A LOOK AT T1 AND Tb THEOREMS ON NON-HOMOGENEOUS SPACES
THROUGH TIME-FREQUENCY ANALYSIS

BY

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DISSERTATION

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Abstract

One of the widely studied topics in singular integral operators is T1 theorem. More precisely, it asks if one can extend a Calderón-Zygmund operator to a bounded operator on L^p. In addition, Tb theorem was raised when one asks if the T1 theorem remains true if the function 1 is substituted by some bounded function b. In this dissertation, we apply time-frequency analysis to T1 theorem and Tb theorem. In particular, the theory of tiles and trees is used to prove T1 theorem on non-homogeneous spaces. This provides an alternative and a more visualized point of view to some parts of the proof. We also verify estimates from $L^p \times L^q$ to L^r for the paraproducts appeared in T1 theorem. Although the paraproduct is specific, the method is applicable to this kind of study. Lastly, an extension to the proof of Tb theorem is established via a different tree from T1 theorem.

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Table of Contents

Chapter 1

Introduction

The theory of Calderón-Zygmund operators is known as one of the most powerful subjects in Mathematics due to relations to PDE, Physics, Engineering, etc. One of the theory is T1 theorem which was later generalized to Tb theorem, see some examples of applications in [DJ84, NTV03]. The T1 theorem gives conditions for Calderón-Zygmund operators to be bounded on L^2 which is enough for the extension to L^p due to weak type estimates of such operators. The Tb theorem was motivated by a problem about the Cauchy integral operator on a Lipschitz graph [AHM+02]. Since the first proof of David and Journé on the classical T1 theorem [DJ84], the theory has been well developed in various ways, for instance extending the range of functions to vector value [Fig90, Hyt14]. Our interest is an extension on the domain. In particular, we consider a metric space endowed with a measure that does not satisfy doubling condition. Such spaces are called non-homogeneous type. This situation can occur even to the Lebesgue measure with an open subset of $\mathbf{R}^N$ having an unusual boundary. Verdera surveys more about the need of non-homogeneity and applications [Ver02]. Undoubtedly, the theory on spaces of homogeneous type, the ones equipped with doubling measures, was fine studied. Furthermore, T1 theorem on non-homogenous space for the Cauchy integral operator was proved in many approaches [NTV97, Tol99, Ver00]. With the use of BMO^p_λ, Nazarov, Treil and Volberg [NTV03] refined their work to Tb theorem in the general setting as the classical statements leading to the fairly complete theory. The generalization includes Cotlar inequalities and weak type $(1,1)$ estimates [NTV98] needed to extend L^2 to L^p boundedness of Calderón-Zygmund operators as in the classical homogeneous spaces where one can consider only L^2 case.

On the other side, the connection between the theory of Carleson measures and the theory of trees and tiles has played an interesting role in bilinear singular integrals, see for example [LT97, LT00, MTT02a]. In the expository article of Auscher, Hoffmann, Muscalu, Tao and Thiele

[AHM$^+$02], they reprove Carleson embedding theorem on trees and hence paraproduct estimates and T1 theorem in this manner though the kernel associated to the operator satisfies so-called perfect conditions where one has stronger smoothness conditions. Also, Tb theorem is proved by T1 theorem, tree selection arguments and size estimates under the Lebesgue measure. In this way, they provide different proofs which are clearer in many senses and obtain some extended results. Therefore, it is interesting to apply these time-frequency analysis to generalized T1 or Tb theory, i.e. in non-homogeneous setting in which we have not seen such extension. This leads to another good point of view to understand the theory. We now introduce the main objects and state the main theorems reproved and proved in this work.

Let d be a positive number not necessary the same as dimension N and let μ be a Borel measure on $\mathbf{R}^N$ satisfying $\mu(B(x,r)) \le r^d$ for all ball $B(x,r)$. Note that the doubling property is not assumed. A function $K : \mathbf{R}^N \times \mathbf{R}^N \backslash \{(x,y) : x = y\}$ is said to be a *Calderón-Zygmund kernel* if it satisfies the following conditions for some constants $C > 0$ and $\alpha > 0$:

- $|K(x,y)| \le C \dfrac{1}{|x-y|^d}$
- there exists $\alpha > 0$ such that

$$|K(x,y) - K(x,y')| \le C \frac{|y-y'|^\alpha}{|x-y|^{d+\alpha}} \quad \text{when} \quad |x-y| > 2|y-y'|,$$

  and

$$|K(x,y) - K(x',y)| \le C \frac{|x-x'|^\alpha}{|x-y|^{d+\alpha}} \quad \text{when} \quad |x-y| > 2|x-x'|\ .$$

A *Calderón-Zygmund operator* is an operator $T : \mathcal{S}(\mathbf{R}^N) \to \mathcal{S}'(\mathbf{R}^N)$ of the form

$$Tf(x) = \int K(x,y) f(x) dx$$

for all $f \in \mathcal{C}_c^\infty$ and $x \notin supp(f)$ with the Calderón-Zygmund kernel K and it can be extended to a bounded operator on L^2. Some authors do not require L^2 boundedness but we will keep this original definition.

Another assumption is weak boundedness of the operators. For general measures, we say the operator T is *weakly bounded* if there exist $\Lambda \ge 1,\ C < \infty$ such that $|\langle T\chi_Q, \chi_Q\rangle| \le C\mu(\Lambda Q)$ for any cube $Q \in \mathbf{R}^N$.

The statement of T1 theorem is the following:

Theorem (David-Journé). *A Calderón-Zygmund operator T extends to a bounded operator on $L^2(\mu)$ if the operator T is weakly bounded and $T1$, T^*1 belong to $BMO(\mu)$.*

Next chapter will say more about BMO space used in this general theory. As for Tb theorem, it can be obtained in a similar manner to our proof of T1 theorem but we desire different approaches. We have noticed the local L^p testing conditions for the theorem in the work of Auscher et al. This type of condition was introduced by Christ with L^∞ control [Chr90]. Both of them are in the homogeneous world. Recently, Lacey and Vähäkangas [LV16] extend this story to non-homogeneous local T1 theorem even with dual exponents. We then prove Tb theorem with L^2 testing conditions since we have not seen any result under this setting. This path also serves as a good start to an open problem of Tb theorem for dual exponents where the number two in the local testing conditions in the assumption below is replaced by conjugate pairs p and q. We now define weak accretivity and provide the statement below.

A bounded function b is *weakly accretive* if there exists $\delta > 0$ such that for any cube Q,

$$\frac{1}{\mu(Q)} \left| \int_Q b(x) \, \mathrm{d}\mu(x) \right| \geq \delta.$$

Thus, we have that $|b| \geq \delta$ μ-almost everywhere.

Theorem. *Let T be a Calderón-Zygmund operator satisfying that there exist weakly accretive functions b_1, b_2 and a constant B such that for all cubes Q in $\mathbf{R}^N$,*

$$\|T(b_1\chi_Q)\|_{L^2(Q)} \leq B\mu(Q)^{1/2} \quad \textit{and} \quad \|T^*(b_2\chi_Q)\|_{L^2(Q)} \leq B\mu(Q)^{1/2}.$$

Then T is bounded on L^2.

The proofs are based on random lattices in which one utilize them to avoid bad parts when it comes to analysis of small pieces. This idea was applied to handle Calderón-Zygmund operators in [NTV97]. As a common procedure, one works with Haar system and, in non-homogeneous spaces, one needs martingale difference and also adapted version of it. We observe all of these with their properties in chapter 2 as well as required known lemmas.

As usual in harmonic analysis, there are many components to handle. We then analyze each component in chapter 3 using knowledge from [NTV03, AHM$^+$02].

In chapter 4, we prove T1 theorem as stated above. Although the way we decompose and control most terms are not new ideas, we can apply time-frequency analysis to achieve embedding theorem and thus boundedness of paraproducts. In addition, we are able to deal with a mistake on one of the considered terms in [NTV03].

In chapter 5, we investigate estimates of the paraproduct $\Pi(f, g)$ risen from proving T1 theorem. It has been a topic of interest to seek paraproduct estimates, e.g. boundedness from $L^p \times L^q$ to L^r, as in [AHM$^+$02, MTT02b, Li08] and such estimates for the paraproduct Π are unknown. With the use of time-frequency analysis technique, Li improves this kind of investigation to r larger than 1/2 instead of 1. We then follow this method and obtain the following result.

Theorem. *The paraproduct* $\Pi(f, g)$ *is bounded from* $L^p \times L^q \to L^r$ *where* $1 < p, q < \infty$ *and* $1/p + 1/q = 1/r$.

In the last chapter, we prove Tb theorem stated above by combining techniques from T1 theorem and [LV16]. In particular, we reduce the problem to study the good part via probabilistic techniques. Then decompose it so that some terms can be treated using estimates as in T1 case. For the paraproduct term, we rewrite it regarding a sparse tree and bound each of them.

Chapter 2

Preliminaries

2.1 BMO spaces

Definition. Let $1 \leq p < \infty$ and $\lambda > 1$. Let $f \in L^1_{loc}(\mu)$. We say $f \in \mathrm{BMO}^p_\lambda(\mu)$ if for any cube Q there exists a constant a_Q such that

$$\left(\int_Q |f - a_Q|^p \, \mathrm{d}\mu \right)^{1/p} \leq C\mu(\lambda Q)^{1/p}.$$

The infimum of such constant C taking over all Q is called the $\mathrm{BMO}^p_\lambda(\mu)$-norm of f. Note that the constant a_Q can be replaced by the average $\langle f \rangle_Q := \mu(Q)^{-1} \int_Q f \, \mathrm{d}\mu$. Also we know from [NTV03] that

$$\mathrm{BMO}^{p_2}_\lambda(\mu) \subset \mathrm{BMO}^{p_1}_\lambda(\mu) \quad \text{if } p_1 < p_2, \text{ and} \quad \mathrm{BMO}^p_\lambda(\mu) \subset \mathrm{BMO}^p_\Lambda(\mu) \quad \text{if } \lambda < \Lambda.$$

2.2 Useful lemmas

Lemma (Comparison Lemma)**.** *Let $F \geq 0$ be a decreasing function on $(0, \infty)$, and let the measure μ satisfy $\mu(B(x_0, r)) \leq r^d$ for a fixed x_0 and for all $r \geq 0$. Then, for $\delta > 0$,*

$$\int_{x : \delta \leq |x - x_0|} F(|x - x_0|) \, \mathrm{d}\mu(x) \leq F(\delta)\delta^d + d \int_\delta^\infty F(t) t^{d-1} dt.$$

Lemma (Schur's Test)**.** *Let $K : X \times Y \to \mathbb{C}$ be a measurable function obeying the bounds*

$$\|K(x, \cdot)\|_{L^1} \leq C$$

for almost every $x \in X$, and

$$\|K(\cdot, y)\|_{L^1} \leq C$$

for almost every $y \in Y$. Then the integral operator T is bounded on L^2.

2.3 Martingale decomposition

In this section, we study decomposition of functions into functions on dyadic cubes.

Definition. A *dyadic cube* I is a cube of the form

$$I = [j2^k, (j+1)2^k]^N$$

where $j, k \in \mathbf{Z}$. Denote a *(standard) dyadic lattice* $\mathcal{I}$ the set of all dyadic cubes.

2.3.1 Decomposition for T1

Let $f \in L^2$. Define the averaging operator E_k by

$$E_k f(x) := \sum_{Q \in \mathcal{S}_k} \fint_Q f \, \mathrm{d}\mu \chi_Q(x) = \sum_{Q \in \mathcal{S}_k} \frac{\chi_Q(x)}{\mu(Q)} \int_Q f \, \mathrm{d}\mu.$$

where $\mathcal{S}_k = \{Q \in \mathcal{I} : l(Q) = 2^{-k}\}$. If $l(Q) = 2^{-k}$, then define $E_Q f := (E_k f)\chi_Q$, $\triangle_k f := E_{k+1} f - E_k f$, and $\triangle_Q f := (\triangle_k f)\chi_Q$.

Proposition 2.1. *1. $\{\triangle_Q f : Q \in \mathcal{I}\}$ is orthogonal.*

2. $\triangle_Q = \triangle_Q^*$ for all $Q \in \mathcal{I}$.

3. $\triangle_Q(\triangle_Q) = \triangle_Q$ for all $Q \in \mathcal{I}$.

Proof. 1. For any $Q, R \in \mathcal{I}$, $\langle \triangle_Q f, \triangle_R f \rangle = 0$ if $Q \cap R = \varnothing$. In case $Q \cap R \neq \varnothing$, we assume that $l(Q) = 2^{-k}, l(R) = 2^{-l}$ and WLOG that $k > l$. Since

$$\langle \triangle_Q f, \triangle_R f \rangle = \int (E_{k+1} f(x) - E_k f(x))(E_{l+1} f(x) - E_l f(x))\chi_Q(x) \, \mathrm{d}\mu,$$

we observe that

$$
\begin{aligned}
E_k f(x) E_l f(x) &= \left(\sum_{Q' \in \mathcal{S}_k} \fint_{Q'} f \, \mathrm{d}\mu \chi_{Q'}(x) \right) \left(\sum_{R \in \mathcal{S}_l} \fint_R f \, \mathrm{d}\mu \chi_R(x) \right) \\
&= \sum_{Q' \in \mathcal{S}_k} \sum_{R \in \mathcal{S}_l} \fint_{Q'} f \, \mathrm{d}\mu \fint_R f \, \mathrm{d}\mu \chi_{Q'}(x)
\end{aligned}
$$

so that

$$\int_Q E_k f(x) E_l f(x) \, \mathrm{d}\mu = \sum_{R \in \mathcal{S}_l} \int_Q f \, \mathrm{d}\mu \fint_R f \, \mathrm{d}\mu.$$

Similarly,

$$\int_Q E_k f(x) E_{l+1} f(x) \, \mathrm{d}\mu = \sum_{R \in \mathcal{S}_{l+1}} \int_Q f \, \mathrm{d}\mu \fint_R f \, \mathrm{d}\mu$$

Also, we have

$$
\begin{aligned}
\int_Q E_{k+1} f(x) E_{l+1} f(x) \, \mathrm{d}\mu &= \sum_{Q' \in ch(Q)} \sum_{R \in \mathcal{S}_{l+1}} \int_{Q'} f \, \mathrm{d}\mu \fint_R f \, \mathrm{d}\mu \\
&= \sum_{R \in \mathcal{S}_{l+1}} \int_Q f \, \mathrm{d}\mu \fint_R f \, \mathrm{d}\mu
\end{aligned}
$$

and similarly,

$$\int_Q E_{k+1} f(x) E_l f(x) \, \mathrm{d}\mu = \sum_{R \in \mathcal{S}_l} \int_Q f \, \mathrm{d}\mu \fint_R f \, \mathrm{d}\mu.$$

Therefore, $\langle \triangle_Q f, \triangle_R f \rangle = 0$.

2. It is straightforward to see that

$$
\begin{aligned}
\langle \triangle_Q f, g \rangle &= \int \Big(\sum_{R \in ch(Q)} \frac{g\chi_R}{\mu(R)} \int_R f \, \mathrm{d}\mu - \frac{g\chi_Q}{\mu(Q)} \int_Q f \, \mathrm{d}\mu \Big) \, \mathrm{d}\mu \\
&= \sum_{R \in ch(Q)} \frac{1}{\mu(R)} \int_R g \, \mathrm{d}\mu \int_R f \, \mathrm{d}\mu - \frac{1}{\mu(Q)} \int_R g \, \mathrm{d}\mu \int_Q f \, \mathrm{d}\mu \\
&= \int \Big(\sum_{R \in ch(Q)} \frac{f\chi_R}{\mu(R)} \int_R g \, \mathrm{d}\mu - \frac{f\chi_Q}{\mu(Q)} \int_Q g \, \mathrm{d}\mu \Big) \, \mathrm{d}\mu \\
&= \langle f, \triangle_Q g \rangle.
\end{aligned}
$$

3. It is also easy to see that

$$\begin{aligned}
\triangle_Q(\triangle_Q f) &= \sum_{R\in ch(Q)} \frac{\chi_R}{\mu(R)}\int_R \triangle_Q f \,\mathrm{d}\mu - \frac{\chi_Q}{\mu(Q)}\int_Q \triangle_Q f \,\mathrm{d}\mu \\
&= \sum_{R\in ch(Q)} (\frac{\chi_R}{\mu(R)}\int_R f \,\mathrm{d}\mu - \frac{\chi_R}{\mu(Q)}\int_Q f \,\mathrm{d}\mu) - 0 \\
&= \sum_{R\in ch(Q)} \frac{\chi_R}{\mu(R)}\int_R f \,\mathrm{d}\mu - \frac{\chi_Q}{\mu(Q)}\int_Q f \,\mathrm{d}\mu \\
&= \triangle_Q f.
\end{aligned}$$

□

Proposition 2.2. *Let $\mathfrak{F}_n$ be the smallest σ-algebra containing $\mathcal{S}_n$ and $\mathfrak{F}$ be the smallest σ-algebra containing $\bigcup_{n\in\mathbf{N}} \mathcal{S}_n$ so that the system of sub-σ-algebra of $\mathfrak{F}$, $\{\mathfrak{F}_n : n \in \mathbf{Z}\}$, is a filtration. Then $\{E_n f, \mathfrak{F}_n\}$ is a martingale for $f \in L^1$.*

Proof. First of, for all n we have $E_n f \in L^1$ and for a fixed P of size 2^{-n} we know that $E_n f(x)$ is constant for all $x \in P$. Therefore, $E_n f$ is $\mathfrak{F}_n$-measurable. Lastly, we need to check that $\mathbb{E}[E_n f|\mathfrak{F}_m] = E_m f$ for $n > m$. Indeed, for all $Q \in \mathfrak{F}_m$, we have $l(Q) \geq 2^{-m} > 2^{-n}$. Thus, $\int_Q E_m f \,\mathrm{d}\mu = \int_Q E_n f \,\mathrm{d}\mu$. □

Lemma 2.3 (Martingale difference decomposition)**.** *Every function $f \in L^2(\mu)$ can be decomposed as*

$$f = \sum_{P\in\mathcal{I}} \triangle_P f$$

in L^2. Moreover,

$$\|f\|_{L^2}^2 = \sum_{P\in\mathcal{I}} \|\triangle_P f\|_{L^2}^2$$

Proof. By Lebesgue Differentiation Theorem as $l \to \infty$,

$$E_l f(x) \to f(x) \text{ in } L^2.$$

On the other hand, $E_l f(x) \to 0$ in L^2 as $l \to -\infty$. In fact, for each x, $|E_l f(x)| = |\mu(Q)|^{-1}|\int_Q f \,\mathrm{d}\mu| \leq$

$\|f\|_2\mu(Q)^{-1/2} \to 0$ as $l \to -\infty$. Thus,

$$\sum_{k=-\infty}^{\infty} \triangle_k f(x) = \lim_{|l|\to\infty} E_{l+1}f(x) - E_l f(x) = f(x).$$

The result about L^2 norm of f follows from the orthogonality of $\triangle_P f$ to each $P \in \mathcal{I}$. $\square$

Lemma 2.4. *For every $f \in L^2$,*

$$f = \sum_{Q\in\mathcal{I}} c_Q(f)\chi_Q$$

where $c_Q(f)$ are constants. Moreover,

$$\|f\|_2^2 = \sum_{Q\in\mathcal{I}} |c_Q(f)|^2\mu(Q).$$

Proof. Set $f^k := \sum_{\substack{Q\in\mathcal{I}\\ l(Q)=2^k}} \triangle_Q f$ Then we observe that for each Q of size 2^k, $\triangle_Q f = \sum_{R\in ch(Q)} \frac{\chi_R}{\mu(R)} \int_R f \,\mathrm{d}\mu - \frac{\chi_Q}{\mu(Q)} \int_Q f \,\mathrm{d}\mu = \sum_{R\in ch(Q)} \left(\frac{1}{\mu(R)} \int_R f \,\mathrm{d}\mu - \frac{1}{\mu(Q)} \int_Q f \,\mathrm{d}\mu\right)\chi_R$. Thus, $f^k = \sum_{\substack{R\in\mathcal{I}\\ l(R)=2^{k-1}}} c_R(f)\chi_R$ so that $f = \sum_{k\in\mathbf{Z}} f^k = \sum_{k\in\mathbf{Z}} \sum_{\substack{R\in\mathcal{P}\\ l(R)=2^{k-1}}} c_R(f)\chi_R$. In addition, by disjointness one can see that

$$\|f\|_2^2 = \sum_k \|f^k\|_2^2 = \sum_k \sum_{\substack{R\in\mathcal{I}\\ l(R)=2^{k-1}}} \|c_R(f)\chi_R\|_2^2 = \sum_{R\in\mathcal{I}} |c_R(f)|^2\mu(R).$$

$\square$

2.3.2 Decomposition for Tb

In this part, we establish tools as in T1 theorem for Tb theorem.

Definition. Given a function $f \in L^2$ and a weakly accretive function b. Define $E_k^b f(x)$ to be

$$E_k^b f(x) := \sum_{Q\in\mathcal{S}_k} \left(\int_Q b \,\mathrm{d}\mu\right)^{-1} \left(\int_Q f \,\mathrm{d}\mu\right) b(x)\chi_Q(x)$$

and $\triangle_k^b f(x) := E_{k+1}^b f(x) - E_k^b f(x)$. For a cube $Q \in \mathcal{S}_k$, define $E_Q^b f(x) := (E_k^b f)\chi_Q(x)$ and $\triangle_Q^b f := (\triangle_k^b f)\chi_Q$. Moreover, define $\triangle_Q^c f(x)$ to be the part that $\triangle_Q^b f(x) = b(x)\cdot\triangle_Q^c f(x)$. Then, observe some properties.

Proposition 2.5. *1.* $\int \triangle_Q^b f \, \mathrm{d}\mu = 0$ *for all* $Q \in \mathcal{I}$.

2. $\triangle_k^b(\triangle_l^b f) = 0$ *for* $k \neq l$.

3. $\triangle_Q^b(\triangle_Q^b) = \triangle_Q^b$ *for all* $Q \in \mathcal{I}$.

4. $\triangle_k^c(E_l^b) = 0$ *for* $k > l$ *and* $\triangle_k^c(\triangle_l^b) = 0$ *for* $k \neq l$.

Proof. 1. Basic computation tells us that

$$\begin{aligned}
\int \triangle_Q^b f \, \mathrm{d}\mu &= \int_Q E_{k+1}^b f(x) - E_k^b f(x) \, \mathrm{d}\mu \\
&= \sum_{Q' \in ch(Q)} \left(\int_{Q'} b \, \mathrm{d}\mu \right)^{-1} \left(\int_{Q'} f \, \mathrm{d}\mu \right) \int_{Q'} b \, \mathrm{d}\mu - \left(\int_Q b \, \mathrm{d}\mu \right)^{-1} \left(\int_Q f \, \mathrm{d}\mu \right) \int_Q b \, \mathrm{d}\mu \\
&= \sum_{Q' \in ch(Q)} \int_{Q'} f \, \mathrm{d}\mu - \int_Q f \, \mathrm{d}\mu = 0.
\end{aligned}$$

2. We first show that $\triangle_k^b(E_n^b f) = 0$ and $E_n^b(\triangle_k^b f) = 0$ for $k \geq n$. Indeed, for $k \geq n$

$$\begin{aligned}
E_{k+1}^b(E_n^b f) &= \sum_{Q \in S_{k+1}} (\int_Q b)^{-1} (\int_Q E_n^b f) b \chi_Q(x) \\
&= \sum_{Q \in S_{k+1}} (\int_Q b)^{-1} \int_Q \Big(\sum_{R \in S_n} (\int_R b)^{-1} (\int_R f) b \chi_R(x) \Big) \, \mathrm{d}\mu b \chi_Q(x) \\
&= \sum_{Q \in S_{k+1}} (\int_R b)^{-1} (\int_R f) b \chi_Q(x), \text{ where } R \supseteq Q
\end{aligned}$$

Similarly,

$$E_k^b(E_n^b f) = \sum_{Q' \in S_k} (\int_R b)^{-1} (\int_R f) b \chi_{Q'}(x), \text{ where } R \supseteq Q'.$$

Since for each $Q \in S_{k+1}$ such that $Q \subset Q'$, Q shares the same R, we have

$$\sum_{Q \in S_{k+1}} (\int_R b)^{-1} (\int_R f) b \chi_Q(x) = \sum_{Q' \in S_k} (\int_R b)^{-1} (\int_R f) b \chi_{Q'}(x), \, R \supseteq Q'.$$

Hence $\triangle_k^b(E_n^b f) = E_{k+1}^b(E_n^b f) - E_k^b(E_n^b f) = 0$. To see the other equality, consider that

$$\begin{aligned}
E_n^b(E_{k+1}^b f) &= \sum_{R\in S_n} (\int_R b)^{-1} (\int_R E_{k+1}^b f) b\chi_R(x) \\
&= \sum_{R\in S_n} (\int_R b)^{-1} \int_R \Big(\sum_{Q\in S_{k+1}} (\int_Q b)^{-1} (\int_Q f) b\chi_Q(x) \Big) \, \mathrm{d}\mu b\chi_R(x) \\
&= \sum_{R\in S_n} (\int_R b)^{-1} \Big(\sum_{Q\subset R} (\int_Q f) \Big) b\chi_R(x) \\
&= \sum_{R\in S_n} (\int_R b)^{-1} (\int_R f) b\chi_R(x) \\
&= E_n^b f.
\end{aligned}$$

Similarly, $E_n^b(E_k^b f) = E_n^b f$. Thus, $E_n^b(\triangle_k^b f) = E_n^b(E_{k+1}^b f) - E_n^b(E_k^b f) = 0$. Then, recall that $\triangle_k^b(\triangle_l^b f) = E_{k+1}^b(\triangle_l^b f) - E_k^b(\triangle_l^b f)$. Since $E_n^b(\triangle_k^b f) = 0$ for $k \geq n$, we have $\triangle_k^b(\triangle_l^b f) = 0$ when $l \geq k-1$. When $l < k$, we see that $\triangle_k^b(\triangle_l^b f) = \triangle_k^b(E_{l+1}^b f) - \triangle_k^b(E_l^b f) = 0$ by what we have shown above as well.

3. By definition and 2, for $Q \in S_k$, $\triangle_Q^b(\triangle_Q^b f) = (E_{k+1}^b(\triangle_Q^b f) - E_k^b(\triangle_Q^b f))\chi_Q = (E_{k+1}^b(\triangle_Q^b f))\chi_Q = \sum\limits_{R\in ch(Q)} (\int\limits_R b)^{-1} (\int\limits_R \triangle_Q^b f) b\chi_R$. Since $\int\limits_R \triangle_Q^b f = \int\limits_R f - (\int\limits_R b)(\int\limits_Q b)^{-1}(\int\limits_Q f)$, it follows that $\triangle_Q^b(\triangle_Q^b f) = \triangle_Q^b f$.

4. Similar calculation as in 2.

□

Lemma 2.6 (Weighted martingale difference decomposition)**.** *Let b be a weakly accretive function, and let $n \in \mathbf{Z}$. Then, any $f \in L^2(\mu)$ can be decomposed as*

$$f = \sum_{\substack{Q\in\mathcal{I} \\ l(Q)\leq 2^n}} \triangle_Q^b f + \sum_{\substack{Q\in\mathcal{I} \\ l(Q)=2^n}} E_Q^b f$$

in L^2. Moreover,

$$\sum_{\substack{Q\in\mathcal{I} \\ l(Q)\leq 2^n}} \left\| \triangle_Q^b f \right\|_{L^2}^2 + \sum_{\substack{Q\in\mathcal{I} \\ l(Q)=2^n}} \left\| E_Q^b f \right\|_{L^2}^2 \leq C(b,\delta)\|f\|_{L^2}^2$$

Proof. Let $\mathcal{I}$ be a dyadic lattice. We will show first that the following set E is dense in L^2,

$$E := \Big\{ \sum_{Q\in\mathcal{S}_k} C_Q \chi_Q b(x) : k \in \mathbf{Z} \Big\},$$

so that it is enough to prove the lemma on this subset. Indeed, for any $\varepsilon > 0$, and $f \in L^2$, there exists a simple function $g = \sum_{i=1}^{n} a_i \chi_{D_i}(x)$, where $a_i \in \mathbf{R}$, $D_i \in \mathfrak{B}$ such that $\|f - g\|_{L^2} \leq \varepsilon$. Now, we observe that $D_i = \bigsqcup_{j=1}^{m} Q_j^i$ where $Q_j^i \in S_k$. Consider $h(x) := \sum_{i=1}^{n} c_i b(x)\chi_{D_i}(x) = \sum_{i=1}^{n}\sum_{j=1}^{m} c_i b(x)\chi_{Q_j^i}(x) \in E$ where $c_i = \frac{a_i}{b(x)}$. Note that for each i, $|b(x)| > 0$ μ-a.e. on D_i and hence on Q_j^i for all j by weak accretivity of b. Thus we have $|\sum_{i=1}^{n} a_i\chi_{D_i}(x) - c_i b(x)\chi_{D_i}(x)| = |\sum_{i=1}^{n}(a_i - c_i b(x))\chi_{D_i}(x)| = 0.$ and hence $\|g - h\|_{L^2} = 0$. Therefore, $\|f - h\|_{L^2} \leq \|f - g\|_{L^2} + \|g - h\|_{L^2} \leq \varepsilon$ as desired for the density.

Now for any fixed $k \in \mathbf{Z}$, let $f = \sum_{R\in S_k} C_R b(x)\chi_R(x)$ we consider the term

$$\sum_{\substack{Q\in\mathcal{I}\\ l(Q)\leq 2^{-k}}} \triangle_Q^b f + \sum_{Q\in\mathcal{S}_k} E_Q^b f.$$

We will show first that $\sum_{\substack{Q\in\mathcal{I}\\ l(Q)\leq 2^{-k}}} \triangle_Q^b f = 0$. Fix $j \in \mathbf{Z}$, and a cube $Q \in \mathcal{S}_j$, for any $j \geq k$. Then we have that

$$\begin{aligned}
E_{j+1}^b f(x) &= \sum_{Q'\in\mathcal{S}_{j+1}} (\int_{Q'} b \,\mathrm{d}\mu)^{-1} (\int_{Q'} f \,\mathrm{d}\mu) b(x)\chi_{Q'}(x) \\
&= \sum_{Q'\in\mathcal{S}_{j+1}} (\int_{Q'} b \,\mathrm{d}\mu)^{-1} (C_R \int_{Q'} b \,\mathrm{d}\mu) b(x)\chi_{Q'}(x), \text{ where } R \supset Q' \\
&= \sum_{Q'\in\mathcal{S}_{j+1}} C_R b(x)\chi_{Q'}(x)
\end{aligned}$$

and similarly

$$\begin{aligned}
E_j^b f(x) &= \sum_{Q\in\mathcal{S}_j} (\int_{Q} b \,\mathrm{d}\mu)^{-1} (\int_{Q} f \,\mathrm{d}\mu) b(x)\chi_{Q}(x) \\
&= \sum_{Q\in\mathcal{S}_j} C_R b(x)\chi_Q(x), \text{ where } R \supseteq Q.
\end{aligned}$$

Hence, $\triangle_Q^b f = \left(E_{j+1}^b f - E_j^b f\right)\chi_Q = \sum_{Q'\in ch(Q)} C_R b\chi_{Q'} - C_R b\chi_Q = C_R b\chi_Q - C_R b\chi_Q = 0$. Since this is true for all $j \geq k$, we proved that $\sum_{\substack{Q\in\mathcal{I}\\ l(Q)\leq 2^{-k}}} \triangle_Q^{b_1} f = 0$.

Now, we investigate the remaining term $\sum_{Q\in\mathcal{S}_k} E_Q^b f$. As seen above, $E_k^b f = \sum_{Q\in\mathcal{S}_k} C_R b(x)\chi_Q(x)$, where $R \supseteq Q$. Since $R, Q \in \mathcal{S}_k$, it must be $E_k^b f = \sum_{Q\in\mathcal{S}_k} C_Q b(x)\chi_Q(x)$ and hence $E_Q^b f = C_Q b(x)\chi_Q(x)$. Thus, $\sum_{Q\in\mathcal{S}_k} E_Q^b f = \sum_{Q\in\mathcal{S}_k} C_Q b(x)\chi_Q(x) = f$. Therefore, $\sum_{\substack{Q\in\mathcal{I}\\ l(Q)\leq 2^{-k}}} \triangle_Q^b f + \sum_{Q\in S_k} E_Q^b f = f$ as desired.

For the estimate, We consider that for each $Q \in \mathcal{S}_n$,

$$\begin{aligned}
\int_Q |E_n^b f(x)|^2 \,\mathrm{d}\mu &= \int_Q \Big| \sum_{Q'\in\mathcal{S}_n} \left(\int_{Q'} b \,\mathrm{d}\mu\right)^{-1} \left(\int_{Q'} f \,\mathrm{d}\mu\right) b(x)\chi_{Q'}(x) \Big|^2 \,\mathrm{d}\mu \\
&\leq \sum_{Q'\in S_n} \int_Q \Big| \left(\int_{Q'} b \,\mathrm{d}\mu\right)^{-1} \Big|^2 \Big| \int_{Q'} f \,\mathrm{d}\mu \Big|^2 |b(x)\chi_{Q'}|^2 \,\mathrm{d}\mu \\
&= \Big| \left(\int_Q b \,\mathrm{d}\mu\right)^{-1} \Big|^2 \Big| \int_Q f \,\mathrm{d}\mu \Big|^2 \int_Q |b(x)|^2 \,\mathrm{d}\mu \\
&\leq \delta^{-2}\mu(Q)^{-2}\|b\|_\infty^2 \mu(Q) \Big| \int_Q f \,\mathrm{d}\mu \Big|^2 \\
&\leq \delta^{-2}\|b\|_\infty^2 \int_Q |f|^2 \,\mathrm{d}\mu
\end{aligned}$$

and thus

$$\sum_{Q\in S_n} \|E_Q^b f\|_{L^2}^2 \leq \delta^{-2}\|b\|_\infty^2 \sum_{Q\in S_n} \int_Q |f|^2 \,\mathrm{d}\mu \leq C(\delta, b)\|f\|_{L^2}^2.$$

This part is to show that $\sum_{Q\in\mathcal{I}, l(Q)\leq 2^n} \|\triangle_Q^b f\|_{L^2}^2 \leq C(\delta, b)\|f\|_{L^2}^2$. First observe that

$$\sum_{Q\in\mathcal{I}, l(Q)\leq 2^n} \|\triangle_Q^b f\|_{L^2}^2 = \sum_{k\leq n} \sum_{Q\in S_k} \|\triangle_Q^b f\|_{L^2}^2 = \sum_{k\leq n} \int |\triangle_k^b f|^2 \,\mathrm{d}\mu = \sum_{k\leq n} \|\triangle_k^b f\|_{L^2}^2$$

by disjointness of same-sized cubes. Consider next that

$$\begin{aligned}
\triangle_k^b f = E_{k-1}^b f - E_k^b f &= \big((E_{k-1}b)^{-1} E_{k-1} f - (E_k b)^{-1} E_k f\big) b \\
&= (E_{k-1}b)^{-1}(E_{k-1}f - E_k f)b + E_k f\big((E_{k-1}b)^{-1} - (E_k b)^{-1}\big)b \\
&= (E_{k-1}b)^{-1}\triangle_k f b - E_k f \frac{\triangle_k b}{E_k b E_{k-1} b} b.
\end{aligned}$$

Since b is weakly accretive,

$$\sum_{k\le n} \|(E_{k-1}b)^{-1}\triangle_k fb\|_{L^2}^2 \lesssim \delta^{-2}\|b\|_\infty^2\|f\|_{L^2}^2.$$

For the second sum, we recall that $\|f\|_{L^2}^2 = \sum\limits_{\substack{Q\in\mathcal{I}\\ l(Q)\le 2^n}} \|\triangle_Q f\|_{L^2}^2 + \sum\limits_{Q\in S_n} \|E_Q f\|_{L^2}^2$ for any $f \in L^2$. Hence, $\sum\limits_{R\subseteq Q} \|\triangle_R b\|_{L^2}^2 \le \int\limits_Q |b|^2 \,\mathrm{d}\mu \le \|b\|_\infty^2 \mu(Q)$ for any $b \in L^\infty$ so that

$$\sum_{R\subseteq Q} a_R\mu(R) := \sum_{R\subseteq Q} \mu(R)^{-1}\|\triangle_R b\|_{L^2}^2\mu(R) \le \|b\|_\infty^2\mu(Q).$$

By the embedding theorem above, we have

$$\sum_{Q\in\mathcal{I}} \|\triangle_Q b\|_{L^2}^2\big(\mu(Q)^{-1}\int_Q f\,\mathrm{d}\mu\big)^2 = \sum_{Q\in\mathcal{I}} a_Q\big(\mu(Q)^{-1}\int_Q f\,\mathrm{d}\mu\big)^2\mu(Q) \le C\|f\|_{L^2}^2.$$

Since b is weakly accretive, $|b(x)/(E_k b(x)E_{k-1}b(x))|^2 \le C(\delta, b)$. Therefore,

$$\begin{aligned}
\sum_{k\le n}\left\|E_k f\frac{\triangle_k b}{E_k b E_{k-1} b}b\right\|_{L^2}^2 &\le C\sum_{k\le n}\|(E_k f)(\triangle_k b)\|_{L^2}^2 \\
&= C\sum_{k\le n}\int\Big|\sum_{Q\in S_k}\frac{\chi_Q(x)}{\mu(Q)}\int\limits_Q f\,\mathrm{d}\mu\triangle_Q b(x)\Big|^2\,\mathrm{d}\mu \\
&= C\sum_{k\le n}\sum_{Q\in S_k}\int\limits_Q\frac{|\int_Q f\,\mathrm{d}\mu|^2}{|\mu(Q)|^2}|\triangle_Q b|^2\,\mathrm{d}\mu \\
&= C\sum_{Q\in\mathcal{I}}\frac{|\int_Q f\,\mathrm{d}\mu|^2}{|\mu(Q)|^2}\|\triangle_Q b\|_{L^2}^2 \\
&\le C\|f\|_{L^2}^2.
\end{aligned}$$

Thus, we proved that $\sum\limits_{\substack{Q\in\mathcal{I}\\ l(Q)\le 2^n}} \|\triangle_Q^b f\|_{L^2}^2 = \sum\limits_{k\le n}\|\triangle_k^b f\|_{L^2}^2 \le C(\delta,b)\|f\|_{L^2}^2$ and hence the lemma. □

Next, we observe another direction to decompose functions and the estimate of this form.

Lemma 2.7. *For every $f \in L^2$,*

$$f = \sum_{Q\in\mathcal{I}} c_Q(f)b\chi_Q$$

where $c_Q(f)$ are constants. Moreover,

$$\sum_{Q\in\mathcal{I}} |c_Q(f)|^2\mu(Q) \lesssim \|f\|_2^2.$$

Proof. Since $\triangle_Q^b f$ can be written as $c_{Q'}(f)b$ where $Q' \in ch(Q)$ and $E_Q^b f$ can be written as $c_Q(f)b$. Hence, we get the decomposition. To see the estimate, one observe that

$$\delta^2 \sum_{Q\in\mathcal{I}} |c_Q(f)|^2\mu(Q) \le \sum_k \sum_{\substack{Q\in\mathcal{I}\\ l(Q)=2^k}} |c_Q(f)|^2\|b\chi_Q\|_2^2 = \sum_k \| \sum_{\substack{Q\in\mathcal{I}\\ l(Q)=2^k}} \triangle_Q^b f\|_2^2 = \sum_k \|\triangle_k^b f\|_2^2 \lesssim \|f\|_2^2$$

where the first inequality follows from the property that $|b| \ge \delta$ μ-almost everywhere and the last from the previous Lemma 2.6. $\square$

Remark. One may notice that the constant $c_{Q'}(f)$ is in fact the value $\triangle_Q^c f(x)$ when $x \in Q'$, the children of Q. Thus, let us state an alternative form of the estimate as the inequality

$$\Big\|\Big(\sum_{k\in\mathbf{Z}} |\triangle_k^c f|^2\Big)^{1/2}\Big\|_2 \lesssim \|f\|_2 \tag{2.1}$$

Next, the well-known Stein's inequality is needed therefore we record it here, see e.g. [Bou86].

Lemma. *For any $1 < p < \infty$ and sequence $(f_k)_{k\in\mathbf{Z}}$ in L^p,*

$$\left\|\Big(\sum_{k\in\mathbf{Z}} |E_k f_k|^2\Big)^{1/2}\right\|_p \lesssim \left\|\Big(\sum_{k\in\mathbf{Z}} |f_k|^2\Big)^{1/2}\right\|_p. \tag{2.2}$$

Last but not least, we have the following somewhat martingale transform inequality in a general measure, i.e. with the adapted martingale difference.

Lemma 2.8. *For any $T \in \mathcal{T}$, functions $f \in L^2$, and constants satisfying* $\sup_{Q\in\mathcal{D}} |\varepsilon_Q| \le 1$,

$$\Big\| \sum_{\substack{Q\in\mathcal{D}\\ \pi_{\mathcal{T}}Q=T}} \varepsilon_Q \triangle_Q^b f\Big\|_2 \lesssim \|f\|_2.$$

One may think of $\mathcal{T}$ as a family of cubes or a tree and $\pi_{\mathcal{T}}Q$ as a parent of Q in the family $\mathcal{T}$. Their definitions will be introduced when we need it to handle the term involving $l(R) < 2^{-r}l(Q)$.

This lemma is proved in [LM16] where the problem is reduced to the bound of maximal truncations

$$\left\|\sup_{\varepsilon>0}\left|\sum_{\substack{Q\in\mathcal{D}\\ \pi_{\mathcal{T}}Q=T\\ l(Q)>\varepsilon}}\varepsilon_Q D_Q f\right|\right\|_2^2$$

where $D_Q f = \triangle_Q^c f$ except one sums over $Q' \in ch(Q)\backslash ch_{\mathcal{T}}T$. To have this kind of estimates, one usually rewrite $\triangle_Q^c f$ into terms, for example, as the sum over $Q' \in ch(Q)\backslash ch_{\mathcal{T}}T$ of

$$\frac{1}{\langle b\rangle_Q}(\langle f\rangle_{Q'}-\langle f\rangle_Q)+(\langle f\rangle_{Q'}-\langle f\rangle_Q)\left(\frac{1}{\langle b\rangle_{Q'}}-\frac{1}{\langle b\rangle_Q}\right)+\langle f\rangle_Q\left(\frac{1}{\langle b\rangle_{Q'}}-\frac{1}{\langle b\rangle_Q}\right)$$

so that the classical martingale transform inequality comes to help. For Lebesgue measure, the inequality as the classical one where one sums over all dyadic cubes holds with help from the perfect Calderón-Zygmund operator [LV14]. However, in our setting, the above lemma is all we need.

2.4 Random dyadic lattices

This section is where probabilistic analysis is in charge. A dyadic lattice randomly shifted from the standard dyadic lattice is introduced to obtain a desired distribution property.

Construction of a random dyadic lattice.

Let $\Omega = [0,1]$, $\mathfrak{B}$ is a Borel $\sigma-$ algebra, and l is the Lebesgue measure so that $(\Omega, \mathfrak{B}, l)$ is a probability space. Let $\eta(\omega) = \omega$ be a random variable uniformly distributed over [0,1). Indeed, if $x < 0$, then $F(x) = l(\omega : \omega = \eta(\omega) < x) = 0$. If $x \in [0,1]$, then $F(x) = l(\omega : \omega = \eta(\omega) < x) = x$. If $x > 1$, then $F(x) = l(\omega : \omega = \eta(\omega) < x) = 1$. So the distribution function is the uniform distribution. Let $\xi_j(\omega)$ be the following random variables for $j \in \mathbf{N}$:

$$\xi_j(\omega) = \begin{cases} 1 & , \omega \in \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right] \text{ for all positive odd number } k < 2^j \\ -1 & \text{otherwise} \end{cases},$$

so that $l\{\omega : \xi_j(\omega) = 1\} = 2^{j-1}\left(\frac{1}{2^j}\right) = \frac{1}{2} = 1 - \frac{1}{2} = l\{\omega : \xi_j(\omega) = -1\}$.

The random lattice $\mathcal{D}(\omega)$ consists of the following cubes (interval in this case):

i) $I_{k=0}(\omega) := [x(\omega)-1, x(\omega)] = [\omega-1,\omega] \in \mathcal{D}(\omega)$. $\widetilde{I_0}(\omega) := its\, siblings\, \in \mathcal{D}(\omega)$

ii) For $k<0$, $I_k(\omega) := \left[\omega - 2^k, \omega\right] \in \mathcal{D}(\omega)$. $\widetilde{I_k}(\omega) := its\, siblings\, \in \mathcal{D}(\omega)$

iii) For $k=1$, first choose one of $\widetilde{I_0}(\omega)$, say I_0. Then put

$$I_1(\omega) := \begin{cases} I_0 \cup \text{ left adjacent sibling of } I_0 & , \xi_1(\omega) = 1 \\ I_0 \cup \text{ right adjacent sibling of } I_0 & , \xi_1(\omega) = -1 \end{cases} \in \mathcal{D}(\omega)$$

and $\widetilde{I_1}(\omega) \in \mathcal{D}(\omega)$.

Inductively, we have all $I_k(\omega)$ and $\widetilde{I_k}(\omega) \in \mathcal{D}(\omega)$ for all $k>0$. In other words, for all $k>0$ we choose one of $\widetilde{I_{k-1}}(\omega)$, say I_{k-1} Then put

$$I_k(\omega) := \begin{cases} I_{k-1} \cup \text{ left adjacent sibling of } I_{k-1} & , \xi_k(\omega) = 1 \\ I_{k-1} \cup \text{ right adjacent sibling of } I_{k-1} & , \xi_k(\omega) = -1 \end{cases} \in \mathcal{D}(\omega)$$

and $\widetilde{I_k}(\omega) \in \mathcal{D}(\omega)$. Hence we get intervals of length 2^k for all $k \in \mathbf{Z}$ in the random lattice $\mathcal{D}(\omega)$.

Lastly, get the random lattice in $\mathbf{R}^N$ by taking a product of N independent random lattices $\mathcal{D}(\omega)$, $\omega \in \Omega$.

Lemma. *The random lattice $\mathcal{D}(\omega)$ in $\mathbf{R}^N$ is uniformly distributed over $\mathbf{R}^N$.*

It means, for any cube $Q \in \mathcal{D}(\omega)$, the probability that a given point $x \in Q$ is in a subcube $Q' \subseteq Q$ of $\mathbf{R}^N$, i.e. the event $E := \{\omega : x \in Q' \subseteq Q \in \mathcal{D}(\omega)\}$, is $\left(\frac{l(Q')}{l(Q)}\right)^N$.

Proof. For first dimension, since $\eta(\omega)$ is uniformly distributed, the probability

$$\mathbf{P}\left(\omega : \eta(\omega) < x\right) = F(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases} \cdot$$

Thus, for $l(Q) = 1$, the probability of E is $l(Q')$. Otherwise, the probability of E is the ratio $\frac{l(Q')}{l(Q)}$. Then we get $\left(\frac{l(Q')}{l(Q)}\right)^N$ in $\mathbf{R}^N$. $\square$

Lemma (Equidistribution property.). *For $x \in \mathbf{R}^N$, $k \in \mathbf{Z}$, the probability that $dist(x, \partial Q) \geq \varepsilon l(Q)$ for some cube of size 2^k is exactly $(1-2\varepsilon)^N$.*

Proof. Consider first in the real line case, i.e. let $x \in \mathbf{R}$. Thus Q is the interval of size 2^k. Let $\mathrm{A} := \{\omega : \exists Q \in \mathcal{D}(\omega), dist(x, \partial Q) \geq \varepsilon l(Q)\}$. We can see also that $A = \{\omega : x \in Q' \subseteq Q \in \mathcal{D}(\omega), l(Q') = l(Q) - 2\varepsilon l(Q)\}$. Since the random lattice $\mathcal{D}(\omega)$ is uniformly distributed, $\mathbf{P}(\mathrm{A})$ is the ratio of the length of A to the size of Q, i.e., $\mathbf{P}(\mathrm{A}) = \left(\frac{l(Q)-2\varepsilon l(Q)}{l(Q)}\right)^N = \left(\frac{(1-2\varepsilon)2^k}{2^k}\right)^N = (1-2\varepsilon)^N$. $\square$

2.5 Bad parts with small probabilities

The purpose of the random lattices is to ignore some bad parts when we decompose functions. So we study bad cubes and bad parts of functions here.

2.5.1 Bad cubes, decomposition and their probabilities

Definition. Let $\gamma = \frac{\alpha}{2\alpha+2d}$ and r be a large quantity chosen later. A cube $Q \in \mathcal{D}(\omega)$ is *bad* if there exists a cube R in $\mathcal{D}'(\omega')$ such that $l(Q) < 2^{-r} l(R)$ and $dist(Q, \partial R) \leq l(Q)^\gamma l(R)^{1-\gamma}$ or $dist(Q, \partial R_k) \leq l(Q)^\gamma l(R_k)^{1-\gamma}$ for some $R_k \in ch(R)$.

Lemma 2.9 (Small probability of bad cubes). *Let r, γ be from the previous definition. Then for any fixed ω and a cube $Q \in \mathcal{D}(\omega)$ we have P:=$\mathbf{P}\{\omega' : Q\, is\, bad\} \leq 4N \frac{2^{-r\gamma}}{1-2^{-\gamma}}$.*

Proof. Given a cube $Q \in \mathcal{D}(\omega)$ where ω is fixed. There exists a cube $R \in \mathcal{D}'(\omega')$ such that $Q \subseteq R$ and $l(R) = 2^k l(Q)$ for all $k \geq r$. We consider events A and B as $\mathrm{A} = \left\{\omega' : dist(Q, \partial R) \leq 2^{k-\gamma k} l(Q)\right\}$ and $\mathrm{B} = \left\{\omega' : dist(Q, \partial R) > 2^{k-\gamma k} l(Q)\right\}$ so that $1 = \mathbf{P}(\mathrm{A} \cup \mathrm{B}) = \mathbf{P}(\mathrm{A}) \cup \mathbf{P}(\mathrm{B})$. We then observe that

$$\mathbf{P}(\mathrm{B}) \geq \mathbf{P}\left\{\omega' : dist(c(Q), \partial R) \geq 2^{k-\gamma k} l(Q) + l(Q) = \left(2^{-\gamma k} + 2^{-k}\right) l(R)\right\}$$

and from equidistribution property,

$$\mathbf{P}\left\{\omega' : dist(c(Q), \partial R) \geq \left(2^{-\gamma k} + 2^{-k}\right) l(R)\right\} = \left(1 - 2\left(2^{-\gamma k} + 2^{-k}\right)\right)^N.$$

Thus, $\mathbf{P}(\mathrm{A}) = 1 - \mathbf{P}(\mathrm{B}) \leq 1 - \left(1 - 2\left(2^{-\gamma k} + 2^{-k}\right)\right)^N$. Now we know that $1 - (1 - 2(x + x^y))^N - N\left(2 + 4x^{-y+1}\right) x^y \leq 0$ for $x, y > 0$ (proved below) so that $\mathbf{P}(\mathrm{A}) \leq N\left(2 + 4\left(2^{-k}\right)^{-\gamma+1}\right) 2^{-\gamma k} =$

$N2^{-\gamma k+1} + N2^{-\gamma k-2k}$. Since $k \geq r \geq 0$, the probability P can be estimated as

$$P = \sum_{k \geq r} \mathbf{P}\left(\mathrm{A}\right) \leq 2N \sum_{k \geq r} \left(2^{-\gamma k} + 2^{-\gamma k-2k-1}\right) \leq 2N \sum_{k \geq r} \left(2^{-\gamma k} + 2^{-\gamma k}\right)$$
$$\leq 4N \sum_{k \geq r} 2^{-\gamma k} = 4N \, \frac{2^{-r\gamma}}{1-2^{-\gamma}}.$$

□

Proof. (of $(1-2(x+x^y))^n \geq 1-n\left(2+4x^{-y+1}\right)x^y$ for $x,\, y > 0$ and $n \in \mathbf{N}$.) We prove it by induction on n. For $n=1$, we see that $1-2(x+x^y)-1+\left(2+4x^{-y+1}\right)x^y = -2x-2x^y+2x^y+4x^{-y+1}x^y = -2x+4x = 2x \geq 0$. Assume that the statement holds for n. Then we consider the statement for $n+1 : (1-2(x+x^y))^{n+1}$:

$$\begin{aligned}
(1-2(x+x^y))^{n+1} &= (1-2(x+x^y))^n(1-2(x+x^y)) \\
&\geq \left(1-n\left(2+4x^{-y+1}\right)x^y\right)(1-2(x+x^y)) \\
&= 1-2x-2x^y-n\left(2+4x^{-y+1}\right)x^y+2n\left(2+4x^{-y+1}\right)x^{y+1}+2n\left(2+4x^{-y+1}\right)x^{2y} \\
&= 1-n\left(2+4x^{-y+1}\right)x^y-\left(2+4x^{-y+1}\right)x^y-2x-2x^y+2n\left(2+4x^{-y+1}\right)x^{y+1} \\
&\quad +2n\left(2+4x^{-y+1}\right)x^{2y}+\left(2+4x^{-y+1}\right)x^y \\
&= 1-(n+1)\left(2+4x^{-y+1}\right)x^y+2x+2n\left(2+4x^{-y+1}\right)x^{y+1}+2n\left(2+4x^{-y+1}\right)x^{2y} \\
&\geq 1-(n+1)\left(2+4x^{-y+1}\right)x^y.
\end{aligned}$$

□

Definition. Let f and g be in L^2 and $\mathcal{D} := \mathcal{D}(\omega)$ be a random dyadic lattice. As in Lemma 2.3 that we can write $f = \sum\limits_{Q \in \mathcal{D}} \triangle_Q f$ in L^2, define

$$f_{bad} := \sum_{\substack{Q \in \mathcal{D} \\ Q \text{ bad}}} \triangle_Q f, \quad \text{and} \quad f_{gd} := \sum_{Q \in \mathcal{G}} \triangle_Q f,$$

so that $f = f_{gd} + f_{bad}$ in L^2.

Next, we will see that it is possible to pick lattices with desired control.

Lemma 2.10. *With probability at least* $\frac{9}{16}$, *for some dyadic lattices we have*

$$\|f_{bad}\|_2 \leq 2^{-3}2^{-2N}\|f\|_2\,, \quad \|g_{bad}\|_{L^2} \leq 2^{-3}2^{-2N}\|g\|_2\,.$$

Remark. More precisely, as ω, ω' determine bad terms of functions, the probability in this lemma just means $\mathbb{P}\left\{(\omega,\omega')\in\Omega^2 : \|f_{bad}\|_{L^2} \leq 2^{-3}2^{-2N}\|f\|_{L^2}\,,\ \|g_{bad}\|_{L^2} \leq 2^{-3}2^{-2N}\|g\|_{L^2}\right\}.$

Proof. In order to estimate $\|f_{bad}\|$, we consider a square function $Sf(x)$ on $\mathbf{R}^N$:

$$S_{\mathcal{D}}f(x) := \sum_{Q\in\mathcal{D}} \|\triangle_Q f\|_2^2\,\mu(Q)^{-1}\chi_Q$$

so that

$$\int_{\mathbf{R}^N} S_{\mathcal{D}}f(x)\,\mathrm{d}\mu(x) = \sum_{Q\in\mathcal{D}} \|\triangle_Q f\|_2^2 = \|f\|_2^2,$$

and

$$S_{\mathcal{D}}f_{bad}(x) := \sum_{\substack{Q\in\mathcal{D}\\ Q\,bad}} \|\triangle_Q f\|_2^2\,\mu(Q)^{-1}\chi_Q(x).$$

We first compute $\mathbb{E}_{\omega'}S_{\mathcal{D}}f_{bad}(x) := \int S_{\mathcal{D}}f_{bad}(x)\,\mathrm{d}\mathbb{P}(\omega')$ where ω' ranges for the existence of bad cubes Q. That is

$$\mathbb{E}_{\omega'}S_{\mathcal{D}}f_{bad}(x) = \int_{\{\omega':\exists\,bad\,Q\,\}} S_{\mathcal{D}}f_{bad}(x)\,\mathrm{d}\mathbb{P}(\omega')$$

Since $S_{\mathcal{D}}f_{bad}$ is taken from $S_{\mathcal{D}}f$, we get

$$\int_{\{\omega':\exists\,bad\,Q\,\}} S_{\mathcal{D}}f_{bad}(x)\,\mathrm{d}\mathbb{P}(\omega') \leq \int_{\{\omega':\exists\,bad\,Q\,\}} S_{\mathcal{D}}f(x)\,\mathrm{d}\mathbb{P}(\omega')$$

and since $S_{\mathcal{D}}f(x)$ is independent of ω', we have

$$\int_{\{\omega':\exists\,bad\,Q\,\}} S_{\mathcal{D}}f(x)\,\mathrm{d}\mathbb{P}(\omega') = \mathbb{P}\left\{\omega' : \exists\,bad\,Q\,\right\}S_{\mathcal{D}}f(x). \leq 2^{-8}2^{-4N}S_{\mathcal{D}}f(x).$$

We have seen that $\mathbb{P}\{\omega' : Q\,is\,bad\} \leq \frac{4N2^{-r\gamma}}{1-2^{-\gamma}}$ by Lemma 2.9 for a cube Q. If one chooses $r \geq \frac{1}{\gamma}\log_2\left(\frac{2^{10}2^{4N}N}{1-2^{-\gamma}}\right)$, we then have $\mathbb{P}\{\omega' : Q\,is\,bad\} \leq 2^{-8}2^{-4N}$. Also the case that there exist bad cubes covers the one that one cube Q is bad. Therefore,

$$\begin{aligned}\mathbb{E}_{\omega'} S_{\mathcal{D}} f_{bad}(x) &\leq \mathbb{P}\left\{\omega' : \exists\, bad\, Q\,\right\} S_{\mathcal{D}} f(x) \\ &\leq \mathbb{P}\left\{\omega' :\ a\, cube\, Q\, is\, bad\right\} S_{\mathcal{D}} f(x) \\ &\leq 2^{-8}2^{-4N} S_{\mathcal{D}} f(x).\end{aligned}$$

By orthogonality, we can see that

$$\mathbb{E}_{\omega'} \left\| f_{bad} \right\|_2^2 = \mathbb{E}_{\omega'} \left(\int S_{\mathcal{D}} f_{bad} \, \mathrm{d}\mu \right).$$

Since $\|f\|_2^2 = \int_{\mathbf{R}^N} S_{\mathcal{D}} f(x) \, \mathrm{d}\mu$,

$$\mathbb{E}_{\omega'} \left(\int S_{\mathcal{D}} f_{bad} \, \mathrm{d}\mu \right) = \int \mathbb{E}_{\omega'} S_{\mathcal{D}} f_{bad} \, \mathrm{d}\mu \leq 2^{-8}2^{-4N} \int_{\mathbf{R}^N} S_{\mathcal{D}} f(x) \, \mathrm{d}\mu = 2^{-8}2^{-4N} \left\| f \right\|_{L^2}^2.$$

Note here that we can change the order of integration since $\mathbb{E}_{\omega'}[\int_{\mathbf{R}^N} S_{\mathcal{D}} f_{bad} \, \mathrm{d}\mu] \leq \mathbb{E}_{\omega'}[\int_{\mathbf{R}^N} S_{\mathcal{D}} f \, \mathrm{d}\mu] \leq \mathbb{E}_{\omega'}[\|f\|_2^2] \leq \|f\|_2^2$. Then,

$$\mathbb{E}_{\omega,\omega'} \left\| f_{bad} \right\|_2^2 := \int_{\{\omega \in \Omega\}} \mathbb{E}_{\omega'} \left\| f_{bad} \right\|_2^2 \, \mathrm{d}\mathbb{P}\omega \leq 2^{-8}2^{-4N} \left\| f \right\|_2^2$$

Hence, using Markov's inequality,

$$\begin{aligned}\mathbb{P}\left\{(\omega,\omega') : \left\| f_{bad} \right\|_2^2 \geq 2^2 2^{-8}2^{-4N} \left\| f \right\|_2^2\right\} &\leq \frac{\mathbb{E}_{\omega,\omega'} \left\| f_{bad} \right\|_2^2}{2^2 2^{-8}2^{-4N} \left\| f \right\|_2^2} \\ &\leq \frac{2^{-8}2^{-4N} \left\| f \right\|_2^2}{2^2 2^{-8}2^{-4N} \left\| f \right\|_2^2} = \frac{1}{4}.\end{aligned}$$

That is with probability at least $\frac{3}{4}$, $\quad \left\| f_{bad} \right\|_{L^2} \leq 2 \cdot 2^{-4} 2^{-2N} \left\| f \right\|_{L^2}$. Similary, we have $g = g_{gd} + g_{bad}$ where cubes for g are in $\mathcal{D}'(\omega')$ and are bad to $\mathcal{D}(\omega)$ satisfying $\left\| g_{bad} \right\|_2 \leq 2 \cdot 2^{-4}2^{-2N} \left\| g \right\|_2$ with probability at least $\frac{3}{4}$. Therefore, with probability at least $\frac{9}{16}$,

$$\left\| f_{bad} \right\|_2 \leq 2^{-3}2^{-2N} \left\| f \right\|_2, \quad \left\| g_{bad} \right\|_2 \leq 2^{-3}2^{-2N} \left\| g \right\|_2,$$

simultaneously. $\square$

2.5.2 Another bad part

In this section, we want to avoid more bad part defined below.

Definition. For a cube $Q \in \mathcal{D}$, define the bad part of Q to be

$$Q_b := Q \cap \left(\bigcup_{\substack{R \in \mathcal{D}' \\ 2^{-r} l(Q) \leq l(R) \leq 2^r l(Q)}} \delta_R \right)$$

where $\delta_R := (1+2\varepsilon)R \backslash (1-2\varepsilon)R$ (i.e. $\varepsilon l(R)$-neighborhood of ∂R).

For a function $f \in L^2(\mu)$ and for each k, define the bad parts f_b^k of f^k to be

$$f_b^k := \sum_{\substack{Q \in \mathcal{D} \\ l(Q)=2^k}} c_Q(f) \chi_{Q_b}.$$

The following lemma says that even more satisfied lattices can be chosen.

Lemma 2.11. *With probability at least $\frac{1}{4}$, for some lattices $\mathcal{D}(\omega), \mathcal{D}'(\omega')$*

$$\|f_{bad}\|_2 \leq 2^{-3} 2^{-2N} \|f\|_2, \quad \|g_{bad}\|_2 \leq 2^{-3} 2^{-2N} \|g\|_2$$

and

$$\sum_k \left\| f_b^k \right\|_2^2 \leq 8 p_\varepsilon \|f\|_2^2, \quad \sum_k \left\| g_b^k \right\|_2^2 \leq 8 p_\varepsilon \|g\|_2^2.$$

where p_ε is defined below. Roughly speaking, we are talking about probability $\mathbb{P}\{(\omega, \omega') \in \Omega^2$: four inequalities above holds simultaneously$\}$ which is greater than 0 so that there exists (ω, ω') giving $\mathcal{D}(\omega), \mathcal{D}'(\omega')$ with such inequalities.

Proof. Given the random dyadic lattice $\mathcal{D} := \mathcal{D}(\omega)$. For a fixed $x \in \mathbf{R}^N, k \in \mathbf{Z}$, consider

$$\mathbb{E}_{\omega'} \left| f_b^k(x) \right|^2 \leq \int \sum_{\substack{Q \in \mathcal{D} \\ l(Q)=2^k}} |c_Q(f)|^2 \chi_{Q_b}(x) d\mathbb{P}(\omega').$$

We want to consider the case that there is $R \in \mathcal{D}'(\omega')$, $l(R) = 2^k$, $x \in \delta_R$ so that $\chi_{Q_b}(x) = 1$. Thus

$$\begin{aligned}
\int \sum_{\substack{Q\in\mathcal{D} \\ l(Q)=2^k}} |c_Q(f)|^2 \chi_{Q_b}(x) \mathrm{d}\mathbb{P}(\omega') &= \int_{\{\omega' : \exists R \in \mathcal{D}'(\omega'), l(R)=2^k, x\in\delta_R\}} \Big| \sum_{\substack{Q\in\mathcal{D} \\ l(Q)=2^k}} c_Q(f) \Big|^2 \mathrm{d}\mathbb{P} \\
&= \int_{\{\omega' : \exists R \in \mathcal{D}'(\omega'), l(R)=2^k, x\in\delta_R\}} \left| f^k(x) \right|^2 \mathrm{d}\mathbb{P} \\
&= p_\varepsilon \left| f^k(x) \right|^2,
\end{aligned}$$

where $p_\varepsilon = \mathbb{P}\{\omega' : \exists R \in \mathcal{D}'(\omega'),\, l(R) = 2^k,\, x \in \delta_R\}$. Note that p_ε does not depend on k. Then, one can consider that

$$\begin{aligned}
\mathbb{E}_{\omega'}\left(\sum_k \|f_b^k\|_2^2 \right) = \sum_k \left(\int_{\mathbf{R}^N} \mathbb{E}_{\omega'} |f_b^k|^2 \mathrm{d}\mu(x) \right) &\le p_\varepsilon \sum_k \left(\int_{\mathbf{R}^N} |f^k(x)|^2 \mathrm{d}\mu(x) \right) \\
&= p_\varepsilon \sum_k \|f^k\|_2^2 \\
&= p_\varepsilon \left\| f \right\|_2^2 .
\end{aligned}$$

Since the above inequality holds for any dyadic grid $\mathcal{D}(\omega)$, we have

$$\mathbb{E}_{\omega,\omega'}\left(\sum_k \left\| f_b^k \right\|_2^2 \right) := \mathbb{E}_\omega \left(\mathbb{E}_{\omega'} \left(\sum_k \left\| f_b^k(x) \right\|_2^2 \right) \right) \le p_\varepsilon \left\| f \right\|_2^2 .$$

Similarly for a given $\mathcal{D}'(\omega')$, $\quad \mathbb{E}_\omega\big(\sum_k \left\| g_b^k \right\|_2^2 \big) \le p_\varepsilon \left\| g \right\|_2^2$. Then,

$$\mathbb{E}_{\omega',\omega}\left(\sum_k \left\| g_b^k \right\|_2^2 \right) \le p_\varepsilon \left\| g \right\|_2^2 .$$

Hence, by Markov's inequality, we have

$$\mathbb{P}_{\omega,\omega'}\left\{ \sum_k \left\| f_b^k \right\|_2^2 \ge 8 p_\varepsilon \left\| f \right\|_2^2 \right\} \le \frac{\mathbb{E}_{\omega,\omega'}\left(\sum_k \left\| f_b^k \right\|_2^2 \right)}{8 p_\varepsilon \left\| f \right\|_2^2} \le \frac{1}{8}.$$

Similarly, $\mathbb{P}_{\omega',\omega}\left\{\sum\limits_{k}\left\|g_b^k\right\|_{L^2}^2 \geq 8p_\varepsilon \left\|g\right\|_{L^2}^2\right\} \leq \frac{1}{8}$.

Since $\mathbb{P}(\mathrm{A}\cap\mathrm{B}\cap\mathrm{C}\cap\mathrm{D}) \geq 1-\mathrm{A}'-\mathrm{B}'-\mathrm{C}'-\mathrm{D}'$ together with the estimate in Lemma 2.10, the probability that $\|f_{bad}\|_2 \leq 2^{-3}2^{-2N}\|f\|_2$, $\|g_{bad}\|_2 \leq 2^{-3}2^{-2N}\|g\|_2$, $\sum\limits_{k}\left\|f_b^k\right\|_2^2 \leq 8p_\varepsilon\|f\|_2^2$, and $\sum\limits_{k}\left\|g_b^k\right\|_2^2 \leq 8p_\varepsilon\|g\|_2^2$ is greater than $1-\frac{1}{4}-\frac{1}{4}-\frac{1}{8}-\frac{1}{8}=\frac{1}{4}$. □

2.5.3 Bad parts for Tb theorem

For Tb theorem, the definition of bad cubes is slightly adapted though important properties are the same [Hyt11]. Since ideas and proofs are explicitly presented for the martingale difference, we only state required lemmas without proofs in this part.

Definition. Given $\mathcal{D}$ and $\mathcal{D}'$ dyadic lattices. A cube $Q \in \mathcal{D}$ is *bad* if there is a cube R in $\mathcal{D}$ or $\mathcal{D}'$ such that $l(Q) \leq 2^{-r}l(R)$ and $dist(Q,\partial R) \leq l(Q)^\gamma l(R)^{1-\gamma}$. Then, by Lemma 2.6, one can consider $f = f_{gd} + f_{bad}$ in L^2 where

$$f_{bad} := \sum_{\substack{Q\in\mathcal{D}\\ Q\ bad}} \triangle_Q^b f, \quad \text{and} \quad f_{gd} := \sum_{Q\in\mathcal{G}} \triangle_Q^b f + \sum_{Q\in\mathcal{S}_n} E_Q^b f.$$

Moreover, for each k, the bad parts f_b^k of f^k can be defined as

$$f_b^k := \sum_{\substack{Q\in\mathcal{D}\\ l(Q)=2^k}} c_Q(f)b\chi_{Q_b}.$$

Lemma 2.12. *With probability at least $\frac{1}{4}$, for some lattices $\mathcal{D}(\omega), \mathcal{D}'(\omega')$*

$$\|f_{bad}\|_2 \leq 2^{-3}2^{-2N}\|f\|_2\,, \quad \|g_{bad}\|_2 \leq 2^{-3}2^{-2N}\|g\|_2$$

and

$$\sum_{k\leq n}\left\|f_b^k\right\|_2^2 \leq 8C(\delta,b)p_\varepsilon\|f\|_2^2\,, \quad \sum_{k\leq n}\left\|g_b^k\right\|_2^2 \leq 8C(\delta,b)p_\varepsilon\|g\|_2^2\,,$$

where p_ε is the probability that there exists $R \in \mathcal{D}'(\omega')$ in which a point x lies in its $\varepsilon l(R)$-nbhd ($Q \in \mathcal{D}(\omega)$ with $\varepsilon l(Q)$-nbhd for g).

Remark. Again, the lemma says about the probability $\mathbb{P}\{(\omega,\omega') \in \Omega^2$: four inequalities above holds simultaneously$\}$ which is greater than 0 so that there exists (ω,ω') giving $D(\omega), D'(\omega')$ with such inequalities. Also, from the estimates, observe that $\|f_{gd}\|_2 = \|f - f_{bad}\|_2 \leq \|f\|_2 + \|f_{bad}\|_2 \lesssim \|f\|_2$, and similar for g.

2.6 Trees and decompositon

In this section, we introduce language of trees. Also, we will see how one can decompose a tree in a useful direction.

Definition. A *tree* is a collection $\mathcal{T} \subseteq \mathcal{I}$ of dyadic cubes (a.k.a. tiles) with a top tile $I_{\mathcal{T}} \in \mathcal{T}$ such that $P \subseteq I_{\mathcal{T}}$ for all $P \in \mathcal{T}$.

The *complete tree* $\mathrm{Tree}(I)$ is the collection $\{P \in \mathcal{I} : P \subseteq I\}$ with the top tile I.

A collection $\mathcal{P} \subseteq \mathcal{I}$ is *convex* if for every pair $P \subseteq P'$ in $\mathcal{P}$, and $I \in \mathcal{I}$ such that $P \subseteq I \subseteq P'$, then $I \in \mathcal{P}$.

Let $a : \mathcal{I} \to \mathbf{R}^+$ be a positive-real-valued function. Define the *size of a on a tree T* by

$$\|a\|_{size(T)} := \frac{1}{\mu(I_T)} \sum_{P \in T} a(P)$$

and the *maximal size of a* by

$$\|a\|_{size^*(\mathcal{I})} := \sup_{T \subset \mathcal{I}} \|a\|_{size(T)}.$$

Given f on $\mathbf{R}$ and $P \in \mathcal{I}$, define

$$\|f\|_{mean(P)} := \frac{1}{\mu(I_P)} \int_{I_P} |f| \, \mathrm{d}\mu$$

and for any collection $\mathcal{P} \subset \mathcal{I}$, define

$$\|f\|_{mean^*(\mathcal{P})} := \sup_{P \in \mathcal{P}} \|f\|_{mean(P)}.$$

Lemma 2.13 (Decompositon for mean)**.** *For $n \in \mathbf{Z}$, given a convex collection $\mathcal{P}_n \subset \mathcal{I}$ and a function $f \in \mathcal{S}$ such that $\|f\|_{mean^*(\mathcal{P}_n)} \leq 2^n$. There exists a disjoint partition $\mathcal{P}_n = \bigcup_{\mathcal{T} \in \mathcal{T}_n} \mathcal{T} \cup \mathcal{P}_{n-1}$*

where $\mathcal{P}_{n-1}$ is a convex collection of tiles such that $\|f\|_{mean^*(\mathcal{P}_{n-1})} \leq 2^{n-1}$, and $\mathcal{T}_n$ is a collection of convex trees $\mathcal{T}$ with disjoint spatial intervals $I_{\mathcal{T}}$ such that

$$\|f\|_{mean(I_{\mathcal{T}})} \sim \|f\|_{mean^*(\mathcal{T})} \sim 2^n$$

for all $\mathcal{T} \in \mathcal{T}_n$. In particular,

$$\sum_{\mathcal{T}\in\mathcal{T}_n} \mu(I_{\mathcal{T}}) \leq 2^{-n+2} \int_{|f|\geq 2^{n-2}} |f| \, \mathrm{d}\mu \leq 2^{-np+2p} \|f\|_p^p$$

for $1 \leq p < \infty$.

Proof. Choose $P \in \mathcal{P}_n$ that is maximal with respect to set inclusion for $\|f\|_{mean(P)} > 2^{n-1}$. If no such P, then set $\mathcal{P}_{n-1} := \mathcal{P}_n$. Otherwise, we collect the complete tree $\mathcal{T} = Tree(P) \cap \mathcal{P}_n$ into $\mathcal{T}_n$ so that $2^{n-1} < \|f\|_{mean(I_{\mathcal{T}})} \leq \|f\|_{mean^*(\mathcal{T})} \leq 2^n$. Remove $\mathcal{T}$ from $\mathcal{P}_n$ and do the same with $\mathcal{P}_n \backslash \mathcal{T}$. Repeat this procedure with the remaining tiles. Set $\mathcal{P}_{n-1} := \mathcal{P}_n \backslash \bigcup_{\mathcal{T}\in\mathcal{T}_n}$. Now we can see that in $\mathcal{T}_n$ the trees in $\mathcal{T}_n$ are disjoint since we choose the maximal top tile. In addition, they are complete w.r.t. $\mathcal{P}_n$ and thus convex. Then we see that $\|f\|_{mean^*(\mathcal{P}_{n-1})} \leq 2^{n-1}$ and $2^{n-1} < \|f\|_{mean(I_{\mathcal{T}})} \leq \|f\|_{mean^*(\mathcal{T})} \leq 2^n$ for all $\mathcal{T} \in \mathcal{T}_n$. This leads to $2^{n-1} \leq \frac{1}{\mu(I_{\mathcal{T}})} \int_{I_{\mathcal{T}}} |f| \, \mathrm{d}\mu$ for all $\mathcal{T} \in \mathcal{T}_n$. That is

$$2^{n-1}\mu(I_{\mathcal{T}}) \leq \int\limits_{\substack{I_{\mathcal{T}} \\ |f|\geq 2^{n-2}}} |f| \, \mathrm{d}\mu + \int\limits_{\substack{I_{\mathcal{T}} \\ |f|\leq 2^{n-2}}} |f| \, \mathrm{d}\mu \leq \int\limits_{\substack{I_{\mathcal{T}} \\ |f|\geq 2^{n-2}}} |f| \, \mathrm{d}\mu + 2^{n-2}\mu(I_{\mathcal{T}})$$

Thus,

$$2^n \mu(I_{\mathcal{T}}) \leq 4 \int\limits_{\substack{I_{\mathcal{T}} \\ |f|\geq 2^{n-2}}} |f| \, \mathrm{d}\mu.$$

Then summing over disjoint $\mathcal{T} \in \mathcal{T}_n$, we have

$$\sum_{\mathcal{T}\in\mathcal{T}_n} \mu(I_{\mathcal{T}}) \leq 4 \cdot 2^{-n} \sum_{\mathcal{T}\in\mathcal{T}_n} \int\limits_{\substack{I_{\mathcal{T}} \\ |f|\geq 2^{n-2}}} |f| \, \mathrm{d}\mu \leq 4 \cdot 2^{-n} \int\limits_{|f|\geq 2^{n-2}} |f| \, \mathrm{d}\mu.$$

To obtain L^p norm of f, first apply Hölder's inequality to get

$$4 \cdot 2^{-n} \int\limits_{|f| \geq 2^{n-2}} |f| \, \mathrm{d}\mu \leq 4 \cdot 2^{-n} \|f\|_p \|\chi_{|f| \, \geq \, 2^{n-2}}\|_q = 4 \cdot 2^{-n} \|f\|_p \mu(\{x : |f(x)| \geq 2^{n-2}\})^{1/q}.$$

Then, by Chebyshev's inequality, one can see that

$$4 \cdot 2^{-n} \|f\|_p \mu(\{x : |f(x)| \geq 2^{n-2}\})^{1/q} \leq 4 \cdot 2^{-n} \|f\|_p 2^{-np+n+2p-2} \|f\|_p^{p-1} = 2^{-np+2p} \|f\|_p^p \, .$$

$\square$

Chapter 3

Prerequisite lemmas

In addition to tools and their properties in the previous chapter, we prepare some estimates and lemmas in this one. One may refer to the next chapter to get some ideas why we deal with this stuff.

3.1 L^2 boundedness of Π and cancellation

It is common to meet some paraproducts in study of singular integral operator. The one we encounter is

$$\Pi(g, T^*1) := \sum_{S\in\mathcal{D}'} \sum_{\substack{P\in\mathcal{D} \\ l(P)=2^{-r}l(S) \\ dist(P,\partial S)\geq\lambda l(P)}} E_S g \cdot \triangle_P^*(T^*1)\ .$$

Lemma 3.1. *Let $h \in BMO_\lambda^2$. Define a function $a : \mathcal{D}' \to \mathbf{R}^+$ by*

$$a(S) := \sum_{\substack{P\in\mathcal{D} \\ l(P)=2^{-r}l(S) \\ dist(P,\partial S)\geq\lambda l(P)}} \|\triangle_P^*(h)\|_2^2.$$

Then we have that the size of a on $\mathcal{T}$ is bounded ,i.e. $\|a\|_{size(T)} < \infty$ for all $\mathcal{T} \in \mathcal{T}_n$ in Lemma 2.13 where $\mathcal{P}_n = \mathcal{D}'$ and hence $\|a\|_{size^*(\mathcal{D}')} < \infty$.

Proof. For all $\mathcal{T} \in \mathcal{T}_n$, consider that

$$\sum_{S\in\mathcal{T}} a(S) \leq \sum_{\substack{P\in\mathcal{D}; P\subseteq I_\mathcal{T} \\ l(P)\leq 2^{-r}l(I_\mathcal{T}) \\ dist(P,\partial I_\mathcal{T})\geq\lambda l(P)}} \|\triangle_P^*(h)\|_2^2.$$

We want to rewrite the last sum to form a collection of Whitney intervals $\mathcal{W} : \mathcal{W} := \bigcup_{i\geq 0} W_i$ where

W_0 is the collection of intervals $P \subset I_{\mathcal{T}}$ such that $l(P) = 2^{-r}l(I_{\mathcal{T}})$ and $dist(P, \partial I_{\mathcal{T}}) \geq \lambda l(P)$ and W_i is the collection of intervals $P \subset I_{\mathcal{T}}$ such that $l(P) = 2^{-r-i}l(I_{\mathcal{T}})$ and $dist(P, \partial I_{\mathcal{T}}) \geq \lambda l(P)$ and $P \cap \bigcup_{j=0}^{i-1} W_j = \varnothing$ for $i = 1, 2, 3, \ldots$ Then we can see that for every $Q \in \mathcal{W}$,

$$\sum_{P \subseteq Q} \|\triangle_P^*(h)\|_2^2 = \sum_{P \subseteq Q} \|\triangle_P^*(h - c_Q)\|_2^2 \leq \int_Q |h - c_Q|^2 \,\mathrm{d}\mu \leq C\mu(\lambda Q),$$

where we use martingale difference properties in the first two steps and that $h \in \mathrm{BMO}_\lambda^2$ in last inequality. Therefore,

$$\sum_{Q \in \mathcal{W}} \sum_{P \subseteq Q} \|\triangle_P^*(h)\|_2^2 \leq C \sum_{Q \in \mathcal{W}} \mu(\lambda Q) = C \sum_{Q \in \mathcal{W}} \int \chi_{\lambda Q} \,\mathrm{d}\mu = C \int_{I_T} \sum_{Q \in \mathcal{W}} \chi_{\lambda Q} \,\mathrm{d}\mu \leq C(\lambda)\mu(I_T)$$

where the last step follows from that for each $x \in I_{\mathcal{T}}$, there are finitely many Q such that $x \in \lambda Q$. Hence,

$$\sum_{S \in \mathcal{T}} a(S) \leq \sum_{\substack{P \in \mathcal{D}; P \subseteq I_{\mathcal{T}} \\ l(P) \leq 2^{-r} l(I_{\mathcal{T}}) \\ dist(P, \partial I_{\mathcal{T}}) \geq \lambda l(P)}} \|\triangle_P^*(h)\|_2^2 = \sum_{Q \in \mathcal{W}} \sum_{P \subseteq Q} \|\triangle_P^*(h)\|_2^2 \leq C(\lambda)\mu(I_{\mathcal{T}})$$

which means $\|a\|_{size(T)} < C(\lambda)$. $\square$

Lemma 3.2 (Carleson's embedding theorem)**.** *Let $h \in BMO_\lambda^2$. For $1 < p < \infty$,*

$$\sum_{S \in \mathcal{D}'} \sum_{\substack{P \in \mathcal{D} \\ l(P) = 2^{-r} l(S) \\ dist(P, \partial S) \geq \lambda(P)}} \|\triangle_P^*(h)\|_2^2 |\langle f \rangle_S|^p \leq C(p, \lambda)\|a\|_{size^*(\mathcal{D}')} \|f\|_p^p$$

for all locally integrable functions f.

Proof. Choose large enough k such that $\|f\|_{mean^*(\mathcal{D}')} \leq 2^k$. Then, by Lemma 2.13, we get $\mathcal{D}' = \bigcup_{\mathcal{T} \in \mathcal{T}_k} \mathcal{T} \cup \mathcal{P}_{k-1}$. Repeatedly decomposing $\mathcal{P}_{k-i}$, we obtain $\mathcal{D}' = \bigcup_{n \leq k} \bigcup_{\mathcal{T} \in \mathcal{T}_n} \mathcal{T}$ where $\mathcal{T}_n$ is a collection of convex trees $\mathcal{T}$ with disjoint spatial interval $I_{\mathcal{T}}$ such that $\|f\|_{mean(I_{\mathcal{T}})} \sim \|f\|_{mean^*(\mathcal{T})} \sim 2^n$. It is then easy to see that

$$\sum_{S \in \mathcal{D}'} a(S)|\langle f \rangle_S|^p = \sum_{n \leq k} \sum_{\mathcal{T} \in \mathcal{T}_n} \sum_{S \in \mathcal{T}} a(S)|\langle f \rangle_{I_S}|^p \lesssim \sum_{n \leq k} \sum_{\mathcal{T} \in \mathcal{T}_n} \sum_{S \in \mathcal{T}} 2^{np} a(S).$$

By definition, we have

$$\sum_{n\le k}\sum_{\mathcal{T}\in\mathcal{T}_n}\sum_{S\in\mathcal{T}} 2^{np}a(S)=\sum_{n\le k}2^{np}\sum_{\mathcal{T}\in\mathcal{T}_n}\mu(I_{\mathcal{T}})\|a\|_{size(\mathcal{T})}\le\sum_{n\le k}2^{np}\sum_{\mathcal{T}\in\mathcal{T}_n}\mu(I_{\mathcal{T}})\|a\|_{size^*(\mathcal{D}')}.$$

From Lemma 2.13,

$$\sum_{n\le k}2^{np}\sum_{\mathcal{T}\in\mathcal{T}_n}\mu(I_{\mathcal{T}})\|a\|_{size^*(\mathcal{D}')}\le\|a\|_{size^*(\mathcal{D}')}\sum_{n\le k}2^{np-n+2}\int_{|f|\ge 2^{n-2}}|f|\,\mathrm{d}\mu$$

and then one can get L^p norm of f as follows. Obtaining the upper bound from the fact that

$$\sum_{n\le k}2^{np-n+2}\lesssim\sum_{n\le k}2^{np-n-2p+2}=\sum_{n\le k}2^{(n-2)(p-1)}$$

As a geometric series with the first term $2^{(k-2)(p-1)}$, we are done since

$$\|a\|_{size^*(\mathcal{D}')}\int_{2^{n-2}\le|f|}\sum_{n\le k}2^{(n-2)(p-1)}|f|\,\mathrm{d}\mu\lesssim\|a\|_{size^*(\mathcal{D}')}\int|f|^{p-1}|f|\,\mathrm{d}\mu=\|a\|_{size^*(\mathcal{D}')}\|f\|_p^p.$$

□

Theorem 3.3. *The paraproduct* $\Pi(g,h)$ *is bounded on* L^2 *when* $h\in BMO^2_\lambda$.

Proof. One just play with the definition and apply the previous embedding as

$$\begin{aligned}\|\Pi(g,h)\|_2^2=\sup_{\|f\|_2=1}|\langle\Pi(g,h),f\rangle|^2&\le\sup_{\|f\|_2=1}\sum_{S\in\mathcal{D}'}\sum_{\substack{P\in\mathcal{D}\\ l(P)=2^{-r}l(S)\\ dist(P,\partial S)\ge\lambda l(P)}}|\langle E_S g\cdot\triangle_P^*(h),f\rangle|^2\\&=\sup_{\|f\|_2=1}\sum_{S\in\mathcal{D}'}\sum_{\substack{P\in\mathcal{D}\\ l(P)=2^{-r}l(S)\\ dist(P,\partial S)\ge\lambda l(P)}}|\langle g\rangle_S\langle\triangle_P^*(h),f\rangle|^2\\&\le\sum_{S\in\mathcal{D}'}|\langle g\rangle_S|^2\sum_{\substack{P\in\mathcal{D}\\ l(P)=2^{-r}l(S)\\ dist(P,\partial S)\ge\lambda l(P)}}\|\triangle_P^*(h)\|_2^2\\&\lesssim\|g\|_2^2.\end{aligned}$$

□

Lemma 3.4. *For any $R \in \mathcal{D}', Q \in \mathcal{D}$ such that $R \subseteq Q$ or $l(Q) \geq 2^{-r} l(R)$, the terms $\langle \triangle_Q f, \Pi \triangle_R g \rangle$ is zero.*

Proof. If $Q \supseteq R$, then $R \subseteq Q \subseteq P \subset S$ so that $E_S(\triangle_R g) = 0$. If $l(Q) \geq 2^{-r} l(R)$ then $l(P) = 2^{-r} l(S) < 2^{-r} l(R) \leq l(Q)$. Since $P, Q \in D$ and $P \subset Q$ (otherwise 0), we have that $\triangle_Q f$ is constant on P as well as $E_S(\triangle_R g)$. Hence $\int \triangle_Q f \cdot E_S(\triangle_R g) \cdot \triangle_P^*(T^* 1) \, \mathrm{d}\mu = c \int \triangle_P^*(T^* 1) \, \mathrm{d}\mu = 0$. □

3.2 Estimates of bilinear forms on some cubes

When one decomposes $\langle Tf, g \rangle$, summands as $\langle \triangle_Q f, \triangle_R g \rangle$ arise. For some relations of Q and R, one gets good estimates so that the summation over such cubes can be controlled. Before starting, denote the long distance $D(Q, R) := dist(Q, R) + l(R) + l(Q)$.

Lemma 3.5. *For $R \in \mathcal{D}'$ and $Q \in \mathcal{G}$ such that $Q \cap R = \varnothing$, $l(Q) < 2^{-r} l(R)$,*

$$|\langle T \triangle_Q f, \triangle_R g \rangle| \leq C \frac{l(Q)^{\alpha/2} l(R)^{\alpha/2}}{D(Q, R)^{d+\alpha}} \mu(Q)^{1/2} \mu(R)^{1/2} \|\triangle_Q f\|_2 \|\triangle_R g\|_2.$$

Proof. To prove this, we first prove that for such Q, R,

$$|\langle T \triangle_Q f, \triangle_R g \rangle| \leq C \frac{l(Q)^{\alpha}}{dist(Q, R)^{d+\alpha}} \mu(Q)^{1/2} \mu(R)^{1/2} \|\triangle_Q f\|_2 \|\triangle_R g\|_2.$$

Indeed, we have $|x - y_0| \geq 2|y - y_0|$ since Q is good leading to $dist(Q, R) \geq l(Q)$. Then using the condition of the operator and Hólder inequality for the last line we get

$$\begin{aligned}
|\langle T \triangle_Q f, \triangle_R g \rangle| &= |\int \int (K(x, y) - K(x, y_0)) \triangle_Q f(y) \triangle_R g(x) \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x)| \\
&\lesssim \int \int \frac{|y - y_0|^{\alpha}}{|x - y_0|^{d+\alpha}} |\triangle_Q f(y)| |\triangle_R g(x)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\
&\leq \frac{l(Q)^{\alpha}}{dist(Q, R)^{d+\alpha}} \|\triangle_Q f\|_{L^1} \|\triangle_R g\|_{L^1} \\
&\leq \frac{l(Q)^{\alpha}}{dist(Q, R)^{d+\alpha}} \mu(Q)^{1/2} \mu(R)^{1/2} \|\triangle_Q f\|_{L^2} \|\triangle_R g\|_{L^2}.
\end{aligned}$$

Next we consider two cases regarding $dist(Q, R)$ and $l(R)$:

If $dist(Q,R) \geq l(R)$, then $D(Q,R) \leq 3dist(Q,R)$. Thus

$$|\langle T\triangle_Q f, \triangle_R g\rangle| \lesssim \frac{l(Q)^\alpha}{D(Q,R)^{d+\alpha}}\mu(Q)^{1/2}\mu(R)^{1/2}\|\triangle_Q f\|_{L^2}\|\triangle_R g\|_{L^2}$$
$$\lesssim \frac{l(Q)^{\alpha/2}l(R)^{\alpha/2}}{D(Q,R)^{d+\alpha}}\mu(Q)^{1/2}\mu(R)^{1/2}\|\triangle_Q f\|_{L^2}\|\triangle_R g\|_{L^2}.$$

In case $dist(Q,R) \leq l(R)$, we use the fact that Q is good and that $\gamma d + \gamma\alpha = \alpha/2$ to have

$$\frac{l(Q)^\alpha}{dist(Q,R)^{d+\alpha}} \leq \frac{l(Q)^\alpha}{(l(Q)^\gamma l(R)^{1-\gamma})^{d+\alpha}} = \frac{l(Q)^\alpha}{l(Q)^{\alpha/2}l(R)^{d+\alpha/2}} = \frac{l(Q)^{\alpha/2}l(R)^{\alpha/2}}{l(R)^{d+\alpha}} = \frac{l(Q)^{\alpha/2}l(R)^{\alpha/2}}{D(Q,R)^{d+\alpha}}$$

and then the result follows from the inequality we first proved. □

Lemma 3.6. *The following term is bounded,*

$$\sum_{R\in\mathcal{D}'}\sum_{\substack{Q\cap R=\varnothing \\ 2^{-r}l(R)>l(Q)}} \frac{l(Q)^{\alpha/2}l(R)^{\alpha/2}}{D(Q,R)^{d+\alpha}}\mu(Q)^{1/2}\mu(R)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2$$
$$\leq C(\sum_{Q\in\mathcal{D}}\|\triangle_Q f\|_2^2)^{1/2}(\sum_{R\in\mathcal{D}'}\|\triangle_R g\|_2^2)^{1/2}$$
$$\lesssim \|f\|_2\|g\|_2.$$

Proof. We rewrite the sum on the left as $\sum_{n>r}\sum_k\sum_{\substack{R\in\mathcal{D}' \\ l(R)=2^{-k}}}\sum_{\substack{Q\cap R=\varnothing \\ l(Q)=2^{-n-k}}}$ so that it suffices to prove that

$$\sum_{\substack{R\in\mathcal{D}' \\ l(R)=2^{-k}}}\sum_{\substack{Q\cap R=\varnothing \\ l(Q)=2^{-n-k}}} \frac{l(Q)^{\alpha/2}l(R)^{\alpha/2}}{D(Q,R)^{d+\alpha}}\mu(Q)^{1/2}\mu(R)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2$$
$$\leq 2^{-n\beta}(\sum_{\substack{Q\cap R=\varnothing \\ l(Q)=2^{-n-k}}}\|\triangle_Q f\|_2^2)^{1/2}(\sum_{\substack{R\in\mathcal{D}' \\ l(R)=2^{-k}}}\|\triangle_R g\|_2^2)^{1/2}$$

for some positive β. In order to prove that we will consider the sum as an integral operator

$$\int\int K_k^{(n)}(x,y)X(x)Y(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$

and show that it is bounded so that the double integral is bounded by $C\|X\|_2\|Y\|_2$. Indeed, we set

$$X := \sum_{\substack{Q:Q\cap R=\varnothing \\ l(Q)=2^{-k-n}}} \frac{\|\triangle_Q f\|_2}{\mu(Q)^{1/2}}\chi_Q, \quad Y := \sum_{\substack{R \\ l(R)=2^{-k}}} \frac{\|\triangle_R g\|_2}{\mu(R)^{1/2}}\chi_R,$$

and

$$K_k^{(n)}(x,y) := \sum_{\substack{R \\ l(R)=2^{-k}}} \sum_{\substack{Q:Q\cap R=\varnothing \\ l(Q)=2^{-k-n}}} \frac{l(Q)^{\alpha/2} l(R)^{\alpha/2}}{D(Q,R)^{d+\alpha}} \chi_Q(x)\chi_R(y).$$

We observe now that for each x, y there is only one non-zero term in $K_k^{(n)}$. By Schur's Test, we just show that the kernel is bounded in L_1. Firstly, we get the geometric sequences as follows:

$$|K_k^{(n)}(x,y)| \le \frac{l(Q)^{\alpha/2} l(R)^{\alpha/2}}{D(Q,R)^{d+\alpha}} = \frac{2^{(-k\alpha-n\alpha)/2}\cdot 2^{-k\alpha/2}}{D(Q,R)^{d+\alpha}} \le \frac{2^{-n\alpha/2}\cdot 2^{-k\alpha}}{(2^{-k}+|x-y|)^{d+\alpha}}$$

where the last inequality holds since $2D(Q,R) \ge 2^{-k} + |x-y|$. Then consider that

$$\int \frac{2^{-k\alpha}}{(2^{-k}+|x-y|)^{d+\alpha}}\,\mathrm{d}\mu \le 2^{-k\alpha} \lim_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} \frac{1}{(2^{-k}+|x-y|)^{d+\alpha}}\,\mathrm{d}\mu.$$

By Comparison Lemma,

$$\begin{aligned}\int\limits_{|x-y|>\varepsilon} \frac{1}{(2^{-k}+|x-y|)^{d+\alpha}}\,\mathrm{d}\mu &\le \frac{\varepsilon^d}{(2^{-k}+\varepsilon)^{d+\alpha}} + d\int_\varepsilon^\infty \frac{t^{d-1}}{(2^{-k}+t)^{d+\alpha}}\,\mathrm{d}t \le d\int_\varepsilon^\infty (2^{-k}+t)^{-\alpha-1}\,\mathrm{d}t \\ &= d\frac{(2^{-k}+t)^{-\alpha}}{-\alpha}\Bigg|_\varepsilon^\infty \\ &= \frac{d}{\alpha}(2^{-k}+\varepsilon)^{-\alpha}.\end{aligned}$$

Taking limit we get the term $\frac{d}{\alpha}2^{k\alpha}$ and hence $\int \frac{2^{-k\alpha}}{(2^{-k}+|x-y|)^{d+\alpha}}\,\mathrm{d}\mu$ is bounded by $\frac{d}{\alpha}$ which is independent of k. $\square$

Lemma 3.7. *For $R \in \mathcal{D}'$ and $Q \in \mathcal{G}$ satisfying $Q \subset R$, $l(Q) < 2^{-r}l(R)$. Then*

$$|\langle (T-\Pi^*)\triangle_Q f, \triangle_R g\rangle| \le C\left(\frac{l(Q)}{l(R)}\right)^{\alpha/2}\left(\frac{\mu(Q)}{\mu(R_l)}\right)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2$$

Proof. We want to write the operator as something easier to deal with but first we recall that $\triangle_R g$ can be written in terms of a step function on the children R_l and R_r of R as $C_l\chi_{R_l} + C_r\chi_{R_r}$ for some constant C_l, C_r. Now fix such R, Q and consider

$$\langle \triangle_Q f, \Pi\triangle_R g\rangle = \sum_{S\in D'} \sum_{\substack{P\in\mathcal{D}\\ l(P)=2^{-r}l(S)\\ dist(P,\partial S)\geq\lambda l(P)}} \langle \triangle_Q f, E_S\triangle_R g\cdot\triangle_P^*(T^*1)\rangle = \sum_{S\subseteq R_l}\sum_P \langle\triangle_Q f, C_l\chi_S\cdot\triangle_p^*(T1)\rangle$$

since S, R are in the same lattice and only non-zero terms of $E_S(\triangle_R g)$ are those of $S\subset R$. These sums are actually simple since Q is one of those P's and they are in the same lattice so only $P = Q$ is left. In fact, one observes that there exists $S \in \mathcal{D}'$ such that $Q\subset S\subseteq R_l \in\ ch(R)$, $l(Q) = 2^{-r}l(S)$, and $dist(Q,\partial S)\geq\lambda l(Q)$. We can choose $S = R_l$ if needed since $dist(Q,\partial R_l)\geq l(Q)^\gamma l(R_l)^{1-\gamma} = l(Q)2^{r(1-r)}\geq\lambda l(Q)$. Therefore,

$$\sum_{S\subseteq R_l}\sum_P \langle\triangle_Q f, C_l\chi_S\cdot\triangle_p^*(T1)\rangle = \langle\triangle_Q f, \triangle_Q^*(T1)\rangle C_l = \langle T\triangle_Q f, 1\rangle C_l.$$

We now consider the inner product as follows:

$$\begin{aligned}|\langle(T-\Pi^*)\triangle_Q f, \triangle_R g\rangle| = |\langle T\triangle_Q f, \triangle_R g\rangle - \langle\triangle_Q f, \Pi\triangle_R g\rangle| &= |\langle T\triangle_Q f, \triangle_R g - C_l\rangle| \\ &= |\langle T\triangle_Q f, C_l\chi_{R_l} - C_l\rangle + \langle T\triangle_Q f, C_r\chi_{R_r}\rangle| \\ &\leq |C_l||\langle T\triangle_Q f, \chi_{R_l} - 1\rangle| + |\langle T\triangle_Q f, C_r\chi_{R_r}\rangle|\end{aligned}$$

For the latter term, we follow Lemma 3.5 together with $D(Q,R_l)\geq l(R_l)$ to get

$$|\langle T\triangle_Q f, C_r\chi_{R_r}\rangle| \leq C\frac{l(Q)^{\alpha/2}l(R_l)^{\alpha/2}}{D(Q,R_l)^{d+\alpha}}\mu(Q)^{1/2}\mu(R_l)^{1/2}\|\triangle_Q f\|_2\|C_r\chi_{R_r}\|_2.$$

Since $l(R) = 2l(R_l)$ and $\chi_{R_l}\cap\chi_{R_r} = \varnothing$,

$$|\langle T\triangle_Q f, C_r\chi_{R_r}\rangle| \leq C\cdot 2^{d+\alpha/2}\frac{l(Q)^{\alpha/2}}{l(R)^{d+\alpha/2}}\mu(Q)^{1/2}\mu(R_l)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2.$$

By property of the measure,

$$
\begin{aligned}
|\langle T\triangle_Q f, C_r\chi_{R_r}\rangle| &\lesssim \left(\frac{l(Q)}{l(R)}\right)^{\alpha/2} \mu(Q)^{1/2}\frac{1}{l(R_l)^{d/2}}\|\triangle_Q f\|_2\|\triangle_R g\|_2 \\
&\lesssim \left(\frac{l(Q)}{l(R)}\right)^{\alpha/2} \left(\frac{\mu(Q)}{\mu(R_l)}\right)^{1/2} \|\triangle_Q f\|_2\|\triangle_R g\|_2.
\end{aligned}
$$

To handle the first term, we see first that for all $x \in Q^c$,

$$
|(T\triangle_Q f)(x)| \leq C\frac{l(Q)^\alpha}{dist(x,Q)^{d+\alpha}}\|\triangle_Q f\|_{L^1}.
$$

Indeed, if $dist(x,Q) \geq l(Q)$, then

$$
\begin{aligned}
|(T\triangle_Q f)(x)| = |\int K(x,y)\triangle_Q f(y)\, \mathrm{d}\mu(y)| &= |\int K(x,y)\triangle_Q f(y)\, \mathrm{d}\mu(y) - K(x,c(Q))\triangle_Q f(y)\, \mathrm{d}\mu(y)| \\
&= |\int \frac{|y-c(Q)|^\alpha}{|x-c(Q)|^{d+\alpha}}\triangle_Q f(y)\, \mathrm{d}\mu(y)| \\
&\leq C\frac{l(Q)^\alpha}{dist(x,Q)^{d+\alpha}}\|\triangle_Q f\|_{L^1}.
\end{aligned}
$$

On the other hand, if $dist(x,Q) \leq l(Q)$, then

$$
\begin{aligned}
|(T\triangle_Q f)(x)| \leq \int |K(x,y)||\triangle_Q f(y)|\, \mathrm{d}\mu(y) \leq \int \frac{|\triangle_Q f(y)|}{|x-y|^d}\, \mathrm{d}\mu(y) &\leq \frac{1}{dist(x,Q)^d}\|\triangle_Q f\|_{L^1} \\
&\leq C\frac{l(Q)^\alpha}{dist(x,Q)^{d+\alpha}}\|\triangle_Q f\|_{L^1}.
\end{aligned}
$$

By the above computation and Hölder's inequality, we bound the term as

$$
\begin{aligned}
|\langle T\triangle_Q f, \chi_{R_l} - 1\rangle| \leq \int_{R_l^c} |T\triangle_Q f|\, \mathrm{d}\mu &\leq Cl(Q)^\alpha\|\triangle_Q f\|_2\mu(Q)^{1/2}\int_{R_l^c} \frac{1}{dist(x,Q)^{d+\alpha}}\, \mathrm{d}\mu \\
&\leq C(\frac{d}{\alpha}+1)l(Q)^\alpha\|\triangle_Q f\|_2\mu(Q)^{1/2}\frac{1}{dist(Q,\partial R_l)^\alpha}
\end{aligned}
$$

where we can apply comparison lemma to estimate the integral in the last step since $dist(x,Q) > dist(Q,\partial R_l)$. Also observe that $\|\triangle_R g\|_2^2 = C_l^2\mu(R_l) + C_r^2\mu(R_r)$ and thus $|C_l| \leq \mu(R_l)^{-1/2}\|\triangle_R g\|_2^2$.

Together we then have

$$|C_l||\langle T\triangle_Q f, \chi_{R_l} - 1\rangle| \le C\frac{l(Q)^\alpha}{dist(Q,\partial R_l)^\alpha}\left(\frac{\mu(Q)}{\mu(R_l)}\right)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2.$$

Since Q is good, $dist(Q,\partial R_l) > l(Q)^\gamma l(R_l)^{1-\gamma} \ge l(Q)^{1/2}l(R_l)^{1/2}$. Finally, we have the estimate

$$\begin{aligned}|C_l||\langle T\triangle_Q f, \chi_{R_l} - 1\rangle| &\le C\left(\frac{l(Q)}{l(R_l)}\right)^{\alpha/2}\left(\frac{\mu(Q)}{\mu(R_l)}\right)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2\\ &\le C\cdot 2^{\alpha/2}\left(\frac{l(Q)}{l(R)}\right)^{\alpha/2}\left(\frac{\mu(Q)}{\mu(R_l)}\right)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2.\end{aligned}$$

□

Lemma 3.8. *The following estimate holds:*

$$\sum_{R\in\mathcal{D}'}\sum_{\substack{Q\subset R\\ l(Q)<2^{-r}l(R)}}\left(\frac{l(Q)}{l(R)}\right)^{\alpha/2}\left(\frac{\mu(Q)}{\mu(R_l)}\right)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2 \le C\Big(\sum_{Q\in\mathcal{D}}\|\triangle_Q f\|_2^2\Big)^{1/2}\Big(\sum_{R\in\mathcal{D}'}\|\triangle_R g\|_2^2\Big)^{1/2}.$$

Proof. We again consider the second sum as layers $l(Q) = 2^{-n}l(R)$ for $n > r$ in order to get the term $2^{-n\beta}, \beta > 0$ on the right side of the inequality so that it converges when we sum all the layers over n.

We first consider the layer $l(Q) = 2^k$ as $\sum\limits_{R\in\mathcal{D}'}\sum\limits_{\substack{Q\subset R\\ l(Q)=2^{-n}l(R)}} = \sum\limits_k\sum\limits_{\substack{R\\ l(R)=2^{n+k}}}\sum\limits_{\substack{Q\subset R\\ l(Q)=2^k}}$ and see that

$$\sum_{\substack{Q\subset R\\ l(Q)=2^k}}\left(\frac{l(Q)}{l(R)}\right)^{\alpha}\left(\frac{\mu(Q)}{\mu(R_l)}\right) \le \left(\frac{l(Q)}{l(R)}\right)^{\alpha}\sum_{R'\in ch(R)}\sum_{\substack{Q\subset R'\\ l(Q)=2^k}}\frac{\mu(Q)}{\mu(R')} \le 2\left(\frac{l(Q)}{l(R)}\right)^{\alpha} = 2^{1-n\alpha}.$$

Hence, by Hölder's inequality,

$$\begin{aligned}&\sum_k\sum_{\substack{R\\ l(R)=2^{n+k}}}\sum_{\substack{Q\subset R\\ l(Q)=2^k}}\left(\frac{l(Q)}{l(R)}\right)^{\alpha/2}\left(\frac{\mu(Q)}{\mu(R_l)}\right)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2\\ &\qquad\le\sum_k\sum_{\substack{R\\ l(R)=2^{n+k}}}\left(\sum_{\substack{Q\subset R\\ l(Q)=2^k}}\left(\frac{l(Q)}{l(R)}\right)^{\alpha}\left(\frac{\mu(Q)}{\mu(R_l)}\right)\right)^{1/2}\left(\sum_{\substack{Q\subset R\\ l(Q)=2^k}}\|\triangle_Q f\|_2^2\right)^{1/2}\|\triangle_R g\|_2.\end{aligned}$$

By the previous calculation and Hölder's inequality again, we bound the RHS by

$$2^{(1-n\alpha)/2}\Bigg(\sum_k \sum_{\substack{R\\ l(R)=2^{n+k}}} \sum_{\substack{Q\subset R\\ l(Q)=2^k}} \|\triangle_Q f\|_2^2\Bigg)^{1/2}\Bigg(\sum_k \sum_{\substack{R\\ l(R)=2^{n+k}}} \|\triangle_R g\|_2^2\Bigg)^{1/2}.$$

Since for each n the inner sum give one full layer of cubes $l(Q)=2^k$ for the first parentheses and $l(R)=2^{n+k}$ for the second ones, the upper bound is the desired one, i.e.

$$2^{(1-n\alpha)/2}\Bigg(\sum_{Q\in D} \|\triangle_Q f\|_2^2\Bigg)^{1/2}\Bigg(\sum_{R\in D'} \|\triangle_R g\|_2^2\Bigg)^{1/2}.$$

□

Lemma 3.9. *Let $K : R\times Q \to \mathbb{C}$ be a Calderón-Zygmund kernel for any cubes R, Q such that $2^{-r}l(Q)\le l(R)\le 2^r l(Q)$, and $dist(Q,R) > \varepsilon \min(l(Q), l(R))$. The Calderón-Zygmund operator T is bounded on L^2.*

Proof. WLOG, we first assume that $l(Q)\le l(R)$. We want to use Schur's test so we consider for all $x\in R$,

$$\|K(x,\cdot)\|_{L^1} \le \int_Q \frac{1}{|x-y|^d}\,\mathrm{d}\mu(y) = \int_{\varepsilon l(Q)<|x-y|<cl(R)} \frac{1}{|x-y|^d}\,\mathrm{d}\mu(y),$$

and the Comparison Lemma implies

$$\begin{aligned}\|K(x,\cdot)\|_{L^1} &\le d\int_{\varepsilon l(Q)}^{cl(R)} \frac{1}{t}dt\\ &= d\log\frac{cl(R)}{\varepsilon l(Q)}\\ &\le d\log\frac{c2^r l(Q)}{\varepsilon l(Q)} = C(r,\varepsilon).\end{aligned}$$

Similarly for all $y\in Q$,

$$\|K(\cdot,y)\|_{L^1} \le \int_R \frac{1}{|x-y|^d}\,\mathrm{d}\mu(x) \le C(r,\varepsilon).$$

By Schur's test, The operator T is bounded on L^2. □

Remark. This lemma also holds when $K : R' \times Q' \to \mathbb{C}$ for any parallelepipeds Q', R' in cubes Q, R, resp. where $2^{-r}l(Q) \le l(R) \le 2^r l(Q)$ such that $R' \cap Q' = \varnothing$ with $dist(Q', R') > \varepsilon \min(l(Q), l(R))$.

Lemma 3.10. *For any cubes $Q \in \mathcal{D}$, $R \in \mathcal{D}'$ such that $dist(Q, R) \le \varepsilon \min(l(Q), l(R))$ and $2^{-r}l(Q) \le l(R) \le 2^r l(Q)$, we have the estimate*

$$|\langle T\chi_Q, \chi_R\rangle| \le C_1 \mu(Q)^{1/2}\mu(R)^{1/2} + C_2 \mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\mu(Q)^{1/2}\mu(R)^{1/2} + \mathcal{M}(n)\left(\|\chi_Q\|_2 \|\chi_{R_b}\|_2 + \|\chi_{Q_b}\|_2 \|\chi_R\|_2\right).$$

Proof. Note first that this includes the cases that one cube contains in the other and $Q \cap R = \varnothing$. We would like to put a random grid G on the set $\triangle := Q \cap R$ regardless its emptiness so that we can get some estimating property. Thus, we consider the following: for any two cubes Q, R such that $2^{-r}l(Q) \le l(R) \le 2^r l(Q)$, let $s = (10\Lambda)^{-1}\varepsilon \min(l(Q), l(R))$ be the size of cubes S in the grid G. Note that we will see how small we pick the ε later so that it is fixed. Again, we want it to be uniformly distributed over $\mathbf{R}^N$. We can shift a fixed grid by $\xi(\omega)$ where ξ is a random vector uniformly distributed over $[0, s)^N$

For $\varepsilon' > 0$, let $G_{\varepsilon'} := \bigcup_{S \in G} S \backslash (1 - 2\varepsilon')S$ be an $\varepsilon' s$-neighborhood of the boundaries of the cubes S in the grid G. Hence, for a fixed point $x \in \mathbf{R}^N$, $\mathbb{P}\{\omega : x \in G_{\varepsilon'}\} = \varphi(\varepsilon')$. Clearly, $\varphi(\varepsilon') \to 0$ as $\varepsilon' \to 0$. Then, $\mathbb{E}(\mu(G_{\varepsilon'} \cap \triangle)) = \iint_{G_{\varepsilon'} \cap \triangle} \mathrm{d}\mu(x)\, \mathrm{d}\mathbb{P}(\omega) = \iint \chi_{G_{\varepsilon'}}(x) \cap \chi_\triangle(x)\, \mathrm{d}\mathbb{P}\, \mathrm{d}\mu = \int \chi_\triangle(x) \int \chi_{G_{\varepsilon'}}(x)\, \mathrm{d}\mathbb{P}\, \mathrm{d}\mu = \varphi(\varepsilon') \int \chi_\triangle(x)\, \mathrm{d}\mu = \varphi(\varepsilon')\mu(\triangle)$. Since $\mathbb{P}\{\omega : \mu(G_{\varepsilon'} \cap \triangle) = \varphi(\varepsilon')\mu(\triangle)\} \ne 0$, we have $\mathbb{P}\{\omega : \mu(G_{\varepsilon'} \cap \triangle) > \varphi(\varepsilon')\mu(\triangle)\} < \mathbb{P}\{\omega : \mu(G_{\varepsilon'} \cap \triangle) \ge \varphi(\varepsilon')\mu(\triangle)\} \le \frac{\mathbb{E}(\mu(G_{\varepsilon'} \cap \triangle))}{\varphi(\varepsilon')\mu(\triangle)} = 1$. Therefore, $\mathbb{P}\{\omega : \mu(G_{\varepsilon'} \cap \triangle) \le \varphi(\varepsilon')\mu(\triangle)\} > 0$. In other words, we can always find a grid G with the inequality for given ε' and $\triangle$.

To estimate $\left|\langle T\chi_Q, \chi_R\rangle\right|$, we split the cubes Q, R into three parts, Q_{sep}, Q_∂, and $\triangle_Q$, defined as the following:

$Q_{sep} := Q \backslash (\triangle \cup \delta_R)$

$Q_\partial := (Q \cap \delta_R) \backslash S$ where S is a small part of $Q \cap \delta_R \cap \triangle$ making boundary hyperplanes of Q_∂ in $\triangle$ go along the boundaries of the grid G

$\triangle_Q := Q \backslash (Q_{sep} \cup Q_\partial)$.

Note here that $Q_\partial \subset Q_b$. Then, we can decompose $\langle T\chi_Q, \chi_R\rangle = \langle T\chi_Q, \chi_{R_{sep}}\rangle + \langle T\chi_Q, \chi_{R_\partial}\rangle +$

$\langle T\chi_Q, \chi_{\triangle_R}\rangle$. For the first term, we have that $Q \cap R_{sep} = \varnothing$ with $dist(Q, R_{sep}) > \varepsilon l(Q)$ thus by Lemma 3.9,

$$\left|\langle T\chi_Q, \chi_{R_{sep}}\rangle\right| \leq C\mu(Q)^{1/2}\mu(R_{sep})^{1/2} \leq C\mu(Q)^{1/2}\mu(R)^{1/2}.$$

For the middle term, since $R_\partial \subset R_b$ together with the definition of $\mathcal{M}(n)$,

$$\langle T\chi_Q, \chi_{R_\partial}\rangle \leq \mathcal{M}(n) \left\|\chi_Q\right\|_{L^2} \|\chi_{R_b}\|_{L^2}.$$

For the last term, we write $\langle T\chi_Q, \chi_{\triangle_R}\rangle = \langle T\chi_{\triangle_Q}, \chi_{\triangle_R}\rangle + \langle T\chi_{Q_\partial}, \chi_{\triangle_R}\rangle + \langle T\chi_{Q_{sep}}, \chi_{\triangle_R}\rangle$.

Similarly to the previous consideration, we have, by definition, the estimate

$$\langle T\chi_{Q_\partial}, \chi_{\triangle_R}\rangle \leq \mathcal{M}(n)\|\chi_{Q_b}\|_{L^2}\|\chi_R\|_{L^2}$$

and by Lemma 3.9, the estimate

$$\langle T\chi_{Q_{sep}}, \chi_{\triangle_R}\rangle \leq C\mu(Q_{sep})^{1/2}\mu(\triangle_R)^{1/2} \leq C\mu(Q)^{1/2}\mu(R)^{1/2}.$$

So, only the first term $\langle T\chi_{\triangle_Q}, \chi_{\triangle_R}\rangle$ is left. We write $\triangle_Q = \triangle'_Q \cup \tilde{\triangle}_Q$ where $\triangle'_Q := \triangle_Q \cap G_{\varepsilon'}$, and $\tilde{\triangle}_Q := \triangle_Q \backslash G_{\varepsilon'}$, and similarly for $\triangle_R$. Recall that we pick G such that $\mu(G_{\varepsilon'} \cap \triangle) \leq \varphi(\varepsilon')\mu(\triangle)$. Hence, we can decompose $\langle T\chi_{\triangle_Q}, \chi_{\triangle_R}\rangle = \langle T\chi_{\triangle'_Q}, \chi_{\triangle_R}\rangle + \langle T\chi_{\tilde{\triangle}_Q}, \chi_{\triangle'_R}\rangle + \langle T\chi_{\tilde{\triangle}_Q}, \chi_{\tilde{\triangle}_R}\rangle$. For the first summand,

$$\begin{aligned}\langle T\chi_{\triangle'_Q}, \chi_{\triangle_R}\rangle \leq \mathcal{M}(n)\|\chi_{\triangle'_Q}\|_{L^2}\|\chi_{\triangle_R}\|_{L^2} &\leq \mathcal{M}(n)\mu(\triangle'_Q)^{1/2}\mu(\triangle_R)^{1/2} \\ &\leq \mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\mu(\triangle)^{1/2}\mu(\triangle)^{1/2} \\ &\leq \mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\mu(Q)^{1/2}\mu(R)^{1/2},\end{aligned}$$

and similarly for the middle one,

$$\langle T\chi_{\tilde{\triangle}_Q}, \chi_{\triangle'_R}\rangle \leq \mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\mu(Q)^{1/2}\mu(R)^{1/2}.$$

For the last term, consider that $\tilde{\triangle}_Q \cup \tilde{\triangle}_R$ consists of finitely many disjoint parallelepipeds S_k. Also, the set $\tilde{\triangle}_Q$ is a union of some of these parallelepipeds, and similarly for $\tilde{\triangle}_R$.

For two disjoint parallelepipeds S_1 and S_2, we have $dist(S_1, S_2) > 2\varepsilon' s$ thus, by Lemma 3.9,

$$|\langle T\chi_{S_1}, \chi_{S_2}\rangle| \leq C\mu(S_1)^{1/2}\mu(S_2)^{1/2} \leq C\mu(Q)^{1/2}\mu(R)^{1/2}.$$

The other case is that $S \in \tilde{\triangle}_Q \cap \tilde{\triangle}_R$. In this case, S must be a cube and hence by the assumption of weak boundedness and the chosen size of the grid G,

$$|\langle T\chi_S, \chi_S\rangle| \leq C\mu(\Lambda S) \leq C\mu(\triangle) \leq C\mu(Q)^{1/2}\mu(R)^{1/2}.$$

Since the number of the parallelepipeds S_k is finite depending on $r, \varepsilon, \Lambda, \varepsilon'$, taking the sum over all the parallelepipeds we have

$$\langle T\chi_{\tilde{\triangle}_Q}, \chi_{\tilde{\triangle}_R}\rangle \leq C\mu(Q)^{1/2}\mu(R)^{1/2}.$$

To summarize, we have estimated all the terms and see that

$$\begin{aligned} |\langle T\chi_Q, \chi_R\rangle| \leq C_1\mu(Q)^{1/2}\mu(R)^{1/2} + \mathcal{M}(n)\left(\|\chi_Q\|_{L^2}\|\chi_{R_b}\|_{L^2} + \|\chi_{Q_b}\|_{L^2}\|\chi_R\|_{L^2}\right) \\ + C_2\mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\mu(Q)^{1/2}\mu(R)^{1/2}. \end{aligned}$$

$\square$

Chapter 4

T1 theorem

In this chapter, we prove T1 theorem. As mentioned, we reduce the problem to bound a good part. To bound it, we decompose such good part in suitable way so that the estimates from previous chapter can apply. Since all hard work is prepared, here we should see a wide picture of the proof.

Theorem 4.1. *A Calderón-Zygmund operator T extends to a bounded operator on $L^2(\mu)$ if and only if the operator T is weakly bounded and $T1$, T^*1 belong to $BMO = BMO(\mu)$*

Proof. Let $X := \left\{f \in L^2(\mu) : \|f\|_2 \leq 1, \exists\, R \text{ such that } supp\, f \subseteq R, l(R) = 2^n\right\}$ and

$$\mathcal{M}(n) := \sup\{|\langle Tf, g\rangle| : f, g \in X\}.$$

Thus, the task is to bound this quantity uniformly in n. By definition of $\mathcal{M}(n)$, we choose functions $f, g \in X$ such that $|\langle Tf, g\rangle| \geq \frac{3}{4}\mathcal{M}(n)$. Let $\mathcal{D}(\omega)$ and $\mathcal{D}'(\omega')$ be random dyadic lattices in Lemma 2.11 so that $\|f_{bad}\|_2 \leq 2^{-3-2N}\|f\|_2$, $\|g_{bad}\|_2 \leq 2^{-3-2N}\|g\|_2$. We notice that f_{bad}, f_{gd} might not be in X since their support is not in the dyadic lattice so it can be bigger. However, for example, Q is the cube of size 2^n supporting f, Q can be covered by at most 2^N dyadic cubes $Q_k \in D(\omega)$, $l(Q_k) = 2^n$ so that f_{bad} and f_{good} are supported by union of Q_k and similarly for g_{bad}. We then have $\frac{f_{bad}\chi_{Q_k}}{\|f_{bad}\|_{L^2}} \in X$. Hence,

$$\left|\left\langle T\left(\frac{f_{bad}}{\|f_{bad}\|_{L^2}}\right), g\right\rangle\right| \leq \sum_{Q_k \in ch\left(\bigcup_{k=1}^{2^N} Q_k\right)} \left|\left\langle T\left(\frac{f_{bad}\chi_{Q_k}}{\|f_{bad}\|_{L^2}}\right), g\right\rangle\right| \leq 2^N\mathcal{M}(n)$$

and then $|\langle Tf_{bad}, g\rangle| \leq 2^N \|f_{bad}\|_{L^2} \mathcal{M}(n) \leq 2^N 2^{-3-2N}\mathcal{M}(n) \leq 2^{-3}\mathcal{M}(n)$. Similarly, $|\langle Tf_{gd}, g_{bad}\rangle| \leq 2^{2N}2^{-3-2N}\mathcal{M}(n) = 2^{-3}\mathcal{M}(n)$.

To estimate $|\langle Tf_{gd}, g_{gd}\rangle|$, we decompose f_{gd} and g_{gd} to get upper bound

$$\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ 2^{-r}l(R)\le l(Q)\le 2^{r}l(R)}}\langle T\triangle_Q f, \triangle_R g\rangle+\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ 2^{-r}l(R)>l(Q)}}\langle T\triangle_Q f, \triangle_R g\rangle+\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ 2^{r}l(R)<l(Q)}}\langle T\triangle_Q f, \triangle_R g\rangle. \tag{4.1}$$

For the second and third terms, we first observe that they are symmetric so that we can consider one case, say the second one. Since Q is good, we can decompose

$$\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ 2^{-r}l(R)>l(Q)}}\langle T\triangle_Q f, \triangle_R g\rangle=\sum_{R\in\mathcal{D}'}\sum_{\substack{Q\in\mathcal{G}\\ Q\cap R=\varnothing\\ 2^{-r}l(R)>l(Q)}}\langle T\triangle_Q f, \triangle_R g\rangle+\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in D\\ Q\subset R\\ 2^{-r}l(R)>l(Q)}}\langle T\triangle_Q f, \triangle_R g\rangle$$

For the first term, we use Lemma 3.5 and 3.6 to get

$$\begin{aligned}\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ Q\cap R=\varnothing\\ 2^{-r}l(R)>l(Q)}}|\langle T\triangle_Q f, \triangle_R g\rangle| &\lesssim \sum_{R\in\mathcal{D}'}\sum_{\substack{Q\in\mathcal{G}\\ Q\cap R=\varnothing\\ 2^{-r}l(R)>l(Q)}}\frac{l(Q)^{\alpha/2}l(R)^{\alpha/2}}{D(Q,R)^{d+\alpha}}\mu(Q)^{1/2}\mu(R)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2\\ &\lesssim \|f\|_2\|g\|_2.\end{aligned}$$

To estimate the latter sum, we use the paraproduct $\Pi(\cdot):=\Pi(\cdot,T^*1)$ to see that for each R

$$\begin{aligned}\sum_{\substack{Q\subset R\\ l(Q)<2^{-r}l(R)}}\langle T\triangle_Q f, \triangle_R g\rangle &= \sum_{\substack{Q\subset R\\ l(Q)<2^{-r}l(R)}}\langle (T-\Pi^*)\triangle_Q f, \triangle_R g\rangle+\sum_{\substack{Q\subset R\\ l(Q)<2^{-r}l(R)}}\langle \triangle_Q f, \Pi\triangle_R g\rangle\\ &= \sum_{\substack{Q\subset R\\ l(Q)<2^{-r}l(R)}}\langle (T-\Pi^*)\triangle_Q f, \triangle_R g\rangle+\langle f, \Pi g\rangle\end{aligned}$$

where the last equality follows from Lemma 3.4. Since T^*1 is in BMO, the paraproduct Π is bounded on L^2 and hence $|\langle f,\Pi g\rangle|\le C\|f\|_2\|g\|_2$. The rest uses Lemma 3.7, 3.8 and 2.3 to get that

$$\begin{aligned}\sum_{R\in\mathcal{D}'}\sum_{\substack{Q\subset R\\ l(Q)<2^{-r}l(R)}}|\langle (T-\Pi^*)\triangle_Q f, \triangle_R g\rangle| &\le C\sum_{R\in\mathcal{D}'}\sum_{\substack{Q\subset R\\ l(Q)<2^{-r}l(R)}}\left(\frac{l(Q)}{l(R)}\right)^{\alpha/2}\left(\frac{\mu(Q)}{\mu(R_1)}\right)^{1/2}\|\triangle_Q f\|_2\|\triangle_R g\|_2\\ &\le C\left(\sum_{Q\in D}\|\triangle_Q f\|_2^2\right)^{1/2}\left(\sum_{R\in D'}\|\triangle_R g\|_2^2\right)^{1/2}\le C\|f\|_2\|g\|_2.\end{aligned}$$

Now we go back to estimate the first sum of comparable size Q, R. For arbitrary $\varepsilon > 0$, we can separate the sum in (4.1) into:

$$\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ 2^{-r}l(R)\le l(Q)\le 2^{r}l(R)\\ dist(Q,R)\ge\varepsilon\min(l(Q),l(R))}}\langle T\triangle_Q f,\triangle_R g\rangle+\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in D\\ 2^{-r}l(R)\le l(Q)\le 2^{r}l(R)\\ dist(Q,R)<\varepsilon\min(l(Q),l(R))}}\langle T\triangle_Q f,\triangle_R g\rangle.$$

For the first term, the sum is included in the cases separate cubes and comparably separated cubes in Tb theorem. Thus, we refer to the cases to bound this term. To estimate the other term, by Lemme 2.4, we have

$$|\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ 2^{-r}l(R)\le l(Q)\le 2^{r}l(R)\\ dist(Q,R)<\varepsilon\min(l(Q),l(R))}}\langle T\triangle_Q f,\triangle_R g\rangle|\le\sum_{R\in\mathcal{D}'}\sum_{\substack{Q\in\mathcal{D}\\ 2^{-r}l(R)\le l(Q)\le 2^{r}l(R)\\ dist(Q,R)<\varepsilon\min(l(Q),l(R))}}|c_Q(f)c'_R(g)\langle T\chi_Q,\chi_R\rangle|.$$

We then estimate the bound applying Lemma 3.10 to get

$$\begin{aligned}&\sum_{R\in\mathcal{D}'}\sum_{R-related\ Q}|c_Q(f)c'_R(g)\langle T\chi_Q,\chi_R\rangle|\\ &\qquad\le\left(C_1+C_2\mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\right)\sum_{R\in\mathcal{D}'}\sum_{R-related\ Q}|c_Q(f)c'_R(g)|\mu(Q)^{1/2}\mu(R)^{1/2}\\ &\qquad\qquad+\mathcal{M}(n)\sum_{R\in\mathcal{D}'}\sum_{R-related\ Q}|c_Q(f)c'_R(g)|\left(\|\chi_Q\|_{L^2}\|\chi_{R_b}\|_{L^2}+\|\chi_{Q_b}\|_{L^2}\|\chi_R\|_{L^2}\right),\end{aligned}$$

where $R-related\ Q$ is $Q\in\mathcal{D}$ such that $2^{-r}l(R)\le l(Q)\le 2^{r}l(R),\ dist(Q,R)<\varepsilon\min(l(Q),l(R))$. Next, we observe that for each $R\in\mathcal{D}'$ there are at most $M(N,r)$ such $R-related\ Q$. Thus we can write the RHS of the above inequality as

$$\begin{aligned}&\left(C_1+C_2\mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\right)\sum_{j=1}^{M(N,r)}\sum_{R\in\mathcal{D}'}|c_{R(j)}(f)c'_R(g)|\mu(R(j))^{1/2}\mu(R)^{1/2}\\ &\qquad+\mathcal{M}(n)\sum_{j=1}^{M(N,r)}\sum_{R\in\mathcal{D}'}|c_{R(j)}(f)c'_R(g)|\left(\|\chi_{R(j)}\|_{L^2}\|\chi_{R_b}\|_{L^2}+\|\chi_{R(j)_b}\|_{L^2}\|\chi_R\|_{L^2}\right).\end{aligned}$$

Then using Cauchy-Schwartz inequality we have

$$\Big(C_1 + C_2\mathcal{M}(n)\sqrt{\varphi(\varepsilon')}\Big) \sum_{j=1}^{M(N,r)} \left(\sum_{R\in\mathcal{D}'} |c_{R(j)}(f)|^2 \mu(R(j))\right)^{1/2} \left(\sum_{R\in\mathcal{D}'} |c'_R(g)|^2\mu(R)\right)^{1/2}$$
$$+ \mathcal{M}(n) \left(\sum_{j=1}^{M(N,r)} \left(\sum_{R\in\mathcal{D}'} |c_{R(j)}(f)|^2 \|\chi_{R(j)}\|_{L^2}^2\right)^{1/2} \left(\sum_{R\in\mathcal{D}'} |c'_R(g)|^2 \|\chi_{R_b}\|_{L^2}^2\right)^{1/2}\right)$$
$$+ \mathcal{M}(n) \left(\sum_{j=1}^{M(N,r)} \left(\sum_{R\in\mathcal{D}'} |c_{R(j)}(f)|^2 \|\chi_{R(j)_b}\|_{L^2}^2\right)^{1/2} \left(\sum_{R\in\mathcal{D}'} |c'_R(g)|^2 \|\chi_R\|_{L^2}^2\right)^{1/2}\right).$$

Since the same-sized cubes are disjoint together with the other estimated terms from Lemma 2.11, we have that $\sum\limits_Q |c_Q(f)|^2\|\chi_{Q_b}\|_2^2 = \sum\limits_k \|f_b^k\|_2^2 \leq 8p_\varepsilon\|f\|_2^2$, $\sum\limits_R |c'_R(g)|^2\|\chi_{R_b}\|_2^2 = \sum\limits_k \|g_b^k\|_2^2 \leq 8p_\varepsilon\|g\|_2^2$. Also recall from Lemma 2.4 that $\sum\limits_Q |c_Q(f)|^2\mu(Q) = \|f\|_2^2$, $\sum\limits_R |c'_R(g)|^2\mu(R) = \|g\|_2^2$. Thus we obtain the upper bound

$$C_1 M(N,r)\|f\|_2\|g\|_2 \;+\; C_2\mathcal{M}(n)\sqrt{\varphi(\varepsilon')}M(N,r)\|f\|_2\|g\|_2 \;+\; 2\sqrt{8p_\varepsilon}\mathcal{M}(n)M(N,r)\|f\|_2\|g\|_2.$$

Then choose $\varepsilon, \varepsilon'$ small enough so that $4\sqrt{2p_\varepsilon}M(N,r) \leq \frac{1}{4}$ and $C_2\sqrt{\varphi(\varepsilon')}M(N,r) \leq \frac{1}{8}$ leading to the bound

$$|\sum_{R\in\mathcal{G}'} \sum_{\substack{Q\in D \\ 2^{-r}l(R)\leq l(Q)\leq 2^r l(R) \\ dist(Q,R)<\varepsilon\min(l(Q),l(R))}} \langle T\triangle_Q f, \triangle_R g\rangle| \leq C\|f\|_2\|g\|_2 + \frac{1}{8}\mathcal{M}(n)\|f\|_2\|g\|_2 + \frac{1}{8}\mathcal{M}(n)\|f\|_2\|g\|_2.$$

To recap, we have finished bounding the term $|\langle Tf_{gd}, g_{gd}\rangle|$ and hence obtain that

$$\begin{aligned}\frac{3}{4}\mathcal{M}(n) \leq |\langle Tf, g\rangle| &\leq |\langle Tf_{good}, g_{good}\rangle| + |\langle Tf_{good}, g_{bad}\rangle| + |\langle Tf_{bad}, g\rangle| \\ &\leq C + \frac{1}{8}\mathcal{M}(n) + \frac{1}{4}\mathcal{M}(n) + \frac{1}{8}\mathcal{M}(n) + \frac{1}{8}\mathcal{M}(n)\end{aligned}$$

which in turn yields boundedness of the quantity $\mathcal{M}(n)$. □

Chapter 5

Paraproduct estimates

In this chaper, we prove that the paraproduct Π arose in the T1 theorem is bounded from $L^p \times L^q$ to L^r where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for $p, q > 1$ and the doubling condition of the measure is assumed. For convenience, let us recall that

$$\Pi(g, T^*1) := \sum_{S \in \mathcal{D}'} \sum_{\substack{P \in \mathcal{D} \\ l(P) = 2^{-r} l(S) \\ dist(P, \partial S) \geq \lambda l(P)}} E_S g \cdot \triangle_P^*(T^*1).$$

To start, we observe L^p boundedness of the averaging operator and use it to prove an inequality that defines a size of a tree.

Lemma. *For $1 \leq p \leq \infty$ and $P \in \mathcal{I}$,*

$$\|E_P f\|_p \leq \|f\|_p.$$

Proof. Obviously, $\|E_P f\|_\infty \leq \|f\|_\infty$. It is also easy to see that

$$\|\frac{\chi_P(x)}{\mu(P)} \int_P f \,\mathrm{d}\mu\|_1 \leq \int_P |f| \,\mathrm{d}\mu \leq \|f\|_1.$$

By Marcinkiewicz's Interpolation, we get the result. $\square$

Lemma 5.1. *For a given tree $\mathcal{T}$ and $P \in \mathcal{T}$,*

$$\|E_P f\|_p \leq \inf_{x \in I_\mathcal{T}} M_p(Mf)(x) \mu(I_\mathcal{T})^{1/p}$$

where $M_p f = (M(|f|^p))^{1/p}$ and M is the dyadic maximal function.

Proof. Consider first that $\|E_P f\|_p \leq \|E_P(f\chi_{I_\mathcal{T}})\|_p + \|E_P(f\chi_{I_\mathcal{T}^C})\|_p$. It is obvious that $E_P(f\chi_{I_\mathcal{T}^C}) = 0$.

On the other hand, the previous lemma tells that

$$\|E_P(f\chi_{I_{\mathcal{T}}})\|_p \le \|f\chi_{I_{\mathcal{T}}}\|_p = (\frac{1}{\mu(I_{\mathcal{T}})}\int_{I_{\mathcal{T}}} |f|^p)^{1/p}\mu(I_{\mathcal{T}})^{1/p}.$$

With the fact that $|f(x)| \le Mf(x)$, it is straightforward to see that

$$(\frac{1}{\mu(I_{\mathcal{T}})}\int_{I_{\mathcal{T}}} |f|^p)^{1/p}\mu(I_{\mathcal{T}})^{1/p} \le \inf_{x\in I_{\mathcal{T}}} M_p f(x)\mu(I_{\mathcal{T}})^{1/p} \le \inf_{x\in I_{\mathcal{T}}} M_p(Mf(x))\mu(I_{\mathcal{T}})^{1/p}.$$

□

Definition. For a tree $\mathcal{T}$, define a square function

$$S_{\mathcal{T}}f(x) := (\sum_{P\in\mathcal{T}} |\triangle_P f|^2)^{1/2} = (\sum_{k=N}^{\infty} |\triangle_k f|^2)^{1/2}$$

and

$$S_n f(x) := (\sum_{k=N}^{n} |\triangle_k f|^2)^{1/2}.$$

Without loss of generality, we can consider $N = 1$ that means the biggest cubes in the tree have size $1/2$. We also need some general definitions along the proofs so let us state them here.

Let $f = (f_1, f_2, ...)$ be a sequence of function. Define the maximal function of the sequence $f^*(x) := \sup_{n\in\mathbf{N}} |f_n(x)|$.

Define the Rademacher functions $r_n(t) := sign\ \sin(2^n\pi t)$ for $n \in \mathbf{N}$.

Define a transform of $Ef = (E_1 f, E_2 f, ...)$ to be $Rf = (R_1 f, R_2 f, ...)$ where $R_n f := \sum_{k=1}^{n} r_k(t)\triangle_k f$.

Similarly to the averaging operator, we need L^p boundedness of the square function. In order to see that, we need the following result from Burkholder on martingale transforms [Bur66].

Lemma. *For each $n \ge 1$, R_n is bounded on L^p.*

Now we are ready to prove L^p boundedness of the square function. In fact, we have the following lemma.

Lemma. *For $1 < p < \infty$, there are positive real numbers M and N such that*

$$M\|Sf\|_p \le \|f\|_p \le N\|Sf\|_p.$$

Proof. We will show that $\|S_n(f)\|_p \lesssim \|E_n f\|_p \lesssim \|S_n(f)\|_p$ then by DCT with the fact that $E_n f \to f$ as $n \to \infty$, we get the result. First, by Khintchine's inequality and Fubini's theorem, one have

$$\|S_n(f)\|_p^p \lesssim \int \mathbb{E}[|\sum_{k=1}^n r_k(t)\triangle_k f|^p] = \mathbb{E}[\int |\sum_{k=1}^n r_k(t)\triangle_k f|^p] \,.$$

Since $k \leq n$ and R_n is bounded on L^p, we see that

$$\mathbb{E}[\int |\sum_{k=1}^n r_k(t)\triangle_k f|^p] = \mathbb{E}[\int |\sum_{k=1}^n r_k(t)\triangle_k (E_n f)|^p] \lesssim \|E_n f\|_p^p \,.$$

To see the other inequality, we use the fact about the Rademacher functions to see that conversely Ef is a transform of Rf under the same Rademacher sequences. In other words, $E_n f = \sum\limits_{k=1}^n r_k(t) d_k f$ where $d_k f := R_{k+1} f - R_k f = r_k(t)\triangle_k f$. Similarly, one have the desired inequality

$$\|E_n f\|_p^p \leq \mathbb{E} \int |\sum_{k=1}^n r_k(t)\triangle_k f|^p = \int \mathbb{E}|\sum_{k=1}^n r_k(t)\triangle_k f|^p \lesssim \int (\sum_{k=1}^n |\triangle_k f|^2)^{p/2} \,.$$

$\square$

As before, we have the following lemma which will define a size of a tree so that the sum of $\mu(I_{\mathcal{T}})$ is controlled by L^p norm of functions.

Lemma 5.2.

$$\|Sf\|_p \leq C \inf_{x \in I_{\mathcal{T}}} M_p(Mf)(x)\mu(I_{\mathcal{T}})^{1/p}.$$

Proof. Again one have $\|Sf\|_p \leq \|S(f\chi_{I_{\mathcal{T}}})\|_p + \|S(f\chi_{I_{\mathcal{T}}^C})\|_p$. Considering as $E_P f$ function, one obtain

$$\begin{aligned}\|S(f\chi_{I_{\mathcal{T}}})\|_p \leq \|f\chi_{I_{\mathcal{T}}}\|_p = (\frac{1}{\mu(I_{\mathcal{T}})}\int_{I_{\mathcal{T}}} |f|^p)^{1/p}\mu(I_{\mathcal{T}})^{1/p} &\leq \inf_{x\in I_{\mathcal{T}}} M_p f(x)\mu(I_{\mathcal{T}})^{1/p} \\ &\leq \inf_{x\in I_{\mathcal{T}}} M_p(Mf(x))\mu(I_{\mathcal{T}})^{1/p}\end{aligned}$$

where the last inequality follows from the fact that $|f(x)| \leq Mf(x)$ a.e. The latter term $S(f\chi_{I_{\mathcal{T}}^C}) = 0$ since Sf is supported on $I_{\mathcal{T}}$. $\square$

Now we define some sizes of a tree so that we can decompose cubes regarding these sizes.

Definition. For a tree $\mathcal{T}$, $1 < p < \infty$, define

$$size_1(\mathcal{T}) := \frac{1}{\mu(I_{\mathcal{T}})^{1/p}} \sup_{P \in \mathcal{T}} \|E_P f\|_p,$$

$$size_2(\mathcal{T}) := \frac{1}{\mu(I_{\mathcal{T}})^{1/p}} \|Sf\|_p,$$

$$size^*(\mathcal{T}) := \sup\{size(\mathcal{S}) : \text{convex tree } \mathcal{S} \subseteq \mathcal{T}\}.$$

Lemma 5.3. *$\|E_P f\|_\infty \leq size_1^*(\mathcal{T})$ for all $P \in \mathcal{T}$.*

Proof. Observe that $\|E_P f\|_\infty \leq \frac{1}{\mu(P)} \int_P |f| \leq \inf_{x \in P} Mf(x) \lesssim \inf_{x \in P} Mf(x)$. Also note that $E_P(E_P f) = E_P f$. Thus, with Hölder's inequality, we see that

$$\|E_P f\|_\infty = \|E_P(E_P f)\|_\infty \leq \inf_{x \in P} M(E_P f)(x) \leq \left(\frac{1}{\mu(P)} \int_P |E_P f|^p \right)^{\frac{1}{p}}.$$

From the definition, this means $\|E_P f\|_\infty \leq size_1^*(\mathcal{T})$.

$\square$

Next, we prepare a collection of trees with some desired properties.

Lemma 5.4. *Given a convex collection $\mathcal{Q}$ of tiles in $\mathcal{P}$, we can decompose it into a collection $\mathcal{S}$ of maximal convex trees $\mathcal{T}$ with respect to set inclusion with disjoint top tiles $I_{\mathcal{T}}$ and*

$$\sum_{\mathcal{T} \in \mathcal{S}} \mu(I_{\mathcal{T}}) \lesssim \frac{1}{size_l^*(\mathcal{Q})^p} \|f\|_p^p$$

for both $l = 1, 2$. The remaining tiles are collected in $\mathcal{Q}_{left}$ such that $size_l^*(\mathcal{Q}_{left}) \leq \dfrac{size_l^*(\mathcal{Q})}{2}$ so that $\mathcal{Q} = \bigcup_{\mathcal{T} \in \mathcal{S}} \mathcal{T} \cup \mathcal{Q}_{left}$.

Proof. Take P in $\mathcal{Q}$ that is maximal such that $size_l(\mathcal{T}) > \frac{size_l^*(\mathcal{Q})}{2}$ where $\mathcal{T} = Tree(P) \cap \mathcal{Q}$ for both $l = 1, 2$. If there is no such P, then $\mathcal{Q}_{\text{left}} = \mathcal{Q}$. Remove the maximal tree with top P from $\mathcal{Q}$. Repeating the procedure with the new collections until no such P and then collecting all the remaining P in $\mathcal{Q}_{\text{left}}$, we get the decomposition.

Next, from the previous lemmas, $\inf_{x \in I_{\mathcal{T}}} M_p(Mf_l)(x) \geq size_l(\mathcal{T}) \geq \frac{size_l^*(\mathcal{Q})}{2}$ for all $\mathcal{T} \in \mathcal{S}$. Thus, we have $\bigcup_{\mathcal{T}\in\mathcal{S}} I_{\mathcal{T}} \subset \{x : M_p(Mf_l)(x) \geq \frac{size_l^*(\mathcal{Q})}{2}\}$. By disjointness, we have

$$\sum_{\mathcal{T}\in\mathcal{S}} \mu(I_{\mathcal{T}}) = \mu(\bigcup_{\mathcal{T}\in\mathcal{S}} I_{\mathcal{T}}) \leq \mu(\{x : M_p(Mf_l)(x) \geq \frac{size_l^*(\mathcal{Q})}{2}\}).$$

From weak L^1 and strong L^p boundedness of M, we respectively have

$$\mu(\{x : M_p(Mf_l)(x) \geq \frac{size_l^*(\mathcal{Q})}{2}\}) \lesssim \frac{1}{size_l^*(\mathcal{Q})^p}\|Mf_l\|_p^p \lesssim \frac{1}{size_l^*(\mathcal{Q})^p}\|f_l\|_p^p.$$

$\square$

Finally, we are able to establish the paraproduct estimate.

Theorem 5.5.

$$\|\Pi(f,g)\|_r \leq C\|f\|_p\|g\|_q$$

where $1/p + 1/q = 1/r$ *and* $p, q > 1$.

Proof. By an interpolation argument in [MTT02a], one need to show that for any measurable set F_1, F_2, F_3, there exists a measurable set F_3' such that $F_3' \subseteq F_3$ and $\mu(F_3') \geq \frac{1}{2}\mu(F_3)$ satisfying

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'}\rangle| \leq C\mu(F_1)^{\frac{1}{p}}\mu(F_2)^{\frac{1}{q}}\mu(F_3')^{\frac{1}{r'}}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. We consider the set

$$F_3' := F_3 \Big\backslash \left(\left\{ x : M_p(M\mathbb{1}_{F_1})(x) > C_0 \frac{\mu(F_1)^{\frac{1}{p}}}{\mu(F_3)^{\frac{1}{p}}} \right\} \bigcup \left\{ x : M_q(M\mathbb{1}_{F_2})(x) > C_0 \frac{\mu(F_2)^{\frac{1}{q}}}{\mu(F_3)^{\frac{1}{q}}} \right\} \right).$$

To check that this F_3' works, we observe using weak L^1 of M that

$$\mu(\left\{ x : M_p(M\mathbb{1}_{F_1})(x) > C_0 \frac{\mu(F_1)^{\frac{1}{p}}}{\mu(F_3)^{\frac{1}{p}}} \right\}) \leq \frac{C\mu(F_3)}{C_0^p\mu(F_1)}\|M\mathbb{1}_{F_1}\|_p^p$$

Therefore, by strong boundedness, we have

$$\frac{C\mu(F_3)}{C_0^p\mu(F_1)}\|M\mathbb{1}_{F_1}\|_p^p \leq \frac{C}{C_0^p}\mu(F_3) \leq \frac{1}{4}\mu(F_3)$$

where we choose C_0 large enough for the last inequality. Similarly for F_2 with q. Therefore, we have $\mu(F_3') \sim \mu(F_3)$.

To obtain the main inequality, we first apply Lemma 5.4 to decompose $\mathcal{D}'$ as $\mathcal{D}' = \bigcup_{\sigma} \bigcup_{\mathcal{T}\in\mathcal{T}_\sigma} \mathcal{T}$. Observe that $size_l^*(\mathcal{D}') = \sup_{\mathcal{T}\subset\mathcal{D}'} size_l(\mathcal{T}) \leq \sup_{\mathcal{T}\subset\mathcal{D}'} \inf_{x\in I_{\mathcal{T}}} M_p(M\mathbb{1}_{F_l})(x) \leq C_0 \frac{\mu(F_l)^{1/p}}{\mu(F_3)^{1/p}}$ since $x \in F_3'$. We thus consider $size_1(\mathcal{T}) \sim \frac{\sigma\mu(F_1)^{\frac{1}{p}}}{\mu(F_3)^{\frac{1}{p}}}$ and $size_2(\mathcal{T}) \sim \frac{\sigma\mu(F_2)^{\frac{1}{q}}}{\mu(F_3)^{\frac{1}{q}}}$ for all $\mathcal{T} \in \mathcal{T}_\sigma$ where σ represent dyadic numbers. Therefore, we have

$$\begin{aligned}\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'}\rangle &= \sum_{S\in\mathcal{D}'} \sum_{P\in\mathcal{D}(S)} \langle E_S\mathbb{1}_{F_1} \cdot \triangle_P \mathbb{1}_{F_2}, \mathbb{1}_{F_3'}\rangle \\ &= \sum_{\sigma} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} \sum_{S\in\mathcal{T}} \sum_{P\in\mathcal{D}(S)} \langle E_S\mathbb{1}_{F_1} \cdot \triangle_P \mathbb{1}_{F_2}, \mathbb{1}_{F_3'}\rangle\end{aligned}$$

where $P \in \mathcal{D}(S)$ stands for $P \in \mathcal{D}$ such that $P \subset S$, $l(P) = 2^{-r}l(S)$ and $dist(P, \partial S) \geq \lambda l(P)$. Since for all $x \in P \subset S$, $E_S\mathbb{1}_{F_1}(x) = \langle \mathbb{1}_{F_1}\rangle_S$ is a constant, we get

$$\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'}\rangle = \sum_{\sigma} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} \sum_{S\in\mathcal{T}} \sum_{P\in\mathcal{D}(S)} \langle \mathbb{1}_{F_1}\rangle_S \langle \triangle_P \mathbb{1}_{F_2}, \mathbb{1}_{F_3'}\rangle.$$

Since $\triangle_P^2 = \triangle_P, \triangle_P^* = \triangle_P$,

$$\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'}\rangle = \sum_{\sigma} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} \sum_{S\in\mathcal{T}} \sum_{P\in\mathcal{D}(S)} \langle \mathbb{1}_{F_1}\rangle_S \langle \triangle_P \mathbb{1}_{F_2}, \triangle_P \mathbb{1}_{F_3'}\rangle.$$

We know by definition that $|\langle \mathbb{1}_{F_1}\rangle_S| = \|E_S\mathbb{1}_{F_1}\|_\infty$ and we have seen that

$$\sum_{S\in\mathcal{T}} \sum_{P\in\mathcal{D}(S)} |\langle \triangle_P \mathbb{1}_{F_2}, \triangle_P \mathbb{1}_{F_3'}\rangle| \leq \sum_{\substack{P\subseteq I_{\mathcal{T}} \\ l(P)\leq 2^{-r}l(I_{\mathcal{T}}) \\ dist(P,\partial I_{\mathcal{T}})\geq\lambda l(P)}} |\langle \triangle_P \mathbb{1}_{F_2}, \triangle_P \mathbb{1}_{F_3'}\rangle|.$$

Together with Lemma 5.3, we get

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'}\rangle| \leq \sum_{\sigma} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} size_1^*(\mathcal{T}) \sum_{\substack{P\subseteq I_{\mathcal{T}} \\ l(P)\leq 2^{-r}l(I_{\mathcal{T}}) \\ dist(P,\partial I_{\mathcal{T}})\geq\lambda l(P)}} \int |\triangle_P \mathbb{1}_{F_2}(x)||\triangle_P \mathbb{1}_{F_3'}(x)| \, \mathrm{d}\mu.$$

We also have seen that we can rewrite the sum in the form of a collection of Whitney intervals $\mathcal{W}$, i.e.

$$\mathcal{W} := \bigcup_{i\geq 0} W_i$$

where W_0 is the collection of intervals $P \subset I_{\mathcal{T}}$ such that $l(P) = 2^{-r} l(I_{\mathcal{T}})$ and $dist(P, \partial I_{\mathcal{T}}) \geq \lambda l(P)$ and W_i is the collection of intervals $P \subset I_{\mathcal{T}}$ such that $l(P) = 2^{-r-i} l(I_{\mathcal{T}})$ and $dist(P, \partial I_{\mathcal{T}}) \geq \lambda l(P)$ and $P \cap \bigcup_{j=0}^{i-1} W_j = \varnothing$ for $i = 1, 2, 3, ...$ so that

$$\sum_{\substack{P\subseteq I_{\mathcal{T}} \\ l(P)\leq 2^{-r} l(I_{\mathcal{T}}) \\ dist(P,\partial I_{\mathcal{T}})\geq \lambda l(P)}} \int |\triangle_P \mathbb{1}_{F_2}(x)||\triangle_P \mathbb{1}_{F_3'}(x)| \, \mathrm{d}\mu = \sum_{Q\in\mathcal{W}} \sum_{P\subseteq Q} \int |\triangle_P \mathbb{1}_{F_2}(x)||\triangle_P \mathbb{1}_{F_3'}(x)| \, \mathrm{d}\mu.$$

Cauchy-Schwartz inequality yields that

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'} \rangle| \leq \sum_{\sigma} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} size_1^*(\mathcal{T}) \sum_{Q\in\mathcal{W}} \int (\sum_{P\subseteq Q} |\triangle_P \mathbb{1}_{F_2}(x)|^2)^{\frac{1}{2}} (\sum_{P\subseteq Q} |\triangle_P \mathbb{1}_{F_3'}(x)|^2)^{\frac{1}{2}} \, \mathrm{d}\mu.$$

Hölder's nequality yields that

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'} \rangle| \leq \sum_{\sigma} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} size_1^*(\mathcal{T}) \sum_{Q\in\mathcal{W}} \|S_Q \mathbb{1}_{F_2}\|_p \|S_Q \mathbb{1}_{F_3}\|_{p'}.$$

Let us observe more that $\|S_Q \mathbb{1}_{F_3'}\|_{p'} \leq C \|\mathbb{1}_{F_3'} \mathbb{1}_Q\|_{p'} \leq C\mu(Q)^{\frac{1}{p'}}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ together with the definition of the sizes we then have

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'} \rangle| \leq \sum_{\sigma} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} size_1^*(\mathcal{T}) \sum_{Q\in\mathcal{W}} size_2(Tree(Q))\mu(Q)^{\frac{1}{p}} \mu(Q)^{\frac{1}{p'}}.$$

Then according to the sizes we chose at the beginning we can write

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'} \rangle| \leq \sum_{\sigma} \frac{\sigma\mu(F_1)^{\frac{1}{p}}}{\mu(F_3)^{\frac{1}{p}}} \frac{\sigma\mu(F_2)^{\frac{1}{q}}}{\mu(F_3)^{\frac{1}{q}}} \sum_{\mathcal{T}\in\mathcal{T}_\sigma} \sum_{Q\in\mathcal{W}} \mu(Q).$$

Since $Q \in \mathcal{W}$ are disjoint, we have that that

$$\sum_{Q \in \mathcal{W}} \mu(Q) \leq \mu(I_{\mathcal{T}}),$$

and therefore

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'} \rangle| \leq \sum_{\sigma} \frac{\sigma \mu(F_1)^{\frac{1}{p}}}{\mu(F_3)^{\frac{1}{p}}} \frac{\sigma \mu(F_2)^{\frac{1}{q}}}{\mu(F_3)^{\frac{1}{q}}} \sum_{\mathcal{T} \in \mathcal{T}_\sigma} \mu(I_{\mathcal{T}}).$$

On the other hand, we can see that

$$\frac{1}{size_l^*(\mathcal{D}')^p} \|\mathbb{1}_{F_l}\|_p^p \leq \frac{1}{size_l^*(\mathcal{T}_\sigma)^p} \|\mathbb{1}_{F_l}\|_p^p \sim \frac{\mu(F_3)}{\sigma^p \mu(F_l)} \mu(F_l) = \frac{1}{\sigma^p} \mu(F_3)$$

where this p represents both p and q. Thus, by the previous Lemma 5.4,

$$|\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'} \rangle| \leq C \sum_{\sigma} \frac{\sigma \mu(F_1)^{\frac{1}{p}}}{\mu(F_3)^{\frac{1}{p}}} \frac{\sigma \mu(F_2)^{\frac{1}{q}}}{\mu(F_3)^{\frac{1}{q}}} \frac{1}{\sigma^p} \mu(F_3)$$

and hence

$$\begin{aligned} |\langle \Pi(\mathbb{1}_{F_1}, \mathbb{1}_{F_2}), \mathbb{1}_{F_3'} \rangle| &\leq C \mu(F_1)^{\frac{1}{p}} \mu(F_2)^{\frac{1}{q}} \mu(F_3)^{1-\frac{1}{p}-\frac{1}{q}} \sum_{\sigma} \sigma^{2-p} \\ &\leq C \mu(F_1)^{\frac{1}{p}} \mu(F_2)^{\frac{1}{q}} \mu(F_3)^{\frac{1}{r'}} \end{aligned}$$

for $p < 2$. Note here that the sum converges because $\sigma \leq C_0$. Indeed, if $\sigma > C_0$, we would get $M_p(M\mathbb{1}_{F_l})(x) \geq \inf_{x \in I_{\mathcal{T}}} M_p(M\mathbb{1}_{F_l})(x) \geq size_l(\mathcal{T}) \sim \frac{\sigma \mu(F_l)^{1/p}}{\mu(F_3)^{1/p}} > C_0 \frac{\mu(F_l)^{1/p}}{\mu(F_3)^{1/p}}$ which contradicts to the fact that $x \in F_3'$.

□

Chapter 6

Tb theorem

In this closing chapter, Tb theorem is proved. We start setting the proof as the T1 theorem and treat each piece in sections. Definitions and lemmas are written in the smallest environments that require them to avoid confusion. Furthermore, we will work on dimension $N = 1$ for simplicity as higher dimensions are considered almost the same.

Theorem 6.1. *Let T be a Calderón-Zygmund operator satisfying that there exist weakly accretive functions b_1, b_2 such that for all cubes Q in $\mathbf{R}$,*

$$\|T(b_1\chi_Q)\|_{L^2(Q)} \leq B\mu(Q)^{1/2} \quad \text{and} \quad \|T^*(b_2\chi_Q)\|_{L^2(Q)} \leq B\mu(Q)^{1/2}.$$

Then T is bounded on L^2.

First, consider a quantity $\mathcal{M} := \sup\{|\langle Tf, g\rangle| : \|f\|_2, \|g\|_2 \leq 1\}$. Pick functions f and g in L^2 such that $\frac{3}{4}\mathcal{M} \leq |\langle Tf, g\rangle|$. Then choose two random dyadic lattice $\mathcal{D}$ and $\mathcal{D}'$ as in Lemma 2.12 leading to inequalities $|\langle Tf_{bad}, g\rangle| \leq \frac{1}{8}\mathcal{M}$ and $|\langle Tf_{gd}, g_{bad}\rangle| \leq \frac{1}{8}\mathcal{M}$. The main story in this chapter is to prove that $|\langle Tf_{gd}, g_{gd}\rangle| \leq C + \frac{1}{4}\mathcal{M}$. Once it is known, we have $\frac{3}{4}\mathcal{M} \leq \frac{1}{8}\mathcal{M} + \frac{1}{8}\mathcal{M} + C + \frac{1}{4}\mathcal{M}$ resulting in boundedness of $\mathcal{M}$.

To estimate $\langle Tf_{gd}, g_{gd}\rangle$, we break the term using Lemma 2.6 into three parts as

$$\langle Tf_{gd}, g_{gd}\rangle = \langle T(\sum_{Q\in\mathcal{G}} \triangle_Q^b f), \sum_{R\in\mathcal{G}'} \triangle_R^b g\rangle + \langle T(E_{Q_0}^b f), g_{gd}\rangle + \langle T(\sum_{Q\in\mathcal{G}} \triangle_Q^b f), E_{R_0}^b g\rangle$$

where Q_0, R_0 contain the support of the measure μ. Note that there is no concern to write just b associated to f and g instead of b_1 and b_2, respectively. Also, we only need to consider the first summand because the other ones can be estimated in the same way. To handle the first term, one splits it as

$$\langle T(\sum_{Q\in\mathcal{G}}\triangle_Q^b f),\sum_{R\in\mathcal{G}'}\triangle_R^b g\rangle=\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ l(Q)\geq l(R)}}\langle T(\triangle_Q^b f),\triangle_R^b g\rangle+\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ l(Q)<l(R)}}\langle T(\triangle_Q^b f),\triangle_R^b g\rangle.$$

This way one can treat only one term since the other can be considered similarly regarding T^*. Next, we divide the sum into three parts with respect to the distant of Q and R, i.e $dist(Q,R)\geq l(R)$, $\varepsilon l(R)\leq dist(Q,R)<l(R)$, and $dist(Q,R)<\varepsilon l(R)$. The rest is devoted to estimate these parts and such ε will be determined in the last one.

6.1 Separated cubes

In this case, we treat the sum in which $dist(Q,R)\geq l(R)$. Before working on the main part, let us post some lemmas first.

Lemma 6.2. *Let $\theta(i)=\lceil\frac{\gamma i+r}{1-\gamma}\rceil$ for $i\in\mathbf{N}_0$. For any cube $Q\in\mathcal{D}$ of size 2^k and any cube $R\in\mathcal{G}'$ of size 2^{k-m} such that $2^n l(Q)<D(Q,R)\leq 2^{n+1}l(Q)$ where $k\in\mathbf{Z}$ and $n,m\in\mathbf{N}_0$. Then $R\subset\pi^{n+\theta(n+m)}Q$.*

Proof. Observe that $n+\theta(n+m)>r$ so that $2^r l(R)\leq 2^r l(Q)<l(\pi^{n+\theta(n+m)}Q)$. Thus, by goodness of R, either $R\subset\pi^{n+\theta(n+m)}Q$ or $R\subset\mathbf{R}\backslash\pi^{n+\theta(n+m)}Q$. For the latter case, again by goodness, $l(R)^\gamma l(\pi^{n+\theta(n+m)}Q)^{1-\gamma}<dist(R,\partial\pi^{n+\theta(n+m)}Q)\leq D(R,Q)\leq 2^{n+1}l(Q)$. Computing the inequality with the size of Q,R brings about $r<1$ which is a contradiction. □

Lemma 6.3. *For any Q,R as in the previous lemma with additional assumption $dist(Q,R)\geq l(R)$. Then, for $x\in R,y\in Q$,*

$$|K(x,y)-K(x_R,y)|\lesssim\frac{2^{-\alpha(n+m)/4}}{\mu(\pi^{n+\theta(n+m)}Q)}.$$

Proof. Set $P:=\pi^{n+\theta(n+m)}Q$. Since $dist(Q,R)\geq l(R)$, the assumption on the kernel gives that $|K(x,y)-K(x_R,y)|\leq\frac{|x-x_R|^\alpha}{|x-y|^{d+\alpha}}$. Then decorate the bound as

$$\frac{|x-x_R|^\alpha}{|x-y|^{d+\alpha}}\leq\frac{2^{dj}l(R)^\alpha}{dist(R,Q)^\alpha(2^j|x-y|)^d}\leq\frac{2^{dj}l(R)^\alpha}{dist(R,Q)^\alpha}\cdot\frac{1}{\mu(B(x,2^j|x-y|))}=:\mathbf{A}\cdot\mathbf{B}$$

where j is any integer. We consider two possible cases.

Case $dist(R,Q) > l(Q)$: let $j = 2 + \theta(n+m)$. Observe that $2^{n+k} < D(R,Q) < 4dist(R,Q)$ so that $2^j|x-y| \geq 2^j dist(R,Q) > 2^{n+k+\theta(n+m)}$. Thus, $B(x, 2^j|x-y|) \supseteq P$ by Lemma 6.2 above and hence $\mathbf{B} \leq \mu(P)^{-1}$. On the other hand, $\mathbf{A} \leq 4^{\alpha} \cdot 2^{-\alpha(n+k)} \cdot 2^{dj+\alpha(k-m)} \lesssim 2^{d\theta(n+m)-\alpha(n+m)} \lesssim 2^{-\alpha(n+m)/4}$.

Case $dist(R,Q) \leq l(Q)$: let $j \in \mathbf{N}$ such that $2^{j-1} < \frac{2^{r(1-\gamma)}l(P)}{l(R)^{\gamma}l(Q)^{1-\gamma}} \leq 2^j$. First, we show that $dist(R,Q) \geq \frac{l(R)^{\gamma}l(Q)^{1-\gamma}}{2^{r(1-\gamma)}}$. Indeed, let otherwise assume. If $l(R) < 2^{-r}l(Q)$, then, by goodness of R, $dist(R,Q) > l(R)^{\gamma}l(Q)^{1-\gamma} \geq \frac{l(R)^{\gamma}l(Q)^{1-\gamma}}{2^{r(1-\gamma)}}$, a contradiction. If $l(R) \geq 2^{-r}l(Q)$, then $dist(Q,R) < l(R)^{\gamma}l(R)^{1-\gamma} = l(R)$, a contradiction. With such inequality, we see that $\frac{l(P)}{dist(Q,R)} \leq \frac{2^{r(1-\gamma)}l(P)}{l(R)^{\gamma}l(Q)^{1-\gamma}} \leq 2^j$ so that $l(P) \leq 2^j dist(R,Q) \leq 2^j|x-y|$. From Lemma 6.2 we know that $R \subset P$. Together we have that $P \subseteq B(x, 2^j|x-y|)$ leading to $\mathbf{B} \leq \mu(P)^{-1}$. Moreover, $\mathbf{A} \leq 2^{dj}l(R)^{\alpha}\frac{2^{\alpha r(1-\gamma)}}{l(R)^{\alpha\gamma}l(Q)^{\alpha(1-\gamma)}} \lesssim 2^{dj}\left(\frac{l(R)}{l(Q)}\right)^{\alpha(1-\gamma)} \lesssim \frac{2^{d(n+\theta(n+m))}}{2^{-dm\gamma}} = \frac{2^{d(m+n+\theta(n+m))}}{2^{m(d+\alpha)(1-\gamma)}}$. Also, it follows from $dist(R,Q) \leq l(Q)$ that $n \leq 1$ since $2^n < \frac{D(Q,R)}{l(Q)} \leq \frac{3l(Q)}{l(Q)}$. With $\gamma = \alpha(2\alpha + 2d)$, we get estimate $\mathbf{A} \lesssim 2^{-\alpha(n+m)/4}$. $\square$

We are ready to begin. Let us break the sum regarding the size of cubes and the long distance $D(Q,R)$ i.e.

$$\sum_{R\in\mathcal{G}'} \sum_{\substack{Q\in\mathcal{G} \\ l(Q)\geq l(R) \\ dist(Q,R)\geq l(R)}} \langle T\triangle_Q^b f, \triangle_R^b g\rangle = \sum_{n\in\mathbf{N}_0}\sum_{m\in\mathbf{N}_0}\sum_{k\in\mathbf{Z}} \sum_{\substack{R\in\mathcal{G}' \\ l(R)=2^{k-m}}} \sum_{\substack{Q\in\mathcal{G} \\ l(Q)=2^k \\ 2^{n+k}<D(Q,R)\leq 2^{n+k+1} \\ dist(Q,R)\geq l(R)}} \langle T\triangle_Q^b f, \triangle_R^b g\rangle.$$

Since $\triangle_Q^c f$ is constant on its children, we get $\langle T\triangle_Q^b f, \triangle_R^b g\rangle = \sum_{Q'\in ch(Q)} \langle \triangle_Q^c f\rangle_{Q'} \langle T(b\chi_{Q'}), \triangle_R^b g\rangle$. Though the number of the children of any cubes depend on the dimension, it is finite so we will not keep track on this sum. We rewrite the term further from the fact that the mean of $\triangle_R^b g$ is zero to obtain $\langle T(b\chi_{Q'}), \triangle_R^b g\rangle = \langle T(b\chi_{Q'}) - T(b\chi_{Q'})(x_R), \triangle_R^b g\rangle$. Now we apply Cauchy-Schwartz inequality to bound

$$\Big|\sum_{k\in\mathbf{Z}} \sum_{\substack{R\in\mathcal{G}' \\ l(R)=2^{k-m}}} \triangle_R^b g(x)\chi_R(x) \sum_{\substack{Q\in\mathcal{G} \\ l(Q)=2^k \\ 2^{n+k}<D(Q,R)\leq 2^{n+k+1} \\ dist(Q,R)\geq l(R)}} \langle \triangle_Q^c f\rangle_{Q'} \left(Tb\chi_{Q'}(x) - Tb\chi_{Q'}(x_R)\right)\Big|$$

by

$$\Big(\sum_{k\in\mathbf{Z}}\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^{k-m}}}|\triangle_R^b g|^2\Big)^{1/2}\Big(\sum_{k\in\mathbf{Z}}\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^{k-m}}}|\chi_R\cdot\sum_{\substack{Q\in\mathcal{G}\\ l(Q)=2^k\\ 2^{n+k}<D(Q,R)\le 2^{n+k+1}\\ dist(Q,R)\ge l(R)}}\langle\triangle_Q^c f\rangle_{Q'}\left(Tb\chi_{Q'}(x)-Tb\chi_{Q'}(x_R)\right)|^2\Big)^{1/2}.$$

Using Cauchy-Schwartz inequality again we bound the integration of the above terms by

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^{k-m}}}|\triangle_R^b g|^2\Big)^{1/2}\Big\|_2\cdot\mathbf{B}$$

where $\mathbf{B}:=\Big\|\Big(\sum_{k\in\mathbf{Z}}\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^{k-m}}}|\chi_R\cdot\sum_{\substack{Q\in\mathcal{G}\\ l(Q)=2^k\\ 2^{n+k}<D(Q,R)\le 2^{n+k+1}\\ dist(Q,R)\ge l(R)}}\langle\triangle_Q^c f\rangle_{Q'}\left(Tb\chi_{Q'}-Tb\chi_{Q'}(x_R)\right)|^2\Big)^{1/2}\Big\|_2$. The left norm can be bounded by Lemma 2.6 as

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^{k-m}}}|\triangle_R^b g|^2\Big)^{1/2}\Big\|_2\lesssim\|g\|_2.$$

Thus, we are left to show that $\sum_{n\in\mathbf{N}_0}\sum_{m\in\mathbf{N}_0}\mathbf{B}\le C$. We will show that $\mathbf{B}\lesssim 2^{-\alpha(m+n)/4}$ and we are done since they are geometric series.

To see that, we consider a cube $S\in\mathcal{D}$ with $l(S)=k+n+\theta(n+m)$, $k\in\mathbf{Z}$ and consider $\mathbf{B}$ using disjointness of R as

$$\mathbf{B}^2=\int\Big|\sum_{k\in\mathbf{Z}}|\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^{k-m}}}\chi_R(x)\sum_{\substack{Q\in\mathcal{G}\\ l(Q)=2^k\\ D(Q,R)\sim 2^{n+k}\\ dist(Q,R)\ge l(R)}}\langle\triangle_Q^c f\rangle_{Q'}\left(Tb\chi_{Q'}(x)-Tb\chi_{Q'}(x_R)\right)|^2\Big|d\mu(x).$$

Then we group the sum over Q regarding a cube $S\in\mathcal{D}$ of size $k+n+\theta(n+m)$ so that

$$\mathbf{B}^2=\int\Big|\sum_{k\in\mathbf{Z}}|\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^{k-m}}}\chi_R(x)\sum_{\substack{S\in\mathcal{D}\\ l(S)=2^{k+n+\theta(n+m)}}}\sum_{\substack{Q\in\mathcal{G}\\ Q\subset S,\, l(Q)=2^k\\ D(Q,R)\sim 2^{n+k}\\ dist(Q,R)\ge l(R)}}\langle\triangle_Q^c f\rangle_{Q'}\left(Tb\chi_{Q'}(x)-Tb\chi_{Q'}(x_R)\right)|^2\Big|d\mu(x)$$

Set

$$K_S(x,y) := \sum_{\substack{R\in\mathcal{G}' \\ l(R)=2^{k-m}}} \sum_{\substack{Q\in\mathcal{G} \\ Q\subset S, l(Q)=2^k \\ 2^{n+k}<D(Q,R)\leq 2^{n+k+1} \\ dist(Q,R)\geq l(R)}} \mu(Q')^{-1}\left(Tb\chi_{Q'}(x)-Tb\chi_{Q'}(x_R)\right)\chi_R(x)\chi_{Q'}(y)$$

then we see that

$$\mathbf{B}^2 = \int\Big|\sum_{k\in\mathbf{Z}}|\sum_{\substack{S\in\mathcal{D} \\ l(S)=2^{k+n+\theta(n+m)}}} \int K_S(x,y)\triangle_k^c f(y)d\mu(y)|^2\Big|d\mu(x)$$

In order to bound K_S, from Lemma 6.3, we have $\left|Tb\chi_{Q'}(x)-Tb\chi_{Q'}(x_R)\right| \lesssim \dfrac{2^{-\alpha(n+m)/4}}{\mu(S)}\mu(Q')$ and thus,

$$|K_S(x,y)| \lesssim \frac{2^{-\alpha(n+m)/4}}{\mu(S)} \sum_{\substack{R\in\mathcal{G}' \\ l(R)=2^{k-m}}} \sum_{\substack{Q\in\mathcal{G} \\ Q\subset S, l(Q)=2^k \\ 2^{n+k}<D(Q,R)\leq 2^{n+k+1} \\ dist(Q,R)\geq l(R)}} \chi_R(x)\chi_{Q'}(y).$$

Then by Lemma 6.2 we have $R\subset S$ for such R and hence $|K_S(x,y)| \lesssim \dfrac{2^{-\alpha(n+m)/4}}{\mu(S)}\chi_S(x)\chi_S(y)$ by disjointness of cubes. Therefore,

$$\begin{aligned}\mathbf{B}^2 &\leq \int\Big|\sum_{k\in\mathbf{Z}}|\sum_{\substack{S\in\mathcal{D} \\ l(S)=2^{k+n+\theta(n+m)}}} \int |K_S(x,y)\triangle_k^c f(y)|d\mu(y)\big|^2\Big|d\mu(x) \\ &\lesssim 2^{-2\alpha(n+m)/4}\int\Big|\sum_{k\in\mathbf{Z}}|\sum_{\substack{S\in\mathcal{D} \\ l(S)=2^{k+n+\theta(n+m)}}} \langle|\triangle_k^c f|\rangle_S\chi_S(x)\big|^2\Big|d\mu(x).\end{aligned}$$

Observe that $\sum_{\substack{S\in\mathcal{D} \\ l(S)=2^{k+n+\theta(n+m)}}} \langle|\triangle_k^c f|\rangle_S\chi_S(x) = E_{k+n+\theta(u+m)}|\triangle_k^c f|(x)$. By Stein's inequality (2.2),

$$\int\Big|\sum_{k\in\mathbf{Z}}|\sum_{\substack{S\in\mathcal{D} \\ l(S)=2^{k+n+\theta(n+m)}}} \langle|\triangle_k^c f|\rangle_S\chi_S(x)\big|^2\Big|d\mu(x) \lesssim \int\Big|\sum_{k\in\mathbf{Z}}|\triangle_k^c f|^2\Big|d\mu(x) \lesssim \|f\|_2^2$$

where the last step is the inequality (2.1). Equivalently, $\mathbf{B}\lesssim 2^{-\alpha(n+m)/4}\|f\|_2$.

6.2 Comparably separated cubes

This section refers to the sums over Q, R such that $\varepsilon l(R) \le dist(Q,R) < l(R)$. We will apply Lemma 3.6 for this term. Thus our work is to get some bound of $\langle T\triangle_Q^b f, \triangle_R^b g\rangle$. First, let us observe a key inequality from goodness of cubes.

Lemma 6.4. *For good cubes Q, R with $dist(Q,R) \ge \varepsilon l(R)$, we have $dist(Q,R) \ge \dfrac{\varepsilon l(R)^\gamma l(Q)^{1-\gamma}}{2^{r(1-\gamma)}}$.*

Proof. If $l(R) > 2^{-r} l(Q)$, then we are done otherwise we would get a contradiction from

$$dist(Q,R) < \frac{\varepsilon l(R)^\gamma l(Q)^{1-\gamma}}{2^{r(1-\gamma)}} < \varepsilon l(R).$$

If $l(R) \le 2^{-r} l(Q)$, by goodness of R, we have

$$dist(Q,R) > l(R)^\gamma l(Q)^{1-\gamma} > \frac{l(R)^\gamma l(Q)^{1-\gamma}}{2^{r(1-\gamma)}} \ge \frac{\varepsilon l(R)^\gamma l(Q)^{1-\gamma}}{2^{r(1-\gamma)}}.$$

$\square$

Also recall that for all x outside the cube R, $|T^*(\triangle_R^b g)(x)| \le \dfrac{Cl(R)^\alpha}{dist(x,R)^{d+\alpha}} \|\triangle_R^b g\|_1$ as we considered in T1 theorem. Now we can see that

$$\begin{aligned}
|\langle T\triangle_Q^b f, \triangle_R^b g\rangle| = |\langle \triangle_Q^b f, T^*\triangle_R^b g\rangle| &\le Cl(R)^\alpha \|\triangle_R^b g\|_1 \int_Q \frac{|\triangle_Q^b f|}{dist(x,R)^{d+\alpha}} d\mu(x) \\
&\le C\frac{l(R)^\alpha}{dist(Q,R)^{d+\alpha}} \|\triangle_R^b g\|_1 \|\triangle_Q^b f\|_1 \\
&\le \frac{C2^{r(1-\gamma)(d+\alpha)}}{\varepsilon^{d+\alpha}} \frac{l(R)^\alpha}{l(R)^{\gamma(d+\alpha)} l(Q)^{(1-\gamma)(d+\alpha)}} \|\triangle_R^b g\|_1 \|\triangle_Q^b f\|_1
\end{aligned}$$

where the previous lemma is used in the last inequality. Recall that $\gamma d + \gamma\alpha = \alpha/2$. Hence,

$$\begin{aligned}
|\langle T\triangle_Q^b f, \triangle_R^b g\rangle| &\le C(r,\gamma,d,\alpha,\varepsilon) \frac{l(R)^\alpha}{l(R)^{\alpha/2} l(Q)^{d+\alpha/2}} \|\triangle_R^b g\|_1 \|\triangle_Q^b f\|_1 \\
&= C(r,\gamma,d,\alpha,\varepsilon) \frac{l(R)^{\alpha/2} l(Q)^{\alpha/2}}{l(Q)^{d+\alpha}} \|\triangle_R^b g\|_1 \|\triangle_Q^b f\|_1
\end{aligned}$$

Since $D(Q,R) \leq 3l(Q)$ in this case, we have

$$\begin{aligned}|\langle T\triangle_Q^b f, \triangle_R^b g\rangle| &\leq C(r,\gamma,d,\alpha,\varepsilon)\frac{l(R)^{\alpha/2}l(Q)^{\alpha/2}}{D(Q,R)^{d+\alpha}}\|\triangle_R^b g\|_1\|\triangle_Q^b f\|_1 \\ &\leq C(r,\gamma,d,\alpha,\varepsilon)\frac{l(R)^{\alpha/2}l(Q)^{\alpha/2}}{D(Q,R)^{d+\alpha}}\mu(R)^{1/2}\mu(Q)^{1/2}\|\triangle_R^b g\|_2\|\triangle_Q^b f\|_2\end{aligned}$$

where Cauchy-Schwartz is applied in the last step. Then similarly to Lemma 3.6, we have

$$\begin{aligned}\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ l(Q)\geq l(R)\\ \varepsilon l(R)\leq dist(Q,R)<l(R)}} |\langle T\triangle_Q^b f, \triangle_R^b g\rangle| &\leq C\left(\sum_{Q\in\mathcal{D}}\|\triangle_Q^b f\|_2^2\right)^{1/2}\left(\sum_{R\in\mathcal{D}'}\|\triangle_R^b g\|_2^2\right)^{1/2} \\ &\leq C\|f\|_2\|g\|_2\end{aligned}$$

in which we apply Lemma 2.6 to finish the proof.

6.3 Nearby and inside cubes

In this last section, we handle the sums of Q,R over $dist(Q,R) < \varepsilon l(R)$. We separate the sum regarding the size of Q and R into $2^{-r}l(Q) \leq l(R)$ $(\leq l(Q))$ and $l(R) < 2^{-r}l(Q)$.

6.3.1 The nearby term $2^{-r}l(Q) \leq l(R) \leq l(Q)$

This term is very similar to one term in T1 theorem except the weighted martingale difference is used here. In addition, the situation is simpler on account of one dimension. One can extend this proof to higher dimension by following T1 theorem. The difference is indeed in the following lemma.

Lemma. *For any cubes Q,R such that $2^{-r}l(Q) \leq l(R) \leq l(Q)$ with $dist(Q,R) < \varepsilon l(R)$,*

$$|\langle T(b\chi_Q), b\chi_R\rangle| \leq C\mu(Q)^{1/2}\mu(R)^{1/2} + \mathcal{M}\left(\|b\chi_Q\|_2\|b\chi_{R_b}\|_2 + \|b\chi_{Q_b}\|_2\|b\chi_R\|_2\right).$$

Proof. The proof is in a similar manner to Lemma 3.10 even without bothering ε'. □

To estimate the sum in this part, we break the summand using what we observe in Lemma 2.7

so that

$$\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\in\mathcal{G}\\ dist(Q,R)<\varepsilon l(R)\\ 2^{-r}l(Q)\le l(R)\le l(Q)}}|\langle T\triangle_Q^b f,\triangle_R^b g\rangle|\le\sum_{R\in\mathcal{D}'}\sum_{\substack{Q\in\mathcal{D}\\ dist(Q,R)<\varepsilon l(R)\\ 2^{-r}l(Q)\le l(R)\le l(Q)}}|c_Q(f)c'_R(g)\langle T(b\chi_Q),b\chi_R\rangle|.$$

Then by the above lemma we have

$$\begin{aligned}\sum_{R\in\mathcal{D}'}\sum_{such\,Q}|c_Q(f)c'_R(g)\langle T(b\chi_Q),b\chi_R\rangle| &\le C\cdot\sum_{R\in\mathcal{D}'}\sum_{such\,Q}|c_Q(f)c'_R(g)|\mu(Q)^{1/2}\mu(R)^{1/2}\\ &\quad+\mathcal{M}\cdot\sum_{R\in\mathcal{D}'}\sum_{such\,Q}|c_Q(f)c'_R(g)|\left(\|b\chi_Q\|_2\|b\chi_{R_b}\|_2+\|b\chi_{Q_b}\|_2\|b\chi_R\|_2\right).\end{aligned}$$

Again, recall the fact that for each $R\in\mathcal{D}'$ there are at most $M(r)$ cubes $Q\in\mathcal{D}$ such that $2^{-r}l(Q)\le l(R)\le l(R)$ and $dist(Q,R)<\varepsilon l(R)$. Thus we may consider the sums on RHS as

$$\begin{aligned}&C\cdot\sum_{j=1}^{M(r)}\sum_{R\in\mathcal{D}'}|c_{R(j)}(f)c'_R(g)|\mu(R(j))^{1/2}\mu(R)^{1/2}\\ &\quad+\mathcal{M}\cdot\sum_{j=1}^{M(r)}\sum_{R\in\mathcal{D}'}|c_{R(j)}(f)c'_R(g)|\left(\|b\chi_{R(j)}\|_2\|b\chi_{R_b}\|_2+\|b\chi_{R(j)_b}\|_2\|b\chi_R\|_2\right)=:\mathbf{I}+\mathbf{II}.\end{aligned}$$

Applying Cauchy-Schwartz inequality we have

$$\mathbf{I}\le C\cdot\sum_{j=1}^{M(r)}\left(\sum_{R\in\mathcal{D}'}|c_{R(j)}(f)|^2\mu(R(j))\right)^{1/2}\left(\sum_{R\in\mathcal{D}'}|c'_R(g)|^2\mu(R)\right)^{1/2}\lesssim\|f\|_2\|g\|_2$$

where the last step follows from Lemma 2.7. Now Cauchy-Schwartz inequality again yields

$$\begin{aligned}\mathbf{II}&\le\mathcal{M}\left(\sum_{j=1}^{M(r)}\left(\sum_{R\in\mathcal{D}'}|c_{R(j)}(f)|^2\|b\chi_{R(j)}\|_2^2\right)^{1/2}\left(\sum_{R\in\mathcal{D}'}|c'_R(g)|^2\|b\chi_{R_b}\|_2^2\right)^{1/2}\right)\\ &\quad+\mathcal{M}\left(\sum_{j=1}^{M(r)}\left(\sum_{R\in\mathcal{D}'}|c_{R(j)}(f)|^2\|b\chi_{R(j)_b}\|_2^2\right)^{1/2}\left(\sum_{R\in\mathcal{D}'}|c'_R(g)|^2\|b\chi_R\|_2^2\right)^{1/2}\right)\end{aligned}$$

Since cubes of the same size are disjoint, we have that $\sum_k\|f_b^k\|_2^2=\sum_{Q\in\mathcal{D}}|c_Q(f)|^2\|b\chi_{Q_b}\|_2^2$ and

$\sum\limits_{k} \|g_b^k\|_2^2 = \sum\limits_{R \in \mathcal{D}'} |c'_R(g)|^2 \|b\chi_{R_b}\|_2^2$. With the estimates from Lemma 2.12, we can conclude that

$$\mathbf{II} \lesssim \sqrt{8C(b,\delta)p_\varepsilon} M(r)\mathcal{M}\left(\|f\|_2\|g\|_2 + \|f\|_2\|g\|_2\right) \lesssim \frac{1}{4}\mathcal{M}$$

where we choose small ε making small enough p_ε for the last inequality.

6.3.2 The inside term $l(R) < 2^{-r}l(Q)$

Note first that "inside" comes from the fact that $dist(R, \partial Q) > l(Q)^\gamma l(R)^{1-\gamma} \geq l(R)$ by goodness of R. Thus, under the case $dist(Q,R) < \varepsilon l(R)$, it is only possible that $R \subset Q$. Also, we make an observation that $R \subset Q'$ for some $Q' \in ch(Q)$ since $l(R) \leq 2^{-r}l(Q')$ and R is good. We denote such child as Q'_R. In what follows, we apply time-frequency techniques constructing a tree which satisfies some desired properties. Also, as needed for the techniques, we replace $\triangle_Q^b f$ with $\triangle_Q^b f_{gd}$. This is fine due to the properties of the weighted martingale difference and only good Q we sum. The same applies to g as well.

Before constructing such tree, we need some notations. Intuitively, we need layers of parents and children in a tree of a cube. For a cube Q, denote its parent by πQ. In addition if a tree $\mathcal{T}$ is given, denote $\pi_\mathcal{T} Q$ to be the smallest cubes in $\mathcal{T}$ containing Q. Let $\pi_\mathcal{T}^1 Q$ be the smallest cube in $\mathcal{T}$ containing $\pi_\mathcal{T} Q$. Define $\pi_\mathcal{T}^k Q$ inductively. Similarly, define $ch_\mathcal{T} Q$ to be the collection of maximal cubes in $\mathcal{T}$ strictly contained in Q. Define $ch_\mathcal{T}^2 Q$ to be the collection of maximal cubes strictly contained in cubes of $ch_\mathcal{T} Q$. Inductively define $ch_\mathcal{T}^k Q$ and we are ready to go now.

We start with the maximal good cubes $Q \subset Q_0$ in $\mathcal{G}$ and put them in the tree $\mathcal{T}$. Next for each cube $T \in \mathcal{T}$, consider the maximal cubes $Q \subset T$ in $\mathcal{D}$ satisfying that $\langle |f_{gd}| \rangle_Q > 4\langle |f_{gd}| \rangle_T$ and that either Q or πQ is good. Then we add such cubes to the tree $\mathcal{T}$. Now we repeat the process at each minimal cubes $T \in \mathcal{T}$. Note that $\mathcal{T}'$ is denoted for the tree constructed by cubes in $\mathcal{D}'$ and g_{gd}. We observe desired properties about Carleson condition and embedding in the lemmas below.

Lemma 6.5. *For each cube $T \in \mathcal{T}$,*

$$\sum_{T' \in ch_\mathcal{T}(T)} \mu(T') \leq \frac{1}{4}\mu(T).$$

Moreover, the tree $\mathcal{T}$ satisfies a Carleson condition

$$\sum_{\substack{S \in \mathcal{T} \\ S \subseteq T}} \mu(S) \lesssim \mu(T)$$

for every $T \in \mathcal{T}$.

Proof. Given $T \in \mathcal{T}$, we have from the first condition that $\frac{4\mu(T')}{\mu(T)} \int_T |f_{gd}| d\mu < \int_{T'} |f_{gd}| d\mu$ for every $T' \in ch_{\mathcal{T}}(T)$. Since children are disjoint, we get $\frac{4}{\mu(T)} \int_T |f_{gd}| d\mu \sum_{T' \in ch_{\mathcal{T}}(T)} \mu(T') < \int_T |f_{gd}| d\mu$ and we are done. In addition, we can consider the sum of $S \in \mathcal{T}$ where $S \subseteq T$ as the sum of $T^{(n)} \in ch^n_{\mathcal{T}}(T)$ for all $n \in \mathbf{N}_0$. From the sparseness of each layer we just proved, we can argue inductively that for each n,

$$\begin{aligned}
\sum_{T^{(n)} \in ch^n_{\mathcal{T}}(T)} \mu(T^n) &\leq \frac{1}{4} \sum_{T^{(n-1)} \in ch^{n-1}_{\mathcal{T}}(T)} \mu(T^{(n-1)}) \\
&\leq \frac{1}{4^2} \sum_{T^{(n-2)} \in ch^{n-2}_{\mathcal{T}}(T)} \mu(T^{(n-2)}) \\
&\vdots \\
&\leq \frac{1}{4^n} \mu(T)
\end{aligned}$$

where $T^0 = T$. Therefore, it is summable over $n \in \mathbf{N}_0$. $\square$

Lemma 6.6. *The following embedding holds*

$$\sum_{T \in \mathcal{T}} \langle |f_{gd}| \rangle_T^2 \mu(T) \lesssim \|f_{gd}\|_2^2 \lesssim \|g\|_2^2.$$

Proof. Given $T \in \mathcal{T}$. Observe an inequality $\mu(T \backslash \bigcup_{T' \in ch_{\mathcal{T}}(T)} T') \geq \mu(T) - \frac{1}{4}\mu(T) = \frac{3}{4}\mu(T)$. Thus,

$$\sum_{T \in \mathcal{T}} \langle |f_{gd}| \rangle_T^2 \mu(T) \leq \frac{4}{3} \sum_{T \in \mathcal{T}} \langle |f_{gd}| \rangle_T^2 \mu(T \backslash \bigcup_{T' \in ch_{\mathcal{T}}(T)} T').$$

Recall a dyadic maximal function $Mf(x) := \sup_Q \langle |f| \rangle_Q$ where $x \in Q \in \mathcal{D}$. Then we have

$$\sum_{T\in\mathcal{T}} \langle |f_{gd}| \rangle_T^2 \mu(T \backslash \bigcup_{T' \in ch_{\mathcal{T}}(T)} T') \le \sum_{T\in\mathcal{T}} \int_{T \backslash \bigcup_{T' \in ch_{\mathcal{T}}(T)} T'} (Mf_{gd})^2 d\mu.$$

Since $T \backslash \bigcup_{T' \in ch_{\mathcal{T}}(T)} T'$ are disjoint for all $T \in \mathcal{T}$,

$$\sum_{T\in\mathcal{T}} \langle |f_{gd}| \rangle_T^2 \mu(T) \le \frac{4}{3} \int_{\mathbf{R}} (Mf_{gd})^2 d\mu.$$

By boundedness of the maximal function on L^p,

$$\sum_{T\in\mathcal{T}} \langle |f_{gd}| \rangle_T^2 \mu(T) \le \frac{4}{3} \int_{\mathbf{R}} |f_{gd}|^2 d\mu$$

as desired. □

We get back to the main track now. First, we decompose

$$\triangle_Q^b f_{gd} = \langle \triangle_Q^c f_{gd} \rangle_{Q'_R} b\chi_{\pi_{\mathcal{T}} Q'_R} - \langle \triangle_Q^c f_{gd} \rangle_{Q'_R} b\chi_{\pi_{\mathcal{T}} Q'_R \backslash Q'_R} + \triangle_Q^b f_{gd} \cdot \chi_{Q \backslash Q'_R}$$

so that $\langle T\triangle_Q^b f_{gd}, \triangle_R^b g_{gd} \rangle$ equals **I−II+III** :=

$$\langle \triangle_Q^c f_{gd} \rangle_{Q'_R} \langle T(b\chi_{\pi_{\mathcal{T}} Q'_R}), \triangle_R^b g_{gd} \rangle - \langle \triangle_Q^c f_{gd} \rangle_{Q'_R} \langle T(b\chi_{\pi_{\mathcal{T}} Q'_R \backslash Q'_R}), \triangle_R^b g_{gd} \rangle + \langle T(\triangle_Q^b f_{gd} \cdot \chi_{Q \backslash Q'_R}), \triangle_R^b g_{gd} \rangle.$$

6.3.3 I The sum of $\langle \triangle_Q^c f_{gd} \rangle_{Q'_R} \langle T(b\chi_{\pi_{\mathcal{T}} Q'_R}), \triangle_R^b g_{gd} \rangle$

First note that $\pi_{\mathcal{T}} Q'_R \in \mathcal{T}$. Thus we rewrite this sum according to $T \in \mathcal{T}$ as it has nice properties:

$$\sum_{R\in\mathcal{G}'} \sum_{\substack{Q\supset R \\ 2^r l(R) < l(Q)}} \langle \triangle_Q^c f_{gd} \rangle_{Q'_R} \langle T(b\chi_{\pi_{\mathcal{T}} Q'_R}), \triangle_R^b g_{gd} \rangle = \sum_{T\in\mathcal{T}} \sum_{R\in\mathcal{G}'} \sum_{\substack{Q\supset R \\ 2^r l(R) < l(Q) \\ \pi_{\mathcal{T}} Q'_R = T}} \langle \triangle_Q^c f_{gd} \rangle_{Q'_R} \langle T(b\chi_T), \triangle_R^b g_{gd} \rangle.$$

On purpose of utilizing a martingale transform, a constant $\varepsilon_{R,T}$ for fixed $R \in \mathcal{G}'$ and $T \in \mathcal{T}$ is defined by

$$\varepsilon_{R,T} := \frac{1}{\langle |f_{gd}| \rangle_T} \sum_{\substack{Q \supset R \\ 2^r l(R) < l(Q) \\ \pi_{\mathcal{T}} Q'_R = T}} \langle \triangle^c_Q f_{gd} \rangle_{Q'_R}.$$

Then we observe an important property as needed to apply the martingale transform inequality.

Lemma 6.7. *The constants defined above are uniformly bounded i.e.* $|\varepsilon_{R,T}| \lesssim 1$ *for any such* R *and* T*.*

Proof. Let us denote $\underline{Q}$, $\overline{Q}$ the minimal and maximal of cubes Q such that $Q \supset R, 2^r l(R) < l(Q)$, $\pi_{\mathcal{T}} Q'_R = T$, and $\mu(Q'_R) \neq 0$. Since $\triangle^c_Q f_{gd}$ is constant on Q'_R, so is on $\underline{Q}'_R$. Thus one can consider that

$$\sum_{\substack{Q \supset R \\ 2^r l(R) < l(Q) \\ \pi_{\mathcal{T}} Q'_R = T}} \langle \triangle^c_Q f_{gd} \rangle_{Q'_R} = \sum_{\substack{Q \in \mathcal{G} \\ \underline{Q} \subseteq Q \subseteq \overline{Q}}} \langle \triangle^c_Q f_{gd} \rangle_{\underline{Q}'_R} = \langle \sum_{\substack{Q \in \mathcal{G} \\ \underline{Q} \subseteq Q \subseteq \overline{Q}}} \triangle^c_Q f_{gd} \rangle_{\underline{Q}'_R}.$$

Since $\triangle^c_Q f_{gd} = 0$ for $Q \notin \mathcal{G}$, the sum can be viewed as all such $Q \in \mathcal{D}$. Now for all $x \in \underline{Q}'_R$, the series is a telescoping ones so that

$$\sum_{\substack{Q \in \mathcal{G} \\ \underline{Q} \subseteq Q \subseteq \overline{Q}}} \triangle^c_Q f_{gd} \cdot \chi_{\underline{Q}'_R} = \left(\left(\int_{\underline{Q}'_R} b \right)^{-1} \int_{\underline{Q}'_R} f_{gd} - \left(\int_{\overline{Q}} b \right)^{-1} \int_{\overline{Q}} f_{gd} \right) \chi_{\underline{Q}'_R}.$$

Therefore,

$$\sum_{\substack{Q \supset R \\ 2^r l(R) < l(Q) \\ \pi_{\mathcal{T}} Q'_R = T}} \langle \triangle^c_Q f_{gd} \rangle_{Q'_R} = \left(\int_{\underline{Q}'_R} b \right)^{-1} \int_{\underline{Q}'_R} f_{gd} - \left(\int_{\overline{Q}} b \right)^{-1} \int_{\overline{Q}} f_{gd}.$$

From accretivity of b and construction of $\mathcal{T}$, we have

$$\begin{aligned} |\varepsilon_{R,T}| \langle |f_{gd}| \rangle_T &\leq \frac{1}{\delta} \left(\frac{1}{\mu(\underline{Q}'_R)} \int_{\underline{Q}'_R} |f_{gd}| + \frac{1}{\mu(\overline{Q})} \int_{\overline{Q}} |f_{gd}| \right) \\ &\leq \frac{1}{\delta} \left(4 | \langle |f_{gd}| \rangle_T + 4 | \langle |f_{gd}| \rangle_T \right) \end{aligned}$$

leading to the conclusion that $|\varepsilon_{R,T}| \leq 8/\delta$.

□

At this point, the term we are considering can be written as

$$\sum_{T\in\mathcal{T}}\sum_{R\in\mathcal{G}'}\sum_{\substack{Q\supset R\\ 2^r l(R)<l(Q)\\ \pi_{\mathcal{T}}Q'_R=T}}\langle\triangle^c_Q f_{gd}\rangle_{Q'_R}\langle T(b\chi_T),\triangle^b_R g_{gd}\rangle=\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T\sum_{R\in\mathcal{G}'}\langle T(b\chi_T),\varepsilon_{R,T}\triangle^b_R g_{gd}\rangle.$$

As a consequence of non-orthogonality of this martingale difference, we need to consider the term in two parts, $\pi_{\mathcal{T}'}R\subset T$ and $\pi_{\mathcal{T}'}R\not\subseteq T$, regarding the other tree $\mathcal{T}'$. Therefore the following family is introduced.

Definition. For $T\in\mathcal{T}$, let $\mathcal{L}(T)$ be the collection of $\pi_{\mathcal{T}'}R$ for possible Q,R. More precisely,

$$\mathcal{L}(T):=\{\pi_{\mathcal{T}'}R\,|\,R\in\mathcal{G}',\,R\subset Q,\,2^r l(R)<l(Q),\text{ and }\pi_{\mathcal{T}}Q'_R=T\text{ for some }Q\in\mathcal{G}\}.$$

Denote $\mathcal{L}^k(T)$, for $k\geq 0$, the layer of cubes in $\mathcal{L}(T)$ for which $\pi^k_{\mathcal{L}(T)}$ of the cubes are maximal in $\mathcal{L}(T)$.

Hence, the sum can be considered as

$$\begin{aligned}\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T\sum_{R\in\mathcal{G}'}\langle T(b\chi_T),\varepsilon_{R,T}\triangle^b_R g_{gd}\rangle&=\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T\sum_{\substack{S\in\mathcal{L}(T)\\ S\not\subseteq T}}\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S}}\langle T(b\chi_T)\chi_S,\varepsilon_{R,T}\triangle^b_R g_{gd}\rangle\\&\quad+\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T\sum_{\substack{S\in\mathcal{L}(T)\\ S\subset T}}\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S}}\langle T(b\chi_T)\chi_S,\varepsilon_{R,T}\triangle^b_R g_{gd}\rangle.\end{aligned}$$

Two sections below show how to bound each term.

The term with $\pi_{\mathcal{T}'}R\not\subseteq T$

Recall that each considered R there exists Q for some size containing it such that $\pi_{\mathcal{T}}Q'_R=T$. Thus, $\pi_{\mathcal{T}}R\subset T$. Hence, we can rewrite the sum in terms of layers of children of T, i.e. for each $T\in\mathcal{T}$

$$\sum_{\substack{S\in\mathcal{L}(T)\\ S\not\subseteq T}}\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S}}\langle T(b\chi_T)\chi_S,\varepsilon_{R,T}\triangle^b_R g_{gd}\rangle=\sum_{t\geq 0}\sum_{\substack{S\in\mathcal{L}(T)\\ S\not\subseteq T}}\sum_{T'\in ch^t_{\mathcal{T}}(T)}\langle T(b\chi_T)\chi_S\chi_{T'},\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}}\varepsilon_{R,T}\triangle^b_R g_{gd}\rangle.$$

Next, let us observe the following lemma to get some finiteness.

Lemma. *For each $T \in \mathcal{T}$ and $S \in \mathcal{L}^k(T)$ such that $S \nsubseteq T$, k must be less than $2(r+1)$.*

Proof. Let $S_0 \in \mathcal{L}^0(T)$ the maximal cube containing S. Thus, $S_0 \cap T \neq \varnothing$ with $T \nsubseteq S_0$. Otherwise, either goodness of S_0 with $dist(S_0, \partial T) = 0$ would give $l(S_0) \leq 2^{-r} l(T)$ or goodness of πS_0 with $dist(\pi S_0, \partial T) = 0$ would give $l(\pi S_0) \leq 2^{-r} l(T)$, a contradiction. Hence, $dist(T, \partial S_0) = 0 = dist(\pi T, \partial S_0)$. Since either T or πT is good, one have either $l(T) > 2^{-r} l(S_0)$ or $l(\pi T) > 2^{-r} l(S_0)$. In other words, $l(S_0) \leq 2^r l(T)$. Therefore, $l(S) \leq 2^{-k} l(S_0) \leq 2^{-k+r} l(T)$. If $k \geq 2(r+1)$, then $2^{r+2} l(S) \leq l(T)$. However, $S \nsubseteq T$ so that $dist(T, \partial S_0) = 0 = dist(\pi T, \partial S_0)$ which contradicts to goodness of T or πT. □

We divide the sum over t respecting to $2(r+1)$ to make use of the fact that $2^t < 2^{2(r+1)}$ for $t \leq 2r+1$. That is to consider, using Cauchy-Schwartz inequality, to see that

$$\begin{aligned}
&\sum_{t \leq 2r+1} \sum_{\substack{S \in \mathcal{L}(T) \\ S \nsubseteq T}} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \Big|\Big\langle T(b\chi_T)\chi_S \chi_{T'}, \sum_{\substack{R \in \mathcal{G}' \\ \pi_{\mathcal{T}'} R = S \\ \pi_{\mathcal{T}} R = T'}} \varepsilon_{R,T} \triangle^b_R g_{gd} \Big\rangle\Big| \\
&\leq \sum_{t \leq 2r+1} \Bigg(\sum_{\substack{S \in \mathcal{L}(T) \\ S \nsubseteq T}} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \left\| T(b\chi_T)\chi_S \chi_{T'} \right\|_2^2 \Bigg)^{1/2} \Bigg(\sum_{\substack{S \in \mathcal{L}(T) \\ S \nsubseteq T}} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \Bigg\| \sum_{\substack{R \in \mathcal{G}' \\ \pi_{\mathcal{T}'} R = S \\ \pi_{\mathcal{T}} R = T'}} \varepsilon_{R,T} \triangle^b_R g_{gd} \Bigg\|_2^2 \Bigg)^{1/2}
\end{aligned}$$

For the term with the operator T with the above lemma, we can write

$$\begin{aligned}
\sum_{\substack{S \in \mathcal{L}(T) \\ S \nsubseteq T}} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \left\| T(b\chi_T)\chi_S \chi_{T'} \right\|_2^2 &= \sum_{k=0}^{2r+1} \sum_{\substack{S \in \mathcal{L}^k(T) \\ S \nsubseteq T}} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \left\| T(b\chi_T)\chi_S \chi_{T'} \right\|_2^2 \\
&\leq \sum_{k=0}^{2r+1} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \left\| T(b\chi_T)\chi_{T'} \right\|_2^2 \\
&\lesssim \sum_{T' \in ch^t_{\mathcal{T}}(T)} \left\| T(b\chi_T)\chi_{T'} \right\|_2^2 \\
&\leq \left\| T(b\chi_T)\chi_T \right\|_2^2
\end{aligned}$$

where the inequalities follow from disjointness of cubes in each layer $\mathcal{L}^k(T)$, finiteness of the sum in k and disjointness of children in each $ch^t_{\mathcal{T}}(T)$, respectively. Then, by the assumption, one have that

$$\left\|T(b\chi_T)\chi_T\right\|_2^2 \leq B^2\mu(T) < 2^{2(r+1)}B^2 2^{-t}\mu(T).$$

Now, let us turn to the other term where $t \geq 2(r+1)$ before treating the term with the martingale difference in the sum. We first adjust the term without changing anything since $\triangle_R^b g_{gd}$ has mean zero as

$$\langle T(b\chi_T)\chi_S\chi_{T'}, \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle = \langle\Big(T(b\chi_T) - T(b\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'})(x_{T'})\Big)\chi_S\chi_{T'}, \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle.$$

Using Cauchy-Schwartz inequality for the triple sums we can get the bound

$$\sum_{t>2r+1}\Bigg(\sum_{\substack{S\in\mathcal{L}(T) \\ S\not\subseteq T}}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\Big\|\Big(T(b\chi_T) - T(b\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'})(x_{T'})\Big)\chi_S\chi_{T'}\Big\|_2^2\Bigg)^{1/2}$$

$$\cdot\Bigg(\sum_{\substack{S\in\mathcal{L}(T) \\ S\not\subseteq T}}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\Big\|\sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}}\varepsilon_{R,T}\triangle_R^b g_{gd}\Big\|_2^2\Bigg)^{1/2}.$$

Again, consider the term with the operator T as above we obtain

$$\sum_{\substack{S\in\mathcal{L}(T) \\ S\not\subseteq T}}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\Big\|\Big(T(b\chi_T) - T(b\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'})(x_{T'})\Big)\chi_S\chi_{T'}\Big\|_2^2$$

$$\lesssim \sum_{T'\in ch_{\mathcal{T}}^t(T)}\Big\|\Big(T(b\chi_T) - T(b\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'})(x_{T'})\Big)\chi_{T'}\Big\|_2^2. \quad (6.1)$$

The above bound on the right side can be written regarding the cubes $\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'$ in $ch_{\mathcal{T}}^{\lceil t/2\rceil}(T)$ as

$$\sum_{T''\in ch_{\mathcal{T}}^{\lceil t/2\rceil}(T)}\sum_{\substack{T'\in ch_{\mathcal{T}}^t(T) \\ \pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'=T''}}\Big\|\big(T(b\chi_T) - T(b\chi_{T\setminus T''})(x_{T'})\big)\chi_{T'}\Big\|_2^2.$$

At each $T'' \in ch_{\mathcal{T}}^{\lceil t/2 \rceil}(T)$, we bound the second sum as

$$\begin{aligned}
&\sum_{\substack{T' \in ch_{\mathcal{T}}^t(T) \\ \pi_{\mathcal{T}}^{\lfloor t/2 \rfloor} T' = T''}} \left\| \left(T(b\chi_T) - T(b\chi_{T \setminus T''})(x_{T'})\right) \chi_{T'} \right\|_2^2 \\
&\qquad \le 2 \cdot \sum_{\substack{T' \in ch_{\mathcal{T}}^t(T) \\ \pi_{\mathcal{T}}^{\lfloor t/2 \rfloor} T' = T''}} \left\| \left(T(b\chi_{T''})\right) \chi_{T'} \right\|_2^2 + 2 \cdot \sum_{\substack{T' \in ch_{\mathcal{T}}^t(T) \\ \pi_{\mathcal{T}}^{\lfloor t/2 \rfloor} T' = T''}} \left\| \left(T(b\chi_{T \setminus T''}) - T(b\chi_{T \setminus T''})(x_{T'})\right) \chi_{T'} \right\|_2^2
\end{aligned}$$

where the first term can be bounded by disjointness of children $T' \subsetneq T''$ and the assumption as

$$\sum_{\substack{T' \in ch_{\mathcal{T}}^t(T) \\ \pi_{\mathcal{T}}^{\lfloor t/2 \rfloor} T' = T''}} \left\| \left(T(b\chi_{T''})\right) \chi_{T'} \right\|_2^2 \le \left\| \left(T(b\chi_{T''})\right) \chi_{T''} \right\|_2^2 \le B^2 \mu(T'').$$

For the second term, observe that $l(T'') \ge 2^{\lfloor t/2 \rfloor} l(T') \ge 2^{r+1} l(T') > 2^r l(T')$ and also recall that either T' or $\pi T'$ is good when collecting cubes in $\mathcal{T}$. If T' is good then, by its goodness, $dist(T', \partial T'') > l(T')$. In case $\pi T'$ is good, we can see that $2^r l(\pi T')$ is still less than $l(T'')$ and hence $dist(T', \partial T'') \ge dist(\pi T', \partial T'') > l(\pi T') > l(T')$ by goodness of $\pi T'$. Since $dist(T', \partial T'') \ge l(T')$ in any cases, for $x \in T'$,

$$|T(b\chi_{T \setminus T''})(x) - T(b\chi_{T \setminus T''})(x_{T'})| \le \int_{T \setminus T''} \frac{|x - x_{T'}|^\alpha}{|x-y|^{d+\alpha}} |b(y)| d\mu(y) \le \|b\|_\infty l(T')^\alpha \int_{T \setminus T''} \frac{1}{|x-y|^{d+\alpha}} d\mu(y).$$

By Comparison Lemma,

$$\begin{aligned}
|T(b\chi_{T \setminus T''})(x) - T(b\chi_{T \setminus T''})(x_{T'})| &\le (\frac{d}{\alpha} + 1) \|b\|_\infty \frac{l(T')^\alpha}{dist(T', \partial T'')^\alpha} \\
&\le (\frac{d}{\alpha} + 1) \|b\|_\infty.
\end{aligned}$$

Lastly, we use disjointness of T' again to get the bound $(\frac{d}{\alpha} + 1)^2 \|b\|_\infty^2 \mu(T'')$ for the second term. Thus, we just showed that

$$\sum_{\substack{T' \in ch_{\mathcal{T}}^t(T) \\ \pi_{\mathcal{T}}^{\lfloor t/2 \rfloor} T' = T''}} \left\| \left(T(b\chi_T) - T(b\chi_{T \setminus T''})(x_{T'})\right) \chi_{T'} \right\|_2^2 \le 2B^2 \mu(T'') + 2(\frac{d}{\alpha} + 1)^2 \|b\|_\infty^2 \mu(T'').$$

Taking sum over T'' we have

$$\sum_{T''\in ch_{\mathcal{T}}^{\lceil t/2\rceil}(T)} \sum_{\substack{T'\in ch_{\mathcal{T}}^{t}(T)\\ \pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'=T''}} \left\| \left(T(b\chi_{T}) - T(b\chi_{T\setminus T''})(x_{T'})\right)\chi_{T'} \right\|_2^2 \leq 2(B^2+(\frac{d}{\alpha}+1)^2\|b\|_\infty^2) \sum_{T''\in ch_{\mathcal{T}}^{\lceil t/2\rceil}(T)} \mu(T'').$$

Applying Lemma 6.5 $\lceil \frac{t}{2}\rceil$ steps so that

$$\sum_{T''\in ch_{\mathcal{T}}^{\lceil t/2\rceil}(T)} \mu(T'') \leq \frac{1}{2^{2\lceil t/2\rceil}}\mu(T) < \frac{1}{2^t}\mu(T).$$

There are two terms which are the same except t left and they should be treated similarly so let us recap and simplify things a bit.

$$\begin{aligned}
&|\sum_{T\in\mathcal{T}} \langle |f_{gd}|\rangle_T \sum_{\substack{S\in\mathcal{L}(T)\\ S\not\subseteq T}} \sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S}} \langle T(b\chi_T)\chi_S, \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle| \\
&\qquad \lesssim \sum_{t\geq 0} 2^{-t/2} \sum_{T\in\mathcal{T}} \langle |f_{gd}|\rangle_T \mu(T)^{1/2} \Bigg(\sum_{\substack{S\in\mathcal{L}(T)\\ S\not\subseteq T}} \sum_{T'\in ch_{\mathcal{T}}^{t}(T)} \Big\| \sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}} \varepsilon_{R,T}\triangle_R^b g_{gd} \Big\|_2^2 \Bigg)^{1/2} \\
&\qquad \leq \sum_{t\geq 0} 2^{-t/2} \Bigg(\sum_{T\in\mathcal{T}} \langle |f_{gd}|\rangle_T^2 \mu(T) \Bigg)^{1/2} \Bigg(\sum_{T\in\mathcal{T}} \sum_{\substack{S\in\mathcal{L}(T)\\ S\not\subseteq T}} \sum_{T'\in ch_{\mathcal{T}}^{t}(T)} \Big\| \sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}} \varepsilon_{R,T}\triangle_R^b g_{gd} \Big\|_2^2 \Bigg)^{1/2}
\end{aligned}$$

where our favorite Cauchy-Schwartz inequality is used in the last step. Lemma 6.6 says that $\sum_{T\in\mathcal{T}} \langle |f_{gd}|\rangle_T^2 \mu(T)$ is bounded by L^2 norm square of f with constant. If the remaining parentheses are bounded by something independent of t (actually by L^2 norm of g with constant), the sum is convergent geometric series. Thus what follows is only to see its boundedness.

First let us adjust the form for fixed $T\in\mathcal{T}$, $S\in\mathcal{L}(T)$ such that $S\not\subseteq T$, and $T'\in ch_{\mathcal{T}}^{t}(T)$ as

$$\Big\| \sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}} \varepsilon_{R,T}\triangle_R^b g_{gd} \Big\|_2^2 = \Big\| \sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S}} \varepsilon_{R,T}\triangle_R^b \Big(\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd} \Big) \Big\|_2^2 \lesssim \Big\| \sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd} \Big\|_2^2$$

where the properties of the weighted martingale difference give the equality (with suitable constants and R added to the first sum in the middle norm) and Lemma 2.8 with Lemma 6.7 yields the second inequality. Then we estimate the form as

$$\Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd} \Big\|_2^2 \lesssim \mu(T'\cap S)\langle|g_{gd}|\rangle_S^2 + \sum_{\substack{S'\in ch_{\mathcal{T}'}(S) \\ \pi_{\mathcal{T}}(\pi S')=T'}} \mu(S')\langle|g_{gd}|\rangle_{S'}^2$$

which is proved in the next four paragraphs.

Set $F_S := S\setminus \cup_{S'\in ch_{\mathcal{T}'}(S)} S'$. We then break the norm and consider that

$$\begin{aligned}\Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd} \Big\|_2^2 &= \Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd}\cdot \chi_{S\,\cap\, T'\setminus F_S\,\cap\, T'} \Big\|_2^2 + \Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd}\cdot \chi_{F_S\,\cap\, T'} \Big\|_2^2 \\ &\le \Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd}\cdot \chi_{S\setminus F_S} \Big\|_2^2 + \Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd}\cdot \chi_{F_S\,\cap\, T'} \Big\|_2^2.\end{aligned}$$

To handle the first term, consider a cube $S'\in ch_{\mathcal{T}'}(S)$ such that $R\in\mathcal{G}'$, $\pi_{\mathcal{T}'}R=S$, $\pi_{\mathcal{T}}R=T'$, and $S'\subset R$ exists. Denote $\underline{R}$ the minimal cube of such cubes and $\overline{R}$ the maximal one. In other words, $\underline{R}\subseteq R\subseteq \overline{R}$. Thus, due to the fact that $\triangle_R^b g_{gd}=0$ if R is bad, we can obtain a telescoping series and see that

$$\Big| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd}\cdot\chi_{S'} \Big| = \Big| \sum_{\substack{R\in\mathcal{D}' \\ \underline{R}\subseteq R\subseteq\overline{R}}} \triangle_R^b g_{gd}\cdot\chi_{S'\,\cap\, T'} \Big| \le \Big| E^b_{\underline{R}'_{S'}} g_{gd} - E^b_{\overline{R}} g_{gd} \Big| \chi_{S'\,\cap\, T'}$$

where $\underline{R}'_{S'}$ is the child of $\underline{R}$ containing S'. Also observe that for any cube R and function g, $E_R^b g$ can be estimated by $|E_R^b g| \le \frac{|b|\chi_R}{|\int_R bd\mu|}|\int_R gd\mu| \le \frac{\|b\|_\infty}{\delta\mu(R)}|\int_R gd\mu|\chi_R \le \frac{\|b\|_\infty}{\delta}\langle|g|\rangle_R\chi_R$. Therefore

$$\Big| E^b_{\underline{R}'_{S'}} g_{gd} - E^b_{\overline{R}} g_{gd} \Big| \chi_{S'\,\cap\, T'} \le \frac{\|b\|_\infty}{\delta}\left(\langle|g_{gd}|\rangle_{\underline{R}'_{S'}} + \langle|g_{gd}|\rangle_{\overline{R}}\right)\chi_{S'\,\cap\, T'}.$$

To exclude the good cube $\overline{R}$ from $\mathcal{T}'$, one must have $\langle|g_{gd}|\rangle_{\overline{R}} \le 4\langle|g_{gd}|\rangle_S$. For the other cube, if $\underline{R}'_{S'}=S'$ which is in $\mathcal{T}'$ then it gives us not more than $\langle|g_{gd}|\rangle_{\underline{R}'_{S'}} = \langle|g_{gd}|\rangle_{S'}$ for S' such that $\pi_{\mathcal{T}}(\pi S')=T'$. If $\underline{R}'_{S'}\neq S'$, then $\langle|g_{gd}|\rangle_{\underline{R}'_{S'}} \le 4\langle|g_{gd}|\rangle_S$ due to goodness of $\underline{R}$ and maximality of S'.

Hence, we get the estimate

$$\Big|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}}\triangle_R^b g_{gd}\cdot\chi_{S'}\Big|^2\lesssim\left(\langle|g_{gd}|\rangle_{S'}\chi_{S'}+\langle|g_{gd}|\rangle_S\chi_{S'\cap T'}\right)^2\lesssim\langle|g_{gd}|\rangle_{S'}^2\chi_{S'}+\langle|g_{gd}|\rangle_S^2\chi_{S'\cap T'}.$$

Then, using disjointness of S', we can estimate our first term as

$$\begin{aligned}\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}}\triangle_R^b g_{gd}\cdot\chi_{S\setminus F_S}\Big\|_2^2&=\sum_{S'\in ch_{\mathcal{T}'}(S)}\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}}\triangle_R^b g_{gd}\cdot\chi_{S'}\Big\|_2^2\\ &\lesssim\sum_{\substack{S'\in ch_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}(\pi S')=T'}}\langle|g_{gd}|\rangle_{S'}^2\mu(S')+\langle|g_{gd}|\rangle_S^2\mu(S\cap T').\end{aligned}$$

Here we look at the latter term. Consider a point $x\in F_S\cap T'$ such that $\lim_{k\to\infty}E_k g_{gd}(x)=g_{gd}(x)$ and all cubes $R\in\mathcal{G}'$ containing x satisfying $\pi_{\mathcal{T}'}R=S$, $\pi_{\mathcal{T}}R=T'$. Again, let $\overline{R}$ be the maximal one. Since $x\in F_S$ and $\triangle_R^b g_{gd}=0$ if R is bad, one can write

$$\Big|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}}\triangle_R^b g_{gd}(x)\Big|=\Big|\sum_{\substack{R\in\mathcal{D}'\\ R\subseteq\overline{R}\\ \pi_{\mathcal{T}}R=T'}}\triangle_R^b g_{gd}(x)\Big|$$

where the condition $\pi_{\mathcal{T}}R=T'$ is kept to determine cases. Indeed, if there is a minimal cube $\underline{R}$ subject to conditions $\pi_{\mathcal{T}}R=T'$, $x\in R\subset\overline{R}$, $R\in\mathcal{G}'$. As in the above paragraph, we obtains

$$\Big|\sum_{\substack{R\in\mathcal{D}'\\ R\subseteq\overline{R}\\ \pi_{\mathcal{T}}R=T'}}\triangle_R^b g_{gd}(x)\Big|\lesssim\langle|g_{gd}|\rangle_{\underline{R}'_x}+\langle|g_{gd}|\rangle_{\overline{R}}$$

where $\underline{R}'_x$ is the children of R containing x. Since the smallest cube in $\mathcal{T}'$ containing x is S, both $\langle|g_{gd}|\rangle_{\underline{R}'_x}$ and $\langle|g_{gd}|\rangle_{\overline{R}}$ are less than $4\langle|g_{gd}|\rangle_S$. If all cubes R satisfy such conditions, then in a similar manner we have

$$\Big|\sum_{\substack{R\in\mathcal{D}'\\ R\subseteq\overline{R}\\ \pi_{\mathcal{T}}R=T'}}\triangle_R^b g_{gd}(x)\Big|=\Big|\lim_{l(R)\to0}E_R^b g_{gd}(x)-E_{\overline{R}}^b g_{gd}(x)\Big|\le\frac{\|b\|_\infty}{\delta}\left(|g_{gd}(x)|+\langle|g_{gd}|\rangle_{\overline{R}}\right).\tag{6.2}$$

What's left is to estimate $|g_{gd}|$ by $\langle |g_{gd}| \rangle_S$. This can be treated in two cases below according to the good cubes belonging to $\mathcal{D}'$ contained in S.

The first case is that there is a minimal cube, say $\underline{R}$, of such good cubes containing x. Let $\underline{R}'_x$ be the child of $\underline{R}$ containing x. One reason that we works with the child is that $\langle g_{gd} \rangle_{\underline{R}'_x} = \langle g_{gd} \rangle_R$ for all $R \subseteq \underline{R}'_x$ in $\mathcal{D}'$ containing x. Hence,

$$|g_{gd}(x)| = \lim_{l(R)\to 0} |\langle g_{gd} \rangle_R| = |\langle g_{gd} \rangle_{\underline{R}'_x}|.$$

Since $\pi_{\mathcal{T}'} \underline{R}'_x = S$ and $\underline{R}$ is good, $|\langle g_{gd} \rangle_{\underline{R}'_x}| \leq 4\langle |g_{gd}| \rangle_S$ as desired. For the other case, there are infinitely many such good cubes. With the same argument as the previous case in the last step, we can see that

$$|g_{gd}(x)| = \lim_{\substack{R\in\mathcal{G}' \\ x\in R\subseteq S \\ l(R)\to 0}} |\langle g_{gd} \rangle_R| \leq \sup\{\langle |g_{gd}| \rangle_R : R \in \mathcal{G}',\, x \in R \subseteq S\} \leq 4\langle |g_{gd}| \rangle_S$$

finishing all the cases.

To recap, we just proved that

$$\Big| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd} \Big| \chi_{F_S \cap T'} \lesssim \langle |g_{gd}| \rangle_S \chi_{F_S \cap T'} \tag{6.3}$$

and hence

$$\Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd} \cdot \chi_{F_S \cap T'} \Big\|_2^2 \lesssim \langle |g_{gd}| \rangle_S^2 \mu(F_S \cap T') \leq \langle |g_{gd}| \rangle_S^2 \mu(S \cap T')$$

completing the proof of the desired estimate

$$\Big\| \sum_{\substack{R\in\mathcal{G}' \\ \pi_{\mathcal{T}'}R=S \\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd} \Big\|_2^2 \lesssim \mu(T' \cap S)\langle |g_{gd}| \rangle_S^2 + \sum_{\substack{S'\in ch_{\mathcal{T}'}(S) \\ \pi_{\mathcal{T}}(\pi S')=T'}} \mu(S')\langle |g_{gd}| \rangle_{S'}^2.$$

Going back to the term which causes a few pages above, i.e.

$$\sum_{T\in\mathcal{T}}\sum_{\substack{S\in\mathcal{L}(T)\\ S\not\subseteq T}}\sum_{T'\in ch^t_{\mathcal{T}}(T)}\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}}\varepsilon_{R,T}\triangle^b_R g_{gd}\Big\|_2^2$$

and using the last estimate above, we are able to bound it by

$$\sum_{S\in\mathcal{T}'}\langle|g_{gd}|\rangle_S^2\sum_{\substack{T\in\mathcal{T}\\ S\not\subseteq T}}\mu(T\cap S)+\sum_{S\in\mathcal{T}'}\sum_{S'\in ch_{\mathcal{T}'}(S)}\sum_{T\in\mathcal{T}}\sum_{\substack{T'\in ch^t_{\mathcal{T}}(T)\\ \pi_{\mathcal{T}}(\pi S')=T'}}\mu(S')\langle|g_{gd}|\rangle_{S'}^2.$$

We are almost there. To bound the first term, we consider the sum over T in two parts

$$\sum_{\substack{T\in\mathcal{T}\\ S\not\subseteq T}}\mu(T\cap S)=\sum_{\substack{T\in\mathcal{T},\,S\not\subseteq T\\ T\subseteq S}}\mu(T\cap S)+\sum_{\substack{T\in\mathcal{T},\,S\not\subseteq T\\ T\not\subseteq S}}\mu(T\cap S).$$

For $T\subseteq S$, we can obtain Carleson condition as in Lemma 6.5 with $T\cap S$ to have that $\sum\mu(T\cap S)\lesssim\mu(S)$. To tackle the sum with $T\not\subseteq S$, fix $S\in\mathcal{T}'$. Then recall that $l(S)\le 2^r l(T)$ as seen in the lemma about $\mathcal{L}^k$. In addition, since either S or πS is good, we must have $l(S)>2^{-r}l(T)$ or $l(\pi S)>2^{-r}l(T)$ for S such that $S\cap T\neq\varnothing$, $S\not\subseteq T$, $T\not\subseteq S$, respectively. In other words, $l(S)\ge 2^{-r}l(T)$. Hence, there are a certain number of such cubes T depending on r and dimension. This leads to the estimate $\sum\mu(T\cap S)\lesssim\mu(S)$ for the second sum and thus for the combined one as well. Applying Lemma 6.6, we can achieve boundedness for the first term

$$\sum_{S\in\mathcal{T}'}\langle|g_{gd}|\rangle_S^2\sum_{\substack{T\in\mathcal{T}\\ S\not\subseteq T}}\mu(T\cap S)\lesssim\sum_{S\in\mathcal{T}'}\langle|g_{gd}|\rangle_S^2\mu(S)\lesssim\|g\|_2^2.$$

The latter and the last term in this part looks frustrating however the sums over T and T' are nothing. This is because, for a cube S', there is none or one of possible T' such that $\pi_{\mathcal{T}}(\pi S')=T'$ and hence one T. Together with our favorite Lemma 6.6, we are done since

$$\sum_{S\in\mathcal{T}'}\sum_{S'\in ch_{\mathcal{T}'}(S)}\sum_{T\in\mathcal{T}}\sum_{\substack{T'\in ch^t_{\mathcal{T}}(T)\\ \pi_{\mathcal{T}}(\pi S')=T'}}\mu(S')\langle|g_{gd}|\rangle_{S'}^2\le\sum_{S\in\mathcal{T}'}\sum_{S'\in ch_{\mathcal{T}'}(S)}\mu(S')\langle|g_{gd}|\rangle_{S'}^2\lesssim\|g\|_2^2.$$

The term with $\pi_{\mathcal{T}'} R \subset T$

In this subsection we deal with the term

$$\sum_{T\in\mathcal{T}} \langle |f_{gd}|\rangle_T \sum_{\substack{S\in\mathcal{L}(T)\\ S\subset T}} \sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S}} \langle T(b\chi_T)\chi_S, \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle.$$

Since S is in $\mathcal{T}'$ and is subset of T, one can consider the family $\mathcal{R}(T')$ of maximal cubes in $\{S\in\mathcal{T}' : \pi_{\mathcal{T}}S = T'\}$ for fixed $T\in\mathcal{T}$, $T'\in ch^t_{\mathcal{T}}(T)$, and $t\geq 0$ though it can be empty. This way the term can be rewrite as

$$\sum_{T\in\mathcal{T}} \langle |f_{gd}|\rangle_T \sum_{t,k\geq 0} \sum_{T'\in ch^t_{\mathcal{T}}(T)} \sum_{S\in\mathcal{R}(T')} \sum_{\substack{S'\in ch^k_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}S'=T'}} \sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}} \langle T(b\chi_T)\chi_{S'}, \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle.$$

As before, we separate the sums over t,k into $0\leq t,k\leq 2r+1$ and $t,k\geq 2(r+1)$ in three cases.

In case $0\leq k,t\leq 2r+1$, let us first fix $T\in\mathcal{T}$ and t. We first use Cauchy-Schwartz inequality twice to get the bound

$$\Big(\sum_{k=0}^{2r+1} \sum_{T'\in ch^t_{\mathcal{T}}(T)} \sum_{S\in\mathcal{R}(T')} \sum_{\substack{S'\in ch^k_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}S'=T'}} \Big\|T(b\chi_T)\chi_{S'}\Big\|_2^2\Big)^{1/2}$$
$$\cdot\Big(\sum_{k=0}^{2r+1} \sum_{T'\in ch^t_{\mathcal{T}}(T)} \sum_{S\in\mathcal{R}(T')} \sum_{\substack{S'\in ch^k_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}S'=T'}} \Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}} \varepsilon_{R,T}\triangle_R^b g_{gd}\Big\|_2^2\Big)^{1/2}.$$

We will tackle the second parentheses later as before. For the first ones, it is clear to see due to disjointness of S' at each k that

$$\sum_{k=0}^{2r+1} \sum_{T'\in ch^t_{\mathcal{T}}(T)} \sum_{S\in\mathcal{R}(T')} \sum_{\substack{S'\in ch^k_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}S'=T'}} \Big\|T(b\chi_T)\chi_{S'}\Big\|_2^2 \leq \sum_{k=0}^{2r+1} \sum_{T'\in ch^t_{\mathcal{T}}(T)} \sum_{S\in\mathcal{R}(T')} \Big\|T(b\chi_T)\chi_S\Big\|_2^2$$

and maximality of S, disjointness of T' that

$$\sum_{k=0}^{2r+1} \sum_{T'\in ch^t_{\mathcal{T}}(T)} \sum_{S\in\mathcal{R}(T')} \Big\|T(b\chi_T)\chi_S\Big\|_2^2 \leq \sum_{k=0}^{2r+1} \sum_{T'\in ch^t_{\mathcal{T}}(T)} \Big\|T(b\chi_T)\chi_{T'}\Big\|_2^2 \leq \sum_{k=0}^{2r+1} \Big\|T(b\chi_T)\chi_T\Big\|_2^2$$

and finite number of k, and the assumption that

$$\sum_{k=0}^{2r+1}\left\|T(b\chi_T)\chi_T\right\|_2^2 \lesssim \left\|T(b\chi_T)\chi_T\right\|_2^2 \lesssim B^2\mu(T) \lesssim 2^{-t}\mu(T).$$

In case $0 \le k \le 2r+1$ and $t \ge 2(r+1)$, we add zero to the term first as

$$\langle T(b\chi_T)\chi_{S'}, \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle = \langle (Tb\chi_T - Tb\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'}(x_{T'}))\chi_{S'}, \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle$$

and get the bound

$$\Big(\sum_{k=0}^{2r+1}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch_{\mathcal{T}'}^k(S)\\ \pi_{\mathcal{T}}S'=T'}}\left\|(Tb\chi_T - Tb\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'}(x_{T'}))\chi_{S'}\right\|_2^2\Big)^{1/2}$$
$$\cdot\Big(\sum_{k=0}^{2r+1}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch_{\mathcal{T}'}^k(S)\\ \pi_{\mathcal{T}}S'=T'}}\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\varepsilon_{R,T}\triangle_R^b g_{gd}\Big\|_2^2\Big)^{1/2}.$$

Again using disjointness of children S', maximality of S,and finite number of k, we have that

$$\sum_{k=0}^{2r+1}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch_{\mathcal{T}'}^k(S)\\ \pi_{\mathcal{T}}S'=T'}}\left\|(Tb\chi_T - Tb\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'}(x_{T'}))\chi_{S'}\right\|_2^2$$
$$\lesssim \sum_{T'\in ch_{\mathcal{T}}^t(T)}\left\|(Tb\chi_T - Tb\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'}(x_{T'}))\chi_{T'}\right\|_2^2.$$

What we got on RHS is the term (6.1) in the previous subsection thus we just cite the result to here that

$$\sum_{k=0}^{2r+1}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch_{\mathcal{T}'}^k(S)\\ \pi_{\mathcal{T}}S'=T'}}\left\|(Tb\chi_T - Tb\chi_{T\setminus\pi_{\mathcal{T}}^{\lfloor t/2\rfloor}T'}(x_{T'}))\chi_{S'}\right\|_2^2 \lesssim 2^{-t}\mu(T).$$

The last case is that $k \ge 2(r+1)$, we adjust the summand as

$$\langle T(b\chi_T)\chi_{S'}, \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle = \langle (Tb\chi_T - Tb\chi_{T\setminus\pi_{\mathcal{T}'}^{\lfloor k/2\rfloor}S'}(x_{S'}))\chi_{S'}, \varepsilon_{R,T}\triangle_R^b g_{gd}\rangle$$

and again bound the sum by

$$\Big(\sum_{k \geq 2(r+1)} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \sum_{S \in \mathcal{R}(T')} \sum_{\substack{S' \in ch^k_{\mathcal{T}'}(S) \\ \pi_{\mathcal{T}} S' = T'}} \Big\| (Tb\chi_T - Tb\chi_{T \setminus \pi^{\lfloor k/2 \rfloor}_{\mathcal{T}'} S'}(x_{S'}))\chi_{S'} \Big\|_2^2 \Big)^{1/2}$$
$$\cdot \Big(\sum_{k \geq 2(r+1)} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \sum_{S \in \mathcal{R}(T')} \sum_{\substack{S' \in ch^k_{\mathcal{T}'}(S) \\ \pi_{\mathcal{T}} S' = T'}} \Big\| \sum_{\substack{R \in \mathcal{G}' \\ \pi_{\mathcal{T}'} R = S'}} \varepsilon_{R,T} \triangle^b_R g_{gd} \Big\|_2^2 \Big)^{1/2}.$$

We then consider that

$$\begin{aligned}
\sum_{\substack{S' \in ch^k_{\mathcal{T}'}(S) \\ \pi_{\mathcal{T}} S' = T'}} \Big\| (Tb\chi_T - Tb\chi_{T \setminus \pi^{\lfloor k/2 \rfloor}_{\mathcal{T}'} S'}(x_{S'}))\chi_{S'} \Big\|_2^2 &\leq \sum_{S' \in ch^k_{\mathcal{T}'}(S)} \Big\| (Tb\chi_T - Tb\chi_{T \setminus \pi^{\lfloor k/2 \rfloor}_{\mathcal{T}'} S'}(x_{S'}))\chi_{S'} \Big\|_2^2 \\
&= \sum_{S'' \in ch^{\lceil k/2 \rceil}_{\mathcal{T}'}(S)} \sum_{\substack{S' \in ch^k_{\mathcal{T}'}(S) \\ \pi^{\lfloor k/2 \rfloor}_{\mathcal{T}'} S' = S''}} \Big\| (Tb\chi_T - Tb\chi_{T \setminus S''}(x_{S'}))\chi_{S'} \Big\|_2^2 \\
&\lesssim 2^{-k} \mu(S)
\end{aligned}$$

where we follow the situation (6.1) again. Maximality of $S \in \mathcal{R}(T')$ leads to

$$\begin{aligned}
\sum_{k \geq 2(r+1)} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \sum_{S \in \mathcal{R}(T')} \sum_{\substack{S' \in ch^k_{\mathcal{T}'}(S) \\ \pi_{\mathcal{T}} S' = T'}} \Big\| (Tb\chi_T - Tb\chi_{T \setminus \pi^{\lfloor k/2 \rfloor}_{\mathcal{T}'} S'}(x_{S'}))\chi_{S'} \Big\|_2^2 \\
\lesssim \sum_{k \geq 2(r+1)} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \sum_{S \in \mathcal{R}(T')} 2^{-k} \mu(S) \\
\leq \sum_{k \geq 2(r+1)} 2^{-k} \cdot \sum_{T' \in ch^t_{\mathcal{T}}(T)} \mu(T').
\end{aligned}$$

Here we recall from Lemma 6.5 that

$$\sum_{T' \in ch^t_{\mathcal{T}}(T)} \mu(T') \leq 2^{-2t} \mu(T) < 2^{-t} \mu(T)$$

and hence

$$\sum_{k \geq 2(r+1)} \sum_{T' \in ch^t_{\mathcal{T}}(T)} \sum_{S \in \mathcal{R}(T')} \sum_{\substack{S' \in ch^k_{\mathcal{T}'}(S) \\ \pi_{\mathcal{T}} S' = T'}} \Big\| (Tb\chi_T - Tb\chi_{T \setminus \pi^{\lfloor k/2 \rfloor}_{\mathcal{T}'} S'}(x_{S'}))\chi_{S'} \Big\|_2^2 \lesssim 2^{-t} \mu(T).$$

We turn to the remaining terms related to martingale difference. In stead of looking into the three cases, we can consider

$$\sum_{k\geq 0}\sum_{T'\in ch^t_{\mathcal{T}}(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch^k_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}S'=T'}}\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\varepsilon_{R,T}\triangle^b_R g_{gd}\Big\|_2^2$$

which bounds the underlying terms in all cases. In fact, one can rewrite the summation over $k\geq 0, S\in\mathcal{R}(T')$, and $S'\in ch^k_{\mathcal{T}'}(S)$ as over $S'\in\mathcal{T}'$. As before, one can simplify the norm using the properties of weighted martingale difference for each $S'\in\mathcal{T}'$ as

$$\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\varepsilon_{R,T}\triangle^b_R g_{gd}\Big\|_2^2=\Big\|\sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S'}}\varepsilon_{R,T}\triangle^b_R\Big(\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\triangle^b_R g_{gd}\Big)\Big\|_2^2.$$

Then by Lemma 2.8 we can get the estimate

$$\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\varepsilon_{R,T}\triangle^b_R g_{gd}\Big\|_2^2\lesssim\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\triangle^b_R g_{gd}\Big\|_2^2=\Big\|\sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S'}}\triangle^b_R g_{gd}\Big\|_2^2$$

to which zero terms are added due to goodness of g_{gd} for the equation. To recap, we proved that

$$\begin{aligned}\sum_{k\geq 0}\sum_{T'\in ch^t_{\mathcal{T}}(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch^k_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}S'=T'}}\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\varepsilon_{R,T}\triangle^b_R g_{gd}\Big\|_2^2&=\sum_{T'\in ch^t_{\mathcal{T}}(T)}\sum_{\substack{S'\in\mathcal{T}'\\ \pi_{\mathcal{T}}S'=T'}}\Big\|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\varepsilon_{R,T}\triangle^b_R g_{gd}\Big\|_2^2\\ &\lesssim\sum_{T'\in ch^t_{\mathcal{T}}(T)}\sum_{\substack{S'\in\mathcal{T}'\\ \pi_{\mathcal{T}}S'=T'}}\Big\|\sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S'}}\triangle^b_R g_{gd}\Big\|_2^2.\end{aligned}$$

What follows is to prepare upper bounds for the latest summation. First, WLOG, we can replace S' by S and consider the sum in two parts as

$$\Big\|\sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S}}\triangle^b_R g_{gd}\Big\|_2^2=\Big\|\sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S}}\triangle^b_R g_{gd}\cdot\chi_{S\setminus F_S}\Big\|_2^2+\Big\|\sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S}}\triangle^b_R g_{gd}\cdot\chi_{F_S}\Big\|_2^2$$

where $F_S:=S\setminus\cup_{S'\in ch_{\mathcal{T}'}(S)}S'$ for each $S\in\mathcal{T}'$. The first term is treated in the following paragraph and the second term is in the next one.

First, observe as before that for any $x \in S'$,

$$\sum_{\substack{R \in \mathcal{D}' \\ \pi_{\mathcal{T}'} R = S}} \triangle_R^b g_{gd}(x) = \sum_{\substack{R \in \mathcal{D}' \\ S' \subset R \subseteq S}} \triangle_R^b g_{gd}(x) = E_{S'}^b g_{gd}(x) - E_S^b g_{gd}(x)$$

as a telescoping series. Thus it can be estimated as

$$\Big| \sum_{\substack{R \in \mathcal{D}' \\ \pi_{\mathcal{T}'} R = S}} \triangle_R^b g_{gd}(x) \Big| \leq \frac{\|b\|_\infty}{\delta} \Big(\langle |g_{gd}| \rangle_{S'} + \langle |g_{gd}| \rangle_S \Big) \leq \Big(1 + \frac{1}{4} \Big) \frac{\|b\|_\infty}{\delta} \langle |g_{gd}| \rangle_{S'}$$

where the last inequality holds by construction of $\mathcal{T}'$. In other words,

$$\Big| \sum_{\substack{R \in \mathcal{D}' \\ \pi_{\mathcal{T}'} R = S}} \triangle_R^b g_{gd} \cdot \chi_{S'} \Big| \lesssim \langle |g_{gd}| \rangle_{S'} \chi_{S'}$$

and hence by disjointness of children S',

$$\Big| \sum_{\substack{R \in \mathcal{D}' \\ \pi_{\mathcal{T}'} R = S}} \triangle_R^b g_{gd} \cdot \chi_{S \setminus F_S} \Big|^2 \lesssim \sum_{S' \in ch_{\mathcal{T}'}(S)} \langle |g_{gd}| \rangle_{S'}^2 \chi_{S'}.$$

Therefore,

$$\Big\| \sum_{\substack{R \in \mathcal{D}' \\ \pi_{\mathcal{T}'} R = S}} \triangle_R^b g_{gd} \cdot \chi_{S \setminus F_S} \Big\|_2^2 \lesssim \sum_{S' \in ch_{\mathcal{T}'}(S)} \langle |g_{gd}| \rangle_{S'}^2 \mu(S').$$

For the other term, consider points x such that $\lim\limits_{k \to \infty} E_k g_{gd}(x) = g_{gd}(x)$. Similarly to consideration in (6.2), we can have that

$$\begin{aligned}
\Big| \sum_{\substack{R \in \mathcal{D}' \\ \pi_{\mathcal{T}'} R = S}} \triangle_R^b g_{gd} \cdot \chi_{F_S}(x) \Big| &= \Big| \lim_{l(R) \to 0} E_R^b g_{gd}(x) - E_S^b g_{gd}(x) \Big| \cdot \chi_{F_S}(x) \\
&\leq \frac{\|b\|_\infty}{\delta} \left(\lim_{l(R) \to 0} \langle |g_{gd}| \rangle_R + \langle |g_{gd}| \rangle_S \right) \cdot \chi_{F_S}(x) \\
&\lesssim (|g_{gd}(x)| + \langle |g_{gd}| \rangle_S) \cdot \chi_{F_S}(x).
\end{aligned}$$

Moreover, we can proceed further and obtain the estimate as in (6.3) as

$$\Big|\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S\\ \pi_{\mathcal{T}}R=T'}} \triangle_R^b g_{gd}\Big|\chi_{F_S} \lesssim \langle|g_{gd}|\rangle_S\chi_{F_S}.$$

Together with that $F_S\subseteq S$, we can bound the underlying term as

$$\Big\|\sum_{\substack{R\in\mathcal{D}'\\ \pi_{\mathcal{T}'}R=S}} \triangle_R^b g_{gd}\cdot\chi_{F_S}\Big\|_2^2 \lesssim \langle|g_{gd}|\rangle_S^2\mu(F_S)\le\langle|g_{gd}|\rangle_S^2\mu(S).$$

At this point, we have treated all important pieces so we finalize things here. First,

$$\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T\sum_{t,k\ge0}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch_{\mathcal{T}'}^k(S)\\ \pi_{\mathcal{T}}S'=T'}}\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\langle T(b\chi_T)\chi_{S'},\varepsilon_{R,T}\triangle_R^b g_{gd}\rangle\Big|\\
&\lesssim\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T\sum_{t\ge0}\Big(2^{-t}\mu(T)\Big)^{1/2}\Big(\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{\substack{S\in\mathcal{T}'\\ \pi_{\mathcal{T}}S=T'}}\Big(\sum_{S'\in ch_{\mathcal{T}'}(S)}\langle|g_{gd}|\rangle_{S'}^2\mu(S')+\langle|g_{gd}|\rangle_S^2\mu(S)\Big)\Big)^{1/2}.
\end{aligned}$$

By switching order of the sums over T and t and applying Cauchy-Schwartz inequality,

$$\begin{aligned}
&\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T\sum_{t\ge0}\Big(2^{-t}\mu(T)\Big)^{1/2}\Big(\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{\substack{S\in\mathcal{T}'\\ \pi_{\mathcal{T}}S=T'}}\Big(\sum_{S'\in ch_{\mathcal{T}'}(S)}\langle|g_{gd}|\rangle_{S'}^2\mu(S')+\langle|g_{gd}|\rangle_S^2\mu(S)\Big)\Big)^{1/2}\\
&\le\sum_{t\ge0}2^{-t/2}\Big(\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T^2\mu(T)\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{\substack{S\in\mathcal{T}'\\ \pi_{\mathcal{T}}S=T'}}\Big(\sum_{S'\in ch_{\mathcal{T}'}(S)}\langle|g_{gd}|\rangle_{S'}^2\mu(S')+\langle|g_{gd}|\rangle_S^2\mu(S)\Big)\Big)^{1/2}.
\end{aligned}$$

We then observe that for each t, the sums over $T\in\mathcal{T}, T'\in ch_{\mathcal{T}}^t(T)$, and $S\in\mathcal{T}'$ such that $\pi_{\mathcal{T}'}S=T'$ are included in the sum over $S\in\mathcal{T}'$. Hence

$$\begin{aligned}
&\sum_{t\ge0}2^{-t/2}\Big(\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T^2\mu(T)\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}}\sum_{T'\in ch_{\mathcal{T}}^t(T)}\sum_{\substack{S\in\mathcal{T}'\\ \pi_{\mathcal{T}}S=T'}}\Big(\sum_{S'\in ch_{\mathcal{T}'}(S)}\langle|g_{gd}|\rangle_{S'}^2\mu(S')+\langle|g_{gd}|\rangle_S^2\mu(S)\Big)\Big)^{1/2}\\
&\qquad\le\sum_{t\ge0}2^{-t/2}\Big(\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T^2\mu(T)\Big)^{1/2}\Big(\sum_{S\in\mathcal{T}'}\Big(\sum_{S'\in ch_{\mathcal{T}'}(S)}\langle|g_{gd}|\rangle_{S'}^2\mu(S')+\langle|g_{gd}|\rangle_S^2\mu(S)\Big)\Big)^{1/2}\\
&\qquad\lesssim\sum_{t\ge0}2^{-t/2}\Big(\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T^2\mu(T)\Big)^{1/2}\Big(\sum_{S\in\mathcal{T}'}\langle|g_{gd}|\rangle_S^2\mu(S)\Big)^{1/2}.
\end{aligned}$$

Lemma 6.6 then bound the term by $\sum_{t\geq 0} 2^{-t/2}\|f\|_2\|g\|_2$. At last, as a convergent geometric series, we conclude that

$$\Big|\sum_{T\in\mathcal{T}}\langle|f_{gd}|\rangle_T \sum_{t,k\geq 0}\sum_{T'\in ch^t_{\mathcal{T}}(T)}\sum_{S\in\mathcal{R}(T')}\sum_{\substack{S'\in ch^k_{\mathcal{T}'}(S)\\ \pi_{\mathcal{T}}S'=T'}}\sum_{\substack{R\in\mathcal{G}'\\ \pi_{\mathcal{T}'}R=S'}}\langle T(b\chi_T)\chi_{S'},\varepsilon_{R,T}\triangle^b_R g_{gd}\rangle\Big| \lesssim \|f\|_2\|g\|_2$$

finishing the proof of part I.

6.3.4 II The sum of $\langle\triangle^c_Q f_{gd}\rangle_{Q'_R}\langle T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R}),\triangle^b_R g_{gd}\rangle$

Here we look at the sum over $R\in\mathcal{G}', Q\in\mathcal{G}$ such that $R\subset Q$ and $2^r l(R)<l(Q)$. We can specify the sum more considering $l(Q)=2^t l(R)$ and summing t from $r+1$ to ∞. For simplicity, we say $Q\in\mathbf{I}_t$ for such conditions. Also we will not use the goodness of f,g here so we just say f,g. Hence consider, for each $t\geq r+1$,

$$\begin{aligned}&\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\langle\triangle^c_Q f\rangle_{Q'_R}\langle T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R}),\triangle^b_R g\rangle|\\ &\qquad=\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\langle\triangle^c_Q f\rangle_{Q'_R}\langle T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R})-T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R})(x_R),\triangle^b_R g\rangle|\end{aligned}$$

due to the zero mean of $\triangle^b_R g=0$. Since $dist(R,\partial Q)>l(R)$, for $x\in R$,

$$|T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R})(x)-T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R})(x_R)|\leq\int_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R}\frac{|x-x_R|^\alpha}{|x-y|^{d+\alpha}}|b(y)|d\mu(y).$$

Bounding the integration and applying Comparison Lemma to get that

$$\begin{aligned}|T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R})(x)-T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R})(x_R)| &\leq\|b\|_\infty l(R)^\alpha\int_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R}\frac{d\mu(y)}{|x-y|^{d+\alpha}}\\ &\leq(\frac{d}{\alpha}+1)\|b\|_\infty\frac{l(R)^\alpha}{dist(R,\partial Q)^\alpha}\\ &\leq(\frac{d}{\alpha}+1)\|b\|_\infty\frac{l(R)^\alpha}{l(R)^{\alpha\gamma}l(Q)^{\alpha-\alpha\gamma}}\\ &\leq(\frac{d}{\alpha}+1)\|b\|_\infty 2^{-t\alpha(1-\gamma)}\end{aligned}$$

using goodness of R and the size of R in terms of Q. Therefore,

$$\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\langle\triangle_Q^c f\rangle_{Q'_R}\langle T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R}),\triangle_R^b g\rangle|\lesssim 2^{-t\alpha(1-\gamma)}\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\langle\triangle_Q^c f\rangle_{Q'_R}|\int|\triangle_R^b g|d\mu(x).$$

Recall that $\triangle_Q^c f$ is constants on its children thus $\langle\triangle_Q^c f\rangle_{Q'_R}=\triangle_Q^c f(x)$ for all $x\in R$. Then we have

$$\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\langle\triangle_Q^c f\rangle_{Q'_R}\langle T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R}),\triangle_R^b g\rangle|\lesssim 2^{-t\alpha(1-\gamma)}\int\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\triangle_Q^c f(x)||\triangle_R^b g(x)|d\mu(x).$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}&\int\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\triangle_Q^c f(x)||\triangle_R^b g(x)|d\mu(x)\\&\qquad\leq\int\left(\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}\chi_R(x)|\triangle_Q^c f(x)|^2\right)^{1/2}\left(\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\triangle_R^b g(x)|^2\right)^{1/2}d\mu(x).\end{aligned}$$

Now observe that R of the same size are disjoint and are covered by the same Q or Q's of the same size leading to $\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}\chi_R(x)|\triangle_Q^c f(x)|^2=\sum_{Q\in\mathbf{I}}|\triangle_Q^c f(x)|^2$. For each R, also, there is only one $Q\in\mathbf{I}_t$ so that $\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\triangle_R^b g(x)|^2=\sum_{R\in\mathcal{G}'}|\triangle_R^b g(x)|^2$. Applying Cauchy-Schwarz inequality again, we get

$$\begin{aligned}\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}|\langle\triangle_Q^c f\rangle_{Q'_R}\langle T(b\chi_{\pi_{\mathcal{T}}Q'_R\backslash Q'_R}),\triangle_R^b g\rangle|&\lesssim 2^{-t\alpha(1-\gamma)}\Big\|\Big(\sum_{Q\in\mathbf{I}}|\triangle_Q^c f|^2\Big)^{1/2}\Big\|_2\Big\|\Big(\sum_{R\in\mathcal{G}'}|\triangle_R^b g|^2\Big)^{1/2}\Big\|_2\\&\lesssim 2^{-t\alpha(1-\gamma)}\end{aligned}$$

by recalling inequality (2.1) and Lemma 2.6 in the last step. Lastly, we sum the last inequality in t from $r+1$ to ∞ proving boundedness of the desired term.

6.3.5 III The sum of $\langle T(\triangle_Q^b f_{gd}\cdot\chi_{Q\backslash Q'_R}),\triangle_R^b g_{gd}\rangle$

For this term, let us note first that $Q\backslash Q'_R=Q'$ in $\mathbf{R}$. For higher dimension, we can just add what we will consider according to the number of children of Q except Q'_R. Again, we replace $f_{gd},\ g_{gd}$

back to $f,\ g$. Now we can start as in the previous one. For each $t \geq r+1$, we rewrite the term as

$$\begin{aligned}\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}\langle T(\triangle_Q^b f\cdot\chi_{Q'}),\triangle_R^b g\rangle &= \sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}\langle\triangle_Q^c f\rangle_{Q'}\langle T(b\chi_{Q'}),\triangle_R^b g\rangle\\ &= \sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}\langle\triangle_Q^c f\rangle_{Q'}\langle T(b\chi_{Q'})-T(b\chi_{Q'})(x_R),\triangle_R^b g\rangle\\ &= \int\sum_{R\in\mathcal{G}'}\triangle_R^b g(x)\sum_{Q\in\mathbf{I}_t}\langle\triangle_Q^c f\rangle_{Q'}(T(b\chi_{Q'})(x)-T(b\chi_{Q'})(x_R))d\mu(x).\end{aligned}$$

Applying Cauchy-Schwartz inequality twice to see that

$$\begin{aligned}&\Big|\sum_{R\in\mathcal{G}'}\sum_{Q\in\mathbf{I}_t}\langle T(\triangle_Q^b f\cdot\chi_{Q'}),\triangle_R^b g\rangle\Big|\\ &\quad=\Big|\int\sum_{R\in\mathcal{G}'}\triangle_R^b g(x)\sum_{Q\in\mathbf{I}_t}\langle\triangle_Q^c f\rangle_{Q'}\left(T(b\chi_{Q'})(x)-T(b\chi_{Q'})(x_R)\right)d\mu(x)\Big|\\ &\quad\leq\int\Big|\Big(\sum_{R\in\mathcal{G}'}|\triangle_R^b g(x)|^2\Big)^{1/2}\Big(\sum_{R\in\mathcal{G}'}\Big|\chi_R(x)\sum_{Q\in\mathbf{I}_t}\langle\triangle_Q^c f\rangle_{Q'}\left(T(b\chi_{Q'})(x)-T(b\chi_{Q'})(x_R)\right)\Big|^2\Big)^{1/2}\Big|d\mu(x)\\ &\quad\leq\Big\|\Big(\sum_{R\in\mathcal{G}'}|\triangle_R^b g|^2\Big)^{1/2}\Big\|_2\Big\|\Big(\sum_{R\in\mathcal{G}'}\Big|\chi_R\sum_{Q\in\mathbf{I}_t}\langle\triangle_Q^c f\rangle_{Q'}\left(T(b\chi_{Q'})-T(b\chi_{Q'})(x_R)\right)\Big|^2\Big)^{1/2}\Big\|_2\\ &\quad\lesssim\|g\|_2\Big\|\Big(\sum_{R\in\mathcal{G}'}\Big|\chi_R\sum_{Q\in\mathbf{I}_t}\langle\triangle_Q^c f\rangle_{Q'}\left(T(b\chi_{Q'})-T(b\chi_{Q'})(x_R)\right)\Big|^2\Big)^{1/2}\Big\|_2\end{aligned}$$

where Lemma 2.6 is used in the last bound. Now we look into each layer of R's for the remaining term and consider it as

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}\Big|\sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^k}}\chi_R\sum_{\substack{Q\supset R\\ l(Q)=2^{k+t}}}\langle\triangle_Q^c f\rangle_{Q'}\left(T(b\chi_{Q'})-T(b\chi_{Q'})(x_R)\right)\Big|^2\Big)^{1/2}\Big\|_2.$$

We can see that $2^r l(R)\leq l(Q')$ and thus $dist(R,\partial Q')>l(R)$ due to goodness of R. Hence we can consider as in Lemma 6.3 in the case that $dist(Q',R)<l(Q')$ to obtain

$$\begin{aligned}\Big|\sum_{\substack{R\subset Q\\ l(R)=2^k}}\chi_R(x)\langle\triangle_Q^c f\rangle_{Q'}\left(T(b\chi_{Q'})(x)-T(b\chi_{Q'})(x_R)\right)\Big| &\lesssim 2^{-t\alpha/4}\Big|\sum_{\substack{R\subset Q\\ l(R)=2^k}}\chi_R(x)\mu(Q)^{-1}\int_{Q'}\triangle_Q^c f\,d\mu\Big|\\ &\leq 2^{-t\alpha/4}\chi_Q(x)\mu(Q)^{-1}\int_Q|\triangle_Q^c f|d\mu.\end{aligned}$$

Then summing over Q we get

$$\sum_{\substack{Q\in\mathcal{G}\\ l(Q)=2^{k+t}}} \Big| \sum_{\substack{R\subset Q\\ l(R)=2^k}} \chi_R(x)\langle \triangle_Q^c f\rangle_{Q'} \left(T(b\chi_{Q'})(x) - T(b\chi_{Q'})(x_R)\right) \Big| \lesssim 2^{-t\alpha/4} \sum_{\substack{Q\in\mathcal{G}\\ l(Q)=2^{k+t}}} \langle |\triangle_Q^c f|\rangle_Q \chi_Q$$
$$\leq 2^{-t\alpha/4} E_{k+t}|\triangle_{k+t}^c f|$$

Therefore,

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}\Big| \sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^k}} \chi_R \sum_{\substack{Q\supset R\\ l(Q)=2^{k+t}}} \langle \triangle_Q^c f\rangle_{Q'} \left(T(b\chi_{Q'}) - T(b\chi_{Q'})(x_R)\right) \Big|^2\Big)^{1/2}\Big\|_2$$
$$\lesssim 2^{-t\alpha/4} \Big\|\Big(\sum_{k\in\mathbf{Z}} \left(E_{k+t}|\triangle_{k+t}^c f|\right)^2\Big)^{1/2}\Big\|_2.$$

From inequalities (2.2) and (2.1), we finally have

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}\Big| \sum_{\substack{R\in\mathcal{G}'\\ l(R)=2^k}} \chi_R \sum_{\substack{Q\supset R\\ l(Q)=2^{k+t}}} \langle \triangle_Q^c f\rangle_{Q'} \left(T(b\chi_{Q'}) - T(b\chi_{Q'})(x_R)\right) \Big|^2\Big)^{1/2}\Big\|_2$$
$$\lesssim 2^{-t\alpha/4} \Big\|\Big(\sum_{k\in\mathbf{Z}} \left(|\triangle_{k+t}^c f|\right)^2\Big)^{1/2}\Big\|_2$$
$$\lesssim 2^{-t\alpha/4}\|f\|_2.$$

Again, the proof is finished by summing t from $r+1$ to ∞ to obtain $\|f\|_2$.

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