APPROXIMATING ROTATION ALGEBRAS AND INCLUSIONS OF C*-ALGEBRAS

BY

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DISSERTATION

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Abstract

In the first part of this thesis, we will follow Kirchberg's categorical perspective to establish new notions of WEP and QWEP relative to a C*-algebra, and develop similar properties as in the classical WEP and QWEP. Also we will show some examples of relative WEP and QWEP to illustrate the relations with the classical cases.

The focus of the second part of this thesis is the approximation of rotation algebras in the quantum Gromov–Hausdorff distance. We introduce the completely bounded quantum Gromov–Hausdorff distance and show that for even dimensions, the higher dimensional rotation algebras can be approximated by matrix algebras in this sense. Finally, we show that for even dimensions, matrix algebras converge to the rotation algebras in the strongest form of Gromov–Hausdorff distance, namely in the sense of Latrémolière's Gromov–Hausdorff propinquity.

To my parents and my grandparents.

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List of Symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathcal{K}$	the space of compact operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}(\mathcal{H})$	the space of bounded operators on a Hilbert space $\mathcal{H}$
$\mathcal{M}(A)$	the multiplier algebra of a C*-algebra A
$\mathbb{F}_n$	free group on n generators
$\mathbb{F}_\infty$	free group on countably infinite generators
r.w.i.	relatively weakly injective
$C_r^*(G)$	the reduced group C*-algebra of a group G
$C^*(G)$	the full group C*-algebra of a group G
$\mathcal{L}(A, B)$	linear operators from a space A to B
$(T_t)_{t\geq 0}$	noncommutative symmetric Markov semigroup
Γ^A	gradient form associated to a generator A
$C(\mathbb{T})$	the space of continuous functions on the torus
M_n	the algebra of $n \times n$ matrices
$\mathcal{A}_\theta$	the 2-dimensional rotation algebra associated to θ
Poly	the vector space of noncommutative polynomials of degree k, $k \geq 1$
Θ	$d \times d$ skew symmetric matrix with entries in $[0, 1)$
$\mathcal{A}_\Theta^d$	the d-dimensional rotation algebra associated to Θ
$\mathcal{R}_\Theta$	the rotation von Neumann algebra associated to Θ
$\vert\vert\vert f \vert\vert\vert$	Lip-norm of f
$d_{oq,R}^{cb}(X, Y)$	R-cb-quantum Gromov–Hausdorff distance of X and Y
$\Lambda_F((A, L_A), (B, L_B))$	Gromov-Hausdorff propinquity between (A, L_A) and (B, L_B)

Chapter 1

Introduction

In the first part of this thesis, we investigate a new notion of operator-valued WEP and QWEP. This part is joint work with Jian Liang [LR14]. Let us recall that the notions of weak expectation property (abbreviated as WEP) were introduced by E. Christopher Lance in his paper [Lan73] of 1973, as a generalization of nuclearity of C*-algebras. In 1993, Eberhard Kirchberg [Kir93] revealed remarkable connections between tensor products of C*-algebras and Lance's weak expectation property. He defined the notion of QWEP as a quotient of a C*-algebra with the WEP, and formulated the famous QWEP conjecture that all C*-algebras are QWEP. He showed a vast amount of equivalences between various open problems in operator algebras. In particular, he showed that the QWEP conjecture is equivalent to an affirmative answer to the Connes Embedding Problem.

It is known that for two QWEP von Neumann algebras M and N, and an amenable C*-subalgebra A, the reduced amalgamated free product $M *_A N$ is also QWEP [Jun05]. But the answer is not known for a general C*-subalgebra A. To reduce the complexity of the problem, we consider the property of being QWEP relative to a C*-algebra A, We are interested to see if in the case where both M and N are QWEP relative to A, whether $M *_A N$ is QWEP relative to A or not.

To study the notion of relative QWEP, first we need to define the relative WEP. Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ be the C*-algebra of bounded operators on $\mathcal{H}$, and $\mathcal{K}$ be the C*-algebra of compact operators on $\mathcal{H}$. Recall that in [Lan73], Lance defined a C*-algebra A to have the WEP, if for $A \subset \mathcal{B}(\mathcal{H})$, A is relatively weakly injective (abbreviated as *r.w.i.*) in $\mathcal{B}(\mathcal{H})$, namely there exists a u.c.p. map from $\mathcal{B}(\mathcal{H})$ to A^{**} such that its restriction to A is the identity. To define the notion of the relative WEP, there are two natural ways of replacing $\mathcal{B}(\mathcal{H})$ in the framework of Hilbert C*-modules. Recall that for a C*-algebra D, any C*-algebra can be regarded as a C*-subalgebra of $\mathcal{L}(\mathcal{H}_D)$, where $\mathcal{H}_D$ is the Hilbert D-module given by the completion of $\mathcal{H} \otimes D$, and $\mathcal{L}(\mathcal{H}_D)$ is the C*-algebra of bounded adjointable D-linear maps on $\mathcal{H}_D$. Another way of representation is to replace $\mathcal{L}(\mathcal{H}_D)$ by the von Neumann algebra $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$. We say that A has the WEP$_1$ (respectively, WEP$_2$) relative to C*-algebra D, if A is *r.w.i.* in $\mathcal{L}(\mathcal{H}_D)$ (respectively, $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$).

In Chapter 3, we investigate basic properties of these two notions. We discover that Kirchberg's method in his seminal work on non-semisplit extensions is functorial, and gives rise to properties as in the classical case. In particular, we establish the tensorial characterization for the two notions. Also we study the relation between the two notions of relative WEP. This leads to a more general question: Let A and B be C*-algebras such that $A \subset B \subset A^{**}$ canonically. Does this imply that B is *r.w.i.* in A^{**}? The answer to this question turns out to be negative in general. However, in the special case where $\mathcal{K}$ is the space of compact operators on $\mathcal{H}$, $A = \mathcal{K} \otimes_{\min} D$ and $B = \mathcal{L}(\mathcal{H}_D)$, B is *r.w.i.* in $A^{**} = (\mathcal{K} \otimes_{\min} D)^{**} = \mathcal{B}(\mathcal{H})\bar{\otimes} D^{**}$. This shows that $D\mathrm{WEP}_1$ implies $D\mathrm{WEP}_2$. We also show that the converse is not true.

Furthermore, we define two notions of WEP relative to a C*-algebra D more generally. Let E_D be a Hilbert D-module, and $\mathcal{L}(E_D)$ be the C*-algebra of bounded adjointable linear operators on E_D. Also let $E_{D^{**}}$ be the weakly closed Hilbert D^{**}-module, and $\mathcal{L}^w(E_{D^{**}})$ be the von Neumann algebra of bounded adjointable linear operators on $E_{D^{**}}$. We say that a C*-algebra A has the $D\mathrm{WEP}_1$ if it is relatively weakly injective in $\mathcal{L}(E_D)$, i.e. for a faithful representation $A \subset \mathcal{L}(E_D)$, there exists a ucp map $\mathcal{L}(E_D) \to A^{**}$, which preserves the identity on A. Respectively, we define the $D\mathrm{WEP}_2$ to be the relatively weak injectivity in $\mathcal{L}(E_{D^{**}})$. By universality of $\mathcal{H}_D$, these definitions of WEP_1 and WEP_2 coincide with what we defined above. After investigating some basic properties, we establish a tensor product characterization of DWEP following Kirchberg's framework. Let $\max_1^D$ be the tensor norm on $A \otimes C^*\mathbb{F}_\infty$ induced from the inclusion $A \otimes C^*\mathbb{F}_\infty \subseteq \mathcal{L}(E_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$ for some universal Hilbert D-module E_D^u and $A \subset \mathcal{L}(E_D^u)$. Then a C*-algebra A has the $D\mathrm{WEP}_1$, if and only if

$$A \underset{\max_1^D}{\otimes} C^*\mathbb{F}_\infty = A \underset{\max}{\otimes} C^*\mathbb{F}_\infty.$$

We have the similar result for $D\mathrm{WEP}_2$, where the $\max_2^D$-norm is defined by replacing E_D^u with some universal weakly closed D^{**}-module $E_{D^{**}}^u$.

Then we define two notions of relative QWEP, derived from two notions of relative WEP. Following Kirchberg's scheme, after developing basic properties of relative QWEP, we show that the two notions are equivalent, unlike the case in the relative WEP.

Finally, we investigate some properties of WEP and QWEP relative to some special classes of C*-algebras, and illustrate the relations with classical results in the theory of WEP and QWEP. In particular our examples show that the property of having DWEP is a tool to tell the C*-algebras apart.

The second part of this thesis is joint work with Marius Junge and Qiang Zeng [JRZ16]. The focus of this part is convergence of matrix algebras to rotation algebras for specific choices of distance. The notion of *Gromov–Hausdorff distance* of metric spaces was first introduced by Gromov [Gro81]. Since then, there has been plenty of research on what the "correct" definition of distance should be. Our goal is to define the right distances that best serve our purposes.

In order to modify Gromov's definition, Rieffel [Rie04b] adapted the definition to *quantum Gromov–Hausdorff distance* corresponding to *quantum metric spaces*. To introduce this distance, Rieffel defined the notion of "compact quantum metric spaces", which was motivated by a similar notion given by Connes in his theory of quantum Riemannian geometry defined by Dirac operators. Rieffel's main motivation to introduce the quantum Gromov–Hausdorff distance came from string theory. Since quantum tori have proved to be very useful in quantum physics, his main example in [Rie04b] involves these objects. He shows that for a consistent choice of "metrics", if a sequence of parameters $(\theta_n)_n$ converges to a parameter θ, then the corresponding sequence of quantum tori, $(\mathcal{A}_{\theta_n})_n$, converges in quantum Gromov–Hausdorff distance to $\mathcal{A}_\theta$. But in his definition of Gromov–Hausdorff distance, Rieffel used the Hausdorff distance of the state spaces, which are not very easy objects to work with. This was the reason that Li introduced the notion of *order-unit quantum Gromov–Hausdorff distance* [Li06] by replacing the state spaces. His main objects are order-unit spaces equipped with a Lipschitz norm. For such an object $(A, |||\cdot|||_A)$ and $r \geq 0$, he introduced $\mathcal{D}_r(A)$ as follows to replace the state spaces in Rieffel's definition:

$$\mathcal{D}_r(A) = \{a \in A : |||a|||_A \leq 1, \|a\| \leq r\}.$$

Li's order-unit quantum Gromov–Hausdorff distance is equivalent to quantum Gromov–Hausdorff distance [Li06], but using these alternative objects makes the arguments a lot smoother.

We used this definition in Chapters 4 and 5 to approximate $C(\mathbb{T})$, the space of continuous functions on the torus, and the rotation algebra $\mathcal{A}_\theta$. In the last section of Chapter 5 we extend this definition to operator spaces by using operator-valued coefficients. We introduce the notion of *completely bounded quantum Gromov–Hausdorff distance* of two operator spaces as follows. Let X and Y be two operator spaces, (X, L) be a *Lip operator space structure*, $R > 0$, $\|\cdot\|$ denote the C*-norm and

$$\mathcal{D}_R(M_n(X)) = \{x \in M_n(X) : |||x|||_{M_n(L)} \leq 1, \|x\|_{M_n(X)} \leq R\}.$$

We denote the *R-cb-quantum Gromov–Hausdorff distance* of X and Y by $d^{cb}_{oq,R}(X, Y)$, and define it by

$$d^{cb}_{oq,R}(X, Y) = \inf \sup_{n \in \mathbb{N}} \{d_H[\mathrm{id} \otimes \iota_X(\mathcal{D}_R(M_n(X))), \mathrm{id} \otimes \iota_Y(\mathcal{D}_R(M_n(Y)))]\},$$

where d_H denotes the Hausdorff distance, and the infimum runs over all operator spaces V and completely isometric embeddings $\iota_X : X \to V$ and $\iota_Y : Y \to V$. This definition seems to be stronger than that of Wu's [Wu06]. We show that there exists a sequence of matrix algebras that converges to the 2-dimensional rotation algebras in this sense. Furthermore, we have the boundedness of the diameter and compactness properties, i.e. we show that the map id : $(A, |||\cdot|||_A) \to (A, \|\cdot\|)$ is completely bounded. Furthermore, we can construct a net of completely bounded finite rank maps that approximate the identity map in the cb sense.

Later we show that there exists a sequence of $n^d \times n^d$ matrix algebras that converges to the d-dimensional rotation algebra in cb quantum Gromov-Hausdorff distance for even d. In fact we can even go further and show that the convergence occurs in the strongest possible form, i.e. in the "propinquity" sense. In [Lat15], Latrémolière introduces the stronger notion of *Gromov–Hausdorff propinquity*. He shows that to prove the "closeness" of two spaces $(A, |||.|||_A)$ and $(B, |||.|||_B)$ in the propinquity sense, it is enough for the spaces to satisfy the following criteria:

There exist two **-homomorphisms* $\pi_A : A \hookrightarrow \mathcal{B}(\mathcal{H})$ *and* $\pi_B : B \hookrightarrow \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$ *such that the following hold:*

1. *For all* $a \in A$ *such that* $|||a|||_A \leq 1$*, there exists* $b \in B$ *such that* $|||b|||_B \leq 1$ *and* $\|\pi_A(a) - \pi_B(b)\|_D < \varepsilon$*,*

2. *For all* $b \in B$ *such that* $|||b|||_B \leq 1$*, there exists* $a \in A$ *such that* $|||a|||_A \leq 1$ *and* $\|\pi_A(a) - \pi_B(b)\|_D < \varepsilon$*.*

In Chapter 6, we use Kirchberg–Blanchard's machinery [Bla97] with nuclearity as its main ingredient, to construct the *-homomorphisms needed to satisfy the criteria above.

In our work, the main objects that we deal with are of the form $(A, \|.\|, |||\cdot|||_A)$. In our approach we do not always use the same Lip-norms as those of Li's. In fact we are taking a dynamic approach by using Markov semigroups $(T_t)_{t\geq 0}$ on the matrix algebras. In particular we are interested in *Poisson* and *heat* semigroups. For instance, in Chapter 5, we study the 2-dimensional case by taking the heat semigroup defined by

$$T_t(u_\theta^j v_\theta^k) = e^{-t(|j|^2+|k|^2)} u_\theta^j v_\theta^k,$$

where u_θ, v_θ are the generators of the rotation algebra associated to θ. Using this semigroup we can define a Lipschitz norm given by the gradient form, Γ, associated to the generator of the semigroup. In general, if $T_t = e^{-tA}$, the gradient form associated to A is given by

$$\Gamma^A(f, g) = \frac{1}{2}[A(f^*)g + f^*A(g) - A(f^*g)],$$

for f, g in the domain of A. If Γ is the gradient form associated to the heat semigroup, one can show that for any $x = u_\theta^j v_\theta^k$ and $y = u_\theta^{j'} v_\theta^{k'}$,

$$\Gamma(x, y) = (jj' + kk')(u_\theta^j v_\theta^k)^* u_\theta^{j'} v_\theta^{k'}.$$

We define the Lipschitz norm on the rotation algebra by $|||f||| = \|\Gamma(f, f)^{1/2}\|_\infty$. This norm turns out to be equivalent to Connes' derivative given by Dirac operators [Con94]. Hence it is the correct choice of Lip-norm for our purposes.

This part of the thesis is organized as follows. In Chapters 4 we give a brief introduction to order-unit spaces and conditionally negative length functions. Then we show some analytic estimates which will provide the main ingredients to prove the convergence in the later chapters. We conclude the chapter by studying the 1-dimensional case, i.e. we find an approximation for $C(\mathbb{T})$, the space of continuous functions on the torus. Here $(T_t)_{t\geq 0}$ is the Poisson semigroup.

In Chapter 5 we consider the 2-dimensional case, and we give an approximation of the 2-dimensional rotation algebras by matrix algebras. In this section we choose $(T_t)_{t\geq 0}$ to be the heat semigroup. Then we introduce the notion of "cb-quantum Gromov-Hausdorff" distance, prove a compactness theorem and show that we have an estimate for the 2-dimensional rotation algebras in the cb-quantum Gromov-Hausdorff distance.

Finally, in Chapter 6 we explore the higher dimensional rotation algebras and approximate them with matrix algebras for even dimensions. Furthermore, for the even dimensions we show that there is a sequence of matrix algebras that converge to the rotation algebras in the sense of Gromov–Hausdorff propinquity.

Chapter 2

Preliminaries

2.1 WEP and QWEP

The notion of WEP is from Lance [Lan73], inspired by Tomiyama's extensive work on conditional expectations. Kirchberg in [Kir93] raises the famous QWEP conjecture and establishes its several equivalences. Here we list some useful results for readers' convenience. Most of the results and proofs can be found in Ozawa's survey paper [Oza04].

Definition 2.1.1. Let A be a unital C*-subalgebra of a unital C*-algebra B. We say A is relatively weakly injective (abbreviated as r.w.i.) in B, if there is a ucp map $\varphi : B \to A^{**}$ such that $\varphi|_A = \mathrm{id}_A$.

For von Neumann algebras $M \subset N$, the relative weak injectivity is equivalent to the existence of a (non-normal) conditional expectation from N to M.

We say a C*-algebra A has the weak expectation property (abbreviated as WEP), if it is relatively weakly injective in $\mathbb{B}(\mathcal{H})$ for a faithful representation $A \subset \mathbb{B}(\mathcal{H})$.

Since $\mathbb{B}(\mathcal{H})$ is injective, the notion of WEP does not depend on the choice of a faithful representation of A. We say a C*-algebra is QWEP if it is a quotient of a C*-algebra with the WEP. The QWEP conjecture raised by Kirchberg in [Kir93] states that all C*-algebras are QWEP.

From the definition of *r.w.i.*, it is easy to see the following transitivity property.

Lemma 2.1.2. *For C*-algebras $A_0 \subseteq A_1 \subseteq A$, such that A_0 is relatively weakly injective in A_1, A_1 is relatively weakly injective in A, then A_0 is relatively weakly injective in A.*

Kirchberg also shows the following local characterization for *r.w.i.* property.

Lemma 2.1.3. *Let $A \subset B$ be C*-algebras. The following are equivalent.*

1. *the C*-algebra A is* r.w.i. *in B;*

2. *for any finite-dimensional subspace $E \subset B$ and any $\varepsilon > 0$, there exists a contraction $\psi : E \to A$ such that $\|\psi|_{E\cap A} - \mathrm{id}\,|_{E\cap A}\| < \varepsilon$.*

By the above lemma, it is easy to see that the property of *r.w.i.* is also closed under direct product.

Lemma 2.1.4. *If $(A_i)_{i\in I}$ is a net of C^*-algebras such that A_i is relatively weakly injective in B_i for all $i \in I$, then $\Pi_{i\in I}A_i$ is relatively weakly injective in $\Pi_{i\in I}B_i$.*

In [Lan73], Lance establishes the following tensor product characterization of the WEP. The proof of the theorem is called *The Trick*, and we will be using this throughout the paper. In the following, let $\mathbb{F}_\infty$ denote the free group with countably many infinite generators, and $C^*\mathbb{F}_\infty$ be the full group C*-algebra of $\mathbb{F}_\infty$.

Theorem 2.1.5. *A C*-algebra A has the WEP, if and only if*

$$A \underset{\max}{\otimes} C^*\mathbb{F}_\infty = A \underset{\min}{\otimes} C^*\mathbb{F}_\infty.$$

As a consequence of the above theorem, we have the following result.

Corollary 2.1.6. *A C*-algebra A has the WEP if and only if for any inclusion $A \subseteq B$, A is relatively weakly injective in B.*

Similar to the WEP, the QWEP is also preserved by the relatively weak injectivity as following.

Lemma 2.1.7. *If a C*-algebra A is relatively weakly injective in a QWEP C*-algebra, then it is QWEP.*

Although the WEP does not pass to the double dual, the QWEP property is more flexible.

Proposition 2.1.8. *A C*-algebra A is QWEP if and only if A^{**} is QWEP.*

As a corollary of the above proposition, $\mathbb{B}(\mathcal{H})^{**}$ is QWEP. Moreover we have the following equivalence.

Corollary 2.1.9. *A C*-algebra A is QWEP if and only if A is relatively weakly injective in $\mathbb{B}(\mathcal{H})^{**}$.*

2.2 Hilbert C*-Modules

The notion of Hilbert C*-modules first appeared in a paper by Irving Kaplansky in 1953 [Kap53]. The theory was then developed by the work of William Lindall Paschke in [Pas73] . In this section we give a brief introduction to Hilbert C*-modules and present some of their fundamental properties which we are going to use throughout this paper.

Definition 2.2.1. Let D be a C*-algebra. An inner-product D-module is a linear space E which is a right D-module with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$, for $x \in E$, $a \in D$, $\lambda \in \mathbb{C}$, and a map $(x, y) \longmapsto \langle x, y\rangle : E \times E \to D$ with the following properties:

1. $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x,\ y,\ z \in E$ and $\alpha,\ \beta \in \mathbb{C}$;

2. $\langle x, ya\rangle = \langle x, y\rangle a$ for $x,\ y \in E$ and $a \in D$;

3. $\langle y, x\rangle = \langle x, y\rangle^*$ for $x,\ y \in E$;

4. $\langle x, x\rangle \geq 0$; if $\langle x, x\rangle = 0$, then $x = 0$.

For $x \in E$, we let $\|x\| = \|\langle x, x\rangle\|^{1/2}$. It is easy to check that if E is an inner-product D-module, then $\|\cdot\|$ is a norm on E.

Definition 2.2.2. An inner-product D-module which is complete with respect to its norm is called a Hilbert D-module or a Hilbert C*-module over the C*-algebra D.

Note that any C*-algebra D is a Hilbert D-module itself with the inner product $\langle x, y\rangle = x^*y$ for x and y in D. Another important example of a Hilbert C*-module is the following:

Example 2.2.3. Let $\mathcal{H}$ be a Hilbert space. Then the algebraic tensor product $\mathcal{H} \otimes_{alg} D$ can be equipped with a D-valued inner-product:

$$\langle \xi \otimes a, \eta \otimes b\rangle = \langle \xi, \eta\rangle a^*b \qquad (\xi, \eta \in \mathcal{H}, a, b \in D).$$

Let $\mathcal{H}_D = \mathcal{H} \otimes D$ be the completion of $\mathcal{H} \otimes_{alg} D$ with respect to the induced norm. Then $\mathcal{H}_D$ is a Hilbert D-module.

Let E and F be Hilbert D-modules. Let t be an adjointable map from E to F, i.e. there exists a map t^* from F to E such that

$$\langle tx, y\rangle = \langle x, t^*y\rangle, \qquad \text{for } x \in E \text{ and } y \in F.$$

One can easily see that t must be right D-linear, that is, t is linear and $t(xa) = t(x)a$ for all $x \in E$ and $a \in D$. It follows that any adjointable map is bounded, but the converse is not true – a bounded D-linear map need not be adjointable. Let $\mathcal{L}(E, F)$ be the set of all adjointable maps from E to F, and we abbreviate $\mathcal{L}(E, E)$ to $\mathcal{L}(E)$. Note that $\mathcal{L}(E)$ is a C*-algebra equipped with the operator norm.

Now we review the notion of compact operators on Hilbert D-modules, as an analogue to the compact operators on a Hilbert space. Let E and F be Hilbert D-modules. For every x in E and y in F, define a map $\theta_{x,y} : E \to F$ by

$$\theta_{x,y}(z) = y\langle x, z\rangle \qquad \text{for } z \in E.$$

One can check that $\theta_{x,y} \in \mathcal{L}(E,F)$ and $\theta^*_{x,y} = \theta_{y,x}$. We denote by $\mathcal{K}(E,F)$ the closed linear subspace of $\mathcal{L}(E,F)$ spanned by $\{\theta_{x,y} : x \in E, y \in F\}$, and we abbreviate $\mathcal{K}(E,E)$ to $\mathcal{K}(E)$. We call the elements of $\mathcal{K}(E,F)$ *compact* operators.

Let E be a Hilbert D-module and Z be a subset of E. We say that Z is a *generating set* for E if the closed submodule of E generated by Z is the whole of E. If E has a countable generating set, we say that E is countably generated.

In [Kas80], Kasparov proves the following theorem known as the *absorption theorem*, which shows the universality of $\mathcal{H}_D$ in the category of Hilbert D-modules.

Theorem 2.2.4. *Let D be a C^*-algebra, $\mathcal{H}$ be an infinite dimensional Hilbert space and E be a countably generated Hilbert D-module. Then $E \oplus \mathcal{H}_D \approx \mathcal{H}_D$, i.e. there exists an element $u \in \mathcal{L}(E \oplus \mathcal{H}_D, \mathcal{H}_D)$ such that $u^*u = 1_{E\oplus\mathcal{H}_D}$ and $uu^* = 1_{\mathcal{H}_D}$.*

Remark 2.2.5. Using the absorption theorem, for an arbitrary Hilbert D-module E, we have $\mathcal{L}(E \oplus \mathcal{H}_D) \simeq \mathcal{L}(\mathcal{H}_D)$. Hence we have an embedding of $\mathcal{L}(E)$ in $\mathcal{L}(\mathcal{H}_D)$ and a conditional expectation from $\mathcal{L}(\mathcal{H}_D)$ to $\mathcal{L}(E)$.

Before we proceed to the main results of Hilbert C*-modules, let us recall the notion of *multiplier algebra* of a C*-algebra.

Definition 2.2.6. Let A and B be C*-algebras. If A is an ideal in B, we call A an essential ideal if there is no nonzero ideal of B that has zero intersection with A. Or equivalently, if $b \in B$ and $bA = \{0\}$, then $b = 0$.

It can be shown that for any C*-algebra A, there is a unique (up to isomorphism) maximal C*-algebra which contains A as an essential ideal, i.e. $A \cap J \neq \emptyset$ for all ideals J. This algebra is called the multiplier algebra of A and is denoted by $\mathcal{M}(A)$.

Theorem 2.2.7. *If E is a Hilbert D-module, then $\mathcal{L}(E) = \mathcal{M}(\mathcal{K}(E))$.*

Note that if $E = D$ for a unital C*-algebra D, then $D = \mathcal{K}(D)$ and $\mathcal{L}(D) = \mathcal{M}(D)$.

In the special case where $E = \mathcal{H}_D$, we have

$$\mathcal{K}(\mathcal{H}_D) \simeq \mathcal{K}(\mathcal{H}) \underset{\min}{\otimes} D = \mathcal{K} \underset{\min}{\otimes} D,$$

where $\mathcal{K} = \mathcal{K}(\mathcal{H})$ is the C*-algebra of the compact operators. Therefore, by Theorem 2.2.7 we have

$$\mathcal{L}(\mathcal{H}_D) \simeq \mathcal{M}(\mathcal{K} \underset{\min}{\otimes} D).$$

In [Kas80] Kasparov introduces a GNS type of construction in the context of Hilbert C*-modules, known as the KSGNS construction (for Kasparov, Stinespring, Gelfand, Neimark, Segal) as follows.

Theorem 2.2.8. *Let A be a C*-algebra, E be a Hilbert D-module and let $\rho : A \to \mathcal{L}(E)$ be a completely positive map. There exists a Hilbert D-module E_ρ, a *-homomorphism $\pi_\rho : A \to \mathcal{L}(E_\rho)$ and an element v_ρ of $\mathcal{L}(E, E_\rho)$, such that*

$$\rho(a) = v_\rho^* \pi_\rho(a) v_\rho \qquad (a \in A),$$

$$\pi_\rho(A) v_\rho E \quad \textit{is dense in} \quad E_\rho.$$

As a consequence of the above theorem, Kasparov shows that given a C*-algebra D, any separable C^*-algebra can be considered as a C*-subalgebra of $\mathcal{L}(\mathcal{H}_D)$. This indicates that $\mathcal{L}(\mathcal{H}_D)$ plays the similar role in the category of Hilbert C*-modules to that of $\mathbb{B}(\mathcal{H})$ in the category of C*-algebras.

Proposition 2.2.9. *Let A be a separable C*-algebra. Then there exists a faithful nondegenerate *-homomorphism $\pi : A \to \mathcal{L}(\mathcal{H}_D)$.*

As we see, $\mathcal{L}(\mathcal{H}_D)$ plays the role of $\mathbb{B}(\mathcal{H})$. Note that $\mathbb{B}(\mathcal{H})$ is also a von Neumann algebra, but $\mathcal{L}(\mathcal{H}_D)$ is not in general. Paschke in [Pas73] introduces self-dual Hilbert C*-modules to play the similar role in the von Neumann algebra context.

Let E be a Hilbert D-module. Each $x \in E$ gives rise to a bounded D-module map $\hat{x} : E \to D$ defined by $\hat{x}(y) = \langle y, x \rangle$ for $y \in E$. We will call E *self-dual* if every bounded D-module map of E into D arises by taking D-valued inner products with some $x \in E$. For instance, if D is unital, then it is a self-dual Hilbert D-module. Any self-dual Hilbert C*-module is complete, but the converse is not true.

For von Neumann algebra N, it is natural to consider the the self-dual Hilbert N-module E_N, because of the following theorem from [JS05].

Theorem 2.2.10. *For a Hilbert C*-module E over a von Neumann algebra N, the following conditions are equivalent:*

1. *The unit ball of E is strongly closed;*
2. *E is principal, or equivalently, E is an ultraweak direct sum of Hilbert C*-modules $q_\alpha N$, for some projections q_α;*
3. *E is self-dual;*
4. *The unit ball of E is weakly closed.*

We denote the algebra of adjointable maps on E_N closed in the weak operator topology by $\mathcal{L}^w(E_N)$.

Remark 2.2.11. According to [Pas73] and the absorption theorem, for a von Neuamann algebra N, we have that $\mathcal{L}^w(E_N) = e\mathbb{B}(\mathcal{H})\bar{\otimes}Ne$ for some projection e.

Remark 2.2.12. Let N be a von Neumann subalgeba of M, such that $N = zM$ for some central projection $z \in M$. Then one can unitize the inclusion map $\iota : \mathbb{B}(\ell_2)\bar{\otimes}N \hookrightarrow \mathbb{B}(\ell_2)\bar{\otimes}M$. Indeed since $\mathbb{B}(\ell_2)$ is a type I_∞ factor, the projection $1 \otimes z : \mathbb{B}(\ell_2)\bar{\otimes}M \to \mathbb{B}(\ell_2)\bar{\otimes}N$ is properly infinite, and hence it is equivalent to identity on $\mathbb{B}(\ell_2)\bar{\otimes}M$ [Tak02]. Let $1 \otimes z = v^*v$, and $\mathrm{id}_{\mathbb{B}(\ell_2)\bar{\otimes}M} = vv^*$. Note that $1 \otimes z) \circ \iota = \mathrm{id}_{\mathbb{B}(\ell_2)\bar{\otimes}N}$. Multiplying by v from left and by v^* from right, we get $v\iota v^* = \mathrm{id}_{\mathbb{B}(\ell_2)\bar{\otimes}N}$.

2.3 Kirchberg's observations on the multiplier algebra

In this section, we explore Kirchberg's seminal paper on non-semisplit extensions in detail. In particular we show the factorization property explicitly for readers' convenience.

Let A, B and C be unital C*-algebras. We say a map $h : A \to B$ *factors through* C *approximately via ucp maps in point-norm topology* if there exist ucp maps $\phi_n : A \to C$ and $\psi_n : C \to B$ such that the following diagram commutes approximately in point-norm topology.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "\phi_n"'] & & B \\ & C \arrow[ur, "\psi_n"'] & \end{tikzcd}$$

i.e. $\|(\psi_n \circ \phi_n)(x) - h(x)\| \to 0$ for all $x \in A$.

The idea of the shortened proof of the following theorem was suggested to us by an anonymous referee.

Theorem 2.3.1. *Let A be a σ-unital C*-algebra and $\mathcal{M}(A)$ be its multiplier algebra. Then the identity map on $\mathcal{M}(A)$ factors through $\ell_\infty(A)$ approximately via ucp maps in point-norm topology.*

Sketch of proof. Since A is σ-unital, given a finite subset $F \subset \mathcal{M}(A)$ and $\varepsilon > 0$, one can find an approximate identity $(e_n)_{n=1}^\infty$ such that $e_0 = 0$, $e_n e_{n+1} = e_n$, and $\|[e_n, x]\| < \delta_n$ for all n and $x \in F$, where $\delta_n > 0$ are chosen so that $\|[x, (e_n - e_{n-1})^{1/2}]\| < 2^{-n}\varepsilon$ for all $n \geq 1$ and $x \in F$. Define

$$\begin{aligned} \phi_{F,\varepsilon} : \mathcal{M}(A) &\to \ell_\infty(A) \\ a &\mapsto (e_{n+1} a e_{n+1})_{n=1}^\infty, \end{aligned}$$

and

$$\begin{aligned}\psi_{F,\varepsilon} : \ell_\infty(A) &\to \mathcal{M}(A)\\ (a_n)_{n=1}^\infty &\mapsto \sum_n (e_n - e_{n-1})^{1/2} a_n (e_n - e_{n-1})^{1/2},\end{aligned}$$

where the series converges strictly. Then for all $x \in F$, we have

$$\|\psi_{F,\varepsilon} \circ \phi_{F,\varepsilon}(x) - x\| < \varepsilon.$$

Now consider $\Lambda = \{(F, \varepsilon) : F \subset \mathcal{M}(A),\ F \text{ finite set}, \varepsilon > 0\}$, equipped with the following order: $(F, \varepsilon) < (F', \varepsilon')$ if and only if $F \subset F'$ and $\varepsilon' < \varepsilon$. We get two nets of ucp maps $(\phi_\alpha)_{\alpha\in\Lambda}$ and $(\psi_\alpha)_{\alpha\in\Lambda}$. Then for any $x \in \mathcal{M}(A)$ and $\varepsilon > 0$, there exists α_0 such that for all $\alpha > \alpha_0$, $\|\psi_\alpha \circ \phi_\alpha(x) - x\| < \varepsilon$. ☐

Using the theorem above, we can establish the following result on the relation between $\mathcal{M}(A)$ and A^{**}.

Corollary 2.3.2. *Suppose A is a C^*-algebra and $\mathcal{M}(A)$ is its multiplier algebra. Then $\mathcal{M}(A)$ is relatively weakly injective in A^{**}.*

Proof. Let ϕ and ψ be as above. Since there is a natural inclusion $\mathcal{M}(A) \subset A^{**}$, we can define $\tilde{\phi} : A^{**} \to \ell_\infty(A^{**})$ as an extension of ϕ, by $\tilde{\phi}(a) = (e_{n+1} a e_{n+1})_{n=1}^\infty$ for all $a \in A^{**}$. Then the folowing diagram commutes locally

$$\begin{tikzcd}
\mathcal{M}(A) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\phi"'] \arrow[dd, hook] & & \mathcal{M}(A) \\
& \ell_\infty(A) \arrow[ur, "\psi"'] & \\
A^{**} \arrow[r, "\tilde{\phi}"] & \ell_\infty(A^{**}) \arrow[u] &
\end{tikzcd}$$

Let ε be arbitrary, F a finite-dimensional subspace of A^{**}, and $F_0 = F \cap \mathcal{M}(A)$. Then we get a net $\Lambda = (F_0, F, \varepsilon)$. Define $\delta_{n,\lambda} : A^{**} \to \mathcal{M}(A)$ locally, as the composition of the following maps

$$F \hookrightarrow A^{**} \xrightarrow{\tilde{\phi}} \ell_\infty(A^{**}) \longrightarrow \ell_\infty(A) \xrightarrow{\psi} \mathcal{M}(A).$$

Then we have

$$\lim_{n,\lambda} \delta_{n,\lambda}(1) = 1.$$

Let $\delta := \lim_{n,\lambda} \delta_{n,\lambda} : A^{**} \to \mathcal{M}(A)^{**}$ in the weak *-topology. Then δ gives the required conditional expectation. Now it follows from Lemma 2.1.3 that $\mathcal{M}(A)$ is *r.w.i.* in A^{**}. ☐

Chapter 3

Operator-valued Kirchberg Theory

3.1 Module version of the weak expectation property

This Chapter is joint work with Jian Liang [LR14]. We will only state the results in this Section, since the proofs have alreay appeared in his thesis (UIUC Ph.D. thesis, 2015).

The notion of *r.w.i.* is a paired relation between a C*-subalgebra and its parent C*-algebra. If the parent C*-algebra is $\mathcal{B}(\mathcal{H})$, the *r.w.i.* property is equivalent to the WEP. By carefully choosing a parent C*-algebra, we can define the notion of WEP relative to a C*-algebra.

Let $\mathcal{C}$ be a collection of inclusions of unital C*-algebras $\{(A \subseteq X)\}$.

For a C*-algebra D, there are two classes of objects that we will discuss throughout this paper.

1. $\mathcal{C}_1 = \{A \subseteq \mathcal{L}(E_D)\}$, where E_D is a Hilbert D-module.

2. $\mathcal{C}_2 = \{A \subseteq \mathcal{L}^w(E_{D^{**}})\}$, where $E_{D^{**}}$ is a self dual Hilbert D^{**}-module.

Definition 3.1.1. A C*-algebra A is said to have the DWEP$_i$ for $i = 1, 2$, if there exists a pair of inclusions $A \subseteq X$ in $\mathcal{C}_i$ such that A is relatively weakly injective in X.

Note the in the case where $D = \mathbb{C}$, WEP coincides with DWEP$_i$, $i = 1, 2$.

Notice that the notion of DWEP is a *r.w.i.* property. By Corollary 2.1.6, the WEP implies the DWEP$_i$, for $i = 1, 2$. Also, inherited from *r.w.i.* property, we have the following lemmas for DWEP.

Lemma 3.1.2. *Let A_0 and A_1 be C*-algebras such that A_0 is relatively weakly injective in A_1. If A_1 has the $DWEP_i$ for $i = 1, 2$, then so does A_0.*

Remark 3.1.3. By the absorption theorem and Remark 2.2.5 and 2.2.11, $\mathcal{L}(E_D)$ is *r.w.i.* in some $\mathcal{L}(\mathcal{H}_D)$ and $\mathcal{L}^w(E_{D^{**}})$ is *r.w.i.* in some $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$. Sometimes it is more convenient to consider the DWEP$_1$ as the relatively weak injectivity in $\mathcal{L}(\mathcal{H}_D)$, and the DWEP$_2$ as the relatively weak injectivity in $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$, because of the concrete structures.

Example 3.1.4. From the above, all WEP algebras have $D\text{WEP}_i$ for arbitrary C*-algebra D. Also, D has the $D\text{WEP}_i$ trivially for 1-dimensional Hilbert space $\mathcal{H}$. Our first nontrivial example of $D\text{WEP}_i$ is $\mathcal{K}\otimes_{\min} D$. For the first class $\mathcal{C}_1$, $\mathcal{K}\otimes_{\min} D$ is a principle ideal of $\mathcal{L}(\mathcal{H}_D)$, and thus is *r.w.i.* in $\mathcal{L}(\mathcal{H}_D)$. For the second class $\mathcal{C}_2$, note that $(\mathcal{K}\otimes_{\min} D)^{**} = \mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$, so $\mathcal{K}\otimes_{\min} D$ is *r.w.i.* in $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$. By universality of $\mathcal{L}(\mathcal{H}_D)$ and $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$, $\mathcal{K}\otimes_{\min} D$ has the $D\text{WEP}_i$ for both $i = 1, 2$.

Because of the injectivity of $\mathcal{B}(\mathcal{H})$, we see that the notion of WEP does not depend on the representation $A \subseteq \mathcal{B}(\mathcal{H})$. By constructing a universal object in the classes $\mathcal{C}_i$, we can define the $D\text{WEP}_i$ independent of inclusions.

Lemma 3.1.5. *A C*-algebra A has the DWEP$_i$ for some inclusion $A \subseteq X$ in $\mathcal{C}_i$, if and only if there exists a universal object X^u and $A \subseteq X^u$ in $\mathcal{C}_i$, such that*

1. *A is relatively weakly injective in X^u;*
2. *If A is relatively weakly injective in some X, then there exists a ucp map from X^u to X, which is identity on A.*

Proof. Note that for all ucp maps $\rho : A \to \mathcal{L}(E_D)$, by KSGNS construction there exists a Hilbert D-module E_ρ and a *-homomorphism $\pi_\rho : A \to \mathcal{L}(E_\rho)$. Let $E_D^u = \bigoplus_\rho E_\rho$. Then any $\mathcal{L}(E_D)$ containing A can be embedded into $\mathcal{L}(E_D^u)$, and there exists a truncation $\mathcal{L}(E_D^u) \to \mathcal{L}(E_D)$. Now suppose A is *r.w.i.* in some $\mathcal{L}(E_D)$. Then it is also *r.w.i.* in $\mathcal{L}(E_D^u)$ □

Following Lance's tensor product characterization Theorem 2.1.5, we have a similar result for the $D\text{WEP}_i$, for $i = 1, 2$. We only present the result for the first class. The other case can be proved similarly.

Let $A \subseteq \mathcal{L}(E_D^u)$ be the universal representation as observed in the proof of Lemma 3.1.5. We define a tensor norm $\max_1^D$ on $A\otimes C^*\mathbb{F}_\infty$ to be the norm induced from the inclusion $A\otimes C^*\mathbb{F}_\infty \subseteq \mathcal{L}(E_D^u)\otimes_{\max} C^*\mathbb{F}_\infty$ isometrically. This induced norm is categorical in the sense that if ϕ is a ucp map from A to B, then $\phi\otimes\mathrm{id}$ extends a ucp map from $A\otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $B\otimes_{\max_1^D} C^*\mathbb{F}_\infty$. Indeed, let ι be the inclusion map from B to its universal representation $\mathcal{L}^B(E_D^u)$, then $\iota\circ\phi$ is a ucp map from A to $\mathcal{L}^B(E_D^u)$. By KSGNS and the construction of $\mathcal{L}^A(E_D^u)$, there exists a ucp map from $\mathcal{L}^A(E_D^u)$ to $\mathcal{L}^B(E_D^u)$ extending the map $\iota\circ\phi$. Hence we have a composition of ucp maps

$$A \underset{\max_1^D}{\otimes} C^*\mathbb{F}_\infty \subseteq \mathcal{L}^A(E_D^u) \underset{\max}{\otimes} C^*\mathbb{F}_\infty \to \mathcal{L}^B(E_D^u) \underset{\max}{\otimes} C^*\mathbb{F}_\infty,$$

whose image is $B\otimes_{\max_1^D} C^*\mathbb{F}_\infty$.

Theorem 3.1.6. *A C^*-algebra A has the the DWEP$_1$, if and only if*

$$A \underset{\max_1^D}{\otimes} C^*\mathbb{F}_\infty = A \underset{\max}{\otimes} C^*\mathbb{F}_\infty.$$

It is natural to explore the relationship between DWEP$_1$ and DWEP$_2$. We have the following.

Theorem 3.1.7. *If a C^*-algebra A has the DWEP$_1$, then it also has the DWEP$_2$.*

In fact, the converse of the above theorem is not true, and we will give a counterexample in Section 5.

The following lemmas are crucial for the proof of the above Theorem.

Lemma 3.1.8. *Suppose that the identity map on a C^*-algebra A factors through a C^*-algebra B approximately via ucp maps in point-norm topology, i.e. there exist two nets of ucp maps $\phi_i : A \to B$ and $\psi_i : B \to A$, such that $\|\psi_i \circ \phi_i(x) - x\| \to 0$ for $x \in A$. If B has the DWEP$_i$, then so does A.*

Another lemma we need is that the DWEP$_i$ property is preserved under the direct product.

Lemma 3.1.9. *If $(A_i)_{i\in I}$ is a net of C^*-algebras with the DWEP$_i$, then $\prod_{i\in I} A_i$ has the DWEP$_i$.*

Kirchberg [Kir93] shows that for a C*-algebra A, the multiplier algebra $\mathcal{M}(A)$ factors through $\ell_\infty(A)$ approximately by ucp maps (Theorem 2.3.1). Using this fact, we have the following.

Corollary 3.1.10. *Suppose that the C^*-algebra A has the DWEP$_i$, for $i = 1, 2$. Then the multiplier algebra $\mathcal{M}(A)$ also has the DWEP$_i$, for $i = 1, 2$.*

Remark 3.1.11. Now we see that DWEP$_1$ implies DWEP$_2$. We also have the following relations between the two notions.

(1) Note that by Remark 2.2.11, $\mathcal{L}^w(\mathcal{H}_N) = e\mathcal{B}(\mathcal{H})\bar{\otimes}Ne$ for some projection e, and hence it is *r.w.i.* in $\mathcal{B}(\mathcal{H}) \otimes N^{**}$ by Remark 2.2.12. Following the same pattern in the proof of Corollary 2.3.2, by extending the inclusion map $\iota : \mathcal{L}^w(\mathcal{H}_N) \to \mathcal{B}(\mathcal{H})\bar{\otimes}N^{**}$ to the map $\mathcal{M}(\mathcal{K} \otimes_{\min} N) \to \mathcal{B}(\mathcal{H})\bar{\otimes}N^{**}$, one can show that $\mathcal{L}^w(\mathcal{H}_N)$ is *r.w.i.* in $\mathcal{M}(\mathcal{K} \otimes_{\min} N) = \mathcal{L}(\mathcal{H}_N)$. Let N be a von Neumann algebra. Then by Remark 2.2.5, we have

$$\mathcal{L}^w(E_N) \overset{r.w.i.}{\hookrightarrow} \mathcal{L}^w(\mathcal{H}_N) \overset{r.w.i.}{\hookrightarrow} \mathcal{M}(\mathcal{K} \otimes_{\min} N) = \mathcal{L}(\mathcal{H}_N).$$

Hence $\mathcal{L}^w(E_N)$ has the NWEP$_1$.

(2) We also have that D^{**}WEP$_1$ implies DWEP$_2$. Indeed having D^{**}WEP$_1$ is equivalent to being *r.w.i.* in $\mathcal{L}(\mathcal{H}_{D^{**}}) = \mathcal{M}(\mathcal{K} \otimes_{\min} D^{**})$, and having DWEP$_2$ is equivalent to being *r.w.i.* in $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$. Note that $\mathcal{K} \otimes_{\min} D^{**}$ is *r.w.i.* in $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$. By Corollary 3.1.10, we have $\mathcal{M}(\mathcal{K} \otimes_{\min} D^{**})$ has the DWEP$_2$ as well.

Now we investigate some properties of module WEP. The first result is that the module WEP is stable under tensoring with a nuclear C*-algebra, similar to the classical case.

Proposition 3.1.12. *For a C*-algebra D, the following properties hold:*

1. *If a C*-algebra A has the DWEP$_1$, and B is a nuclear C*-algebra, then $A \otimes_{\min} B$ has the DWEP$_1$ as well.*

2. *If the von Neumann algebras M and N have the CWEP$_2$ and DWEP$_2$ respectively, then $M \bar{\otimes} N$ has the $(C \otimes_{\min} D)$WEP$_2$.*

As a consequence of Corollary 3.1.10, we have the transitivity property of DWEP.

Proposition 3.1.13. *If A has the BWEP$_i$, and B has the CWEP$_i$, then A has the CWEP$_i$, for $i = 1, 2$.*

Corollary 3.1.14. *If A has the DWEP$_1$, and D has the WEP, then A has the WEP.*

Remark 3.1.15. The previous result is not necessarily true for the WEP$_2$ case, since $\mathcal{B}(\ell_2)\bar{\otimes}D^{**}$ may not have the WEP, for instance for $D = \mathcal{B}(\ell_2)$. See Example 3.3.1 for the proof.

In his Habilitation [Jun99], Junge shows the following finite dimensional characterization of the WEP.

Theorem 3.1.16. *The C*-algebra A has the WEP if and only if for arbitrary finite dimensional subspaces $F \subset A$ and $G \subset A^*$, and $\varepsilon > 0$, there exist matrix algebra M_m and ucp maps $u : F \to M_m$, $v : M_m \to A/G^{\perp}$, such that*

$$\|v \circ u - q_G \circ \iota_F\| < \varepsilon,$$

where $\iota_F : F \to A$ is the inclusion map and $q_G : A \to A/G^{\perp}$ is the quotient map.

We have a similar result for the module WEP as follows.

Theorem 3.1.17. *The C*-algebra A has the DWEP$_1$ if and only if for arbitrary finite dimensional subspaces $F \subset A$ and $G \subset A^*$, and $\varepsilon > 0$, there exist matrix algebra $M_m(D)$ and ucp maps $u : A \to M_m(D)$, $v : M_m(D) \to A/G^{\perp}$, such that*

$$\|v \circ u|_F - q_G \circ \iota_F\| < \varepsilon,$$

where $\iota_F : F \to A$ is the inclusion map and $q_G : A \to A/G^{\perp}$ is the quotient map.

For the DWEP$_2$ case, a similar result holds when we replace the matrix algebra $M_m(D)$ by $M_m(D^{**})$.

3.2 Module version of QWEP

Definition 3.2.1. A C^*-algebra B is said to be DQWEP$_i$ if it is the quotient of a C^*-algebra A with DWEP$_i$ for $i = 1, 2$.

Similar to the DWEP$_i$, we have a tensor characterization for DQWEP$_i$ for $i = 1, 2$ as follows. First we need the following result due to Kirchberg.

Lemma 3.2.2 ([Kir93] Corollary 3.2 (v))**.** *If $\phi : A \to B^{**}$ is a ucp map such that ϕ maps the multiplicative domain* $\mathrm{md}(\phi)$ *of ϕ onto a C^*-subalgerba C of B^{**} containing B as a subalgebra, then the C^*-algebra* $\mathrm{md}(\phi) \cap \phi^{-1}(B)$ *is relatively weakly injective in A.*

We only prove the tensor characterization for DQWEP$_1$. The proof of the other case is similar.

Theorem 3.2.3. *Let $C^*\mathbb{F}_\infty \subset \mathcal{L}(\mathcal{H}_D^u)$ be the universal representation. The following statements are equivalent for a C^*-algebra B:*

(i) B is $DQWEP_1$;

(ii) For any ucp map $u : C^*\mathbb{F}_\infty \to B$, the map $u \otimes \mathrm{id}$ extends to a continuous map from $C^*\mathbb{F}_\infty \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $B \otimes_{\max} C^*\mathbb{F}_\infty$, where $\max_1^D$ is the induced norm from the inclusion $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty \subseteq \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$.

Proof. (i)$\Rightarrow$(ii): Suppose B is DQWEP$_1$. Then $B = A/J$ for some C^*-algebra A with DWEP$_1$. Let $u : C^*\mathbb{F}_\infty \to B$ be a ucp map, and $\pi : A \to B$ be the quotient map. Since $C^*\mathbb{F}_\infty$ has the lifting property, there exists a ucp map $\varphi : C^*\mathbb{F}_\infty \to A$ which lifts u, i.e. the following diagram commutes

$$\begin{tikzcd} C^*\mathbb{F}_\infty \arrow[r, "u"] \arrow[d, "\varphi"'] & B \\ A \arrow[ur, "\pi"'] & \end{tikzcd}$$

By Theorem 3.1.6, we have $A \otimes_{\max_1^D} C^*\mathbb{F}_\infty = A \otimes_{\max} C^*\mathbb{F}_\infty$. Therefore, we have the following continuous maps

$$C^*\mathbb{F}_\infty \underset{\max_1^D}{\otimes} C^*\mathbb{F}_\infty \overset{\varphi\otimes\mathrm{id}}{\longrightarrow} A \underset{\max_1^D}{\otimes} C^*\mathbb{F}_\infty = A \underset{\max}{\otimes} C^*\mathbb{F}_\infty \overset{\pi\otimes\mathrm{id}}{\longrightarrow} B \underset{\max}{\otimes} C^*\mathbb{F}_\infty.$$

Note that $(\pi \otimes \mathrm{id}) \circ (\varphi \otimes \mathrm{id})|_{C^*\mathbb{F}_\infty \otimes 1_{C^*\mathbb{F}_\infty}} = u$ by the lifting property. Therefore, $u \otimes \mathrm{id}$ extends to a continuous map from $C^*\mathbb{F}_\infty \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $B \otimes_{\max} C^*\mathbb{F}_\infty$.

(ii)⇒(i): Let $u : C^*\mathbb{F}_\infty \to B$ be the quotient map. We have the following diagram

$$\begin{tikzcd}
C^*\mathbb{F}_\infty \underset{\max_1^D}{\otimes} C^*\mathbb{F}_\infty \arrow[r, "u\otimes id"] \arrow[d, hook] & B \underset{\max}{\otimes} C^*\mathbb{F}_\infty \arrow[r] & \mathcal{B}(\mathcal{H}) \\
\mathcal{L}(\mathcal{H}_D^u) \underset{\max}{\otimes} C^*\mathbb{F}_\infty \arrow[urr] & &
\end{tikzcd}$$

where $\mathcal{B}(\mathcal{H})$ is the universal representation of B. By Arveson's extension theorem, there exists a ucp map $\Phi : \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} C^*\mathbb{F}_\infty \to \mathcal{B}(\mathcal{H})$. Using *The Trick* (see proof of Theorem 3.1.6), we get a map $\phi : \mathcal{L}(\mathcal{H}_D^u) \to B^{**}$. Let $\mathrm{md}(\phi)$ be the multiplicative domain of ϕ. Note that $C^*\mathbb{F}_\infty \subset \mathrm{md}(\phi)$. Therefore, ϕ maps $\mathrm{md}(\phi)$ onto a C^*-subalgebra of B^{**} containing B. Let $A = \mathrm{md}(\phi) \cap \phi^{-1}(B)$. Then by Lemma 3.2.2, A is *r.w.i.* in $\mathcal{L}(\mathcal{H}_D^u)$, so A has the DWEP$_1$. Hence B as a quotient of A is DQWEP$_1$. □

Remark 3.2.4. In the proof of the above Theorem, we showed that the second statement is equivalent to the statement that for any ucp maps $u : C^*\mathbb{F}_\infty \to B$, $w : C^*\mathbb{F}_\infty \to B^{\mathrm{op}}$, the map $u \otimes w$ extends to a continuous map from $C^*\mathbb{F}_\infty \otimes_{\max_1^D} C^*\mathbb{F}_\infty$ to $B \otimes_{\max} B^{\mathrm{op}}$.

Now let us investigate some basic properties of the DQWEP. We have the following proposition similar to the DWEP case.

Proposition 3.2.5. *The following hold:*

1. *If a C^*-algebra B is DQWEP$_1$ and C is nuclear, then $C \otimes_{\min} B$ is also DQWEP$_1$.*
2. *If von Neumann algebras M and N are CQWEP$_2$ and DQWEP$_2$, respectively, then $M\bar{\otimes}N$ is $(C \otimes_{\min} D)$QWEP$_2$.*

Proof. (1) Suppose B is DQWEP$_1$, then $B = A/J$ for some C*-algebra A with the DWEP$_1$. Since C is nuclear, it is also exact. Therefore, we have

$$C \underset{\min}{\otimes} B = C \underset{\min}{\otimes} (A/J) \cong \frac{C \otimes_{\min} A}{C \otimes_{\min} J}.$$

But $C \otimes_{\min} A$ has the DWEP$_1$ by Proposition 3.1.12(1). Therefore, $C \otimes_{\min} B$ is DQWEP$_1$.

(2) Since M is CQWEP$_2$, it is *r.w.i.* in $\mathcal{L}^w(\mathcal{H}_{C^{**}})^{**}$. Similarly, N is *r.w.i.* in $\mathcal{L}^w(\mathcal{H}_{D^{**}})^{**}$. Therefore, we have ucp maps

$$M\bar{\otimes}N \overset{r.w.i.}{\hookrightarrow} \mathcal{L}^w(\mathcal{H}_{C^{**}})^{**}\bar{\otimes}\mathcal{L}^w(\mathcal{H}_{D^{**}})^{**} \overset{r.w.i.}{\hookrightarrow} \mathcal{L}^w(\mathcal{H}_{C^{**}\bar{\otimes}D^{**}})^{**}.$$

Note that by the same argument as in the proof of Proposition 3.1.12 (2), $\mathcal{L}^w(\mathcal{H}_{C^{**}\bar{\otimes}D^{**}})^{**}$ is *r.w.i.* in

$\mathcal{L}^w(\mathcal{H}_{(C\otimes_{\min}D)^{**}})^{**}$. Hence $M\bar{\otimes}N$ is *r.w.i.* in $\mathcal{L}^w(\mathcal{H}_{(C\otimes_{\min}D)^{**}})^{**}$. Therefore, $M\bar{\otimes}N$ is $(C\otimes_{\min}D)$QWEP$_2$. $\square$

By Theorem 3.1.7, DWEP$_1$ implies DWEP$_2$, and hence DQWEP$_1$ implies DQWEP$_2$. In Section 5 we will show that there exist C^*-algebras with DWEP$_2$ which do not have DWEP$_1$. However in the QWEP context, the two concepts coincide. To see this, we need the following lemmas in which we use Kirchberg's categorical method. The next lemma shows that DQWEP$_i$, for $i = 1, 2$, is stable under the direct products.

Lemma 3.2.6. *Suppose $(B_i)_{i\in I}$ is a net of C^*-algebras in $\mathcal{B}(\mathcal{H})$. If B_i is DQWEP$_i$, for all $i \in I$, then so is $\Pi_{i\in I}B_i$.*

Proof. Since B_i is DQWEP$_i$, it is a quotient of a C^*-algebra A_i with DWEP$_i$. By Lemma 3.1.9, $\Pi_{i\in I}A_i$ has the DWEP$_i$. Therefore, $\Pi_{i\in I}B_i$ is DQWEP$_i$. $\square$

Lemma 3.2.7. *Let B be a DQWEP$_i$ C^*-algebra, for $i = 1, 2$, and B_0 a C^*-subalgebra of B which is relatively weakly injective in B. Then B_0 is also a DQWEP$_i$ C^*-algebra.*

Proof. If B is DQWEP$_i$, then it is a quotient of a C^*-algebra A with DWEP$_i$. Let $\pi : A \to B$ be the quotient map, $B = A/J$ and $A_0 = \pi^{-1}(B_0)$. Then A_0 is *r.w.i.* in A. In fact this follows from the fact that

$$A_0^{**} = J^{**} \oplus B_0^{**} \subset J^{**} \oplus B^{**} = A^{**}.$$

Now by Lemma 3.1.2, $A_0 = \pi^{-1}(B_0)$ has the DWEP$_i$. Hence B_0 is DQWEP$_i$. $\square$

Lemma 3.2.8. *Let A and B be unital C^*-algebras. Suppose there exists a ucp map $\psi : A \to B$ which maps the closed unit ball of A onto the closed unit ball of B. If A has the DWEP$_i$, then B is DQWEP$_i$, for $i = 1, 2$.*

Proof. Let $A_0 \subset A$ be the multiplicative domain of ψ. Since ψ maps the closed unit ball of A onto that of B, the restriction of ψ on A_0 is a surjective $*$-homomorphism onto B. Let $\pi = \psi|_{A_0}$.

By Lemma 3.2.2, we have A_0 is *r.w.i.* in A and hence it has the DWEP$_i$ by Lemma 3.1.2. Since B is a quotient of A_0, B is D-QWEP$_i$. $\square$

Corollary 3.2.9. *Let B and C be C^*-algebras. Suppose B is DQWEP$_i$, and $\psi : B \to C$ is a ucp map that maps the closed unit ball of B onto that of C. Then C is DQWEP$_i$.*

Proof. Since B is DQWEP$_i$, there exists a C^*-algebra A with the DWEP$_i$, and a surjective *-homomorphism $\pi : A \to B$. Notice that π maps closed unit ball of A onto that of B. Hence the composition $\psi \circ \pi$ maps the closed unit ball of A onto that of C. By Lemma 3.2.8, C is DQWEP$_i$. $\square$

Lemma 3.2.10. *Suppose $(B_i)_{i\in I}$ is an increasing net of C^*-algebras in $\mathcal{B}(\mathcal{H})$. If all B_i are $DQWEP_i$, then $\overline{\cup B_i}$ and $(\cup B_i)''$ are $DQWEP_i$.*

Proof. Let $B = \cup B_i$. It suffices to show that B'' is DQWEP$_i$. Since B_i is DQWEP$_i$, there exists a C^*-algebra A_i with DWEP$_i$, and a surjective *-homomorphism $\pi_i : A_i \to B_i$. Let J be a directed set containing I. By Lemma 3.1.9, $\prod_{j\in J} A_j$ has the DWEP$_i$. Fix a free ultrafilter $\mathcal{U}$ on the net J. Define a ucp map $\varphi : \prod_{j\in J} A_j \to B''$ by $\varphi((x_j)_{j\in J}) = \lim_{j\to\mathcal{U}} \pi(x_j)$ in the ultraweak topology. By Kaplansky's density theorem, if J is large enough, then φ maps the closed unit ball of $\prod_{j\in J} A_j$ onto that of B''. Now by Lemma 3.2.8, B'' is DQWEP$_i$. □

The next corollary shows that unlike the DWEP case, the DQWEP of a C*-algebra and its double dual are equivalent.

Corollary 3.2.11. *A C^*-algebra B is $DQWEP_i$ if and only if B^{**} is $DQWEP_i$ for $i = 1, 2$.*

Proof. The "if" direction follows directly from Lemma 3.2.7 since B is *r.w.i.* in B^{**}. For the other direction, we can apply Lemma 3.2.10 to B together with its universal representation. □

Lemma 3.2.12. *Suppose B and C are C^*-algebras, and B factors through C approximately via ucp maps in the point-weak* topology. If C is $DQWEP_i$, then so is B.*

Proof. Since B factors through C, there are families of ucp maps $\alpha_i : B \to C$ and $\beta_i : C \to B$, $i \in I$ such that $\beta_i \circ \alpha_i$ converges to the identity map on B in the point-weak* topology, i.e.

$$\lim_{x,\mathcal{U}} (\beta_i \circ \alpha_i)(x)(x^*) = x^*(x)$$

for $x \in B$, $x^* \in B^*$ and an ultrafilter $\mathcal{U}$. Define $\alpha : B \to \prod_{i\in I} C$ by $\alpha(x) = (\alpha_i(x))_{i\in I}$, for $x \in B$. Let $\beta : \prod_{i\in I} C \to B^{**}$, $\beta = \lim_{i\to\mathcal{U}} \beta_i$. Define $\beta^\# : B^* \to \prod_{\mathcal{U}} C^*$, by $\beta^\#(x^*) = (\beta_i^*(x^*))^\bullet$. In fact $\beta^\# = \beta^*|_{B^*}$. Then we have

$$B^* \xrightarrow{\beta^\#} (\textstyle\prod_{\mathcal{U}} C)^*.$$

By taking the duals, we have

$$B \xrightarrow{\alpha} C^{**} \xrightarrow{(\beta^\#)^*} B^{**}.$$

This gives a conditional expectation from C^{**} to B^{**} which is identity on B. Therefore, B is *r.w.i.* in C^{**}. By Corollary 3.2.11, C^{**} is DQWEP$_i$. Hence so is B. □

Corollary 3.2.13. *If a C^*-algebra B is $DQWEP_i$, for $i = 1, 2$, then so is $\mathcal{M}(B)$.*

Proof. Note that by Theorem 2.3.1, the identity map on $\mathcal{M}(B)$ factors through $\ell_\infty(B)$ approximately via ucp maps in point weak *-topology. Since B is $D\mathrm{QWEP}_i$, by Lemma 3.2.6, so is $\ell_\infty(B)$. Therefore, by Lemma 3.2.12, $\mathcal{M}(B)$ is $D\mathrm{QWEP}_i$. $\square$

We have the following transitivity result for $D\mathrm{QWEP}_i$. We only show the $D\mathrm{QWEP}_1$ case. The proof of the other case is similar. First we need the following lemma.

Lemma 3.2.14. *Let D be a C^*-algebra. If D is $CQWEP_i$ for $i = 1, 2$, then so are $\mathcal{L}(\mathcal{H}_D)$ and $\mathcal{L}^w(\mathcal{H}_{D^{**}})$.*

Proof. If D is $C\mathrm{QWEP}_1$, then by Proposition 3.2.5(1), so is $\mathcal{K} \otimes_{\min} D$. By Theorem 2.3.1, $\mathcal{L}(\mathcal{H}_D) = \mathcal{M}(\mathcal{K} \otimes D)$ factors through $\ell_\infty(\mathcal{K} \otimes_{\min} D)$, and therefore, it is $D\mathrm{QWEP}_1$, by Lemma 3.2.12. Hence it is also $D\mathrm{QWEP}_2$. For the other case, it suffices to show that $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$ is $D\mathrm{QWEP}_1$. Note that $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**} = (\mathcal{K} \otimes_{\min} D)^{**}$ and $\mathcal{K} \otimes_{\min} D$ is $D\mathrm{QWEP}_1$. By Corollary 3.2.11, $(\mathcal{K} \otimes_{\min} D)^{**}$ is $D\mathrm{QWEP}_1$, and hence it is $D\mathrm{QWEP}_2$. $\square$

The following result shows the transitivity of the $D\mathrm{QWEP}_i$ for $i = 1, 2$.

Corollary 3.2.15. *Let B, C and D be C^*-algebras such that B is $DQWEP_i$, and D is $CQWEP_i$. Then B is $CQWEP_i$.*

Proof. We only show this for $i = 1$. Let $C^*\mathbb{F}_\infty \subset \mathcal{L}(\mathcal{H}_D^u)$ be the universal representation. Since B is $D\mathrm{QWEP}_1$, by Theorem 3.2.3, for all ucp maps $u : C^*\mathbb{F}_\infty \to B$, the map $u \otimes \mathrm{id} : C^*\mathbb{F}_\infty \otimes_{\max_1^D} C^*\mathbb{F}_\infty \to B \otimes_{\max} C^*\mathbb{F}_\infty$ is continuous, where $\max_1^D$ is the norm induced from the inclusion $C^*\mathbb{F}_\infty \otimes C^*\mathbb{F}_\infty \subset \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$. Since D is $C\mathrm{QWEP}_1$, so is $\mathcal{L}(\mathcal{H}_D^u)$ by Lemma 3.2.14. Now by the tensor characterization of $C\mathrm{QWEP}_1$, the map $w \otimes id : C^*\mathbb{F}_\infty \otimes_{\max_1^C} C^*\mathbb{F}_\infty \to \mathcal{L}(\mathcal{H}_D^u) \otimes_{\max} C^*\mathbb{F}_\infty$ is continuous for all ucp maps $w : C^*\mathbb{F}_\infty \to \mathcal{L}(\mathcal{H}_D^u)$. Now let w be a faithful representation $C^*\mathbb{F}_\infty \to \mathcal{L}(\mathcal{H}_D^u)$. Then we have the following diagram

$$\begin{tikzcd}
C^*\mathbb{F}_\infty \underset{\max_1^D}{\otimes} C^*\mathbb{F}_\infty \arrow[r, hook] \arrow[d, "u\otimes\mathrm{id}"'] & \mathcal{L}(\mathcal{H}_D^u) \underset{\max}{\otimes} C^*\mathbb{F}_\infty \\
B \underset{\max}{\otimes} C^*\mathbb{F}_\infty & C^*\mathbb{F}_\infty \underset{\max_1^C}{\otimes} C^*\mathbb{F}_\infty \arrow[u, "w\otimes\mathrm{id}"'] \arrow[ul]
\end{tikzcd}$$

Note that the image of $w \otimes \mathrm{id}$ is $C^*\mathbb{F}_\infty \otimes_{\max_1^D} C^*\mathbb{F}_\infty$. Therefore, we get a continuous map from $C^*\mathbb{F}_\infty \otimes_{\max_1^C} C^*\mathbb{F}_\infty$ to $B \otimes_{\max} C^*\mathbb{F}_\infty$. This proves that B is $C\mathrm{QWEP}_1$. $\square$

Now we are ready to establish the equivalence between the $D\mathrm{QWEP}$ notions by observing the following result.

Theorem 3.2.16. *For a C^*-algebra B, the following conditions are equivalent:*

1. *B is $DQWEP_1$;*

2. *B is $DQWEP_2$;*

3. *B^{**} is $D^{**}QWEP_1$;*

4. *B^{**} is $D^{**}QWEP_2$.*

Proof. (1)⇒(2): This follows from the fact that $D\mathrm{WEP}_1$ implies $D\mathrm{WEP}_2$.

(2)⇒(3): Suppose B is $D\mathrm{QWEP}_2$. Therefore, B is the quotient of a C^*-algebra A which is *r.w.i.* in $\mathcal{L}^w(E_{D^{**}})$. By Remark 3.1.11(1), $\mathcal{L}^w(E_{D^{**}})$ has the $D^{**}\mathrm{WEP}_1$. Hence A has the $D^{**}\mathrm{WEP}_1$, and therefore, B is $D^{**}\mathrm{QWEP}_1$.

(3)⇒(4): Follows from (1)⇒(2).

(4)⇒(1): Suppose B^{**} is $D^{**}\mathrm{QWEP}_2$, and therefore so is B by Corollary 3.2.11. Then B is the quotient of a C^*-algebra A which is *r.w.i.* in $\mathcal{L}^w(E_{D^{****}})$. We have

$$A \overset{r.w.i.}{\subset} \mathcal{L}^w(E_{D^{****}}) \overset{r.w.i.}{\subset} \mathbb{B}(\ell_2)\bar{\otimes}D^{****} = (\mathcal{K} \underset{\min}{\otimes} D^{**})^{**}.$$

Therefore, it suffices to show that $\mathcal{K} \otimes_{\min} D^{**}$ is $D\mathrm{QWEP}_1$. Notice that $\mathcal{K} \otimes_{\min} D^{**}$ factors through $\prod_n M_n(D^{**})$ approximately via ucp maps in point-norm topology, since $\cup M_n(D^{**})$ is norm-dense in $\mathcal{K} \otimes_{\min} D^{**}$. Now since D has the $D\mathrm{WEP}_1$, D^{**} is $D\mathrm{QWEP}_1$. Therefore, by Proposition 3.2.5, so is $M_n(D^{**}) = M_n \otimes_{\min} D^{**}$. Hence by Lemma 3.2.12, $\mathcal{K} \otimes_{\min} D^{**}$ is $D\mathrm{QWEP}_1$. This finishes the proof. □

3.3 Illustrations

In Section 3, we showed that $D\mathrm{WEP}_1$ implies $D\mathrm{WEP}_2$. Our first example will show the converse is not true, and hence the two notions of DWEP are not equivalent.

Example 3.3.1. Let $D = \mathcal{B}(\ell_2)$. Note that $\mathcal{L}(\mathcal{H}_D) = \mathcal{M}(\mathcal{K} \otimes_{\min} \mathcal{B}(\ell_2))$, and $\mathcal{K} \otimes_{\min} \mathcal{B}(\ell_2)$ has the WEP, and so does $\mathcal{M}(\mathcal{K} \otimes_{\min} \mathcal{B}(\ell_2))$. Therefore the two notions of $D\mathrm{WEP}_1$ and WEP coincide. On the other hand, the $D\mathrm{WEP}_2$ of a C^*-algebra is the same as being *r.w.i.* in $\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{B}(\ell_2)^{**}$. Notice that $\mathcal{B}(\mathcal{H})\bar{\otimes}\mathcal{B}(\ell_2)^{**} = (\mathcal{K} \otimes \mathcal{B}(\ell_2))^{**}$ is QWEP. Therefore by Proposition 2.1.9, $D\mathrm{WEP}_2$ is equivalent to QWEP. Hence if A is a QWEP C^*-algebra without the WEP, for instance $C^*_r\mathbb{F}_n$, then A has the $D\mathrm{WEP}_2$ but not the $D\mathrm{WEP}_1$, for $D = \mathcal{B}(\ell_2)$.

Now we are ready to see some examples of relative WEP and QWEP over special classes of C*-algebras.

Proposition 3.3.2. *Let D be a nuclear C^*-algebra. Then a C^*-algebra A has the DWEP$_i$ for $i = 1, 2$ if and only if it has the WEP.*

Proof. Suppose A has the WEP. Therefore A has the DWEP$_1$, and hence the DWEP$_2$.

Now assume A has the DWEP$_2$, i.e. it is *r.w.i.* in $\mathcal{B}(\ell_2)\bar{\otimes}D^{**}$. Since D is nuclear, D^{**} is injective. Hence we have $D^{**} \subseteq \mathcal{B}(\mathcal{H}) \xrightarrow{\mathbb{E}} D^{**}$, where $\mathbb{E}$ is a conditional expectation. Let $CB(A, B)$ be the space of completely bounded maps from A to B. Therefore we have

$$CB(S_1, D^{**}) \overset{\pi}{\hookrightarrow} CB(S_1, \mathcal{B}(\mathcal{H})) \xrightarrow{\varphi} CB(S_1, D^{**}),$$

where S_1 is the algebra of trace class operators, π is a *-homomorphism, and φ acts by composing the maps in $CB(S_1, \mathcal{B}(\mathcal{H}))$ and $\mathbb{E}$. Note that by operator space theory $CB(S_1, D^{**}) \simeq \mathcal{B}(\ell_2)\bar{\otimes}D^{**}$ and $CB(S_1, \mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\ell_2)\bar{\otimes}\mathcal{B}(\mathcal{H}) = \mathcal{B}(\ell_2 \otimes \mathcal{H})$. Hence we have the maps $\mathcal{B}(\ell_2)\bar{\otimes}D^{**} \xrightarrow{\pi} \mathcal{B}(\ell_2)\bar{\otimes}\mathcal{B}(\mathcal{H}) = \mathcal{B}(\ell_2 \otimes \mathcal{H}) \xrightarrow{\varphi} \mathcal{B}(\ell_2)\bar{\otimes}D^{**}$. Now by Remark 2.2.12 we can unitize these two maps. Therefore A is *r.w.i.* in $\mathcal{B}(\ell_2 \otimes \mathcal{H})$, and hence it has the WEP. □

After nuclear C*-algebras, it is natural to consider the relative WEP for an exact C^*-algebra D. For convenience, we consider the following stronger version of weak exactness property. A von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ is said to be *algebraically weakly exact*, (a.w.e. for short), if there exists a weakly dense exact C^*-algebra D in M. By [Kir95], we know that the a.w.e. implies the weak exactness.

Notice that the unitization trick works better in $\mathcal{C}_2$ category, and hence we have the following.

Proposition 3.3.3. *A C^*-algebra has the DWEP$_2$ for some exact C^*-algebra D if and only if it is relatively weakly injective in an a.w.e. von Neumann algebra.*

Proof. Suppose a C^*-algebra A has the DWEP$_2$, then A is *r.w.i.* in $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$. Since both $\mathcal{K}$ and D are exact C*-algebras, so is $\mathcal{K} \otimes_{\min} D$. Note that $\mathcal{K} \otimes_{\min} D$ is weakly dense in $(\mathcal{K} \otimes_{\min} D)^{**} = \mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$. We have $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$ is a.w.e.

For the other direction, suppose A is *r.w.i.* in an a.w.e von Neumann algebra M. Let D be an exact C^*-algebra with $D'' = M$. Then there exists a central projection z in D^{**} such that $M = zD^{**}$. Hence we have completely positive maps $M \hookrightarrow D^{**} \to M$, which preserves the identity on M. Therefore by unitization M is *r.w.i.* in $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$ for some infinite dimensional Hilbert space $\mathcal{H}$. Hence if A is *r.w.i.* in M, then it is also *r.w.i.* in $\mathcal{B}(\mathcal{H})\bar{\otimes}D^{**}$, and therefore it has the DWEP$_2$. □

As we showed, the nuclear-WEP is equivalent to the WEP. But the exact-WEP is different.

Example 3.3.4. Let $\mathbb{F}_2$ be the free group on two generators. Then $\mathbb{F}_2$ is exact and hence $C_r^*\mathbb{F}_2$ is exact and $L\mathbb{F}_2$ is weakly exact. Since $C_r^*\mathbb{F}_2$ is *r.w.i.* in $L\mathbb{F}_2$, by Proposition 3.3.3, $C_r^*\mathbb{F}_2$ has the DWEP$_2$ for $D = C_r^*\mathbb{F}_2$. But $C_r^*\mathbb{F}_2$ does not have the WEP, since the WEP of a reduced group C*-algebra is equivalent to the amenability of the group (see Proposition 3.6.9 in [BO08]).

Now we consider the full group C^*-algebra of free group $C^*\mathbb{F}_\infty$. Since it is universal in the sense that for any unital separable C^*-algebra A, we have a quotient map $q : C^*\mathbb{F}_\infty \to A$. By the unitization trick, we have the following.

Proposition 3.3.5. *Let A be a unital separable C^*-algebra. Then it has the DWEP$_2$ for $D = C^*\mathbb{F}_\infty$.*

Proof. Since we have a quotient map $q : C^*\mathbb{F}_\infty \to A$, there exists a central projection z in $C^*\mathbb{F}_\infty{}^{**}$ such that $A^{**} = zC^*\mathbb{F}_\infty{}^{**}$. Hence we have an embedding $A^{**} \hookrightarrow \mathcal{B}(\mathcal{H})\bar{\otimes}C^*\mathbb{F}_\infty{}^{**}$ with a completely positive map from $\mathcal{B}(\mathcal{H})\bar{\otimes}C^*\mathbb{F}_\infty{}^{**}$ to A^{**} by multiplying $1 \otimes z$. By the unitization trick in Remark 2.2.12, A^{**} has the DWEP$_2$ for $D = C^*\mathbb{F}_\infty$ and so does A, since A is *r.w.i.* in A^{**}. □

It is natural and even more interesting to ask whether the full group C*-algebra $C^*\mathbb{F}_\infty$ has DWEP, for D is the reduced group C*-algebra $C_r^*\mathbb{F}_2$. In fact, this is related to the QWEP conjecture. If $C^*\mathbb{F}_\infty$ has the DWEP$_1$ for some WEP algebra D, then it has the WEP by Corollary 3.1.14 of transitivity. If $C^*\mathbb{F}_\infty$ does not have the DWEP$_1$ for some C*-algebra D, then it does not have the WEP either. At the time of writing this paper, we do not have an answer for this question.

Now let us discuss some properties of being module QWEP relative to some special classes of C*-algebras. In the rest of this section, we will examine the relation between one of the equivalent statements of Theorem 3.2.16 (for example statement (1), B is DQWEP$_i$), and the statement that B^{**} is D^{**}WEP$_i$, for either $i = 1$ or 2.

Proposition 3.3.6. *Let B be a C^*-algebra. If B^{**} has the D^{**}WEP$_i$, then B is DQWEP$_i$, for $i = 1, 2$.*

Proof. Suppose B^{**} has the D^{**}WEP$_i$, and hence B^{**} is D^{**}QWEP$_i$ by the trivial quotient. By Theorem 3.2.16, B is DQWEP$_i$. □

For some C*-algebra D, the four equivalent statements in Theorem 3.2.16 are equivalent to the statement that B^{**} has the D^{**}WEP$_i$. But this is not true in general. We will show examples of both circumstances.

Example 3.3.7. Let $D = \mathcal{B}(\ell_2)$. Then a C^*-algebra B is DQWEP$_i$ if and only if B^{**} has the D^{**}WEP$_i$, since they are both equivalent to B being QWEP. Indeed, if B is DQWEP$_1$, then $B = A/J$ and A has the DWEP$_1$. Since $\mathcal{L}(\mathcal{H}_D)$ has the WEP as shown in Example 3.3.1, so does A, and hence B is QWEP. On the other hand, having $\mathcal{B}(\ell_2)^{**}\mathrm{WEP}_1$ is equivalent to being *r.w.i.* in $\mathcal{M}(\mathcal{K} \otimes_{\min} \mathcal{B}(\ell_2)^{**})$, which is QWEP. Hence B^{**} is QWEP. By Proposition 2.1.8, B is QWEP as well.

Example 3.3.8. Let D be a nuclear C*-algebra. Then the above statements are not equivalent. Indeed, it follows from Proposition 3.3.2 that a C*-algebra is $D\mathrm{QWEP}_i$ if and only if it is QWEP. On the other hand, assume that B^{**} has the $D^{**}\mathrm{WEP}_1$. Note that $D^{**}\mathrm{WEP}_1$ implies $D\mathrm{WEP}_2$ by Remark 3.1.11(2), which is equivalent to WEP by Proposition 3.3.2, and B^{**} has the WEP if and only if it is injective. Therefore the fact that a C*-algebra B is $D\mathrm{QWEP}_i$ does not imply that B^{**} has the $D^{**}\mathrm{WEP}_1$.

Example 3.3.9. For a von Neumann algebra M, let us compare the properties $M\mathrm{QWEP}_1$ of B and the $M^{**}\mathrm{WEP}_1$ of B^{**}. We have the following partial results.

Case (i): M is of type I_n. Then M is subhomogeneous, which is equivalent to nuclearity. By Example 3.3.8, these two statements are not equivalent.

Case (ii): M is of type I_∞, then $\mathcal{B}(\ell_2)\bar{\otimes}M$ is *r.w.i.* in M. Suppose B is $M\mathrm{QWEP}_1$, then B is a quotient of a C^*-algebra A which is *r.w.i.* in $\mathcal{B}(\ell_2)\bar{\otimes}M$. Hence B^{**} is *r.w.i.* in A^{**} and hence in $(\mathcal{B}(\ell_2)\bar{\otimes}M)^{**}$, and hence in M^{**}. Since M^{**} is isomorphic to $\mathcal{L}(\mathcal{H}_{M^{**}})$ for 1-dimensional Hilbert space $\mathcal{H}$, it follows that B^{**} has the $M^{**}\mathrm{WEP}_1$.

Case (iii): M is of type II_∞ or III, then $\mathcal{B}(\ell_2)\bar{\otimes}M \simeq M$. By similar argument in Case (ii), we have the same conclusion.

Case (iv): M is of type II_1 and a McDuff factor, i.e. $M\bar{\otimes}R \simeq M$. Then we have a completely postive map from $M\bar{\otimes}\mathcal{B}(\ell_2)$ to M by the following:

$$M\bar{\otimes}\mathcal{B}(\ell_2) \to M\bar{\otimes}\prod_{n=1}^{\infty} M_n \to M\bar{\otimes}R\bar{\otimes}L_\infty[0,1] \subseteq M\bar{\otimes}R\bar{\otimes}R \simeq M\bar{\otimes}R \simeq M.$$

with a completely positive left inverse from M to $M\bar{\otimes}\mathcal{B}(\ell_2)$, namely $M\bar{\otimes}\mathcal{B}(\ell_2)$ factors through M by completely positive maps. Therefore $M\bar{\otimes}\mathcal{B}(\ell_2)$ is *r.w.i.* in M^{**}. By the same argument as above, the equivalence is established.

At the time of writing this paper, we do not have an affirmative answer for the case where M is a non-Mcduff II_1 factor.

Chapter 4

Gromov-Hausdorff Convergence for $C(\mathbb{T})$

4.1 Order-unit Spaces and Forms of Gromov–Hausdorff Convergence

In this section we briefly review the notions of order-unit spaces and Gromov-Hausdorff distance. We will be mostly following [Rie04a, Li06, Lat15, Lat16] in this chapter.

The abstract characterization of order-unit spaces is due to Kadison [Kad51].

Definition 4.1.1. An order-unit space is a real partially ordered vector space, A, with a distinguished element e (the order unit) satisfying:

(1) (Order unit property): For each $a \in A$, there is $r \in \mathbb{R}$ such that $a \leq re$;

(2) (Archimedean property): For $a \in A$, if $a \leq re$ for all $r \in \mathbb{R}$ with $r > 0$, then $a \leq 0$. On an order-unit space (A, e), we can define a norm as

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leq a \leq re\}.$$

Furthermore, we require that $A^+ \cap (-A^+) = 0$, where A^+ denotes the positive cone of A. This condition ensures that for $a \in A$, if $a \leq 0$ and $a \geq 0$, then $\|a\| = 0$.

Then A becomes a normed vector space and we can consider its dual, A', consisting of the bounded linear functionals, equipped with the dual norm $\|.\|'$. By a state of an order-unit space (A, e), we mean a $\mu \in A'$ such that $\mu(e) = \|\mu\|' = 1$. Denote the set of all states of A by $S(A)$. For an order-unit space (A, e) and a seminorm L on A, we can define an ordinary metric, ρ_L, on $S(A)$ by

$$\rho_L(x, y) = \sup_{L(f) \leq 1} |f(x) - f(y)|. \tag{4.1.1}$$

Then we say L is a *Lipschitz* seminorm on A if it satisfies:

(1) For $a \in A$, we have $L(a) = 0$ if and only if $a \in \mathbb{R}e$.

We call L a Lip-norm, and call the pair (A, L) a *compact quantum metric space* if L also satisfies:

(2) The topology on $S(A)$ induced by the metric L is the weak-* topology.

For two metric spaces Y, Z inside a metric space (X, d), let their Hausdorff distance in X be denoted by $\mathrm{dist}_H^d(Y, Z)$. We may drop d when it is clear from the context what metric we are using.

For any two compact metric spaces X and Y, their *Gromov–Hausdorff distance* was introduced by Gromov [Gro81] as follows

$$\begin{aligned}\mathrm{dist}_{GH}(X, Y) := \inf\{&\mathrm{dist}_H(h_X(X), h_Y(Y)) \mid h_X : X \to Z, h_Y : Y \to Z \\ &\text{are isometric embeddings into some metric space } Z\}.\end{aligned}$$

The weakest form of Gromov–Hausdorff convergence that we are using in this thesis is convergence in the sense of *quantum Gromov–Hausdorff*. This notion was first introduced by Rieffel [Rie04a]. Let A be an order-unit space. By a quotient (π, B) of A, we mean an order-unit space B and a surjective linear positive map $\pi : A \to B$ preserving the order-unit. Let (A, L_A) and (B, L_B) be compact quantum metric spaces. The direct sum $A \oplus B$, of vector spaces, with (e_A, e_B) as order-unit, and with the natural order structure is also an order-unit space. We call a Lip-norm L on $A \oplus B$ admissible if it induces L_A and L_B under the natural quotient maps $A \oplus B \to A$ and $A \oplus B \to B$. Let ρ_L be as defined in (4.1.1). Then the *quantum Gromov–Hausdorff distance* is defined by

$$\mathrm{dist}_q(A, B) = \inf\{\mathrm{dist}_H^{\rho_L}(S(A), S(B)) : L \text{ is an admissible norm on } A \oplus B\}.$$

For a compact quantum metric space (A, L) and $r \geq 0$, let

$$\mathcal{D}_r(A) := \{a \in A : L(a) \leq 1, \|a\| \leq r\}.$$

The following definition is due to Li [Li06].

Definition 4.1.2. Let (A, L_A) and (B, L_B) be compact quantum metric spaces and $R \geq 0$. The R-order-unit quantum Gromov–Hausdorff distance between them, denoted by $\mathrm{dist}_{oq}^R(A, B)$, is defined by

$$\mathrm{dist}_{oq}(A, B) := \inf\{\max(\mathrm{dist}_H(h_A(\mathcal{D}_R(A)), h_B(\mathcal{D}_R(B))), \|h_A(Re_A) - h_B(Re_B))\|\},$$

where the infima are taken over all triples (V, h_A, h_B) consisting of a real normed vector space V and linear isometric embeddings $h_A : A \to V$ and $h_B : B \to V$.

Note that the term $\|h_A(Re_A) - h_B(Re_B))\|$ is chosen to take care of the order-units. One may omit these terms and require $h_A(e_A) = h_B(e_B)$. Then we can immediately get the following results.

Lemma 4.1.3. *Let $\varphi_j : A \to B_j$ be linear isometric embeddings of normed spaces (over $\mathbb{R}$ or $\mathbb{C}$) for $j \in J$, where J is an index set. Then there is a normed space C and linear isometric embeddings $\psi_j : B_j \to C$ such that $\psi_j \circ \varphi_j = \psi_k \circ \varphi_k$ for all $j, k \in J$.*

Hence we get the triangle inequality.

Lemma 4.1.4. *For $R \geq 0$ and any quantum compact metric spaces (A, L_A), (B, L_B) and (C, L_C) we have*

$$dist_{oq}^R(A, C) \leq dist_{oq}^R(A, B) + dist_{oq}^R(B, C).$$

Note that it was shown in [Li06] that for $R \geq 0$, the following holds:

$$\frac{1}{2} dist_{oq}^R \leq dist_q \leq \frac{5}{2} dist_{oq}^R.$$

The following Theorem is due to Li [Li06]. We are using this Theorem to find an approximation for the space of continuous functions on the torus and an approximation for the rotation algebras in this chapter and the next. First we recall the notion of a *continuous field*. Let T be a topological space. A *continuous field of Banach spaces* over T is a family $E(t)_{t\in T}$ of Banach spaces, with a set $\Gamma \subset \prod_{t\in T} E(t)$ of vector fields such that:

1. Γ is (complex) linear subset of $\prod_{t\in T} E(t)$;

2. For every $t \in T$ the set of $x(t)$ for $t \in \Gamma$ is dense in $E(t)$;

3. For every $x \in \Gamma$ the function $x \to \|x(t)\|$ is continuous;

4. Let $x \in \prod_{t\in T} E(t)$ be a vector field. If for every $t \in T$ and every $\varepsilon > 0$, there exists an $x' \in \Gamma$ such that $\|x(t) - x'(t)\| \leq \varepsilon$ throughout some neighborhood of t, then $x \in \Gamma$.

Theorem 4.1.5. *([Li06]) Let $(\{(A_t, L_t)\}, \Gamma)$ be a continuous field of quantum compact metric spaces over a locally compact Hausdorff space T. Let $R \geq 0$. Let $t_0 \in T$ and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in Γ such that $(f_n)_{t_0} \in \mathcal{D}_R(A_{t_0})$ for each $n \in \mathbb{N}$ and the set $\{(f_n)_{t_0} : n \in \mathbb{N}\}$ is dense in $\mathcal{D}_R(A_{t_0})$. Then the following are equivalent:*

1. *$dist_{oq}^R(A_t, A_{t_0}) \to 0$ as $t \to t_0$;*

2. *$dist_{GH}(\mathcal{D}_R(A_t), \mathcal{D}_R(A_{t_0})) \to 0$ as $t \to 0$;*

3. *for any $\varepsilon > 0$, there is an N such that the open ε-balls in A_t centered at $(f_1)_t, ..., (f_N)_t$ cover $\mathcal{D}_R(A_t)$ for all t in some neighborhood $\mathcal{U}$ of t_0.*

Lemma 4.1.6. *([Li06]) Let A and B be normed spaces (over $\mathbb{R}$ or $\mathbb{C}$). Let X be a linear subspace of A, and let $\varepsilon > 0$. Let $\varphi : X \to B$ be a linear map with $(1-\varepsilon)\|x\| \leq \|\varphi(x)\| \leq (1+\varepsilon)\|x\|$ for all $x \in X$. Then there are a normed space V and linear isometric embeddings $h_A : A \hookrightarrow V$ and $h_B : B \hookrightarrow V$ such that $\|h_A(x) - (h_B \circ \varphi)(x)\| \leq \varepsilon\|x\|$ for all $x \in X$.*

The strongest form of Gromov–Hausdorff convergence is convergence in the sense of *quantum Gromov–Hausdorff propinquity*, which was introduced by Latrémolière in [Lat15]. Before we introduce this notion, we need to give some definitions.

Definition 4.1.7. ([Lat16]) Let A and B be two unital C*-algebras. A bridge $\gamma = (D, \omega, \pi_A, \pi_B)$ is given by a unital C*-algebra D, two unital *-monomorphisms $\pi_A : A \hookrightarrow D$ and $\pi_B : B \hookrightarrow D$ and $\omega \in D$ such that the set $S(A|\omega) := \{\varphi \in S(A) : \forall d \in D, \varphi(d) = \varphi(d\omega) = \varphi(\omega d)\}$ is not empty, where $S(A)$ denotes the state space of A.

In the following let $F : [0,\infty)^4 \to [0,\infty)$ be defined by $F(x, y, l_x, l_y) = xl_y + yl_x$, for $x, y, l_x, l_y \in [0,\infty)$.

For a C*-algebra A, let $sa(A)$ denote the self-adjoint elements of A. Let uA denote the unitization of A. Recall ([Rie98]) that a *Lipschitz pair* (A, L) is a pair of a C*-algebra and a seminorm L on a dense subspace $\mathrm{dom}(L)$ of $sa(uA)$ and such that

$$\{a \in sa(uA) : L(a) = 0\} = \mathbb{R}1_A.$$

Definition 4.1.8. ([Lat15]) A *F-quasi-Leibniz pair* (A, L) is a Lipschitz pair such that:

(1) the domain $\mathrm{dom}(L)$ of L is a dense Jordan-Lie subalgebra of $sa(A)$,

(2) for all $a, b \in \mathrm{dom}(L)$, we have:

$$L(a \circ b) \leq F(\|a\|_A, \|b\|_A, L(a), L(b)) \quad \text{and} \quad L(\{a, b\}) \leq F(\|a\|_A, \|b\|_A, L(a), L(b))$$

A *Leibniz pair* [Lat16] (A, L_A) is a Lipschitz pair such that such that:

1. the domain $\mathrm{dom}(L)$ is a Jordan-Lie subalgebra of $sa(A)$,

2. for all $a, b \in \mathrm{dom}(L)$, we have

$$L(a \circ b) \leq \|a\|_A L(b) + L(A)\|b\|_A$$

and

$$L(\{a, b\}) \leq \|a\|_A L(b) + L(a)\|b\|_A.$$

Remark 4.1.9. A Leibniz pair is a F-quasi Leibniz pair for F defined as above.

We say a quasi-Leibniz pair (A, L) is an *F-quasi-Leibniz quantum compact metric space* [Lat16] when:

1. (A, L) is a compact quantum metric space,

2. L is lower semicontinuous,

3. (A, L) is an F-quasi-Leibniz pair.

Definition 4.1.10. ([Lat16]) Let $\mathcal{C}$ be a nonempty class of F-quasi-Leibniz quantum compact metric spaces and let $(A, L_A), (B, L_B) \in \mathcal{C}$. A *$\mathcal{C}$-trek* from (A, L_A) to (B, L_B) is a finite family:

$$\Gamma = (A_j, L_k, \gamma_{j+1}, L_{j+1} : j = 1, ..., n)$$

where:

1. for all $j \in \{1, ..., n+1\}$ we have $(A_j, L_j) \in \mathcal{C}$,

2. we have $(A_1, L_1) = (A, L_A)$ and $(A_{n+1}, L_{n+1}) = (B, L_B)$,

3. for all $j \in \{1, ..., n+1\}$, we are given a bridge γ_j from (A_j, L_j) to (A_{j+1}, L_{j+1})

The Gromov–Hausdorff $\mathcal{C}$-propinquity [Lat16], $\Lambda_{\mathcal{C}}((A, L_A), (B, L_B))$ between (A, L_A) and (B, L_B) is defined by:

$$\inf\{\lambda(\Gamma) : \Gamma \text{ is a } \textit{trek} \text{ from } (A, L_A) \text{ to } (B, L_B)\}.$$

Then we have the following refined criteria for convergence in the Gromov–Hausdorff propinquity sense [Lat15].

Lemma 4.1.11. *Let $(A, |||.|||_A)$ and $(B, |||.|||_B)$ be two F-quasi-Leibniz compact quantum metric spaces. If there exist two *-homomorphisms $\pi_A : A \hookrightarrow \mathcal{B}(\mathcal{H})$ and $\pi_B : B \hookrightarrow \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ such that the following hold:*

1. *For all $a \in A$ such that $|||a|||_A \leq 1$, there exists $b \in B$ such that $|||b|||_B \leq 1$ and $\|\pi_A(a) - \pi_B(b)\|_D < \varepsilon$,*

2. *For all $b \in B$ such that $|||b|||_B \leq 1$, there exists $a \in A$ such that $|||a|||_A \leq 1$ and $\|\pi_A(a) - \pi_B(b)\|_D < \varepsilon$,*

then $\Lambda_F((A, |||.|||_A), (B, |||.|||_B)) \leq \varepsilon$.

We will use this Lemma later in Chapter 6 to approximate the rotation algebras in higher dimensions in the sense of Gromov–Hausdorff propinquity.

4.2 Conditionally negative length functions on groups

Although the objects we study in this thesis are C*-algebras (more precisely order-unit spaces), we will use various estimates in noncommutative L_p spaces. To this end, we need to work in the context of von Neumann algebras. We refer to e.g. [BO08, JMP14, JZ15] and the references therein for the unexplained facts in the following. Let $(\mathcal{N},\tau)$ be a noncommutative W^* probability space. Here $\mathcal{N}$ is a finite von Neumann algebra and τ is a normal faithful tracial state. Let $(T_t)_{t\geq 0}$ be a pointwise weak* continuous semigroup acting on $(\mathcal{N},\tau)$ such that every T_t is unital, normal, completely positive and self-adjoint in the sense that $\tau(T_t(f)g)=\tau(fT_t(g))$ for every $f,g\in\mathcal{N}$. We will call a semigroup satisfying these conditions a noncommutative symmetric Markov semigroup. One can extend T_t to a strongly continuous semigroup of contractions on $L_2(\mathcal{N},\tau)$ (actually on $L_p(\mathcal{N},\tau)$ for all $1\leq p<\infty$). Here the noncommutative $L_p(\mathcal{N},\tau)$ space is the closure of $\mathcal{N}$ in the norm $\|f\|_p=[\tau(f^*f)^{p/2}]^{1/p}$ for $1\leq p<\infty$ and $\|f\|_\infty=\|f\|=\|f\|_{\mathcal{N}}$, the operator norm. We denote by A the infinitesimal generator of T_t, i.e., $T_t=e^{-tA}$. We define the gradient form associated to A (Meyer's "carré du champ") by

$$\Gamma^A(f,g)=\frac{1}{2}[A(f^*)g+f^*A(g)-A(f^*g)], \tag{4.2.1}$$

for f,g in the domain of A. Our major examples involve groups with conditionally negative length functions.

Let G be a countable discrete group. Let $\lambda:G\to B(\ell_2(G))$ be the left regular representation of G given by $\lambda(x)\delta_y=\delta_{xy}$ for $x,y\in G$, where $(\delta_x)_{x\in G}$ is the natural unit vectors of $\ell_2(G)$, the natural Hilbert space associated to G. Let $C_r^*(G)$ and LG denote the reduced C*-algebra and von Neumann algebra of G, respectively. They are the norm closure and weak* closure of $\lambda(G)$ in $B(\ell_2(G))$, respectively. There is a canonical normal faithful tracial state τ_G on $\mathrm{C}_r^*(G)$ and LG given by $\tau_G(f)=\langle\delta_e,f\delta_e\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\ell_2(G)$ and e is the identity of G. A function $\psi:G\to\mathbb{R}_+$ is called a length function if $\psi(e)=0$ and $\psi(x)=\psi(x^{-1})$. A length function ψ is said to be conditionally negative if $\sum_x\beta_x=0$ implies that $\sum_{x,y}\bar{\beta}_x\beta_y\psi(x^{-1}y)\leq 0$. By Schoenberg's theorem, a conditionally negative length function ψ gives rise to a completely positive semigroup $(T_t)_{t\geq 0}$ acting on LG, which is defined by $T_t\lambda(x)=e^{-t\psi(x)}\lambda(x)$. It is well known that T_t thus defined is a noncommutative symmetric Markov semigroup and its generator is given by $A\lambda(x)=\psi(x)\lambda(x)$. The Gromov form K in this context is defined as

$$K(x,y)=\frac{1}{2}[\psi(x)+\psi(y)-\psi(x^{-1}y)],\ \text{for } x,y\in G.$$

It is well known [BO08] that ψ is conditionally negative if and only if K is positive semidefinite as a matrix.

We can write the gradient form as

$$\Gamma^{\psi}(f,g) = \sum_{x,y} \bar{\hat{f}}(x)K(x,y)\hat{g}(y)\lambda(x^{-1}y) \tag{4.2.2}$$

for $f = \sum_x \hat{f}(x)\lambda(x) \in LG$ and $g = \sum_y \hat{g}(y)\lambda(y) \in LG$ being finite linear combinations. In the following, we will frequently ignore the superscript A and ψ in the notation of gradient form for short.

In this paper, we will mainly work with $G = \mathbb{Z}^d$ or $G = \mathbb{Z}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$. In this paragraph we write λ and λ_n for the left regular representations of $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. Using the Fourier transform, we can identify $\lambda(k)$ with $e^{2\pi \mathrm{i} k\cdot}$ for $k \in \mathbb{Z}$, identify $C_r^*(\mathbb{Z})$ with $C(\mathbb{T})$, the continuous functions on the torus $\mathbb{T} = \hat{\mathbb{Z}}$, and identify $L\mathbb{Z}$ with $L_\infty(\mathbb{T})$. Since the dual group of $\mathbb{Z}_n$ is $\mathbb{Z}_n$, we can identify $\lambda_n(j)$ with $\exp(\frac{2\pi \mathrm{i} j\cdot}{n})$ for $j \in \mathbb{Z}_n$ and $\mathrm{C}_r^*(\mathbb{Z}_n) = L(\mathbb{Z}_n) \simeq L_\infty(\mathbb{Z}_n) = \ell_\infty(n)$. Here the induced trace on $\ell_\infty(n)$ is the normalized trace on the $n \times n$ matrix algebra M_n where $\ell_\infty(n)$ is regarded as the diagonal subalgebra of M_n. In other words, $\lambda_n(j)$ is identified with

$$u_j(n) = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2\pi \mathrm{i} j}{n}} & & & \\ & & e^{\frac{2\pi \mathrm{i} j 2}{n}} & & \\ & & & \ddots & \\ & & & & e^{\frac{2\pi \mathrm{i} j(n-1)}{n}} \end{pmatrix} \in \ell_\infty(n) \tag{4.2.3}$$

We will consider two types of conditionally negative length functions on $\mathbb{Z}$ and $\mathbb{Z}_n$, namely

$$\psi(k) = |k|, \text{ for } k \in \mathbb{Z} \text{ and } \psi_n(k) = |k|_n = \min\{k, n-k\}, \text{ for } k \in \mathbb{Z}_n = \{0, 1, ..., n-1\}.$$

It is known that the word length functions are conditionally negative; see e.g. [JZ13, JPPP13]. To unify our notation, we will write $\mathbb{Z} = \mathbb{Z}_\infty$, $\psi = \psi_\infty$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. We will call the semigroup generated by ψ_n the Poisson semigroup on $\mathrm{C}_r^*(\mathbb{Z}_n)$ (or $L(\mathbb{Z}_n)$) for $n \in \overline{\mathbb{N}}$. This corresponds to the semigroup generated by $(-\mathrm{d}^2/\mathrm{d}x^2)^{1/2}$ on $C(\mathbb{T})$ in Fourier analysis. A more natural operator to consider is $-\mathrm{d}^2/\mathrm{d}x^2$, the 1-dimensional Laplacian. The corresponding conditionally negative length function on $\mathbb{Z}$ is $\psi(k) = k^2$ for $k \in \mathbb{Z}$. On $\mathbb{Z}_n$ for $n \in \mathbb{N}$, it is tempting to consider $\psi(k) = k^2$ for $|k| \leq n/2$. (Note that here and in what follows we may replace k by $k - n$ if $k > n/2$.) However, it is easy to check that this length function on $\mathbb{Z}_n$

for finite n is not conditionally negative. Instead, we consider

$$\psi_n(k) = \frac{n^2}{2\pi^2}[1-\cos(\frac{2\pi k}{n})], \text{ for } k \in \mathbb{Z}_n, n \in \mathbb{N}. \tag{4.2.4}$$

One can check that ψ_n defined in (4.2.4) is conditionally negative by noting that $\exp(\frac{2\pi \mathrm{i} \cdot}{n})$ is a positive semidefinite function on $\mathbb{Z}_n$. Note that

$$\psi_n(k) = \frac{n^2}{\pi^2}\sin^2(\frac{\pi k}{n}) \text{ if } k \neq 0 \text{ in } \mathbb{Z}_n.$$

Using

$$\frac{2}{\pi} \leq |\frac{\sin x}{x}| \leq 1, \ \forall x \in (0, \pi/2),$$

we see that

$$\frac{4}{\pi^2}k^2 \leq \psi_n(k) \leq k^2,$$

whenever $|k| \leq \frac{n}{2}$. Since $\lim_{n\to\infty}\psi_n(k) = k^2$ for any fixed k, we have

$$\psi_n(k) \sim k^2 \text{ for } |k| \leq n/2. \tag{4.2.5}$$

Here and in the following $a_k \sim b_k$ for two sequences (a_k) and (b_k) means that there exists an absolute constant $C \geq 1$ such that $C^{-1} \leq a_k/b_k \leq C$. We also define $\psi_\infty(k) = k^2$ for $k \in \mathbb{Z}$ and call the semigroup generated by ψ_n defined by (4.2.4) the heat semigroup on $\mathrm{C}^*_r(\mathbb{Z}_n)$ (or $L(\mathbb{Z}_n)$) for $n \in \overline{\mathbb{N}}$. Once we know ψ_n for $n \in \overline{\mathbb{N}}$, we write $\Gamma = \Gamma^{\psi_\infty}$ and $\Gamma^n = \Gamma^{\psi_n}$. We also denote by $\|\cdot\|_\infty$ the supremum norm on both $C(\mathbb{T})$ and $\mathrm{C}^*_r(\mathbb{Z}_n)$.

Let us now introduce the terminology and notation of compact quantum metric spaces. Our references here are [Rie04a, Li06]. Given a unital C*-algebra $\mathcal{A}$, we denote by $\mathcal{A}_{sa}$ the set of self-adjoint elements in $\mathcal{A}$. Then $\mathcal{A}_{sa}$ is an order-unit space in the sense of [Li06] with the identity of $\mathcal{A}$ as its order unit. Let L be a (densely) defined Lip-norm on $\mathcal{A}_{sa}$ and write $\mathcal{A} = \{f \in \mathcal{A}_{sa} : L(f) < \infty\}$. By definition, $\mathcal{A}$ is a dense order-unit subspace of $\mathcal{A}_{sa}$ and $(\mathcal{A}, L)$ is a compact quantum metric space; see [Li06]. Let $S(\mathcal{A})$ denote the state space of $\mathcal{A}$. For $r \geq 0$, recall

$$\mathcal{D}_r(\mathcal{A}) = \{a \in \mathcal{A} : L(a) \leq 1, \|a\| \leq r\}.$$

For a (separable) Hilbert space H, we write H^c and H^r for its associated column and row operator space, respectively. We denote by S_p^m (resp. S_p) the Schatten p class on ℓ_2^m (resp. ℓ_2).

4.3 Some analytic estimates

In this section we collect some analytic estimates which we will need later. Let us define

$$L_p^0(\mathcal{N}) = \{f \in L_p(\mathcal{N}) : \lim_{t\to\infty} T_t f = 0\}$$

for $1 \le p \le \infty$. Here the limit is taken in $\|\cdot\|_p$ for $1 \le p < \infty$ and in the weak* topology for $p = \infty$. Following [JM10], we define the (mean zero) Lorentz spaces $L_{r,s}^0(\mathcal{N}) = [L_p^0(\mathcal{N}), L_q^0(\mathcal{N})]_{\theta,s}$, where $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. See e.g. [BL76, PX03] for the interpolation spaces. Note that in our case for the generator A of the semigroup $(T_t)_{t\ge 0}$, we have $\mathrm{Ker}(A^{1/2}) = \{1\}$.

Proposition 4.3.1. *Let $T_t = e^{-tA}$ be a noncommutative symmetric Markov semigroup on $(\mathcal{N}, \tau)$. Suppose*

$$\|T_t : L_1^0(\mathcal{N}, \tau) \to L_\infty(\mathcal{N}, \tau)\|_{\mathrm{cb}} \le C t^{-m/2}. \tag{4.3.1}$$

Then $\|A^{-\alpha} : L_p^0(\mathcal{N}, \tau) \to L_\infty^0(\mathcal{N})\|_{\mathrm{cb}} \le C(m, \alpha)$ for $\alpha > \frac{m}{2p}$, where $C(m, \alpha) < \infty$ only depends on m and α.

Proof. The argument modifies from [JM10]; see also [JZ15]*Corollary 4.22. Let $\alpha = \frac{m}{2s}$. The argument in [JM10]*Lemma 1.1.3 can be trivially generalized to prove the complete boundedness. Hence, we have

$$\|A^{-\alpha} : L_{s,1}^0(\mathcal{N}) \to L_\infty(\mathcal{N})\|_{\mathrm{cb}} \le C(m, \alpha).$$

We know from the interpolation theory that $L_p^0(\mathcal{N}) \hookrightarrow L_{s,1}^0(\mathcal{N})$ if $p > s$. The assertion follows. $\square$

Let us consider the rotation C*-algebra $\mathcal{A}_\Theta$, where $\Theta = (\theta_{ij})$ is a $d \times d$ skew symmetric matrix with $\theta_{ij} \in [0, 1)$. By definition, $\mathcal{A}_\Theta$ is the universal C*-algebra generated by unitaries $u_1, ..., u_d$ with the commutation relations

$$u_k u_l = e^{2\pi \mathrm{i} \theta_{kl}} u_l u_k, \quad k, l = 1, ..., d.$$

It is well known that $\mathcal{A}_\Theta$ admits a faithful canonical tracial state τ such that $\tau(u_1^{k_1} \cdots u_d^{k_d}) = 1$ if and only if $k_1 = \cdots = k_d = 0$; see e.g. [Rie90]. In order to work with noncommutative L_p spaces of von Neumann algebras, we recall that $\mathcal{R}_\Theta = \mathcal{A}_\Theta''$ is the rotation von Neumann algebra associated to Θ, which is the weak* closure of $\mathcal{A}_\Theta$ acting on the GNS Hilbert space $L_2(\mathcal{A}_\Theta, \tau)$. The linear combinations of $u_1^{k_1} \cdots u_d^{k_d}$ form a weakly dense subspace of $\mathcal{R}_\Theta$. We will frequently use the following *-homomorphism:

$$\pi : \mathcal{R}_\Theta \to L(\mathbb{Z}^d)\overline{\otimes}\mathcal{R}_\Theta, \quad \pi(u_1^{k_1}\cdots u_d^{k_d}) = e^{2\pi \mathrm{i}\langle \vec{k},\cdot\rangle} u_1^{k_1}\cdots u_d^{k_d}. \tag{4.3.2}$$

Note that π is trace preserving. Let ψ be a conditionally negative length function on $\mathbb{Z}^d$ and $\tilde{T}_t$ the semigroup on $L(\mathbb{Z}^d)$ generated by ψ. We define a semigroup on $\mathcal{R}_\Theta$

$$T_t(u_1^{k_1}\cdots u_d^{k_d}) = e^{-t\psi(k_1,\dots,k_d)} u_1^{k_1}\cdots u_d^{k_d}.$$

Then $(\tilde{T}_t \otimes \mathrm{id})\circ \pi = \pi \circ T_t$. We see that T_t is a noncommutative symmetric Markov semigroup on $\mathcal{R}_\Theta$; see also [JZ15]*Proposition 5.10. Thanks to Schoenberg's Theorem, T_t is a completely positive map.

Corollary 4.3.2. *Let A be the infinitesimal generator of T_t defined as above. Assume that there exist $D > 0$ and $\gamma \ge 0$ such that*

$$\#\{\vec{k} \in \mathbb{Z}^d : \psi(\vec{k}) = j\} \le D j^\gamma, \text{ for all } j \in \mathbb{Z}_{>0}.$$

Then

$$\|T_t : L_1^0(\mathcal{R}_\Theta) \to L_\infty(\mathcal{R}_\Theta)\|_{\mathrm{cb}} \le C t^{-(\gamma+1)},$$

where C only depends on D and γ (and independent of n). Therefore, $A^{-\alpha} : L_p^0(\mathcal{R}_\Theta) \to L_\infty^0(\mathcal{R}_\Theta)$ is completely bounded for $\alpha > \frac{\gamma+1}{p}$. In particular, if $\psi(\vec{k}) \sim |k_1| + \cdots + |k_d|$, we can choose $\gamma = d-1$; and if $\psi(\vec{k}) \sim |k_1|^2 + \cdots + |k_d|^2$, we have actually a better bound $\|T_t : L_1^0(\mathcal{R}_\Theta) \to L_\infty(\mathcal{R}_\Theta)\|_{\mathrm{cb}} \le C t^{-d/2}$.

Proof. Let $x = \sum_{\psi(\vec{k})>0} a_{\vec{k}} \otimes u_1^{k_1}\cdots u_d^{k_d} \in M_m(L_1^0(\mathcal{R}_\Theta))$ be a finite linear combination. Then $(\mathrm{id}\otimes T_t)(x) = \sum_{\vec{k}} e^{-t\psi(\vec{k})} a_{\vec{k}} \otimes u_1^{k_1}\cdots u_d^{k_d}$. Consider the linear functional

$$\phi : L_1(\mathcal{R}_\Theta) \to \mathbb{C}, \quad \phi(f) = \tau(f\cdot (u_1^{k_1}\cdots u_d^{k_d})^*).$$

We have $\|\phi\|_{\mathrm{cb}} = \|\phi\|$ and thus $\|a_{\vec{k}}\|_{M_m} \le \|x\|_{M_m(L_1)}$. It follows that

$$\begin{aligned}\|(\mathrm{id}\otimes T_t)x\|_{M_m(\mathcal{R}_\Theta)} &\le \sum_{\psi(\vec{k})>0} \|a_{\vec{k}}\|_{M_m} e^{-t\psi(\vec{k})}\|u_1^{k_1}\cdots u_d^{k_d}\| \le \|x\|_{M_m(L_1(\mathcal{R}_\Theta))} \sum_{\psi(\vec{k})>0} e^{-t\psi(\vec{k})}\\ &\le D\|x\|_{M_m(L_1)} \int_0^\infty s^\gamma e^{-ts}\mathrm{d}s = D\Gamma(\gamma+1)\|x\|_{M_m(L_1)} t^{-(\gamma+1)}.\end{aligned}$$

This yields $\|T_t : L_1^0(\mathcal{R}_\Theta) \to L_\infty(\mathcal{R}_\Theta)\|_{\mathrm{cb}} \le c t^{-\gamma-1}$. We deduce from Proposition 4.3.1 with $m = 2(\gamma+1)$ that $A^{-\alpha} : L_p^0(\mathcal{R}_\Theta) \to L_\infty(\mathcal{R}_\Theta)$ is completely bounded for $\alpha > \frac{\gamma+1}{p}$.

It remains to check the value of γ. For $\psi(\vec{k}) \sim |k_1| + \cdots + |k_d|$, we have

$$\#\{\vec{k} : \psi(\vec{k}) = j\} \leq Dj^{d-1}.$$

For $\psi(\vec{k}) \sim |k_1|^2 + \cdots + |k_d|^2$, we can of course take $\gamma = d-1$. But the most interesting value of t is $0 < t < 1$. So we want a smaller value of γ. Let $x = \sum_{\vec{k}} a_{\vec{k}} \otimes u_1^{k_1} \cdots u_d^{k_d} \in M_m(L_1^0(\mathcal{R}_\Theta))$. Then

$$\begin{aligned} \|(\mathrm{id} \otimes T_t)x\|_{M_m(\mathcal{R}_\Theta)} &\leq \sum_{\vec{k}} \|a_{\vec{k}}\|_{M_m} e^{-t\psi(\vec{k})} \\ &\leq C\|x\|_{M_m(L_1)} \Big(\int_0^\infty e^{-cts^2} \mathrm{d}s \Big)^d \leq Ct^{-d/2} \|x\|_{M_m(L_1)}. \end{aligned}$$

Hence $\|T_t : L_1^0(\mathcal{R}_\Theta) \to L_\infty(\mathcal{R}_\Theta)\|_{\mathrm{cb}} \leq Ct^{-d/2}$. □

For notational convenience, let us introduce the following norms for $2 \leq p \leq \infty$. Let $\mathcal{N}$ be a von Neumann algebra with a trace τ and H a separable Hilbert space. Recall from [Pis03] that $H^c[p] = (H^c, H^r)_{1/p}$ and $H^r[p] = (H^r, H^c)_{1/p}$. We define $L_p(\mathcal{N}, H^c[p])$ as a subspace of $L_p(B(H)\overline{\otimes}\mathcal{N})$ with the norm

$$\|x\|_{L_p(\mathcal{N}, H^c[p])} = \|\langle x, x\rangle_{\mathcal{N}}^{1/2}\|_{L_p(\mathcal{N},\tau)}, \quad x \in H \otimes \mathcal{N}. \tag{4.3.3}$$

Here $\langle \cdot, \cdot \rangle_{\mathcal{N}}$ is the $\mathcal{N}$-valued inner prodcut given by $\langle a \otimes x, b \otimes y\rangle_{\mathcal{N}} = \langle a, b\rangle_H x^* y$. Similarly, we define $L_p(\mathcal{N}, H^r[p])$ as a subspace of $L_p(\mathcal{N}\overline{\otimes}B(H))$ with the norm

$$\|x\|_{L_p(\mathcal{N}, H^r[p])} = \|x^*\|_{L_p(\mathcal{N}, H^c[p])}.$$

Note that $L_\infty(\mathcal{N}, H^c) = \mathcal{N} \otimes_{\min} H^c$, $L_\infty(\mathcal{N}, H^r) = \mathcal{N} \otimes_{\min} H^r$ and

$$\|x\|_{L_p(\mathcal{N}, H^c[p] \cap H^r[p])} = \max\{\|x\|_{L_p(\mathcal{N}, H^c[p])}, \|x\|_{L_p(\mathcal{N}, H^r[p])}\}.$$

Let us turn to the group case. Let ψ be a conditionally negative length function on G. Recall that ψ determines a 1-cocycle $b : G \to H_\psi$ with values in a real unitary representation (α, H_ψ). Here H_ψ is a real Hilbert space and $\langle b(g), b(h)\rangle_{H_\psi} = K(g,h)$. One has

$$b(gh) = b(g) + \alpha_g(b(h)) \quad \text{and} \quad \psi(g) = \|b(g)\|^2,$$

for $g, h \in G$. See [BO08] for more details. We define $\mathcal{H} = H_\psi \otimes LG$ to be a LG-LG bimodule with the left action

$$\lambda(g)(b(h)\otimes\lambda(s)) = \alpha_g(b(h))\otimes\lambda(gs)$$

and the right action $(b(h)\otimes\lambda(s))\lambda(g) = b(h)\otimes\lambda(sg)$ for $s,g,h\in G$. Let $\delta : LG\to\mathcal{H}$ be (densely) defined by

$$\delta(\lambda(g)) = b(g)\otimes\lambda(g). \tag{4.3.4}$$

One can check that δ is a (densely defined) derivation on LG. Moreover, we have

$$\Gamma(x,y) = \langle\delta(x),\delta(y)\rangle_{LG}$$

for x,y in the domain of Γ. Here $\langle\cdot,\cdot\rangle_{LG}$ is the LG-valued inner product of $\mathcal{H}$. One can naturally extend δ to $M_m(LG)$ by defining $\delta(a_g\otimes\lambda(g)) = b(g)\otimes a_g\otimes\lambda(g)$ for $a_g\in M_m$. In terms of (4.3.3), we may choose $\mathcal{N} = M_m(LG)$ and $H = H_\psi$. Extending the semigroup generated by ψ to the matrix level, we can define the gradient form Γ on $M_m(LG)$. Then we have

$$\|\Gamma(x,x)^{1/2}\|_{L_p(M_m(LG))} = \|\delta(x)\|_{L_p(M_m(LG),H_\psi^c[p])}$$

for $x\in M_m(LG)$. Note that $L_p(M_m(LG)) = S_p^m L_p(LG)$. For our later c.b. estimates of the Riesz transform, we wish to completely embed $L_\infty(LG,H_\psi^c)$ into $L_p(LG,H_\psi^c[p])$. To this end, we have to consider $H_\psi^c\cap H_\psi^r$ and $H_\psi^c[p]\cap H_\psi^r[p]$.

Lemma 4.3.3. *If G is abelian, then*

$$\|\Gamma(x^*,x^*)^{1/2}\|_{L_p(M_m(LG))} = \|\delta(x)\|_{L_p(M_m(LG),H_\psi^r[p])}$$

for $x\in M_m(LG)$.

Proof. Let $x = \sum_g a_g\otimes\lambda(g)$ where $a_g\in M_m$. We define a linear map

$$J : H_\psi\to H_\psi,\quad J(b(g)) = b(g^{-1}).$$

Then thanks to commutativity,

$$\begin{aligned}\langle b(g),b(h)\rangle = K(g,h) &= \frac{1}{2}[\psi(g)+\psi(h)-\psi(g^{-1}h)]\\ &= \frac{1}{2}[\psi(g^{-1})+\psi(h^{-1})-\psi(gh^{-1})] = \langle b(g^{-1}),b(h^{-1})\rangle.\end{aligned}$$

Namely, J preserves the inner product of H_ψ. Note that

$$\delta(x^*)^* = \Big(\sum_g b(g^{-1}) \otimes a_g^* \otimes \lambda(g^{-1})\Big)^* = \sum_g b(g^{-1})^* \otimes a_g \otimes \lambda(g) = (J \otimes \mathrm{id} \otimes \mathrm{id})\delta(x).$$

Here we used $b(g^{-1})^*$ to specify that we view $b(g^{-1})$ as a row vector. Since J is an isometry, we have

$$\begin{aligned}\|\Gamma(x^*,x^*)^{1/2}\|_{L_p(M_m(LG))} &= \|\delta(x^*)\|_{L_p(M_m(LG),H_\psi^c[p])} = \|\delta(x^*)^*\|_{L_p(M_m(LG),H_\psi^r[p])}\\ &= \|(J\otimes \mathrm{id}\otimes\mathrm{id})\delta(x)\|_{L_p(M_m(LG),H_\psi^r[p])} = \|\delta(x)\|_{L_p(M_m(LG),H_\psi^r[p])}.\end{aligned}$$

□

Let us return to the rotation von Neumann algebra $\mathcal{R}_\Theta$. Recall the homomorphism π as defined in (4.3.2). Let $\delta : L(\mathbb{Z}^d) \to H_\psi \otimes L(\mathbb{Z}^d)$ be the derivation given in (4.3.4). Considering $(\mathrm{id}\otimes\delta)\circ\pi$, we extend the derivation δ to $M_m(\mathcal{R}_\Theta)$ by

$$\delta(a_{\vec{k}} \otimes u_1^{k_1}\cdots u_d^{k_d}) = b(\vec{k}) \otimes a_{\vec{k}} \otimes u_1^{k_1}\cdots u_d^{k_d}. \tag{4.3.5}$$

Note that the derivation is constructed so that the following diagram commutes in the matrix level:

$$\begin{tikzcd} \mathcal{R}_\Theta \arrow[r, "\pi"] \arrow[d, dotted, "\delta"] & L(\mathbb{Z}^d)\overline{\otimes}\mathcal{R}_\Theta \arrow[d, "\delta\otimes\mathrm{id}"] \\ H_\psi \otimes \mathcal{R}_\Theta \arrow[r, "\mathrm{id}\otimes\pi"] & H_\psi \otimes L(\mathbb{Z}^d) \otimes \mathcal{R}_\Theta \end{tikzcd}$$

Extending T_t to $\mathrm{id}_{M_m}\otimes T_t$ on $M_m(\mathcal{R}_\Theta)$, we can define the gradient form Γ on $M_m(\mathcal{R}_\Theta)$ associated to the generatorid$_{M_m}\otimes A$. Then we have $\Gamma(x,y) = \langle \delta(x), \delta(y)\rangle_{M_m(\mathcal{R}_\Theta)}$ for x, y in the domain of Γ. It follows that

$$\|\Gamma(x,x)^{1/2}\|_{L_p(M_m(\mathcal{R}_\Theta))} = \|\delta(x)\|_{L_p(M_m(\mathcal{R}_\Theta),H_\psi^c[p])}$$

for $x \in M_m(\mathcal{R}_\Theta)$. Using similar argument to that of Lemma 4.3.3, we have the following result.

Lemma 4.3.4. *Let $x = \sum_{\vec{k}\in\mathbb{Z}^d} a_{\vec{k}} \otimes u_1^{k_1}\cdots u_d^{k_d}$ be a finite sum where $a_{\vec{k}} \in M_m$. Then*

$$\|\Gamma(x^*,x^*)^{1/2}\|_{L_p(M_m(\mathcal{R}_\Theta))} = \|\delta(x)\|_{L_p(M_m(\mathcal{R}_\Theta),H_\psi^r[p])}.$$

Proof. Observing (4.3.5), we may define for clarity,

$$\delta^c(u_1^{k_1}\cdots u_d^{k_d}) = b(\vec{k}) \otimes u_1^{k_1}\cdots u_d^{k_d} \in H_\psi^c \otimes \mathcal{R}_\Theta, \quad \delta^r(u_1^{k_1}\cdots u_d^{k_d}) = b(\vec{k}) \otimes u_1^{k_1}\cdots u_d^{k_d} \in H_\psi^r \otimes \mathcal{R}_\Theta.$$

As in (4.3.5), we may extend δ^c and δ^r to matrix levels. Then

$$\delta^c(x^*) = \sum_{\vec{k}} b(-\vec{k}) \otimes a_{\vec{k}}^* \otimes (u_1^{k_1} \cdots u_d^{k_d})^*.$$

Since $\langle b(-\vec{k}), b(-\vec{k'})\rangle_{H_\psi} = \langle b(\vec{k}), b(\vec{k'})\rangle_{H_\psi}$, we have

$$\begin{aligned}
&\|\Gamma(x^*, x^*)\|_{L_p(M_m(\mathcal{R}_\Theta))} = \|\delta^c(x^*)\|_{L_p(M_m(\mathcal{R}_\Theta), H_\psi^c[p])} \\
&= \|\sum_{\vec{k},\vec{k'}} \langle b(-\vec{k}), b(-\vec{k'})\rangle_{H_\psi} a_{\vec{k}} a_{\vec{k'}}^* \otimes (u_1^{k_1} \cdots u_d^{k_d})(u_1^{k_1'} \cdots u_d^{k_d'})^* \|_p \\
&= \|\sum_{\vec{k},\vec{k'}} \langle b(\vec{k}), b(\vec{k'})\rangle_{H_\psi} a_{\vec{k}} a_{\vec{k'}}^* \otimes (u_1^{k_1} \cdots u_d^{k_d})(u_1^{k_1'} \cdots u_d^{k_d'})^* \|_p \\
&= \|\delta^r(x)\|_{L_p(M_m(\mathcal{R}_\Theta), H_\psi^r[p])}.
\end{aligned}$$

□

Let us introduce more notations to formulate our complete embedding results. For $2 \le p \le \infty$, let $\nabla_p(\mathcal{R}_\Theta)$ be a subspace of $L_p(\mathcal{R}_\Theta)$ with the semi-norm defined by

$$\|x\|_{\nabla_p(\mathcal{R}_\Theta)} = \|\delta(x)\|_{L_p(\mathcal{R}_\Theta, H_\psi^c[p] \cap H_\psi^r[p])}.$$

Then by Lemma 4.3.4 we have

$$\|x\|_{S_p^m(\nabla_p(\mathcal{R}_\Theta))} = \max\{\|\Gamma(x,x)^{1/2}\|_p, \|\Gamma(x^*,x^*)^{1/2}\|_p\} \tag{4.3.6}$$

for any x in the domain of $\Gamma^{\mathrm{id}_{M_m} \otimes A}$.

For notational convenience, let us define for x in the domain of $\Gamma^{\mathrm{id}_{M_m} \otimes A}$,

$$|||x|||_m = \max\{\|\delta^c(x)\|_{M_m \otimes_{\min} \mathcal{R}_\Theta \otimes H_\psi^c}, \|\delta^r(x)\|_{M_m \otimes_{\min} \mathcal{R}_\Theta \otimes H_\psi^r}\} = \|x\|_{M_m(\nabla_\infty(\mathcal{R}_\Theta))}. \tag{4.3.7}$$

Then $|||x|||_m$ is a Lip-norm. We usually ignore the subscript m and write $|||x|||$ if the underlying space is clear from context. We will also use frequently the notation $L(x) := |||x|||$, especially when we consider a continuous field of quantum metric spaces.

Corollary 4.3.5. *With the notation above, we have* $\| \mathrm{id} : \nabla_\infty(\mathcal{R}_\Theta) \to \nabla_p(\mathcal{R}_\Theta)\|_{\mathrm{cb}} \le C_p$ *for some constant* C_p.

Proof. Writing c.c. and c.b. for completely contractive and completely bounded isomorphisms, respectively, we consider the following diagram:

$$\begin{tikzcd}
\mathcal{N} \otimes_{\min} (H^c_\psi \cap H^r_\psi) \arrow[r, hook, "\text{c.b.}"] \arrow[d, dotted] & \mathcal{N} \otimes_{\min} L(\mathbb{F}_\infty) \arrow[r, hook, "\text{c.c.}"] & \mathcal{N}\overline{\otimes}L(\mathbb{F}_\infty) \arrow[d, "\text{c.c.}"] \\
L_p(\mathcal{N}, H^c_\psi[p] \cap H^r_\psi[p]) \arrow[rr, hook, "\text{c.b.}"] & & L_p(\mathcal{N}\overline{\otimes}L(\mathbb{F}_\infty))
\end{tikzcd}$$

Here $\mathcal{N}$ can be any finite von Neumann algebra. In particular we take $\mathcal{N} = \mathcal{R}_\Theta$. From [Pis03]*Theorem 9.7.1, we know that $H^c_\psi \cap H^r_\psi \hookrightarrow L(\mathbb{F}_\infty)$ completely isomorphically and the first line of the diagram follows. Also, by Corollary 9.7.2 and 9.8.8 in [Pis03], $H^c_\psi[p] \cap H^r_\psi[p]$ completely embeds into $L_p(L(\mathbb{F}_\infty))$ and the second line of the diagram follows. But $\mathcal{N}\overline{\otimes}L(\mathbb{F}_\infty) \hookrightarrow L_p(\mathcal{N}\overline{\otimes}L(\mathbb{F}_\infty))$ is completely contractive. We deduce that there is a complete contraction form $\mathcal{N} \otimes_{\min} (H^c_\psi \cap H^r_\psi)$ to $L_p(\mathcal{N}, H^c_\psi[p] \cap H^r_\psi[p])$. Combining this with the definition of $\nabla_p(\mathcal{R}_\Theta)$, we find that $\nabla_\infty(\mathcal{R}_\Theta)$ completely embeds into $\nabla_p(\mathcal{R}_\Theta)$. □

Remark 4.3.6. The above procedure works not only for $\mathcal{N} = \mathcal{R}_\Theta$, it also works for $\mathcal{N} = M_{n^d}$, the $n^d \times n^d$ dimensional matrix algebra, by choosing $2d$ generators of M_{n^d}. To see this, we simply define the homomorphism π as in (4.3.2) and the derivation δ as in (4.3.5) using $L(\mathbb{Z}^d_n)$ instead of $L(\mathbb{Z}^d)$. The notation $\nabla_p(\mathcal{N})$ will be used to represent $\nabla_p(\mathcal{R}_\Theta)$ or $\nabla_p(M_{n^d})$.

Suppose the semigroup $T_t = e^{-tA}$ on $\mathcal{N}$ satisfies $\Gamma_2 \geq 0$, where $\Gamma_2(f, g) = \frac{1}{2}[\Gamma(Af, g) + \Gamma(f, Ag) - A\Gamma(f, g)]$. Then $\mathrm{id}_{M_m} \otimes T_t$ also satisfies $\Gamma_2 \geq 0$; see [JM10, JZ15] for more detailed discussion on this condition. Hence, we deduce from [JM10] the complete boundedness of Riesz transforms

$$\|A^{1/2} : \nabla_p(\mathcal{N}) \to L^0_p(\mathcal{N})\|_{\mathrm{cb}} \leq K_p. \tag{4.3.8}$$

Combining this with Corollary 4.3.5, we obtain the following crucial ingredient in our argument for approximation in cb Gromov–Hausdorff convergence. Recall that we may take $\mathcal{N} = \mathcal{R}_\Theta$ or $\mathcal{N} = M_{n^d}$ as in Remark 4.3.6.

Corollary 4.3.7. *Suppose T_t satisfies $\Gamma_2 \geq 0$ on $\mathcal{N}$. Then we have $\|A^{1/2} : \nabla_\infty(\mathcal{N}) \to L^0_p(\mathcal{N})\|_{\mathrm{cb}} \leq C_p$ for some constant C_p.*

Recall that for a given function $\varphi : G \to \mathbb{C}$, the Fourier multiplier T_φ on LG is defined by extending $T_\varphi(\lambda(s)) = \varphi(s)\lambda(s)$ for $s \in G$. φ is called a Herz–Schur multiplier if T_φ is completely bounded; see e.g. [BO08].

Lemma 4.3.8. *Let φ be a Herz–Schur multiplier on G and Γ be the gradient form associated to* $\mathrm{id} \otimes A$ *defined in* (4.2.2). *Let $f \in M_m(LG)$ and assume* $(\mathrm{id} \otimes T_\varphi)f$ *belongs to the domain of the generator* $\mathrm{id} \otimes A$, *then*

$$\|\Gamma((\mathrm{id}\otimes T_\varphi)f,(\mathrm{id}\otimes T_\varphi)f)\|_{M_m(LG)} \le \|T_\varphi\|_{\mathrm{cb}}^2 \|\Gamma(f,f)\|_{M_m(LG)}.$$

Moreover, if φ is a Herz–Schur multiplier on $\mathbb{Z}^d$, then for any finite sum $f=\sum_{\vec{k}} a_{\vec{k}}\otimes u_1^{k_1}\cdots u_d^{k_d}\in M_m(\mathcal{R}_\Theta)$, we have

$$\|\Gamma((\mathrm{id}\otimes T_\varphi)f,(\mathrm{id}\otimes T_\varphi)f)\|_{M_m(\mathcal{R}_\Theta)} \le \|T_\varphi\|_{\mathrm{cb}}^2 \|\Gamma(f,f)\|_{M_m(\mathcal{R}_\Theta)}.$$

Proof. For $f=\sum_s a_s\otimes\lambda(s)$ in the domain of $\mathrm{id}\otimes A$, since the multiplier commutes with the generator A, we have

$$\begin{aligned}\|\Gamma((\mathrm{id}\otimes T_\varphi)f,&(\mathrm{id}\otimes T_\varphi)f)^{1/2}\|_{M_m(LG)} = \|\delta[(\mathrm{id}\otimes T_\varphi)f]\|_{L_\infty(M_m(LG),H_\psi^c)}\\ &= \|(\mathrm{id}_{M_m}\otimes \mathrm{id}_{H_\psi}\otimes T_\varphi)\delta(f)\|_{L_\infty(M_m(LG),H_\psi^c)} \le \|T_\varphi\|_{\mathrm{cb}}\|\delta(f)\|_{L_\infty(M_m(LG),H_\psi^c)}.\end{aligned}$$

We get the first assertion. The "moreover" part follows the same argument using the trace preserving *-homomorphism given in (4.3.2). □

Remark 4.3.9. Similar to Remark 4.3.6, by considering $G=\mathbb{Z}^d$ (resp. $G=\mathbb{Z}_n^d$) and using the homomorphism (4.3.2), we find that Lemma 4.3.8 still holds if we replace LG by $\mathcal{R}_\Theta$ (resp. M_{n^d}). This shows that $T_\varphi:(\mathcal{R}_\Theta,|||\cdot|||)\to(\mathcal{R}_\Theta,|||\cdot|||)$ (resp. $T_\varphi:(M_{n^d},|||\cdot|||)\to(M_{n^d},|||\cdot|||)$) is completely bounded.

Lemma 4.3.10. *Let $\psi: G\to\mathbb{Z}$ be a conditionally negative length function. Suppose ψ has at most polynomial growth, i.e. $\#\{g\in G:\psi(g)=0\}<\infty$ and for all $l\ge 1$, $\#\{g\in G:\psi(g)=l\}\le Dl^\gamma$ for some constants γ and $D\ge 1$. Then for any $\varepsilon>0$ and $k\in\mathbb{N}$, there exists a Herz–Schur multiplier $\varphi_{k,\varepsilon}$ and $m=m(k)>k$ such that*

(i) $\|T_{\varphi_{k,\varepsilon}}\|_{\mathrm{cb}}\le 1+\varepsilon$;

(ii) the image of $T_{\varphi_{k,\varepsilon}}$ is contained in $\mathrm{span}\{\lambda(g)\in G:\psi(g)\le m\}$;

(iii) $|\varphi_{k,\varepsilon}(g)-1|\le\varepsilon$ *for $\psi(g)\le k$;*

(iv) there exists $\varepsilon_0<\varepsilon$ such that for any $r\in\mathbb{N}$, $1\le p\le q\le\infty$, $\eta\in(0,\varepsilon_0)$ and $x=\sum_{g:\psi(g)\le k} a_g\otimes\lambda(g)\in S_q^r(L_p(LG))$,

$$\|(\mathrm{id}\otimes T_{\varphi_{k,\eta}})(x)-x\|_{S_q^r(L_q(LG))}\le\varepsilon\|x\|_{S_q^r(L_p(LG))}.$$

Therefore, if we define $P_k(\sum_{g\in G}\hat{f}_g\lambda(g))=\sum_{g:\psi(g)\le k}\hat{f}_g\lambda(g)$, then $\|(T_{\varphi_{k,\eta}}-\mathrm{id})P_k: L_p(LG)\to L_q(LG)\|_{\mathrm{cb}}\le\varepsilon$ for $1\le p\le q\le\infty$.

Proof. Let us define

$$\varphi_\alpha(g) = e^{-\psi(g)/\alpha} 1_{[\psi(g)\le m]}, \quad g \in G. \tag{4.3.9}$$

We know from Schoenberg's theorem, $\phi_\alpha(g) := e^{-\psi(g)/\alpha}$ gives a completely positive Fourier multiplier T_{ϕ_α} on LG. We have $\|T_{\phi_\alpha}\|_{\mathrm{cb}} = \|T_{\phi_\alpha}(1)\| = 1$. Given any $x = \sum_g a_g \otimes \lambda(g) \in S_q(L_p(LG))$, we claim that for $1 \le p, q \le \infty$,

$$\|a_g\|_{S_q^r} \le \|x\|_{S_q(L_p(LG))}. \tag{4.3.10}$$

Indeed, similar to the argument of Corollary 4.3.2, we define

$$\varrho : L_p(LG) \to \mathbb{C}, \quad y \mapsto \varrho(y) = \tau_G(y\lambda(g)^*).$$

We have $\|\varrho\|_{\mathrm{cb}} = \|\varrho\| \le 1$. By [Pis03]*Lemma 1.7, we also have for any $1 \le q \le \infty$,

$$\|\varrho\|_{\mathrm{cb}} = \sup_r \| \mathrm{id} \otimes \varrho : S_q^r(L_p(LG)) \to S_q^r\|.$$

Hence, we have

$$\|a_g\|_{S_q^r} = \| \mathrm{id} \otimes \varrho(x)\|_{S_q^r} \le \|x\|_{S_q^r(L_p(LG))}.$$

Using (4.3.10) with $p = q = \infty$, we have

$$\|(\mathrm{id} \otimes T_{\phi_\alpha})(x) - (\mathrm{id} \otimes T_{\varphi_\alpha})(x)\|_{M_r(LG)} \le \sum_{\psi(g)\ge m} \|a_g\|_{M_r} e^{-\psi(g)/\alpha} \le \varepsilon \|x\|_{M_r(LG)}$$

for α large enough and thus $\|T_{\varphi_\alpha}\|_{\mathrm{cb}} \le 1+\varepsilon$. Given ε, k, we can choose $m > k$ and α large enough in (4.3.9), and define $\varphi_{k,\varepsilon} = \varphi_\alpha$ such that

$$|\varphi_{k,\varepsilon}(g) - 1| \le \varepsilon \quad \text{for} \quad \psi(g) \le k < m$$

and supp $\varphi_{k,\varepsilon} \subset \{g \in G : \psi(g) \le m\}$. Clearly, the image of $T_{\varphi_{k,\varepsilon}}$ is contained in span$\{\lambda(g) : \psi(g) \le m\}$. Let $S_k = |\psi^{-1}(0)| + 1 + 2^\gamma + \cdots + k^\gamma$, where $|\psi^{-1}(0)|$ is the number of zeros of ψ, and let $\varepsilon_0 = \frac{\varepsilon}{DS_k}$. Using (4.3.10) again, we have for any $\eta \in (0, \varepsilon_0)$ and $x = \sum_{g:\psi(g)\le k} a_g \otimes \lambda(g) \in S_q^r(L_q(LG))$,

$$\|(\mathrm{id} \otimes T_{\varphi_{k,\eta}})(x) - x\|_{S_q^r(L_q(LG))} \le \sum_{\psi(g)\le k} \|a_g\|_{S_q^r} |\varphi_{k,\eta}(g) - 1| \le \varepsilon \|x\|_{S_q^r(L_p(LG))}.$$

This inequality implies the last assertion by using [Pis03]*Lemma 1.7 again. ∎

The target space $\mathbb{Z}$ of the length function ψ in the above may be replaced by some other countable discrete set, for instance, when we consider the length function (4.2.4). The proof can be modified easily to deal with this case.

To motivate our following discussion, let us fix a conditional negative length function ψ on $\mathbb{Z}_n$ for $n \in \overline{\mathbb{N}}$. Let A_n denote the generator of the semigroup associated to ψ and assume $2 \le p < \infty$. Following the notation above, we define

$$\nabla_p(L(\mathbb{Z}_n)) = \{x \in L_p(L(\mathbb{Z}_n)) : \max\{\|\Gamma^n(x,x)^{1/2}\|_p, \|\Gamma^n(x^*,x^*)^{1/2}\|_p\} < \infty\}.$$

Let $\frac{1}{2} = \alpha + \beta$ for some fixed $\alpha, \beta > 0$. Consider the following chain of maps:

$$\mathcal{D}_R(\mathbb{C}[\mathbb{Z}_n]) \subset \nabla_p(L(\mathbb{Z}_n)) \xrightarrow{A_n^{1/2}} L_p^0(L\mathbb{Z}_n) \xrightarrow{A_n^{-\beta}} L_p^0(L\mathbb{Z}_n) \xrightarrow{A_n^{-\alpha}} L_\infty^0(L\mathbb{Z}_n).$$

Here $\mathbb{C}[\mathbb{Z}_n]$ is the group algebra of $\mathbb{Z}_n$. Note that by the boundedness of Riesz transform (4.3.8), we have $\|A_n^{1/2} : \nabla_p(L(\mathbb{Z}_n)) \to L_p^0(L(\mathbb{Z}_n))\| \le K_p$. Suppose A_n has a spectral gap, by [JM10]*Proposition 1.1.5,

$$\|A_n^{-\beta} : L_p^0 \to L_p^0\|_{\mathrm{cb}} \le C_p. \tag{4.3.11}$$

Using Proposition 4.3.1, we can show that $A_n^{-\alpha} : L_p^0 \to L_\infty^0$ is bounded for $p > 1/\alpha$. Then

$$\mathrm{id} = A_n^{-\alpha} \circ A_n^{-\beta} \circ A_n^{1/2} : \mathring{\mathcal{D}}_R(\mathbb{C}[\mathbb{Z}_n]) \to L_\infty^0,$$

where $\mathring{\mathcal{D}}_R(\mathbb{C}[\mathbb{Z}_n])$ consists of the mean zero elements of $\mathcal{D}_R(\mathbb{C}[\mathbb{Z}_n])$. It will become clear later that these maps will help to establish crucial norm estimates.

For $\Delta \subset \mathbb{Z}_n$, we define

$$L_p^\Delta(L(\mathbb{Z}_n)) = \{f \in L_p(L(\mathbb{Z}_n)) : f = \sum_{k\in\Delta} \hat{f}(k)\lambda(k)\}.$$

For $k \le n/2$ and $n \in \mathbb{N}$, we define $\Lambda_k = \{0, \pm1, ..., \pm k\} \subset \mathbb{Z}_n$ and $\Lambda_k^c = \{\pm(k+1), ..., \pm[\frac{n}{2}]\} \subset \mathbb{Z}_n$. For $n = \infty$, we let $\Lambda_k^c = \{j \in \mathbb{Z} : |j| > k\}$. Let us define the projection

$$Q_k : L_p(L(\mathbb{Z}_n)) \to L_p^{\Lambda_k^c}(L(\mathbb{Z}_n)), \quad Q_k\Big(\sum_j \hat{f}(j)\lambda(j)\Big) = \sum_{|j|>k} \hat{f}(j)\lambda(j).$$

Lemma 4.3.11. *For $1 < p < \infty$ and $n > 2k$ or $n = \infty$,*

$$\|Q_k : L_p(L(\mathbb{Z}_n)) \to L_p^{\Lambda_k^c}(L(\mathbb{Z}_n))\|_{\mathrm{cb}} \leq C_p$$

for some constant C_p independent of n, k.

Proof. It is well known (see e.g. [Bou86, PX03]) that every projection $P : L_p(L\mathbb{Z}) \to L_p^{\Delta}(L\mathbb{Z})$ is completely bounded for any subinterval $\Delta \subset \mathbb{Z}$. The case $n = \infty$ follows. Assume $n \in \mathbb{N}$. Let tr denote the normalized trace on the $n \times n$ matrix algebra M_n. It is well known that there exists an injective trace preserving *-homomorphism $\rho : L(\mathbb{Z}_n) \to (M_n, tr)$ given by

$$\lambda(j) \mapsto \begin{pmatrix} 0 & I_j \\ I_{n-j} & 0 \end{pmatrix}$$

where the first 1 in the first column appears in the $(j+1)^{\mathrm{st}}$ row, the first 1 in the first row appears in the $(n-j+1)^{\mathrm{st}}$ column, and the matrix entries are constant along diagonals. Fix k and put

$$\begin{aligned} B_1 &= \{(i,j) : i \geq k+2, j \leq i-k\}, \\ B_2 &= \{(i,j) : j \geq 2, i \leq j-1\}, \\ B_3 &= \{(i,j) : j \geq k+2, i \leq j-k\}. \end{aligned}$$

Let P_B denote the projection on M_n given by

$$P_B([a_{ij}]_{1 \leq i,j \leq n}) = \sum_{(i,j) \in B} a_{ij} \otimes e_{ij}$$

where e_{ij} is the matrix unit of M_n. Then $Q_k = P_{B_1} + P_{B_2} - P_{B_3}$. It is well known (see e.g. [Bou86]*Corollary 19, [PX97]) that for any triangular projection P_B and $1 < p < \infty$,

$$\|P_B : S_p \to S_p\|_{\mathrm{cb}} \leq C_p.$$

The assertion follows immediately. ☐

Lemma 4.3.12. *Let $2 \leq p < \infty$. Then*

$$\|A_n^{-\beta} : L_p^{\Lambda_k^c}(L(\mathbb{Z}_n)) \to L_p(L(\mathbb{Z}_n))\|_{\mathrm{cb}} \leq C_p \psi(k)^{-\beta/(p-1)}$$

uniformly for $n > 2k$ or $n = +\infty$.

Proof. Let $q = 2p$ and $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$. Then $\theta = \frac{1}{p-1}$. By (4.3.11) and Lemma 4.3.11, we have

$$\|A_n^{-\beta} Q_k : L_q(L(\mathbb{Z}_n)) \to L_q(L(\mathbb{Z}_n))\|_{\mathrm{cb}} \le C_p.$$

Since $\|A_n^{-\beta} Q_k : L_2(L(\mathbb{Z}_n)) \to L_2(L(\mathbb{Z}_n))\|_{\mathrm{cb}} \le \psi(k)^{-\beta}$, by the Riesz–Thorin theorem, we have

$$\|A_n^{-\beta} Q_k : L_p(L(\mathbb{Z}_n)) \to L_p(L(\mathbb{Z}_n))\|_{\mathrm{cb}} \le C_p \psi(k)^{-\beta\theta},$$

which yields the assertion. □

4.4 Approximation for $C(\mathbb{T})$

Unless otherwise specified, in this section we consider the Poisson semigroups on $L(\mathbb{Z}_n)$ defined in Section 4.2; that is, the generator $A_n\lambda(k) = |k|\lambda(k)$ for $|k| \le n/2$. Following the notation of [Li06], for $n \in \overline{\mathbb{N}}$, we define

$$L_n(f) = \|\Gamma^n(f,f)^{1/2}\|_\infty \text{ for } f \in C_r^*(\mathbb{Z}_n)_{sa}.$$

We also write $\Gamma := \Gamma^\infty$ and $L(f) := L_\infty(f)$. It was proved in [JM10, JMP14] that L and L_n are Lip-norms[1]. Clearly, $L_n(f) < \infty$ for $f \in C_r^*(\mathbb{Z}_n)_{sa}$ for $n \in \mathbb{N}$. Note that L is only defined in a dense subspace of $C(\mathbb{T})$. We define $\mathcal{A}_\infty = \{f \in C(\mathbb{T};\mathbb{R}) : L(f) < \infty\}$. Here $C(\mathbb{T};\mathbb{R})$ is the set of real-valued continuous functions on $\mathbb{T}$, which corresponds to $\mathrm{C}_r^*(\mathbb{Z})_{sa}$. One can simply take $\mathcal{A}_\infty = \mathbb{C}(\mathbb{Z})_{sa}$ which can be identified with the self-adjoint trigonometric polynomials on $[0,1]$. We also write $\mathcal{A}_n = C_r^*(\mathbb{Z}_n)_{sa}$ for short. Then $(\mathcal{A}_n, L_n), n \in \overline{\mathbb{N}}$ are compact quantum metric spaces in the sense of [Rie04b, Li06]. Our first task is to check that they form a continuous field of compact quantum metric spaces.

Define $\pi_n : \mathrm{C}_r^*(\mathbb{Z}) \to \mathrm{C}_r^*(\mathbb{Z}_n)$ to be the linear map sending $\lambda(k)$ to $\lambda(k \bmod n)$. Since $\mathbb{Z}_n$ is abelian, its universal C*-algebra coincides with its reduced C*-algebras and therefore π_n is a *-homomorphism extended from $\lambda(1) \mapsto \lambda(1)$ by universality. To describe π_n in the function spaces, we have

$$\pi_n : C(\mathbb{T}) \to \ell_\infty(n), \quad f \mapsto \pi_n(f) = (f(j/n))_{j=0}^n,$$

and $\pi_n(e^{2\pi \mathrm{i} k \cdot})(j) = e^{\frac{2\pi \mathrm{i} kj}{n}}$.

[1] There are different versions of definitions of compact quantum metric spaces. While L_n defined here satisfies more conditions than the one in [Li06], our proof of convergence in the quantum Gromov–Hausdorff distance only requires the conditions listed in [Li06].

Lemma 4.4.1. *Let $f = \sum_{k=-m}^{m} a_k e^{2\pi \mathrm{i} k \cdot}$ and $m \le n/2$. Then $\pi_n A f = A_n \pi_n f$. Therefore,*

$$\Gamma^n(\pi_n f, \pi_n f) = \pi_n \Gamma(f, f)$$

Proof. Note that

$$\pi_n A f = \pi_n\Big(\sum_{k=-m}^{m} a_k |k| e^{2\pi \mathrm{i} k \cdot}\Big) = \sum_{k=-m}^{m} a_k |k| e^{\frac{2\pi \mathrm{i} k \cdot}{n}}.$$

Since $m \le n/2$, we get

$$A_n \pi_n f = A_n\Big(\sum_{k=-m}^{m} a_k e^{\frac{2\pi \mathrm{i} k \cdot}{n}}\Big) = \sum_{k=-m}^{m} a_k |k| e^{\frac{2\pi \mathrm{i} k \cdot}{n}}.$$

Therefore, $\pi_n A f = A_n \pi_n f$. Now since π_n is a *-homomorphism, we have

$$\begin{aligned}\Gamma^n(\pi_n f, \pi_n f) &= \frac{1}{2}(A_n(\pi_n f^*)\pi_n f + \pi_n f^* A_n(\pi_n f) - A_n(\pi_n f^* \pi_n f)) \\ &= \frac{1}{2}(\pi_n(A f^* f) + \pi_n(f^* A f) - \pi_n A(f^* f)) \\ &= \pi_n \Gamma(f, f).\end{aligned}$$

□

Proposition 4.4.2. *Let $f = \sum_{k=-m}^{m} \hat{f}(k) e^{2\pi \mathrm{i} k \cdot}$. Then*

$$\lim_{n\to\infty} \|\pi_n f\|_\infty = \|f\|_\infty,$$

and

$$\lim_{n\to\infty} \|\Gamma^n(\pi_n f, \pi_n f)\|_\infty = \|\Gamma(f, f)\|_\infty.$$

Proof. By Lemma 4.4.1, when n is large, $\Gamma^n(\pi_n f, \pi_n f) = \pi_n \Gamma(f, f)$. Let $h = \Gamma(f, f)$. Note that since f is a smooth function, so is h. By continuity of h, there exists $t_0 \in [0, 1]$ such that $\|h\|_\infty = h(t_0)$. Let $j \in \mathbb{N}$ be such that $|\frac{j}{n} - t_0| < \frac{1}{2n}$. Using the mean value theorem, we get

$$0 \le h(t_0) - h\Big(\frac{j}{n}\Big) \le \|h'\|_\infty \Big|\frac{j}{n} - t_0\Big|.$$

By (4.2.2), we may assume $h = \sum_{k=-l}^{l} a_k e^{2\pi \mathrm{i} k \cdot}$ for some finite l which only depends on m. Then $h'(x) = \sum_{k=-l}^{l} 2\pi \mathrm{i} k a_k e^{2\pi \mathrm{i} k x}$ and thus

$$\sup_{x\in[0,1]} |h'(x)| \le \sum_{k=-l}^{l} 2\pi |k| |a_k| \le C_l \|h\|_1 \le C_m \|h\|_\infty,$$

for some constant C_m only depending on m. This proves that $\lim_{n\to\infty}\|\Gamma^n(\pi_n f,\pi_n f)\|_\infty = \|\Gamma(f,f)\|_\infty$. The first assertion follows similarly. $\square$

Proposition 4.4.3. *Let $S = C(\overline{\mathbb{N}};\prod_{n\in\overline{\mathbb{N}}}\mathcal{A}_n)$ denote the continuous sections of $\prod_{n\in\overline{\mathbb{N}}}\mathcal{A}_n$. Then $(\{\mathcal{A}_n, L_n\}_{n\in\overline{\mathbb{N}}}, S)$ is a continuous field of compact quantum metric spaces (see [Li06]*Definition 6.4).*

Proof. Note that the continuity at $n\in\mathbb{N}$ is trivial and that $1=(1_n)$ is clearly in S. Here 1_n is the identity of $\mathcal{A}_n$. An element of S can be written as $(\pi_n(f))_{n\in\overline{\mathbb{N}}}$ for some $f\in\mathcal{A}_\infty$. Then Proposition 4.4.2 verifies that (a) $(\{\bar{\mathcal{A}}_n\}_{n\in\overline{\mathbb{N}}}, S)$ is a continuous field of order-unit spaces and (b) $(\{\mathcal{A}_n, L_n\}_{n\in\overline{\mathbb{N}}}, S)$ is a continuous field of compact quantum metric spaces. Here $\bar{\mathcal{A}}_n$ is the norm closure of $\mathcal{A}_n$. $\square$

Our next goal is to show that $\mathcal{A}_n$ converges to $\mathcal{A}_\infty$ in the quantum Gromov–Hausdorff distance. In light of Li's criterion [Li06], we need to find a "uniform" cover of $\mathcal{D}_R(\mathcal{A}_n)$ for n large enough. We will achieve this by using the approximation properties of $\mathbb{Z}$ and going through various estimates in L_p spaces. Recall that a Fourier multiplier T_ϕ on $L(\mathbb{Z}_n)$ is defined as

$$T_\phi(\sum_j a_j\lambda(j)) = \sum_j a_j\phi(j)\lambda(j).$$

Lemma 4.4.4. *Let $\varepsilon>0$ and $k\in\mathbb{N}$. Then there exist $m=m(k)>k$ and Herz–Schur multipliers $\varphi^n_{k,\varepsilon}$ on $\mathbb{Z}_n$ for $n>2m$ (including $n=\infty$) such that*

(i) $\|T_{\varphi^n_{k,\varepsilon}}\|_{\mathrm{cb}}\le 1+\varepsilon$;

(ii) the image of $T_{\varphi^n_{k,\varepsilon}}$ *is contained in* $\mathrm{span}\{\lambda(j):|j|\le m\}$;

(iii) $|\varphi_{k,\varepsilon}(j)-1|\le\varepsilon$ *for* $|j|_n\le k$;

(iv) for x in $\mathrm{span}\{\lambda(j):|j|_n\le k\}$ *and* $\eta\in(0,\frac{\varepsilon}{2(k+1)})$,

$$\|T_{\varphi^n_{k,\eta}}x - x\|_\infty \le \varepsilon\|x\|_2. \tag{4.4.1}$$

Proof. Note that $\#\{j\in\mathbb{Z}_n : |j|_n = k\}\le 2$ for $k\ge 1$. Applying Lemma 4.3.10 first to $G=\mathbb{Z}$ (so we have $D=2,\gamma=0$), we get m and a multiplier $\varphi_{k,\varepsilon}$ on $\mathbb{Z}$. Then applying Lemma 4.3.10 again to $G=\mathbb{Z}_n$ for $n>2m$, we find multipliers $\varphi^n_{k,\varepsilon}$ on $\mathbb{Z}_n$, which satisfy $\varphi^n_{k,\varepsilon}(j)=\varphi_{k,\varepsilon}(j)$ for $|j|\le m$ because the proof of Lemma 4.3.10 does not depend on n once we choose m. The assertion follows by taking $p=2,q=\infty,r=1$. $\square$

Lemma 4.4.5. *Let $T_t^n = e^{-tA_n}$ be the Poisson semigroup associated with ψ_n acting on $L(\mathbb{Z}_n)$ defined in Section 4.2. Then $A_n^{-\alpha} : L_p^0(L(\mathbb{Z}_n)) \to L_\infty^0(L(\mathbb{Z}_n))$ is completely bounded uniformly in $n \in \overline{\mathbb{N}}$ for $\alpha > \frac{1}{p}$.*

Proof. The argument is the same as for Corollary 4.3.2 with $\gamma = 0$. □

Lemma 4.4.6. *Let $\varepsilon > 0$. Then there exist $k = k(\varepsilon), m = m(k)$ and Herz–Schur multipliers $\varphi_{k,\eta}^n$, $\eta \in (0, \frac{\varepsilon}{2(k+1)})$ on $\mathbb{Z}_n$ for $n > 2m$ (including $n = \infty$) such that*

$$\|x - T_{\varphi_{k,\eta}^n}(x)\|_\infty \leq \varepsilon[\|x\|_2 + L_n(x)]$$

for $n > 2m$ (including $n = \infty$).

Proof. Let $k \in \mathbb{N}$ be a large number which will be determined later. We choose m and $\varphi_{k,\eta}^n$ as in Lemma 4.4.4. Since $\|(1 - Q_k)x\|_2 \leq \|x\|_2$, by (4.4.1) we have

$$\|(1 - Q_k)(x - T_{\varphi_{k,\eta}^n}(x))\|_\infty = \|(1 - Q_k)x - T_{\varphi_{k,\eta}^n}((1 - Q_k)x)\|_\infty \leq \|(1 - Q_k)x\|_2 \varepsilon \leq \varepsilon\|x\|_2.$$

Note that Q_k and A_n commute. Using Lemma 4.4.5, equation (4.3.11), Lemma 4.3.12 and the boundedness of Riesz transforms [JM10], we have for $p > 1/\alpha$,

$$\begin{aligned}
&\|A_n^{-\alpha} A_n^{-\beta} A_n^{1/2} Q_k(x - T_{\varphi_{k,\eta}^n}(x))\|_\infty \leq c_\alpha \|A_n^{-\beta} Q_k A_n^{1/2}(x - T_{\varphi_{k,\eta}^n}(x))\|_p \\
&\leq c_\alpha C_p k^{-\beta/(p-1)} \|A_n^{1/2}(x - T_{\varphi_{k,\eta}^n}(x))\|_p, \\
&\leq c_\alpha K_p C_p k^{-\beta/(p-1)} (\|\Gamma^n(x, x)^{1/2}\|_p + \|\Gamma^n(T_{\varphi_{k,\eta}^n}(x), T_{\varphi_{k,\eta}^n}(x))^{1/2}\|_p)
\end{aligned}$$

where $c_\alpha = \|A_n^{-\alpha} : L_p^0(L(\mathbb{Z}_n)) \to L_\infty(L(\mathbb{Z}_n))\|$, K_p is the L_p bound of Riesz transforms, and $C_p k^{-\beta/(p-1)}$ is the bound in Lemma 4.3.12. By Lemma 4.3.8, we have

$$\|Q_k(x - T_{\varphi_{k,\eta}^n}(x))\|_\infty \leq (2 + \varepsilon) c_\alpha K_p C_p k^{-\beta/(p-1)} \|\Gamma^n(x, x)^{1/2}\|_\infty \leq \varepsilon L_n(x)$$

by choosing k large enough. The claim follows. □

Proposition 4.4.7. *Let $\varepsilon > 0$ and $R \geq 0$. There exist $N > 0$ and $x_1, ..., x_r$ in $\mathcal{D}_R(\mathcal{A}_\infty)$ such that the open ε-balls in $\mathcal{A}_n$ centered at $\pi_n(x_1), ..., \pi_n(x_r)$ cover $\mathcal{D}_R(\mathcal{A}_n)$ for all $n > N$ (including $n = \infty$).*

Proof. The case $R = 0$ is trivial. Assume $R > 0$. Let m and $\varphi_{k,\eta}^n$ be given by Lemma 4.4.6. For $n > 2m$, let us define

$$\mathcal{D}_R^m(\mathcal{A}_n) = \{x \in \mathcal{D}_R(\mathcal{A}_n) : x = \sum_{|j|\le m} a_j \lambda(j)\}.$$

Since $\mathcal{D}_R^m(\mathcal{A}_\infty)$ is compact, we can find $x_1, ..., x_r \in \mathcal{D}_R(\mathcal{A}_\infty)$ such that for all $y \in \mathcal{D}_R^m(\mathcal{A}_\infty)$ there exists an $s \in \{1, ..., r\}$ with $\|y - x_s\|_\infty \le \varepsilon$. By Lemma 4.3.8 and 4.4.4, we know that $\frac{1}{1+\varepsilon} T_{\varphi_{k,\eta}^n}(x)$ belongs to $\mathcal{D}_R^m(\mathcal{A}_\infty)$ for every $x \in \mathcal{D}_R(\mathcal{A}_\infty)$. Thus we can find $s \in \{1, ..., r\}$ such that $\|\frac{1}{1+\varepsilon} T_{\varphi_{k,\eta}^n}(x) - x_s\|_\infty \le \varepsilon$. By Lemma 4.4.6,

$$\|x - x_s\|_\infty \le \|x - \frac{1}{1+\varepsilon} x\|_\infty + \frac{1}{1+\varepsilon}\|x - T_{\varphi_{k,\eta}^n}(x)\|_\infty + \|\frac{1}{1+\varepsilon} T_{\varphi_{k,\eta}^n}(x) - x_s\|_\infty \le 2(R+1)\varepsilon.$$

This shows that $(x_i)_{i=1,\dots,r}$ is an $2(R+1)\varepsilon$-net of $\mathcal{D}_R(\mathcal{A}_\infty)$.

Let $n > 2m$ and $y^n \in \mathcal{D}_R(\mathcal{A}_n)$. We may write $T_{\varphi_{k,\eta}^n}(y^n) = \sum_{|j|\le m} a_j e^{\frac{2\pi i j \cdot}{n}}$. Since the coefficients (a_j) are uniquely determined by y^n and $\varphi_{k,\eta}^n$, we may define $\hat{y} = \sum_{|j|\le m} a_j e^{2\pi i j \cdot}$ in $\mathcal{A}_\infty$. Then

$$\pi_n(\hat{y}) = T_{\varphi_{k,\eta}^n}(y^n). \tag{4.4.2}$$

But by Proposition 4.4.2, we have

$$\lim_{n\to\infty} \sup_{f\in K} |\|\pi_n(f)\|_\infty - \|f\|_\infty| = 0, \quad \lim_{n\to\infty} \sup_{f\in K} |\|\Gamma^n(\pi_n(f), \pi_n(f))\|_\infty - \|\Gamma(f,f)\|_\infty| = 0$$

for any compact subset K of $L\mathbb{Z}$. Note that

$$\|\hat{y}\|_2 = \|T_{\varphi_{k,\eta}^n}(y^n)\|_2 \le \|T_{\varphi_{k,\eta}^n}(y^n)\|_\infty \le (1+\varepsilon)\|y^n\|_\infty \le (1+\varepsilon)R.$$

Since $\hat{y}$ falls in a finite-dimensional space with bounded L_2 norm, the set

$$\{\hat{y} : y^n \in \mathcal{D}_R(\mathcal{A}_n)\}$$

is pre-compact in $L\mathbb{Z}$. This yields that there exists $N > 2m$ such that

$$\|\hat{y}\|_\infty \le (1+\varepsilon)\|\pi_n(\hat{y})\|_\infty \quad \text{and} \quad \|\Gamma(\hat{y}, \hat{y})\|_\infty \le (1+\varepsilon)\|\Gamma^n(\pi_n(\hat{y}), \pi_n(\hat{y}))\|_\infty \tag{4.4.3}$$

for all $y^n \in \mathcal{D}_R(\mathcal{A}_n)$ and all $n > N$. It follows from (4.4.2) that

$$\|\hat{y}\|_\infty \le (1+\varepsilon)\|T_{\varphi_{k,\eta}^n}(y^n)\|_\infty \le (1+\varepsilon)^2\|y^n\|_\infty \le (1+\varepsilon)^2 R.$$

By Lemma 4.3.8,

$$\|\Gamma^n(T_{\varphi^n_{k,\eta}}(y^n), T_{\varphi^n_{k,\eta}}(y^n))\|_\infty \leq (1+\varepsilon)^2 \|\Gamma^n(y^n, y^n)\|_\infty.$$

Thus by (4.4.2) and (4.4.3), $\|\Gamma(\hat{y}, \hat{y})\|_\infty \leq (1+\varepsilon)^3$. We find $\frac{1}{(1+\varepsilon)^2}\hat{y} \in \mathcal{D}_R(\mathcal{A}_\infty)$. Hence there exists an x_s in the $2(R+1)\varepsilon$-net $(x_i)_{i=1,\ldots,r}$ of $\mathcal{D}_R(\mathcal{A}_\infty)$ such that $\|\frac{1}{(1+\varepsilon)^2}\hat{y} - x_s\|_\infty \leq 2(R+1)\varepsilon$. Then we deduce from (4.4.2) that

$$\begin{aligned}\|T_{\varphi^n_{k,\eta}}(y^n) - \pi_n(x_s)\|_\infty &\leq \|\hat{y} - x_s\|_\infty \\ &\leq \|\hat{y} - \frac{1}{(1+\varepsilon)^2}\hat{y}\|_\infty + \|\frac{1}{(1+\varepsilon)^2}\hat{y} - x_s\|_\infty \leq (5R+2)\varepsilon.\end{aligned}$$

Using Lemma 4.4.6, we have

$$\|y^n - \pi_n(x_s)\|_\infty \leq \|y^n - T_{\varphi^n_{k,\eta}}(y^n)\|_\infty + \|T_{\varphi^n_{k,\eta}}(y^n) - \pi_n(x_s)\|_\infty \leq (6R+3)\varepsilon.$$

Replacing ε with $\frac{\varepsilon}{6R+3}$ in the very beginning, we complete the proof. □

Theorem 4.4.8. *$(\mathcal{A}_n, L_n)$ converges to $(\mathcal{A}_\infty, L)$ in the quantum Gromov–Hausdorff distance.*

Proof. By Proposition 4.4.3, $(\{\mathcal{A}_n, L_n\}_{n\in\overline{\mathbb{N}}}, S)$ is a continuous field of compact quantum metric spaces in the sense of [Li06]. Let $\varepsilon = 1/m$. By Proposition 4.4.7, we find

$$x_1(m), \cdots, x_{r_m}(m) \in \mathcal{D}_R(\mathcal{A}_\infty)$$

such that for any $x \in \mathcal{D}_R(\mathcal{A}_\infty)$ there exists $x_s(m)$ so that $\|x - x_s(m)\| \leq 1/m$. Then the set

$$\Lambda := \cup_{m=1}^\infty \{x_1(m), \cdots, x_{r_m}(m)\}$$

is dense in $\mathcal{D}_R(\mathcal{A}_\infty)$. Give an ordering on Λ as follows: $x_i(m) < x_j(m)$ if i $< j$ and $x_i(m) < x_j(m')$ if $m < m'$. Then Λ is totally ordered and we can list the elements of Λ according to this ordering. Identify $x \in \mathcal{D}_R(\mathcal{A}_\infty)$ with a section $x = (\pi_n(x))_{n\in\overline{\mathbb{N}}}$ such that $\pi_n(x) \in \mathcal{A}_n$. By our construction, for any $\varepsilon > 0$, there exist r and N such that the open ε-balls in $\mathcal{A}_n$ centered at $\pi_n(x_1), \cdots, \pi_n(x_r)$ cover $\mathcal{D}_R(\mathcal{A}_n)$ for all $n > N$, where $x_i \in \Lambda$ for all i. In other words, Λ satisfies Condition (iii) in [Li06]*Theorem 7.1. Hence $\mathcal{A}_n$ converges to $\mathcal{A}_\infty$ in $\mathrm{dist}^R_{\mathrm{oq}}$, the R-variant order-unit quantum Gromov–Hausdorff distance by the same theorem. The assertion follows from [Li06]*Theorem 1.1. □

Remark 4.4.9. In Theorem 4.4.8, we used the Poisson semigroup on $L(\mathbb{Z}_n)$ to define the Lip-norm. In fact, the same approximation result remains true if we use the heat semigroup on $L(\mathbb{Z}_n)$ and the proof is slightly more direct. Indeed, thanks to (4.2.5), we would get $m = 1$ in (4.3.1), which allows to choose $p = 2$ and $\frac{1}{4} < \alpha < \frac{1}{2}$ to replace Lemma 4.4.5. Then certain L_p estimates reduce to L_2 estimates. We leave this to the interested reader.

Chapter 5

Gromov-Hausdorff Convergence for Rotation Algebras

5.1 Matrix algebras converge to noncommutative tori

Similar to the previous section, we need to define a Lipschitz norm and a semigroup action on M_n, and show that the family of matrix algebras together with these Lip-norms form a continuous field of compact quantum metric spaces. Now we have to introduce some notation. Let $n \in \mathbb{N}$. Then $M_n \simeq \ell_\infty(n) \rtimes_\alpha \mathbb{Z}_n = \{u_j(n), v_k(n)\}''$, where $u_j(n)$ is defined in (4.2.3) and $v_k(n) = \lambda_n(k)$, the left regular representation of $\mathbb{Z}_n$. The action α is given by

$$\alpha_k(u_j(n)) = v_k(n)^* u_j(n) v_k(n).$$

Then we have the following relations

$$u_j(n)e_p = e^{\frac{2\pi \mathrm{i} jp}{n}} e_p \quad \text{and} \quad v_k(n)e_l = e_{k+l},$$

where $\{e_j\}_{j=1}^n$ is the standard orthonormal basis for $\mathbb{C}^n$. It follows that

$$u_1(n)v_1(n) = e^{\frac{2\pi \mathrm{i}}{n}} v_1(n)u_1(n).$$

We expect that $u_j(n)$ and $v_k(n)$ commute in the limit.

5.1.1 Norm Estimates for Trigonometric Polynomials

For $n \in \mathbb{N}$, we define T_t^n to be the semigroup acting on M_n by $T_t^n(u_j(n)v_k(n)) = e^{-t(\psi_n(j)+\psi_n(k))} u_j(n)v_k(n)$, where ψ_n is given by (4.2.4),

$$\psi_n(k) = \frac{n^2}{2\pi^2}[1 - \cos(\frac{2\pi k}{n})].$$

Then by Schoenberg's Theorem T_t^n is a completely positive map. Note that $u_j(n) = [u_1(n)]^j$ and $v_k(n) = [v_1(n)]^k$. So here we are using $u_1(n), v_1(n)$ as the generators of M_n when we define the semigroup T_t^n. In fact, as we shall see later, we may use any fixed pair of generators of M_n or any prime powers of these generators as the generators of M_n, but we always define T_t^n as if they were $u_1(n), v_1(n)$. For example, $u_p(n), v_q(n)$ also generate M_n as long as $(pq, n) = 1$; see e.g. [Dav96]. In this case, we may define

$$T_t^n([u_p(n)]^j[v_q(n)]^k) = e^{-t(\psi_n(j)+\psi_n(k))}[u_p(n)]^j[v_q(n)]^k. \tag{5.1.1}$$

For simplicity, we may just write $u_1(n)$ and $v_1(n)$ for $u_p(n)$ and $v_q(n)$ by abuse of notation. The semigroup we are using should be clear from context. Note that $\psi_n(j)+\psi_n(k)$ on $\mathbb{Z}_n^2$ is conditionally negative. Clearly, for fixed k we have

$$\psi_n(k) \sim \begin{cases} k^2 & \text{if } |k| \le \frac{n}{2}, \\ (n-k)^2 & \text{if } |k| > \frac{n}{2}. \end{cases}$$

Note that

$$\frac{4}{\pi^2}k^2 \le \psi_n(k) \le k^2, \text{ if } |k| \le \frac{n}{2},$$

and

$$\frac{4}{\pi^2}(n-k)^2 \le \psi_n(k) \le (n-k)^2, \text{ if } \frac{n}{2} < |k| \le n.$$

Let u and v be the generators of $M_\infty := \mathcal{A}_\theta$. Intuitively, since $\lim_{n\to\infty}\psi_n(k) = k^2 =: \psi_\infty(k)$, we would expect the heat semigroup in the limit

$$T_t(u^jv^k) := T_t^\infty(u^jv^k) = e^{-t(|j|^2+|k|^2)}u^jv^k \tag{5.1.2}$$

acting on $\mathcal{A}_\theta$. We define the gradient form Γ^n associated to the generators

$$A_n(u_j(n)v_k(n)) = (\psi_n(j)+\psi_n(k))u_j(n)v_k(n)$$

as in (4.2.1) for $n \in \overline{\mathbb{N}}$. Without loss of generality, from now on we always assume that n is large enough and $|j|, |k| \le n/2$. For $n \in \overline{\mathbb{N}}$, we define $L_n(f) = \|\Gamma^n(f,f)^{1/2}\|_\infty$. Write $\Gamma := \Gamma^\infty$ and $L(f) := L_\infty(f)$. Note that $M_n \simeq C_r^*(\mathbb{Z}_n \rtimes_\alpha \mathbb{Z}_n)$ for $n \in \overline{\mathbb{N}}$. It follows from [JM10, JMP14] that L_n and L are Lip-norms on M_n and $\mathcal{A}_\theta$, respectively. Since the heat semigroup T_t on $L(\mathbb{Z}_n) \rtimes_\alpha \mathbb{Z}_n$ is a symmetric Markov semigroup, the following result follows the same argument as for Corollary 4.3.2.

Lemma 5.1.1. *Let A_n be the generator of the heat semigroup acting on $L(\mathbb{Z}_n) \rtimes_\alpha \mathbb{Z}_n$ defined as above. Then $A_n^{-\alpha} : L_p^0(M_n) \to L_\infty^0(M_n)$ is completely bounded uniformly in $n \in \mathbb{N}$ for $\alpha > \frac{1}{p}$.*

Similar to (4.3.2), for $n \in \mathbb{N}$, we define a *-homomorphism

$$\pi : M_n \to \ell_\infty(\mathbb{Z}_n^2) \otimes M_n, \quad u_j(n)v_k(n) \mapsto \lambda(j,k) \otimes u_j(n)v_k(n), \quad (j,k) \in \mathbb{Z}_n^2.$$

Here $\lambda(j,k)$ is the left regular representation of $\mathbb{Z}_n^2$. We also define a *-homomorphism for $0 < \theta < 1$

$$\pi : \mathcal{A}_\theta \to L(\mathbb{Z}^2) \otimes \mathcal{A}_\theta, \quad u_\theta^j v_\theta^k \mapsto \lambda(j,k) \otimes u_\theta^j v_\theta^k, \quad (j,k) \in \mathbb{Z}^2.$$

Here u_θ, v_θ are the generators of $\mathcal{A}_\theta$. It is easy to check that π is trace preserving. If we understand $M_\infty = \mathcal{A}_\theta$ and $u_\theta^j = u_j(\infty), v_\theta^k = v_k(\infty)$, we can define the Fourier multipliers for $n \in \overline{\mathbb{N}}$ by

$$\tilde{T}_\phi(\lambda(j,k)) = \phi(j,k)\lambda(j,k), \quad T_\phi(u_j(n)v_k(n)) = \phi(j,k)u_j(n)v_k(n). \tag{5.1.3}$$

Note that $\pi \circ T_\phi = (\tilde{T}_\phi \otimes \mathrm{id}) \circ \pi$. We immediately have the following useful co-representation transference technique.

Lemma 5.1.2. *For any $n \in \overline{\mathbb{N}}$ and $1 \le p \le \infty$, we have*

$$\|T_\phi : L_p(M_n) \to L_p(M_n)\|_{\mathrm{cb}} \le \|\tilde{T}_\phi : L_p(\mathbb{Z}_n^2) \to L_p(\mathbb{Z}_n^2)\|_{\mathrm{cb}}.$$

Let us consider $\phi(j,k) = e^{-t\psi(j,k)}$ in (5.1.3) for a conditionally negative length function ψ on $\mathbb{Z}_n^2$. For instance, we may take $\psi(j,k) = \psi_n(j) + \psi_n(k)$ on $\mathbb{Z}_n^2$ where ψ_n is defined in (4.2.4). This gives a symmetric Markov semigroup on M_n, which coincides with the semigroup T_t defined in (5.1.1) and (5.1.2). Again, let Γ denote the gradient form associated to T_t. For the development of next section, we may extend T_t to $M_m \otimes_{\min} M_n$ by $\mathrm{id}_{M_m} \otimes T_t$ for any $m \in \mathbb{N}$ even though we only need $m = 1$ in this section. The following result is a special case of (4.3.8).

Proposition 5.1.3. *Let $2 \le p < \infty$. For any $m \in \mathbb{N}$, $a_{j,k} \in M_m$ and a finite sum $f = \sum_{j,k} a_{j,k} \otimes u_j(n)v_k(n)$, we have*

$$\|(\mathrm{id}_{M_m} \otimes A)^{1/2}(f)\|_{L_p(M_m(M_n))} \le C_p \max\{\|\Gamma(f,f)^{1/2}\|_{L_p(M_m(M_n))}, \ \|\Gamma(f^*,f^*)^{1/2}\|_{L_p(M_m(M_n))}\}$$

where C_p is independent of $m \in \mathbb{N}$ and $n \in \overline{\mathbb{N}}$. Therefore, $A^{1/2} : \nabla_\infty(M_n) \to L_p^0(M_n)$ is completely bounded.

Proof. The conditionally negative length function ψ gives the positive semidefinite Gromov form K on $\mathbb{Z}_n^2$. By the Schur product theorem, we know that $K \bullet K$ is also positive semidefinite, where $\bullet$ denotes the Schur product of matrices. It follows $\Gamma_2 \geq 0$ on $L(\mathbb{Z}_n^2)$; see e.g. [JZ15]. This transfers to $\Gamma_2 \geq 0$ on M_n by our definition of T_t on M_n, which further extends to $M_m \otimes_{\min} M_n$. Now we can apply (4.3.8) and then Corollary 4.3.7. □

Let $Q_l^1, Q_l^2 : L_p(M_n) \to L_p(M_n)$, $n > 2l$, $n \in \overline{\mathbb{N}}$, be the projections defined as

$$Q_l^1(\sum_{j,k} a_{jk} u_j(n) v_k(n)) = \sum_{\substack{|j|>l,\\ k}} a_{jk} u_j(n) v_k(n),$$

$$Q_l^2(\sum_{j,k} a_{jk} u_j(n) v_k(n)) = \sum_{\substack{|k|>l,\\ j}} a_{jk} u_j(n) v_k(n).$$

Let $\Delta \subset \mathbb{Z}_n^2$. We define

$$L_p^{\Delta}(M_n) = \{f \in L_p(M_n) : f = \sum_{(j,k)\in\Delta} a_{jk} u_j(n) v_k(n)\}.$$

Let

$$\Lambda_l^2 = \{0, \pm 1, ..., \pm l\} \times \{0, \pm 1, ..., \pm l\}. \tag{5.1.4}$$

Observe that Q_l^1 and Q_l^2 commute and the idempotent P_l defined by $P_l = (1-Q_l^1)(1-Q_l^2)$ projects $L_p(M_n)$ on $L_p^{\Lambda_l^2}(M_n)$.

Lemma 5.1.4. *For $1 < p < \infty$, $n \in \overline{\mathbb{N}}$ such that $n > 2l$,*

$$\|Q_l^1 : L_p(M_n) \to L_p(M_n)\|_{\mathrm{cb}} \leq C_p, \quad \|Q_l^2 : L_p(M_n) \to L_p(M_n)\|_{\mathrm{cb}} \leq C_p$$

$$\|P_l : L_p(M_n) \to L_p(M_n)\|_{\mathrm{cb}} \leq C_p,$$

for some constant C_p independent of n, l.

Proof. As we proved in Lemma 4.3.11, Q_l^1 and Q_l^2 are completely bounded operators on $L_p(L\mathbb{Z}_n)$. Therefore they are also completely bounded on $L_p(L\mathbb{Z}_n \overline{\otimes} L\mathbb{Z}_n)$. This implies that Q_l^1 and Q_l^2 are completely bounded on $L_p(M_n)$ for $n > 2l$ and $n \in \overline{\mathbb{N}}$ by Lemma 5.1.2. Here is another argument for $n = \infty$. Note that we have for $a_{j,k} \in M_m$,

$$(\mathrm{id}_{M_m} \otimes Q_l^1 \otimes \mathrm{id}_{L(\mathbb{Z}^2)})(\sum_{j,k} a_{j,k} \otimes u_\theta^j v_\theta^k \otimes \lambda(j,k))$$

$$= (\mathrm{id}_{M_m} \otimes \mathrm{id}_{\mathcal{R}_\theta} \otimes Q_l^1)(\sum_{j,k} a_{j,k} \otimes u_\theta^j v_\theta^k \otimes \lambda(j,k)).$$

We deduce that Q_l^1 is completely bounded on $L_p(\mathcal{R}_\theta \otimes L(\mathbb{Z}^2))$ and the assertion for Q_l^1 follows. The case of Q_l^2 is similar. As a consequnce, P_l is also completely bounded in L_p. □

Proposition 5.1.5. *Let $1 < p < \infty$, $\beta > 0$ and $n > 2l$. Let ψ be a conditionally negative length function on $\mathbb{Z}_n$ satisfying $\psi(l) \leq \psi(j)$ for $|l| \leq |j|$. Then for any $m \in \mathbb{N}$ and $a_{ij} \in M_m, i, j \in \mathbb{Z}_n$, we have*

$$\Big\| \sum_{\substack{l \leq |j| \leq n/2 \\ \psi(j) > 0}} \psi(j)^{-\beta} a_{jk} \otimes u_j(n) v_k(n) \Big\|_{L_p(M_m(M_n))} \tag{5.1.5}$$

$$\leq c_p \psi(l)^{-\beta} \Big\| \sum_{j,k} a_{jk} \otimes u_j(n) v_k(n) \Big\|_{L_p(M_m(M_n))},$$

for some constant c_p independent of m, n and l.

Proof. Let $2 < p < p_0$ be such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}$ for some $0 < \theta < 1$. We define $F_j(z) = (\frac{\psi(l)}{\psi(j)})^{z\alpha} e^{(z-\theta)^2}$, for some α large enough so that $\theta\alpha = \beta$. Define a new operator T by

$$T(z)(\sum_{j,k} a_{jk} \otimes u_j(n) v_k(n)) = \sum_{|j| \geq l} F_j(z) a_{jk} \otimes u_j(n) v_k(n).$$

Let $z = \mathrm{i}t$. Consider the Fourier multiplier

$$A_\psi(\sum_{j,k} a_{jk} \otimes u_j(n) v_k(n)) = \sum_{j,k} \psi(j) a_{jk} \otimes u_j(n) v_k(n).$$

By [JM12]*Corollary 5.4 (see also [Cow83]), we have $\|A_\psi^{\mathrm{i}s} f\|_{p_0} \leq C_{p_0} e^{c_{p_0}|s|} \|f\|_{p_0}$ for all $s \in \mathbb{R}$ and $f \in L_{p_0}(M_m(M_n))$. Then

$$\|T(\mathrm{i}t) : L_{p_0} \to L_{p_0}\| \leq C_{p_0} e^{c_{p_0}\alpha|t| - t^2} \|Q_l^1 : L_{p_0} \to L_{p_0}\|,$$

for some constants C_{p_0} and c_{p_0} independent of n and k. By Lemma 4.3.11, $T(\mathrm{i}t)$ is bounded. Now let $z = 1 + \mathrm{i}t$. Since $|\frac{\psi(l)}{\psi(j)}| \leq 1$, we have

$$\|T(1 + \mathrm{i}t) : L_2 \to L_2\| \leq |e^{(1+\mathrm{i}t-\theta)^2}| \leq e^{-t^2 + (\theta-1)^2}.$$

Therefore, $T(1 + \mathrm{i}t)$ is also bounded. For $z = \theta$, the assertion follows from Stein's interpolation theorem [Ste56]. By duality, the result holds for $1 < p \leq 2$ as well. □

Proposition 5.1.6. *Let $1 < p < \infty$ and $\beta > 0$. For any conditionally negative length function ψ on $\mathbb{Z}_n$, any $m \in \mathbb{N}$ and $a_{ij} \in M_m, i, j \in \mathbb{Z}_n$, we have*

$$\Big\| \sum_{\substack{j,k \\ \psi(j)+\psi(k)>0}} \Big(\frac{\psi(j)}{\psi(j)+\psi(k)}\Big)^{\beta} a_{jk} \otimes u_j(n) v_k(n) \Big\|_{L_p(M_m(M_n))} \tag{5.1.6}$$
$$\leq c_p \Big\| \sum_{j,k} a_{jk} \otimes u_j(n) v_k(n) \Big\|_{L_p(M_m(M_n))},$$

for some constant c_p independent of m and n.

Proof. It follows from the same argument as for Proposition 5.1.5 applied to $F_{j,k}(z) = (\frac{\psi(j)}{\psi(j)+\psi(k)})^{z\alpha} e^{(z-\theta)^2}$. □

Let $A_n^{(1)}(u_j(n)v_k(n)) = \psi_n(j)u_j(n)v_k(n)$ and $A_n^{(2)}(u_j(n)v_k(n)) = \psi_n(k)u_j(n)v_k(n)$. Then $A_n = A_n^{(1)} + A_n^{(2)}$. Here we allow ψ_n to be any conditionally negative length function with $\psi_n(k) \leq \psi(l)$ if $|k| \leq |l|$. By (5.1.3), $A_n^{(1)}, A_n^{(2)}$ and A_n are all generators of certain semigroups on M_n.

Corollary 5.1.7. *Let $1 < p < \infty, \beta > 0$ and $n \in \overline{\mathbb{N}}$ such that $n > 2l$. Then*

$$\|A_n^{-\beta}(1 - P_l) : L_p(M_n) \to L_p(M_n)\|_{\mathrm{cb}} \leq C_p \psi_n(l)^{-\beta},$$

where C_p is independent of $n, l \in \mathbb{N}$.

Proof. By (5.1.5) and (5.1.6), we have for any $m \in \mathbb{N}$ and any finite sum $x = \sum_{j,k} a_{j,k} \otimes u_j(n)v_k(n) \in M_m \otimes M_n$,

$$\begin{aligned}\|\mathrm{id} \otimes Q_l^1(x)\|_{L_p(M_m(M_n))} &\leq c_p \psi_n(l)^{-\beta} \|\mathrm{id} \otimes (A_n^{(1)})^{\beta} x\|_{L_p(M_m(M_n))} \\ &\leq c_p \psi_n(l)^{-\beta} \|\mathrm{id} \otimes (A_n^{(1)} + A_n^{(2)})^{\beta} x\|_{L_p(M_m(M_n))}.\end{aligned}$$

Similar inequality holds for Q_l^2. Using Lemma 5.1.4, we get

$$\begin{aligned}\|\mathrm{id} \otimes (1 - P_l)(x)\|_{L_p(M_m(M_n))} &= \|\mathrm{id} \otimes [Q_l^1 + Q_l^2(1 - Q_l^1)](x)\|_{L_p(M_m(M_n))} \\ &\leq (c_p \psi_n(l)^{-\beta} + \tilde{c}_p \psi_n(l)^{-\beta}) \|\mathrm{id} \otimes (A_n)^{\beta} x\|_{L_p(M_m(M_n))} \\ &= C_p \psi_n(l)^{-\beta} \|\mathrm{id} \otimes (A_n)^{\beta}(x)\|_{L_p(M_m(M_n))},\end{aligned}$$

for some constants c_p, $\tilde{c}_p$ and C_p independent of m, n and l. □

We remark that the previous complete boundedness results for matrix algebras can be alternatively proved using Lemma 5.1.2 in the same way as what we did in Lemma 5.1.4.

5.1.2 Continuous Fields of Compact Quantum Metric Spaces

Let $\mathcal{A}_\theta$ denote the rotation C*-algebra associated to $\theta \in [0, 1)$. It is well known that $\mathcal{A}_0 = C(\mathbb{T}^2)$. Let $(M_n)_{sa}$ denote the algebra of self-adjoint elements of M_n. In this section we show that

$$(\{(M_n)_{sa}, L_n\}_{n\in\overline{\mathbb{N}}}, S)$$

is a continuous field of compact quantum metric spaces. Here S consists of suitable continuous sections and $M_\infty = \mathcal{A}_\theta$. In order to establish this, we have to consider two cases, namely $\theta = 0$ and θ a non-zero rational.

Approximation in the commutative case

A key tool is the following map, defined by comultiplication:

$$\begin{aligned} \rho_n : C(\mathbb{T}^2) = C(\mathbb{T}) \otimes_{\min} C(\mathbb{T}) &\to M_n \\ \lambda_j \otimes \lambda_k &\mapsto u_j(n)v_k(n) \end{aligned} \tag{5.1.7}$$

Note that for a fixed n, ρ_n is not a *-homomorphism. Therefore, we need to introduce a *-homomorphism ρ_ω as follows. First we recall the ultraproduct construction; see, e.g., [BO08]. Let ω be a free ultrafilter on $\mathbb{N}$. Note that the Banach space $\prod_\omega X_n$ is defined as a quotient of $\prod_n X_n$ by the subspace

$$I_\omega = \{(x_n) \in \prod_n X_n : \lim_{n\to\omega} \|x_n\| = 0\}$$

with respect to the norm

$$\|(x_n)^\bullet\| = \lim_{n\to\omega} \|x_n\|_{X_n}.$$

If (X_n) are C*-algebras, we obtain a new C*-algebra $\prod X_n/I_\omega$, since I_ω is an ideal. If in addition (X_n) are von Neumann algebras with finite traces, then the von Neumann algebra ultraproduct $(X_n)^\omega$ is defined to be $\prod X_n/I_{\tau_\omega}$, where

$$I_{\tau_\omega} = \{(x_n) \in \prod_n X_n : \lim_{n\to\omega} \tau(x_n^* x_n) = 0\}.$$

Note that $I_\omega \subset I_{\tau_\omega}$ and we obtain a quotient *-homomorphism

$$\sigma_\omega : \prod\nolimits_\omega X_n \to (X_n)^\omega.$$

Now we focus on $X_n = M_n$. We define the maps $\pi_1, \pi_2 : C(\mathbb{T}) \to \prod_\omega M_n$ as follows:

$$\pi_1(\lambda_j) = (\pi_n^{(1)}(\lambda_j))^\bullet, \quad \text{where} \quad \pi_n^{(1)}(\lambda_j) = u_j(n),$$

and

$$\pi_2(\lambda_k) = (\pi_n^{(2)}(\lambda_k))^\bullet, \quad \text{where} \quad \pi_n^{(2)}(\lambda_k) = v_k(n).$$

Suppose $\sum_k f^k \otimes g^k$ is a tensor of polynomials in $C(\mathbb{T}^2)$. Then

$$\rho_n(\sum_k f^k \otimes g^k) = \sum_k \pi_n^{(1)}(f^k)\pi_n^{(2)}(g^k)$$

is a densely-defined linear map. The maps π_1 and π_2 are *-homomorphisms with commuting ranges. In fact we have

$$\begin{aligned} \|[\pi_1(\lambda_1), \pi_2(\lambda_1)]\| &= \lim_{n\to\omega} \|[u_1(n), v_1(n)]\| = \lim_{n\to\omega} \|u_1(n)v_1(n) - v_1(n)u_1(n)\| \\ &= \lim_{n\to\omega} \|(e^{\frac{2\pi i}{n}} - 1)v_1(n)u_1(n)\| = \lim_{n\to\infty} |e^{\frac{2\pi i}{n}} - 1| = 0. \end{aligned}$$

It follows that the map $\rho_\omega := (\rho_n)^\bullet$ extends to the universal C* algebra $C(\mathbb{T}) \otimes_{\max} C(\mathbb{T})$ and

$$\rho_\omega : C(\mathbb{T}^2) = C(\mathbb{T}) \otimes_{\min} C(\mathbb{T}) = C(\mathbb{T}) \otimes_{\max} C(\mathbb{T}) \to \prod\nolimits_\omega M_n$$

is a well-defined *-homomorphism. Let $\pi_\omega = \sigma_\omega\rho_\omega$. Then $\pi_\omega : C(\mathbb{T}^2) \to (M_n)^\omega$ is also a *-homomorphism.

Lemma 5.1.8. *The maps π_ω and ρ_ω are faithful. In particular,* $\lim_{n\to\infty} \|\rho_n(f)\|_{M_n} = \|f\|_{\min}$.

Proof. Let τ_n be the normalized trace on M_n and $\tau_\omega = \lim_{\tau\to\omega} \tau_n$. Then since $u_j(n)$ is a diagonal matrix and $v_k(n)$ is a shift matrix, we have

$$\tau_\omega(\sigma_\omega\rho_\omega(\lambda(j) \otimes \lambda(k))) = \lim_{n\to\omega} \tau_n(u_j(n)v_k(n)) = \delta_{j0}\delta_{k0} = (\tau \otimes \tau)(\lambda(j) \otimes \lambda(k)),$$

where τ is the canonical trace on $C_r^*(\mathbb{Z}) \simeq C(\mathbb{T})$. This proves that π_ω is trace preserving. Now let $x \in C(\mathbb{T}^2)$, and $\pi_\omega(x) = 0$. Then since π_ω is trace preserving, we have

$$\tau_\omega(\pi_\omega(x^*)\pi_\omega(x)) = \tau_\omega(\pi_\omega(x^*x)) = \tau \otimes \tau(x^*x) = 0.$$

Since the trace on $C(\mathbb{T}^2)$ is faithful, this proves that π_ω is faithful and so is ρ_ω. We deduce that $\lim_{n\to\omega} \|\rho_n(f)\|_{M_n} = \|f\|_{\min}$. But the ultrafilter ω is arbitrary, hence the assertion follows. □

Let $\mathrm{Poly}(x, y)$ denote the vector space of noncommutative polynomials on two variables. That is

$$\mathrm{Poly}(x, y) = \bigcup_{k\geq 1} \{p : p = \sum_{|i|,|j|\leq k} a_{ij} x^i y^j\}.$$

If u, v are the canonical unitary generators of $C(\mathbb{T}^2)$, we have $\mathring{\mathrm{Poly}}(u, v) \subset C(\mathbb{T}^2)$. For instance, we may take $u = \lambda_1 \otimes 1$ and $v = 1 \otimes \lambda_1$. Let

$$S = \{\rho_n(x) : x \in (C(\mathbb{T}^2))_{sa} \cap \mathrm{Poly}(u, v), n \in \overline{\mathbb{N}}\}.$$

Here and in the following we understand $\rho_\infty = \mathrm{id}$.

Proposition 5.1.9. *Let Γ^n be the gradient form associated to A_n on M_n. Then*

$$\lim_{n\to\infty} \|\Gamma^n(\rho_n(x), \rho_n(x)) - \rho_n(\Gamma(x, x))\|_{M_n} = 0$$

for $x = \sum_{j,k} a_{jk} u^j v^k \in Poly(u, v)$. Therefore, $(\{(M_n)_{sa}, L_n\}_{n\in\overline{\mathbb{N}}}, S)$ is a continuous field of compact quantum metric spaces.

Proof. Note that $\rho_n(x) = \sum_{j,k} a_{jk} u_j(n) v_k(n)$. As usual, we assume all $|j|, |k| \leq n/2$. Using the commutation relation, we have

$$\begin{aligned}\Gamma_n(\rho_n(x), \rho_n(x)) =& \frac{1}{2}\Big[\sum_{j,j',k,k'} [\psi_n(-j) + \psi_n(-k) + \psi_n(j') + \psi_n(k') \\ & - e^{-\frac{2\pi \mathrm{i}(j'-j)k}{n}} (\psi_n(j'-j) + \psi_n(k'-k))] \bar{a}_{jk} a_{j'k'} u_{j'-j}(n) v_{k'-k}(n)\Big],\end{aligned}$$

and $\rho_n(\Gamma(x, x))$ has a similar expression. The first assertion follows from the triangle inequality and taking limit. Together with Lemma 5.1.8, we get the second claim. □

Approximation for rational θ

Let $0 < \theta < 1$ be a rational number. Then $\mathcal{A}_\theta \simeq C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$. On the other hand $\mathcal{A}_\theta$ is the universal C*-algebra generated by two unitaries u and v, which commute according to the following rule

$$uv = e^{2\pi \mathrm{i}\theta} vu.$$

Now we extend the map ρ_n defined previously, from $\theta = 0$ to θ rational. In the following, we embed $\mathcal{A}_\theta$ in $M_m(C(\mathbb{T}^2))$ using the unitaries $u_j(n)$ and $v_k(n)$ which were introduced in the previous section. Since θ is rational, we can write $\theta = \frac{p}{m}$, such that $(p, m) = 1$. Note that m is fixed. We define a *-homomorphism

$$\begin{aligned} \sigma : \mathcal{A}_\theta &\to M_m \otimes_{\min} C(\mathbb{T}) \otimes_{\min} C(\mathbb{T}) \\ u^j &\mapsto u_j(m) \otimes \lambda_j \otimes 1 \\ v^k &\mapsto v_{kp}(m) \otimes 1 \otimes \lambda_k \end{aligned}$$

Recall that the canonical trace τ on $\mathcal{A}_\theta$ is faithful (see [Boc01]). Since σ is trace preserving, it is injective. Now let $\rho_n^\theta = (\mathrm{id} \otimes \rho_n) \circ \sigma$, i.e.,

$$\begin{aligned} \rho_n^\theta : \mathcal{A}_\theta &\to M_m(M_n) \\ u^j &\mapsto u_j(m) \otimes u_j(n) =: U_j(n) \\ v^k &\mapsto v_{kp}(m) \otimes v_k(n) =: V_k(n) \end{aligned} \tag{5.1.8}$$

It suffices to check the commutation relations for $U_1(n)$ and $V_1(n)$. We have

$$(u_1(m) \otimes u_1(n)) \cdot (v_p(m) \otimes v_1(n)) = e^{2\pi \mathrm{i}\theta + \frac{2\pi \mathrm{i}}{n}} (v_p(m) \otimes v_1(n)) \cdot (u_1(m) \otimes u_1(n)).$$

This means $U_1(n)V_1(n) = e^{2\pi \mathrm{i}\eta_n} V_1(n)U_1(n)$, where $\eta_n = \theta + \frac{1}{n} = \frac{pn+m}{mn}$. In order for $U_1(n)$, $V_1(n)$ to generate M_n, we need to write η_n as $\frac{a}{n}$ for some a, such that $(a, n) = 1$; see e.g. [Dav96]. For this, choose a subsequence $n = m^{k_n}$ for some exponents k_n. Then $\eta_{k_n} = \frac{pm^{k_n-1}+1}{m^{k_n}} = \frac{a}{n}$. Suppose q is a prime number which divides both $pm^{k_n-1} + 1$ and m^{k_n}. So q divides m. This implies that q divides m^{k_n-1} and hence it divides pm^{k_n-1}. But q also divides $pm^{k_n-1} + 1$. Hence q divides 1 which is a contradiction. Therefore, $R_n = m^{k_n}$ does the job, and it suffices to take the subsequence $n_k = m^k$. Let us state what we have found so far:

Lemma 5.1.10. *The map $\rho^\theta_{n_k} : \mathcal{A}_\theta \to M_{n_k}$ is surjective.*

The Lemma above says that $C^*(\rho^\theta_{n_k}(\mathcal{A}_\theta)) = M_{n_k}$, where $C^*(\rho^\theta_{n_k}(\mathcal{A}_\theta))$ denotes the C*-algebra generated by $\rho^\theta_{n_k}(\mathcal{A}_\theta)$. We next check the continuity at infinity. Let $M_\infty = \mathcal{A}_\theta$ and $S = \{\rho^\theta_{n_k}(x) : x \in (\mathcal{A}_\theta)_{sa} \cap \mathrm{Poly}(u,v), k \in \overline{\mathbb{N}}\}$. As usual, we define $A_n(U_j(n)V_k(n)) = (\psi_n(j) + \psi_n(k))U_j(n)V_k(n)$ and $L_n(f) = \|\Gamma^n(f,f)\|_\infty$ on M_n.

Proposition 5.1.11. *Choose $n_k \in \mathbb{N}$ as above. Then $(\{(M_{n_k})_{sa}, L_{n_k}\}_{k\in\overline{\mathbb{N}}}, S)$ is a continuous field of compact quantum metric spaces.*

Proof. We follow an argument similar to that of Proposition 5.1.9. Since $\rho_\omega : C(\mathbb{T}^2) \to \prod_\omega M_n$ is a trace preserving *-homomorphism, it extends to a trace preserving *-homomorphism on $M_m(C(\mathbb{T}^2))$. For any $\varepsilon > 0$, we have

$$(1-\varepsilon)\|f\|_{\mathcal{A}_\theta} \le \|\rho^\theta_{n_k}(f)\|_{M_{n_k}} \le (1+\varepsilon)\|f\|_{\mathcal{A}_\theta}$$

for all $f \in \mathrm{Poly}(u,v)$ and k large enough. Namely, $\lim_{k\to\infty} \|\rho^\theta_{n_k}(f)\|_{M_{n_k}} = \|f\|_{\mathcal{A}_\theta}$. Then by a direct calculation, one can show that $\Gamma^{n_k}(\rho^\theta_{n_k}(f), \rho^\theta_{n_k}(f))$ and $\rho^\theta_{n_k}[\Gamma(f,f)]$ coincide in the large k limit, which concludes the proof. $\square$

From now on, with abuse of notation, when we use ρ^θ_n for $\mathcal{A}_\theta$, we always mean $\rho^\theta_{n_k}$. We still need to consider the case when θ is irrational. In fact, we now deal with a more general situation.

Continuous field for the higher dimensional case

In the following, let $\mathcal{A}^d_\Theta$ denote the d-dimensional noncommutative torus which was introduced in Section 4.3. Recall that $\Theta = (\theta_{ij})$ is a $d \times d$ skew symmetric matrix. We will discuss $\mathcal{A}^d_\Theta$ in Chapter 6 in more depth. In this section we only show that they form a continuous field of compact quantum metric spaces.

Recall that for a compact Hausdorff space X, a $C(X)$-algebra is a C*-algebra A endowed with a unital morphism from $C(X)$ of continuous functions on X into the center of the multiplier algebra $M(A)$ of A; see [Kas88]. In the following we are going to derive some results about the rotation algebras using the Heisenberg group $\mathbb{H}_B = \mathbb{Z}^m \times_B \mathbb{Z}^d$, where $m = \frac{d(d-1)}{2}$ and $B : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^m$ is a skew-symmetric bilinear map. For $u = (u_i)_i$, $u' = (u'_j)_j$ in $\mathbb{Z}^d$ and z, z' in $\mathbb{Z}^m$, the multiplication on $\mathbb{H}_B$ is defined by

$$(z,u)(z',u') = (z + z' + B(u,u'), u + u')$$

where $[B(u,u')]_{rs} = u_r u'_s - u'_r u_s$ for $r, s = 1, ..., d$. Here we have identified $[B(u,u')]_{rs}$ as a vector in $\mathbb{Z}^m$. Indeed, since B is skew-symmetric, the diagonal of $B(u,u')$ is 0 and the upper triangular submatrix has

$\frac{d(d-1)}{2}$ entries. One can check that this construction gives a group structure. Note that for $(z,u) \in \mathbb{H}_B$, we have $(z,u)^{-1} = (-z - B(u,u), -u)$. For definiteness, we use the upper triangular submatrix to represent B. Let $C^*(\mathbb{H}_B)$ and $C^*_r(\mathbb{H}_B)$ be the universal C*-algebra and the reduced C*-algebra of $\mathbb{H}_B$, respectively.

Lemma 5.1.12. *$C^*(\mathbb{H}_B)$ is a $C(\mathbb{T}^m)$-algebra.*

Proof. Note that $C^*(\mathbb{H}_B) = C^*_r(\mathbb{H}_B)$ since $\mathbb{H}_B$ is amenable. Let $\lambda(k,j) \in C^*(\mathbb{H}_B)$ be the left regular representation. The left regular representation on $\mathbb{Z}^m$ induces a representation on $\ell_1(\mathbb{Z}^m)$ given by

$$\lambda : \ell_1(\mathbb{Z}^m) \to C(\mathbb{T}^m), \quad f = \sum_{l \in \mathbb{Z}^m} f(l) e_l \mapsto \lambda(f) = \sum_l f(l)\lambda(l,0),$$

where $(e_l)_l$ is the standard orthonormal basis of $\ell_2(\mathbb{Z}^m)$. Let $f \in \ell_1(\mathbb{Z}^m)$. Then we have

$$\lambda(f)\lambda(k,j) = \sum_{l \in \mathbb{Z}^m} f(l)\lambda(l,0)\lambda(k,0)\lambda(0,j) = \sum_{l \in \mathbb{Z}^m} f(l)\lambda(l+k,0)\lambda(0,j) = \lambda(k,j)\lambda(f).$$

By density, this shows that $C(\mathbb{T}^m)$ is in the center of $C^*_r(\mathbb{H}_B)$. Since $C^*(\mathbb{H}_B)$ is unital, $M(C^*(\mathbb{H}_B)) = C^*(\mathbb{H}_B)$. Hence $C^*(\mathbb{H}_B)$ is a $C(\mathbb{T}^m)$-algebra. $\square$

Let $I_\Theta = \{fx : x \in C^*(\mathbb{H}_B), f \in C(\mathbb{T}^m), f(\Theta) = 0\}$ be an ideal in $C^*(\mathbb{H}_B)$ and define $C_\Theta = C^*(\mathbb{H}_B)/I_\Theta$. More generally, let A be a $C(X)$-algebra. Consider the evaluation map $\mathrm{ev}_x : C(X) \to \mathbb{C}$ at x. Denote by A_x the quotient of A by the closed ideal

$$I_x = \{fa : f \in C(X), a \in A, f(x) = 0\} = A \cdot \mathrm{Kernel}(\mathrm{ev}_x),$$

and by a_x the image of an element $a \in A$ in the fibre A_x. Recall that the $C(X)$-algebra A is said to be a continuous field of C*-algebras over X if the function $\pi_a : X \to \mathbb{C}$ defined by $\pi_a(x) = \|a_x\|$ is continuous for every $a \in A$. In fact, the function $x \mapsto \|a_x\|$ is always upper semi-continuous; see [Bla97, Rie89, Dix77] and the references therein.

Lemma 5.1.13. *$C_\Theta \simeq \mathcal{A}^d_{2\Theta}$ and $\mathcal{A}^d_\Theta$ is a continuous field of C*-algebras.*

Proof. Let $(e_i)_{i=1}^d$ be the canonical generators of $\mathbb{Z}^d$. Note that for all r and s, we have $\lambda(0,e_s)\lambda(0,e_r) = \lambda(B(e_s,e_r), e_s + e_r)$, and

$$\begin{aligned}\lambda(0,e_r)\lambda(0,e_s) &= \lambda(B(e_r,e_s), e_r + e_s)\\ &= \lambda(2B(e_r,e_s),0)\lambda(B(e_s,e_r), e_r + e_s)\end{aligned}$$

$$= \lambda(2B(e_r, e_s), 0)\lambda(0, e_s)\lambda(0, e_r). \tag{5.1.9}$$

Let us fix $\Theta^0 \in \mathbb{T}^m$. Recall that $B(e_r, e_s) = e_{r,s}$ for $r < s$, where $e_{r,s}$ is a vector in $\mathbb{Z}^m$. Then the map f defined by $f(\Theta) = e^{4\pi \mathrm{i}\theta_{rs}} - e^{4\pi \mathrm{i}\theta^0_{rs}}$ is in I_{Θ^0}. Note that the image of $\lambda(2e_{r,s}, 0)$ in the quotient C_{Θ^0} is simply $e^{4\pi \mathrm{i}\theta^0_{rs}}$. Considering (5.1.9) in C_{Θ^0}, we find that the unitaries $\lambda(0, e_r)$ in C_{Θ^0} satisfy the commutation relations of $\mathcal{A}^d_{2\Theta^0}$. This means that one can define a *-homomorphism

$$\sigma : \mathcal{A}^d_{2\Theta^0} \to C_{\Theta^0}, \quad \sigma(u_r) = \lambda(0, e_r) + I_{\Theta^0} \in C_{\Theta^0},$$

where $(u_r)^d_{r=1}$ are generators of $\mathcal{A}^d_{2\Theta^0}$.

To identify C_Θ with $\mathcal{A}^d_{2\Theta}$, we define for $k = (k_1, ..., k_d) \in \mathbb{Z}^d$

$$\tilde{\lambda}(0, k) = \lambda(0, k_1 e_1) \cdots \lambda(0, k_d e_d).$$

Let $L(\mathbb{H}_B)$ be the von Neumann algebra of $\mathbb{H}_B$. For $i, j, k \in \mathbb{Z}^d$ and $f \in C(\mathbb{T}^m)$, by Lemma 5.1.12 and (5.1.9) we have

$$\begin{aligned}
\langle \tilde{\lambda}(0, k)\tilde{\lambda}(0, i) f,\, \tilde{\lambda}(0, j)\rangle_{L_2(L(\mathbb{H}_B),\tau)} &= \tau\left(\tilde{\lambda}(0, j)^* \tilde{\lambda}(0, k)\tilde{\lambda}(0, i) f\right) \\
&= \tau\left(f\lambda\left(\sum_{\alpha<\beta} 2 i_\alpha k_\beta B(e_\beta, e_\alpha), 0\right)\tilde{\lambda}(0, j)^*\tilde{\lambda}(0, k + i)\right) \\
&= \delta_{j,k+i}\int_{\mathbb{T}^m} f(\Theta)\exp\left(-4\pi \mathrm{i}\sum_{\alpha<\beta} i_\alpha k_\beta \theta_{\alpha\beta}\right)\mu(\mathrm{d}\Theta),
\end{aligned}$$

where μ is the normalized Haar measure on $\mathbb{T}^m$. Let $f_n^{\Theta^0} \in C(\mathbb{T}^m), n \geq 1$, be a sequence of positive functions such that $\int f_n^{\Theta^0}\mathrm{d}\mu = 1$ and $\lim_{n\to\infty}\int f_n^{\Theta^0}(\Theta)g(\Theta)\mu(\mathrm{d}\Theta) = g(\Theta^0)$. Let ω be a free ultrafilter on $\mathbb{N}$. We consider the ultrapower of Hilbert spaces

$$L_2(L(\mathbb{H}_B), \tau)^\omega = \ell_2(\mathbb{H}_B)^\omega = L_2(\mathbb{T}^m, \ell_2(\mathbb{Z}^d))^\omega$$

and ultrapower of von Neumann algebra $[L(\mathbb{H}_B)]^\omega$. We may regard each element of $C^*(\mathbb{H}_B)$ as an element of $[L(\mathbb{H}_B)]^\omega$. Then for $g \in C(\mathbb{T}^m)$ and $(\sqrt{f_n^{\Theta^0}} 1_{L(\mathbb{H}_B)})^\bullet \in L_2(L(\mathbb{H}_B), \tau)^\omega$, we have

$$\langle g\tilde{\lambda}(0, k)\tilde{\lambda}(0, i)(\sqrt{f_n^{\Theta^0}})^\bullet,\ \tilde{\lambda}(0, j)(\sqrt{f_n^{\Theta^0}})^\bullet\rangle_{L_2(L(\mathbb{H}_B),\tau)^\omega} = \delta_{j,k+i}\exp\left(-4\pi \mathrm{i}\sum_{\alpha<\beta} i_\alpha k_\beta \theta^0_{\alpha\beta}\right) g(\Theta^0)$$

$$= g(\Theta^0)\langle u_1^{k_1}\cdots u_d^{k_d}u_1^{i_1}\cdots u_d^{i_d}, u_1^{j_1}\cdots u_d^{j_d}\rangle_{L_2(\mathcal{A}_{2\Theta^0},\tau)}. \tag{5.1.10}$$

Let $K = \text{span}\{(\sqrt{f_n^{\Theta^0}}x)^\bullet : x \in C^*(\mathbb{H}_B)\}$. Then K is a subspace of $L_2(L(\mathbb{H}_B),\tau)^\omega$. We consider a special representation of $C^*(\mathbb{H}_B)$ on K defined by

$$w_{\Theta^0}(x)(\sqrt{f_n^{\Theta^0}}y)^\bullet = (\sqrt{f_n^{\Theta^0}}xy)^\bullet, \quad x, y \in C^*(\mathbb{H}_B).$$

Then by (5.1.10) we have

$$\tau_\omega([w_{\Theta^0}(x)(\sqrt{f_n^{\Theta^0}}y)^\bullet]^*[w_{\Theta^0}(x)(\sqrt{f_n^{\Theta^0}}y)^\bullet]) = \lim_{n\to\omega}\tau(y^*x^*xyf_n^{\Theta^0}) = 0,$$

for $x \in I_{\Theta^0}$. Thus w_{Θ^0} factors through C_{Θ^0}: If we denote the quotient map by $q_{\Theta^0} : C^*(\mathbb{H}_B) \to C_{\Theta^0}$ and define $v_{\Theta^0}(x + I_{\Theta^0}) = w_{\Theta^0}(x)$, then $w_{\Theta^0} = v_{\Theta^0}q_{\Theta^0}$. We define a linear operator $\alpha : L_2(\mathcal{A}_{2\Theta^0}) \to K$ by

$$\alpha(u_1^{k_1}\cdots u_d^{k_d}) = (\sqrt{f_n^{\Theta^0}}\tilde{\lambda}(0,k))^\bullet.$$

Note that α has dense range and preserves the inner product by (5.1.10). Then α is unitary. We define $\phi : \mathcal{B}(K) \to \mathcal{B}(L_2(\mathcal{A}^d_{2\Theta^0}))$ by $\phi(x) = \alpha^*x\alpha$. One directly checks that $\phi[w_{\Theta^0}(\tilde{\lambda}(0,k))] = u_1^{k_1}\cdots u_d^{k_d}$. We define $\pi_{\Theta^0} = \phi\circ v_{\Theta^0}$. Then

$$\pi_{\Theta^0} : C_{\Theta^0} \to \mathcal{A}^d_{2\Theta^0}, \quad \pi_{\Theta^0}(\tilde{\lambda}(0,k) + I_{\Theta^0}) = u_1^{k_1}\cdots u_d^{k_d}.$$

It follows that $\pi_{\Theta^0}\circ\sigma = \text{id}$, and σ, π_{Θ^0} are trace preserving isomorphisms. We can represent our argument here in a commutative diagram

$$\begin{tikzcd}
\mathcal{A}^d_{2\Theta} \arrow[r, "\sigma"] & C_\Theta \arrow[r, "\pi_\Theta"] \arrow[dr, "v_\Theta"] & \mathcal{A}^d_{2\Theta} \arrow[r, hook] & \mathcal{B}(L_2(\mathcal{A}^d_{2\Theta})) \\
& C^*(\mathbb{H}_B) \arrow[u, "q_\Theta"] \arrow[r, "w_\Theta"] & \mathcal{B}(K) \arrow[ur, "\phi"] &
\end{tikzcd}$$

Now we prove the lower semi-continuity of $\Theta \mapsto \|q_\Theta(x)\|$ for $x \in C^*(\mathbb{H}_B)$. First note that $\pi_\Theta\circ\sigma = \text{id}$. Hence π_Θ is injective. The map

$$\mathcal{A}^d_{2\Theta} \hookrightarrow \mathcal{B}(L_2(\mathcal{A}^d_{2\Theta}))$$

is also injective. Therefore, v_Θ is an isometry. Note that $\|q_\Theta(x)\| = \|v_\Theta[q_\Theta(x)]\| = \|w_\Theta(x)\|$. Let $x, \xi, \eta \in$

$C^*(\mathbb{H}_B)$. We may write

$$x = \sum_{k\in\mathbb{Z}^m,\ j\in\mathbb{Z}^d} x_{kj} e^{2\pi\mathrm{i}\langle k,\cdot\rangle}\tilde{\lambda}(0,j) \in C^*(\mathbb{H}_B),$$

$$\xi = \sum_{k\in\mathbb{Z}^m,\ j\in\mathbb{Z}^d} a_{kj} e^{2\pi\mathrm{i}\langle k,\cdot\rangle}\tilde{\lambda}(0,j), \quad \eta = \sum_{k\in\mathbb{Z}^m,\ j\in\mathbb{Z}^d} b_{kj} e^{2\pi\mathrm{i}\langle k,\cdot\rangle}\tilde{\lambda}(0,j).$$

We deduce from (5.1.10) that $\langle w_\Theta(x)(\sqrt{f_n^\Theta}\xi)^\bullet, (\sqrt{f_n^\Theta}\eta)^\bullet\rangle$ is a continuous function of Θ. Hence,

$$\begin{aligned} |\langle w_\Theta(x)(\sqrt{f_n^\Theta}\xi)^\bullet, (\sqrt{f_n^\Theta}\eta)^\bullet\rangle| &= \liminf_{\Xi\to\Theta} |\langle w_\Xi(x)(\sqrt{f_n^\Xi}\xi)^\bullet, (\sqrt{f_n^\Xi}\eta)^\bullet\rangle| \\ &\le \liminf_{\Xi\to\Theta} \|w_\Xi(x)\|\|(\sqrt{f_n^\Xi}\xi)^\bullet\|\|(\sqrt{f_n^\Xi}\eta)^\bullet\|. \end{aligned}$$

Note that

$$\|q_\Theta(x)\| = \sup\{|\langle w_\Theta(x)(\sqrt{f_n^\Theta}\xi)^\bullet, (\sqrt{f_n^\Theta}\eta)^\bullet\rangle| : \|(\sqrt{f_n^\Theta}\xi)^\bullet\| \le 1, \|(\sqrt{f_n^\Theta}\eta)^\bullet\| \le 1\}.$$

It follows that $\|q_\Theta(x)\| \le \liminf_{\Xi\to\Theta} \|w_\Xi(x)\|$ and the proof is complete. □

In particular, we obtained the following:

Lemma 5.1.14. *Let ρ_n^θ be as defined previously. For any θ_0 we have*

$$\lim_{\substack{\theta\to\theta_0\\ \theta\in\mathbb{Q}}} \lim_{j\to\infty} \|\rho^\theta_{n_j(\theta)}(x_\theta)\|_{M_{n_j(\theta)}} = \|x_{\theta_0}\|_{\mathcal{A}_{\theta_0}}.$$

Here $x_\theta = \sum_{j,k} a_{jk} u_\theta^j v_\theta^k$ for any given $x = x_{\theta_0} = \sum_{j,k} a_{jk} u_{\theta_0}^j v_{\theta_0}^k \in \mathcal{A}_{\theta_0}$ and $n_j(\theta)$ is chosen according to Lemma 5.1.10 for any given rational θ.

We also need to show that the same result as above holds for the Lip-norm. This can be done for d-dimensional noncommutative tori. For simplicity, we restrict our attention to the 2-dimensional case. In order to complete the proof of the continuity of the field of compact quantum metric spaces, we construct suitable derivations. In fact, this was done in (4.3.5). Here we give two concrete cases.

Case 1: Poisson semigroup.

Consider the Hilbert $\mathcal{A}_\theta$-module $\mathcal{H} = (\ell_2(\mathbb{Z})\oplus\ell_2(\mathbb{Z}))\otimes_{\min}\mathcal{A}_\theta$. Let $h_k = (\sum_{|i|\le k} e_i)$ and define a derivation δ by

$$\delta(u_\theta^j v_\theta^k) = (h_j \oplus h_k) \otimes u_\theta^j v_\theta^k.$$

We have

$$\langle\delta(u_\theta^j v_\theta^k),\delta(u_\theta^{j'} v_\theta^{k'})\rangle_{\mathcal{A}_\theta} = \langle h_j\oplus h_k, h_{j'}\oplus h_{k'}\rangle (u_\theta^j v_\theta^k)^*(u_\theta^{j'} v_\theta^{k'})$$
$$= (\langle h_j, h_{j'}\rangle + \langle h_k, h_{k'}\rangle)(u_\theta^j v_\theta^k)^*(u_\theta^{j'} v_\theta^{k'}),$$

showing that

$$\Gamma(u_\theta^j v_\theta^k, u_\theta^{j'} v_\theta^{k'}) = K((j,j'),(k,k'))(u_\theta^j v_\theta^k)^* u_\theta^{j'} v_\theta^{k'} = \langle\delta(u_\theta^j v_\theta^k),\delta(u_\theta^{j'} v_\theta^{k'})\rangle_{\mathcal{A}_\theta},$$

where $K((j,j'),(k,k')) = K_1(j,j') + K_2(k,k')$. Here K_1 and K_2 are the Gromov forms defined in Section 4.2.

Case 2: Heat semigroup. Consider the Hilbert $\mathcal{A}_\theta$-module $\mathcal{H} = \mathbb{R}^2\otimes\mathcal{A}_\theta$ and define a derivation δ by

$$\delta(u_\theta^j v_\theta^k) = (j,k)\otimes u_\theta^j v_\theta^k.$$

Let $x = u_\theta^j v_\theta^k$ and $y = u_\theta^{j'} v_\theta^{k'}$. We have

$$\Gamma(x,y) = \frac{1}{2}[(j^2+k^2) + (j')^2 + (k')^2 - (j-j')^2 - (k-k')^2](u_\theta^j v_\theta^k)^* u_\theta^{j'} v_\theta^{k'}$$
$$= (jj' + kk')(u_\theta^j v_\theta^k)^* u_\theta^{j'} v_\theta^{k'}.$$

Therefore, we get

$$\langle\delta(u_\theta^j v_\theta^k),\delta(u_\theta^{j'} v_\theta^{k'})\rangle_{\mathcal{A}_\theta} = \langle (j,k)\otimes u_\theta^j v_\theta^k, (j',k')\otimes u_\theta^{j'} v_\theta^{k'}\rangle$$
$$= \langle (j,k),(j',k')\rangle (u_\theta^j v_\theta^k)^*(u_\theta^{j'} v_\theta^{k'})$$
$$= (jj'+kk')(u_\theta^j v_\theta^k)^*(u_\theta^{j'} v_\theta^{k'}).$$

Note that both $\mathbb{R}^d$ and $\oplus_{i=1}^d \ell_2(\mathbb{Z})$ embed into the column space ℓ_2^c, we may take $\mathcal{H} = \ell_2^c\otimes_{\min}\mathcal{A}_\theta$. Let $p(x,y) = \sum_{j,k} a_{jk} x^j y^k \in \mathrm{Poly}(x,y)$ be a noncommutative polynomial. Then by Lemma 5.1.13,

$$\lim_{\theta'\to\theta}\|\delta p(u_{\theta'}, v_{\theta'})\|_{\mathcal{H}} = \|\delta p(u_\theta, v_\theta)\|_{\mathcal{H}} = \|\sum_{j,k} a_{jk}\xi_{jk}\otimes u_\theta^j v_\theta^k\|_{\mathcal{H}}$$
$$= \|\sum_{j,k,j',k'} \bar{a}_{j',k'} a_{jk}\langle\xi_{j'k'},\xi_{jk}\rangle (u_\theta^{j'} v_\theta^{k'})^* u_\theta^j v_\theta^k\|_{\mathcal{A}_\theta}^{1/2}$$

for some $\xi_{jk}\in\ell_2^c$ and the coefficients a_{jk} are independent of θ. In particular we have proven the following:

Lemma 5.1.15. *For any $p = p(x, y) \in Poly(x, y)$, the map*

$$\theta \mapsto \|\delta p(u_\theta, v_\theta)\|_{\mathcal{H}} = \|\Gamma(p(u_\theta, v_\theta), p(u_\theta, v_\theta))^{1/2}\|_{\mathcal{A}_\theta}$$

is continuous.

Let $u^1_\Theta, u^2_\Theta, ..., u^d_\Theta$ be the generators of $\mathcal{A}^d_\Theta$. Let $x_\Theta = p(u^1_\Theta, u^2_\Theta, ..., u^d_\Theta)$ and $L_\Theta(x_\Theta) = \|\Gamma(x_\Theta, x_\Theta)\|_{\mathcal{A}_\Theta}$. We define the set of continuous sections $S = \{p(u^1_\Theta, u^2_\Theta, ..., u^d_\Theta) : p \in \text{Poly}(x_1, x_2..., x_d)\}$. The following summarizes the arguments of this section.

Proposition 5.1.16. $(\{(\mathcal{A}^d_\Theta)_{sa}, L_\Theta\}_{\Theta \in \mathbb{T}^{d(d-1)/2}}, S)$ *forms a continuous field of compact quantum metric spaces.*

5.1.3 Matrix Algebras Converge to the Rotation Algebras

Let us define the following maps. Let $\{u_\theta, v_\theta\}$ be the generators of $\mathcal{A}_\theta$. Observe that $\text{Poly}(x, y) \subset C(\mathbb{T}) * C(\mathbb{T})$ as a vector space. Define a linear map

$$\begin{aligned} \sigma_\theta : \text{Poly}(x, y) &\to \mathcal{A}_\theta \\ p(x, y) &\mapsto p(u_\theta, v_\theta) \end{aligned}$$

with dense range.

Lemma 5.1.17. *For any $\theta \in [0, 1)$, there exists sequences $(\theta_j) \subset \mathbb{Q} \cap [0, 1)$ and $(n_j) \subset \mathbb{N}$ such that*

(i) $\lim_{j\to\infty} \theta_j = \theta$;

(ii) $(n_j)_j$ *is increasing to infinity;*

(iii) $(\{(M_{n_j})_{sa}, L_{n_j}\}_{j\in\overline{\mathbb{N}}}, S)$ *is a continuous field of compact quantum metric spaces, where* $S = \{\rho^{\theta_j}_{n_j}(\sigma_{\theta_j}(p)) : j \in \overline{\mathbb{N}}, p(x, y) \in \text{Poly}(x, y)_{sa}\}$ *and ρ^θ_n is defined in* (5.1.8).

Proof. If θ is rational, we simply take $\theta_j \equiv \theta$ and choose n_j as in Proposition 5.1.11. Suppose θ is irrational. Let (θ_j) be a sequence of rational numbers such that $\lim_{j\to\infty} \theta_j = \theta$. Put $\theta_\infty = \theta$. Let $p(x, y) \in \text{Poly}(x, y)$. Using Lemma 5.1.14 and 5.1.15, for each θ_j, we can choose an n_j such that $n_j \le n_{j+1}$,

$$|\|\rho^{\theta_j}_{n_j}(p(u_{\theta_j}, v_{\theta_j}))\|_{M_{n_j}} - \|p(u_\theta, v_\theta)\|_{\mathcal{A}_\theta}| < \frac{1}{j}$$

and

$$|\|\Gamma^{n_j}(\rho_{n_j}^{\theta_j}(p(u_{\theta_j},v_{\theta_j})),\rho_{n_j}^{\theta_j}(p(u_{\theta_j},v_{\theta_j})))\|_{M_{n_j}}-\|\Gamma(p(u_\theta,v_\theta),p(u_\theta,v_\theta))\|_{\mathcal{A}_\theta}|<\frac{1}{j}.$$

This means that $(\{(M_{n_j})_{sa},L_{n_j}\}_{j\in\overline{\mathbb{N}}},S)$ is a continuous field of compact quantum metric spaces. $\square$

The following is an analog of Lemma 4.4.6 for M_n and $\mathcal{A}_\theta$.

Lemma 5.1.18. *Let $\varepsilon>0$. Then there exist $k=k(\varepsilon)$, $m=m(k)$, and multipliers $\phi_{k,\eta}^n$, $\eta\in(0,\frac{\varepsilon}{2(2k+1)^2})$, on M_n for $n>2m$ (including $n=\infty$) such that*

$$\|x-T_{\phi_{k,\eta}^n}(x)\|_{M_n}\le\varepsilon[\|x\|_2+L_n(x)]$$

for $n>2m$ (including $n=\infty$). Here $T_{\phi_{k,\eta}^n}$ is induced by $\tilde{T}_{\phi_{k,\eta}^n}$ as defined in (5.1.3).

Proof. Let $k\in\mathbb{N}$ be a large number which will be determined later. We choose m and $\varphi_{k,\eta}^n$ on $\mathbb{Z}_n$ as in Lemma 4.4.4 for $n>2m$. Here we actually use the heat length function ψ_n as defined by (4.2.4) in Lemma 4.4.4. But since $(\frac{2}{\pi})^2j^2\le\psi_n(j)\le j^2$ and

$$\#\{j:|j|_n^2\le k\}\le\#\{j:|j|_n\le k\},$$

we may still choose $\eta\in(0,\frac{\varepsilon}{2(2k+1)^2})$ and the conclusion of Lemma 4.4.4 remains valid. Let $\phi_{k,\eta}^n(j,l)=\varphi_{k,\eta}^n(j)\varphi_{k,\eta}^n(l)$ for $(j,l)\in\mathbb{Z}_n^2$. Note that for the Fourier multiplier $\phi_{k,\eta}^n$,

$$\|\tilde{T}_{\phi_{k,\eta}^n}\|_{\mathrm{cb}}\le\|T_{\varphi_{k,\eta}^n}\|_{\mathrm{cb}}^2\le(1+\varepsilon)^2.$$

By Lemma 5.1.2, we have

$$\|T_{\phi_{k,\eta}^n}\|_{\mathrm{cb}}\le(1+\varepsilon)^2.$$

According to our choice of $\phi_{k,\eta}^n$, we have $\operatorname{supp}\phi_{k,\eta}^n\subset[-m,m]^2$. By choosing $\eta\le\varepsilon/(2k+1)^2$ and using Lemma 4.3.10, we have

$$|\phi_{k,\eta}^n(j,l)-1|\le\frac{\varepsilon}{(2k+1)^2},\quad (j,l)\in[-k,k]^2. \tag{5.1.11}$$

Then for any $x=\sum_{|j|,|l|\le k}a_{j,l}u_j(n)v_l(n)$, we have

$$\|T_{\phi_{k,\eta}^n}(x)-x\|_{M_n}\le\sum_{|j|,|l|\le k}|a_{j,l}||\phi_{k,\eta}^n(j,l)-1|\le\|x\|_2\varepsilon.$$

Since $\|P_k y\|_2 \leq \|y\|_2$ for any $y \in M_n$, $n > 2m$ (including $n = \infty$), we have

$$\|P_k(y - T_{\phi^n_{k,\eta}}(y))\|_{M_n} = \|P_k y - T_{\phi^n_{k,\eta}}(P_k y)\|_{M_n} \leq \|y\|_2 \varepsilon \tag{5.1.12}$$

Using Lemma 5.1.1, Corollary 5.1.7, and the boundedness of Riesz transforms

$$\begin{aligned}
&\|A_n^{-\alpha-\beta} A_n^{1/2}(1 - P_k)(y - T_{\phi^n_{k,\eta}}(y))\|_\infty \\
&\qquad \leq c_\alpha \|A_n^{-\beta} A_n^{1/2}(1 - P_k)(y - T_{\phi^n_{k,\eta}}(y))\|_p \\
&\qquad \leq c_\alpha C_p k^{-2\beta} \|A_n^{1/2}(y - T_{\phi^n_{k,\eta}}(y))\|_p \\
&\qquad \leq c_\alpha C_p K_p k^{-2\beta} (\|\Gamma^n(y,y)^{1/2}\|_p + \|\Gamma^n(T_{\phi^n_{k,\eta}}(y), T_{\phi^n_{k,\eta}}(y))^{1/2}\|_p),
\end{aligned}$$

where $c_\alpha = \|A_n^{-\alpha} : L_p^0 \to L_\infty\|$, $C_p k^{-2\beta}$ is the bound in Corollary 5.1.7 and K_p is the bound of the noncommutative Riesz transforms. Using Lemma 4.3.8 and choosing k large enough in the beginning, we have

$$\|(1 - P_k)(y - T_{\phi^n_{k,\eta}}(y))\|_\infty \leq (2 + 2\varepsilon + \varepsilon^2) c_\alpha C_p K_p k^{-2\beta} \|\Gamma^n(y,y)^{1/2}\|_\infty \leq \varepsilon L_n(y).$$

The proof is complete. □

For notational simplicity, we also write $M_n^{sa} = (M_n)_{sa}$ and $\mathcal{A}_\theta^{sa} = (\mathcal{A}_\theta)_{sa}$ in the following.

Proposition 5.1.19. *Let $\varepsilon > 0$ and $R \geq 0$. Then there exists N and $p_1, ..., p_r \in \mathrm{Poly}(x,y)_{sa}$ with the following properties:*

(i) $\sigma_\theta(p_j) \in \mathcal{D}_R(\mathcal{A}_\theta^{sa})$;

(ii) for any $j > N$ and any $y \in \mathcal{D}_R(M_{n_j}^{sa})$, there exists $s \in \{1, ..., r\}$ such that $\|y - \rho_{n_j}^{\theta_j}(\sigma_{\theta_j}(p_s))\|_{M_{n_j}} \leq \varepsilon$. Here (n_j) are chosen as in Lemma 5.1.17.

Proof. The argument is similar to that of Proposition 4.4.7. The case $R = 0$ is trivial. Let $\min\{R, 1\} \gg \varepsilon > 0$ and $R > 0$ be given. We choose m and $\phi^n_{k,\eta}$ as in Lemma 5.1.18 for $n > 2m$. We define

$$B = \{y \in L_2^{\Lambda_m^2}(\mathcal{A}_\theta^{sa}) : \|y\|_{\mathcal{A}_\theta} \leq R, \|\Gamma(y,y)^{1/2}\|_{\mathcal{A}_\theta} \leq 1\}.$$

Since $B \subset \ell_2([-m,m]^2)$, B is pre-compact. Therefore, there exists an ε-net $\{y_1, ..., y_r\}$ which covers B. Without loss of generality, we may choose $(y_i)_{i=1}^r$ from $\mathrm{Poly}(u_\theta, v_\theta)$. In this way we obtain noncommutative polynomials $p_1, ..., p_r \in \mathrm{Poly}(x,y)$ such that $\sigma_\theta(p_j) = y_j$ and $\sigma_\theta(p_j) \in \mathcal{D}_R(\mathcal{A}_\theta^{sa})$ for $j = 1, ..., r$.

Let $M_n^{\Lambda_m^2}$ (resp. $\mathcal{A}_\theta^{\Lambda_m^2}$) denote the elements of M_n (resp. $\mathcal{A}_\theta$) which are linear combinations of $U_j(n)V_l(n)$ (resp. $u_\theta^j v_\theta^l$) for $(j,l) \in \Lambda_m^2$. Since ρ_n^θ is injective, we introduce a locally defined map $s_{n,m}$ as follows

$$\begin{aligned} s_{n,m} : M_n^{\Lambda_m^2} &\to \mathcal{A}_\theta^{\Lambda_m^2} \\ y &\mapsto (\rho_n^\theta)^{-1}(y). \end{aligned}$$

Note that $T_{\phi_{k,\eta}^n}(y)$ is supported in Λ_m^2 for $y \in M_n$. We define $\hat{y} = s_{n,m}(T_{\phi_{k,\eta}^n}(y))$. Then

$$\rho_n^\theta(\hat{y}) = T_{\phi_{k,\eta}^n}(y). \tag{5.1.13}$$

Note that so far our argument is independent of θ and is valid for any $n > 2m$. Now we restrict our discussion to (n_j) in order to use the continuous field of compact quantum metric spaces. By Lemma 5.1.17, we have

$$\lim_{j\to\infty} \sup_{\sigma_\theta(p)\in K} |\|\rho_{n_j}^{\theta_j}(\sigma_{\theta_j}(p))\|_{M_{n_j}} - \|\sigma_\theta(p)\|_{\mathcal{A}_\theta}| = 0,$$

and

$$\lim_{j\to\infty} \sup_{\sigma_\theta(p)\in K} |\|\Gamma^{n_j}(\rho_{n_j}^{\theta_j}(\sigma_{n_j}(p)), \rho_{n_j}^{\theta_j}(\sigma_{n_j}(p)))\|_{M_{n_j}} - \|\Gamma(\sigma_\theta(p), \sigma_\theta(p))\|_{\mathcal{A}_\theta}| = 0$$

for any compact subset K of $\mathcal{A}_\theta^{sa}$. Since $\hat{y} = s_{n,m}(T_{\phi_{k,\eta}^n}(y))$ is in $\ell_2([-m,m])$, we have

$$\|\hat{y}\|_2 = \|\rho_n^\theta(\hat{y})\|_2 \le \|T_{\phi_{k,\eta}^n}(y)\|_{M_n} \le (1+\varepsilon)^2 \|y\|_{M_n}.$$

It follows that the set $\{\hat{y} : y \in \mathcal{D}_R(M_{n_j}^{sa})\}$ is pre-compact. Then we can choose $N > 2m$ large enough so that for any $j > N$ and $y \in \mathcal{D}_R(M_{n_j}^{sa})$

$$\begin{aligned} (1+\varepsilon)^{-1}\|\hat{y}\|_{\mathcal{A}_\theta} \le &\|\rho_{n_j}^{\theta_j}(\hat{y})\|_{M_{n_j}} \le (1+\varepsilon)\|\hat{y}\|_{\mathcal{A}_\theta}, \\ (1+\varepsilon)^{-1}\|\Gamma(\hat{y},\hat{y})\|_{\mathcal{A}_\theta} \le &\|\Gamma^{n_j}(\rho_{n_j}^{\theta_j}(\hat{y}), \rho_{n_j}^{\theta_j}(\hat{y}))\|_{M_{n_j}} \le (1+\varepsilon)\|\Gamma(\hat{y},\hat{y})\|_{\mathcal{A}_\theta}. \end{aligned} \tag{5.1.14}$$

Hence, we have

$$\|\hat{y}\|_{\mathcal{A}_\theta} \le (1+\varepsilon)\|\rho_{n_j}^{\theta_j}(\hat{y})\|_{M_{n_j}} = (1+\varepsilon)\|T_{\phi_{k,\eta}^{n_j}}(y)\|_{M_{n_j}} \le (1+\varepsilon)^3\|y\|_{M_{n_j}} \le (1+\varepsilon)^3 R$$

and by Lemma 4.3.8 and (5.1.13),

$$\|\Gamma(\hat{y},\hat{y})\|_{\mathcal{A}_\theta} \leq (1+\varepsilon)\|\Gamma^{n_j}(\rho^{\theta_j}_{n_j}(\hat{y}), \rho^{\theta_j}_{n_j}(\hat{y}))\|_{M_{n_j}} \leq (1+\varepsilon)^5 \|\Gamma^{n_j}(y,y)\|_{M_{n_j}}$$

for all $y \in \mathcal{D}_R(M_{n_j})$ and $j > N$. Since $\frac{1}{(1+\varepsilon)^3}\hat{y} \in B$, there exists $p_s \in \{p_1, ..., p_r\}$ such that $\|\sigma_\theta(p_s) - \frac{1}{(1+\varepsilon)^3}\hat{y}\| \leq \varepsilon$. By (5.1.13) and (5.1.14), we have for $j > N$

$$\Big\|\rho^{\theta_j}_{n_j}(\sigma_{\theta_j}(p_s)) - \frac{T_{\phi^{n_j}_{k,\eta}}(y)}{(1+\varepsilon)^3}\Big\|_{M_n} \leq (1+\varepsilon)\varepsilon,$$

because $\sigma_{\theta_j}(p_s) - \frac{\hat{y}}{(1+\varepsilon)^3} \in \mathcal{D}_R(M^{sa}_{n_j})$.

Finally, for any $y \in \mathcal{D}_R(M_{n_j})$ and $j > N$, we have

$$\begin{aligned}
&\|T_{\phi^{n_j}_{k,\eta}}(y) - \rho^{\theta_j}_{n_j}(\sigma_{\theta_j}(p_s))\|_{M_{n_j}} \\
\leq &\Big\|T_{\phi^{n_j}_{k,\eta}}(y) - \frac{1}{(1+\varepsilon)^3}T_{\phi^{n_j}_{k,\eta}}(y)\Big\|_{M_{n_j}} + \Big\|\frac{1}{(1+\varepsilon)^3}T_{\phi^{n_j}_{k,\eta}}(y) - \rho^{\theta_j}_{n_j}(\sigma_{\theta_j}(p_s))\Big\|_{M_{n_j}} \leq (4R+2)\varepsilon.
\end{aligned}$$

By Lemma 5.1.18, we have

$$\|y - \rho^{\theta_j}_{n_j}(\sigma_{\theta_j}(p_s))\|_{M_{n_j}} \leq \|y - T_{\phi^{n_j}_{k,\eta}}(y)\|_{M_{n_j}} + \|T_{\phi^{n_j}_{k,\eta}}(y) - \rho^{\theta_j}_{n_j}(\sigma_{\theta_j}(p_s))\|_{M_{n_j}} \leq (5R+3)\varepsilon.$$

Replacing ε by $\frac{\varepsilon}{5R+3}$ in the beginning completes the proof. □

Theorem 5.1.20. *Let $\theta \in [0,1)$ and (n_j) be given in Lemma 5.1.17. Then $((M_{n_j})_{sa}, L_{n_j})$ converges to $(\mathcal{A}_\theta, L)$ in the quantum Gromov–Hausdorff distance.*

Proof. In Lemma 5.1.16 we proved that $(\{(M_{n_j})_{sa}, L_{n_j}\}_{n\in\overline{\mathbb{N}}}, S)$ is a continuous field of compact quantum metric spaces in the sense of [Li06]. Let $\varepsilon = 1/m$ and $R \geq 0$. By Proposition 5.1.19, we can find $N \in \mathbb{N}$ and

$$y^m_1 = \sigma_\theta(p^m_1), ..., y^m_{r_m} = \sigma_\theta(p^m_{r_m}) \in \mathcal{D}_R(\mathcal{A}^{sa}_\theta),$$

where $(p^m_{r_s})^m_{s=1} \subset \mathrm{Poly}(x,y)$, so that for any $z \in \mathcal{D}_R(M^{sa}_{n_j})$, $j > N$, there exists a $p^m_{r_s} \in \{p^m_{r_1}, ..., p^m_{r_m}\}$ with

$$\|z - \rho^{\theta_j}_{n_j}(\sigma_{\theta_j}(p^m_{r_s}))\|_\infty \leq \varepsilon.$$

The set

$$\Lambda := \cup^\infty_{m=1}\{y^m_1, ..., y^m_{r_m}\} = \sigma_\theta(\cup^\infty_{m=1}\{p^m_1, ..., p^m_{r_m}\})$$

is dense in $\mathcal{D}_R(\mathcal{A}_\theta)$. Give an ordering as in Theorem 4.4.8. By our construction, for any $\varepsilon > 0$, there exist m and r such that the open ε-balls in M_{n_j} centered at

$$\rho_{n_j}^{\theta_j}(\sigma_{\theta_j}(p_1)), ..., \rho_{n_j}^{\theta_j}(\sigma_{\theta_j}(p_r))$$

cover $\mathcal{D}_R(M_{n_j})$ for all $n_j > m$, where $\sigma_\theta(p_i)_{i=1}^m \in \Lambda$ for all i. In other words, Λ satisfies condition (iii) in [Li06]*Theorem 7.1. Hence M_n converges to $\mathcal{A}_\theta$ in the order-unit quantum Gromov–Hausdorff distance by the same theorem. The assertion follows from [Li06]*Theorem 1.1. ☐

So far we have dealt with the heat semigroup on $\mathcal{A}_\theta$. The following indicates that the approximation can also be done using the Poisson semigroup.

Lemma 5.1.21. *Let B_n denote the discrete Poisson semigroup and A_n denote the discrete heat semigroup on M_n. Then we have*

$$\|A_n^{\beta/2}x\|_p \sim \|B_n^\beta x\|_p,$$

for $1 < p < \infty$.

Proof. Note that for fixed j, k such that $|j|, |k| \leq \frac{n}{2}$, $j^2 + k^2 = (|j|_n + |k|_n)^2$, where $|\cdot|_n$ is as defined in Section 4.2. Let p_0 be such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{2}$ for $0 < \theta < 1$ and $\beta = \theta\alpha$. Now since the maps

$$B_n^{\alpha \mathrm{i}t} A_n^{-2\alpha \mathrm{i}t} : L_{p_0} \to L_{p_0},$$

$$B_n^{\alpha(1+\mathrm{i}t)} A_n^{-2\alpha(1+\mathrm{i}t)} : L_2 \to L_2$$

are bounded, the assertion follows from Stein's interpolation theorem in the same way as the proof of Proposition 5.1.5. ☐

Remark 5.1.22. Lemma 5.1.1 has a variant for the Poisson semigroup. Together with Lemma 5.1.21, one can prove Proposition 5.1.19 for the Poisson semigroup, which in turn yields the approximation result. In fact, one can even prove the convergence in quantum Gromov–Hausdorff distance using some exotic semigroups. For example, one may consider the semigroup defined by $T_t(u^j v^k) = e^{-t(|j|+|k|^2)} u^j v^k$.

5.2 Completely Bounded Quantum Gromov–Hausdorff Convergence

In this section we introduce the notion of completely bounded quantum Gromov–Hausdorff distance. The final goal is to show that the continuous fields of compact quantum metric spaces which we presented earlier in this paper, converge in this sense.

Definition 5.2.1. Let X be an operator space. We say (X, L) is a *Lip operator space structure*, if

1. $L \subset X$ is a dense subspace;
2. there exists a subspace $N \subset L$ such that L/N carries an additional operator space structure, which will also be referred to as Lip structure.

In particular, on the first matrix level the Lip structure induces a semi-norm on L. The matrix semi-norms on L will be denoted by $|||x|||_{M_n(L)}$ or simply $|||x|||$ if it is clear that $x \in M_n(X)$. We also use the notation $L(x) := |||x|||$, especially when we consider a continuous field of quantum metric spaces.

We define the completely bounded quantum Gromov–Hausdorff distance of two operator spaces as follows.

Definition 5.2.2. Let X and Y be two operator spaces. Let $R > 0$ and

$$\mathcal{D}_R(M_n(X)) = \{x \in M_n(X) : |||x|||_{M_n(L)} \leq 1, \|x\|_{M_n(X)} \leq R\}.$$

We denote the *R-cb-quantum Gromov–Hausdorff distance* of X and Y by $d^{cb}_{oq,R}(X, Y)$, and define it by

$$d^{cb}_{oq,R}(X,Y) = \inf \sup_{n \in \mathbb{N}} \{d_H[\mathrm{id} \otimes \iota_X(\mathcal{D}_R(M_n(X))), \mathrm{id} \otimes \iota_Y(\mathcal{D}_R(M_n(Y)))]\},$$

where d_H denotes the Hausdorff distance, and the infimum runs over all operator spaces V and completely isometric embeddings $\iota_X : X \to V$ and $\iota_Y : Y \to V$. If in addition X and Y are unital with units e_X and e_Y, respectively, we modify the definition as follows:

$$d^{cb}_{oq,R}(X,Y) = \inf \sup_{n \in \mathbb{N}} \{\max\{d_H[\mathrm{id} \otimes \iota_X(\mathcal{D}_R(M_n(X))), \mathrm{id} \otimes \iota_Y(\mathcal{D}_R(M_n(Y)))], \|\iota_X(Re_X) - \iota_Y(Re_Y)\|\}\}.$$

Remark 5.2.3. The definition above seems stronger than the one introduced in [Wu06].

Remark 5.2.4. Let $\mathcal{K}_1$ denote the unitalization of $\mathcal{K}$, the space of compact operators on a Hilbert space. For two operator spaces X and Y, we are particularly interested in $d^{cb}_{oq,R}(\mathcal{K}_1 \otimes X, \mathcal{K}_1 \otimes Y)$.

Now we prove the triangle inequality. The proof follows the same idea as that of Lemma 4.5 in [Li06].

Lemma 5.2.5. *Let $\iota_j : A \to B_j$ be linear completely isometric embeddings of operator spaces for $j \in \{1, 2\}$. Then there is an operator space C and linear completely isometric embeddings $\psi_j : B_j \to C$ such that $\psi_1 \circ \iota_1 = \psi_2 \circ \iota_2$.*

Proof. Let ψ_j be as defined in the proof of Lemma 4.5 in [Li06]. The same argument extends easily to the matrix levels. Then $\psi_1 \circ \iota_1 = \psi_2 \circ \iota_2$ and ψ_1, ψ_2 are complete isometries. $\square$

Lemma 5.2.6. *Let X, Y and Z be operator spaces. Then the following holds*

$$d^{cb}_{oq,R}(X, Z) \leq d^{cb}_{oq,R}(X, Y) + d^{cb}_{oq,R}(Y, Z).$$

Proof. The triangle inequality follows immediately from applying Lemma 5.2.5 with $A = Y$. $\square$

Let $k \geq 0$. Recall the notation Λ^2_k in (5.1.4). Let $x \in M_m(\mathcal{A}_\theta^{\Lambda^2_k})$ and δ be the derivation of $\mathcal{A}_\theta$ into a Hilbert C*-module $\mathcal{H}_{\mathcal{A}_\theta} := H_\psi \otimes \mathcal{A}_\theta$ as defined in (4.3.5) (for the case $m = 1$). We define the matrix Lip-norm as follows

$$|||x||| = \max\{\|(\mathrm{id} \otimes \delta)(x)\|_{M_m(\mathcal{H}_{\mathcal{A}_\theta})}, \|(\mathrm{id} \otimes \delta)(x^*)\|_{M_m(\mathcal{H}_{\mathcal{A}_\theta})}\}.$$

This is exactly the definition (4.3.7) in the two dimensional case restricted to rotation C*-algebras. Note that if x is self-adjoint, the matrix Lip-norm $|||x|||$ introduced here is just the matrix extension of the Lip-norm used in Proposition 5.1.16 for $d = 2$. We may write $L_\theta(x)$ or $L_\infty(x)$ for $|||x|||$ when considering continuous fields of quantum metric spaces.

Lemma 5.2.7. *Let X and Y be two operator spaces. Let $\varepsilon \geq 0$ and $\varphi : X \to Y$ be a $1 + \varepsilon$ cb-isometry and a $1 + \varepsilon$ Lip-isometry, i.e. for any m and any $\hat{x} \in M_m(X)$, we have*

$$(1 - \varepsilon)\|\hat{x}\|_{M_m(X)} \leq \|(\mathrm{id} \otimes \varphi)(\hat{x})\|_{M_m(Y)} \leq (1 + \varepsilon)\|\hat{x}\|_{M_m(X)},$$

and

$$(1 - \varepsilon)|||\hat{x}||| \leq |||(\mathrm{id} \otimes \varphi)(\hat{x})||| \leq (1 + \varepsilon)|||\hat{x}|||.$$

Then we have

$$d^{cb}_{oq,R}(X, \varphi(X)) \leq 2R\varepsilon.$$

Proof. Let $N = \{(a, -\varphi(a), \varepsilon a) : a \in X\}$. Then $N \subset X \oplus_1 Y \oplus_1 X$. Here $X \oplus_1 Y \oplus_1 X$ is the ℓ_1-sum of X, Y and X in the sense of operator spaces. Let

$$V = \{(x, \varphi(x'), 0) + N : x, x' \in X\} \subset (X \oplus_1 Y \oplus_1 X)/N.$$

Then

$$\|(x, y, 0) + N\|_{(X \oplus_1 Y \oplus_1 X)/N} = \inf\{\|x - a\| + \|y + \varphi(a)\| + \varepsilon\|a\| : a \in X\}.$$

Thus X and Y embed isometrically into V (see Lemma 7.2 in [Li06]). We claim that the embeddings are actually completely isometric. Indeed, since $S_1((X \oplus_1 \varphi(X) \oplus_1 X)/N) = S_1(X) \oplus_1 S_1(\varphi(X)) \oplus_1 S_1(X)/S_1(N)$, we have

$$S_1(V) \subset S_1(X) \oplus_1 S_1(\varphi(X)) \oplus_1 S_1(X)/S_1(N).$$

Hence for $\hat{x} \in S_1(X)$, we have

$$\begin{aligned} \|(\hat{x}, 0, 0) + S_1(N)\|_{S_1(V)} =& \|\hat{x}\|_{S_1(X)}, \\ \|(0, (\mathrm{id} \otimes \varphi)\hat{x}, 0) + S_1(N)\|_{S_1(V)} =& \|(\mathrm{id} \otimes \varphi)\hat{x}\|_{S_1(Y)}. \end{aligned}$$

Note that by a result of Pisier (see [Pis98]*Lemma 1.7), if $u : X \to Y$ is a completely bounded map, for every $1 \le p \le \infty$, we have

$$\|u\|_{\mathrm{cb}} = \sup_m \|\,\mathrm{id} \otimes u : S_p^m(X) \to S_p^m(Y)\|.$$

Therefore, by applying the above for $p = 1$ and $p = \infty$, we find that

$$\iota_1 : X \to (X, 0, 0) + N \subset V \quad \text{and} \quad \iota_2 : \varphi(X) \to (0, \varphi(X), 0) + N \subset V$$

are completely isometric embeddings. Note that the maps

$$\begin{aligned} \iota : X \to X \oplus_1 Y \oplus_1 X, &\quad x \mapsto (0, 0, x), \\ q : X \oplus_1 Y \oplus_1 X \to (X \oplus_1 Y \oplus_1 X)/N, &\quad (x, y, z) \mapsto (x, y, z) + N \end{aligned}$$

are completely contractive. For any $\hat{x} \in M_m(X)$, we have

$$\begin{aligned} &\|(\mathrm{id} \otimes \iota_1)\hat{x} - \mathrm{id} \otimes (\iota_2 \circ \varphi)\hat{x}\|_{M_m(V)} \le \|(\hat{x}, 0, 0) - (0, \mathrm{id} \otimes \varphi(\hat{x}), 0) + N\|_{M_m(V)} \qquad (5.2.1) \\ &= \|(\hat{x}, -\mathrm{id} \otimes \varphi(\hat{x}), 0) - (\hat{x}, -\mathrm{id} \otimes \varphi(\hat{x}), \varepsilon\hat{x}) + N\|_{M_m(V)} \le \varepsilon\|\hat{x}\|_{M_m(X)}. \end{aligned}$$

Now let $\hat{x} \in \mathcal{D}_R(M_m(X))$, i.e. $\hat{x} \in M_m(X)$, $\|\hat{x}\|_{M_m(X)} \leq R$ and $|||\hat{x}||| \leq 1$. Then by the assumption, we have $\|\frac{1}{1+\varepsilon}(\mathrm{id} \otimes \varphi)\hat{x}\|_{M_m(Y)} \leq R$ and $|||\frac{1}{1+\varepsilon}(\mathrm{id} \otimes \varphi)\hat{x}||| \leq 1$. This means $\frac{1}{1+\varepsilon}(\mathrm{id} \otimes \varphi)\hat{x} \in \mathcal{D}_R(M_m(Y))$. Using (5.2.1), we find

$$d^{cb}_{oq,R}(X, \varphi(X)) \leq \varepsilon R + \frac{\varepsilon R}{1+\varepsilon}\|\varphi\|_{\mathrm{cb}} \leq 2R\varepsilon.$$

5.2.1 CB-continuous fields of compact quantum metric spaces

In this section we investigate an operator space version of continuous fields of compact quantum metric spaces, and show that the continuous fields of compact quantum metric spaces which we introduced earlier form cb-continuous fields of compact quantum metric spaces with appropriate operator space Lip-norms defined on them.

Definition 5.2.8. Let T be a locally compact Hausdorff space and let $(\{\mathcal{A}_t, L_t\}_{t\in T}, S_0)$ be a continuous field of order-unit spaces in the sense of [Li06], where S_0 is a dense subset of S, the space of continuous sections, containing the unit. We say $(\{\mathcal{A}_t, L_t\}, S_0)$ is a *cb-continuous field of order-unit spaces* if for any finite subset $\Delta \subset S_0$, s_0 and $\varepsilon > 0$, there exists a neighborhood $\mathcal{U}(s_0) > 0$, such that for any $s \in \mathcal{U}(s_0)$, $m \geq 1$, $f \in \Delta$ and matrix coefficients $a_f \in M_m$, we have the following

$$\begin{aligned}\frac{1}{1+\varepsilon}\|\sum_{f\in\Delta} a_f \otimes f(s)\|_{M_m(\mathcal{A}_s)} &\leq \|\sum_{f\in\Delta} a_f \otimes f(s_0)\|_{M_m(\mathcal{A}_{s_0})} \\ &\leq (1+\varepsilon)\|\sum_{f\in\Delta} a_f \otimes f(s)\|_{M_m(\mathcal{A}_s)}.\end{aligned}$$

We call $(\{\mathcal{A}_t, L_t\}_{t\in T}, S_0)$ a *cb-continuous fields of compact quantum metric spaces* if $(\{\mathcal{A}_t, L_t\}_t, S_0)$ is a continuous field of compact quantum metric spaces and in addition, we have

$$\begin{aligned}\frac{1}{1+\varepsilon}|||\sum_{f\in\Delta} a_f \otimes f(s)|||_{M_m(\mathcal{A}_s)} &\leq |||\sum_{f\in\Delta} a_f \otimes f(s_0)|||_{M_m(\mathcal{A}_{s_0})} \\ &\leq (1+\varepsilon)|||\sum_{f\in\Delta} a_f \otimes f(s)|||_{M_m(\mathcal{A}_s)}.\end{aligned}$$

Recall the map $\rho_n : C(\mathbb{T}^2) \to M_n$ as defined in (5.1.7): $\rho_n(u^j v^k) = u_j(n)v_k(n)$, where u, v are the generators of $C(\mathbb{T}^2)$ and $u_j(n)v_k(n)$ are defined in Section 5.1. Let $C^{\Lambda^2_k}(\mathbb{T}^2)$ (resp. $M_n^{\Lambda^2_k}$) denote the elements in $C(\mathbb{T}^2)$ spanned by $u^j v^l$ (resp. $u_j(n)v_l(n)$) for $(j,l) \in \Lambda^2_k$. Note that $C^{\Lambda^2_k}(\mathbb{T}^2)$ and $M_n^{\Lambda^2_k}$ are operator spaces.

Proposition 5.2.9. *For any $\varepsilon > 0$ and $k \geq 0$, there exists $N > 0$ such that for any $n > N$, the map $\rho_n|_{C^{\Lambda^2_k}(\mathbb{T}^2)} : C^{\Lambda^2_k}(\mathbb{T}^2) \to M_n^{\Lambda^2_k}$ is a $(1+\varepsilon)$ cb-isometry.*

Proof. By Lemma 5.1.8, the map $(\rho_n)^\bullet : C(\mathbb{T}^2) \to \prod_\omega M_n$ is a faithful *-homomorphism, and in particular a complete isometry. Therefore, for any $\varepsilon > 0$ and $k \geq 0$, there exists $N > 0$, such that for $n > N$, $C^{\Lambda_k^2}(\mathbb{T}^2)$ is $(1+\varepsilon)$-isometric to $M_n^{\Lambda_k^2}$ via ρ_n; i.e. we have for scalar coefficients $a_{j,l}$

$$(1-\varepsilon)\Big\| \sum_{|j|,|l|\leq k} a_{j,l} u^j v^l \Big\|_{C(\mathbb{T}^2)} \leq \Big\| \sum_{|j|,|l|\leq k} a_{j,l} u_j(n) v_l(n) \Big\|_{M_n} \leq (1+\varepsilon) \Big\| \sum_{|j|,|l|\leq k} a_{j,l} u^j v^l \Big\|_{C(\mathbb{T}^2)}.$$

Note that if X is an operator space, and B_{X^*} denotes the unit ball of X^*, then the *min-structure* on X is given by the image of the map $\iota_{\min} : X \to C(B_{X^*})$ defined by $\iota_{\min}(x) = f_x$, where $f_x(x^*) = x^*(x)$, for all $x \in X$ and $x^* \in X^*$. Let $\min(M_n)$ denote the min-structure on M_n. Since $\min(M_n)$ is commutative, we have

$$\|\rho_n : C(\mathbb{T}^2) \to \min(M_n)\|_{\mathrm{cb}} = \|\rho_n : C(\mathbb{T}^2) \to \min(M_n)\|.$$

It follows that for any m and $a_{j,l} \in M_m$

$$(1-\varepsilon)\Big\| \sum_{|j|,|l|\leq k} a_{j,l} \otimes u^j v^l \Big\|_{M_m(C(\mathbb{T}^2))} \leq \Big\| \sum_{|j|,|l|\leq k} a_{j,l} \otimes u_j(n) v_l(n) \Big\|_{M_m(\min(M_n))}. \tag{5.2.2}$$

Recalling that $\min(M_n)$ is the smallest operator space norm on M_n, we deduce from (5.2.2) that for any m

$$\begin{aligned}\Big\| \sum_{|j|,|l|\leq k} a_{j,l} \otimes u^j v^l \Big\|_{M_m(C(\mathbb{T}^2))} &\leq \frac{1}{1-\varepsilon} \Big\| \sum_{|j|,|l|\leq k} a_{j,l} \otimes u_j(n) v_l(n) \Big\|_{M_m(\min(M_n))} \\ &\leq \frac{1}{1-\varepsilon} \Big\| \sum_{|j|,|l|\leq k} a_{j,l} \otimes u_j(n) v_l(n) \Big\|_{M_m(M_n)}.\end{aligned} \tag{5.2.3}$$

Now by a result of Haagerup–Rørdam [HR95], there exists a Hilbert space $\mathcal{H}$ and $u(\theta), v(\theta) \in \mathcal{B}(\mathcal{H})$ such that the following hold

1. For any θ, $\mathrm{C}^*(u(\theta), v(\theta)) \simeq \mathcal{A}_\theta$,

2. There is a constant $c > 0$, such that for any θ', $\max\{\|u(\theta) - u(\theta')\|, \|v(\theta) - v(\theta')\|\} \leq c|\theta - \theta'|^{1/2}$.

This implies that there exists $\delta > 0$ such that for $|\theta - \theta'| < \delta$ and $|j| \leq k, |l| \leq k$, we have

$$\sup_{j,l} \|u^j(\theta) v^l(\theta) - u^j(\theta') v^l(\theta')\| \leq 2ck|\theta - \theta'|^{1/2}.$$

Let d_{cb} denote the Banach-Mazur distance of two operator spaces. Then there exists $\delta = \delta(\varepsilon, k) > 0$ such that $d_{\mathrm{cb}}(\mathcal{A}_\theta^{\Lambda_k^2}, \mathcal{A}_{\theta'}^{\Lambda_k^2}) < 1 + \varepsilon$ for any $|\theta - \theta'| < \delta$; see [Pis03]*Section 2.13. We may find a complete bounded map ϕ sending $u(\theta)v(\theta)$ to $u(\theta')v(\theta')$ such that

$$\|\phi : \mathcal{A}_\theta^{\Lambda_k^2} \to \mathcal{A}_{\theta'}^{\Lambda_k^2}\|_{\mathrm{cb}} \le 1+\varepsilon.$$

It follows that for all matrix coefficients $a_{j,l}$ we get

$$\left| \Big\| \sum_{|j|,|l|\le k} a_{j,l} \otimes u^j(\theta)v^l(\theta) \Big\| - \Big\| \sum_{|j|,|l|\le k} a_{j,l} \otimes u^j(\theta')v^l(\theta') \Big\| \right| \le \varepsilon \Big\| \sum_{|j|,|l|\le k} a_{j,l} \otimes u^j(\theta)v^l(\theta) \Big\|.$$

Setting $\theta' = \frac{1}{n} < \delta$ and $\theta = 0$, we have for any m

$$\| \sum_{|j|,|l|\le k} a_{j,l} \otimes u^j(1/n)v^l(1/n) \|_{M_m(\mathcal{A}_{1/n})} \le (1+\varepsilon) \| \sum_{|j|,|l|\le k} a_{j,l} \otimes u^j v^l \|_{M_m(C(\mathbb{T}^2))}. \tag{5.2.4}$$

But $u_1(n)$ and $v_1(n)$ verify the commutation relation of $\mathcal{A}_{1/n}$. By universality of $\mathcal{A}_{1/n}$ we have for any m

$$\| \sum_{|j|,|l|\le k} a_{j,l} \otimes u_j(n)v_l(n) \|_{M_m(M_n)} \le \| \sum_{|j|,|l|\le k} a_{j,l} \otimes u^j(1/n)v^l(1/n) \|_{M_m(\mathcal{A}_{1/n})}. \tag{5.2.5}$$

By combining the estimates (5.2.3), (5.2.4) and (5.2.5), we complete the proof. □

Proposition 5.2.10. *For any $\varepsilon > 0$ and $k \ge 0$, there exists $N > 0$ and a family of maps $\rho_n^\theta : \mathcal{A}_\theta^{\Lambda_k^2} \to M_n^{\Lambda_k^2}$ such that for $n > N$, ρ_n^θ is a $1+\varepsilon$ cb-isometry and a $1+\varepsilon$ Lip-isometry.*

Proof. Note that if we know ρ_n^θ is a $1+\varepsilon$ cb-isometry then by the same argument as for Lemma 4.3.8 it is also a $1+\varepsilon$ Lip-isometry on $\mathcal{A}_\theta^{\Lambda_k^2}$. Therefore it suffices to show that ρ_n^θ is a $1+\varepsilon$ cb-isometry. If $\theta = 0$, then the result follows immediately from Proposition 5.2.9. Let $\theta = \frac{p}{q}$ be rational. Recall from Lemma 5.1.10 that we have a surjective map $\rho_{n_l}^\theta : \mathcal{A}_\theta^{\Lambda_k^2} \to M_{n_l}^{\Lambda_k^2}$ for suitable n_l. We show that this map is a $1+\varepsilon$ cb-isometry. As we observed in Section 5.1.2, there is a trace preserving *-homomorphisms $\sigma : \mathcal{A}_\theta \to M_q \otimes_{\min} C(\mathbb{T}^2)$. By Proposition 5.2.9, there exists $N > 0$ such that the map $\rho_m : C^{\Lambda_k^2}(\mathbb{T}^2) \to M_m^{\Lambda_k^2}$ is a $1+\varepsilon$ cb-isometry for $m > N$. Hence, so is the map $\mathrm{id} \otimes \rho_{n_l} : M_q \otimes_{\min} C^{\Lambda_k^2}(\mathbb{T}^2) \to M_q \otimes_{\min} M_{n_l}^{\Lambda_k^2}$. Note that the specific choice of the subsequence $n_l = q^{l+1}$ guarantees that $\rho_{n_l}^\theta(\mathcal{A}_\theta) = M_{n_l}$. Therefore, the restriction of $\rho_{n_l}^\theta = (\mathrm{id} \otimes \rho_{n_l}) \circ \sigma$ to $\mathcal{A}_\theta^{\Lambda_k^2}$ is also a $1+\varepsilon$ cb-isometry. This gives the following diagram

$$\begin{tikzcd}
M_q \otimes_{\min} C^{\Lambda_k^2}(\mathbb{T}^2) \arrow[r, "\mathrm{id} \otimes \rho_{n_l}"] & M_q \otimes_{\min} M_{n_l}^{\Lambda_k^2} \\
\mathcal{A}_\theta^{\Lambda_k^2} \arrow[u, "\sigma"] \arrow[r, "\rho_{n_l}^\theta"] & M_{n_l}^{\Lambda_k^2} \arrow[u, hook]
\end{tikzcd}$$

which proves the rational case. Finally, let θ be irrational. Then there exists a sequence $\theta_s = \frac{p_s}{q_s}$ of rational numbers converging to θ. We may assume $\theta_s - \theta$ is small enough so that we may apply the result of Haagerup–Rørdam [HR95] to get a $1 + \frac{\varepsilon}{3}$ cb-isometry $\phi_s : \mathcal{A}_\theta^{\Lambda_k^2} \to \mathcal{A}_{\theta_s}^{\Lambda_k^2}$ in the same way as in the proof of Proposition 5.2.9. Then by what we proved above, we may choose n_s large enough such that the map $\rho_{n_s}^{\theta_s} : \mathcal{A}_{\theta_s}^{\Lambda_k^2} \to M_{n_s}^{\Lambda_k^2}$ is a $1 + \frac{\varepsilon}{3}$ cb-isometry. Let $\rho_{n_s}^{\theta} = \rho_{n_s}^{\theta_s} \circ \phi_s$. Then $\rho_{n_s}^{\theta} : \mathcal{A}_\theta^{\Lambda_k^2} \to M_{n_s}^{\Lambda_k^2}$ is a $1 + \varepsilon$ cb-isometry. We can illustrate the argument using the following diagram

$$
\begin{tikzcd}
\mathcal{A}_\theta^{\Lambda_k^2} \arrow[d, "\phi_s"] \arrow[dr, dotted, "\rho_{n_s}^{\theta}"] & \\
\mathcal{A}_{\theta_s}^{\Lambda_k^2} \arrow[r, "\rho_{n_s}^{\theta_s}"'] \arrow[d, "\sigma"] & M_{n_s}^{\Lambda_k^2} \arrow[d, hook] \\
M_{q_s} \otimes C^{\Lambda_k^2}(\mathbb{T}^2) \arrow[r, "\mathrm{id} \otimes \rho_{n_s}"] & M_{q_s} \otimes M_{n_s}^{\Lambda_k^2}.
\end{tikzcd}
$$

□

Let $(\{\mathcal{A}_n, L_n\}_{n \in \overline{\mathbb{N}}}, S)$ denote either of the two continuous fields of compact quantum metric spaces which were introduced in Sections 4.4 and 5.1. The following result follows immediately from Proposition 5.2.10.

Proposition 5.2.11. *$(\{\mathcal{A}_n, L_n\}_{n \in \overline{\mathbb{N}}}, S)$ is a cb-continuous field of compact quantum metric spaces.*

5.2.2 Approximations for $C(\mathbb{T})$ and $\mathcal{A}_\theta$

Here we only present a formal proof of the approximation for $\mathcal{A}_\theta$. The argument modifies easily to the case of $C(\mathbb{T})$. Before we prove the main result, we show the following estimate.

Theorem 5.2.12. *Let $\varepsilon > 0$. Then there exist $k = k(\varepsilon)$, $m = m(k)$ and multipliers $\phi_{k,\eta}^n$, $\eta \in (0, \frac{\varepsilon}{4(2k+1)^2})$ on M_n for $n > 2m$ (including $n = \infty$) such that*

$$\|T_{\phi_{k,\eta}^n} - \mathrm{id} : (M_n, |||\cdot|||) \to (M_n, \|\cdot\|)\|_{\mathrm{cb}} \le \varepsilon.$$

Here $T_{\phi_{k,\eta}^n}$ is induced by $\tilde{T}_{\phi_{k,\eta}^n}$ as defined in (5.1.3).

Proof. We follow the proof of Lemma 5.1.18, but we have to get rid of the L_2 norm this time. Let k be a large number which will be determined later. Fix α, β such that $\alpha + \beta = \frac{1}{2}$. Similar to Lemma 5.1.18, we may choose multipliers $\phi^n_{k,\eta}$, $\eta \in (0, \frac{\varepsilon}{D(2k+1)^2})$ for some D to be determined later, such that

$$|\phi^n_{k,\eta}(j,l) - 1| \leq \frac{\varepsilon}{D(2k+1)^2}, \quad (j,l) \in [-k,k]^2. \tag{5.2.6}$$

Note that

$$\begin{aligned} T_{\phi^n_{k,\eta}} - \mathrm{id} &= A^{-\alpha}A^{-\beta}(T_{\phi^n_{k,\eta}} - \mathrm{id})A^{1/2} \\ &= A^{-\alpha}A^{-\beta}(T_{\phi^n_{k,\eta}} - \mathrm{id})P_k A^{1/2} + A^{-\alpha}A^{-\beta}(T_{\phi^n_{k,\eta}} - \mathrm{id})(\mathrm{id} - P_k)A^{1/2}. \end{aligned}$$

By Proposition 5.1.3, we know $\|A^{1/2} : (M_n, |||\cdot|||) \to L^0_p(M_n)\|_{\mathrm{cb}} = K_p < \infty$. Using (5.2.6) and Lemma 5.1.4, we may extend (5.1.12) to matrix levels as in Lemma 4.3.10 (but with $q = p \geq 2$ here) and obtain

$$\|(T_{\phi^n_{k,\eta}} - \mathrm{id})P_k : L^0_p(M_n) \to L^0_p(M_n)\|_{\mathrm{cb}} \leq \frac{2\varepsilon}{D}.$$

By (4.3.11), we know $\|A^{-\beta} : L^0_p(M_n) \to L^0_p(M_n)\|_{\mathrm{cb}} = c'_\beta < \infty$. And by Lemma 5.1.1, $\|A^{-\alpha} : L^0_p(M_n) \to L^0_\infty(M_n)\|_{\mathrm{cb}} = c_\alpha < \infty$. Therefore, we find

$$\|A^{-\alpha}A^{-\beta}(T_{\phi^n_{k,\eta}} - \mathrm{id})P_k A^{1/2} : (M_n, |||\cdot|||) \to (M_n, \|\cdot\|)\|_{\mathrm{cb}} \leq \frac{2c_\alpha c'_\beta K_p \varepsilon}{D} \leq \frac{\varepsilon}{2}$$

by choosing D large enough. By Corollary 5.1.7, we have $\|A^{-\beta}(1 - P_k) : L_p(M_n) \to L_p(M_n)\|_{\mathrm{cb}} = C_p k^{-2\beta}$. It follows that

$$\begin{aligned} &\|T_{\phi^n_{k,\eta}} - \mathrm{id} : (M_n, |||\cdot|||) \to (M_n, \|\cdot\|)\|_{\mathrm{cb}} \\ \leq\ &\|(T_{\phi^n_{k,\eta}} - \mathrm{id})P_k : (M_n, |||\cdot|||) \to (M_n, \|\cdot\|)\|_{\mathrm{cb}} \\ &\quad + \|(T_{\phi^n_{k,\eta}} - \mathrm{id})A^{-\alpha}A^{-\beta}(1 - P_k)A^{1/2} : (M_n, |||\cdot|||) \to (M_n, \|\cdot\|)\|_{\mathrm{cb}} \\ \leq\ &\frac{\varepsilon}{2} + c_\alpha C_p k^{-2\beta} K_p \|(T_{\phi^n_{k,\eta}} - \mathrm{id}) : L_\infty(M_n) \to L_\infty(M_n)\|_{\mathrm{cb}}. \end{aligned}$$

But the cb-norm of $T_{\phi^n_{k,\eta}} - \mathrm{id} : L_\infty(M_n) \to L_\infty(M_n)$ is less than $2 + \eta$, by the construction of $\phi^n_{k,\eta}$. The assertion follows by choosing k large enough. $\square$

Theorem 5.2.13. *There exists a sequence $n_j \to \infty$ such that $(\mathcal{A}_{n_j}, L_{n_j})$ converges to $(\mathcal{A}_\infty, L_\infty)$ in the R-cb-quantum Gromov–Hausdorff distance.*

Proof. Let $0 < \varepsilon < 1, R > 0$. In this proof we simply write n for n_j. We choose m and $\phi^n_{k,\eta}$ as in Lemma

5.2.12. By Lemma 5.2.6, we have

$$d_{oq,R}^{cb}(\mathcal{A}_\infty,\mathcal{A}_n)\le d_{oq,R}^{cb}(\mathcal{A}_\infty,\mathcal{A}_\infty^{\Lambda_m^2})+d_{oq,R}^{cb}(\mathcal{A}_\infty^{\Lambda_m^2},\mathcal{A}_n^{\Lambda_m^2})+d_{oq,R}^{cb}(\mathcal{A}_n^{\Lambda_m^2},\mathcal{A}_n). \tag{5.2.7}$$

By Proposition 5.2.10, we may choose n large enough such that the map $\rho_n^\theta:\mathcal{A}_\theta^{\Lambda_m^2}\to M_n^{\Lambda_m^2}$ defined by $u_j^n v_l^n\mapsto u_j(n)v_l(n)$ is a $1+\varepsilon$ cb-isometry and $1+\varepsilon$ Lip-isometry. Hence by Lemma 5.2.7,

$$d_{oq,R}^{cb}(\mathcal{A}_\infty^{\Lambda_m^2},\mathcal{A}_n^{\Lambda_m^2})\le 2R\varepsilon.$$

By Lemma 5.1.18, we have $\|T_{\phi_{k,\eta}^n}\|_{\mathrm{cb}}\le(1+\varepsilon)^2$. Together with Lemma 4.3.8, we deduce that $\frac{1}{(1+\eta)^2}(\mathrm{id}\otimes T_{\phi_{k,\eta}^n})x\in\mathcal{D}_R(M_p(\mathcal{A}_n^{\Lambda_m^2}))$ for all $x\in\mathcal{D}_R(M_p(\mathcal{A}_n))$ and n large enough (including $n=\infty$). By Theorem 5.2.12, we have $\|x-(\mathrm{id}\otimes T_{\phi_{k,\eta}^n})x\|<\varepsilon$. This shows that

$$d_H(\mathcal{D}_R(M_p(\mathcal{A}_n)),\mathcal{D}_R(M_p(\mathcal{A}_n^{\Lambda_m^2})))<\varepsilon+\Big[1-\frac{1}{(1+\varepsilon)^2}\Big]R\|T_{\phi_{k,\eta}^n}\|_{\mathrm{cb}}\le(3R+1)\varepsilon.$$

Hence $d_{oq,R}^{cb}(\mathcal{A}_n,\mathcal{A}_n^{\Lambda_m^2})<(3R+1)\varepsilon$. Hence, by (5.2.7), we conclude that

$$d_{oq,R}^{cb}(\mathcal{A}_\infty,\mathcal{A}_n)<8(R+1)\varepsilon.$$

This completes the proof. $\square$

Chapter 6

Approximation for Higher Dimensional Quantum Tori

6.1 Completely Bounded Quantum Gromov–Hausdorff Distance for Higher Dimensional Quantum Tori

In this section we explore the convergence of matrix algebras to the noncommutative tori in higher dimensions. In the following, let $m = \frac{d(d-1)}{2}$ and $\mathcal{A}_\Theta^d$ denote the rotation algebra with d generators which was introduced in Section 4.3. Recall that by Proposition 5.1.16, $(\mathcal{A}_\Theta)_\Theta$ form a continuous field of compact quantum metric spaces. The following is an analog of Haagerup and Rørdam's result in higher dimensions.

Theorem 6.1.1. *There exists a Hilbert space $\mathcal{H}$, such that for all Θ, there exist unitaries $u_1(\Theta), ..., u_d(\Theta) \in \mathcal{B}(\mathcal{H})$ such that $C^*(u_1(\Theta), ..., u_d(\Theta)) \simeq \mathcal{A}_\Theta^d$ and $\lim_{\Theta' \to \Theta} \|u_k(\Theta') - u_k(\Theta)\|_{\mathcal{B}(\mathcal{H})} = 0$ for $k = 1, ..., d$, where $\Theta' \to \Theta$ in $\mathbb{R}^{d(d-1)/2}$.*

Proof. We recall the Heisenberg group $\mathbb{H}_B$ as defined in Subsection 5.1.2. To shorten the notation, we will write $\mathbb{H}$ for $\mathbb{H}_B$ in the following. Note that since $\mathbb{H}$ is amenable, $C^*(\mathbb{H})$ is a nuclear $C(\mathbb{T}^m)$-algebra. Therefore, by Theorem 3.2 in [Bla97] we get a unital monomorphism of $C(\mathbb{T}^m)$-algebras $\alpha : C^*(\mathbb{H}) \hookrightarrow \mathcal{O}_2 \otimes C(\mathbb{T}^m)$ and a unital $C(\mathbb{T}^m)$-linear completely positive map $E : \mathcal{O}_2 \otimes C(\mathbb{T}^m) \to C^*(\mathbb{H})$ such that $E \circ \alpha = \mathrm{id}_{C^*(\mathbb{H})}$. Here $\mathcal{O}_2$ is the Cuntz algebra generated by two orthogonal isometries. Let $\mathcal{O}_2 \subset \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then for all $x \in C^*(\mathbb{H})$, $\alpha(x) \in C(\mathbb{T}^m, \mathcal{B}(\mathcal{H}))$. We define $J_\Theta = \{g \in C(\mathbb{T}^m) : g(\Theta) = 0\}$. Then J_Θ is a closed ideal of $C(\mathbb{T}^m)$. Recall from Lemma 5.1.13 the quotient $C_\Theta = C^*(\mathbb{H})/I_\Theta$. We consider the following diagram

$$\begin{tikzcd} C^*(\mathbb{H}) \arrow[r, "\alpha"] \arrow[d, "q_\Theta"] & C(\mathbb{T}^m, \mathcal{B}(\mathcal{H})) \arrow[d, "\tilde{q}_\Theta"] \\ C_\Theta \arrow[r, dotted] & C(\mathbb{T}^m, \mathcal{B}(\mathcal{H}))/J_\Theta \otimes_{\min} \mathcal{B}(\mathcal{H}). \end{tikzcd}$$

Since α is $C(\mathbb{T}^m)$-linear, the kernel of q_Θ and that of $\tilde{q}_\Theta \circ \alpha$ coincide. We define $\pi_\Theta = \tilde{q}_\Theta \circ \alpha \circ q_\Theta^{-1}$. Then π_Θ is a well-defined monomorphism and the above diagram commutes. Note that for $f \in C(\mathbb{T}^m, \mathcal{B}(\mathcal{H}))$, we have $\tilde{q}_\Theta(f) = f(\Theta)$ and

$$\|f(\Theta)\|_{\mathcal{B}(\mathcal{H})} = \|f + J_\Theta \otimes_{\min} \mathcal{B}(\mathcal{H})\|_{C(\mathbb{T}^m,\mathcal{B}(\mathcal{H}))/J_\Theta\otimes_{\min}\mathcal{B}(\mathcal{H})}.$$

Then $\lim_{\Theta'\to\Theta} \|\alpha(x)(\Theta') - \alpha(x)(\Theta)\|_{\mathcal{B}(\mathcal{H})} = 0$ for $x \in C^*(\mathbb{H})$. It follows that

$$\lim_{\Theta'\to\Theta} \|\pi_{\Theta'}(\hat{x} + I_{\Theta'}) - \pi_\Theta(\hat{x} + I_\Theta)\|_{\mathcal{B}(\mathcal{H})} = 0, \quad \hat{x} + I_\Theta \in C_\Theta.$$

By Lemma 5.1.13, $\mathcal{A}^d_{2\Theta} \simeq C_\Theta$ and $\lambda(0, e_1) + I_\Theta, ..., \lambda(0, e_d) + I_\Theta$ generates C_Θ, where $(e_k)_{k=1}^d$ are canonical generators of $\mathbb{Z}^d$. Let $u_k(\Theta) = \pi_\Theta(\lambda(0, e_k) + I_\Theta), k = 1, ..., d$ and note that $\pi_\Theta(C_\Theta) \subset \mathcal{B}(\mathcal{H})$. The proof is complete. □

We now consider approximations of $\mathcal{A}^{2d}_\Theta$ by matrix algebras. We want to use finite dimensional versions of rotation algebras and we have to determine their center. In order to use induction we have to introduce a new form of action. We consider an action σ of $\mathbb{Z}^2$ on a unital C*-algebra B. Then we can construct the universal crossed product $B \rtimes_\sigma \mathbb{Z}^2$. In particular, if B is faithfully represented on $\mathcal{H}$, we may choose a special representation π of B on $\mathcal{H} \otimes \ell_2(\mathbb{Z}^2)$ such that the left regular representation of $\mathbb{Z}^2$ spatially implements the action σ, i.e.

$$(1 \otimes \lambda_{(j,k)})\pi(b)(1 \otimes \lambda^*_{(j,k)}) = \pi(\sigma_{(j,k)}(b)), \quad b \in B;$$

see e.g. [BO08]. Let u, v denote the universal generators of $\mathcal{A}_\theta$. We define a representation of $\mathcal{A}_\theta$ by

$$\gamma : \mathcal{A}_\theta \to \mathcal{B}(\mathcal{H}) \otimes L(\mathbb{Z}^2) \otimes \mathcal{A}_\theta, \qquad u^j v^k \mapsto 1 \otimes \lambda_{(j,k)} \otimes u^j v^k.$$

It follows that for $b \in B$,

$$\gamma(u^j v^k)(\pi(b) \otimes 1)\gamma(u^j v^k)^* = (1 \otimes \lambda_{(j,k)})(\pi(b) \otimes 1)(1 \otimes \lambda_{(j,k)})^* \otimes u^j v^k(u^j v^k)^* = \pi(\sigma_{(j,k)}(b)) \otimes 1.$$

Therefore, the $\mathbb{Z}^2$-action σ and the representations π and γ satisfy

$$\gamma(u)(\pi(b) \otimes 1)\gamma(u)^* = \pi(\sigma_{(1,0)}(b)), \tag{6.1.1}$$
$$\gamma(v)(\pi(b) \otimes 1)\gamma(v)^* = \pi(\sigma_{(0,1)}(b)), \quad \gamma(u)\gamma(v) = e^{2\pi i\theta}\gamma(v)\gamma(u).$$

In the following, we use the notation $\langle D : R\rangle$ to denote the universal C*-algebra generated by D with relations R. We may even ignore R for short if the relations are clear from context. We define

$$\mathcal{A}_\theta(n) = \langle U, V : U^n = 1 = V^n, UV = e^{2\pi \mathrm{i}\theta} VU, U \text{ and } V \text{ unitaries}\rangle,$$

and

$$B \rtimes_\sigma \mathcal{A}_\theta = \langle b, U, V : b \in B, UbU^* = \sigma_{1,0}(b), VbV^* = \sigma_{0,1}(b), UV = e^{2\pi \mathrm{i}\theta} VU, U \text{ and } V \text{ unitaries}\rangle,$$

Similarly, if we start with a $\mathbb{Z}_n^2$ action σ on B, where $\theta = \frac{q}{n}$, we may find representations π of B and γ of $\mathcal{A}_\theta(n)$ as follows.

$$\gamma : \mathcal{A}_\theta(n) \to \mathcal{B}(\mathcal{H}) \otimes L(\mathbb{Z}_n^2) \otimes \mathcal{A}_\theta(n), \qquad u_p(n)v_q(n) \mapsto 1 \otimes \lambda_{(j,k)} \otimes u_p(n)v_q(n),$$

where the generators $u_p(n), v_q(n)$ of $\mathcal{A}_\Theta(n)$ are as given in equation (5.1.1). Similarly, it follows that for $b \in B$, the $\mathbb{Z}^2$-action σ and the representations π and γ satisfy

$$\gamma(u_p(n))(\pi(b) \otimes 1)\gamma(u_p(n))^* = \pi(\sigma_{(1,0)}(b)), \tag{6.1.2}$$
$$\gamma(v_q(n))(\pi(b) \otimes 1)\gamma(v_q(n))^* = \pi(\sigma_{(0,1)}(b)), \quad \gamma(u_p(n))\gamma(v_q(n)) = e^{2\pi \mathrm{i}\theta}\gamma(v_q(n))\gamma(u_p(n)).$$

For simplicity, in the following we will write $\sigma_{1,0}$ and $\sigma_{0,1}$ for $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$, respectively.

Definition 6.1.2. Suppose B is a unital C*-algebra. We define

$$\begin{aligned} B \rtimes_\sigma \mathcal{A}_\theta(n) = \langle b, U, V : b \in B, UbU^* = \sigma_{1,0}(b), VbV^* = \sigma_{0,1}(b), \\ U^n = 1 = V^n, UV = e^{2\pi \mathrm{i}\theta} VU, U \text{ and } V \text{ unitaries}\rangle, \end{aligned}$$

where $\theta = \frac{q}{n}$, $q, n \in \mathbb{N}$, and σ is an action of $\mathbb{Z}_n^2$ on B.

Thanks to (6.1.1) and (6.1.2), the universal objects defined above exist. Note that for $\mathcal{A}_\theta(n)$ we have necessarily $\theta = \frac{q}{n}$ for some $q \in \mathbb{Z}$. If q and n are coprime, then it is well known that $\mathcal{A}_\theta(n) \simeq M_n$. By universality and using the notation introduced here, we can rewrite the noncommutative torus $\mathcal{A}_\Theta^{2d}$ iteratively as

$$\mathcal{A}_\Theta^{2d} = \mathcal{A}_{\theta_{12}} \rtimes_{\sigma^2} \mathcal{A}_{\theta_{34}} \rtimes_{\sigma^3} \cdots \rtimes_{\sigma^d} \mathcal{A}_{\theta_{2d-1,2d}}, \tag{6.1.3}$$

where the $\mathbb{Z}^2$-action $\sigma^k, k = 2, ..., d$, is defined by

$$\sigma^k_{1,0}(u_1) = e^{-2\pi \mathrm{i}\theta_{1,2k-1}} u_1,\ ...,\ \sigma^k_{1,0}(u_{2k-2}) = e^{-2\pi \mathrm{i}\theta_{2k-2,2k-1}} u_{2k-2},$$
$$\sigma^k_{0,1}(u_1) = e^{-2\pi \mathrm{i}\theta_{1,2k}} u_1,\ ...,\ \sigma^k_{0,1}(u_{2k-2}) = e^{-2\pi \mathrm{i}\theta_{2k-2,2k}} u_{2k-2}.$$

Indeed, note that by the definition of B_θ, we have

$$\sigma^k_{1,0}(u_1) = u_{2k-1} u_1 u^*_{2k-1},\ ...,\ \sigma^k_{1,0}(u_{2k-2}) = u_{2k-1} u_{2k-2} u^*_{2k-1},$$
$$\sigma^k_{0,1}(u_1) = u_{2k} u_1 u^*_{2k},\ ...,\ \sigma^k_{0,1}(u_{2k-2}) = u_{2k} u_{2k-2} u^*_{2k}.$$

Then (6.1.3) follows from universality of $\mathcal{A}^{2d}_\Theta$ and $\mathcal{A}_{\theta_{12}} \rtimes_{\sigma^2} \mathcal{A}_{\theta_{34}} \rtimes_{\sigma^3} \cdots \rtimes_{\sigma^d} \mathcal{A}_{\theta_{2d-1,2d}}$.

Proposition 6.1.3. *Let $\theta = \frac{q}{n}$ and q, n are coprime. Let σ be an action of $\mathbb{Z}^2_n$ on a C^*-algebra B. Then $B \rtimes_\sigma \mathcal{A}_\theta(n) \simeq M_n(B)$.*

Proof. We first consider the case for which the action σ is inner, i.e. there exist unitaries w_1 and w_2 in B such that $\sigma_{1,0}(x) = w_1 x w_1^*$, $\sigma_{0,1}(x) = w_2 x w_2^*$ and $[w_1, w_2] = 0$. Let u_n and v_n be the generators of $\mathcal{A}_\theta(n)$. We consider a special representation π_0 of $B \rtimes_\sigma \mathcal{A}_\theta(n)$ defined by $\pi_0(b) = b \otimes 1$ for $b \in B$, $\pi_0(U) = w_1 \otimes u_n$ and $\pi_0(V) = w_2 \otimes v_n$. It can be directly checked that π_0 is indeed a representation. Then we have $w_1^n = 1 = w_2^n$,

$$\pi_0(U)\pi_0(V) = e^{2\pi i\theta}\pi_0(V)\pi_0(U)$$

and since $C^*(u_n, v_n) = \mathcal{A}_\theta(n) = M_n$, we have

$$\begin{aligned} \pi_0(B \rtimes_\sigma \mathcal{A}_\theta(n)) &= \pi_0(\langle b, U, V : b \in B\rangle) \\ &= C^*(b \otimes 1, w_1 \otimes u_n, w_2 \otimes v_n : b \in B) = B \otimes_{\min} M_n. \end{aligned} \tag{6.1.4}$$

Now let $\pi_u : B \rtimes_\sigma \mathcal{A}_\theta(n) \to \mathcal{B}(\mathcal{H}_u)$ be the universal representation of $B \rtimes_\sigma \mathcal{A}_\theta(n)$. We show that in this case, we can also write $\pi_u(U)$ and $\pi_u(V)$ as tensors. Note that $\mathcal{A}_\theta(n)$ has dimension at most n^2. Thanks to the image of U and V under π_0, we know that $C^*(\pi_u(U), \pi_u(V)) = M_n$. Therefore we may take $\mathcal{H}_u = \mathcal{K} \otimes \ell^n_2$ for some Hilbert space $\mathcal{K}$. Let us define $u = \pi_u(w_1^*)\pi_u(U)$ and $v = \pi_u(w_2^*)\pi_u(V)$. Then for $x \in \pi_u(B)$,

$$x = \sigma^{-1}_{1,0}[\sigma_{1,0}(x)] = \pi_u(w_1^*)\pi_u(U) x \pi_u(U)^* \pi_u(w_1) = u x u^*.$$

Thus $ux = xu$. Similarly, $vx = xv$. We deduce that $\pi_u(B) \subset C^*(u, v)' \cap \mathcal{B}(\mathcal{H}_u)$. Since w_1 and w_2 commute, plugging in $x = \pi_u(w_1), \pi_u(w_2)$, we find $\pi_u(U)\pi_u(w_i) = \pi_u(w_i)\pi_u(U)$ and $\pi_u(V)\pi_u(w_i) = \pi_u(w_i)\pi_u(V)$ for $i = 1, 2$. It follows that $\pi_u(w_i) \in M_n' \cap \mathcal{B}(\mathcal{H}_u)$. Moreover, u and v also satisfy the conditions $uv = e^{2\pi i\theta} vu$ and $u^n = 1 = v^n$. Therefore,

$$C^*(u, v) \simeq M_n, \quad \pi_u(B) \subset \mathcal{B}(\mathcal{K}) \otimes \mathbb{C} \quad \text{and} \quad u, v \in \pi_u(B)' \cap \mathcal{B}(\mathcal{H}_u).$$

We may write $u = a \otimes \tilde{u}$ for some $a \in \pi_u(B)' \cap \mathcal{B}(\mathcal{K})$ and $\tilde{u} \in M_n$, and $\pi_u(w_1) = \pi_{\mathcal{K}}(w_1) \otimes z$ for some $z \in \mathbb{C}$ where $\pi_{\mathcal{K}}$ is the restriction of π_u on $\mathcal{K}$. Hence,

$$\pi_u(U) = \pi_u(w_1)u = \pi_{\mathcal{K}}(w_1)a \otimes z\tilde{u}.$$

Similarly, we can write $\pi_u(V)$ as a tensor. By (6.1.4), $B \rtimes_\sigma \mathcal{A}_\theta(n) \simeq M_n(B)$.

Now we consider σ to be a general action. We define a $\mathbb{Z}_n^2$ action $\hat{\sigma}$ on $B \rtimes_\sigma \mathcal{A}_\theta(n)$: For $x = \sum_{k,l} b_{kl} U^k V^l \in B \rtimes_\sigma \mathcal{A}_\theta(n)$,

$$\hat{\sigma}_{1,0}(x) = \sum_{k,l} \sigma_{1,0}(b_{kl}) U^k V^l, \quad \hat{\sigma}_{0,1}(x) = \sum_{k,l} \sigma_{0,1}(b_{kl}) U^k V^l.$$

Similarly, we define a $\mathbb{Z}_n^2$ action, still denoted by $\hat{\sigma}$, on the universal crossed product $B \rtimes_\sigma \mathbb{Z}_n^2$: For $x = \sum_{k,l} b_{kl} \lambda(k,l) \in B \rtimes_\sigma \mathbb{Z}_n^2$,

$$\hat{\sigma}_{1,0}(x) = \sum_{k,l} \sigma_{1,0}(b_{kl}) \lambda(k,l), \quad \hat{\sigma}_{0,1}(x) = \sum_{k,l} \sigma_{0,1}(b_{kl}) \lambda(k,l).$$

Then by universality we have $(B \rtimes_\sigma \mathcal{A}_\theta(n)) \rtimes_{\hat{\sigma}} \mathbb{Z}_n^2 = (B \rtimes_\sigma \mathbb{Z}_n^2) \rtimes_{\hat{\sigma}} \mathcal{A}_\theta(n)$. By the crossed product construction, the action $\hat{\sigma}$ on $B \rtimes_\sigma \mathbb{Z}_n^2$ is spatially implemented by $w_1 = 1 \otimes \lambda(1,0)$ and $w_2 = 1 \otimes \lambda(0,1)$. More precisely,

$$\pi(\hat{\sigma}_{(1,0)}(x)) = (1 \otimes w_1)\pi(x)(1 \otimes w_1^*), \quad \pi(\hat{\sigma}_{(0,1)}(x)) = (1 \otimes w_2)\pi(x)(1 \otimes w_2^*),$$

where $\pi(x) = \oplus_{g \in \mathbb{Z}_n^2} \sigma_{g^{-1}}(x)$; see e.g. [BO08] for more details. By what we proved in the first paragraph, we find that $(B \rtimes_\sigma \mathbb{Z}_n^2) \rtimes_{\hat{\sigma}} \mathcal{A}_\theta(n) \simeq M_n(B \rtimes_\sigma \mathbb{Z}_n^2)$. But $M_n(B \rtimes_\sigma \mathbb{Z}_n^2) = M_n(B) \rtimes_\sigma \mathbb{Z}_n^2$ where we have denoted the inflated action $\mathrm{id} \otimes \sigma$ still by σ. It is well known that there exists a faithful conditional expectation $E : M_n(B) \rtimes_\sigma \mathbb{Z}_n^2 \to M_n(B)$. Recall that we have the canonical embedding $\iota : B \rtimes_\sigma \mathcal{A}_\theta(n) \hookrightarrow (B \rtimes_\sigma \mathcal{A}_\theta(n)) \rtimes_{\hat{\sigma}} \mathbb{Z}_n^2$.

We have the following diagram

$$\begin{tikzcd}
(B \rtimes_\sigma \mathcal{A}_\theta(n)) \rtimes_{\hat\sigma} \mathbb{Z}_n^2 \arrow[r, "\simeq"] & M_n(B) \rtimes_\sigma \mathbb{Z}_n^2 \arrow[d, "E"] \\
B \rtimes_\sigma \mathcal{A}_\theta(n) \arrow[u, "\iota"] \arrow[r, dotted] & M_n(B)
\end{tikzcd}$$

Note that the multiplicative domain of E is $M_n(B)$, restricted on which E is a *-homomorphism. Moreover, $B \rtimes_\sigma \mathcal{A}_\theta(n)$ is contained in the multiplicative domain of E and clearly $E(B \rtimes_\sigma \mathcal{A}_\theta(n)) = M_n(B)$. But $E \circ \iota$ is faithful. Hence, we find that $B \rtimes_\sigma \mathcal{A}_\theta(n) \simeq M_n(B)$. $\square$

In the following we show the convergence of the matrix algebras to the rotation algebra $\mathcal{A}_\Theta^{2d}$. Similar to the 2 dimensional case, we define the matrix level Lip-norms on (a dense subspace of) $\mathcal{A}_\Theta$ as in (4.3.7):

$$|||x|||_m = \max\{\| \mathrm{id} \otimes \delta(x)\|_{M_m \otimes_{\min} \mathcal{A}_\Theta \otimes_{\min} H_\psi^c},\ \| \mathrm{id} \otimes \delta(x)\|_{M_m \otimes_{\min} \mathcal{A}_\Theta \otimes_{\min} H_\psi^r}\}.$$

Similarly, by Remark 4.3.6 we may define the matrix level Lip-norms on M_{n^d} once we choose a set of generators of M_{n^d}. The Lip-norms on $\mathcal{A}_\Theta$ and M_{n^d} will also be denoted by $L_\infty(\cdot)$ and $L_n(\cdot)$, respectively, especially when we consider continuous fields of compact quantum metric spaces. We follow the same plan as in Section 5.2. Let $u_1(\Theta), ..., u_{2d}(\Theta)$ be the generators of $\mathcal{A}_\Theta^{2d}$. In particular, $u_1(0), ..., u_{2d}(0)$ generate $C(\mathbb{T}^{2d})$. Following Definition 6.1.2, we consider the C*-algebra

$$\mathcal{A}_{1/n}^{2d} := \mathcal{A}_{\theta_{1,2}}(n) \rtimes_{\sigma_{34}} \mathcal{A}_{\theta_{3,4}}(n) \rtimes_{\sigma_{56}} \cdots \rtimes_{\sigma_{2d-1,2d}} \mathcal{A}_{\theta_{2d-1,2d}}(n), \tag{6.1.5}$$

where the action $\sigma_{2k-1,2k} = (\alpha_k, \beta_k), k = 2, ..., d$ is defined by

$$\alpha_k(u_1) = u_{2k-1}^* u_1 u_{2k-1} = e^{2\pi \mathrm{i} \theta_{1,2k-1}} u_1,\ ...,\ \alpha_k(u_{2k-2}) = u_{2k-1}^* u_{2k-2} u_{2k-1} = e^{2\pi \mathrm{i} \theta_{2k-2,2k-1}} u_{2k-2},$$
$$\beta_k(u_1) = u_{2k}^* u_1 u_{2k} = e^{2\pi \mathrm{i} \theta_{1,2k}} u_1,\ ...,\ \beta_k(u_{2k-2}) = u_{2k}^* u_{2k-2} u_{2k} = e^{2\pi \mathrm{i} \theta_{2k-2,2k}} u_{2k-2},$$
$$u_i^q = 1,\ i = 1, ..., 2d, \qquad \theta_{i,j} = \frac{1}{n},\ 1 \le i < j \le n.$$

Then by Proposition 6.1.3, we have $\mathcal{A}_{1/n}^{2d} \simeq M_{n^d}$. For definiteness, let us fix the generators in the iterated crossed product and define $v_1(n) = u_1, ..., v_{2d}(n) = u_{2d}$. Then, we have

$$v_i(n) v_j(n) = e^{\frac{2\pi \mathrm{i}}{n}} v_j(n) v_i(n), \quad 1 \le i < j \le 2d. \tag{6.1.6}$$

We define a map $\rho_n : C(\mathbb{T}^{2d}) \to M_{n^d}$ by $\rho_n(u_1(0)^{i_1} \cdots u_{2d}(0)^{i_{2d}}) = v_1(n)^{i_1} \cdots v_{2d}(n)^{i_{2d}}$. Let $\Lambda_k^d = \{0, \pm 1, ..., \pm k\}^d$ and

$$C^{\Lambda_k^{2d}}(\mathbb{T}^{2d}) = \Big\{ x \in C(\mathbb{T}^{2d}) : x = \sum_{|i_1| \le k, ... |i_{2d}| \le k} a_i u_1(0)^{i_1} \cdots u_{2d}(0)^{i_{2d}}, a_i \in \mathbb{C} \Big\}$$

Similarly, in the following, we will consider

$$M_{n^d}^{\Lambda_k^{2d}} = \Big\{ x \in M_{n^d} : x = \sum_{|i_1| \le k, ... |i_{2d}| \le k} a_i u_1(n)^{i_1} \cdots u_{2d}(n)^{i_{2d}}, a_i \in \mathbb{C} \Big\}.$$

and

$$\mathcal{A}_\Theta^{\Lambda_k^{2d}} = \Big\{ x \in \mathcal{A}_\Theta^{2d} : x = \sum_{|i_1| \le k, ... |i_{2d}| \le k} a_i u_1(\Theta)^{i_1} \cdots u_{2d}(\Theta)^{i_{2d}}, a_i \in \mathbb{C} \Big\}.$$

Lemma 6.1.4. *For any $\varepsilon > 0$ and $k \ge 0$, there exists $N > 0$ such that for any $n > N$, the map $\rho_n|_{C^{\Lambda_k^{2d}}(\mathbb{T}^{2d})} : C^{\Lambda_k^{2d}}(\mathbb{T}^{2d}) \to M_{n^d}^{\Lambda_k^{2d}}$ is a $1+\varepsilon$ cb-isometry and a $1+\varepsilon$ Lip-isometry.*

Proof. By the definition of ρ_n and the commutation relations (6.1.6), we can generalize directly Lemma 5.1.8 to get a faithful *-homomorphism $(\rho_n)^\bullet : C(\mathbb{T}^{2d}) \to \prod_\omega M_{n^d}$, where $\prod_\omega M_{n^d}$ is the von Neumann algebra ultraproduct. Now we repeat the proof of Proposition 5.2.9 with the result of Haagerup–Rørdam replaced by Theorem 6.1.1. The claim of $1+\varepsilon$ Lip-isometry follows the same argument as for Lemma 4.3.8. We leave the details to the reader. □

Suppose $\theta_{rs} = \frac{p_{rs}}{q}, 1 \le r < s \le q$. We consider the iterated crossed product following the notation introduced in Definition 6.1.2

$$\mathcal{A}_\Theta^{2d}(q) := \mathcal{A}_{\theta_{12}}(q) \rtimes_{\sigma_{34}} \mathcal{A}_{\theta_{34}}(q) \rtimes_{\sigma^3} \cdots \rtimes_{\sigma^d} \mathcal{A}_{\theta_{2d-1,2d}}(q), \tag{6.1.7}$$

where the action $\sigma^k, k = 2, ..., d$ is defined by

$$\begin{gathered}\sigma_{1,0}^k(u_1) = u_{2k-1} u_1 u_{2k-1}^* = e^{-2\pi i \theta_{1,2k-1}} u_1, \ ..., \ \sigma_{1,0}^k(u_{2k-2}) = u_{2k-1} u_{2k-2} u_{2k-1}^* = e^{-2\pi i \theta_{2k-2,2k-1}} u_{2k-2}, \\ \sigma_{0,1}^k(u_1) = u_{2k} u_1 u_{2k}^* = e^{-2\pi i \theta_{1,2k}} u_1, \ ..., \ \sigma_{0,1}^k(u_{2k-2}) = u_{2k} u_{2k-2} u_{2k}^* = e^{-2\pi i \theta_{2k-2,2k}} u_{2k-2}, \\ u_i^q = 1, \quad i = 1, ..., 2d.\end{gathered}$$

For definiteness, in the following result the generators of $\mathcal{A}_\Theta^{2d}(q)$ will be denoted by $u_j^\Theta(q)$, $j = 1, ..., 2d$.

Proposition 6.1.5. *Let $\Theta = (\theta_{rs})_{r,s=1}^{2d}$ and $\Theta^n = (\theta_{rs}^n)_{r,s=1}^{2d}$ be two skew symmetric matrices such that*

$\theta_{rs} = \frac{p_{rs}}{q}$ *and* $\theta^n_{rs} = \theta_{rs} + \frac{1}{n}$. *For any* $\varepsilon > 0$ *and* $k \geq 0$, *there exists* $N > 0$ *such that for* $n > N$ *the maps* $\rho^\Theta_n : \mathcal{A}^{\Lambda^{2d}_k}_\Theta \to [\mathcal{A}^{2d}_{\Theta^n}(nq)]^{\Lambda^{2d}_k}$ *defined by*

$$\rho^\Theta_n(u_1(\Theta)^{i_1} \cdots u_{2d}(\Theta)^{i_{2d}}) = [u^{\Theta^n}_1(qn)]^{i_1} \cdots [u^{\Theta^n}_{2d}(qn)]^{i_{2d}}, \quad |i_j| \leq k, j = 1, ..., 2d$$

is a $1+\varepsilon$ *cb-isometry and a* $1+\varepsilon$ *Lip-isometry. Moreover, we have* $\mathcal{A}^{2d}_{\Theta^{q^{l+1}}}(q^{l+2}) \simeq M_{q^{(l+1)d}}$.

Proof. Similar to (5.1.8), we define

$$\sigma : \mathcal{A}^{2d}_\Theta \to \mathcal{A}^{2d}_\Theta(q) \otimes_{\min} C(\mathbb{T}^{2d})$$
$$\sigma(u_1(\Theta)^{k_1} \cdots u_{2d}(\Theta)^{k_{2d}}) = u^\Theta_1(q)^{k_1} \cdots u^\Theta_{2d}(q)^{k_{2d}} \otimes u_1(0)^{k_1} \cdots u_{2d}(0)^{k_{2d}}.$$

Since the canonical trace on $\mathcal{A}^{2d}_\Theta$ is faithful (see e.g. [Rie90]) and σ is trace preserving, σ is a faithful *-homomorphism. Recall that by (6.1.5) and Proposition 6.1.3, $\mathcal{A}^{2d}_{1/n}$ is a matrix algebra. We define

$$\rho^\Theta_n = (\mathrm{id} \otimes \rho_n) \circ \sigma : \mathcal{A}^{2d}_\Theta \to \mathcal{A}^{2d}_\Theta(q) \otimes_{\min} \mathcal{A}^{2d}_{1/n}$$
$$u_1(\Theta)^{k_1} \cdots u_{2d}(\Theta)^{k_{2d}} \mapsto u^\Theta_1(q)^{k_1} \cdots u^\Theta_{2d}(q)^{k_{2d}} \otimes v_1(n)^{k_1} \cdots v_{2d}(n)^{k_{2d}} =: \hat{u}^{k_1}_1 \cdots \hat{u}^{k_{2d}}_{2d}.$$

By Lemma 6.1.4, for any $\varepsilon > 0$ and $k \geq 0$ there exists $N > 0$ such that for all $n > N$

$$\rho^\Theta_n|_{\mathcal{A}^{\Lambda^{2d}_k}_\Theta} : \mathcal{A}^{\Lambda^{2d}_k}_\Theta \to \mathcal{A}^{2d}_\Theta(q) \otimes \mathcal{A}^{2d}_{1/n}$$

is a $1+\varepsilon$ cb-isometry and a $1+\varepsilon$ Lip-isometry onto its image. To identify the image of ρ^Θ_n, note that by (6.1.6) we have

$$\hat{u}_r \hat{u}_s = e^{2\pi \mathrm{i}(\theta_{rs} + \frac{1}{n})} \hat{u}_s \hat{u}_r, \quad 1 \leq r < s \leq 2d.$$

If we let $\theta^n_{rs} = \theta_{rs} + \frac{1}{n} = \frac{np_{rs}+q}{nq}$, then we may define the iterated crossed product $\mathcal{A}_{\Theta^n}(nq)$ in the same way as (6.1.7), where the entries of Θ^n are given by θ^n_{rs}. Although $\mathcal{A}_{\Theta^n}(nq)$ is universally defined, dimension counting shows that we can take $\hat{u}_1, ..., \hat{u}_{2d}$ as its universal generators. Therefore, we have $\rho^\Theta_n(\mathcal{A}^{2d}_\Theta) = \mathcal{A}^{2d}_{\Theta^n}(nq)$, and $\hat{u}_j = u^{\Theta^n}_j(qn)$, $j = 1, ..., 2d$.

Similar to the case of the 2-dimensional tori, we can choose a subsequence n_l so that $\hat{u}_1, ..., \hat{u}_{2d}$ generate $M_{n^d_l}$. Indeed, since $\hat{u}_r \hat{u}_s = e^{2\pi \mathrm{i}(\theta_{rs} + \frac{1}{n_l})} \hat{u}_s \hat{u}_r$ for all $r < s$, by Proposition 6.1.3, we just need $(\theta_{rs} n_l + 1, n_l) = 1$

for all r, s to verify that $C^*(\rho^{\Theta}_{n_l}(\mathcal{A}^{2d}_{\Theta})) \simeq M_{n_l^d}$. But $\theta_{rs} = \frac{p_{rs}}{q}$, it suffices to take $n_l = q^{l+1}$ as in Lemma 5.1.10. Then we find that

$$\theta^{n_l}_{rs} = \frac{q^l p_{rs} + 1}{q^{l+1}} \quad \text{and} \quad \mathcal{A}^{2d}_{\Theta^{n_l}}(n_l q) = \mathcal{A}^{2d}_{\Theta^{n_l}}(n_l) \simeq M_{n_l^d}.$$

Note that the generators of $M_{n_l^d}$ may be different from $v_1(n_l), ..., v_{2d}(n_l)$. Thus $M^{\Lambda^{2d}_k}_{n_l^d}$ refers to the subspace of $M_{n_l^d}$ generated by $\hat{u}_1^{j_1} \cdots \hat{u}_{2d}^{j_{2d}}$ for $|j_i| \leq k, i = 1, ..., 2d$. $\square$

Suppose θ_{rs} may not be written as $\frac{p_{rs}}{q}$. Note that

$$\{(\frac{p_{rs}}{q})_{1 \leq r < s \leq 2d}, 0 \neq q \in \mathbb{Z}, p_{rs} \in \mathbb{Z}, |p_{rs}| \leq q, (p_{rs}, q) = 1\}$$

is dense in $[-1, 1]^{d(2d-1)}$. Following the same argument as that of Proposition 5.2.10 with the result of Haagerup–Rørdam replaced by Theorem 6.1.1, we find maps

$$\rho^{\Theta}_n : \mathcal{A}^{\Lambda^{2d}_k}_{\Theta} \to [\mathcal{A}^{2d}_{\tilde{\Theta}^n}(nq)]^{\Lambda^{2d}_k} \tag{6.1.8}$$

which is a $1+\varepsilon$ cb-isometry and a $1+\varepsilon$ Lip-isometry. Here $\tilde{\Theta}^n$ and q are chosen such that $\tilde{\Theta} = (\tilde{\theta}_{rs} = \frac{p_{rs}}{q})^n_{r,s=1}$ is close to Θ and $\tilde{\Theta}^n = (\tilde{\theta}^n_{rs} = \frac{p_{rs}}{q} + \frac{1}{n})^n_{r,s=1}$. Moreover, we have $\mathcal{A}^{2d}_{\tilde{\Theta}^{q^{l+1}}}(q^{l+2}) \simeq M_{q^{(l+1)d}}$.

Let $S = \{\rho^{\Theta}_{n_l^d}(x) : x = \sum_{\vec{j}=(j_1,...,j_{2d}) \in \Lambda^{2d}_k} a_{\vec{j}} u_1(\Theta)^{j_1} \cdots u_{2d}(\Theta)^{j_{2d}}, k \in \mathbb{N}, l \in \bar{\mathbb{N}}\}$. Here $\rho^{\Theta}_{n_l^d}$ is $1 + \frac{1}{l}$ cb-isometry and $1 + \frac{1}{l}$ Lip-isometry found in Lemma 6.1.4, Proposition 6.1.5 and (6.1.8). Here $n_l = q^{l+1}$ for some q chosen appropriately as above. The following is a consequence of these results.

Proposition 6.1.6. *$(\{(M_{n_l^d})_{sa}, L_{n_l}\}_{l \in \bar{\mathbb{N}}}, S)$ is a cb-continuous field of compact quantum metric spaces.*

We consider a conditionally negative length function ϕ_n on $\mathbb{Z}^{2d}_n$ for $n \in \bar{\mathbb{N}}$ as in Section 4.3. For example, we can take $\phi_n(k_1, ..., k_{2d}) = \psi_n(k_1) + \cdots + \psi_n(k_{2d})$ where ψ_n is given in (4.2.4). We find a symmetric Markov semigroup on $L(\mathbb{Z}^{2d}_n)$, which induces a symmetric Markov semigroup on M_{n^d} as in (5.1.3). Lemma 5.1.2 and Proposition 5.1.3 extend directly to the current situation.

Theorem 6.1.7. *There exists a sequence of matrix algebras $M_{n_j^d}$ converging to $\mathcal{A}^{2d}_{\Theta}$ in the R-cb quantum Gromov–Hausdorff distance.*

Proof. First we need a tail estimate which is an extension of Theorem 5.2.12. This follows the same proof as that of Theorem 5.2.12. Indeed, similar to the proof of Lemma 5.1.18, given $\varepsilon > 0$, we may choose k and then define $\phi^n_{k,\eta}(j_1, ..., j_{2d}) = \varphi^n_{k,\eta}(j_1) \cdots \varphi^n_{k,\eta}(j_{2d})$, where $\varphi^n_{k,\eta}(\cdot)$ is the multiplier found in Lemma 4.4.4 and this time we take $\eta \in (0, \frac{\varepsilon}{2d(2k+1)^{2d}})$. Then we use (possibly extended versions of) Lemma 5.1.1, Lemma

5.1.4, Corollary 5.1.7, Proposition 5.1.3 (or Corollary 4.3.7) and Remark 4.3.9 as explained above. The rest of the argument is a simple extension of the proof of Theorem 5.2.13. $\square$

6.2 Application to Gromov–Hausdorff propinquity

In this section we follow [Lat15] to generalize Latrémolière's result on convergence of matrix algebras to rotation algebras in the sense of Gromov–Hausdorff propinquity. For a fixed permissible function F (see Definition 2.18 in [Lat15]), denote by $\Lambda_F((A, L_A), (B, L_B))$ the Gromov–Hausdorff propinquity between two compact quantum metric spaces, in the sense of [Lat15]; see Definition 3.54. Recall that according to [Lat15], if A and B are two unital C*-algebras, a bridge $\gamma = (D, \omega, \pi_A, \pi_B)$ is given by a unital C*-algebra D, two unital *-monomorphisms $\pi_A : A \hookrightarrow D$ and $\pi_B : B \hookrightarrow D$ and $\omega \in D$ such that the set $S(A|\omega) := \{\varphi \in S(A) : \forall d \in D, \varphi(d) = \varphi(d\omega) = \varphi(\omega d)\}$ is not empty, where $S(A)$ denotes the state space of A; see Definition 3.42. In the following let $F : [0,\infty)^4 \to [0,\infty)$ be defined by $F(x, y, l_x, l_y) = xl_y + yl_x$, for $x, y, l_x, l_y \in [0,\infty)$ (see Definition 2.18 in [Lat15]).

Lemma 6.2.1. *Let $(A, |||.|||_A)$ and $(B, |||.|||_B)$ be two F-quasi-Leibniz compact quantum metric spaces in the sense of [Lat15]; see Definition 2.44. If there exist two *-homomorphisms $\pi_A : A \hookrightarrow \mathcal{B}(\mathcal{H})$ and $\pi_B : B \hookrightarrow \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ such that the following hold:*

1. *For all $a \in A$ such that $|||a|||_A \leq 1$, there exists $b \in B$ such that $|||b|||_B \leq 1$ and $\|\pi_A(a) - \pi_B(b)\|_D < \varepsilon$,*
2. *For all $b \in B$ such that $|||b|||_B \leq 1$, there exists $a \in A$ such that $|||a|||_A \leq 1$ and $\|\pi_A(a) - \pi_B(b)\|_D < \varepsilon$,*

then $\Lambda_F((A, |||.|||_A), (B, |||.|||_B)) \leq \varepsilon$.

Proof. We refine the proof of Lemma 3.79 in [Lat15] by taking a *trek* (see Definition 3.49 in [Lat15]) consisting of a single *bridge* (see Definition 3.42 in [Lat15]), namely $\gamma = (\mathcal{B}(\mathcal{H}), \mathrm{id}, \pi_A, \pi_B)$. Note that in this case, since any state on A or B can be extended to a state on $\mathcal{B}(\mathcal{H})$, and for $\omega = \mathrm{id}$, $S(A|\omega) = S(A)$, with the notation of [Lat15], we have the *height* $\zeta(\gamma||||.|||_A, |||.|||_B) = 0$ (see Definition 3.46 in [Lat15]). On the other hand, if (1) and (2) hold, then by definition, the *reach* $\rho(\gamma||||.|||_A, |||.|||_B) = \varepsilon$ (see Definition 3.45 in [Lat15]). Now by Definitions 3.47 and 3.54 in the aforementioned paper, we have $\Lambda_F((A, |||.|||_A), (B, |||.|||_B)) \leq \varepsilon$. $\square$

Definition 6.2.2. Let $(A_n, |||.|||_{A_n})$ and $(B, |||.|||_B)$ be F-quasi-Leibniz quantum compact metric spaces in the sense of [Lat15]. We say $(A_n, |||.|||_{A_n})$ converges to $(B, |||.|||_B)$ in the strong Gromov–Hausdorff propinquity sense, if the unitization of $(\mathcal{K} \otimes A_n, |||.|||_{\mathcal{K}\otimes A_n})$ converges to the unitization of $(\mathcal{K} \otimes B, |||.|||_{\mathcal{K}\otimes B})$ in the Gromov–Hausdorff propinquity sense, where $\mathcal{K}$ is the space of compact operators on ℓ_2.

Recall the definition of $\mathcal{A}^{2d}_{\Theta^n}(nq)$ in Proposition 6.1.5. By Definitions 2.21 in [Lat15] and the existence of a derivation δ as defined in (4.3.5), $(\mathcal{A}^{2d}_{\Theta}, |||.|||)$ and $(\mathcal{A}^{2d}_{\Theta^n}(nq), |||.|||)$ are *Leibniz pairs*. Indeed, the conditions in the definition were proved in [JM10, JMP14]; see also [Zen14] for more remarks on the Lip-norms. Furthermore, let δ^c and δ^r denote the column and row structure derivations, respectively (see Lemma 4.3.4). Note that Remark 4.3.6 applies to the algebra $\mathcal{A}^{2d}_{\Theta^n}(nq)$ as well by choosing $2d$ generators. Then by (4.3.7), for $x \in \mathcal{A}^{2d}_{\Theta^n}(nq)$, $n \in \overline{\mathbb{N}}$, $|||x||| = \max\{\|\delta^c(x)\|, \|\delta^r(x)\|\}$. Choose the multiplier $\varphi^n_{k,\eta}$ on $\mathbb{Z}^d_n$ as $\phi^n_{k,\eta}$ in the proof of Theorem 6.1.7. Note that

$$\|\delta^c(x)\| = \sup_{k\geq 1, \eta>0} \|\delta^c(T_{\varphi^n_{k,\eta}}(x))\|.$$

But $T_{\varphi^n_{k,\eta}}$ is a finite rank map and δ^c is continuous on a fixed finite dimensional space. Similar argument holds true for $\|\delta^r(x)\|$. It follows that $|||\cdot|||$ is a lower semicontinuous Lip-norm. Therefore, by Definition 2.44 in [Lat15] and by the choice of F, $\mathcal{A}^{2d}_{\Theta}(n)$ and $\mathcal{A}^{2d}_{\Theta}$ are F-quasi-Leibniz quantum compact metric spaces. Here, in fact they are Leibniz quantum compact metric spaces. For notational convenience, we will write $\mathcal{A}^{2d}_{\Theta}(n)$ or even $\mathcal{A}_{\Theta}(n)$ for $\mathcal{A}^{2d}_{\Theta^n}(nq)$ in the following by abuse of notation.

Let $u^{\Theta}_1(n), ..., u^{\Theta}_{2d}(n)$ denote the generators of $\mathcal{A}^{2d}_{\Theta}(n)$ and $u^{\Theta}_1, ..., u^{\Theta}_{2d}$ denote the generators of $\mathcal{A}^{2d}_{\Theta}$. In the following let

$$l = (l_1, ..., l_{2d}) \in \mathbb{Z}^{2d}, \quad \lambda^{\Theta}_n(l) = u^{\Theta}_1(n)^{l_1} ... u^{\Theta}_{2d}(n)^{l_{2d}} \quad \text{and} \quad \lambda^{\Theta}(l) = (u^{\Theta}_1)^{l_1} ... (u^{\Theta}_{2d})^{l_{2d}}.$$

We understand that $\mathcal{A}^{2d}_{\Theta}(\infty) = \mathcal{A}^{2d}_{\Theta}$ and $u^{\Theta}_i(\infty) = u^{\Theta}_i$, for $1 \leq i \leq 2d$, are the generators of $\mathcal{A}^{2d}_{\Theta}$.

Lemma 6.2.3. *Let $m > 0$ and ψ be the length function associated with the heat semigroup that was introduced previously. There exists a constant $C = C(m, \psi)$ such that for $n > 2m$ (including $n = \infty$) and all $y \in \mathcal{K} \otimes \mathcal{A}^{\Lambda^{2d}_m}_{\Theta}(n)$, we have $\|\mathring{y}\|_{\mathcal{K} \otimes \mathcal{A}^{\Lambda^{2d}_m}_{\Theta}(n)} \leq C |||\mathring{y}|||_{\mathcal{K} \otimes \mathcal{A}^{\Lambda^{2d}_m}_{\Theta}(n)}$.*

Proof. Recall from Section 4.3 the definition of $\nabla_p(\mathcal{A}^{2d}_{\Theta}(n))$. For $x \in S_p(\nabla_p(\mathcal{A}^{2d}_{\Theta}(n)))$, by (4.3.6) we have

$$\|x\|_{S_p(\nabla_p(\mathcal{A}^{2d}_{\Theta}(n)))} = \max\{\|\Gamma(x, x)^{1/2}\|_p,\ \|\Gamma(x^*, x^*)^{1/2}\|_p\}.$$

Let $p = 2$ and $x = \sum_k a_k \otimes \lambda^{\Theta}_n(k) \in S_2(\nabla_2(\mathcal{A}^{2d}_{\Theta}(n)))$. Then we have

$$\|x\|^2_{S_2(\nabla_2(\mathcal{A}^{2d}_{\Theta}(n)))} = \sum_k \|a_k\|^2_{S_2} \psi(k).$$

Similar to (4.3.10), we define for fixed k,

$$\phi : \nabla_2(\mathcal{A}_\Theta^{2d}(n)) \to \mathbb{C}, \quad \sum_l a_l \lambda_n^\Theta(l) \mapsto a_k \psi(k).$$

Then we have $\|\phi\|_{\text{cb}} = \|\phi\| \leq 1$. Note that by [Pis98]*Lemma 1,7, we have

$$\|\phi\|_{\text{cb}} = \| \operatorname{id}_{S_2} \otimes \phi : S_2(\nabla_2(\mathcal{A}_\Theta^{2d}(n))) \to S_2\| = \| \operatorname{id}_{\mathcal{K}} \otimes \phi : \mathcal{K} \otimes_{\min} \nabla_2(\mathcal{A}_\Theta^{2d}(n)) \to \mathcal{K}\|.$$

Hence, we have for $x = \sum_k a_k \otimes \lambda_n^\Theta(k) \in \mathcal{K} \otimes_{\min} \nabla_2(\mathcal{A}_\Theta^{2d}(n))$,

$$\sup_k \psi(k)^{1/2} \|a_k\|_{\mathcal{K}} \leq \|x\|_{\mathcal{K} \otimes_{\min} \nabla_2(\mathcal{A}_\Theta^{2d}(n))}. \tag{6.2.1}$$

Let $y = \sum_{k \in \Lambda_m^{2d}} b_k \lambda_n^\Theta(k) \in \nabla_2^0(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n))$, where $\nabla_2^0(\mathcal{A}_\Theta^{2d}(n))$ consists of the mean-zero elements of $\nabla_2(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n))$. Define a map $\nu : \nabla_2^0(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n)) \to \ell_\infty(\Lambda_m^{2d})$ by $\nu(y) = (b_k)_{k \in \Lambda_m^{2d}}$ and let μ be the inverse of η. We have the following chain of maps

$$(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n), |||\cdot|||) \xrightarrow{\text{id}} \nabla_2^0(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n)) \xrightarrow{\nu} \ell_\infty(\Lambda_m^{2d}) \xrightarrow{\mu} (\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n), \|\cdot\|).$$

By Corollary 4.3.5, we have $\| \operatorname{id} : (\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n), |||\cdot|||) \to \nabla_2^0(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n))\|_{\text{cb}} \leq c$ for some constant c. We deduce from (6.2.1) that

$$\|\nu : \nabla_2^0(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n)) \to \ell_\infty(\Lambda_m^{2d})\|_{\text{cb}} \leq \frac{1}{\inf_{\substack{k \in \Lambda_m^{2d} \\ \psi(k) \neq 0}} \psi(k)^{1/2}}.$$

Moreover, $\|\mu : \ell_\infty(\Lambda_m^{2d}) \to \mathcal{A}_\Theta^{\Lambda_m^{2d}}(n)\|_{\text{cb}} \leq (2m+1)^d$, since the cardinality of Λ_m^{2d} is $(2m+1)^d$. This proves that $\|\mu \circ \nu \circ \operatorname{id}\|_{\text{cb}} \leq C$, for $n > 2m$ and some $C = C(m, \psi)$. □

Suppose $\varepsilon > 0$, $k \in \mathbb{N}$ and $\varphi_{k,\eta}^n$ is the multiplier on $\mathbb{Z}_n^d$ chosen as $\phi_{k,\eta}^n$ in the proof of Theorem 6.1.7, which is supported on Λ_m^{2d}. We define the following multipliers for $n > 2m, n \in \overline{\mathbb{N}}$

$$T_{\varphi_{k,\eta}^n}(\lambda_n^\Theta(l)) = \varphi_{k,\eta}^n(l) \lambda_n^\Theta(l),$$

such that for $n > 2m$ we have

$$\|T_{\varphi_{k,\eta}^n} : (\mathcal{A}_\Theta^d(n), \|.\|) \to (\mathcal{A}_\Theta^d(n), \|.\|)\|_{\text{cb}} \leq 1 + \eta,$$

and

$$\|T_{\varphi_{k,\eta}} : (\mathcal{A}_\Theta^d(n), |||.|||) \to (\mathcal{A}_\Theta^d(n), |||.|||)\|_{\mathrm{cb}} \le 1+\eta,$$

for all $n > 2m, n \in \overline{\mathbb{N}}$.

Corollary 6.2.4. *There exists $N > 0$ such that the identity map $(\mathcal{A}_\Theta^{2d}(n), |||.|||) \to (\mathcal{A}_\Theta^{2d}(n), \|.\|)$ is completely bounded uniformly for $n > N$ including $n = \infty$.*

Proof. Let $\varepsilon > 0$. By Theorem 5.2.12, there exist $\eta < \varepsilon$, $k = k(\varepsilon)$ and a multiplier $\varphi_{k,\eta}^n$ and $N > 0$ such that $\|T_{\varphi_{k,\eta}^n}\|_{\mathrm{cb}} \le 1+\eta$. For $n > N$, we have

$$\|\operatorname{id} - T_{\varphi_{k,\eta}^n} : (\mathcal{A}_\Theta^{2d}(n), |||.|||) \to (\mathcal{A}_\Theta^{2d}(n), \|.\|)\|_{\mathrm{cb}} \le \varepsilon.$$

Let $\varepsilon = 1$ and $\operatorname{supp}(\varphi_{k,\eta}^n) = \Lambda_m^{2d}$, for some $m = m(k,\eta)$ independent of n. Then by Lemma 6.2.3, we have

$$\|(\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n), |||.|||) \to (\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n), \|.\|)\|_{\mathrm{cb}} \le C(m,\psi),$$

where $C(m,\psi)$ is the constant in the Lemma. Then for the maps $\operatorname{id} : (\mathcal{A}_\Theta^{2d}(n), |||.|||) \to (\mathcal{A}_\Theta^{2d}(n), \|.\|)$, $T_{\varphi_{k,\eta}^n} : (\mathcal{A}_\Theta^{2d}(n), |||.|||) \to (\mathcal{A}_\Theta^{2d}(n), |||.|||)$, we have

$$\begin{aligned}
\|\operatorname{id}\|_{\mathrm{cb}} &\le \|\operatorname{id} - T_{\varphi_{k,\eta}^n}\|_{\mathrm{cb}} + \|T_{\varphi_{k,\eta}^n}\|_{\mathrm{cb}} \\
&\le 1 + \|\operatorname{id}|_{\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n)} : (\mathcal{A}_\Theta^{2d}(n), |||.|||) \to (\mathcal{A}_\Theta^{2d}(n), \|.\|)\|_{\mathrm{cb}} \|T_{\varphi_{k,\eta}^n}\|_{\mathrm{cb}} \\
&\le 1 + C(m,\psi)(1+\varepsilon).
\end{aligned}$$

Hence

$$\sup_n \|\operatorname{id} : (\mathcal{A}_\Theta^{2d}(n), |||.|||) \to (\mathcal{A}_\Theta^{2d}(n), \|.\|)\|_{\mathrm{cb}} \le c,$$

for some constant c independent of n. □

Let n_j be the subsequence we found in the proof of Proposition 6.1.5. Then we have $C^*(\rho_{n_j}^\Theta(\mathcal{A}_\Theta^{2d})) = M_{n_j^d}$, and $\mathcal{A}_\Theta^{2d}(n_j) = M_{n_j^d}$. Let B_{n_j} and B_∞ denote the spaces $\mathcal{A}_\Theta^{2d}(n_j)$ and $\mathcal{A}_\Theta^{2d}$, respectively. In the following we use the index n instead of n_j, for simplicity. For any $m > 0$, let B_n^m and B_∞^m denote the subspaces $\mathcal{A}_\Theta^{\Lambda_m^{2d}}(n)$ and $\mathcal{A}_\Theta^{\Lambda_m^{2d}}$, respectively.

In the following, we consider the vector space

$$\mathrm{Poly} = \bigcup_{k \ge 1} \{p : p = \sum_{|i_1|,\ldots,|i_{2d}| \le k} a_{i_1 \ldots i_{2d}} x_1^{i_1} \ldots x_{2d}^{i_{2d}}\}.$$

For simplicity we denote an element $x = \sum_{|i_1|,...,|i_{2d}| \leq k} a_{i_1 ... i_{2d}} x_1^{i_1} ... x_{2d}^{i_{2d}} \in \text{Poly}$ by $\sum_{i \in \Lambda_k^{2d}} a_i x(i)$. Let g_1,..., g_{2d} denote the generators of the full group C*-algebra $C^*(\mathbb{F}_{2d})$. Define the following *-homomorphisms

$$\sigma_\Theta^n : C^*(\mathbb{F}_{2d}) \to B_n, \quad \sigma_\Theta : C^*(\mathbb{F}_{2d}) \to B_\infty$$

by $\sigma_\Theta(g_i) = u_i^\Theta$ and $\sigma_\Theta^n(g_i) = u_i^\Theta(n)$, for $1 \leq i \leq 2d$ and $n \in \mathbb{N}$. Then we get a *-homomorphism

$$\sigma_\Theta^\bullet : C^*(\mathbb{F}_{2d}) \to \prod_n B_n = M_\infty$$

defined by $\sigma_\Theta^\bullet = (\sigma_\Theta^n)_n$. Note that $I = c_0(\{B_n\})$ is an ideal in M_∞. Hence we get the quotient map $q : M_\infty \to M_\infty / I$. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Since by Proposition 6.1.6, $\{(B_n)_{n \in \overline{\mathbb{N}}}\}$ is a continuous field over $\overline{\mathbb{N}}$, we have

$$\|q \circ \sigma_\Theta^\bullet(x)\| = \|\sigma_\Theta(x)\|. \tag{6.2.2}$$

Define $\hat{B} = q \circ \sigma_\Theta^\bullet(C^*(\mathbb{F}_{2d}))$. By (6.2.2), since norms on $\hat{B}$ and B_∞ coincide, $\hat{B}$ is isomorphic to B_∞. Let

$$B = q^{-1}(\hat{B}) = \{\sigma_\Theta^\bullet(a) + z : a \in C^*(\mathbb{F}_{2d}), z \in I\}.$$

B is the C*-algebra generated by $c_0(\{B_n\})$ and $\sigma_\Theta^\bullet(C^*(\mathbb{F}_{2d}))$. Then B is a $C(\overline{\mathbb{N}})$-algebra with fiber maps

$$\eta_n : B \to B_n \quad \text{and} \quad \eta_\infty = q|_B : B \to B_\infty,$$

where η_n is the projection of B onto B_n. That is, $\eta_n((x_j)_j + z) = x_n$, for $(x_j)_j \in M_\infty$ and $z \in I$, and $\eta_\infty(\sigma_\Theta^\bullet(x) + y) = \sigma_\Theta(x)$, for $x \in C^*(\mathbb{F}_{2d})$, $y \in I$. Then the following sequence is exact

$$0 \to I \to B \to \hat{B} \cong B_\infty \to 0.$$

Note that both I and B_∞ are nuclear C*-algebras (recall that B_∞ is an iterative cross product). Hence B is nuclear (see Proposition 10.1.3 in [BO08]). Therefore, similar to Theorem 6.1.1, by the aforementioned result of Kirchberg and Blanchard (see Theorem 3.2 in [Bla97]), there exists a Hilbert space $\mathcal{H}$ and a *-homomorphism $\pi : B \to C(\overline{\mathbb{N}}) \otimes \mathcal{B}(\mathcal{H})$. Note that π maps $C(\overline{\mathbb{N}})$ to $C(\overline{\mathbb{N}})$ canonically. Let $\iota_n : B_n \to B$ be defined by $\iota_n(y) = (y_l)_{l\in\mathbb{N}}$, where

$$y_l = \begin{cases} y & l = n, \\ 0 & \text{else} \end{cases}$$

for $y \in B_n$. Therefore, we get the following maps

$$\pi_n : B_n \to C(\overline{\mathbb{N}}) \otimes \mathcal{B}(\mathcal{H}), \quad \sigma : C^*(\mathbb{F}_{2d}) \to B \quad \text{and} \quad \pi_\infty : B_\infty \to C(\overline{\mathbb{N}}) \otimes \mathcal{B}(\mathcal{H})$$

given by $\pi_n = \pi \circ \iota_n$, $\sigma = \iota_n \circ \sigma^n_\Theta$ and define $\pi_\infty(x)$ as follows. For any $a \in \text{Poly}$, let $a_n = \pi_n(\sigma^n_\Theta(a))$. Then $(a_n)_n$ is a convergent sequence in $C(\overline{\mathbb{N}})\otimes\mathcal{B}(\mathcal{H})$. Hence $\lim_n a_n = a_\infty$, for some a_∞. Define $\pi_\infty(\sigma_\Theta(a)) := a_\infty$. The following diagram summarizes this argument

$$\begin{tikzcd}
& & M_\infty \arrow[rr, "q"] & & M_\infty/I & & \\
& & & C^*(\mathbb{F}_{2d}) \arrow[ul, "\sigma_\Theta^\bullet"'] \arrow[ur, "q\circ\sigma_\Theta^\bullet"] \arrow[dl, "\sigma_\Theta^\bullet"'] \arrow[dr, "q\circ\sigma_\Theta^\bullet"] & & & \\
0 \arrow[r] & I \arrow[r] & B \arrow[uu, hook] \arrow[rr] & & \hat{B} \arrow[uu, hook] \arrow[r, equal] & B_\infty \arrow[r] & 0
\end{tikzcd}$$

where the last row is short exact.

Lemma 6.2.5. *With the notation above, the following hold*

1. $\lim_{n\to\infty} \|\pi_n(\lambda^\Theta_n(l)) - \pi_\infty(\lambda^\Theta(l))\| = 0,$

2. *Let $\varepsilon > 0$ and $m \in \mathbb{N}$. There exists $N \in \mathbb{N}$, such that for all $n > N$ and $x = \sum_{l\in\Lambda^{2d}_m} a_l \otimes x(l) \in \mathcal{K}\otimes Poly$, we have*

$$\|\operatorname{id}\otimes(\pi_n \circ \sigma^n_\Theta)(x) - \operatorname{id}\otimes(\pi_\infty \circ \sigma_\Theta)(x)\|_{\mathcal{B}(\ell_2\otimes\mathcal{H})} < \varepsilon \sup_{l\in\Lambda^{2d}_m} \|a_l\|_{\mathcal{K}}.$$

Proof. To prove (1), let $g = g_1^{l_1}...g_{2d}^{l_{2d}}$ and π, σ, σ^n_Θ be as above. Let $(y_n)_n = \pi \circ \sigma(g) \in C(\overline{\mathbb{N}}) \otimes \mathcal{B}(\mathcal{H})$, for some $(y_n)_{n\in\overline{\mathbb{N}}} \in C(\overline{\mathbb{N}})$. Therefore, $\lim_n y_n = y_\infty$. Moreover, we have

$$y_n = \pi(\sigma(g)1_{\{n\}}) = \pi(\iota_n(\sigma^n_\Theta(g))) = \pi_n\sigma^n_\Theta(g),$$

where $1_{\{n\}}$ denotes the characteristic function of $\{n\}$. Since $\lim_{n\to\infty} y_n = y_\infty$ in $C(\overline{\mathbb{N}}) \otimes \mathcal{B}(\mathcal{H})$, this implies that

$$\lim_{n\to\infty} \|\pi_n(\sigma^n_\Theta(g)) - \pi_\infty(\sigma_\Theta(g))\| = 0.$$

Hence, we get

$$\lim_{n\to\infty} \|\pi_n(\lambda_n^{\Theta}(l)) - \pi_\infty(\lambda^{\Theta}(l))\| = 0,$$

for all $l \in \mathbb{Z}^d$.

To prove (2), let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{(2m+1)^{2d}}$. Using (1) and the triangle inequality, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$\begin{aligned}\|\operatorname{id}\otimes(\pi_n\circ\sigma_\Theta^n)(x) - \operatorname{id}\otimes(\pi_\infty\circ\sigma_\Theta)(x)\| &= \|\sum_{l\in\Lambda_m^{2d}} a_l\otimes(\pi_n(\lambda_n^{\Theta}(l)) - \pi_\infty(\lambda^{\Theta}(l)))\| \\ &\le (\sum_{l\in\Lambda_m^{2d}}\|a_l\|)\delta \le (2m+1)^{2d}\sup_{l\in\Lambda_m^{2d}}\|a_l\|\frac{\varepsilon}{(2m+1)^{2d}} = \varepsilon\sup_{l\in\Lambda_m^{2d}}\|a_l\|,\end{aligned}$$

which proves the assertion. □

In the following, let π_n and π_∞ be as defined above.

Theorem 6.2.6. *For every noncommutative torus $\mathcal{A}_\Theta^{2d}$, there exists a sequence of matrix algebras with suitable Lip-norms converging to $\mathcal{A}_\Theta^{2d}$ in the sense of strong Gromov–Hausdorff propinquity.*

Proof. Consider the bridge $\gamma = (\mathcal{B}(\ell_2\otimes\mathcal{H}), \operatorname{id}\otimes\pi_\infty, \operatorname{id}\otimes\pi_n, \operatorname{id})$. Then by Lemma 6.2.1, it suffices to show that there exists a subsequence n_j such that for any $\varepsilon > 0$ and the unitization of $\mathcal{K}\otimes B_\infty$ and $\mathcal{K}\otimes B_n$ (which we will also denote by $\mathcal{K}\otimes B_\infty$ and $\mathcal{K}\otimes B_n$, respectively), the following hold:

1. For all $a \in \mathcal{K}\otimes B_\infty$ such that $|||a||| \le 1$, there exists $b \in \mathcal{K}\otimes B_n$ such that $|||b||| \le 1$ and $\|\operatorname{id}\otimes\pi_\infty(a) - \operatorname{id}\otimes\pi_{n_j}(b)\| < \varepsilon$,
2. For all $b \in \mathcal{K}\otimes B_n$ such that $|||b||| \le 1$, there exists $a \in \mathcal{K}\otimes B_\infty$ such that $|||a||| \le 1$ and $\|\operatorname{id}\otimes\pi_\infty(a) - \operatorname{id}\otimes\pi_{n_j}(b)\| < \varepsilon$.

Let $\varepsilon > 0$. By Theorem 5.2.12, there exist $0 < \eta < \varepsilon$, $k = k(\varepsilon)$ and multipliers $\varphi_{k,\eta}^n$ on B_n supported on Λ_m^{2d}, for some $m = m(k,\eta)$ independent of n, such that $\|T_{\varphi_{k,\eta}^n}\|_{\mathrm{cb}} \le 1+\eta$ and for all $n > 2m$ we have

$$\|\operatorname{id} - T_{\varphi_{k,\eta}^n} : (B_n, |||.|||) \to (B_n, \|.\|)\|_{\mathrm{cb}} \le \frac{\varepsilon}{4}. \tag{6.2.3}$$

In the following, by abuse of notation, for all $n \in \overline{\mathbb{N}}$, we denote $\operatorname{id}\otimes T_{\varphi_{k,\eta}^n} : (\mathcal{K}\otimes B_n, |||\cdot|||) \to (\mathcal{K}\otimes B_n, |||\cdot|||)$ by $T_{\varphi_{k,\eta}^n}$. For any x in $\mathcal{K}\otimes B_n^m$ or $\mathcal{K}\otimes B_\infty^m$, let $\hat{x}$ denote the corresponding element in $\mathcal{K}\otimes\mathrm{Poly}$. Let $\delta < \frac{\varepsilon}{4C(m,\psi)}$, where $C(m,\psi)$ is the constant from Lemma 6.2.3. Using Lemma 6.2.5 (2) and Lemma 6.2.3, we can choose a subsequence n_j such that for all $x \in \mathcal{K} \otimes B_{n_j}$, by denoting $\tilde{x} = T_{\varphi_{k,\eta}^{n_j}}(\mathring{x})$, we have

$$\begin{aligned} &\| \operatorname{id} \otimes (\pi_{n_j} \circ \sigma_\Theta^{n_j})(\hat{\mathring{x}}) - \operatorname{id} \otimes (\pi_\infty \circ \sigma_\Theta)(\hat{\mathring{x}}) \|_{\mathcal{B}(\ell_2 \otimes \mathcal{H})} \\ &\qquad \leq \delta \sup_{l \in \Lambda_m^2} \|a_l\|_{\mathcal{K}} \leq \frac{\varepsilon}{4} |||\tilde{x}|||_{\mathcal{K} \otimes B_{n_j}^m}, \end{aligned} \tag{6.2.4}$$

where $\mathring{T}_{\varphi_{k,\eta}^{n_j}}(x)$ denotes the mean-zero part of $T_{\varphi_{k,\eta}^{n_j}}(x)$ and a_l are the coefficients of $T_{\varphi_{k,\eta}^{n_j}}(x) = \sum_{l \in \Lambda_m^{2d}} a_l \otimes \lambda^\Theta(l)$. From now on we abuse the notation and drop the index j of n_j.

To prove (1), let $a \in \mathcal{K} \otimes B_\infty$ such that $|||a||| \leq 1$. Let $x = T_{\varphi_{k,\eta}}(a) \in \mathcal{K} \otimes B_\infty^m$. Hence $\hat{x} \in \mathcal{K} \otimes \text{Poly}$. Let $b' = \operatorname{id} \otimes \sigma_\Theta^n(\hat{x}) \in \mathcal{K} \otimes B_n$. Then by (6.2.3), (6.2.4) we have

$$\begin{aligned} \| \operatorname{id} \otimes \pi_\infty(a) - \operatorname{id} \otimes \pi_n(b') \| &= \| \operatorname{id} \otimes \pi_\infty(\mathring{a}) - \operatorname{id} \otimes (\pi_n \circ \sigma_\Theta^n)(\hat{\mathring{x}}) \| \\ &\leq \| \operatorname{id} \otimes (\pi_\infty \circ \sigma_\Theta)(\hat{\mathring{a}}) - \operatorname{id} \otimes (\pi_\infty \circ \sigma_\Theta)(\hat{\mathring{x}}) \| \\ &+ \| \operatorname{id} \otimes (\pi_\infty \circ \sigma_\Theta)(\hat{\mathring{x}}) - \operatorname{id} \otimes (\pi_n \otimes \sigma_\Theta^n)(\hat{\mathring{x}}) \| \\ &\leq \| \mathring{a} - T_{\varphi_{k,\eta}}(\mathring{a}) \| + \frac{\varepsilon}{4} |||T_{\varphi_{k,\eta}}(\mathring{a})||| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4}(1 + \eta) \\ &\leq \frac{3\varepsilon}{4}. \end{aligned}$$

Let $b = \frac{b'}{|||b'|||}$. Then $|||b||| \leq 1$. Recall from Proposition 6.1.5 that for $\varepsilon' > 0$, the map $\rho_n^\Theta : B_\infty^m \to B_n^m$ is a $1 + \varepsilon'$ Lip-isometry. Let $\varepsilon' = \eta$. Note that $b' = \operatorname{id} \otimes \rho_n^\Theta(\sigma_\Theta(\hat{x}))$. Hence, $|||b'||| \leq (1 + \eta)|||x||| \leq (1 + \eta)^2$ and we have

$$\|b' - b\| = (\frac{|||b'||| - 1}{|||b'|||})\|b'\| \leq ((1 + \eta)|||x||| - 1)\frac{\|b'\|}{|||b'|||} \leq (\eta^2 + 2\eta)\frac{\|b'\|}{|||b'|||}.$$

By Corollary 6.2.4, $\frac{\|b'\|}{|||b'|||} \leq K$ for some $K > 0$. Therefore, if we choose η small enough, we have

$$\| \operatorname{id} \otimes \pi_n(b) - \operatorname{id} \otimes \pi_n(b') \| \leq \|b - b'\| \leq K(\eta^2 + 2\eta) \leq \frac{\varepsilon}{4},$$

which proves (1).

To prove (2), let a subsequence n_j which we will denote by n, be chosen as above. Let $b \in \mathcal{K} \otimes B_n$ be such that $|||b||| \leq 1$. Let $\mathring{b}$ denote the mean-zero part of b. Therefore, we can write $b = t_n 1 + \mathring{b}$. Let

$$b' = T_{\varphi_{k,\eta}^n}(\mathring{b}) = \sum_{0 \neq l \in \Lambda_m^{2d}} a_l \otimes \lambda_n^\Theta(l) \in \mathcal{K} \otimes B_n.$$

Then by (6.2.3), we have $\|\mathring{b} - b'\| \leq \frac{\varepsilon}{4}|||b|||$. Let $\hat{b'} = \sum_{l\in\Lambda_m^2} a_l \otimes x(l)$ be the corresponding element in $\mathcal{K} \otimes \mathrm{Poly}$. Choose $a' = \sigma_\Theta(\hat{b'}) \in \mathcal{K} \otimes B_\infty$. Then using (6.2.4), we get

$$\begin{aligned}
\| \mathrm{id} \otimes \pi_\infty(a') - \mathrm{id} \otimes \pi_n(b)\| &\leq \| \mathrm{id} \otimes (\pi_\infty \circ \sigma_\Theta)(\hat{b'}) - \mathrm{id} \otimes (\pi_n \circ \sigma_\Theta^n)(\hat{b'}))\| \\
&\quad + \| \mathrm{id} \otimes (\pi_n \circ \sigma_\Theta^n)(\hat{b'}) - \mathrm{id} \otimes (\pi_n \circ \sigma_\Theta^n)(\hat{b})\| \\
&\leq \frac{\varepsilon}{4}|||b'||| + \|b - b'\| \\
&\leq (\frac{\varepsilon}{4}(1+\eta) + \frac{\varepsilon}{4})|||b|||
\end{aligned}$$

Now let $a = \frac{a'}{|||a'|||}$. Similar to (1), using the fact that $(\rho_n^\Theta)^{-1}$ is a $1+\eta$ Lip-isometry, we get $\|a' - a\| \leq K(\eta^2 + 2\eta)$. Therefore, choosing η small enough, we get

$$\| \mathrm{id} \otimes \pi_\infty(a) - \mathrm{id} \otimes \pi_\infty(\tilde{a})\| \leq \|a - a'\| \leq K(\eta^2 + 2\eta) \leq \frac{\varepsilon}{4}.$$

Hence (2) follows.

(1) and (2) together with Lemma 6.2.1 prove that $\Lambda_F(\mathcal{K} \otimes B_\infty, \mathcal{K} \otimes B_n) < \varepsilon$, which proves the assertion. □

Remark 6.2.7. In Sections 4.4 and 5.1, we chose $p > 2$ for our estimates. Note that in the higher-dimensional case, the choice of p depends on the dimension of the rotation algebra and the choice of the semigroup.

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