANI, M., D. FUDENBERG, D. K. LEVINE, AND W. PESENDORFER (1996): "Maintaining a Reputation against a Long-Lived Opponent," *Econometrica*, 64(3), 691–704.

COVER, T., AND J. THOMAS (2012): *Elements of Information Theory*. Wiley.

CRANDALL, M. G., H. ISHII, AND P.-L. LIONS (1992): "Users Guide to Viscosity Solutions of Second Order Partial Differential Equations," *Bulletin of the American Mathematical Society*, 27(1), 1–67.

CRIPPS, M. W., G. J. MAILATH, AND L. SAMUELSON (2004): "Imperfect Monitoring and Impermanent Reputations," *Econometrica*, 72(2), 407–432.

——— (2007): "Disappearing Private Reputations in Long-Run Relationships," *Journal of Economic Theory*, 134(1), 287–316.

CRIPPS, M. W., K. M. SCHMIDT, AND J. P. THOMAS (1996): "Reputation in Perturbed Repeated Games," *Journal of Economic Theory*, 69(2), 387–410.

EKMEKCI, M., O. GOSSNER, AND A. WILSON (2012): "Impermanent Types and Permanent Reputations," *Journal of Economic Theory*, 147(1), 162–178.

ELY, J., D. FUDENBERG, AND D. K. LEVINE (2008): "When is Reputation Bad?," *Games and Economic Behavior*, 63(2), 498–526.

ELY, J. C., AND J. VÄLIMÄKI (2003): "Bad Reputation," *The Quarterly Journal of Economics*, pp. 785–814.

EVANS, L. (2010): *Partial Differential Equations*, Graduate Studies in Mathematics. American Mathematical Society.

FAINGOLD, E., AND Y. SANNIKOV (2011): "Reputation in Continuous-Time Games," *Econometrica*, 79(3), 773–876.

FEEHAN, P., AND C. POP (2015): "Stochastic Representation of Solutions to Degenerate Elliptic and Parabolic Boundary Value and Obstacle Problems with Dirichlet Boundary Conditions," *Transactions of the American Mathematical Society*, 367(2), 981–1031.

FREIDLIN, M. I. (1985): *Functional Integration and Partial Differential Equations.(AM-109)*, vol. 109. Princeton university press.

FRIEDMAN, A. (1974): "Nonattainability of a Set by a Diffusion Process," *Transactions of the American Mathematical Society*, 197, 245–271.

FUDENBERG, D., D. M. KREPS, AND E. S. MASKIN (1990): "Repeated Games with Long-Run and Short-Run Players," *The Review of Economic Studies*, 57(4), 555–573.

FUDENBERG, D., D. LEVINE, AND E. MASKIN (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica*, pp. 997–1039.

FUDENBERG, D., AND D. K. LEVINE (1989): "Reputation and Equilibrium Selection in Games with a Patient Player," *Econometrica*, pp. 759–778.

——— (1992): "Maintaining a Reputation When Strategies are Imperfectly Observed," *The Review of Economic Studies*, 59(3), 561–579.

——— (1994): "Efficiency and Observability in Games with Long-Run and Short-Run Players," *Journal of Economic Theory*, 62.

——— (2007): "Continuous Time Limits of Repeated Games with Imperfect Public Monitoring," *Review of Economic Dynamics*, 10(2), 173–192.

——— (2009): "Repeated Games with Frequent Signals," *The Quarterly Journal of Economics*, pp. 233–265.

GOSSNER, O. (2011): "Simple Bounds on the Value of a Reputation," *Econometrica*, 79(5), 1627–1641.

KALAI, E., AND E. LEHRER (1993): "Rational Learning Leads to Nash Equilibrium," *Econometrica*, pp. 1019–1045.

KARATZAS, I., AND S. SHREVE (2012): *Brownian Motion and Stochastic Calculus*, vol. 113. Springer Science & Business Media.

KREPS, D. M., AND R. WILSON (1982): "Reputation and Imperfect Information," *Journal of Economic Theory*, 27(2), 253–279.

LEHRER, E., AND R. SMORODINSKY (1996): "Compatible Measures and Merging," *Mathematics of Operations Research*, 21(3), 697–706.

——— (1997): "Repeated Large Games with Incomplete Information," *Games and Economic Behavior*, 18(1), 116–134.

MAILATH, G. J., AND L. SAMUELSON (2006): *Repeated Games and Reputations: Long-Run Relationships*. Oxford university press.

——— (2013): "Reputations in Repeated Games," *The Handbook of Game Theory*, 4.

MILGROM, P., AND J. ROBERTS (1982): "Predation, Reputation, and Entry Deterrence," *Journal of Economic Theory*, 27(2), 280–312.

PHELAN, C. (2006): "Public Trust and Government Betrayal," *Journal of Economic Theory*, 130(1), 27–43.

SANNIKOV, Y. (2007): "Games with Imperfectly Observable Actions in Continuous Time," *Econometrica*, 75(5), 1285–1329.

——— (2008): "A Continuous-Time Version of the Principal-Agent Problem," *The Review of Economic Studies*, 75(3), 957–984.

SCHMIDT, K. M. (1993): "Reputation and Equilibrium Characterization in Repeated Games with Conflicting Interests," *Econometrica*, pp. 325–351.

SORIN, S. (1999): "Merging, Reputation, and Repeated Games with Incomplete Information," *Games and Economic Behavior*, 29(1), 274–308.

STAUDIGL, M. (2015): "A New Characterization of Perfect Public Equilibrium Payoffs in Repeated Games with Imperfect Public Monitoring in Continuous Time," .

STROOCK, D., AND S. VARADHAN (1972): "On Degenerate Elliptic-Parabolic Operators of Second Order and Their Associated Diffusions," *Communications on Pure and Applied Mathematics*, 25(6), 651–713.

WATSON, J. (1993): "A "Reputation" Refinement without Equilibrium," *Econometrica*, 61(1), 199–205.

APPENDIX: PROOFS

Proof of Corollary 1

Proof. First, by the definition of $\mu^{\theta_t}(a_t, \bar{b}_t)$,

$$dX_t - \mu^{\theta_t}(a_t, \bar{b}_t)dt = \left\{ dX_t - \mu(a_t, \bar{b}_t)dt \right\} + \left\{ \mu(a_t, \bar{b}_t) - \mu^{\theta_t}(a_t, \bar{b}_t) \right\} dt$$

Since $dZ_t^n = \sigma^{-1}(\bar{b}_t)\big(dX_t - \mu(a_t, \bar{b}_t)dt\big)$, for each $k \in \{1, ..., K\}$,

$$\gamma_k(a_t, \bar{b}_t, \theta_t) \cdot \sigma^{-1}(\bar{b}_t)\big(dX_t - \mu(a_t, \bar{b}_t)dt\big) = \gamma_k(a_t, \bar{b}_t, \theta_t) \cdot dZ_t^n \tag{17}$$

Furthermore, since $\gamma_0(a_t, \bar{b}_t, \theta_t) = \theta_0 \sigma^{-1}(\bar{b}_t)\big(\mu(a_t, \bar{b}_t) - \mu^{\theta_t}(a_t, \bar{b}_t)\big)$,

$$\sigma^{-1}(\bar{b}_t)\left\{ \mu(a_t, \bar{b}_t) - \mu^{\theta}(a_t, \bar{b}_t) \right\} dt = \frac{\gamma_0(a_t, \bar{b}_t, \theta_t)}{\theta_0} dt \tag{18}$$

Therefore,

$$\gamma_k(a_t, \bar{b}_t, \theta_t) \cdot \sigma^{-1}(\bar{b}_t)\left\{ \mu(a_t, \bar{b}_t) - \mu^{\theta_t}(a_t, \bar{b}_t) \right\} dt = \frac{\gamma_k(a_t, \bar{b}_t, \theta_t) \cdot \gamma_0(a_t, \bar{b}_t, \theta_t)}{\theta_0} dt \tag{19}$$

By (18) and (20),

$$d\theta_{k,t} = \frac{\gamma_k \cdot \gamma_0}{x_0} dt + \gamma_k \cdot dZ_t^n$$

■

Proof of Proposition 5

Proof. By the definition of $U(\theta_t) = W_t$, $dU(\theta_t) = dW$. From the drift terms,

$$r\big(U(\theta_t) - g(a_o, \bar{b})\big) = \frac{1}{2}\sum_{i,j=1}^{K} \gamma_i \cdot \gamma_j \frac{\partial^2 U(\theta_t)}{\partial \theta_{i,t} \partial \theta_{j,t}} + \sum_{i=1}^{K} \frac{\gamma_0 \cdot \gamma_i}{\theta_{0,t}} \frac{\partial U(\theta_t)}{\partial \theta_{i,t}}$$

, hence the optimality equation is derived. From the dispersion terms,

$$r\beta_t = \sigma^{-1}(\bar{b}_t) \sum_{j=1}^{K} \frac{\partial U(\theta_t)}{\partial \theta_{i,t}} \gamma_j.$$

■

Proof of Lemma 1

Proof. Let $\theta_t = (\theta_{1,t}, \theta_{2,t}, ..., \theta_{K,t}) \in \Delta^{K-1}$ and $\Gamma = (\Gamma_{ij}) = (\gamma_i \cdot \gamma_j)$ be a $K \times K$ matrix.

$$\begin{aligned}
\theta_t^T \cdot \Gamma \cdot \theta_t &= \theta_t^T \begin{pmatrix} \gamma_1 \cdot \gamma_1 & \gamma_1 \cdot \gamma_2 & \cdots & \gamma_1 \cdot \gamma_K \\ \gamma_2 \cdot \gamma_1 & \gamma_2 \cdot \gamma_2 & \cdots & \gamma_2 \cdot \gamma_K \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_K \cdot \gamma_1 & \gamma_K \cdot \gamma_2 & \cdots & \gamma_K \cdot \gamma_K \end{pmatrix} \begin{pmatrix} \theta_{1,t} \\ \theta_{2,t} \\ \vdots \\ \theta_{K,t} \end{pmatrix} \\
&= (\theta_{1,t}, \theta_{2,t}, ..., \theta_{K,t}) \begin{pmatrix} \gamma_1 \cdot \sum_{i=1} \gamma_i \theta_{i,t} \\ \gamma_2 \cdot \sum_{i=1} \gamma_i \theta_{i,t} \\ \vdots \\ \gamma_K \cdot \sum_{i=1} \gamma_i \theta_{i,t} \end{pmatrix} = \theta_t \cdot \begin{pmatrix} \gamma_1 \cdot \prod \\ \gamma_2 \cdot \prod \\ \vdots \\ \gamma_K \cdot \prod \end{pmatrix} \\
&= \theta_{1,t}\gamma_1 \cdot \prod + \cdots + \theta_{K,t}\gamma_K \cdot \prod \\
&= \prod \cdot \prod \geq 0
\end{aligned}$$

where $\prod = \sum_{i=1}^{K} \gamma_i \theta_{i,t}$.

Furthermore, when $\mu^{\theta_t} = \mu_0$,

$$\begin{aligned}
\sum_{i=1}^{K} \gamma_i &= \sum_{k=1}^{K} \theta_{k,t} \mu(a_k^*, \bar{b}) - \sum_{k=1}^{K} \theta_{k,t} \mu^{\theta_t} \\
&= \sigma^{-1}(\bar{b})\big(\mu^{\theta_t} - \theta_{0,t}\mu_0 - \mu^{\theta_t}(1-\theta_{0,t})\big) \\
&= \sigma^{-1}(\bar{b})\theta_{0,t}\big(\mu^{\theta_t} - \mu_0\big) = 0
\end{aligned}$$

, which implies that $\prod = 0$ with $\theta_t = \left(\frac{1-\theta_{0,t}}{K}, ..., \frac{1-\theta_{0,t}}{K}\right) \in \Delta^{K-1}$ because $\theta_{0,t}\mu_0 + \frac{1-\theta_{0,t}}{K}\sum_{k=1}^{K}\mu_k = \mu_0$, that is, $\frac{1}{K}\sum_{k=1}^{K}\mu_k = \mu_0$. Therefore $\left(\frac{1-\theta_{0,t}}{K}, ..., \frac{1-\theta_{0,t}}{K}\right) \cdot \Gamma \cdot \left(\frac{1-\theta_{0,t}}{K}, ..., \frac{1-\theta_{0,t}}{K}\right) = \prod^2 \geq 0$ and hence Γ is a positive semi-definite matrix.

By the definition of γ_i,

$$\begin{aligned}
\sum_{i=1}^{K} \theta_{i,t}\gamma_i = \sum_{i=1}^{K} \theta_{i,t}^2 \frac{\mu_i - \mu^{\theta_t}}{\sigma} &= \frac{1}{\sigma}\left\{\sum_{i=1}^{K} \theta_{i,t}^2 \mu_i - \mu^{\theta_t} \sum_{i=1}^{K} \theta_{i,t}^2\right\} = 0 \\
\Leftrightarrow \sum_{i=1}^{K} \theta_{i,t}^2 \mu_i &= \mu^{\theta_t} \sum_{i=1}^{K} \theta_{i,t}^2 \\
\Leftrightarrow \sum_{i=1}^{K} \left\{\frac{\theta_{i,t}^2}{\sum_{i=1}^{K}\theta_{i,t}^2} - \theta_{i,t}\right\}\mu_i &= \theta_{0,t}\mu_0
\end{aligned}$$

Define $A^i = \frac{1}{\theta_{0,t}}\left\{\frac{\theta_{i,t}^2}{\sum_{i=1}^{K}\theta_{i,t}^2} - \theta_{i,t}\right\}$. Since $\sum_{i=1}^{K} A^i = \frac{1}{\theta_{0,t}}\left(1 - \sum_{i=1}^{K}\theta_{i,t}\right) = \frac{1}{\theta_{0,t}}\{1-(1-\theta_{0,t})\} = 1$, $\prod = 0$ is implied by that μ_0 is represented by a linear combination of $\{\mu_1, ..., \mu_K\}$ with weight of $\{A^1, ..., A^K\}$.

We can check that $A^1 < 0$ when $\theta_0 = \frac{1}{5}, \theta_1 = \frac{1}{10}, \theta_2 = \frac{1}{2}, \theta_3 = \frac{1}{5}$, and $\theta_j = 0$ for $j \geq 4$. Therefore, $\theta_t^T \cdot \Gamma \cdot \theta_t = \prod \cdot \prod > 0$ under the Condition 2, which implies that Γ is positive definite in Δ^{K-1}. ■

Proof of Proposition 6

It is enough to show that $M = \Delta^{K-1} \cup \partial M$ is *invariant*, which is defined below, for the diffusion process $\{\theta_t\}_{t\geq 0}$ with a prior $p \in \Delta^{K-1}$:

Definition 8 *We say a set $M \in \mathbb{R}^n$ is invariant for the diffusion process $\{X_t^x\}_{t\geq 0}$ if and only if*

$$X_0 = x \in M \ \ \textit{implies} \ \ X_t^x \in M, \ \ \mathbb{P}_x - \textit{a.s. for all } t \geq 0$$

Let $M = \overline{\Delta^{K-1}}$. Since M is a compact set with a piecewise smooth boundary, to show the invariance of S, we follow Cannarsa, Da Prato, and Frankowska (2010)[37]. Denote $M_i = \{x \in \mathbb{R}^K : x_i \geq 0\}$ for some $i \in \{1,...,K\}$ and $M_0 = \{x \in \mathbb{R}^K : \sum_{i=1}^K x_i \leq 1\}$. It is trivial that every M_j for $j \in \{0,1,...,K\}$ is a closed set of class $C^{2,1}$.[38] Each M_j is where, from the view of the normal type, small players certainly believe that the large player is not the type T_j. For any $K \in \mathbb{N}$, we can represent the belief space with an intersection of closed domain as below:

$$M = \overline{\Delta^{K-1}} = \cap_{j=0}^K M_j$$

Let $d_{\partial M_j}(\theta)$ be the Euclidean distant to M_j from $\theta \in \mathbb{R}^K$. The oriented distance function $\delta_{M_j}(\theta)$ to M_j from $\theta \in \mathbb{R}^K$ is defined as below:

$$\delta_{M_j}(\theta) = \begin{cases} d_{\partial M_j}(\theta), & \text{if } x \in M_j; \\ -d_{\partial M_j}(\theta), & \text{if } x \in M_j^c. \end{cases}$$

[37] See Theorem 3.2. in Cannarsa, Da Prato, and Frankowska (2010)

[38] In other words, it is a closed connected subset such that, for all point $x \in M_j$, there exist $r > 0$ and a function $\phi : B(x,r) \to \mathbb{R}$ which is twice differentiable on $B(x,r)$ with bounded Lipschitz second derivatives such that $\partial M_j \cap B(x,r) = \{y \in B(x,r) : \phi(y) = 0\}$. For each M_j, ∂M_j is a $(K-1)$-dimensional hyperplane in K-dimensional space, it can be represented with a linear function ϕ_j. Therefore, for each $j \in \{0,1,...,K\}$, ∂M_j is smooth.

By the definition, it is also clear that $\nabla\delta_{M_j}(\theta) = -\nu_{M_j}(\bar{\theta})$ where $\nu_{M_j}(\bar{\theta})$ is the outward normal to M_j at $\bar{\theta} \in \partial M_j$ such that $\delta_{M_j}(\theta) = \left|\theta - \bar{\theta}\right|$. For each $j \in \{0,1,...,K\}$, let $\mathscr{N}_\varepsilon^j = \left\{x \in \mathbb{R}^K : \left|\delta_{M_j}\right| < \varepsilon\right\}$. Then, since every boundary of M is included in a hyperplane that is belong to some M_j, for any j, there exists $\varepsilon_1 > 0$ such that

$$\mathrm{proj}_{\partial M_j}(\theta) \in \partial M$$

for all $\theta \in \Delta^{K-1} \cap M_j^{\varepsilon_1}$ where $\mathrm{proj}_{\partial M_j}(\theta)$ is the projection of θ to ∂M_j.

Fix $i \in \{1,...,K\}$. For any $\theta \in \Delta^{K-1}$, the distant function to M_i from $\theta \in \Delta^{K-1}$ is defined by $\delta_{M_i}(\theta) = \theta_i$. Therefore, on $\partial M_i \cap M$, the outward normal vector, ν is defined as $\nu^T = (\nu_1,...,\nu_i,...,\nu_K) = (0,...,0,1,0,...,0) = \nabla\delta_{M_i}(\theta)\big|_{\theta\in\partial M_i}$. For $j = 0$, the outward normal vector $\nu^T = (1,...,1) = \nabla\delta_{M_0}(\theta)\big|_{\theta\in\partial M_0}$. Therefore, for each $j \in \{0,1,...,K\}$, $0 \neq \nabla\delta_{M_0}(\theta)\big|_{\theta\in\partial M_j}$

Every $\theta \in \partial M$ belongs to ∂M_j for at most one $j \in \{0,1,...,K\}$. Fix $k \in \{1,...,K\}$. For such a $\theta \in \partial M_k$, $\gamma_k(\theta) = 0$ because $\theta_k = 0$. Therefore,

$$\left\langle \Gamma\nabla\delta_{M_k}(\theta)\big|_{\theta\in\partial M_k}, \nabla\delta_{M_k}(\theta)\big|_{\theta\in\partial M_k} \right\rangle = \sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta)\nu_i^k(\theta)\nu_j^k(\theta) = \gamma_k^2(\theta) = 0.$$

where $\langle\cdot,\cdot\rangle$ is the inner product defined on $\mathbb{R}^K \times \mathbb{R}^K$ and ν^k is the outward normal to ∂M_k at θ. Furthermore, for $\theta \in \partial M_i$,

$$\begin{aligned}
\frac{1}{2}\sum_{i,j=1}^{K}\gamma_i(\theta)\gamma_j(\theta)\frac{\partial^2\delta_{M_k}(\theta)}{\partial\theta_i\partial\theta_j}\bigg|_{\theta\in\partial M_k} &+ \sum_{i=1}^{K}\frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}\frac{\partial\delta_{M_k}(\theta)}{\partial\theta_i}\bigg|_{\theta\in\partial M_k} \\
&= \sum_{i=1}^{K}\frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}\frac{\partial\delta_{M_k}(\theta)}{\partial\theta_i}\bigg|_{\theta\in\partial M_k} \\
&= \frac{\gamma_k(\theta)\gamma_0(\theta)}{\theta_0} \\
&= 0.
\end{aligned}$$

For $j = 0$ and $\theta \in \partial M_0$,

$$\left\langle \Gamma \nabla \delta_{M_0}(\theta)\big|_{\theta \in \partial M_0}, \nabla \delta_{M_0}(\theta)\big|_{\theta \in \partial M_0} \right\rangle = \sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta)\nu_i^0(\theta)\nu_j^0(\theta)$$

$$= \left\{ \sum_{i=1}^{K} \gamma_i(\theta) \right\}^2$$

$$= \gamma_0(\theta)^2 = 0$$

because $\gamma_0(\theta) = 0$ for $\theta \in \partial M_0$. In the similar way,

$$\frac{1}{2}\sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta)\frac{\partial^2 \delta_{M_0}(\theta)}{\partial\theta_i \partial\theta_j}\bigg|_{\theta \in \partial M_0} + \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}\frac{\partial \delta_{M_0}(\theta)}{\partial \theta_i}\bigg|_{\theta \in \partial M_0}$$

$$= \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}\frac{\partial \delta_{M_0}(\theta)}{\partial \theta_i}\bigg|_{\theta \in \partial M_0}$$

$$= \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}$$

$$= 0.$$

Therefore, M is an inaccessible set. Since $\partial M = \cup_{i=0}^{K} \{ \partial M_i \cap M \}$, we conclude that $\partial \Delta^{K-1} = \Psi$ is a non-attainable set and the ε-regular boundary set $\partial \Delta^{\varepsilon} = \emptyset$ is empty.

Proof of Proposition 7

We use the Berg's Maximum theorem. First, note that both A and B are compact sets. Under Assumption 2, $\bar{b} \in \Delta(B)$ is mass-distribution[39], $\Delta(B)$ is also a compact set. Note the definition

[39] In other words, it is centered on each $b \in B$.

of $\mathcal{N}(\cdot,\cdot)$,

$$\mathcal{N}(\theta,r)=\Bigg\{(a,\bar{b}):\, a\in \text{argmax}_{a'\in A}g(a',\bar{b})+\left(\sigma(\bar{b})\cdot\sigma(\bar{b})^T\right)^{-1}\cdot z^T\cdot\begin{pmatrix}\mu_1-\mu^{\theta_t}\\ \vdots\\ \mu_K-\mu^{\theta_t}\end{pmatrix}\mu(a',\bar{b})$$

$$b\in \text{argmax}_{b'\in B}\sum_{i=0}^{K}\theta_i h(a_i,b',\bar{b})\ \forall b\in \text{supp}\,\bar{b}\Bigg\}$$

where $z^T=\frac{1}{r}\left(\theta_{1,t}U_{\theta_{1,t}},...,\theta_{K,t}U_{\theta_{K,t}}\right)$.

For any given (θ,r), it is trivial that $g(a',\bar{b})+\left(\sigma(\bar{b})\cdot\sigma(\bar{b})^T\right)^{-1}\cdot z^T\cdot\begin{pmatrix}\mu_1-\mu^{\theta_t}\\ \vdots\\ \mu_K-\mu^{\theta_t}\end{pmatrix}\mu(a',\bar{b})$ is continuous in (θ,r) because it is a linear in r and is quadratic in θ. Also, $\sum_{i=0}^{K}\theta_{i,t}h(a_i,b',\bar{b})$ is continuous in (θ,r) because it is linear in θ. Furthermore, for each (θ,r), there is no restriction on available $(a,\bar{b})$.

Then, by the Berge's Maximum Theorem, we conclude that both $g(a',\bar{b})+\left(\sigma(\bar{b})\cdot\sigma(\bar{b})^T\right)^{-1}\cdot z^T\cdot\begin{pmatrix}\mu_1-\mu^{\theta_t}\\ \vdots\\ \mu_K-\mu^{\theta_t}\end{pmatrix}\mu(a',\bar{b})$ and $\sum_{i=0}^{K}\theta_{i,t}h(a_i,b',\bar{b})$ are continuous in (θ,r). Moreover, $\mathcal{N}(\theta,r)$ is non-empty, compact-valued, and upper-hemicontinuous. Under Assumption 2, since $\mathcal{N}$ is single-valued, it is trivial that the upper-hemicontinuous correspondence $\mathcal{N}$ is continuous in (θ,r).

Proof of Proposition 8

For $\theta^* = (0,...,0)$, $z^* = 0$, and hence

$$\mathcal{N}(z^*,r) = \Big\{(a,\bar{b}) : a \in \text{argmax}_{a' \in A} g(a',\bar{b}) \\ b \in \text{argmax}_{b' \in B} h(a,b',\bar{b}) \ \forall b \in \text{supp}\,\bar{b}\Big\}$$

For $\theta^* = (\theta_1,...,\theta_K) = (0,...,0,1,0,...,0) \in \Delta^{K-1}$ where $\theta_i = 1$ for some $i \in \{1,...,K\}$, the reputation parameter $z^* = \left(0,...,0,\frac{1}{r}\frac{\partial U(\theta)}{\partial \theta_i}\big|_{\theta_i=1},0,...,0\right)$. By the definition of $\mathcal{N}(\cdot,\cdot)$,

$$\mathcal{N}(z^*,r) = \Big\{(a,\bar{b}) : a \in \text{argmax}_{a' \in A} g(a',\bar{b}) \\ b \in \text{argmax}_{b' \in B} h(a_i^*,b',\bar{b}) \ \forall b \in \text{supp}\,\bar{b}\Big\}$$

Therefore,

$$U_r(\theta^*) \in g\big(\mathcal{N}(z^*,r)\big)$$

Proof of Theorem 1[40]

Proposition 6 shows that $M = \overline{\Delta^{K-1}}$ is invariant for $\left\{\theta_t \in \Delta^{K-1}\right\}_{t\geq 0}$. Since the Markov equilibrium payoff function should satisfy the optimality equation (14), it is sufficient to show that there exists an unique solution to the equation (14). By Proposition 8, we do not need to assign any boundary condition. Under Condition 2, for any compact subset $K \subset \Delta^{K-1}$, the

[40]This proof is based on the proof of Theorem 4.4 in Cannarsa, Da Prato, and Frankowska (2010)

second-order operator L is uniformly elliptic. In other words,

$$\det \Gamma(\theta) > 0 \text{ for any } \theta \in K$$

This is equivalent with that $\Gamma(\cdot)$ is positive definite on Δ^{K-1}, which was guaranteed by Condition 2. Denote $f(\theta) = rg\big(a_0(\theta), \bar{b}(\theta)\big)$. For any $\theta \in M$ and given f,

$$U_r^f(\theta) = \int_0^\infty e^{-rs} \mathscr{P}_s f(\theta) ds$$

where $\mathscr{P}_t$ is the transition semigroup such that $\mathscr{P}_s f(\theta) = \mathbb{E}\big[f(\theta_s^p)\big]$ with a initial prior $\theta_0 = p \in M$ and a continuous function $f(\theta)$ defined on M.

It is well-known[41] that

$$L = \lim_{h \to 0+} \frac{\mathscr{P}_h - I}{h}$$

By the Hille-Yosida theorem[42], it is known that $U_r^f(\theta)$ satisfies the following:

$$LU_r^f(\theta) - rU_r^f(\theta) = -f$$

in M and $U_r^f(\theta) \in D(L)$ where $D(L) = \Big\{h \in C(M) : h \in \mathscr{W}_{loc}^2(\Delta^{K-1}), \text{ and } Lh \in C(M)\Big\}$ with $C(M)$ is the space of continuous function defined on M.

Next, check the regularity. Since M is a compact set of class $C^{2,1}$, there is a sequence of compact domains $\{M^i\}_{i \in \mathbb{N}}$ of class $C^{2,1}$ such that

$$M^i \subset \dot{M}^{i+1} \text{ and } \cup_{i=1}^\infty M^i = \Delta^{K-1}$$

[41] See Cannarsa, Da Prato, and Frankowska (2010).
[42] See Evans (2010).

where $\dot{A}$ is the interior of A. For a sufficiently large $i \in \mathbb{N}$, define

$$U_{r,i}^{f} = \int_0^{\infty} e^{-rs} \mathscr{P}_s^i f(\theta) ds$$

for $\theta \in M^i$ and a stopped transition semigroup $\mathscr{P}_s^i$ such that: for $\tau_i(p) = \inf\{t > 0 : \theta_t^p \in \partial M^i \text{ and } \theta_0^p = p\}$,

$$\mathscr{P}_s^i f(\theta) = \mathbb{E}\left[f(\theta_s^p) \chi_{\{t \le \tau_i(p)\}} \right]$$

Since $\mathscr{P}_s f(\theta) = \mathbb{E}\left[f(\theta_s^p)\chi_{\{t \le \tau\}}\right]$ where $\tau = \inf\{t > 0 : \theta_t^p \in \partial M \text{ and } \theta_0^p = p\}$, for any bounded and continuous function f on M,

$$\lim_{s \to \infty} \mathscr{P}_s^i f(\theta) = \mathscr{P}_s f(\theta)$$

Therefore, for any $\theta \in \Delta^{K-1}$, as $i \to \infty$,

$$U_{r,i}^{f} \to U_r^f$$

By Condition 2, $\{\theta_t\}$ is non-degenerate in M^i for each i. By the Hille-Yosida theorem, it is known that $U_{r,i}^f$ satisfies the following partial differential equation with boundary conditions:

$$\begin{aligned} LU_{r,i}^f(\theta) - rU_{r,i}^f(\theta) = -f \text{ in } M^i \\ U_{r,i}^f = 0 \text{ on } \partial M^i \end{aligned}$$

and $U_{r,i}^f \in \mathscr{W}^2(M^i)$. Since $U_{r,i}^f \to U_r^f$ and $U_{r,i}^f \in \mathscr{W}^2(M^i)$, we can conclude that $U_r^f \in \mathscr{W}(\Delta^{K-1})$. From now on, denote $U_r^f = U_r$.

For the uniqueness of U_r, it is sufficient to show that the solution to the following

$$LU_r(\theta) - rU_r(\theta) = 0$$

for $\theta \in M$ is $U_r = 0$ on M. Since we already know that $U_r \in C(M)$, $\|U_r\|_\infty$ where $\|\cdot\|_\infty$ is the supreme norm is bounded. Let $B = \|U_r\|_\infty \geq \infty$ and define:

$$V_r(\theta) = \frac{U_r(\theta)}{r(1+B)} - \frac{1}{r}$$

Since $U_r \in \mathscr{W}^2_{loc}(\Delta^{K-1})$, it is trivial $V_r \in \mathscr{W}^2_{loc}(\Delta^{K-1})$ and $V_r(\theta) > 0$ for any $\theta \in \Delta^{K-1}$. Furthermore,

$$LV_r(\theta) - rV_r(\theta) = -1$$

because $LU_r(\theta) - rU_r(\theta) = 0$.

For $f(\theta) = -1$, $\mathscr{P}_s f(\theta) = \mathbb{E}\big[f(\theta^p_s)\big] = -1$. Hence, on M, $U_r = \int_0^\infty e^{-rs}\mathscr{P}_s f(\theta)ds = -\int_0^\infty e^{-rs}ds = -\frac{1}{r}$. By the definition of $V_r(\theta)$,

$$V_r(\theta) \leq U_{r,i}(\theta)$$

for any $\theta \in M^i$ and sufficiently large $i \in \mathbb{N}$ because we know that $U_{r,i}(\theta) \to U_r(\theta) = -\frac{1}{r}$. Therefore,

$$V_r(\theta) \leq -\frac{1}{r}$$

for any $\theta \in M$. This implies that, for any $\theta \in M$,

$$U_r(\theta) \leq 0 \tag{20}$$

Next, consider the solution to the following

$$rU_r(\theta) - LU_r(\theta) = 0$$

for $\theta \in M$. Define:

$$V_r(\theta) = \frac{1}{r} - \frac{U_r(\theta)}{r(1+B)}$$

Since $U_r \in \mathscr{W}^2_{loc}(\Delta^{K-1})$, it is trivial $V_r \in \mathscr{W}^2_{loc}(\Delta^{K-1})$ and $V_r(\theta) > 0$ for any $\theta \in \Delta^{K-1}$. Furthermore,

$$rV_r(\theta) - LV_r(\theta) = 1$$

because $LU_r(\theta) - rU_r(\theta) = 0$.

For $f(\theta) = 1$, $P_s f(\theta) = \mathbb{E}\left[f(\theta_s^p)\right] = 1$. Hence, on M, $U_r = \int_0^\infty e^{-rs}\mathscr{P}_s f(\theta)ds = \int_0^\infty e^{-rs}ds = \frac{1}{r}$. By the definition of $V_r(\theta)$,

$$V_r(\theta) \leq U_{r,i}(\theta)$$

for any $\theta \in M^i$ and sufficiently large $i \in \mathbb{N}$ because we know that $U_{r,i}(\theta) \to U_r(\theta) = -\frac{1}{r}$. Therefore,

$$V_r(\theta) \leq \frac{1}{r}$$

for any $\theta \in M$. This implies that, for any $\theta \in M$,

$$U_r(\theta) \geq 0 \tag{21}$$

Therefore, by (21) and (22), the solution to the following problem:

$$LU_r(\theta) - rU_r(\theta) = 0$$

is $U_r(\theta) \equiv 0$ for any $\theta \in M$.

Proof of Lemma 2

Under Condition 2, the optimality equation becomes a non-degenerate second-order elliptic partial differential equation on Δ^{K-1} because Γ is positive definite. In other words, for any nonzero vector $x \in \mathbb{R}^K$ and belief $\theta \in \Delta^{K-1}$,

$$\sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta)x_i x_j > 0.$$

Since $\partial D^\delta/\Omega_\delta$ is a smooth part of boundary ∂D^δ, we can define the outward normal vector of which direction cosines are defined and three times continuously differentiable on $\partial D^\delta/\Omega_\delta$. This implies that, for any outward normal vector ν to $\partial D^\delta/\Omega_\delta$ at $\theta \in \partial D^\delta/\Omega_\delta$

$$\sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta)\nu_i(\theta)\nu_j(\theta) > 0.$$

Therefore, by Freidlin (1985)[43], for any $\varepsilon > 0$, every point of $\partial D^\delta/\Omega_\delta$ is a ε-regular point, which is a strongly regular point. By Proposition 6, it is trivial that Ω_δ is an non-attainable set.

[43] See Theorem 3.4.2 in Freidlin (1985)

Proof of Proposition 9

For any given $\delta > 0$, we can construct an increasing sequence of subsets $\{D^{\delta}\}$ of Δ^{K-1} that converges to Δ^{K-1} as δ goes to zero. Fix a sequence $\{\delta_l\}_{l\in\mathbb{N}}$ satisfying this conditions. We know that the optimality equation on D^{δ} is elliptic second-order partial differential equation and degenerate only on the boundary points that belongs to Ω_{δ}. Since Ω_{δ} is an non-attainable set in the boundary of D^{δ}, it is enough to assign a boundary condition only on the ε-regular set, $\partial D^{\delta}\backslash\Omega_{\delta}$[44].

Therefore, define the Dirichlet problem on D^{δ} as following:

$$
\begin{aligned}
LU^{\delta}(\theta) - rU^{\delta}(\theta) &= -rg\big(a(\theta),\bar{b}(\theta)\big) \text{ on } D^{\delta} \\
U^{\delta}(\theta) &= g\big(a(\theta),\bar{b}(\theta)\big) \text{ on } \partial\Delta^{\varepsilon}_{\delta}
\end{aligned}
$$

where $g(\cdot,\cdot)$ is a continuous function on Δ^{K-1}. On D^{δ_l}, for any give $t > 0$, by Dynkin's formula,

$$
\begin{aligned}
U^{\delta_l}(\theta) &= \mathbb{E}_{\theta}\left[g\big(a(\theta_{t\wedge\tau^{\delta}}),\bar{b}(\theta_{t\wedge\tau^{\delta}})\big)\exp\{-r(t\wedge\tau_{\delta})\}\right] + r\mathbb{E}_{\theta}\int_0^{t\wedge\tau^{\delta}} g\big(a_0(\theta_s),\bar{b}(\theta_s)\big)\exp\{-rs\}ds \\
&= I + II
\end{aligned}
$$

For the second term II, it is clear that

$$
\lim_{t\to\infty} r\mathbb{E}_{\theta}\int_0^{t\wedge\tau^{\delta}} g\big(a_0(\theta_s),\bar{b}(\theta_s)\big)\exp\{-rs\}ds = r\mathbb{E}_{\theta}\int_0^{\tau^{\delta}} g\big(a_0(\theta_s),\bar{b}(\theta_s)\big)\exp\{-rs\}ds
$$

[44] See Theorem 5.2 in Stroock and Varadhan (1972).

We rewrite the first term, I, as following:

$$
\begin{aligned}
&\mathbb{E}_\theta\left[g\big(a(\theta_{t\wedge\tau^\delta}),\bar{b}(\theta_{t\wedge\tau^\delta})\big)\exp\{-r(t\wedge\tau_\delta)\}\right]\\
&\quad=\mathbb{E}_\theta\left[g\big(a(\theta_{\tau^\delta}),\bar{b}(\theta_{\tau^\delta})\big)\exp\{-r\tau_\delta\}\chi_{\{\tau_\delta\le t\}}\right]+\mathbb{E}_\theta\left[g\big(a(\theta_t),\bar{b}(\theta_t)\big)\exp\{-rt\}\chi_{\{\tau_\delta>t\}}\right]\\
&\quad=III+IV
\end{aligned}
$$

By the uniform boundedness of $g(\cdot,\cdot)$, both collections of random variables in III and IV:

$$
\left\{g\big(a(\theta_{\tau^\delta}),\bar{b}(\theta_{\tau^\delta})\big)\exp\{-r\tau_\delta\}\chi_{\{\tau_\delta\le t\}}:t\ge 0\right\}\text{ and }\left\{g\big(a(\theta_t),\bar{b}(\theta_t)\big)\exp\{-rt\}\chi_{\{\tau_\delta>t\}}:t\ge 0\right\}
$$

are uniformly integrable.

Therefore,

$$
\begin{aligned}
\lim_{t\to\infty}\mathbb{E}_\theta\left[g\big(a(\theta_{\tau^\delta}),\bar{b}(\theta_{\tau^\delta})\big)\exp\{-r\tau_\delta\}\chi_{\{\tau_\delta\le t\}}\right]&=\mathbb{E}_\theta\left[g\big(a(\theta_{\tau^\delta}),\bar{b}(\theta_{\tau^\delta})\big)\exp\{-r\tau_\delta\}\right]\\
\lim_{t\to\infty}\mathbb{E}_\theta\left[g\big(a(\theta_t),\bar{b}(\theta_t)\big)\exp\{-rt\}\chi_{\{\tau_\delta>t\}}\right]&=0
\end{aligned}
$$

We conclude that, for any $\theta\in D^\delta$, the solution is given by:

$$
U^\delta(\theta)=\mathbb{E}_\theta\left[g\big(a(\theta_{\tau^\delta}),\bar{b}(\theta_{\tau^\delta})\big)\exp\{-r\tau_\delta\}\right]+r\mathbb{E}_\theta\int_0^{\tau^\delta}g\big(a_0(\theta_s),\bar{b}(\theta_s)\big)\exp\{-rs\}ds
$$

Proof of Theorem 2

Fix $\{\delta_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty}\delta_i=0$. For each δ_i, by Proposition 10, there exists a solution U^{δ_i} to the approximate problem that is defined on D^{δ_i} for each $i\in\mathbb{N}$. Consider the sequence of solutions, $\{U^{\delta_i}\}_{i\in\mathbb{N}}$ each of whom is bounded, measurable, and continuous almost everywhere on each D^{δ_i}. By the definition of equilibrium payoff, it is also uniformly bounded.

For each $\delta_i > 0$, we construct an approximate Dirichlet problem on D^{δ_i}. Therefore, it is sufficient to show that the sequence of solution, $\left\{U^{\delta_i}\right\}_{i\in\mathbb{N}}$, for each approximate Dirichlet problem has a pointwise limit, which is the approximate solution to the Dirichlet problem in Δ^{K-1}. We use the following lemmas. First, not that, by the way of constructing, for any $\delta_1 > \delta_2 > 0$, $D_{\delta_1} \subseteq D_{\delta_2} \subseteq \Delta^{K-1}$. Then, the first hitting time with respect to D^{δ} converges to τ as δ goes to zero.

Lemma 3 $\lim_{\varepsilon\to 0}\tau_\varepsilon(\omega) = \tau(\omega)$ *for any* $\omega \in \Omega^* = \left\{\omega' \in \Omega \,\middle|\, \tau \textit{ is achieved.}\right\}$. *For any* $\delta_1 > \delta_2 > 0$ *and* $\omega \in \Omega_{\varepsilon_2}$, $\tau_{\delta_1}(\omega) < \tau_{\delta_2}(\omega)$ *with probability 1.*

Proof. Consider ω for which τ could be achieved. Since the diffusion process θ_t is continuous with probability 1 from the view of the normal type large player and D_{δ_i} increases when the sequence $\{\delta_i\}$ converges to zero as i goes to ∞, it is trivial that $\left\{\tau_{\delta_i}(\omega)\right\}$ is an increasing sequence of random variables bounded by $\tau(\omega)$.

Supposed that $\lim_{\varepsilon\to 0}\tau_\varepsilon(\omega) = \tau'(\omega)$ for some $\tau'(\omega)$ and $P\left\{\tau'(\omega) < \tau(\omega)\right\} > 0$. Then, there is a $t_0 > 0$ such that:

$$P\left\{\tau'(\omega) < t_0 < \tau(\omega)\right\} > 0$$

Therefore, with $\theta_0^p = p$,

$$\theta_{t_0}^p \notin \cup_{i=1}^{\infty} D^{\delta_i} \; P-\text{a.s. on } \left\{\tau'(\omega) < t_0 < \tau(\omega)\right\}$$

This implies that:

$$\theta_{t_0} \in \partial\Delta^{K-1} \; P-\text{a.s. on } \left\{\tau'(\omega) < t_0 < \tau(\omega)\right\}$$

, which is a contradiction to the definition of τ. Therefore, $\lim_{\varepsilon\to 0}\tau_\varepsilon(\omega) = \tau(\omega)$ for any $\omega \in$

$\Omega^* = \{\omega' \in \Omega \mid \tau \text{ is achieved.}\}$. ■

Lemma 4 *Under Condition 1,* $P_\theta\left\{\omega \in \Omega \mid \theta^\theta_{\tau_\delta}(\omega) \in \partial D^\delta \backslash \Omega_\delta\right\} = 1$ *for any* $\delta > 0$.

Proof. By the result of Faingold and Sannikov (2011), under Condition 1, we can conclude that $\lim_{t\to\infty} \theta_{0,t} = 1$ with probability 1 from the perspective of the normal type. For any $\delta > 0$, we can construct α-neighborhood of $0 \in \Delta^{K-1}$ that is contained in Δ^{K-1}:

$$B_\alpha(0)\Big|_{\Delta^{K-1}} = \left\{\theta \in \Delta^{K-1} \mid |\theta| < \alpha\right\}$$

such that $\overline{B_\alpha(0)\big|_{\Delta^{K-1}}} \cap \overline{D^\delta} = \emptyset$ with $\alpha > 0$. Define $d = d(\alpha,\delta) = \text{dist}(B_\alpha(0)\big|_{\Delta^{K-1}}, D^\delta) > 0$ that is a distant between $B_\alpha(0)\big|_{\Delta^{K-1}}$ and D^δ.

Since $\lim_{t\to\infty} \theta_{0,t} = 1$ with probability 1, we can find $T_\alpha > 0$ such that for all $t > T_\alpha$, $\theta_{0,t} \in B_\alpha(0)\big|_{\Delta^{K-1}}$ with probability 1. Therefore, with probability 1, we can conclude that the belief process θ_t hit the boundary $\partial D^{\delta+d}/\Omega_{\delta+d}$. Since δ is arbitrary, we conclude that for any $\delta > 0$, $P_\theta\{\omega \in \Omega \big| \theta^\theta_{\tau_\delta}(\omega) \in \partial D^\delta/\Omega_\delta\} = 1$. ■

By the Proposition 9, for any given $\delta_i > 0$, the solution to an Dirichlet problem on D^{δ_i} is given by:

$$\begin{aligned}
U^{\delta_i}(\theta) &= \mathbb{E}_\theta\left[g\big(a(\theta_{\tau^{\delta_i}}), \bar{b}(\theta_{\tau^{\delta_i}})\big)\exp\{-r\tau_{\delta_i}\}\right] + r\mathbb{E}_\theta \int_0^{\tau^{\delta_i}} g\big(a_0(\theta_s), \bar{b}(\theta_s)\big)\exp\{-rs\}ds \\
&= I + II
\end{aligned}$$

For I, since θ_t is a continuous process with probability 1 and $g\big(a(\cdot), \bar{b}(\cdot)\big)e^{-r\tau_\delta}$ is a continuous and uniformly bounded function on D^{δ}, the limit as $\delta \to 0$ exists and it is given by:

$$\lim_{\delta \to 0} g\big(a(\theta_{\tau^{\delta}}), \bar{b}(\theta_{\tau^{\delta}})\big) \exp\{-r\tau_{\delta}\} = g\big(a(\theta_{\tau}), \bar{b}(\theta_{\tau})\big) \exp\{-r\tau\} \text{ a.s.}$$

Since $g(\cdot,\cdot)$ is uniformly bounded, the following collection of random variables

$$\left\{ g\big(a(\theta_{\tau^{\delta_i}}), \bar{b}(\theta_{\tau^{\delta_i}})\big) \exp\{-r\tau_{\delta_i}\} : i \in \mathbb{N} \right\}$$

is uniformly integrable. Therefore,

$$\lim_{i \to \infty} \mathbb{E}_{\theta} \left[g\big(a(\theta_{\tau^{\delta_i}}), \bar{b}(\theta_{\tau^{\delta_i}})\big) \exp\{-r\tau_{\delta_i}\} \right] = \mathbb{E}_{\theta} \left[g\big(a(\theta_{\tau}), \bar{b}(\theta_{\tau})\big) \exp\{-r\tau\} \right]$$

For each $\theta \in D^{\delta}$ such that $\theta = (\theta_{1,0}, ..., \theta_{K,0})$ and $\omega \in \Omega^*$, since $\int_0^{\infty} e^{-rs} g\big(a(\theta_s), \bar{b}(\theta_s)\big) ds < \infty$ is uniformly bounded, $\left\{ \nu_{\delta}(\theta) = \int_0^{\tau_{\delta}} e^{-rs} g(a(\theta_s, \bar{b}(\theta_s)) ds \;\middle|\; \delta > 0 \right\}$ is also a uniformly bounded and increasing sequence on the real line by Lemma 5 when δ goes to zero.

Therefore, by Bonzano-Weierstrass theorem, it has a pointwise limit. In other words,

$$\lim_{i \to \infty} \nu_{\delta_i}(\theta) = \int_0^{\lim_{i \to \infty} \tau_{\delta_i}} e^{-rs} g\big(a(\theta_s), \bar{b}(\theta_s)\big) ds = \int_0^{\tau} e^{-rs} g\big(a(\theta_s), \bar{b}(\theta_s)\big) ds$$

by Lemma 4.

Again, by the dominated convergence theorem and Lemma 5,

$$
\begin{aligned}
\lim_{i\to\infty} r\mathbb{E}_\theta \int_0^{\tau^{\delta_i}} g\big(a_0(\theta_s),\bar{b}(\theta_s)\big)\exp\{-rs\}ds &= r\mathbb{E}_\theta\bigg\{\lim_{i\to\infty}\int_0^{\tau_{\delta_i}} e^{-rs} g\big(a(\theta_s),\bar{b}(\theta_s)\big)ds\bigg\} \\
&= r\mathbb{E}_\theta\bigg\{\int_0^{\lim_{i\to\infty}\tau_{\delta_i}} e^{-rs} g\big(a(\theta_s),\bar{b}(\theta_s)\big)ds\bigg\} \\
&= r\mathbb{E}_x\bigg\{\int_0^{\tau} e^{-rs} g\big(a(\theta_s),\bar{b}(\theta_s)\big)ds\bigg\}
\end{aligned}
$$

Therefore, we can conclude that for any given $\theta \in \Delta^{K-1}$, $\{U^{\delta_i}(\theta)\}_{\delta_i\in\mathbb{N}}$ has a pointwise limit as $i \to \infty$, which is given by:

$$
\begin{aligned}
\lim_{i\to\infty} U^{\delta_i}(\theta) &= \mathbb{E}_\theta\Big[g\big(a(\theta_\tau),\bar{b}(\theta_\tau)\big)\exp\{-r\tau\}\Big] + r\mathbb{E}_\theta\int_0^\tau g\big(a_0(\theta_s),\bar{b}(\theta_s)\big)\exp\{-rs\}ds \\
&\equiv U(\theta)
\end{aligned}
$$

This $U(\theta)$ is the approximate Markov equilibrium payoff to the normal type large player when the prior is given at $\theta \in \Delta^{K-1}$.

Proof of Proposition 10

We use Proposition 5.5.22 in Karatzas and Shreve (2012) to prove this proposition. Let $I_\delta = (\delta, 1-\delta)$ for $0 < \delta < \frac{1}{2}$. First, under Condition 1, $\gamma_0(\theta') = \theta_0'\sigma^{-1}(\bar{b})\big(\mu_0 - \mu^{\theta'}\big) \neq 0$ and hence $\gamma_0^2(\theta') > 0$ for any $\theta' \in \Delta_\delta^0 = \Big\{\theta \in \Delta^0 \big| \delta < \theta < 1-\delta\Big\}$. For each $\theta_0 \in I_\delta$, $\gamma_0(\theta') = \gamma_0(1-\theta_0) = \big[\gamma_0(\theta')\big](\theta_0)$ is a function in $\theta_0 \in I_\delta$ because $\theta_0 + \theta' = 1$.

We can find a constant $A > 0$ and $\varepsilon > 0$, such that

$$
\int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon} \frac{1+\big|\big[\gamma_0^2(\theta')\big](y)/y\big|}{\big[\gamma_0^2(\theta')\big](y)}dy < A\int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon}\frac{1}{y} = A\Big\{\log(\theta_0+\varepsilon) - \log(\theta_0-\varepsilon)\Big\} < \theta_0
$$

where $\theta' \in \Delta_y^0$ for each $y \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$.

Let $S_\delta = \inf\{t \geq 0 : \theta_{0,t} \notin I_\delta\}$ be the first hitting time when the belief of normal type escapes I_δ. For any $t > 0$, under Condition 1,

$$\mathrm{P}\left\{\int_0^{t\wedge S_\delta} \left[\gamma_0^2(\theta')\right](\theta_{0,s})\big(1+\frac{1}{\theta_{0,s}}\big)ds < \infty\right\} = 1$$

because $\lim_{t\to\infty}\theta_{0,t} = 1$ with probability 1 under the normal type large player.

For the weak solution $\theta_0 = \{\theta_{0,t}\}_{t\geq 0}$ to the stochastic differential equation:

$$d\theta_{0,t} = \frac{\gamma_0(\theta_{1,t})\cdot\gamma_0(\theta_{1,t})}{\theta_{0,t}}dt + \gamma_0(\theta_{1,t})\cdot dZ_t^n$$

, it is trivial that

$$\mathrm{P}\left\{\theta_{t\wedge S_\delta} = \theta_{0,0} + \int_0^t \frac{\left[\gamma_0(\theta_{1,s})\right](\theta_{0,s})\cdot\left[\gamma_0(\theta_{1,s})\right](\theta_{0,s})}{\theta_{0,s}}\chi_{\{s\leq S_\delta\}}ds \right.$$
$$\left. + \int_0^t \left[\gamma_0(\theta_{1,s})\right](\theta_{0,s})\cdot\chi_{\{s\leq S_\delta\}}dZ_s^n \;\middle|\; 0 \leq t < \infty\right\} = 1$$

Since the score function $s(x) = \int_c^x e^{-\int_c^y \frac{2}{z}dz}dy = -\frac{c^2}{x} + c$ for any $c > 0$, on $\theta_0 \in I_\delta$, $s(\delta+) > -\infty$ and $s\big((1-\delta)-\big) < \infty$ where $s\big((1-\delta)-\big) = \lim_{\theta_0 \nearrow (1-\delta)} s(\theta_0)$ and $s(\delta+) = \lim_{\theta_0 \searrow \delta} s(\theta_0)$. Therefore, as $\delta \to 0$,

$$\begin{aligned}\mathrm{P}_{\theta_0}\left\{\lim_{t\to S_\delta}\theta_{0,t} = \delta\right\} &= 1 - \mathrm{P}_{\theta_0}\left\{\lim_{t\to S_\delta}\theta_{0,t} = 1-\delta\right\} \\ &= \frac{s\big((1-\delta)-\big) - s(\theta_0)}{s\big((1-\delta)-\big) - s(\delta+)} \\ &= \frac{\delta(1-\delta-\theta_0)}{\theta_0(1-2\delta)} \to 0\end{aligned}$$

This implies that $\mathrm{P}_{\theta_0}\left\{\inf_{0\le t\le S_\delta}\theta_t>\delta\right\}\to 1$ as $\delta\to 0$ for any $\theta_0\in I_\delta$.

For $\theta_0\in I=(0,1)$, since $s(0+)=-\infty$ and $s(1-)<\infty$, by Proposition 5.5.22 in Karatzas and Shreve (2012),

$$\mathrm{P}_{\theta_0}\left\{\lim_{t\to S}\theta_{0,t}=1\right\}=\mathrm{P}_{\theta_0}\left\{\inf_{0\le t<S}\theta_{0,t}>0\right\}=1$$

where $S=T_0\wedge T_1$.

Proof of Corollary 2

By Proposition 10,

$$\mathrm{P}_{\theta_0}\left\{\lim_{t\to S}\theta_{0,t}=1\right\}=\mathrm{P}_{\theta_0}\left\{\inf_{0\le t<S}\theta_{0,t}>0\right\}=1$$

This implies that, at the first hitting time $\tau>0$, the belief process does not touch the point that is sufficiently close to where $\theta_1=1$ before it touch some point near $\theta_1=0$. Therefore,

$$\mathbb{E}_{\theta_0}\left[g\big(a(\theta_\tau),\bar b(\theta_\tau)\big)e^{-r\tau}\right]=g\big(a(\theta_\tau),\bar b(\theta_\tau)\big)\mathbb{E}_{\theta_0}\left[e^{-r\tau}\right]=g\big(a^*,\bar b^*\big)\mathbb{E}_{\theta_0}\left[e^{-r\tau}\right]$$

where $\left(a^*,\bar b^*\right)\in\mathscr{N}(0,r)$. Therefore, by Theorem 2,

$$U(\theta)=g(a^*,\bar b^*)\mathbb{E}_\theta\left[\exp\{-r\tau\}\right]+r\mathbb{E}_\theta\int_0^\tau g\big(a_0(\theta_s),\bar b(\theta_s)\big)\exp\{-rs\}ds \tag{22}$$

where

$$(a^*,\bar b^*)\in\left\{(a,\bar b):a\in\operatorname{argmax}_{a'\in A}g(a',\bar b),\text{ and }b\in\operatorname{argmax}_{b'\in B}h(a,b',\bar b)\ \forall b\in\operatorname{supp}\bar b\right\}$$

Proof of Proposition 11

Fix $\theta_t \in \Delta^{K-1}$ and $r > 0$. By definition of $\mathcal{N}(\theta_t, r)$,

$$\mathcal{N}(\theta_t, r) = \left\{ (a, \bar{b}_t) : a \in \operatorname{argmax}_{a' \in A} g(a', \bar{b}_t) + \left(\sigma(\bar{b}_t) \cdot \sigma(\bar{b}_t)^T\right)^{-1} \cdot z^T \cdot \begin{pmatrix} \mu_1 - \mu^{\theta_t} \\ \vdots \\ \mu_K - \mu^{\theta_t} \end{pmatrix} \mu(a', \bar{b}_t) \right.$$

$$\left. b \in \operatorname{argmax}_{b' \in B} \sum_{i=0}^{K} \theta_i h(a_{i,t}, b', \bar{b}_t) \ \forall b \in \operatorname{supp} \bar{b}_t \right\}$$

where $z^T = \frac{1}{r}\left(\theta_{1,t} U_{\theta_{1,t}}, ..., \theta_{K,t} U_{\theta_{K,t}}\right)$, for $\left(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right) \in \mathcal{N}(\theta_t, r)$ and any $i \in \{1, ..., K\}$,

$$\begin{aligned} g\big(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\big) - \theta_{i,t} g\big(a_i^*, \bar{b}_t(\theta_t)\big) \\ &\geq \left(\sigma(\bar{b}_t) \cdot \sigma(\bar{b}_t)^T\right)^{-1} \\ &\quad \cdot \frac{1}{r} \sum_{i=1}^{K} \theta_{i,t} \frac{\partial U(\theta_t)}{\partial \theta_{i,t}} \big(\mu_i(\theta_t) - \mu^{\theta_t}(\theta_t)\big) \cdot \big(\theta_{i,t}\mu_i(\theta_t) - \mu_0(\theta_t)\big) \end{aligned}$$

where $\mu_i(\theta_t) = \mu\left(a_i^*, \bar{b}_t(\theta_t)\right)$ and $\mu_0(\theta_t) = \mu\left(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right)$. For the sake of simplicity, denote $g_0(\theta_t) = g\left(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\right)$ and $g_i(\theta_t) = g\left(a_i^*, \bar{b}_t(\theta_t)\right)$. Hence,

$$\begin{aligned} g\big(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\big) - \sum_{i=0}^{K} \theta_{i,t} g\big(a_i^*, \bar{b}_t(\theta_t)\big) \\ &\geq \left(\sigma(\bar{b}_t) \cdot \sigma(\bar{b}_t)^T\right)^{-1} \\ &\quad \cdot \frac{1}{r} \sum_{i=1}^{K} \theta_{i,t} \frac{\partial U(\theta_t)}{\partial \theta_{i,t}} \big(\mu_i(\theta_t) - \mu^{\theta_t}(\theta_t)\big) \cdot \big(\mu^{\theta_t}(\theta_t) - \mu_0(\theta_t)\big) \end{aligned}$$

Denote $g^{\theta_t}(\theta_t) = \sum_{i=0}^{K} \theta_{i,t} g\big(a_i^*, \bar{b}_t(\theta_t)\big)$. Then,

$$\begin{aligned}
g\big(a_{0,t}(\theta_t), \bar{b}_t(\theta_t)\big) - g^{\theta_t}(\theta_t) \geq{}& \big(\sigma(\bar{b}_t)\cdot\sigma(\bar{b}_t)^T\big)^{-1} \\
&\cdot \frac{1}{r}\sum_{i=1}^{K}\theta_{i,t}\frac{\partial U(\theta_t)}{\partial \theta_{i,t}}\big(\mu_i(\theta_t) - \mu^{\theta_t}(\theta_t)\big)\cdot\big(\mu^{\theta_t}(\theta_t) - \mu_0(\theta_t)\big) \\
\geq{}& \big(\sigma(\bar{b}_t)\cdot\sigma(\bar{b}_t)^T\big)^{-1} \\
&\cdot \frac{C_1}{r}\sum_{i=1}^{K}\theta_{i,t}\big(\mu_i(\theta_t) - \mu^{\theta_t}(\theta_t)\big)\cdot\big(\mu^{\theta_t}(\theta_t) - \mu_0(\theta_t)\big) \\
\geq{}& \big(\sigma(\bar{b}_t)\cdot\sigma(\bar{b}_t)^T\big)^{-1} \\
&\cdot \frac{C_1}{r}\big(\mu^{\theta_t}(\theta_t) - \mu_0(\theta_t)\big)\cdot\big(\mu^{\theta_t}(\theta_t) - \theta_{0,t}\mu_0 - (1-\theta_{0,t})\mu^{\theta_t}(\theta_t)\big) \\
\geq{}& \big(\sigma(\bar{b}_t)\cdot\sigma(\bar{b}_t)^T\big)^{-1}\frac{C_1\theta_{0,t}}{r}\left|\mu^{\theta_t}(\theta_t) - \mu_0(\theta_t)\right|^2 \\
\geq{}& \big(\sigma(\bar{b}_t)\cdot\sigma(\bar{b}_t)^T\big)^{-1}\frac{C_1C_2\theta_{0,t}}{r}\left|\theta_{0,t}a_{0,t}(\theta_t) + \sum_{i=1}^{K}\theta_{i,t}a_i^* - a_{0,t}(\theta_t)\right|^2 \\
\geq{}& (\sigma(\bar{b}_t)\cdot\sigma(\bar{b}_t)^T)^{-1}\theta_{0,t}(1-\theta_{0,t})\frac{C_1C_2}{r}\left|a_{0,t}(\theta_t) - \sum_{i=1}^{K}\frac{\theta_{i,t}}{1-\theta_{0,t}}a_i^*\right|^2
\end{aligned}$$

where the second inequality is from Condition 3(a) and the fifth inequality is from Condition 3(b).

Let $\overline{g} = \max_{\theta_t\in\Delta^{K-1}} g(a_t(\theta_t), \bar{b}_t(\theta_t))$ and $\underline{g} = \min_{\theta_t\in\Delta^{K-1}} g(a_t(\theta_t), \bar{b}_t(\theta_t))$ for any uniformly bounded $g(\cdot,\cdot)$. Then, the right-hand side is bounded by $\overline{g} - \underline{g}$. Therefore, as r goes to zero,

$$\left|a_{0,t}(\theta_t) - \sum_{i=1}^{K}\frac{\theta_{i,t}}{1-\theta_{0,t}}a_i^*\right|^2 \;\to\; 0$$

This convergence result holds for any $\theta_t \in \Delta^{K-1}$. Hence, as the normal type large player becomes sufficiently patient, the equilibrium action converges to a convex combination of all the other commitment types' actions.

Proof of Theorem 3

By Condition $5(a)$, $\beta \in \Delta^{K-1}$. Let $F_i = \left\{ \theta \in \mathbb{R}^{K-1} : \beta_i \le \theta_i \le 1 \right\}$ for $i \in \{1, ..., K\}$ and $F_0 = \left\{ \theta \in \mathbb{R}^{K-1} : \sum_{i=1}^{K} \theta_i \le 1 \right\}$. It is trivial that $F = \cap_{i=1}^{K} F_i$ is a compact subset in $\mathbb{R}^{K-1}$. For each $\theta \in F$ and $i \in \{1, ..., K\}$, the distance from θ to F_i, $\delta_{F_i}(\theta)$, is $\theta_i - \beta_i$. Therefore, $\nabla \delta_{F_i}(\theta)^T = (0, ..., 0, 1, 0, ..., 0)$ where the i-th component is 1.

Every $\theta \in \partial F$ belongs to ∂F_j for at most one $j \in \{0, 1, ..., K\}$. Fix $k \in \{1, ..., K\}$. For such a $\theta \in \partial F_k$, $\gamma_k(\theta) = 0$ by Condition 5(b). Therefore,

$$\left\langle \Gamma \nabla \delta_{F_k}(\theta)\big|_{\theta \in \partial F_k}, \nabla \delta_{F_k}(\theta)\big|_{\theta \in \partial F_k} \right\rangle = \sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta) \nu_i^k(\theta) \nu_j^k(\theta) = \gamma_k^2(\theta) = 0.$$

where $\langle \cdot, \cdot \rangle$ is the inner product defined on $\mathbb{R}^K \times \mathbb{R}^K$ and ν^k is the outward normal to ∂F_k at θ. Furthermore, for $\theta \in \partial F_k$,

$$\begin{aligned}
\frac{1}{2} \sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta) \frac{\partial^2 \delta_{F_k}(\theta)}{\partial \theta_i \partial \theta_j}\bigg|_{\theta \in \partial F_k} + \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0} \frac{\partial \delta_{F_k}(\theta)}{\partial \theta_i}\bigg|_{\theta \in \partial F_k} & \\
&= \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0} \frac{\partial \delta_{F_k}(\theta)}{\partial \theta_i}\bigg|_{\theta \in \partial F_k} \\
&= \frac{\gamma_k(\theta)\gamma_0(\theta)}{\theta_0} \\
&= 0.
\end{aligned}$$

For $j=0$ and $\theta \in \partial F_0$,

$$\left\langle \Gamma \nabla \delta_{F_0}(\theta)\big|_{\theta\in\partial F_0}, \nabla \delta_{F_0}(\theta)\big|_{\theta\in\partial F_0} \right\rangle = \sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta)v_i^0(\theta)v_j^0(\theta)$$
$$= \left\{ \sum_{i=1}^{K} \gamma_i(\theta) \right\}^2$$
$$= \gamma_0(\theta)^2 = 0$$

because $\gamma_0(\theta) = 0$ for $\theta \in \partial F_0$.

In the similar way,

$$\frac{1}{2}\sum_{i,j=1}^{K} \gamma_i(\theta)\gamma_j(\theta)\frac{\partial^2 \delta_{F_0}(\theta)}{\partial \theta_i \partial \theta_j}\bigg|_{\theta\in\partial F_0} + \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}\frac{\partial \delta_{F_0}(\theta)}{\partial \theta_i}\bigg|_{\theta\in\partial F_0}$$
$$= \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}\frac{\partial \delta_{F_0}(\theta)}{\partial \theta_i}\bigg|_{\theta\in\partial F_0}$$
$$= \sum_{i=1}^{K} \frac{\gamma_i(\theta)\gamma_0(\theta)}{\theta_0}$$
$$= 0.$$

Therefore, F is an invariant set, and hence, the interior of F denoted by $\dot{F}$ is also an invariant set. This implies that for any $\theta \in \dot{F}$:

$$\mathbb{P}\left\{ \theta_t \in \dot{F} : \forall t \geq 0 \right\} = 1$$

In the similar way of Theorem 1, we can conclude that there exist a unique Markov equilibrium payoff function, $U_{\dot{F}}(\cdot)$, that satisfies the optimality equation on $\dot{F}$. Suppose Condition 3 is imposed on the $U_{\dot{F}}(\cdot)$. For any $\theta \in \dot{F}$, denote $\alpha = \min\{d_1, ..., d_K\}$ where $d_i = \left|A^*(\theta) - a_i^*\right|$. Since F is strictly included in Δ^{K-1}, $d_i > 0$ for any $i \in \{1, ..., K\}$. Therefore, for any $j \in$

$\{1, ..., K\}$ and $\theta \in \dot{F}$,

$$\left| A^*(\theta) - a_j^* \right| \geq \alpha > 0.$$