COMPACTNESS OF THE SPACE OF MARKED GROUPS AND EXAMPLES OF L^2-BETTI NUMBERS OF SIMPLE GROUPS

BY

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THESIS

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ABSTRACT

This paper contains two parts. The first part will introduce $\mathcal{G}_n$ space and will show it's compact. I will give two proofs for the compactness, the first one is due to Rostislav Grigorchuk [1], which refers to geometrical group theory and after the first proof I will give a more topological proof. In the second part, our goal is to prove a theorem by Denis Osin and Andreas Thom [2]: for every integer $n \geq 2$ and every $\epsilon \geq 0$ there exists an infinite simple group Q generated by n elements such that $\beta_1^{(2)}(Q) \geq n - 1 - \epsilon$. As a corollary, we can prove that for every positive integer n there exists a simple group Q with $d(Q) = n$. In the proof of this theorem, I added the details to the original proof. Moreover, I found and fixed an error of the original proof in [2], although it doesn't affect the final result.

To my parents, for their love and support.

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CHAPTER 1

COMPACTNESS OF THE SPACE OF MARKED GROUPS

In this part, I will show the space of marked groups ($\mathcal{G}_n$) (the definition will be showed in section 1.2.2) is compact, which is originally proved by Rostislav Grigorchuk from his paper Degrees of growth of finitely generated groups. And I will introduce two proofs, the first proof is from Rostislav Grigorchuk [1], which refers to geometric group theory and after the first proof I will give a more topological proof. Before the first proof, I need to introduce a Lemma and a Theorem from Magnus's book [3], which shows when a graph is isomorphic to the graph of a group.

1.1 Magnus Lemma and Theorem

First we need to have some definitions:

1.1.1 Singular graph, connected graph with a regular colouring of n colours and orientation M

A graph that consists of a single point, which is the initial and end point of all edges, is called *singular*. The singular graph with $2n$ edges:$s_1, {s_1}^{-1}, s_2, {s_2}^{-1}, ...s_n, {s_n}^{-1}$ is denoted by S_n.

A graph Γ is called *connected* if, for any two of the points, A, B, there exists a path whose initial point is A and whose end point is B.

A graph can be coloured with n colours and oriented" by associating with each of its edges, an edge of S_n. This leads to the following definition:

A *colouring* (*with n colours*) *and orientation* of a graph Γ is a mapping M of the edges of Γ into the edges of S_n with the following properties:

$(A1)$ For each point P in Γ, the edges of Γ with initial point P are mapped bijectively to all edges of S_n by M, which means that one edge of every colour and orientation begins at each point P of Γ.

$(A2)$ For each edge E in Γ, $M(E^{-1}) = [M(E)]^{-1}$, which means that colour of (E^{-1}) is the same with E, while the orientation is reversed.

Clearly, a graph of the group on n generators $a_1, a_2, ...a_n$ has a colouring and orientation, i.e. maps the edge $(g_1, g_2; {a_i}^{\epsilon})$ into ${s_i}^{\epsilon}$, where $g_2 = g_1{a_i}^{\epsilon}$. Also note, here ϵ=1 or -1 determines the orientation and g_2, g_2 are the edge's initial and end point which corresponds to two elements in the group.

If M is a colouring and orientation of Γ and $\pi = E_1...E_n$ is a path in Γ, we define $M(\pi) = M(E_1)...M(E_n)$, and we say that the path π *covers* the path $M(\pi)$.

A colouring and orientation M of the graph Γ is called regular if for any two paths π, π' of Γ such that $M(\pi) = M(\pi')$, π is closed if and only if π' is closed.

1.1.2 The statements and proofs

Lemma 1.1.1 ([3], Lemma 1.1) If M is a regular colouring (with n colours) and orientation of Γ, then for each point P in Γ, M is a bijection between the paths in Γ with initial point P and all paths in S_n(i.e words in $s_1, s_2, ...s_n$)

Proof: We need to show to any two paths π', π with the same initial point P, s.t. $M(\pi) = M\pi'$ then π'=π. And if σ=${s_{v_1}}^{\epsilon_1}...{s_{v_r}}^{\epsilon_r}$ is any path in S_n, then there exits a path $\pi = E_1...E_n$ in Γ with initial point P s.t. $M(\pi) = \sigma$. Note by definition, M preserves the number of edges in a path, thus here we may assume that $\pi = E_1...E_r$, $\pi' = {E_1}'...{E_r}'$. To prove the results, we need to take the induction on r. When r=1, the results are straight from $(A1)$. Now we assume both results hold for r.

Assume $M(E_1...E_rE,\ \pi') = M({E_1}'...{E_r}'E')$, then $M(E_1) = M({E_1}')$, since we defined $M(E_1...E_n) = M(E_1)...(E_n)$. So we can get $M(E_2...E_rE,\ \pi') = M({E_2}'...{E_r}'E')$. Thus by induction, we get $E_1 = {E_1}'$. Also, by the paths expression $E_1...E_rE$, ${E_1}'...{E_r}'E'$, we know the initial points of $E_2...E_rE, {E_2}'...{E_r}'E'$ are the endpoints of $E_1, {E_1}'$ respectively, i.e. they are the same. In this sense, we can apply the induction on r, then we can get $E_2...E_rE = {E_2}'...{E_r}'E'$, therefore $E_1...E_rE = {E_1}'...{E_r}'E'$.

Now suppose σ=${s_{v_1}}^{\epsilon_1}...{s_{v_r}}^{\epsilon_r}{s_v}^{\epsilon}$. By induction, there exists a path π=$E_1...E_r$ in Γ with initial point P. If Q is the end point of π, by $(A1)$, we can find an edge E with initial point Q which covers ${s_v}^{\epsilon}$. Therefore, $E_1...E_rE$ is the required path π in Γ with initial point P, s.t. $M(E_1...E_rE) = \sigma$. $\square$

Theorem 1.1.2 ([3], Theorem 1.6) Let Γ be a connected graph with a regular colouring of n colours and orientations M. Then Γ is isomorphic to the graph of a group G of n generators $a_1, ...a_n$.

Proof: First we construct the group G, by giving the presentation with generators $a_1, a_2, ...a_n$. And to any word $W(a_1, a_2, ...a_n)$, there is a corresponding word $W(s_1, s_2, ...s_n)$ from S_n, i.e. replace each a_i with the corresponding s_i, and define a word $W(a_1, a_2, ...a_n)$ as a relation if the path π in Γ which covers the corresponding $W(s_1, s_2, ...s_n)$ of $W(a_1, a_2, ...a_n)$ is closed (here, in particular, we define the path π that covers $W(s_1, s_2, ...s_n)$ with initial point P_0 in Γ, as the corresponding path of the word $W(a_1, a_2, ...a_n)$). Now define G is the group of the given presentation.

Now we need to show the graph of G is isomorphic to Γ. First we need to define a map ϕ that maps the elements of G onto the points of Γ, and the edges $(g_v, g_l; {a_i}^{\epsilon})$ onto the edges of Γ. For this purpose, let $g_v \in G$ defined by a word $W_v(a_1, a_2, ...a_n)$, and let π_v be the corresponding path of $W_v(a_1, a_2, ...a_n)$; then we define the map ϕ by $\phi(g_v) := P_v \in \Gamma$, where P_v is the end point of the path π_v, and since the images of g_k, g_l are determined, the image of the edge $(g_v, g_l; {a_i}^{\epsilon})$ of ϕ in Γ is determined, i.e. the path with initial point P_v that covers ${s_i}^{\epsilon}$.

We need to show ϕ is well-defined. If ${W_v}'(a_1, a_2, ...a_n)$ is another word defining g_v. Note ${W_v}'(a_1, a_2, ...a_n)$ can be obtained from $W_v(a_1, a_2, ...a_n)$ by inserting or deleting a finite number of relations $R(a_1, a_2, ...a_n)$ or trivial relations. Therefore, we need to show the paths corresponding to the words: $K(a_1, a_2, ...a_n)T(a_1, a_2, ...a_n)$, $K(a_1, a_2, ...a_n){a_\lambda}^{\epsilon}{a_\lambda}^{-\epsilon}T(a_1, a_2, ...a_n)$, $K(a_1, a_2, ...a_n)R(a_1, a_2, ...a_n)T(a_1, a_2, ...a_n)$ have the same end point.

Let's first consider the path π corresponding to the product $U(a_1, ...a_n)V(a_1, ...a_n)$, it can be obtained as follows: Let π_1 be the path with initial point P_0 covering $U(s_1, s_2, ...s_n)$, and let π_2 be the path with initial point at the end point of π_1 and covering $V(s_1, s_2, ...s_n)$; then $\pi = \pi_1\pi_2$. In this sense, we can define the corresponding path of $K(a_1, a_2, ...a_n)$
$T(a_1, a_2, ...a_n)$ to be ${\pi_1}'{\pi_2}'$, and name the end point of π_1 as Q. Note $R(a_1, ...a_n)$ and ${a_\lambda}^{\epsilon}{a_\lambda}^{-\epsilon}$ are relations, thus by what we defined above, the corresponding $R(s_1, ...s_n)$ and ${s_\lambda}^{\epsilon}{s_\lambda}^{-\epsilon}$ are covered by closed paths with initial point P_0. And since Γ is regular, so the paths π_1, π_2 with initial point Q that cover $R(s_1, s_2, ...s_n)$ and ${s_\lambda}^{\epsilon}{s_\lambda}^{-\epsilon}$ respectively are closed. Also, it's easy to see, by the way we showed above, we can obtain that the paths corresponding to $K(a_1, a_2, ...a_n)R(a_1, a_2, ...a_n)T(a_1, a_2, ...a_n)$ and $K(a_1, a_2, ...a_n){a_\lambda}^{\epsilon}{a_\lambda}^{-\epsilon}$
$T(a_1, a_2, ...a_n)$ are ${\pi_1}'\pi_1{\pi_2}'$, ${\pi_1}'\pi_2{\pi_2}'$ respectively. Therefore the corresponding paths of $K(a_1, a_2, ...a_n)T(a_1, a_2, ...a_n)$, $K(a_1, a_2, ...a_n){a_\lambda}^{\epsilon}{a_\lambda}^{-\epsilon}T(a_1, a_2, ...a_n)$, $K(a_1, a_2, ...a_n)$
$R(a_1, a_2, ...a_n)T(a_1, a_2, ...a_n)$ have the same end point, hence the map π is well-defined.

Now we focus on showing that π is an isomorphism. First we can show π is onto. Since Γ is connected, there is a path π from P_0 to any point, namely, P of Γ, by Lemma 1.1, we can get π covers some word $W(s_1, s_2, ...s_n)$, then the element $g \in G$ defined by $W(a_1, a_2, ...a_n)$ is mapped to P. Now we can show π is injective. Let $g_v, g_v{}'$ be mapped to the same point P of Γ. Let $W_v(a_1, a_2, ...a_n)$, $W_v{}'(a_1, a_2, ...a_n)$ are the words defining $g_v, g_v{}'$, with corresponding paths π_v, $\pi_v{}'$. Then $\pi_v{}'\pi_v{}^{-1}$ is a closed path with initial point P_0. Hence by the definition of relation, $W_v{}'(a_1, a_2, ...a_n)W_v(a_1, a_2, ...a_n)^{-1}$ is a relation in G, therefore $g_v = g_v{}'$, that finishes the proof. $\square$

1.2 Grigorchuk's proof that $\mathcal{G}_n$ is compact

1.2.1 Topology from neighbourhoods

A neighbourhood topology on a set X assigns to each element $x \in X$ a non empty set $\mathcal{N}(x)$ of subsets of X, called neighbourhoods of x, with the properties:

1. If N is a neighbourhood of x then $x \in N$.
2. If M is a neighbourhood of x and $M \subseteq N \subseteq X$, then N is a neighbourhood of x.
3. The intersection of two neighbourhoods of x is a neighbourhood of x.
4. If N is a neighbourhood of x, then N contains a neighbourhood M of x such that N is a neighbourhood of each point of M.

1.2.2 Topology of $\mathcal{G}_n$

Let $\mathcal{G}_n$ be the set consisting of all pairs of the form (G, S), where G is a group and $S = \{a_1, ...a_n\}$ is the generating set with n elements of G. Define the neighbourhoods $N_n(G, S)$ of point (G, S), by defining $N_n(G, S)$ to be the set of pairs (G_α, S_α), where S_α=$\{a_1{}^{(\alpha)}, ...a_n{}^{(\alpha)}\}$ and the map 1$\to$1,$a_1 \to a_1{}^{(\alpha)}$,...$a_n \to a_n{}^{(\alpha)}$ extends to a bijective map θ of the ball of radius m in the group G onto the ball of radius m in the group G_α, with

$$(1)\theta(g_1g_2) = \theta(g_1)\theta(g_2)$$

,

for any two elements $g_1, g_2 \in G$, s.t. $length(g_1) + length(g_2) \leq$m, where the ball of radius m is the set of elements of length $\leq$m in the group. Note, here $N_{n+1}(G,S) \subset N_n(G,S)$, and if $(G',S') \in N_n(G,S)$, then $(G,S) \in N_n(G',S')$ and $N_n(G',S') = N_n(G,S)$, thus as $N_n(G,S)$ is a neighbourhood of (G,S), and $N_n(G,S)$ contains a neighbourhood $N_{n+1}(G,S)$ of (G,S), and to any $(G',S') \in N_{n+1}(G,S)$, we can get $(G',S') \in N_{n+1}(G,S) \subset N_n(G,S)$, which implies $N_n(G',S') = N_n(G,S)$ by above, thus $N_n(G,S)$ is a neighbourhood of (G',S'). That verifies the 4th property of Topology from neighbourhoods.) The topology so defined is called the weak topology. The weak topology in the space $\mathcal{G}_n$ can be described by metric:

$$d(G_1,G_2) := \sum_{n=1}^{\infty} \frac{1}{2^n}(d(B_1(n), B_2(n))$$

where $B_1(n), B_2(n)$ are the balls of radius n, and where $d(B_1, B_2) = 0$ if the map $1 \to 1$,${a_1}^{(1)} \to {a_1}^{(2)}$,...${a_n}^{(1)} \to {a_n}^{(2)}$ extends to a bijective map θ satisfies (1), while $d(G_1, G_2) = 1$ if there is no such extension.

1.2.3 $\mathcal{G}_n$ is compact

We want to show $\mathcal{G}_n$ is compact by showing it's subsequently compact. Let G_n, n=1,2,3.. be a sequence of groups with each group's generating set contains n elements. If there is an element c such that $|G_n| < c$ for all n. Then the existence of a convergent subsequence is obvious.

Now we assume there's no such a c. Call $B_{i_1}(r)$,$B_{i_2}(r)$ isomorphic if there is a bijection θ satisfies $\theta(g_1g_2) = \theta(g_1)\theta(g_2)$. Among balls $B_n(1)$,n=1,2,3... then there exists infinitely many isomorphism between each other, where $B_n(r)$ is the set of elements of length $\leq$ r in the group G_n. Let $n_1(i)$ be a sequence of numbers, for each, the balls with radius 1 i.e. $B_{n_1(i)}(1)$are isomorphic. (Note $\{G_n\}_{n=1,2,3...}$ has infinite elements, while the cardinality of the generating sets is fixed as n, which means the number of combinations of reduced words with length $\leq$l is finite up to isomorphism, thus such a sequence always exists.) From this subsequence, with the same reason, we can pick up a subsequence, call it as $\{n_2(i)\}$, for which the balls $B_{n_2(i)}(2)$ are isomorphic.Repeat it by induction, we can find a subsequence $n_{k+1}(i)$ of sequence $n_k(i)$ and the balls $B_{n_{k+1}(i)}(k+1)$ are isomorphic for i=1,2...

Denote $\Gamma(G_{n_{k(i)}})$ for the subgraph of the Cayley graph of $G_{n_{k(i)}}$(for all i) that consists of the vertices at distance at most k from identity.

Consider the graph with a chosen vertex e (namely Γ), s.t. for any k, its subgraph consisting of the vertices at the distance at most k from e is isomorphic to $\Gamma(G_{n_{k(i)}})$ (for any i), i.e. the graph Γ is just $lim_{k\to\infty}\Gamma(G_{n_{k(k)}})$. For this graph, the condition of Theorem 1.1.2 is satisfied, therefore Γ is a graph of a certain group G with the cardinal number of generating set n. Now we can see Γ as the Cayley graph of G, i.e. Γ= $\Gamma(G)$. So we can get $lim_{k\to\infty}\Gamma(G_{n_{k(k)}})$=$\Gamma$= $\Gamma(G)$. Since the subgraph of Γ ($\Gamma(G)$) consisting of the vertices at the distance at most k from e is isomorphic to $\Gamma(G_{n_{k(k)}})$, we can get that $G \cap B_G(k)$ is isomorphic to $G_{n_{k(k)}} \cap B_{G_{n_{k(k)}}}(k)$, where $B_G(k), B_{G_{n_{k(k)}}}(k)$ are the balls of radius k of G and $G_{n_{k(k)}}$ respectively. Therefore we can get: $lim_{k\to\infty}G_{n_{k(k)}} \cap B_G(k) = lim_{k\to\infty}G_{n_{k(k)}}$ is isomorphic to $lim_{k\to\infty}G \cap B_G(k) = G$. So we conclude $lim_{k\to\infty}G_{n_{k(k)}} = G$, that finishes the poof. $\square$

1.3 A more topological proof

Here is a more topological proof to show $\mathcal{G}_n$ is compact, which doesn't refer to geometric group theory.

Proof: $\mathcal{G}_n$ is the set consisting of all pairs of the form (G, S), thus it induces 1-to-1 epimorphisms: $F_n \to G$, where F_n is the free group of rank n, thus $\mathcal{G}_n$ can be identified with the set of all normal subgroups of F_n. We can define the metric of any two subsets A, C of F_n by:
$d(A,C) = inf\{2^{-k} | k \in Z^* : A \cap B_{F_n}(k) = C \cap B_{F_n}(k)\}$. Note, since S is ordered, we can make an order of the elements in F_n lexicographically by the order of S. Therefore, since A is an elements of 2^{F_n}, it can be seen in the form of $(A(1), A(2)...A(n)...)$, where $A(i)$ equals to 0 or 1, here 1 means the i^{th} element of F_n is in A, and 0 means the i^{th} element of F_n is not in A. And the topology on 2^{F_n} is the product topology of the product space of the discrete space {0,1}.

Lemma 1: The topology induced by the metric showed above on 2^{F_n} is the same as the direct topology on 2^{F_n}.
Proof: Any $G \subseteq F_n$, can be seen as $(G(1), G(2)...G(n)...) \in 2^{F_n}$, where $G(i)$ is 1 or 0. And the open ball centred in G of radius k, induced by the metric is: $B(G,k) = \{H \in F_n | H \cap B_{F_n}(k) = G \cap B_{F_n}(k)\}$. Now assume the elements of $B_{F_n}(k)$ are from the first one of F_n to the m^{th} element of F_n by order defined above. Therefore $B(G,k) = \{G(1)\} \times \{G(2)\} \times ... \times \{G(m)\} \times \{0,1\} \times \{0,1\}...$ which is open in the product topology since there are only m ones that are not $\{0,1\}$ in the product. Therefore any open set in the sense of the topology induced by the metric is also open in the sense of direct topology on 2^{F_n}.
By definition to any element K of basis of 2^{F_n} is in the form of $\prod_{i \in I} U_i$, where each U_i is open in $\{0,1\}$ and $U_i \neq \{0,1\}$ for only finitely many i. So we may assume the last $U_i \neq \{0,1\}$ is U_t. Note $U_1 \times U_2 \times ... \times U_t$ is a finite union of product of t single points i.e. $\prod_{i=1}^{t} U_i = \bigcup_{j \in J} \prod_{i=1}^{t} \{H_j(i)\}$, where each $H_j(l)$ equals to 0 or 1. So $K = \bigcup_{j \in J} \prod_{i=1}^{t} \{H_j(i)\} \times \{0,1\}^{\infty}$. Now assume the elements of $B_{F_n}(r)$ are from the

first one of F_n to the q^{th} element of F_n, where $q > t$. And

$$\begin{aligned}\prod_{i=1}^{t}\{H_j(i)\} \times \{0,1\}^{\infty} &= \prod_{i=1}^{t}\{H_j(i)\} \times \{0,1\}^{q-t} \times \{0,1\}^{\infty} \\ &= \bigcup_{v=1,2\ldots q-t}\prod_{i=1}^{t}\{H_j(i)\} \times \prod_{n=1}^{q-t}\{H_j(j,n)\} \times \{0,1\}^{\infty}\end{aligned}$$

where $\{H_j(j,n)\}$ =0 or 1.
So K can be seen as another union:

$$\bigcup_{j\in J}\bigcup_{v=1,2\ldots q-t}\prod_{n=1}^{t}\{H_j(n)\} \times \prod_{h=1}^{q-t}\{H_j(j,h)\} \times \{0,1\}^{\infty}.$$

For brevity, we may denote the union element $\prod_{n=1}^{t}\{H_j(n)\} \times \prod_{h=1}^{q-t}\{H_j(j,h)\} \times \{0,1\}^{\infty}$ as $G' \in F_n$. Observe to any union element, we have:

$$\begin{aligned}G' &= \{P \in 2^{F_n} | G' \cap B_{F_n}(r) \\ &= P \cap B_{F_n}(r)\} \\ &= B(G', r).\end{aligned}$$

Thus G' is open in the topology induced by the metric. Therefore, we get K is the union of the open sets in topology induced by the metric. So we can conclude that any open set in the sense of the direct topology on 2^{F_n} is also open in the sense of the topology induced by the metric. Therefore these two topologies are the same. □

Lemma 2: 2^{F_n} is compact in the sense of the topology induced by the metric.
Proof: 2^{F_n} is compact in the product topology by Tychonoff's theorem, and by Lemma 1, we get 2^{F_n} is also compact in the topology induced by the metric. □

Lemma 3: $\mathcal{G}_n$ (here we mean the set of all normal subgroups of F_n) is closed in 2^{F_n}
Proof: Assume $\{G_i\}$ to be any convergent sequence in $\mathcal{G}_n$, and assume it converges

to G in F_n, we need to show G is also in $\mathcal{G}_n$. To any k $\in Z^*$, since $B_{F_n}(k)$ contains only finite elements, and $\{G_i\}$ has infinite elements, thus we can always find a subsequence, say, $\{G_{i_j}\}$, s.t. $d(G, G_{i_j}) \leq 2^{-i}$, i.e. $G_{i_j} \cap B_{F_n}(k) = G \cap B_{F_n}(k)$ for all i, j. In this sense, we can always find a subsequence of $\{H_j\}$ of $\{G_i\}$ s.t.to any k $\in Z^*$, $d(G, H_k) \leq 2^{-k}$, i.e. $H_k \cap B_{F_n}(k) = G \cap B_{F_n}(k)$. To any $g \in G$, $|g|$=m (here,$|g|$ denotes the length of g), we have $g \in G \cap B_{F_n}(m)$. So we can get $g \in G \cap B_{F_n}(m) = H_m \cap B_{F_n}(m) \subseteq G \cap B_{F_n}(m + 2h) = H_{m+2h} \cap B_{F_n}(m + 2h)$ for any h $\in Z^*$. To any $j \in G$, we may assume $|j| = h$. It's easy to see $jgj^{-1} \in H_{m+2h} \cap B_{F_n}(m + 2h) = G \cap B_{F_n}(m + 2h)$ since H_{m+2h} is normal. Thus we can get $jgj^{-1} \in G$. Therefore we can conclude that G is normal since g, j are arbitrary elements in G. So we have showed $G \in \mathcal{G}_n$, i.e. $\mathcal{G}_n$ is closed in 2^{F_n}. □

By these 3 Lemmas, we have showed $\mathcal{G}_n$ is compact, since the closed subset of a compact space is compact, that finishes the proof. □

CHAPTER 2

A THEOREM OF SIMPLE GROUPS WITH POSITIVE L^2-BETTI NUMBERS

In this part, we will mainly deal with the Theorem 1.1. from Denis Osin and Andreas Thom's paper: Normal generation and l^2-Betti numbers of groups [2], which is about infinite simple group's l^2-Betti number.

2.1 Preliminaries

First we need to give some definitions:

A group is *hyperbolic* if it admits a finite presentation with linear isoperimetric function. Similarly a group G is *hyperbolic relative* to a collection of subgroups $\{H_\lambda | \lambda \in \Lambda\}$ if it admits a finite relative presentation with linear isoperimetric function.

A group is called *elementary* if it contains a cyclic subgroup of finite index. We also say that an element $g \in G$ is *parabolic* if it is conjugate to an element of H_λ for some $\lambda \in \Lambda$. Otherwise g is said to be *hyperbolic*.

An element $g \in G$ is called *loxodromic* if it has infinite order and is *hyperbolic*.

Let G be a relatively hyperbolic group (i.e. G is *hyperbolic relative* to a collection of subgroups $\{H_\lambda | \lambda \in \Lambda\}$). We call an element $g \in G$ *special* if it is *loxodromic* and $E_G(g) = \langle g \rangle$ (the definition of $E_G(g)$ will be given later) .

If G is an *ordinary hyperbolic* group, it can be thought of as *hyperbolic relative to the trivial subgroup*. Then the same definition applies. In this case *loxodromic* simply means of infinite order.

A ray in an infinite graph is a semi-infinite simple path; that is, it is an infinite sequence of vertices $v_0, v_1, v_2, ...$ in which each vertex appears at most once in the sequence and each two consecutive vertices in the sequence are the two endpoints of an edge in the graph.
A one-sided infinite path in an infinite graph X is called a ray. Two rays are said **equivalent**, if one of the following equivalent conditions hold:
(1) There is a third ray which has infinitely many vertices in common with each.
(2) For every finite vertex set F the two rays are eventually contained in the same connected component of $X - F$.
(3) There are infinitely many disjoint paths in X joining the two rays.

In addition, we may defnie two rays V, W are inequivalent as below:
Define $d(v_i, W)$=min $\{d(v_i, w_j)|j \geq 0\}$ and similarly for $d(V, w_j)$. To say that two rays are inequivalent means that $d(v_i, W) \to \infty$ as $i \to \infty$, which is equivalent to $d(v_j, W) \to \infty$ as $j \to \infty$.

Definition Let $||\Gamma - B(n)||$ be the number of connected unbounded components in the complement of $||\Gamma - B(n)||$, a ball of radius n is the number of vertices from the center to the boundary of the ball, centered around some vertex of Γ

Definition (Ends of a graph) Let Γ be a connected, locally finite graph, and let $B(n)$ be the ball of radius n about a fixed vertex v$\in V(\Gamma)$. Then the number of ends of Γ is $e(\Gamma) :=$ lim$||\Gamma - B(n)||$, denoted by $Ends(\Gamma)$.

Definition (The Ends of a Group) Let G be a group and let Γ be its Cayley graph with respect to a finite generating set. We define the Ends of G, $Ends(G) :=$

$Ends(\Gamma)$.

The two theorems below are well-known:

Theorem 2.1.1 ([4], Proposition 6.9). Let Γ be a finitely generated group. Γ has 0,1,2 or infinitely many ends.

Theorem 2.1.2 ([4], Theorem 6.10).An infinite group is virtually cyclic if and only if it is finitely generated and has exactly two ends.

Here is a well-known application of Theorem 2.1.2, and it will be used in Theorem 2.3.8 later:

Theorem 2.1.3 $Z_p * Z_p * Z_p * Z_p * ... * Z_p$ is virtually cyclic except the case $Z_2 * Z_2$.

Sketch Proof: It's easy to see the Cayley graph of $Z_2 * Z_2$ has two ends, hence by theorem 2.1.2, $Z_2 * Z_2$ is virtually cyclic. Again by theorem 2.1.2, for the other cases, we just need to show that there are more than two ends. If we can show $Z_p * Z_p$ $(p > 2)$ has more than two ends then the case of $Z_p * Z_p * Z_p * Z_p ... * Z_p$ follows immediately, so we can reduce the problem to the case of $Z_p * Z_p$ $(p > 2)$. We can see $Z_p * Z_p$ $(p > 2)$ as $\{a, b | a^p = b^p = 1\}$ and we can see the paths:

$$1 \to a \to ab \to aba \to abab \to ababa...$$

,

$$1 \to b \to ba \to bab \to baba \to babab \to bababa...$$

,

$$1 \to a^{-1} \to a^{-1}b \to a^{-1}ba^{-1} \to a^{-1}ba^{-1}ba^{-1} \to a^{-1}ba^{-1}ba^{-1}b \to ...$$

are not equivalent. Indeed, we may see they are not equivalent by transforming the corresponding Cayley graph to Bass-Serre tree (Name the Cayley graph of $Z_p * Z_p$

($p > 2$) as Γ, and the graph of Bass-Serre tree as T.) in two steps, that is, first lifting Γ to a transition graph, namely Γ_1, second, projecting Γ_1 to T. For the lifting, we replace each vertex of Γ by two vertices $v, w \in T$ connected by an edge labelled e, where the a and $\bar{a}$ edges of Γ are re-attached to v and the b and $\bar{b}$ edges are re-attached to w. For the projection, we collapse every a, $\bar{a}$ edge and every b, $\bar{b}$ edge of Γ_1 to a point, in other words, only the e edges are left. Now we need to take the rays in Γ, lift them to Γ_1, and project them to T. We may see the rays in T would go off in different directions after several steps; and since T is a tree, there are no embedded circles, so two embedded paths's intersection is connected, hence once those rays depart then they can never touch again, i.e. $d(v_i, W) \to \infty$ as $i \to \infty$, where V, W are any two corresponding paths of the three in T, so they are inequivalent in T, and we can prove T and Γ are quasi-isometric, so $d(v_i', W') \to \infty$ as $i \to \infty$, where V', W' are any two corresponding paths of the three in Γ, i.e. the corresponding paths in Γ are inequivalent, hence the three paths defined above are not equivalent. So there are more than two ends. Therefore by Stalling's theorem, $Z_p * Z_p (p > 2)$ is not virtually cyclic. That finishes the sketch of the proof. $\square$

Here are the sketches of

$$1 \to a \to ab \to aba \to abab \to ababa...$$

,

$$1 \to a^{-1} \to a^{-1}b \to a^{-1}ba^{-1} \to a^{-1}ba^{-1}ba^{-1} \to a^{-1}ba^{-1}ba^{-1}b \to ...$$

in Γ, Γ_1, T for the case $Z_3 * Z_3$.

The next lemma is a simplified version of Lemma 2.27 from [5].

Lemma 2.1.4 ([5], Lemma 2.27). Suppose that G is a group hyperbolic relative to a collection of subgroups $\{H_\lambda | \lambda \in \Lambda\}$. Then there exists a constant $K > 0$ and subsets $\Omega_\lambda \subseteq H_\lambda$ such that the following conditions hold:
(1) The union $\Omega = \Omega_\lambda$ is $finite$. $\lambda \in \Lambda$.

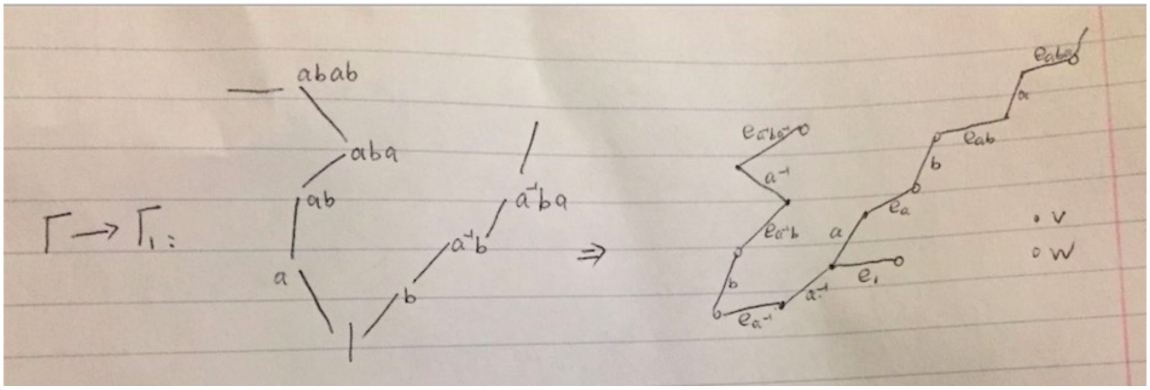

Figure 2.1

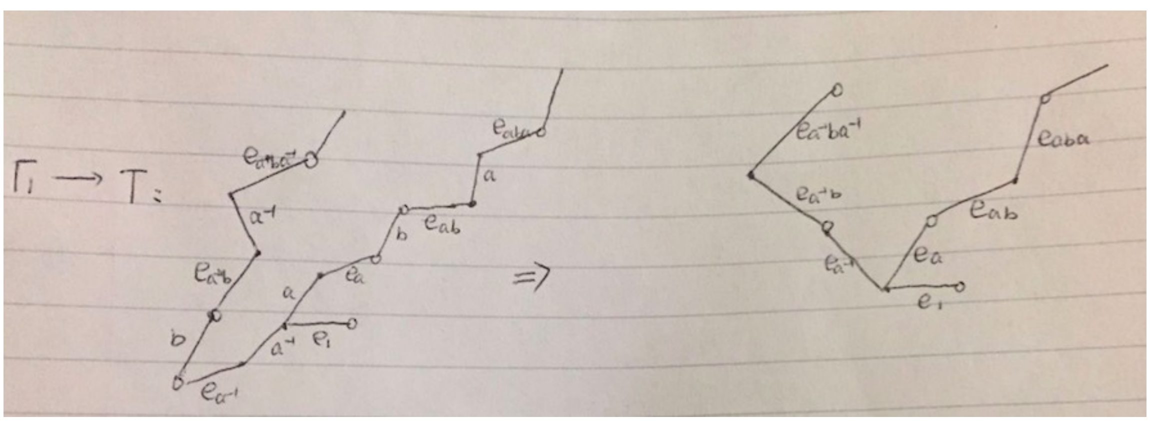

Figure 2.2

(2) Let q be a cycle in $\Gamma(G, X \cup H)$, $p_1, ..., p_k$ a set of isolated H_λ-components of q for some $\lambda \in \Lambda$, $g_1, ..., g_k$ the elements of G represented by the labels of $p_1, ..., p_k$ respectively. Then for any $i = 1, ..., k$, g_i belongs to the subgroup $\langle \Omega_\lambda \rangle \leq G$ and the lengths of g_i with respect to Ω_λ satisfying the inequality: $\sum_{i=1}^{k} |g_i|_{\Omega_\lambda} \leq Kl(q)$.

Throughout this paper we fix a group G hyperbolic relative to a collection of subgroups $\{H_\lambda | \lambda \in \Lambda\}$, a $finite$ relative generating set $X = X^{-1}$ of G with respect to $\{H_\lambda | \lambda \in \Lambda\}$, and the set Ω provided by Lemma 2.1.4.

Theorem 2.1.5, ([5], Theorem 1.6). Let G be a group, $\{H_\lambda | \lambda \in \Lambda\}$ a collection of subgroups of G. Suppose that G is finitely presented with respect to $\{H_\lambda | \lambda \in \Lambda\}$ and the Denh function of G with respect to $\{H_\lambda | \lambda \in \Lambda\}$ is finite for all values of the argument. Then the following conditions hold:
(1) For any $g \in G$, the intersection $H_\lambda^g \cap H_\mu$ is finite whenever $\lambda \neq \mu$.
(2) The intersection $H_\lambda^g \cap H_\lambda$ is finite for any $g \notin H_\lambda$, where H_λ^g is the conjugation of H_λ by g.

Lemma 2.1.6 ([5], Corollary 1.17) If $g \in G$ is hyperbolic and $f^{-1} g^m f = g^n$ for some $f \in G$, then $m = \pm n$.

Lemma 2.1.7 ([6], Lemma 4.1). For any hyperbolic element of infinite order $g \in G$, there exists a constant $C = C(g)$ such that if $f^{-1} g^n f = g^n$ for some $f \in G$ and some $n \in N$, then there are $m \in Z$ and $h \in \langle X \cup \Omega \rangle$ such that $f = hg^m$ and $|h|_{X \cup \Omega} \leq C$.

Theorem 2.1.8 ([6], Theorem 4.3) For any loxodromic element $g \in G$, we set $E_G(g) = \{f \in G : f^{-1} g^n f = g^{\pm n} \text{ for some } n \in N\}$. Every hyperbolic element $g \in G$ is contained in a unique maximal elementary subgroup, namely in $E_G(g)$.

Proof: For any loxodromic element $g \in G$, we set:

$$E_+(g) = \{f \in G : f^{-1}g^n f = g^n, \forall n \in N\}$$

. Note, since X, Ω are finite, hence there are only finite h satisfying $|h|_{X\cup\Omega} \leq C$. And by Lemma 2.1.7, $E_+(g)/\langle g\rangle$=$\{\overline{h} : |h|_{X\cup\Omega} \leq C\}$, therefore $\langle g\rangle$ has finite index in $E_+(g)$. It's easy to see that: $E_G(g)/E_+(g)$=$\{\overline{f}, e\}$ where $f \in G$ s.t. $f^{-1}g^n f = g^{-n}$, i.e. the index of $E_+(g)$ in $E_G(g)$ is 2. Hence $\langle g\rangle$ has finite index in $E_G(g)$, and we can conclude that $E_G(g)$ is an elementary group containing g.

Now we need to show $E_G(g)$ is the maximal elementary group containing g, i.e. $\forall$ elementary group H that contains g is in $E_G(g)$. First we need to show H contains a normal cyclic subgroup with finite index n: Since H is elementary, it contains a cyclic subgroup with finite index, namely D. Consider the core of D in H(i.e.the intersection of its conjugates in H), namely $core_H(D)$, which is normal in H and contained in D. We need to show it's not trivial: Assume the index of D in H is m, then $|H/D|$=m. Then the action of H on H/D by left translation induces a map f: $H \to S_m$, where S_m is the permutation of elements of H/D. Note to $g \in H$, $ghD = hD$ $\forall h$ iff $g \in hDh^{-1}$, it implies that $kerf$=$core_H(D)$. Therefore the induced map f': $H/core_H(D) \to S_m$ is injective. Since H is infinite, $core_H(D)$ is infinite by the injection, so $core_H(D)$ is not trivial. Now we can assume the subgroup $core_H(D)$ of D is in the form of $\langle b^t\rangle$, where we assume D is in the form of $\langle b\rangle$. So we know $|D : core_H(D)|$=t, since $\forall g \in D$, g^t is in $core_H(D)$. And we know D has finite index in H, and $core_H(D)$ is contained in D, so $core_H(D)$ has finite index in H, name the index as n. Now we replace D by $core_H(D)$, so we can get H contains a cyclic normal subgroup with finite index n, namely D.

Now we may assume that $\langle s\rangle$ is the normal cyclic subgroup with finite index in H. And since $\langle s\rangle$ is of finite index, we can get $\exists k \in \mathbb{Z}\setminus\{0\}$, s.t. $g^k \in \langle s\rangle$, i.e.$\exists l \in \mathbb{Z}\setminus\{0\}$, s.t. $g^k = s^l$

In particular, we can show s is hyperbolic: Indeed, if $s \in H_\lambda^a$ (i.e.not hyperbolic), for some $\lambda \in \Lambda$, $a \in G$, then $\langle s^l \rangle \in H_\lambda^a$. Assume $b \in H_\lambda^a$, s.t.$a^{-1}ba = s^l$, so $g^{-1}a^{-1}bag = g^{-1}s^lg = s^l$, since $g^k = s^l$(i.e.s^l commutes with g), hence $s^l \in H_\lambda^{ag}$, therefore we can get $\langle s^l \rangle \in H_\lambda^a \cap H_\lambda^{ag} = (H_\lambda \cap H_\lambda^{aga^{-1}})^a$. Since $\langle s^l \rangle$ is infinite, we can get $(H_\lambda \cap H_\lambda^{aga^{-1}})^a$ is infinite, hence $H_\lambda \cap H_\lambda^{aga^{-1}}$ is infinite. So by (2) of Lemma 2.5, aga^{-1} has to be in H_λ, hence g is not hyperbolic, contradiction. Therefore, we get s is hyperbolic.

Since $\langle s \rangle$ is normal, for $\forall t \in H$, we have $t^{-1}st = s^m$ and some $m \in \mathbb{Z}$ and by Lemma 3, we have $m = \pm 1$. Hence $t^{-1}g^kt = t^{-1}s^lt = s^{\pm l} = g^{\pm k}$. Therefore, by the definition of $E_G(g)$, we get $t \in E_G(g)$, i.e. $H \in E_G(g)$. This finishes the proof. $\square$

Theorem 2.1.9 (Kurosh subgroup theorem) Let $G = A * B$ be the of groups A and B and let $H \leq G$ be a of G. Then there exist a family $(A_i)_{i \in I}$ of subgroups $(A_i) \in A$,$(B_j)_{j \in J}$ of subgroups $(B_i) \in B$, families $g_i, i \in I$ and $f_j, j \in J$ of elements of G, and a subset $X \subseteq G$ such that: $H = F(X) * (*_{i \in I} g_i A_i g_i^{-1}) * (*_{j \in J} f_j B_j f_j^{-1})$.
This means that X freely generates a subgroup of G isomorphic to the free group $F(X)$ with free basis X and that, moreover, $g_i A_i g_i^{-1}, f_j B_j f_j^{-1}$ and X generate H in G as a free product of the above form.

Theorem 2.1.10 The group $G = Z_p * Z_p * Z_p * Z_p ... * Z_p$ doesn't contain finite normal subgroups, where p is prime.
Proof: Case(1), when $G = Z_p * Z_p = \{a, b | a^p = b^p\} = A * B$, where $A = \{a | a^p\}, B = \{b | b^p\}$. Assume there exists such a finite normal subgroup H, then by Theorem 2.9, $H = F(X) * (*_{i \in I} g_i A_i g_i^{-1}) * (*_{j \in J} f_j B_j f_j^{-1})$. Since H is finite, thus the free group $F(X)$ is trivial. Here A, B are both Z_p, and the subgroups of Z_p are Z_p and the trivial group. Thus if H is not trivial, then $H = gAg^{-1}$, $H = gBg^{-1}$ or $H = gAg^{-1} * tBt^{-1}$, however H is finite, thus $H = gAg^{-1}$ or $H = gBg^{-1}$, without losing generality, we may assume $H = gAg^{-1}$. Since H is normal, thus $A = g^{-1}Hg = H$. To any $b \in B$, $bHb^{-1} = H$, however $bab^{-1} \notin H = A$, contradiction. Thus G doesn't contain finite normal subgroups.

Case(2): $G = Z_p * Z_p * Z_p * Z_p ... * Z_p$ (at least 3 copies). And we can reduce it by considering the case of $G = Z_p * Z_p * Z_p$, since if the case of $G = Z_p * Z_p * Z_p$ works, then the other cases follow. We can see $\mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p$ as $\langle a, b, c \mid a^p = b^p = c^p = 1 \rangle$. Note, any non-trivial element c can be represented in the normal form: $c_1 c_2 \cdots c_k$, where $k \geq 1$, and each c_i is in the form of x^i with $x \in \{a, b, c\}$, $1 \leq i < p$, and adjacent c_i are powers of different generators. Without losing generality, we may assume c_1 is in the form of a^i. Then for any normal form word w in the infinite group $\langle b, c \rangle$, the term c_1 cannot cancel when reducing the word wcw^{-1} to normal form, so its normal form has the prefix wc. In this sense, we can get c has infinitely many distinct conjugates and cannot lie in a finite normal subgroup. Therefore, we have all the normal subgroups of G are infinite. $\square$

2.2 Special elements of hyperbolic groups and some results to be used for 2.3

In this section, our goal is to prove a theorem, which is the proposition 3.4 of Denis Osin and Andreas Thom [2]: Let G be a hyperbolic group without nontrivial finite normal subgroups. Then for every nontrivial element a∈G and every x∈G, there exists a special element g $\in x \ll a \gg^G$.

Now, with these definitions above, in order to prove the Theorem, we need to have two Lemmas, the first Lemma is proved by Denis Osin and Andreas Thom [2] , and the second Lemma is proved by Olshanskii [7].

Lemma 2.2.1 ([2], Lemma 3.2) Let G be a relatively hyperbolic group, $h \in G$ a special element. Then for every $a \notin E_G(h)$, there exists a positive integer n such that the element $g = ah^n$ is special.

Lemma 2.2.2 ([7]) Let G be a hyperbolic group, $H \leq G$ a non-elementary subgroup (where a group is called elementary if it contains a cyclic subgroup of finite index), then there exists an element $h \in$H of infinite order such that $E_G(h) = \langle h \rangle \times E_G(H)$.

Now Let us go back to the theorem:

Theorem 2.2.3 ([2], proposition 3.4) Let G be a hyperbolic group without nontrivial finite normal subgroups. Then for every nontrivial element a$\in$G and every x$\in$G, there exists a special element $g \in x \ll a \gg^G$.

Proof of Theorem 2.2.3: First we need the following proposition:

Proposition 2.2.4 Define H to be $\ll a \gg^G$. Then H is also non-elementary.

Proof of proposition 2.2.4: Otherwise, if H is elementary, then H contains a cyclic normal subgroup with finite index. (By the same proof we gave in Theorem 2.1.8, now we can claim H contains a cyclic normal subgroup with finite index n, namely D.(Note, since D is normal in H with index n, $\forall h \in H$, $D = (hD)^n = h^n D$, it implies that $\forall h \in H$, h^n is contained in D.)

Therefore, H contains an infinite cyclic characteristic subgroup (where characteristic subgroup of H is a subgroup which is invariant under every automorphism of H) (Proof: Define C=$\langle h^n | h \in H \rangle$ (it implies $C \subseteq D$). Then to $\forall h^n$, with $h \in H$ and $\forall f \in Aut(H)$, $f(h^n) = f(h)^n \in C$ which implies $f(C) \subseteq C$, then consider f^{-1}, we can get $C = f^{-1}f(C) \subseteq f^{-1}(C) \subseteq C$, thus $f^{-1}(C) = C$, it implies $f(C) = C$, so C is characteristic. Note, $\forall h \in H$, $h^n \in C$, so $|H/C| \leq n$, and $C \subseteq D$, thus C is the infinite cyclic characteristic subgroup of H.)

Then we can get the centralizer $C_G(C)$ has finite index in G (Proof: First, since H is normal in G, C is characteristic in H, it's easy to see C is normal in G. And since C is normal in G. To $\forall m \in C_G(C), g \in G, h \in C, (gmg^{-1})h(gm^{-1}g^{-1}) = gm(g^{-1}hg)m^{-1}g^{-1} = g(g^{-1}hg)mm^{-1}g^{-1} = h$, so $gmg^{-1} \in C_G(C)$, so $C_G(C)$ is normal in G. Now we consider the the map $f : G/C_G(C) \to Aut(C)$, by conjugating C by elements of $G/C_G(C)$.Note $aCa^{-1} = C$ iff $a \in C_G(C)$, i.e. f is injective. Therefore $G/C_G(C)$ is isomorphic to a subgroup of $Aut(C)$. Also, C is infinite cyclic, so $C \cong Z$, therefore, $Aut(C) \cong Z_2$. So $G/C_G(C)$ is finite, i.e. $C_G(C)$ has finite index in G.)

However, C has finite index in $C_G(C)$. (Proof: Since C is cyclic, we assume it's C=$\langle h \rangle$. And $C_G(C) = \{g \in G | gc = cg, \forall c \in G\}$, so $\forall g \in C_G(C)$, gh=hg, therefore $C_G(C) \subseteq C_G(\{h\})$, so $C_G(C) = C_G(\{h\})$. Since G is hyperbolic, h has infinite order, then $C_G(C)$ contains $\langle h \rangle$ as a finite index subgroup (This fact is from Alessandro Sisto's note [8]))

Now, we can conclude that the cyclic subgroup C has finite index in G, hence G is elementary. A contradiction. This finishes the proof of proposition 2.2.4.$\square$

Since H is also non-elementary, by the proof of Lemma of [2], we get $E_G(H)$ is a finite subgroup of G. As we know, $E_G(H) = \cap_{h \in H^0} E_G(h)$, where H^0 denotes the set of all elements of H of infinite order. Since H is normal by definition and conjugation keeps the order, $\forall g \in G$, $g^{-1}hg \in H^0$, $\forall h \in H^0$. By Theorem 1 ([8], Theorem 4.3), $E_G(h) = \{f \in G : f^{-1}h^n f = h^{\pm n}$ for some $n \in N\}$. $g^{-1}E_G(H)g = \cap_{h \in H^0} g^{-1}E_G(h)g$. To any $g \in G, h \in H^0$, define $h' = ghg^{-1} \in H^0$, and to any $f \in G$, we may assume $f^{-1}h^n f = h^{\pm n}$, $f^{-1}h'^l f = h'^{\pm l}$ for some $n, l \in N$, then we can get: $f^{-1}h^m f = h^{\pm m}$, $f^{-1}h'^m f = h'^{\pm m}$, where $m = nl$. So $g^{-1}f^{-1}gh^m g^{-1}fg = g^{-1}f^{-1}h'^m fg = g^{-1}h'^{\pm m}g = h^{\pm m}$. It means, if $f \in E_G(h)$, then for any $g \in G$, $g^{-1}fg \in E_G(h)$, i.e. $E_G(h) = g^{-1}E_G(h)g$. Moreover, we get $g^{-1}E_G(H)g = E_G(H)$, i.e. $E_G(H)$ is normal in G. Therefore $E_G(H)$ is trivial by the definition of G. By Lemma 2.2.2, there is an $h \in H$ with infinite order s.t. $E_G(h) = \langle h \rangle \times E_G(H) = \langle h \rangle$. Now since G is an ordinary hyperbolic group, h is of infinite order and $E_G(h) = \langle h \rangle$, thus h is a special element of G. If $x \in E_G(h) = \langle h \rangle \leq H = \ll a \gg^G$, then we can take the required special element g to be h, so $g = h \in \langle h \rangle \leq H = xH$. Otherwise, by applying Lemma 2.2.1, we can get a special element $g = xh^n \in xH$. That finishes the proof. $\square$

2.3 Simple groups with positive l^2-Betti numbers

In this section, we want to prove the result of Osin-Thom for every integer $n \geq 2$ and every $\epsilon \geq 0$ there exists an infinite simple group Q generated by n elements such that $\beta_1^{(2)}(Q) \geq n - 1 - \epsilon$.

An *irreducible torsion presentation* $\mathcal{P}$ is of the form: $\langle X | R_1^{n_1}, R_2^{n_2}, ..., R_k^{n_k} \rangle$ (1). By this, we define $\sigma(\mathcal{P}) := \sum_{n=1}^{k} 1/n_i$

The next result is from J. Peterson and A. Thom [9].

Theorem 2.3.1 (Theorem 3.2 in [9]). Let G be a group given by an irreducible torsion presentation (1), where $|X| < \infty$. Then $\beta_1^{(2)}(G) \geq |X| - 1 - \sigma(\mathcal{P})$.

The next three theorems are well-known.

Theorem 2.3.2 (Theorem 1.4 in [5]). Suppose a group G is hyperbolic relative to a collection of subgroups $\{H_\lambda | \lambda \in \Lambda\}$. Let g be a loxodromic element of G. Then the following conditions hold:
(a) There is a unique maximal elementary subgroup $EG(g) \leqslant G$ containing g.
(b) $E_G(g) = \{h \in G | \ \exists m \in \mathbb{N} \text{ s.t. } h^{-1} g^m h = g^m\}$.
(c) The group G is hyperbolic relative to the collection $\{H_\lambda | \lambda \in \Lambda\} \cup \{E_G(g)\}$.

Lemma 2.3.3 ([7]). Let G be a non-elementary group hyperbolic relative to a collection of proper subgroups. Suppose also that G has no nontrivial finite normal subgroups. Then G contains a special element.

Theorem 2.3.4 ([10]). Let G be group hyperbolic relative to a collection of sub-

groups $\{H_\lambda | \lambda \in \Lambda\}$. Then for every finite subset $\mathcal{A} \subseteq G$, there exists a finite subset $\mathcal{F} \in G - \{1\}$ such that for any collection of subgroups $\mathcal{N}$=$\{N_\lambda | \lambda \in \Lambda\}$ satisfying $N_\lambda \lhd H_\lambda$ and $N_\lambda \cap \mathcal{F} = \varnothing$ for all $\lambda \in \Lambda$, the following hold.
(a) Let $N =<< \bigcup_{\lambda\in\Lambda} N_\lambda >>$ be the normal closure of $\bigcup_{\lambda\in\Lambda} N_\lambda$ in G. Then for every $\lambda \in \Lambda$, the natural map $H_\lambda/N_\lambda \to G/N$ is injective (equivalently, $H_\lambda \cap N = N_\lambda$).
(b) G/N is hyperbolic relative to $\{H_\lambda/N_\lambda | \lambda \in \Lambda\}$.
(c) The natural homomorphism $G \to G/N$ is injective on $\mathcal{A}$.

Lemma 2.3.5 (Theorem 4.3 and Corollary 1.7 in [5]). Suppose that a group G is hyperbolic relative to a finite collection of hyperbolic subgroups. Then G is hyperbolic itself.

Lemma 2.3.6. Any infinite cyclic subgroup of an infinite elementary group has finite index.

Proof: To any infinite cyclic subgroup $K = \langle k \rangle$ of a infinite elementary group G, we assume $H = \langle h \rangle$ is the infinite cyclic subgroup of G with finite index, we claim $H \cap K$ is not trivial. The reason is since H has finite index, so there exits $i < j$ s.t. $k^i H = k^j H$, therefore $k^{j-i} \in H$, i.e. $H \cap K$ is not trivial. Then we can conclude that $H \cap K$ has finite index in H, thus it has finite index in G. Also $H \cap K$ has finite index in K, therefore K has finite index in G. $\square$

Lemma 2.3.7 (Theorem 2.40, in [5]). Let G be a group hyperbolic relative to a collection of subgroups $\{H_\lambda | \lambda \in \Lambda\}$.Then for every $\lambda \in \Lambda$ and $g \in G - H_\lambda$, we have $|H_\lambda \cap {H_\lambda}^g| < \infty$.

Now let us come to the main theorem:

Theorem 2.3.8 (Theorem 1.1 in [2]) For every integer $n \geq 2$ and every $\epsilon \geq 0$ there exists an infinite simple group Q generated by n elements such that $\beta_1^{(2)}(Q) \geq n - 1 - \epsilon$.

Proof: We split the proof in several steps

Step1: Basic group G_0 and property (a),(b) and (c)

To any $n \in \mathbb{N}$, define $X = \{x_1, ..., x_n\}$ and $G_0 = \langle X | {x_1}^p, {x_2}^p, ...{x_n}^p \rangle$, where p is a prime satisfying $n/p < \epsilon$. In particular, by Theorem 2.2.1, we can get $\beta_1^{(2)}(Q) > n - 1 - \epsilon$. We enumerate all elements of $X \times (G_0 \backslash \{1\}) = \{(x_{m1}, g_1), (x_{m2}, g_2), ...\}$.

Now we can construct a sequence of groups and epimorphisms: $G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \rightarrow$ ... as follows:

Below we will use the same symbols to denote elements of G_i and their images in G_{i+1}. At step i we assume that the group G_i is already constructed and satisfies the following conditions:

(a) G_i is a non-elementary hyperbolic group without finite normal subgroups.

(b) G_i has an irreducible torsion presentation $\mathcal{P}_i$ with $\sigma(\mathcal{P}_i) < \epsilon$.

(c) If $i \geq 1$,then for every $j = 1, ..., i$, we either have $g_j = 1$ in G_i or $x_{mj} \in \ll g_j \gg^{G_i}$.

Let us first consider the case $i = 0$. For (a): First, note that G_0 is the free product of n finite groups, which are all isomorphic to Z_p, and we know the finite groups are hyperbolic, moreover, the free product of two hyperbolic groups is hyperbolic, hence G_0 is hyperbolic. And G_0 does not contain finite normal subgroups by theorem 2.10. And G_0 is not virtually cyclic/non-elementary except for the case of $\langle X | {x_1}^2, {x_2}^2 \rangle$. For (b): the first half is direct from the definition G_0, the second half is true, because $\sigma(\mathcal{P}_0) = n/p < \epsilon$. And (c) is trivially true.

Step 2: Construction of G_i

Now we can construct the group G_{i+1} from G_i. If If $g_{i+1} = 1$ in G_i, then we set $G_{i+1} = G_i$ and $f_i = id$. Otherwise, by (a) and Theorem 2.2.3, there exists a special element $h_i \in x_{m_{i+1}} \ll g_{i+1} \gg^{G_i}$. Note h_i is special, i.e. h_i is loxodromic and $E_G(h_i) = \langle h_i \rangle$, and by (a) G_i is hyperbolic, so it it can be thought of as hyperbolic relative to the trivial subgroup. So by theorem 2.3.2, G_i is hyperbolic relative to the $\{e\} \cup \{E_G(h_i)\} = \langle h_i \rangle$. Rename $\langle h_i \rangle$ as $H_{i,1}$, then we can get G_i is hyperbolic relative to $H_{i,1}$. By Lemma 3.3, we can find another special element $t_i \in G_i$, which is considered as a group hyperbolic relative to $H_{i,1}$. So by theorem 2.3.2 again, we can get G_i is hyperbolic relative to $\{H_{i,1}, H_{i,2}\}$, where $H_{i,2} = \langle t_i \rangle$.

Let $\langle X | \mathcal{A}_i \rangle$ be the irreducible torsion presentation $\mathcal{P}_i$ of G_i, where $\mathcal{A}_i$ is the set of all powers of elements of G_i represented by $R_1, ..., R_k$, i.e. $\mathcal{A}_i = \{R_1^{n_1}, R_2^{n_2}, ..., R_k^{n_k}\}$. Since the order of each R_i is finite, thus the set of combinations of $R_1, ..., R_k$ is finite, i.e. $\mathcal{A}_i$ is finite. Now apply theorem 2.3.4 on G_i which is hyperbolic relative to $\{H_{i,1}, H_{i,2}\}$, let $\mathcal{F}_i$ be the be the finite set corresponding to $\mathcal{F}$ in theorem 2.3.4. And since h_i is special (it implies h_i is loxodromic, thus by the definition of 'loxodromic', h_i has infinite order) $\mathcal{F}_i$ is finite, we can always choose a prime $q_i > p$ such that the subgroup $N_{i,1} = \langle {h_i}^{q_i} \rangle$ does not contain elements of $\mathcal{F}_i$ and $\sigma(\mathcal{P}_i) + 1/q_i < \epsilon$ (Reason: the elements of $\mathcal{F}_i \cup H_{i,1}$ is in the form of ${h_i}^t$, and since $\mathcal{F}$ is finite, thus the powers t are upper bounded, therefore we can require q_i larger than these powers). Let G_{i+1} be the group given by the presentation $\mathcal{P}_{i+1} = \langle \mathcal{P}_i | {h_i}^{q_i} = 1 \rangle$

Step 3: Proof that G_i satisfyies (a),(b), and (c)

Now we define $N_{i,2} = \{1\} \lhd H_{i,2}$, $N_i = << N_{i,1} \cup N_{i,2} >>$, then we can get $G_{i+1} = G_i / N_i$. So we have $N_{i,j} \lhd H_{i,j}$ and $N_{i,j} \cap \mathcal{F}_i = \varnothing$ for j=1,2 and $H_{i,1}/N_{i,1} \cong \mathbb{Z}/q_i\mathbb{Z}$. So we obtain $H_{i,1}/N_{i,1}$ and $H_{i,2}$ naturally embed in G_{i+1}, by theorem 2.3.4-(b), we get $G_{i+1} = G_i/N_i$ is hyperbolic relative to $\{H_{i,1}/N_{i,1}, H_{i,2}\}$. Since $H_{i,1}/N_{i,1}, H_{i,2}$ are both cyclic groups, $H_{i,1}/N_{i,1}, H_{i,2}$ are hyperbolic. Therefore by lemma 2.3.5 G_{i+1} is hyperbolic. If G_{i+1} is elementary, then by lemma 2.3.6, we can get the infinite cyclic subgroup $H_{i,2}$ has finite index in G_{i+1}. Hence by the proof of proposition 2.2.4, we can see there is a finite index subgroup C contained in $H_{i,2}$ and is normal in G_{i+1}. In particular, $C^{h_i} \cap C = C$, where C^{h_i} is the conjugation of C by h_i. Then by C is infinite, we can get $|H_{i,2} \cap H_{i,2}^{h_i}|$ is not finite, hence by Lemma 2.3.7, we can get

$h_i \in H_{i,2}$, which implies h_i has infinite order. Hence we get a contradiction, since h_i is non-trivial and has finite order q_i in G_{i+1}. So we get G_{i+1} is not elementary. Now if G_{i+1} has a finite normal subgroup, namely K, let G_{i+1} acts on K by conjugation. It induces a homomorphism from G_{i+1} to $Aut(K)$. The kernel is just centralizer of K, i.e. $C_{G_{i+1}}(K)$. So we get $|G_{i+1}/C_{G_{i+1}}(K)| < |Aut(K)|$. And since K is finite, $Aut(K)$ is finite. So the index of $C_{G_{i+1}}(K)$ is finite. Then we can conclude that K is centralized by a finite index subgroup of G_{i+1}, rename $C_{G_{i+1}}(K)$ as P. In particular, since P has finite index, by considering the cosets ${t_i}^1P, {t_i}^2P, {t_i}^3P...$, we can see there must be some $j > 1$ s.t. ${t_i}^jP = P$ i.e. ${t_i}^j \in P$, i.e. K is centralized by a nontrivial element $h = {t_i}^j$ of $\langle t_i \rangle$=$H_{i,2}$. It implies that to any $k \in K$, $\langle h \rangle \subseteq |H_{i,2} \cap H_{i,2}^k|$, hence $|H_{i,2} \cap H_{i,2}^k|$ is not finite, therefore by Lemma 4.7, we can get $k \in H_{i,2}$. So we can conclude $K \subseteq H_{i,2}$. Note as a infinite cyclic group, $H_{i,2}$ doesn't contain non-trivial finite subgroups, therefore K is trivial. Thus part (a) of the inductive assumption holds for G_{i+1}.

For part (b), since we have already had $\mathcal{F}_i$ and $\sigma(\mathcal{P}_i) + 1/q_i < \epsilon$, thus it suffices to show $\mathcal{P}_{i+1}$ is an irreducible torsion presentation. Indeed, by part (c) of Theorem 2.2.4, there is a natural homomorphism $G_i \to G_i/N_i = G_{i+1}$, which is injective on $\mathcal{A}_i$, where $\mathcal{A}_i$ is the set of all powers of elements of G_i represented by $R_1, ..., R_k$, so the irreducibility is ensured.

For part(c), if $g_{i+1} \neq 1$ in G_{i+1}, by what we showed above, $h_i \in x_{m_{i+1}} \ll g_{i+1} \gg^{G_i}$, then in $G_{i+1}/\ll g_{i+1} \gg^{G_{i+1}}$, $h_i = x_{m_{i+1}}$. Also by definition of G_0, we have ${x_{m_{i+1}}}^p = 1$ in G_0, therefore ${x_{m_{i+1}}}^p = 1$ in $G_{i+1}/\ll g_{i+1} \gg^{G_{i+1}}$. Note, we require $q_i > p$ above, hence $x_{m_{i+1}} = 1$ in $G_{i+1}/\ll g_{i+1} \gg^{G_{i+1}}$, i.e. $x_{m_{i+1}}$ is in $\ll g_{i+1} \gg^{G_{i+1}}$. For the other G_j, if $x_{mj} \in \ll g_j \gg^{G_i}$, then it's easy to see $x_{mj} \in \ll g_j \gg^{G_{i+1}}$ since if $x_{mj} \in N \lhd G_i$, then $\bar{x}_{mj} \in \bar{N} \lhd \bar{G}_i = G_{i+1}$, also we know the preimage of normal subgroup of quotient group under quotient epimorphism is normal, thus the normal subgroup of G_{i+1} is in the form of $\bar{N} \lhd G_{i+1}$, where $N \lhd G_i$. Therefore, with the inductive assumption this implies (c) for G_{i+1}. Thus the inductive step is completed.

Step 4: Construction of simple group Q and proof of the equality

Now let Q be $G_0/\cup_{i=1}^{\infty} F_i Ker(f_i ... f_0)$, note, since every f_i is onto, Q is the limit of the groups G_i in the topology of marked group presentations as described in section 1.2.2 above and hence $\beta_1^{(2)}(Q) \geq \limsup_{i\to\infty} \beta_1^{(2)}(G_i) \geq n-1-\epsilon$. Here the first inequality follows from semicontinuity of the first l^2-Betti number [11] and the second one follows from (b) and Theorem 2.3.1, i.e. by $\beta_1^{(2)}(G_i) \geq |X|-1-\sigma(\mathcal{P}_i)$ with $\sigma(\mathcal{P}_i) < \epsilon$. Now we need to show Q is infinite. If Q was finite, it would be finitely presented and hence by the construction of each G_j, all relations of Q would be contained in some G_i, then it violates (a), since every G_i is not finite. Finally we need to show that Q is simple. Indeed, let $q \in Q$ be a nontrivial element. Since $\{(x_{m1}, g_1), (x_{m2}, g_2), ...\}$ enumerates all pairs of elements in $X \times (G_0\backslash\{1\})$, hence it contains every (x_i, q). Thus part (c) of the inductive assumption ensures that $x_1, ..., x_n \in \ll q \gg^Q$, hence $Q = \ll q \gg^Q$. And to every non-trivial normal subgroup N of Q, it must contain a nontrivial element p, by the result above, $Q = \ll p \gg^Q \subseteq N \subseteq Q$, hence Q is simple. □

Corollary 2.3.9 (Corollary 1.2 in [2]) For every positive integer n there exists a simple group Q with $d(Q) = n$, where $d(G)$ denotes the minimal number of generators of G.

Sketch proof: By using the Morse inequality [12]: For every finitely generated group, G, let X be a free G-CW-complex of finite type. Let α_p be the number of p-cells in $G\backslash X$. Then we get for $n \geq 0$:

$$\sum_{p=0}^{n}(-1)^{n-p}\beta_p^{(2)}(G) \leq \sum_{p=0}^{n}(-1)^{n-p}\alpha_p$$

Now we can define X to be the Eilenberg-MacLane space: $K(G,1)$ space of G, constructed in this way: there is one 0-cell, $d(G)$ 1-cells corresponding to generators, and several 2-cells corresponding to relations and several higher dimensional cells to kill off the higher homotopy. Moreover, we take the n to be 1, and note since G is infinite, we have $\beta_0^{(2)}(G) = 0$, therefore we can get a well-known inequality:

$$\beta_1^{(2)}(G) \leq d(G) - 1$$

Also by Theorem 2.3.8, for any n, and $\epsilon = 1/2$, there exists a simple group Q generated by n elements (which implies $d(Q) \leq n$) s.t. $\beta_1^{(2)}(Q) \geq n - 1 - \epsilon$. Since Q is finitely generated, apply the inequality above, we get $\beta_1^{(2)}(Q) \leq d(Q) - 1$, i.e. $d(Q)-1 \geq \beta_1^{(2)}(Q) \geq n-1-\epsilon$ i.e. $d(Q) \geq n-\epsilon = n-1/2$. So we have $d(Q) \geq n-1/2$ and $d(Q) \leq n$, which implies $d(Q) = n$. $\square$

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