STABILIZING SPECTRAL FUNCTORS OF EXACT CATEGORIES

BY

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DISSERTATION

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ABSTRACT

We define and study the K-theory of exact categories with coefficients in endofunctors of spectra in analogy with Mitchell's homology of categories. Generalizing computations of McCarthy, we determine, for a discrete ring R, the K-theory of the exact category of finitely-generated projective R-modules with coefficients in the n-fold smash product functor. This computation allows us to analyze the effects of applying this functorial construction to the Goodwillie Taylor tower of a homotopy endofunctor of spectra. In the case of $\Sigma^\infty \Omega^\infty$, the associated tower recovers the Taylor tower of relative K-theory as computed by Lindenstrauss and McCarthy.

A María, Torne y mi Papá, por enseñarme a siempre seguir luchando.

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CHAPTER 1

INTRODUCTION

Whether algebraic K-theory [19] properly belongs to the field of algebraic topology, algebraic geometry, or geometric topology is a matter of debate within some mathematical circles. The applications in seemingly disparate fields are numerous. Since its creation, algebraic K-theory's input has continuously changed, from algebro-geometric objects like rings and schemes to their categorical counterparts, exact categories, Waldhausen categories, and even recently ∞-categories. This rising tide of "homotopical richness" has also been reflected in K-theory's output, from abelian groups to spaces and spectra (see [60] and [18] for historical and modern perspectives, respectively). In any case, there is some consensus: this "supped up" version of linear algebra is *very* difficult to compute; even knowledge of $K(\mathbb{Z})$ would be welcome news. New approaches to tackling these invariants are thus highly sought after.

One approach that has proven fruitful is the use of trace methods [40]. Mapping out of algebraic K-theory into more computable invariants has been systematically studied over the last six decades with ever greater information being encoded by and in the map and target, respectively. By working with homotopical invariants like Hochschild homology (HH) and cyclic homology (HC) [36], and later their topological analogues, topological Hochschild homology (THH) and topological cyclic homology (TC) [9], the trace approach culminated in the proof of Goodwillie's conjecture (Conjecture 9 [24]) by Dundas and McCarthy ([13] and [45], respectively):

Theorem (Theorem 7.2.2.1 [14]) *Let $f : B \to A$ be a map of connective ring spectra inducing a surjection $\pi_0(B) \to \pi_0(A)$ with nilpotent kernel. Then the square induced by the naturality of the cyclotomic trace*

$$
\begin{array}{ccc}
K(B) & \longrightarrow & TC(B) \\
\downarrow & & \downarrow \\
K(A) & \longrightarrow & TC(A)
\end{array}
$$

is homotopy cartesian.

This result has been the backbone of subsequent computations, notably the work of Hesselholt and Madsen [27]. An intermediate step in their computations involved the invariant $TR(R)$ (Section 4.1 [27]) defined from the pieces making up TC, and which recovered on π_0 the (p-typical) Witt vectors [11] of a ring. This ring was known to be related to the reduced Grothendieck group of endomorphisms by work of Almkvist [3]. Motivated by these results, Lindenstrauss and McCarthy [34] generalized the construction to include an R-bimodule variable M. Though it broke the cyclic symmetry of $TR(R)$, this new invariant $W(R; M)$, defined by a tower of "truncated" Witt vectors, still behaved like in the classical setting. Lindenstrauss and McCarthy showed that their *topological* Witt vector tower recovered, for connected bimodules, the Goodwillie Taylor tower of the (suitably defined) reduced K-theory of endomorphisms (Theorem 9.2 [34]):

$$\tilde{K}(R; M) \simeq W(R; M)$$

thereby lifting Almkvist's result to higher K-theory. More importantly, the equivalences of Goodwillie n-excisive approximations $P_n(K(R; -))(M) \simeq W_n(R; M)$ presented a computationally feasible way to analyze $K(R; M)$. Indeed, their building blocks, denoted $U^n(R; M)$, are very tractable, with several computations already in the literature (see [33]). By working with the $W_n(R; M)$, Lindenstrauss and McCarthy were able to successfully analyze the K-theory of formal power series [35].

Another approach that has garnered significant attention recently is the study of algebraic K-theory's universal properties. Though undoubtedly of a more categorical flavor, these characterizations of algebraic K-theory as encoding universal higher additive invariants have shed light on its structural properties. For example, work of McCarthy [44] in the exact category setting and Blumberg, Gepner and Tabuada [7] in the ∞-categorical one makes it clear why K-theory is best studied by mapping *out* of it, rather than to it. Specifically, they show that algebraic K-theory is the universally initial functor satisfying Waldhausen additivity. Furthermore, in the case of [44], an additivization procedure is described, providing a concrete model for the universal additive approximation of any functor.

The aim of this thesis is to try and understand better the relationship between these two approaches. We do so by mixing the models of K-theory used in the former (developed by Dundas and McCarthy in [17]), and the stabilization procedures of the [44]. We define a K-theory of exact categories with coefficients in endofunctors of spectra (4.1.2) and study

the traditional case of finitely generated projective R-modules. We use Goodwillie's calculus of homotopy functors ([22],[25],[26]) to approximate our functors by means of a Taylor tower, and then apply this new K-theory construction to it. Our main result is that when applied to the endofunctor $\Sigma^\infty\Omega^\infty$, this procedure recovers the Lindenstrauss-McCarthy tower relating $\tilde{K}(R;M)$ and $W(R;M)$:

Theorem 4.2.5 *Let R be an associative and unital ring, M an R-bimodule, and $n \geq 1$. There is an equivalence of $\mathbb{S}$-bimodules*

$$K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty\Omega^\infty)) \simeq W_n(R;M)$$

compatible with restrictions from n to $n-1$. Therefore the tower associated to the K-theory with coefficients in the endofunctor $\Sigma^\infty\Omega^\infty$ is weakly equivalent to the Lindenstrauss-McCarthy Taylor tower of relative K-theory:

$$
\begin{array}{ccccccccc}
& & & & \tilde{K}(R;M) & & & & \\
& & & \swarrow \swarrow & \downarrow & \searrow \searrow & & & \\
\cdots \longleftarrow & W_{n+1}(R;M) & \longrightarrow & & W_n(R;M) & & \longrightarrow & W_{n-1}(R;M) & \longrightarrow \cdots \\
& \uparrow & & & \uparrow & & & \uparrow & \\
& U^{n+1}(R;M)_{hC_{n+1}} & & & U^n(R;M)_{hC_n} & & & U^{n-1}(R;M)_{hC_{n-1}} &
\end{array}
$$

The crux of the proof is a detailed analysis of the case of n-homogeneous functors. In the case of chain complexes, this had already been done in [46], with an element-based proof that does not generalize to the spectral setting. We re-derive the result with one that does:

Theorem 4.2.2 *Let R be an associative and unital ring, M an R-bimodule, and $n \geq 1$. There is a feeble equivalence of Σ_n-bimodules*

$$K(\mathcal{P}; \mathfrak{D}(M); (-)^{\wedge n}) \simeq_{f\Sigma_n} U^n(R;M) \wedge_{\mathbb{C}_n} \Sigma_n$$

This thesis is organized as follows. In Chapter 2 we give a new proof of the main result in [46]. In Chapter 3 we show that this proof generalizes well to the spectral setting (Theorem 3.3.7). In Chapter 4 we introduce K-theory with coefficients in endofunctors, prove the structural results that arise from Goodwillie calculus, and use the result from the previous chapters to recover the tower from [34]. In the appendices we give the necessary background used throughout the chapters.

CHAPTER 2

THE ALGEBRAIC CASE

In this chapter we recall the main results of [46], giving an alternate proof to the main result there (Theorem 3.4), which generalizes to the spectral setting.

2.1 Exact Categories, Additivity and $S_\bullet$

Let $\mathcal{E}$ be an exact category (§2.1 [53]) considered as a category with cofibrations (1.1 [59]). Let $S_\bullet\mathcal{E}$ be Waldhausen's $S_\bullet$-construction (1.3 [59]). Let **Ext** be the category of (small) exact categories, $s\mathbf{Ab}$ the category of simplicial abelian groups, and $s\mathbf{Mod}$-R the category of simplicial (right) R-modules for R a discrete ring. We give $s\mathbf{Mod}$-R the projective model structure (as described in 2.3 [29]) along the forgetful functor to simplicial sets (see Appendix A).

Notation If X is an n-multi-simplicial set, abelian group, or R-module we will denote by $\mathfrak{d}X$ its diagonal.

Let $F : \mathbf{Ext} \to s\mathbf{Mod}$-$R$ be a reduced functor (see Appendix C.3). Suppose $\mathcal{E}_*$ is a simplicial exact category, that is, a simplicial object in **Ext** (so face and degeneracy maps are exact functors). Then $F(\mathcal{E}_*)_\bullet$ is a bisimplicial R-module, which we can consider as an R-module by taking diagonals $\mathfrak{d}(F(\mathcal{E}_*)_\bullet)$.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two exact categories. There is a natural exact category $\mathcal{E} \times \mathcal{E}'$ corresponding to the product in **Ext**, with natural projections (exact functors) $\rho_\mathcal{E}$ and $\rho_{\mathcal{E}'}$. This induces a surjective map of simplicial R-modules $F(\mathcal{E} \times \mathcal{E}') \xrightarrow{\left(F(\rho_\mathcal{E}), F(\rho_{\mathcal{E}'})\right)} F(\mathcal{E}) \times F(\mathcal{E}')$ (in particular, it's a fibration). We say a functor F is *product preserving* if $F(\rho)$ (for $\rho := (\rho_\mathcal{E}, \rho_{\mathcal{E}'})$) is a weak equivalence. We say F is a *p-product* functor if $F(\rho)$ is $(p + 1)$-connected (for connectivity conventions see 2.6 [49], specially Remark 2.6.8).

Definition 2.1.1 Using the notation of the previous paragraph, the *second cross effect* of F, denoted cr_2F, is a bi-functor $\mathbf{Ext} \times \mathbf{Ext} \to s\mathbf{Mod}\text{-}R$, given by:

$$cr_2F(\mathcal{E}, \mathcal{E}') := \mathrm{hofib}(F(\rho))$$

$$= \mathrm{hofib}\big(F(\mathcal{E} \times \mathcal{E}') \xrightarrow{\big(F(\rho_\mathcal{E}), F(\rho_{\mathcal{E}'})\big)} F(\mathcal{E}) \times F(\mathcal{E}')\big)$$

(since simplicial groups are Kan complexes we use the uncorrected homotopy fiber; see Appendix B.3 for details).

Observation cr_2F is a symmetric bi-functor reduced in each variable. F is a p-product functor if and only if cr_2F takes values in p-connected simplicial R-modules.

Let $\mathcal{E}$ be any exact category. Consider the exact category of short exact sequences in $\mathcal{E}$, $S_2(\mathcal{E})$, and the exact functor $(d_2, d_0) : S_2(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ sending

$$(A \rightarrowtail B \twoheadrightarrow C) \mapsto (A, C)$$

Waldhausen's additivity theorem (Theorem 1.4.2 [59]) says that, upon applying the $S_\bullet$-construction, we get a homotopy equivalence (of the classifying spaces):

$$S_\bullet\big(S_2(\mathcal{E})\big) \xrightarrow{\simeq} S_\bullet(\mathcal{E} \times \mathcal{E})$$

Since $S_\bullet$ preserves products, $S_\bullet(\mathcal{E} \times \mathcal{E}) \xrightarrow{\cong} S_\bullet\mathcal{E} \times S_\bullet\mathcal{E}$ as simplicial exact categories. Any functor F will preserve simplicial homotopy equivalences, so we get weak equivalences of bisimplicial R-modules:

$$F\big(S_\bullet\big(S_2(\mathcal{E})\big)\big) \xrightarrow{\simeq} F\big(S_\bullet(\mathcal{E} \times \mathcal{E})\big) \xrightarrow{\cong} F\big(S_\bullet\mathcal{E} \times S_\bullet\mathcal{E}\big)$$

Combining with the maps

$$F\big(S_\bullet\mathcal{E} \times S_\bullet\mathcal{E}\big) \xrightarrow{\big(F(\rho_{S_\bullet\mathcal{E}}), F(\rho_{S_\bullet\mathcal{E}})\big)} F(S_\bullet\mathcal{E}) \times F(S_\bullet\mathcal{E})$$

we get maps of bisimplicial R-modules, denoted the same by abuse of notation:

$$F\big(S_\bullet\big(S_2(\mathcal{E})\big)\big) \xrightarrow{(d_2, d_0)} F(S_\bullet\mathcal{E}) \times F(S_\bullet\mathcal{E})$$

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Observation If F is a p-product functor then by a spectral sequence argument the first map is $(p+1)$-connected, so that the second is also $(p+1)$-connected.

Definition 2.1.2 We say a functor F is *additive* if the map $F(S_2(-)) \xrightarrow{(d_2,d_0)} F(-) \times F(-)$ is a weak equivalence. We say F is *p-additive* if it is $(p+1)$-connected.

Observation If F is a product preserving functor, then by the previous remark $\mathfrak{d}F(S_\bullet(-))$ is an additive functor. More generally, if F is a p-product functor, then $\mathfrak{d}F(S_\bullet(-))$ is a p-additive functor. In general, a functor need not be additive.

Lemma 2.1.3 *Let $n \geq 1$. Then $\mathfrak{d}F(S_\bullet^{(n)}-) : \mathbf{Ext} \to s\mathbf{Mod}\text{-}R$ is a reduced $(2n-1)$-product functor (and also $(2n-1)$-additive). Furthermore, if F is a p-product functor, then $\mathfrak{d}F(S_\bullet^{(n)}-) : \mathbf{Ext} \to s\mathbf{Mod}\text{-}R$ is a $(p+2n-1)$-product functor.*

Proof. This is an application of the Eilenberg-Zilber theorem. Here it is crucial that F is reduced (see Lemma 1.3 [44]). $\qquad\square$

Notation Let $\Omega_R := \Omega : s\mathbf{Mod}\text{-}R \to s\mathbf{Mod}\text{-}R$ be the model for the simplicial loop space induced from $\underline{s\mathbf{Sets}_*}(S^1, -) : \underline{s\mathbf{Sets}_*} \to \underline{s\mathbf{Sets}_*}$ given the point-wise simplicial module structure (see Appendix A.3). Note that if F is an additive functor, then so is ΩF.

Definition 2.1.4 Given a reduced functor $F : \mathbf{Ext} \to s\mathbf{Mod}\text{-}R$, define a new functor $F^{\mathrm{st}} : \mathbf{Ext} \to s\mathbf{Mod}\text{-}R$ given by

$$F^{\mathrm{st}}(\mathcal{E}) := \operatorname*{hocolim}_{n} \Omega^{(n)}\big(\mathfrak{d}F\big(S_\bullet^{(n)}(\mathcal{E})\big)\big)$$

(since F need not take cofibrant values, we use the corrected homotopy colimit; see Appendix B.2 for details).

Remark By Lemma 2.1.3, F^{st} is a reduced, product preserving, additive functor. If we let $\alpha : F \to F^{\mathrm{st}}$ be the natural transformation induced from the homotopy colimit system, then one can show F is an additive functor if and only if α is an equivalence for all exact categories. Furthermore, combining the previous two observations, we see that if F is a product preserving functor, then $\Omega(\mathfrak{d}F(S_\bullet(-)))$ is an additive functor. Therefore, $\alpha : \Omega\mathfrak{d}FS_\bullet \to (\Omega\mathfrak{d}FS_\bullet)^{\mathrm{st}}$ is an equivalence. On the other hand, for *any* functor F, $F^{\mathrm{st}} \to (\Omega\mathfrak{d}FS_\bullet)^{\mathrm{st}}$ is an equivalence by homotopy cofinality. So we conclude that for a product preserving functor F, $F^{\mathrm{st}} \simeq \Omega\mathfrak{d}FS_\bullet$.

Example 2.1.5 Consider the functor $\mathbf{R} : \mathbf{Ext} \to s\mathbf{Mod}\text{-}R$ which takes an exact category $\mathcal{E}$ to the reduced free R-module generated by the set of objects of $\mathcal{E}$ (thought of as a constant simplicial module). That is,

$$\mathbf{R}(\mathcal{E}) := \operatorname{coker}\big(R[0] \to R[\operatorname{Obj}(\mathcal{E})]\big)$$

Then $\mathbf{R}^{\mathrm{st}}$ is the stable homology functor with coefficients in R:

$$\pi_*(\mathbf{R}^{\mathrm{st}}(\mathcal{E})) \cong H_*(\mathbf{K}(\mathcal{E}); R)$$

where $\mathbf{K}(\mathcal{E})$ is the algebraic K-theory spectrum of $\mathcal{E}$.

Example 2.1.6 Consider the functor $\mathbf{Hom} : \mathbf{Ext} \to s\mathbf{Ab}$ which takes an exact category $\mathcal{E}$ to the constant simplicial group of endomorphisms of objects in $\mathcal{E}$. That is,

$$\mathbf{Hom}(\mathcal{E}) := \bigoplus_{E \in \operatorname{Obj}(\mathcal{E})} Hom_{\mathcal{E}}(E, E)$$

This functor is utilizing the "linear" structure of the category, i.e. the fact that every exact category is Ab-enriched. The Dundas-McCarthy theorem (Section 2 [15] and Erratum [16]) then states that stabilization of $\mathbf{Hom}$ is the (simplicial abelian group version of the) topological Hochschild homology of the category:

$$\mathbf{Hom}^{\mathrm{st}}(\mathcal{E}) := \operatorname*{hocolim}_n \Omega^{(n)}\big(\mathfrak{d}\mathbf{Hom}\big(S_{\bullet}^{(n)}(\mathcal{E})\big)\big)$$
$$\simeq THH(\mathcal{E})$$

Notation Let $\mathbf{E}$ be a subcategory of $\mathbf{Ext}$ which contains 0 (the trivial exact category), is closed under isomorphisms, and is closed under the $S_\bullet$-construction. That is, if $\mathcal{E} \in \operatorname{Obj}(\mathbf{E})$, then $S_\bullet(\mathcal{E})$ is a simplicial object in $\mathbf{E}$. Given these constraints, the stabilization functor,

$$(-)^{\mathrm{st}} : \operatorname{Func}_*\big(\mathbf{Ext}, s\mathbf{Mod}\text{-}R\big) \to \operatorname{Func}_*\big(\mathbf{Ext}, s\mathbf{Mod}\text{-}R\big)$$

gives a well-defined endofunctor on the subcategory of reduced functors $\operatorname{Func}_*\big(\mathbf{E}, s\mathbf{Mod}\text{-}R\big)$.

Given an exact category $\mathcal{E}$ we can form such a subcategory $\mathcal{S}_{\mathcal{E}}$ of $\mathbf{Ext}$ by taking the smallest subcategory containing $\mathcal{E}$ closed under isomorphisms and $S_\bullet$. Every object in this (skeletally small) subcategory is equivalent to $S_{[n_1]}S_{[n_2]}\ldots S_{[n_t]}\mathcal{E}$ for some finite sequence. Suppose

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$T : \mathcal{E} \to \mathcal{E}'$ is an exact functor. Then we get an induced functor on the subcategories of **Ext**, $\mathcal{S}_T : \mathcal{S}_\mathcal{E} \to \mathcal{S}_{\mathcal{E}'}$, given by:

$$S_{[n_1]}S_{[n_2]} \dots S_{[n_t]}(T) : S_{[n_1]}S_{[n_2]} \dots S_{[n_t]}\mathcal{E} \to S_{[n_1]}S_{[n_2]} \dots S_{[n_t]}\mathcal{E}'$$

on each object (so $\mathcal{S}_{(-)}$ is a functor from **Ext** to the category of small subcategories of all exact categories). Say $\mathcal{A} \in \mathrm{Obj}(\mathcal{S}_\mathcal{E})$, so $\mathcal{A} \cong S_{[n_1]}S_{[n_2]} \dots S_{[n_t]}\mathcal{E}$ for some finite non-negative sequence $(n_1, n_2, \dots, n_t)$. Let $\mathcal{S}_T|_\mathcal{A}$ be the functor $\mathcal{A} \to \mathcal{S}_T(\mathcal{A})$ induced by $\mathcal{S}_T$ (e.g. $S_{[n_1]}S_{[n_2]} \dots S_{[n_t]}(T)$ above). Let $A \in \mathrm{Obj}(\mathcal{A})$. Abusing notation, set $\mathcal{S}_T(A) := \mathcal{S}_T|_\mathcal{A}(A) \in \mathrm{Obj}(\mathcal{S}_T(\mathcal{A}))$.

Example 2.1.7 Let M be an R-bimodule and $\mathcal{P}$ be the category of finitely generated projective right R-modules. Both $\mathcal{P}$ and **Mod**-R can be considered exact categories by taking short exact sequences in them (the former is split-exact), and there are two distinguished exact functors:

$$I : \mathcal{P} \to \textbf{Mod-}R \qquad \text{and} \qquad T_M : \mathcal{P} \to \textbf{Mod-}R$$

which are the (fully faithful) inclusion functor, and tensoring with M, $(-) \underset{R}{\otimes} M$. We get induced functors: $\mathcal{S}_I, \mathcal{S}_{T_M} : \mathcal{S}_\mathcal{P} \to \mathcal{S}_{\textbf{Mod-}R}$. Define a functor $\textbf{Hom}^M : \mathcal{S}_\mathcal{P} \to s\textbf{Ab}$ which takes an exact category $\mathcal{E}$ to the constant simplicial group of endomorphisms of objects in $\mathcal{E}$ *with coefficients in* M. That is,

$$\textbf{Hom}^M(\mathcal{E}) := \bigoplus_{E \in \mathrm{Obj}(\mathcal{E})} Hom_{\mathcal{S}_I(\mathcal{E})}(\mathcal{S}_I(E), \mathcal{S}_{T_M}(E))$$

Then (Section 3 [15] and [16]) states that this stabilized **Hom** at $\mathcal{P}$ is the topological Hochschild homology of R with coefficients in M:

$$\left(\textbf{Hom}^M\right)^{\mathrm{st}}(\mathcal{P}) \simeq THH(R; M)$$

2.2   Homology of a Category and Local Coefficient Systems

In fact, in [15], $\left(\textbf{Hom}^M\right)^{\mathrm{st}}(\mathcal{P})$ is taken to be the definition of $THH(R; M)$. However, they show it is naturally weak homotopy equivalent to the space given by geometrically realizing

the simplicial abelian group

$$F_p(\mathcal{P}; M) = \bigoplus_{\vec{P} \in N_p \mathcal{P}} Hom_{\mathbf{Mod\text{-}}R}(P_0, P_p \underset{R}{\otimes} M) \qquad \vec{P} := P_0 \leftarrow \cdots \leftarrow P_p$$

with the homotopy equivalence coming from inclusion of the 0-simplices by degeneracies. Work of Waldhausen and Pirashvili shows (Theorem 3.2 [51]) that this space is weakly equivalent to Bökstedt's definition (see [8]) of topological Hochschild homology of R with coefficients in M.

Remark This example highlights a more general relationship between stabilized functors and the Hochschild-Mitchell homology of categories defined by Baues and Wirsching [6] that we will now discuss.

Definition 2.2.1 Let $\mathcal{C}$ be a small category and let $D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ be a bi-functor. We let $F_*(\mathcal{C}; D)$ be the simplicial abelian group defined by:

$$F_p(\mathcal{C}; D) = \bigoplus_{\vec{C} \in N_p \mathcal{C}} D(C_0, C_p) \qquad \vec{C} := C_0 \overset{\alpha_1}{\leftarrow} C_1 \overset{\alpha_2}{\leftarrow} \ldots \overset{\alpha_{p-1}}{\leftarrow} C_{p-1} \overset{\alpha_p}{\leftarrow} C_p$$

with face and degeneracy maps given by:

$$d_i(\,(g; \alpha_1, \ldots, \alpha_p)\,) = \begin{cases} (D(\alpha_1, \mathrm{id}_{C_p})(g); \alpha_2, \ldots, \alpha_p) & i = 0 \\ (g; \alpha_1, \ldots, \alpha_i \alpha_{i+1}, \ldots, \alpha_p) & 1 \le i \le p - 1 \\ (D(\mathrm{id}_{C_0}, \alpha_p); \alpha_1, \ldots, \alpha_{p-1}) & i = p \end{cases}$$

$$s_i(\,(g; \alpha_1, \ldots, \alpha_p)\,) = (g; \alpha_1, \ldots, \alpha_i, \mathrm{id}_{C_i}, \alpha_{i+1}, \ldots, \alpha_p) \; 0 \le i \le p$$

The homotopy groups of $F_*(\mathcal{C}; D)$ are called the (Hochschild-Mitchell) *homology of the category $\mathcal{C}$ with coefficients in the bi-functor D*, denoted $H_*(\mathcal{C}; D)$.

Remark Suppose $D' : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ is another bi-functor, and $\eta : D \Rightarrow D'$ a natural transformation of bi-functors. Then we get induced maps of simplicial abelian groups $F_*(\mathcal{C}; D) \to F_*(\mathcal{C}; D')$, and therefore maps of Hochschild-Mitchell homologies $H_*(\mathcal{C}; D) \to H_*(\mathcal{C}; D')$.

Example 2.2.2 Let G be an abelian group, $\mathcal{C}$ a category, and let $D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ be the constant functor with value G. Then $H_*(\mathcal{C}; D) \cong H_*(B\mathcal{C}; G)$, the singular homology of the classifying space $B\mathcal{C} = |N(\mathcal{C})|$ (see Appendix A.1.4) with coefficients in G. If $\phi : G \to G'$

is a group homomorphism, and D' the constant bi-functor associated to G', the map on Hochschild-Mitchell homologies from the previous remark is simply the change of coefficients for singular homology under the previous isomorphism.

Remark Let C, C' be small categories, $\Lambda : C \to C'$ a functor, and $D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ and $D' : \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}' \to Ab$ be bi-functors. Consider the functor $\tilde{\Lambda} := (\Lambda^{\mathrm{op}}, \Lambda) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}'$. Suppose we are given a natural transformation $\eta : D \Rightarrow D' \circ \tilde{\Lambda}$ of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$. Then, applying $\eta_{(C_0, C_p)}$ to each factor of the p-simplices, we get induced maps of simplicial abelian groups $F_*(\mathcal{C}; D) \to F_*(\mathcal{C}'; D')$, and therefore maps of Hochschild-Mitchell homologies $H_*(\mathcal{C}; D) \to H_*(\mathcal{C}'; D')$.

Definition 2.2.3 Let $\mathcal{E}$ be an exact category. A *local coefficient system D at $\mathcal{E}$* associates to each $\mathcal{C} \in \mathcal{S}_{\mathcal{E}}$ a bi-functor $D_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ to abelian groups such that:

i $D_{\mathcal{C}}$ is bi-reduced, that is, $D_{\mathcal{C}}(C, 0) = D_{\mathcal{C}}(0, C) = 0$ for all $C \in \mathcal{C}$.

ii $D_{(-)}$ is "natural". Given a morphism $\Lambda : \mathcal{C} \to \mathcal{C}'$ in $\mathcal{S}_{\mathcal{E}}$, let $\tilde{\Lambda} := (\Lambda^{\mathrm{op}}, \Lambda) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}'$. Then we have a natural transformation $D_{\Lambda} : D_{\mathcal{C}} \Rightarrow D_{\mathcal{C}'} \circ \tilde{\Lambda}$ of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ satisfying:

(a) $D_{id_{\mathcal{C}}} = id_{D_{\mathcal{C}}}$.

(b) If $\Lambda' : \mathcal{C}' \to \mathcal{C}''$ is another morphism in $\mathcal{S}_{\mathcal{E}}$, we have $D_{\Lambda'} \circ \tilde{\Lambda} \circ D_{\Lambda} = D_{\Lambda' \circ \Lambda}$.

Example 2.2.4 The illuminating example to keep in mind is that of Hom. Indeed, let $\mathcal{E}$ be an exact category. Then $Hom_{\mathcal{E}}(-, -) : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to Ab$ is a bi-functor, reduced in each variable, and can be extended by naturality to all of $\mathcal{S}_{\mathcal{E}}$. Therefore, every exact category carries a natural local coefficient system.

Example 2.2.5 Let R be a ring, and let $M_1, \ldots, M_n$ be R-bimodules. Define a local coefficient system $D(M_1, \ldots, M_n)$ at $\mathcal{P}$ by setting, for $\mathcal{C} \in \mathcal{S}_{\mathcal{P}}$ and $C, C' \in \mathcal{C}$,

$$D(M_1, \ldots, M_n)_{\mathcal{C}}(C, C') = \bigotimes_{i=1}^n Hom_{\mathcal{S}_I(\mathcal{C})}(\mathcal{S}_I(C), \mathcal{S}_{T_{M_i}}(C'))$$

Definition 2.2.6 Let $\mathcal{E}$ be an exact category, and D and D' be local coefficient systems at $\mathcal{E}$. A *morphism of local coefficient systems $\mathcal{H} : D \to D'$* consists of a natural transformation $\mathcal{H}_{\mathcal{C}} : D_{\mathcal{C}} \Rightarrow D'_{\mathcal{C}}$ of bi-functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ for each $\mathcal{C} \in \mathcal{S}_{\mathcal{E}}$ such that, for each morphism

$\Lambda : \mathcal{C} \to \mathcal{C}'$ in $\mathcal{S}_{\mathcal{E}}$, the following square of natural transformations commutes:

$$
\begin{array}{ccc}
D_{\mathcal{C}} & \overset{D_{\Lambda}}{\Longrightarrow} & D_{\mathcal{C}'} \circ \tilde{\Lambda} \\
\Big\Downarrow {\scriptstyle \mathcal{H}_{\mathcal{C}}} & & \Big\Downarrow {\scriptstyle \mathcal{H}_{\mathcal{C}'} \circ \tilde{\Lambda}} \\
D'_{\mathcal{C}} & \overset{D'_{\Lambda}}{\Longrightarrow} & D'_{\mathcal{C}'} \circ \tilde{\Lambda}
\end{array}
$$

Example 2.2.7 Let $D(M_1, \ldots, M_n)$ be the local coefficient system at $\mathcal{P}$ of Example 2.2.5. Let Σ_n be the symmetric group on n letters, and let $\tau \in \Sigma_n$. For $\mathcal{C} \in \mathcal{S}_{\mathcal{P}}$ set $(\tau_*)_{\mathcal{C}}$ to be the natural transformation $D(M_1, \ldots, M_n)_{\mathcal{C}} \to D(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{\mathcal{C}}$ given by reordering the tensor factors, i.e. $\alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \alpha_{\tau^{-1}(1)} \otimes \cdots \otimes \alpha_{\tau^{-1}(n)}$. Then $\tau_* : D(M_1, \ldots, M_n) \to D(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})$ is a morphism of local coefficient systems.

Example 2.2.8 Let $\mathcal{E}, \mathcal{E}'$ be exact categories, D' a local coefficient system at $\mathcal{E}'$, and let $T : \mathcal{E} \to \mathcal{E}'$ be an exact functor. As before, T induces a functor $\mathcal{S}_T : \mathcal{S}_{\mathcal{E}} \to \mathcal{S}_{\mathcal{E}'}$, and for $\mathcal{C} \in \mathcal{S}_{\mathcal{E}}$ the assignment

$$
(T^*(D'))_{\mathcal{C}} := D'_{\mathcal{S}_T(\mathcal{C})} \circ \widetilde{\mathcal{S}_T|_{\mathcal{C}}}
$$

defines a local coefficient system on $\mathcal{E}$, called the *pullback along T*. Pullbacks behave nicely under composition. That is, suppose $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ are exact categories, with $T : \mathcal{E} \to \mathcal{E}'$ and $T' : \mathcal{E}' \to \mathcal{E}''$ exact functors, and that D'' is a local coefficient system on $\mathcal{E}''$. Then $T^*\big((T')^*(D'')\big) \cong (T' \circ T)^*(D'')$.

Remark Pullbacks allow us to compare local coefficient systems over different categories. Indeed, as above, let $\mathcal{E}, \mathcal{E}'$ be exact categories, D' a local coefficient system at $\mathcal{E}'$, and $T : \mathcal{E} \to \mathcal{E}'$ an exact functor. Then, by the remark following 2.2.2 we get a natural transformation of functors $\mathcal{S}_{\mathcal{E}} \to sAb$:

$$
F_*(-; (T^*(D'))_{(-)}) \Rightarrow F_*(\mathcal{S}_T(-); D'_{\mathcal{S}_T(-)})
$$

Example 2.2.9 Let $\mathcal{E}$ be an exact category, and D, D' local coefficient systems at $\mathcal{E}$. We can define a new local coefficient system at $\mathcal{E}$, denoted $D \oplus D'$. For $\mathcal{C} \in \mathcal{S}_{\mathcal{E}}$ it is given by:

$$
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \overset{\Delta}{\to} (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \times (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \xrightarrow{D_{\mathcal{C}} \times D'_{\mathcal{C}}} Ab \times Ab \overset{\oplus}{\to} Ab
$$

Pullbacks preserve sums of local coefficient systems. That is, if we have $T : \mathcal{E} \to \mathcal{E}'$ an exact functor, and D, D' local coefficient systems at $\mathcal{E}'$, then $T^*(D \oplus D') \cong T^*(D) \oplus T^*(D')$ at $\mathcal{E}$.

Example 2.2.10 Let $\mathcal{E}$ be an exact category and D, D' be local coefficient systems at $\mathcal{E}$.

We can define a new local coefficient system at $\mathcal{E}$, denoted $D \underset{\mathbb{Z}}{\otimes} D'$. For $\mathcal{C} \in \mathcal{S}_\mathcal{E}$ it is given by:

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\Delta} (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \times (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \xrightarrow{D_\mathcal{C} \times D'_\mathcal{C}} Ab \times Ab \xrightarrow{\otimes} Ab$$

In the case of 2.2.5, as local coefficient systems over $\mathcal{P}$, $D(M_1, \ldots, M_n) \cong D(M_1) \underset{\mathbb{Z}}{\otimes} \ldots \underset{\mathbb{Z}}{\otimes} D(M_n)$.

Remark Let D be a local coefficient system at $\mathcal{E}$. For each $\mathcal{C} \in \mathcal{S}_\mathcal{E}$ we get a bi-functor $D_\mathcal{C}$ to which we can associate a Hochschild-Mitchell homology, $F_*(\mathcal{C}; D_\mathcal{C})$. By condition ii in the definition of local coefficient systems and by the remark above it, $F_*(-; D_{(-)})$ becomes a functor from $\mathcal{S}_\mathcal{E}$ to simplicial abelian groups. Furthermore, if D' is another local coefficient system at $\mathcal{E}$ and $\mathcal{H} : D \to D'$ is a morphism of local coefficient systems, then $\mathcal{H}$ induces a natural transformation $F_*(-; D_{(-)}) \Rightarrow F_*(-; D'_{(-)})$ of functors $\mathcal{S}_\mathcal{E} \to sAb$.

Example 2.2.11 The morphism $\tau_* : D(M_1, \ldots, M_n) \to D(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})$ of local coefficient systems over $\mathcal{P}$ defined in 2.2.7 induces a natural transformation of functors $\mathcal{S}_\mathcal{P} \to sAb$:

$$F_*(-; D(M_1, \ldots, M_n)_{(-)}) \Rightarrow F_*(-; D(M_{\tau^{-1}(1)} \ldots, M_{\tau^{-1}(n)})_{(-)})$$

In the case that $M_1 = \cdots = M_n = M$ this gives $F_*(-; D(M, \ldots, M)_{(-)})$ a (right) Σ_n-action.

Notation In Section 3 [46], the simplicial abelian group $F_*(\mathcal{C}; D(M_1, \ldots, M_n)_\mathcal{C})$ is denoted F^n.

Example 2.2.12 In the case of 2.2.9, we have a canonical natural isomorphism $F_*(-; D_{(-)}) \oplus F_*(-; D'_{(-)}) \cong F_*(-; D \oplus D'_{(-)})$ of functors $\mathcal{S}_\mathcal{E} \to sAb$.

Observation Let $\mathcal{C}$ be a small category and $D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ a bi-functor. Considering $F_0(\mathcal{C}; D)$ as a constant simplicial group, we have a map of simplicial groups $\delta : F_0(\mathcal{C}; D) \to F_*(\mathcal{C}; D)$ given by inclusion by degeneracies. Explicitly, at simplicial level k, $\delta_k : \underset{C \in \mathrm{Obj}(\mathcal{C})}{\bigoplus} D(C, C) \to \underset{\overrightarrow{C} \in N_k \mathcal{C}}{\bigoplus} D(C_0, C_p)$ is given by

$$(g; C) \mapsto (g; \overbrace{\mathrm{id}_C, \mathrm{id}_C, \ldots, \mathrm{id}_C}^{k-\mathrm{times}}).$$

The following proposition (Proposition 2.1 [46]) says the degeneracies capture the (stabilized) homotopy type:

Proposition 2.2.13 *Let D be a local coefficient system at $\mathcal{E}$. Consider the natural transformation of functors $\mathcal{S}_\mathcal{E} \to s\mathbf{Ab}$*

$$\delta : F_0(-; D_{(-)}) \to F_*(-; D_{(-)})$$

If we iterate the $S_\bullet$-construction, the natural transformation of functors $\mathcal{S}_\mathcal{E} \to s\mathbf{Ab}$

$$\delta(S_\bullet^{(n)}) : \mathfrak{d}F_0\left(S_\bullet^{(n)}(-); D_{S_\bullet^{(n)}(-)}\right) \to \mathfrak{d}F_*\left(S_\bullet^{(n)}(-); D_{S_\bullet^{(n)}(-)}\right)$$

is $2n-1$-connected, and hence δ^{st} is an equivalence.

Remark We can rewrite both Example 2.1.6 and 2.1.7 in terms of this new construction. Let R be a ring, and M an R-bimodule. We can construct two local coefficient systems at $\mathcal{P}$, $D(R)$ and $D(M)$ (in the notation of 2.2.5). Then:

$$F_0(-; D(R)_{(-)}) = \mathbf{Hom}(-) \qquad \text{and} \qquad F_0(-; D(M)_{(-)}) = \mathbf{Hom}^M(-)$$

The proposition recovers the Dundas-McCarthy results mentioned earlier, that

$$\left(F_0(-; D(R)_{(-)})\right)^{\mathrm{st}}(\mathcal{P}) \simeq THH(R) \qquad \text{and} \qquad \left(F_0(-; D(M)_{(-)})\right)^{\mathrm{st}}(\mathcal{P}) \simeq THH(R; M)$$

Observation In the case of 2.2.11, the stabilizations of $F_*(-; D(M, \ldots, M)_{(-)})$ and its 0-simplices inherit a (right) Σ_n-action as well, and the map δ^{st} is Σ_n-equivariant as δ was originally.

The stabilization procedure can greatly simplify the homotopy type of a functor and their local coefficient systems. Indeed, let $\mathcal{E}, \mathcal{E}'$ be exact categories and D, D' local coefficient systems over them, respectively. Let $\pi_\mathcal{E} : \mathcal{E} \times \mathcal{E}' \to \mathcal{E}$ and $\pi_{\mathcal{E}'} : \mathcal{E} \times \mathcal{E}' \to \mathcal{E}'$ be the exact projection functors from the product category. We may form the new local coefficient system $\pi_\mathcal{E}^*(D) \underset{\mathbb{Z}}{\otimes} \pi_{\mathcal{E}'}^*(D')$ over $\mathcal{E} \times \mathcal{E}'$ by pulling back along projections. Then, evaluating on $\mathcal{E} \times \mathcal{E}'$ we get

$$F_0(\mathcal{E} \times \mathcal{E}'; \pi_\mathcal{E}^*(D) \underset{\mathbb{Z}}{\otimes} \pi_{\mathcal{E}'}^*(D')_{\mathcal{E} \times \mathcal{E}'}) = \bigoplus_{(E, E') \in \mathrm{Obj}(\mathcal{E} \times \mathcal{E}')} D_\mathcal{E}(E, E) \underset{\mathbb{Z}}{\otimes} D'_{\mathcal{E}'}(E', E')$$

Though this abelian group may not be 0, stabilization makes it contractible. We show this in two steps.

Lemma 2.2.14 *Let $X_{\bullet,\dots,\bullet}$ be a n-multi-simplicial abelian group. Suppose that in simplicial direction i, X is m_i-reduced (that is, $X_{k_1,\dots,k_{i-1},k_i,k_{i+1},\dots,k_n} = 0$ for $k_i \leq m_i$). Then $\mathfrak{d}X$ is $(n - 1 + \sum_{i=1}^n m_i)$-connected.*

Proof. Recall that we may compute the homotopy groups of $X_{\bullet,\dots,\bullet}$ as the homologies of a chain complex by taking the associated Moore chain complex in each simplicial direction (ending up with an n-multidimensional chain complex), and then taking Tot (see I.1.1 and I.1.4 [21]). An m-chain in this complex lives in

$$\bigoplus_{k_1+\dots+k_n=m} X_{k_1,\dots,k_n}$$

The first possible non-zero summand is $X_{m_1+1,\dots,m_n+1}$. $\qquad\qquad\square$

Lemma 2.2.15 *Let $\mathcal{E}, \mathcal{E}'$ be exact categories and let D, D' be local coefficient systems over them, respectively. Then*

$$\left(F_0\left(-; \pi_{\mathcal{E}}^*(D) \underset{\mathbb{Z}}{\otimes} \pi_{\mathcal{E}'}^*(D')_{(-)}\right)^{\mathrm{st}}(\mathcal{E} \times \mathcal{E}') \simeq 0$$

Proof. We work with the associated chain complexes. First, recall that the n-th chain complex in the homotopy colimit system defining the stabilization iterates the $S_\bullet$-construction on $\mathcal{E} \times \mathcal{E}'$ n-times. Recall that the chain complex $F_0\left(S_\bullet^{(n)}(\mathcal{E} \times \mathcal{E}'); \pi_{\mathcal{E}}^*(D) \underset{\mathbb{Z}}{\otimes} \pi_{\mathcal{E}'}^*(D')_{S_\bullet^{(n)}(\mathcal{E} \times \mathcal{E}')}\right)$ is obtained by considering the n-multi-simplicial abelian group with $[k_1] \times \dots \times [k_n]$ simplices given by

$$F_0\left(S_{[k_1]}\dots S_{[k_n]}(\mathcal{E} \times \mathcal{E}'); \pi_{\mathcal{E}}^*(D) \underset{\mathbb{Z}}{\otimes} \pi_{\mathcal{E}'}^*(D')_{S_{[k_1]}\dots S_{[k_n]}(\mathcal{E} \times \mathcal{E}')}\right)$$

The resulting chain complex is chain homotopy equivalent to the associated Moore chain complex of the diagonal simplicial abelian group. Note, however, that

$$S_{[k_1]}\dots S_{[k_n]}(\mathcal{E} \times \mathcal{E}') \cong S_{[k_1]}\dots S_{[k_n]}(\mathcal{E}) \times S_{[k_1]}\dots S_{[k_n]}(\mathcal{E}')$$

and therefore our chain complex $F_0\left(S_\bullet^{(n)}(\mathcal{E} \times \mathcal{E}'); \pi_{\mathcal{E}}^*(D) \underset{\mathbb{Z}}{\otimes} \pi_{\mathcal{E}'}^*(D')_{S_\bullet^{(n)}(\mathcal{E} \times \mathcal{E}')}\right)$ is actually chain homotopy equivalent to the associated Moore chain complex of the diagonal of a $2n$-multi-simplicial abelian group. This $2n$-multi-simplicial abelian group is equal to, in simplicial degree $[k_1] \times \dots \times [k_n] \times [t_1] \times \dots \times [t_n]$,

$$F_0\left(S_{[k_1]}\dots S_{[k_n]}(\mathcal{E}) \times S_{[t_1]}\dots S_{[t_n]}(\mathcal{E}'); \pi_{\mathcal{E}}^*(D) \underset{\mathbb{Z}}{\otimes} \pi_{\mathcal{E}'}^*(D')_{(S_{[k_1]}\dots S_{[k_n]}(\mathcal{E}) \times S_{[t_1]}\dots S_{[t_n]}(\mathcal{E}'))}\right)$$

$$= \bigoplus_{(E,E')\in\mathrm{Obj}\left(S_{[k_1]}\ldots S_{[k_n]}(\mathcal{E})\right)\times\mathrm{Obj}\left(S_{[t_1]}\ldots S_{[t_n]}(\mathcal{E}')\right)} D_{S_{[k_1]}\ldots S_{[k_n]}(\mathcal{E})}(E,E) \underset{\mathbb{Z}}{\otimes} D'_{S_{[t_1]}\ldots S_{[t_n]}(\mathcal{E}')}(E',E')$$

Since D, D' and $\underset{\mathbb{Z}}{\otimes}$ are all bi-reduced functors, this $2n$-multi-simplicial abelian group is 0-reduced in each simplicial direction. By 2.2.14 its associated Moore chain complex is $(2n-1)$-connected. Therefore $\Omega^{(n)}\left(F_0\left(S_\bullet^{(n)}(\mathcal{E}\times\mathcal{E}'); \pi_\mathcal{E}^*(D)\underset{\mathbb{Z}}{\otimes}\pi_{\mathcal{E}'}^*(D')_{S_\bullet^{(n)}(\mathcal{E}\times\mathcal{E}')}\right)\right)$ is $(n-1)$-connected. $\qquad\square$

The stabilization functor

$$(-)^{\mathrm{st}} : \mathrm{Func}_*\left(\mathbf{Ext}, s\mathbf{Mod}\text{-}R\right) \to \mathrm{Func}_*\left(\mathbf{Ext}, s\mathbf{Mod}\text{-}R\right)$$

takes natural transformations of functors $F \Rightarrow G$ to natural transformations $F^{\mathrm{st}} \Rightarrow G^{\mathrm{st}}$. This is homotopically well-behaved:

Lemma 2.2.16 *If $\eta : F \Rightarrow G$ is natural transformation of functors $\mathbf{Ext} \to s\mathbf{Mod}\text{-}R$ which is a point-wise weak equivalence (that is, that $F(\mathcal{E}) \xrightarrow{\eta_\mathcal{E}} G(\mathcal{E})$ is a weak equivalence of simplicial (right) R-modules for all $\mathcal{E} \in \mathbf{Ext}$), then so is $\eta^{\mathrm{st}} : F^{\mathrm{st}} \Rightarrow G^{\mathrm{st}}$.*

Proof. For each finite sequence $(k_1, \ldots, k_t)$ of non-negative integers we have a weak equivalence

$$F(S_{[k_1]}S_{[k_2]}\ldots S_{[k_t]}\mathcal{E}) \xrightarrow{\eta} G(S_{[k_1]}S_{[k_2]}\ldots S_{[k_t]}\mathcal{E})$$

and hence we get, for each $\mathcal{E} \in \mathbf{Ext}$, a weak equivalence $\mathfrak{d}F(S_\bullet^{(n)}\mathcal{E}) \xrightarrow{\eta} \mathfrak{d}G(S_\bullet^{(n)}\mathcal{E})$. Since simplicial groups are always fibrant, we get an induced weak equivalence $\Omega^{(n)}\left(\mathfrak{d}F(S_\bullet^{(n)}\mathcal{E})\right) \to \Omega^{(n)}\left(\mathfrak{d}G(S_\bullet^{(n)}\mathcal{E})\right)$. The induced map

$$\mathop{\mathrm{hocolim}}_n \Omega^{(n)}\left(\mathfrak{d}F\left(S_\bullet^{(n)}(\mathcal{E})\right)\right) \xrightarrow{\eta^{\mathrm{st}}} \mathop{\mathrm{hocolim}}_n \Omega^{(n)}\left(\mathfrak{d}G\left(S_\bullet^{(n)}(\mathcal{E})\right)\right)$$

is then a weak equivalence by construction. $\qquad\square$

When $\eta : F \Rightarrow G$ is not a point-wise weak equivalence it can be difficult to get a hold on their stabilizations. However, when we restrict to the subcategory $\mathcal{S}_\mathcal{E}$ (for some $\mathcal{E} \in \mathbf{Ext}$), we need only check their homotopy types on a single value:

Lemma 2.2.17 *If $\eta : F \Rightarrow G$ is natural transformation of functors $\mathbf{Ext} \to s\mathbf{Mod}\text{-}R$ such that for some $\mathcal{E} \in \mathbf{Ext}$ $\eta_\mathcal{E}^{\mathrm{st}} : F^{\mathrm{st}}(\mathcal{E}) \to G^{\mathrm{st}}(\mathcal{E})$ is a weak equivalence, then $\eta^{\mathrm{st}}|_{\mathcal{S}_\mathcal{E}}$ is point-wise weak equivalence.*

Proof. Let $k \geq 1$, $1 \leq i \leq k$, and $J_i : S_{[k]}\mathcal{E} \to \mathcal{E}$ be the exact functor $J_i = d_0^{(i-1)} \circ d_{i+1} \circ d_{i+2} \circ \cdots \circ d_k$. Set $\mathfrak{r} := \prod\limits_{i=1}^{k} J_i : S_{[k]}\mathcal{E} \to \mathcal{E}^{\times k}$. A corollary of Waldhausen's additivity theorem is that $S_\bullet \mathfrak{r} : S_\bullet(S_{[k]}\mathcal{E}) \to S_\bullet(\mathcal{E}^{\times k})$ is a weak equivalence. Therefore $\Omega F(S_\bullet(S_{[k]}\mathcal{E})) \xrightarrow{\simeq} \Omega F(S_\bullet(\mathcal{E}^{\times k}))$ is a weak equivalence. Iterating the $S_\bullet$-construction and using the realization lemma we get that $F^{\mathrm{st}}(S_{[k]}\mathcal{E}) \to F^{\mathrm{st}}(\mathcal{E}^{\times k})$ is a weak equivalence. By 2.1.3 F^{st} is product preserving, and so we get $F^{\mathrm{st}}(S_{[k]}\mathcal{E}) \xrightarrow{\cong} F^{\mathrm{st}}(\mathcal{E})^{\times k}$. Let $\mathcal{C} \in \mathcal{S}_\mathcal{E}$. Then there is a sequence $k_1, \ldots, k_t$ of non-negative integers such that $\mathcal{C}$ is equivalent to $S_{[k_1]}S_{[k_2]} \ldots S_{[k_t]}\mathcal{E}$, and so $F^{\mathrm{st}}(\mathcal{C}) \simeq F^{\mathrm{st}}(S_{[k_1]}S_{[k_2]} \ldots S_{[k_t]}\mathcal{E})$. The naturality of $\eta : F \Rightarrow G$ gives a commutative diagram:

$$
\begin{array}{ccc}
F^{\mathrm{st}}(S_{[k_1]}S_{[k_2]} \ldots S_{[k_t]}\mathcal{E}) & \xrightarrow{\simeq} & F(\mathcal{E})^{\times k_1 k_2 \ldots k_t} \\
\downarrow{\scriptstyle \eta^{\mathrm{st}}_{S_{[k_1]}S_{[k_2]} \cdots S_{[k_t]}\mathcal{E}}} & & \downarrow{\scriptstyle \prod \eta^{\mathrm{st}}_\mathcal{E}} \\
G^{\mathrm{st}}(S_{[k_1]}S_{[k_2]} \ldots S_{[k_t]}\mathcal{E}) & \xrightarrow{\simeq} & G(\mathcal{E})^{\times k_1 k_2 \ldots k_t}
\end{array}
$$

Since the product of weak equivalences is a weak equivalence, if $\eta^{\mathrm{st}}_\mathcal{E}$ is a weak equivalence, then so is the left hand map, and hence $\eta^{\mathrm{st}}_\mathcal{C} : F^{\mathrm{st}}(\mathcal{C}) \xrightarrow{\simeq} G^{\mathrm{st}}(\mathcal{C})$. $\square$

Lastly, we get that stabilization is well-behaved with colimits:

Lemma 2.2.18 *Let $\mathcal{I}$ be a small category, and $\mathcal{F} : \mathcal{I} \to \mathrm{Func}_*\big(\mathbf{Ext}, s\mathbf{Mod}\text{-}R\big)$ a functor. Then we have an isomorphism of simplicial R-modules:*

$$
\operatorname*{colim}_{i \in \mathcal{I}}\big((\mathcal{F}(i, -))^{\mathrm{st}}(\mathcal{E})\big) \cong \big(\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}(i, -)\big)^{\mathrm{st}}(\mathcal{E})
$$

Proof. Expressing the stabilization as a coend by the Bousfield-Kan formula, we obtain the desired result from the fact that colimits commute. $\square$

2.3 n-fold Tensor Product

For the rest of this section, we will concentrate on trying to understand what happens when we stabilize the functor $F_*(-; D(M_1, \ldots, M_n)_{(-)}) : \mathcal{S}_\mathcal{P} \to s\mathbf{Ab}$ and evaluate at $\mathcal{P}$. By Proposition 2.2.13, up to homotopy, we need only consider $\big(F_0(-; D(M_1, \ldots, M_n)_{(-)})\big)^{\mathrm{st}}$.

Definition 2.3.1 Let $R, M_1, \ldots, M_n$ be as before. We will define a local coefficient system, denoted $D^{1_n}(M_1, \ldots, M_n)$, at $\mathcal{P}^{\times n}$ by setting $D^{1_n}(M_1, \ldots, M_n)_{\mathcal{P}^{\times n}} : (\mathcal{P}^{\times n})^{\mathrm{op}} \times (\mathcal{P}^{\times n}) \cong$

$(\mathcal{P}^{\mathrm{op}})^{\times n} \times (\mathcal{P}^{\times n}) \to Ab$ to be given by:

$$(C_1, \ldots, C_n) \times (C'_1 \ldots, C'_n) \mapsto Hom_{\mathbf{Mod}\text{-}R}(C_1, C'_1 \underset{R}{\otimes} M_1) \underset{\mathbb{Z}}{\otimes} \ldots \underset{\mathbb{Z}}{\otimes} Hom_{\mathbf{Mod}\text{-}R}(C_n, C'_n \underset{R}{\otimes} M_n)$$

and extending by naturality to all of $\mathcal{S}_{\mathcal{P}^{\times n}}$. Let $\sigma, \sigma' \in \Sigma_n$. Consider the functor: $\sigma \times \sigma' : (\mathcal{P}^{\times n})^{\mathrm{op}} \times (\mathcal{P}^{\times n}) \to (\mathcal{P}^{\times n})^{\mathrm{op}} \times (\mathcal{P}^{\times n})$ that permutes the contravariant and covariant variables by σ and σ', respectively (we act on the left, so $(C_1, \ldots, C_n) \times (C'_1, \ldots, C'_n) \mapsto (C_{\sigma(1)}, \ldots, C_{\sigma(n)}) \times (C_{\sigma'(1)}, \ldots, C_{\sigma'(n)}))$. We define another local coefficient system at $\mathcal{P}^{\times n}$ by setting $D^{\sigma, \sigma'}(M_1, \ldots, M_n)_{\mathcal{P}^{\times n}} := D^{1_n}(M_1, \ldots, M_n)_{\mathcal{P}^{\times n}} \circ \sigma \times \sigma'$ and extending by naturality. Finally, set $D^{\Sigma_n}(M_1, \ldots, M_n) := \bigoplus_{\sigma \in \Sigma_n} D^{1_n \times \sigma}(M_1, \ldots, M_n)$ at $\mathcal{P}^{\times n}$, that is, where we only permute the covariant variables. By abuse of notation, we will often write $D^\sigma(M_1, \ldots, M_n)$ in place of $D^{1_n \times \sigma}(M_1, \ldots, M_n)$, when it is clear we are only permuting the covariant variables.

Example 2.3.2 Let $n = 2$, and $R = M_1 = M_2 = \mathbb{Z}$ (so $\mathcal{P}$ is the category of free abelian groups of finite rank). Then $D^{\Sigma_2} := D^{\Sigma_2}(M_1, M_2) := \bigoplus_{\sigma \in \Sigma_2} D^\sigma(M_1, M_2)$ sends:

$$(C_1, C_2) \times (C'_1, C'_2) \mapsto \left(Hom_{Ab}(C_1, C'_1) \underset{\mathbb{Z}}{\otimes} Hom_{Ab}(C_2, C'_2)\right) \oplus \left(Hom_{Ab}(C_1, C'_2) \underset{\mathbb{Z}}{\otimes} Hom_{Ab}(C_2, C'_1)\right)$$

Example 2.3.3 Let $R, M_1, \ldots, M_n$ be as before. Consider the (exact) diagonal functor $\mathcal{P} \xrightarrow{\Delta} \mathcal{P}^{\times n}$. For any $\sigma \in \Sigma_n$ we have that $\Delta^*(D^\sigma(M_1, \ldots, M_n)) \cong D(M_1, \ldots, M_n)$ as local coefficient systems at $\mathcal{P}$, recovering 2.2.5. By our previous remarks we get a natural transformation of functors $\mathcal{S}_{\mathcal{P}} \to sAb$:

$$F_*(-; (D(M_1, \ldots, M_n))_{(-)}) \Rightarrow F_*(\mathcal{S}_\Delta(-); D^\sigma(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})$$

Taking the coproduct of each such natural transformation of functors, produces:

$$F_*(-; (D(M_1, \ldots, M_n))_{(-)}) \Rightarrow \bigoplus_{\sigma \in \Sigma_n} F_*(\mathcal{S}_\Delta(-); D^\sigma(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})$$

$$\cong F_*(\mathcal{S}_\Delta(-); \bigoplus_{\sigma \in \Sigma_n} D^\sigma(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})$$

$$= F_*(\mathcal{S}_\Delta(-); D^{\Sigma_n}(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})$$

Denote by $\diamond$ the induced natural transformation of functors $\mathcal{S}_{\mathcal{P}} \to sAb$:

$$F_*(-; (D(M_1, \ldots, M_n))_{(-)}) \Rightarrow F_*(\mathcal{S}_\Delta(-); D^{\Sigma_n}(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})$$

Notation In Section 3 [46], the simplicial abelian group $F_*(\mathcal{S}_\Delta(\mathcal{C}); D^{\Sigma_n}(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(\mathcal{C})})$ (for $\mathcal{C} \in \mathcal{S}_\mathcal{P}$) is denoted F^{Σ_n}.

Example 2.3.4 Let $\tau \in \Sigma_n$. We define a morphism of local coefficient systems at $\mathcal{P}^{\times n}$, by reordering tensor factors. That is, for $\mathcal{C} \in \mathcal{S}_\mathcal{P}$ set $(\tau_*)_\mathcal{C}$ to be the natural transformation

$$D^\sigma(M_1, \ldots, M_n)_\mathcal{C} \to D^{\tau^{-1} \times \sigma\tau^{-1}}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_\mathcal{C}$$

given by $\alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \alpha_{\tau^{-1}(1)} \otimes \cdots \otimes \alpha_{\tau^{-1}(n)}$. Taking the coproduct of each such morphism for every $\sigma \in \Sigma_n$ produces a morphism of local coefficient systems:

$$\tau_* : D^{\Sigma_n}(M_1, \ldots, M_n) = \bigoplus_{\sigma \in \Sigma_n} D^\sigma(M_1, \ldots, M_n) \to \bigoplus_{\sigma \in \Sigma_n} D^{\tau^{-1} \times \sigma\tau^{-1}}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})$$

As we've seen, this induces a natural transformation of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \to sAb$:

$$F_*(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F_*\Big(-; \bigoplus_{\sigma \in \Sigma_n} D^{\tau^{-1} \times \sigma\tau^{-1}}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)}\Big)$$

However, by permuting the index set of the nerve, we get that as simplicial abelian groups,

$$F_*\Big(-; \bigoplus_{\sigma \in \Sigma_n} D^{\tau^{-1} \times \sigma\tau^{-1}}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)}\Big)$$

$$\cong F_*\Big(-; \bigoplus_{\sigma \in \Sigma_n} D^{1_n \times \tau\sigma\tau^{-1}}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)}\Big)$$

$$= F_*\Big(-; \bigoplus_{\sigma' \in \Sigma_n} D^{1_n \times \sigma'}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)}\Big)$$

$$= F_*(-; D^{\Sigma_n}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)})$$

Therefore $\tau \in \Sigma_n$ induces a natural transformation of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \to sAb$:

$$F_*(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F_*(-; D^{\Sigma_n}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)})$$

which by abuse of notation we also denote τ_*. For example, in degree 0, we have:

$$\big((C_1, \ldots, C_n); \sigma; \alpha_1 \otimes \cdots \otimes \alpha_n\big) * \tau := \big((C_{\tau^{-1}(1)}, \ldots, C_{\tau^{-1}(n)}); \tau\sigma\tau^{-1}; \alpha_{\tau^{-1}(1)} \otimes \cdots \otimes \alpha_{\tau^{-1}(n)}\big)$$

In the case that $M_1 = \cdots = M_n = M$ this gives $F_*(-; D^{\Sigma_n}(M, \ldots, M)_{(-)})$ a (right) Σ_n-action.

Observation The choice of natural transformation τ_* is compatible with our pullback comparison map $\diamond$. That is, the following square of natural transformations $\mathcal{S}_{\mathcal{P}} \to sAb$ commutes:

$$
\begin{array}{ccc}
F_*(-; D(M_1, \ldots, M_n)_{(-)}) & \overset{\diamond}{\Longrightarrow} & F_*(\mathcal{S}_\Delta(-); D^{\Sigma_n}(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)}) \\[2mm]
\Big\Downarrow {\scriptstyle \tau_*} & & \Big\Downarrow {\scriptstyle \tau_*} \\[2mm]
F_*(-; D(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)}) & \overset{\diamond}{\Longrightarrow} & F_*(\mathcal{S}_\Delta(-); D^{\Sigma_n}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{\mathcal{S}_\Delta(-)})
\end{array}
$$

Therefore, in the case that $M_1 = \cdots = M_n$, the map $\diamond$ is equivariant.

Example 2.3.5 Continuing the previous example, by making a choice of direct sum for every n-tuple in $\mathcal{P}$, we get an exact functor $\mathcal{P}^{\times n} \overset{\oplus}{\to} \mathcal{P}$. We can pull back $D(M_1, \ldots, M_n)$ to get a local coefficient system over $\mathcal{P}^{\times n}$, $\bigoplus^*(D(M_1, \ldots, M_n))$. It is defined, as $(\mathcal{P}^{\times n})^{\mathrm{op}} \times (\mathcal{P}^{\times n}) \to Ab$, by:

$$
(C_1, \ldots, C_n) \times (C_1' \ldots, C_n') \mapsto
$$

$$
Hom_{\mathbf{Mod}\text{-}R}\Big(\bigoplus_{i=1}^n C_i, \bigoplus_{i=1}^n (C_i' \underset{R}{\otimes} M_1)\Big) \underset{\mathbb{Z}}{\otimes} \ldots \underset{\mathbb{Z}}{\otimes} Hom_{\mathbf{Mod}\text{-}R}\Big(\bigoplus_{i=1}^n C_i, \bigoplus_{i=1}^n (C_i' \underset{R}{\otimes} M_n)\Big)
$$

where we've extended by naturality to all of $S_{\mathcal{P}^{\times n}}$. From our previous remarks we know this induces a natural transformation of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \to sAb$:

$$
F_*\Big(-; \bigoplus{}^*(D(M_1, \ldots, M_n))_{(-)}\Big) \Rightarrow F_*(\mathcal{S}_\oplus(-); D(M_1, \ldots, M_n)_{\mathcal{S}_\oplus(-)})
$$

We will now construct a morphism of local coefficient systems (at $\mathcal{P}^{\times n}$), $\phi_\sigma : D^\sigma(M_1, \ldots, M_n) \to \bigoplus^*(D(M_1, \ldots, M_n))$ as follows: for $(C_1, \ldots, C_n) \times (C_1' \ldots, C_n') \in (\mathcal{P}^{\times n})^{\mathrm{op}} \times (\mathcal{P}^{\times n})$

$$
\alpha_1 \otimes \cdots \otimes \alpha_n \in Hom_{\mathbf{Mod}\text{-}R}(C_1, C_{\sigma(1)}' \underset{R}{\otimes} M_1) \underset{\mathbb{Z}}{\otimes} \ldots \underset{\mathbb{Z}}{\otimes} Hom_{\mathbf{Mod}\text{-}R}(C_n, C_{\sigma(n)}' \underset{R}{\otimes} M_n)
$$

$$
\mapsto \tilde{\alpha}_1 \otimes \cdots \otimes \tilde{\alpha}_n
$$

where $\tilde{\alpha}_j : \bigoplus_{i=1}^n C_i \to \bigoplus_{i=1}^n (C_i' \underset{R}{\otimes} M_j)$ is given by $\tilde{\alpha}_j := \iota_{\sigma(j)} \circ \alpha_j \circ \pi_j$, that is, it is α_j on C_j and

0's elsewhere.

$$
\begin{array}{ccc}
C_1 & & C_1' \otimes M_j \\
\oplus & & \oplus \\
\vdots & & \\
\oplus & & \vdots \\
C_j \xrightarrow{\ \pi_j\ } C_j & & \\
\oplus & \searrow^{\alpha_j} & \oplus \\
\vdots & C_{\sigma(j)}' \otimes M_j \xleftarrow{\ \iota_{\sigma(j)}\ } C_{\sigma(j)}' \otimes M_j & \\
& & \oplus \\
& & \vdots \\
\oplus & & \oplus \\
C_n & & C_n' \otimes M_j
\end{array}
$$

One easily checks that ϕ_σ is natural, preserves the chosen $\oplus-$action, and therefore extends to a morphism of local coefficient systems. Then $\phi := \Sigma \phi_\sigma$ is a morphism of local coefficient systems (at $\mathcal{P}^{\times n}$),

$$
\phi : D^{\Sigma_n}(M_1,\ldots,M_n) = \bigoplus_{\sigma \in \Sigma_n} D^\sigma(M_1,\ldots,M_n) \to \bigoplus{}^*(D(M_1,\ldots,M_n))
$$

Composing the induced map on F_* with the previous one produces a natural transformation of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \to sAb$, denoted Φ:

$$
F_*(-; D^{\Sigma_n}(M_1,\ldots,M_n)_{(-)}) \Rightarrow F_*(\mathcal{S}_\oplus(-); D(M_1,\ldots,M_n)_{\mathcal{S}_\oplus(-)})
$$

Remark Composing the maps $\diamond$ and Φ we get a natural transformation of functors: $\mathcal{S}_{\mathcal{P}} \to sAb$

$$
F_*(-; (D(M_1,\ldots,M_n))_{(-)}) \xRightarrow{\Phi \circ \diamond} F_*(-; (D(M_1,\ldots,M_n))_{(-)})
$$

and a natural transformation of functors: $\mathcal{S}_{\mathcal{P}^{\times n}} \to sAb$

$$
F_*(-; D^{\Sigma_n}(M_1,\ldots,M_n)_{(-)}) \xRightarrow{\diamond \circ \Phi} F_*(-; D^{\Sigma_n}(M_1,\ldots,M_n)_{(-)})
$$

Proposition 2.3.6 *The natural transformations $\Phi \circ \diamond$ and $\diamond \circ \Phi$ are stably equivalent to the identity, that is, $\diamond$ and Φ are stably weak homotopy inverses.*

Proof. First, by 2.2.13 we need only check on simplicial degree 0, F_0. That is, we need to

check that

$$F_0(-; (D(M_1, \ldots, M_n))_{(-)})^{\mathrm{st}} \xRightarrow{(\Phi \circ \diamond)^{\mathrm{st}}} F_0(-; (D(M_1, \ldots, M_n))_{(-)})^{\mathrm{st}}$$

and

$$F_0(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})^{\mathrm{st}} \xRightarrow{(\diamond \circ \Phi)^{\mathrm{st}}} F_0(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})^{\mathrm{st}}$$

are equivalent to the identity, as natural transformations of $\mathcal{S}_\mathcal{P} \to sAb$ and $\mathcal{S}_{\mathcal{P}^{\times n}} \to sAb$, respectively. To that end, we will factor our natural transformations through $S_{[2]}\mathcal{P}$ and $S_{[2]}\mathcal{P}^{\times n}$ and exploit additivity. Let us deal with the former first. Consider the functor $S_{[2]} : \mathcal{S}_\mathcal{P} \to \mathcal{S}_{S_{[2]}\mathcal{P}}$, sending $\mathcal{C} \mapsto S_{[2]}\mathcal{C}$. Define a natural transformation $\rho : F_0(-; (D(M_1, \ldots, M_n))_{(-)}) \to F_0(S_{[2]}(-); (D(M_1, \ldots, M_n))_{S_{[2]}(-)})$ of functors $\mathcal{S}_\mathcal{P} \to sAb$ given, on $\mathcal{C} \in \mathcal{S}_\mathcal{P}$, by:

$$(C; \alpha_1 \otimes \cdots \otimes \alpha_n) \mapsto (\widehat{C}; \widehat{\alpha_1} \otimes \cdots \otimes \widehat{\alpha_n})$$

where $\widehat{C} \in S_{[2]}\mathcal{C}$ is the short exact sequence $C \xrightarrow{\Delta} C^{\oplus n} \xrightarrow{\pm} C^{\oplus(n-1)}$:

where Δ is the diagonal map and $\pm$ the alternating multiplication by 1 or -1, and $\widehat{\alpha}_i \in$

$Hom(\widehat{C}, \widehat{C} \underset{R}{\otimes} M_i)$ is the map of short exact sequences:

$$
\begin{array}{ccc}
C & \xrightarrow{\;\alpha_i\;} & C \otimes M_i \\
{\scriptstyle \Delta}\big\downarrow & & {\scriptstyle \Delta}\big\downarrow \\
C^{\oplus n} & \xrightarrow{\;\sum\limits_{j=1}^{n} \iota_j \circ \alpha_i \circ \pi_i\;} & (C \otimes M_i)^{\oplus n} \\
{\scriptstyle \pm}\big\downarrow & & {\scriptstyle \pm}\big\downarrow \\
C^{\oplus(n-1)} & \xrightarrow{\quad 0 \quad} & (C \otimes M_i)^{\oplus(n-1)}
\end{array}
$$

A calculation shows that ρ is in fact a natural transformation of functors (i.e. respecting the face and degeneracy maps of $S_\bullet$-constructions which generate the morphisms of $\mathcal{S}_\mathcal{P}$). Now, let (d_2, d_0) be the induced natural transformation (of functors $\mathcal{S}_\mathcal{P} \to sAb$) $F_0(S_{[2]}(-); (D(M_1, \ldots, M_n))_{S_{[2]}(-)}) \to F_0(-; (D(M_1, \ldots, M_n))_{(-)}) \times F_0(-; (D(M_1, \ldots, M_n))_{(-)})$ by taking the source and target of the short exact sequences, and let

$$d_1 : F_0(S_{[2]}(-); (D(M_1, \ldots, M_n))_{S_{[2]}(-)}) \to F_0(-; (D(M_1, \ldots, M_n))_{(-)})$$

be the total map. Then, for each $\mathcal{C} \in \mathcal{S}_\mathcal{P}$ we have a non-commutative diagram:

$$
F_0(\mathcal{C}; D(M_1, \ldots, M_n)_\mathcal{C}) \xrightarrow{\;\rho\;} F_0(S_{[2]}\mathcal{C}; D(M_1, \ldots, M_n)_{S_{[2]}\mathcal{C}}) \xrightarrow{(d_2, d_0)} F_0(\mathcal{C}; D(M_1, \ldots, M_n)_\mathcal{C})^{\times 2}
$$

$$
\big\downarrow {\scriptstyle d_1} \qquad\qquad\qquad {\scriptstyle +}
$$

$$
F_0(\mathcal{C}; D(M_1, \ldots, M_n)_\mathcal{C})
$$

where the last map is the sum map. The outside composite $+ \circ (d_2, d_0) \circ \rho$ is the identity, whilst $d_1 \circ \rho$ is the map

$$
(C; \alpha_1 \otimes \cdots \otimes \alpha_n) \mapsto \big(C^{\oplus n}; (\sum_{j=1}^n \iota_j \circ \alpha_1 \circ \pi_1) \otimes \cdots \otimes (\sum_{j=1}^n \iota_j \circ \alpha_n \circ \pi_n)\big)
$$

$$
= \big(C^{\oplus n}; \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n (\iota_{j_1} \circ \alpha_1 \circ \pi_1) \otimes \cdots \otimes (\iota_{j_n} \circ \alpha_n \circ \pi_n)\big)
$$

$$
= \big(C^{\oplus n}; \sum_{\lambda \in \mathcal{M}_n} \tilde{\alpha}_1^{\lambda(1)} \otimes \cdots \otimes \tilde{\alpha}_n^{\lambda(n)}\big)
$$

where $\mathcal{M}_n$ is the set of all functions $\{1, \ldots, n\} \to \{1, \ldots, n\}$ and $\tilde{\alpha}_i^{\lambda(i)} := \iota_{\lambda(i)} \circ \alpha_i \circ \pi_i$.

Observe that by Proposition 1.3.2 [59], $(d_1)^{\mathrm{st}} \simeq (+ \circ (d_2, d_0))^{\mathrm{st}}$ and therefore:

$$
\begin{aligned}
(d_1 \circ \rho)^{\mathrm{st}} &= (d_1)^{\mathrm{st}} \circ \rho^{\mathrm{st}} \\
&\simeq (+ \circ (d_2, d_0))^{\mathrm{st}} \circ \rho^{\mathrm{st}} \\
&= (+ \circ (d_2, d_0) \circ \rho)^{\mathrm{st}} \\
&= id^{\mathrm{st}} = id
\end{aligned}
$$

So the map $F_0(-; (D(M_1, \ldots, M_n))_{(-)}) \to F_0(-; (D(M_1, \ldots, M_n))_{(-)})$ given by $(C; \alpha_1 \otimes \cdots \otimes \alpha_n) \mapsto \sum_{\lambda \in \mathcal{M}_n} \tilde{\alpha}_1^{\lambda(1)} \otimes \cdots \otimes \tilde{\alpha}_n^{\lambda(n)}$ is stably equivalent to the identity. Let's analyze this natural transformation. Suppose $\lambda \in \mathcal{M}_n$ isn't surjective. Pick $u \in \{1, \ldots, n\}$ not in the image. Consider the natural transformation $\rho^\lambda : F_0(-; (D(M_1, \ldots, M_n))_{(-)}) \to F_0(S_{[2]}(-); (D(M_1, \ldots, M_n))_{S_{[2]}(-)})$ of functors $\mathcal{S}_\mathcal{P} \to sAb$ given, on $\mathcal{C} \in \mathcal{S}_\mathcal{P}$, by:

$$
(C; \alpha_1 \otimes \cdots \otimes \alpha_n) \mapsto (\bar{C}; \bar{\alpha}_1 \otimes \cdots \otimes \bar{\alpha}_n)
$$

where $\bar{C} \in S_{[2]}\mathcal{C}$ is the short exact sequence $C^{\oplus(n-1)} \rightarrowtail C^{\oplus n} \xrightarrow{\pi_u} C$, where the first map is inclusion into the non$-u$ coordinates, and $\bar{\alpha}_i \in Hom(\bar{C}, \bar{C} \underset{R}{\otimes} M_i)$ is the map of short exact sequences:

$$
\begin{array}{ccc}
C^{\oplus(n-1)} \xrightarrow{\tilde{\alpha}_i^{\bar{\lambda}(i)}} (C \otimes M_i)^{\oplus(n-1)} & C^{\oplus(n-1)} \xrightarrow{\tilde{\alpha}_{i-1}^{\bar{\lambda}(i)}} (C \otimes M_i)^{\oplus(n-1)} & C^{\oplus(n-1)} \xrightarrow{\ 0\ } (C \otimes M_i)^{\oplus(n-1)} \\
\downarrow \qquad\qquad \downarrow & \downarrow \qquad\qquad \downarrow & \downarrow \qquad\qquad \downarrow \\
C^{\oplus n} \xrightarrow{\tilde{\alpha}_i^{\lambda(i)}} (C \otimes M_i)^{\oplus n} & C^{\oplus n} \xrightarrow{\tilde{\alpha}_i^{\lambda(i)}} (C \otimes M_i)^{\oplus n} & C^{\oplus n} \xrightarrow{\tilde{\alpha}_i^{\lambda(i)}} (C \otimes M_i)^{\oplus n} \\
{\scriptstyle \pi_u} \downarrow \qquad\qquad {\scriptstyle \pi_u} \downarrow & {\scriptstyle \pi_u} \downarrow \qquad\qquad {\scriptstyle \pi_u} \downarrow & {\scriptstyle \pi_u} \downarrow \qquad\qquad {\scriptstyle \pi_u} \downarrow \\
C \xrightarrow{\ 0\ } C \otimes M_i & C \xrightarrow{\ 0\ } C \otimes M_i & C \xrightarrow{\ 0\ } C \otimes M_i
\end{array}
$$

for $i < u$, $i > u$, and $i = u$, respectively. Here $\bar{\lambda}(i)$ is $\lambda(i)$ if $\lambda(i) < u$ and $\lambda(i) - 1$ otherwise. Again, a calculation shows that ρ^λ is a natural transformation of functors $\mathcal{S}_\mathcal{P} \to sAb$, and

we have, for each $\mathcal{C} \in \mathcal{S}_{\mathcal{P}}$ a non-commutative diagram:

$$F_0(\mathcal{C}; D(M_1, \ldots, M_n)_{\mathcal{C}}) \xrightarrow{\rho^{\lambda}} F_0(S_{[2]}\mathcal{C}; D(M_1, \ldots, M_n)_{S_{[2]}\mathcal{C}}) \xrightarrow{(d_2, d_0)} F_0(\mathcal{C}; D(M_1, \ldots, M_n)_{\mathcal{C}})^{\times 2}$$

$$\downarrow{d_1} \qquad\qquad +$$

$$F_0(\mathcal{C}; D(M_1, \ldots, M_n)_{\mathcal{C}})$$

The outside composite $+ \circ (d_2, d_0) \circ \rho^{\lambda}$ is 0, whilst $d_1 \circ \rho^{\lambda}$ is the map $(C; \alpha_1 \otimes \cdots \otimes \alpha_n) \mapsto (C^{\oplus n}; \tilde{\alpha}_1^{\lambda(1)} \otimes \cdots \otimes \tilde{\alpha}_n^{\lambda(n)})$. This map, therefore, is stably equivalent to 0. So the map $F_0(-; (D(M_1, \ldots, M_n))_{(-)}) \rightarrow F_0(-; (D(M_1, \ldots, M_n))_{(-)})$ given by $(C; \alpha_1 \otimes \cdots \otimes \alpha_n) \mapsto (C^{\oplus n}; \sum_{\lambda \in \mathcal{M}_n} \tilde{\alpha}_1^{\lambda(1)} \otimes \cdots \otimes \tilde{\alpha}_n^{\lambda(n)})$ is stably equivalent to the one given by only surjective maps

$$(C; \alpha_1 \otimes \cdots \otimes \alpha_n) \mapsto \left(C^{\oplus n}; \sum_{\sigma \in \Sigma_n} \tilde{\alpha}_1^{\sigma(1)} \otimes \cdots \otimes \tilde{\alpha}_n^{\sigma(n)}\right)$$

$$= (\Phi \circ \diamond)\left(C; \alpha_1 \otimes \cdots \otimes \alpha_n\right)$$

Therefore $id = id^{\mathrm{st}} \simeq (d_1 \circ \rho)^{\mathrm{st}} \simeq (\Phi \circ \diamond)^{\mathrm{st}}$ on $F_0(-; (D(M_1, \ldots, M_n))_{(-)})^{\mathrm{st}}$. $\qquad\square$

Notation Let X, Y be simplicial sets with Σ_n-action. We say X and Y are *feebly* Σ_n-weak equivalent, denoted $X \simeq_{f\Sigma_n} Y$, is there is a zig-zag of weak equivalences of simplicial sets between X and Y, each of which is a Σ_n equivariant map. Since we make no claim on restriction to subgroups, this *not* the same as a weak equivalence in the equivariant homotopy theory setting.

Remark By 2.3.6 and 2.3.4, we have a feeble Σ_n-weak equivalence between the stabilization of $F_*(-; (D(M_1, \ldots, M_n))_{(-)})$ and $F_*(\mathcal{S}_\Delta(-); D^{\Sigma_n}(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})$ when $M_1 = \cdots = M_n = M$. Thus, we will now concentrate on analyzing the latter.

Definition 2.3.7 Let $R, M_1, \ldots, M_n$ be as before and let $\mathcal{S}_n \subset \Sigma_n$ be the subset of full length cycles. Define a local coefficient system at $\mathcal{P}^{\times n}$ by:

$$D^{\mathcal{S}_n}(M_1, \ldots, M_n) := \bigoplus_{\sigma \in \mathcal{S}_n} D^{1_n \times \sigma}(M_1, \ldots, M_n)$$

There is an inclusion of local coefficient systems $D^{\mathcal{S}_n}(M_1, \ldots, M_n) \hookrightarrow D^{\Sigma_n}(M_1, \ldots, M_n)$ inducing an injective natural transformation of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \rightarrow sAb$:

$$F_*(-; D^{\mathcal{S}_n}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F_*(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})$$

Since $\mathcal{S}_n$ is invariant under conjugation, for each $\tau \in \Sigma_n$ the maps τ_* descend to give a commutative diagram:

$$
\begin{array}{ccc}
F_*(-; D^{\mathcal{S}_n}(M_1, \ldots, M_n)_{(-)}) & \Longrightarrow & F_*(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)}) \\
\Big\| \tau_* & & \Big\| \tau_* \\
F_*(-; D^{\mathcal{S}_n}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)}) & \Longrightarrow & F_*(-; D^{\Sigma_n}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})_{(-)})
\end{array}
$$

Therefore, in the case that $M_1 = \cdots = M_n$, the inclusion $F_*(-; D^{\mathcal{S}_n}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F_*(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})$ is Σ_n-equivariant. We now seek to show that this inclusion is an equivalence upon stabilization.

Proposition 2.3.8 *The map of simplicial abelian groups* $F_*(-; D^{\mathcal{S}_n}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F_*(-; D^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})$ *is a weak equivalence after stabilization.*

Proof. First, by 2.2.13 we need only case about the stabilization of the degree 0-simplices. Second, by 2.2.12 we can equivalently show that for $\sigma \in \Sigma_n \setminus \mathcal{S}_n$, $F_0(-; D^\sigma(M_1, \ldots, M_n)_{(-)})$ is stably contractible. If $\sigma \in \Sigma_n \setminus \mathcal{S}_n$, then there exist distinct $\tau, \tau' \in \Sigma_n$, disjoint cycles of shorter length, such that $\sigma = \tau \cdot \tau'$. Let $s = |\tau|, s' = |\tau'|$ (so that $s + s' = n$), and let $u = \mathrm{Im}(\tau), u' = \mathrm{Im}(\tau')$ be the images of the disjoint cycles in $\underline{n}$. There is an exact natural isomorphism $A : \mathcal{P}^{\times n} \cong \mathcal{P}^{\times s} \times \mathcal{P}^{\times s'}$ such that $A \circ \sigma = \mathrm{id}_{\mathcal{P}^{\times s} \times \mathcal{P}^{\times s'}}$ which extends to a natural isomorphism $\mathcal{S}_A : \mathcal{S}_{\mathcal{P}^{\times n}} \cong \mathcal{S}_{\mathcal{P}^{\times s}} \times \mathcal{S}_{\mathcal{P}^{\times s'}}$. Let π_1 and π_2 be the projection functors $\mathcal{P}^{\times s} \times \mathcal{P}^{\times s'} \to \mathcal{P}^{\times s}$ and $\mathcal{P}^{\times s} \times \mathcal{P}^{\times s'} \to \mathcal{P}^{\times s'}$, respectively. We then get a natural isomorphism of functors

$$
F_0(-; D^\sigma(M_1, \ldots, M_n)_{(-)}) \cong F_0(-; \pi_1^*(D^{1_s}(M_{a_1}, \ldots, M_{a_s})) \underset{\mathbb{Z}}{\otimes} \pi_2^*(D^{1_{s'}}(M_{b_1}, \ldots, M_{b_{s'}}))_{(-)})
$$

where $u = \{a_1, \ldots, a_s\}$ and $u' = \{b_1, \ldots, b_{s'}\}$. By 2.2.15 we get our desired conclusion. $\square$

Remark Let $\omega = (n \ldots 2\, 1) \in \Sigma_n$ be the standard full-length cycle. Then $\mathcal{S}_n = \{\tau \omega \tau^{-1} \mid \tau \in \Sigma_n\}$. When $M_1 = \cdots = M_n$ the local coefficient system $D^\omega(M_1, \ldots, M_n)$ acquires a C_n-action, and so $F_*(\mathcal{C}; D^\omega(M_1, \ldots, M_n)_{\mathcal{C}})$ becomes a simplicial $\mathbb{Z}[C_n]$-module. Since C_n is abelian, the inclusion map $F_*(\mathcal{C}; D^\omega(M_1, \ldots, M_n)_{\mathcal{C}}) \hookrightarrow F_*(\mathcal{C}; D^{\mathcal{S}_n}(M_1, \ldots, M_n)_{\mathcal{C}})$ is C_n-equivariant. Therefore restriction/extension-of-scalars adjunction gives a map of simplicial (right) $\mathbb{Z}[\Sigma_n]$-modules

$$
F_*(\mathcal{C}; (D^\omega(M_1, \ldots, M_n))_{\mathcal{C}}) \underset{\mathbb{Z}[C_n]}{\otimes} \mathbb{Z}[\Sigma_n] \xrightarrow{\nu} F_*(\mathcal{C}; D^{\mathcal{S}_n}(M_1, \ldots, M_n)_{\mathcal{C}})
$$

which is natural in $\mathcal{C} \in \mathcal{S}_{\mathcal{P}^{\times n}}$. By analyzing the Σ_n-action from 2.3.4, the following claim is easily checked:

Claim 2.3.9 *The map of simplicial right* $\mathbb{Z}[\Sigma_n]$-*modules* ν *is an isomorphism.*

Combining this with 2.2.18 it follows that we have a Σ_n-isomorphism of the associated stabilizations:

$$\left(F_*(-;(D^\omega(M_1,\dots,M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{C}) \underset{\mathbb{Z}[C_n]}{\otimes} \mathbb{Z}[\Sigma_n] \cong \left(F_*(-;(D^{\mathcal{S}_n}(M_1,\dots,M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{C})$$

Altogether, in the case when $M_1 = \cdots = M_n$, we get the following chain of feeble Σ_n-weak equivalences:

$$
\begin{aligned}
\left(F_*(-;(D(M_1,\dots,M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{P}) &\simeq \left(F_*(-;(D^{\Sigma_n}(M_1,\dots,M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{P}^{\times n}) \\
&\simeq \left(F_*(-;(D^{\mathcal{S}_n}(M_1,\dots,M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{P}^{\times n}) \\
&\simeq \left(F_*(-;(D^\omega(M_1,\dots,M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{P}^{\times n}) \underset{\mathbb{Z}[C_n]}{\otimes} \mathbb{Z}[\Sigma_n] \\
&\simeq \Omega\left(\mathfrak{d}F_*(S_\bullet\mathcal{P}^{\times n};D^\omega(M_1,\dots,M_n)_{S_\bullet\mathcal{P}^{\times n}})\right) \underset{\mathbb{Z}[C_n]}{\otimes} \mathbb{Z}[\Sigma_n] \\
&\simeq F_*(\mathcal{P}^{\times n};D^\omega(M_1,\dots,M_n)_{\mathcal{P}^{\times n}}) \underset{\mathbb{Z}[C_n]}{\otimes} \mathbb{Z}[\Sigma_n]
\end{aligned}
$$

where the last two weak equivalences follow from Corollary 3.7 [46]. This recovers the main result of McCarthy in [46], which we generalize in the next chapter.

CHAPTER 3

THE SPECTRUM CASE

In this chapter we translate many of the algebraic concepts from the previous section to the case of spectra. Though many of the definitions and propositions carry through directly; we will be careful in those places where we feel confusion may arise. Our model of stable homotopy will be functors with stabilization (FST) and functor with smash product (FSP), as in [34] (see Appendix E for details).

3.1 Additivity Revisited

Let $F : \textbf{Ext} \to \mathbb{S}\text{-BiMod}$ be a reduced functor. We can define *p-product* and *product preserving* functors just as in Chapter 2, and mimic the construction of 2.1.1 for the *second cross effect* of F, cr_2F. Just as in the algebraic case we get, for each exact category $\mathcal{E}$, a natural map of $\mathbb{S}$-bimodules $F(S_2(\mathcal{E})) \xrightarrow{(d_2,d_0)} F(\mathcal{E}) \times F(\mathcal{E})$.

Definition 3.1.1 A functor $F : \textbf{Ext} \to \mathbb{S}\text{-BiMod}$ is *additive* if the map $F(S_2(-)) \xrightarrow{(d_2,d_0)} F(-) \times F(-)$ is a weak equivalence. We say F is *p-additive* if it's $(p+1)$-connected.

Observation If F is a product preserving functor, then $\mathfrak{d}F(S_\bullet(-))$ is an additive functor. More generally, if F is a *p*-product functor, then $\mathfrak{d}F(S_\bullet(-))$ is a *p*-additive functor. In general, a functor need not be additive, however, just as before, there is a universal additive approximation:

Definition 3.1.2 Given a reduced functor $F : \textbf{Ext} \to \mathbb{S}\text{-BiMod}$ define a new functor $F^{\mathrm{st}} : \textbf{Ext} \to \mathbb{S}\text{-BiMod}$ given by

$$F^{\mathrm{st}}(\mathcal{E}) := \operatorname*{hocolim}_{n} \Omega^{(n)}\big(\mathfrak{d}F\big(S_\bullet^{(n)}(\mathcal{E})\big)\big)$$

(we use the corrected homotopy colimit; see Appendix B.2).

Remark By our choice of homotopy colimit, F^{st} is a reduced functor. By lemmas 5.7 and 5.8 in [44], F^{st} is a product preserving and additive functor, and the natural transformation $\alpha : F \to F^{\mathrm{st}}$ is universally initial (up to homotopy) among additive functors. Exactly as the algebraic case, if F is a product preserving functor, then $\Omega(\eth F(S_\bullet(-)))$ is an additive functor. Therefore, $\alpha : \Omega \eth F S_\bullet \to (\Omega \eth F S_\bullet)^{\mathrm{st}}$ is an equivalence. Combining this with the cofinality isomorphism for product preserving functors, $F^{\mathrm{st}} \to (\Omega \eth F S_\bullet)^{\mathrm{st}}$, we conclude that for a product preserving functor $F^{\mathrm{st}} \simeq \Omega \eth F S_\bullet$.

Example 3.1.3 Analogous to the case of Example 2.1.5, let E be an FST, and consider the functor $\mathbf{E} : \mathbf{Ext} \to \mathbb{S}\text{-BiMod}$ given by $\mathcal{E} \mapsto \mathrm{Obj}(\mathcal{E}) \wedge E$. Then by the Hurewicz isomorphism, we have

$$\pi_i(\mathbf{E}^{\mathrm{st}}(\mathcal{E})) \cong H_i(\mathbf{K}(\mathcal{E}); \mathbf{E})$$

In particular, if $E = \mathbb{S}$ is the sphere FST, then $\mathbf{S}^{\mathrm{st}}(\mathcal{E}) \simeq \mathbf{K}(\mathcal{E})$

Example 3.1.4 Since exact categories are *Ab*-enriched (and therefore, by prolongation, also *sAb*-enriched), we can use the Eilenberg-Maclane functor construction to give it an enrichment over $\mathbb{S}$-BiMod (see Appendix E.4.4). If $\mathcal{E}$ is an exact category, we denote the associated FSP over $\mathcal{E}$ by $\tilde{\mathcal{E}}$. We define a functor $\underline{\mathbf{Hom}} : \mathbf{Ext} \to \mathbb{S}\text{-BiMod}$ by

$$\underline{\mathbf{Hom}}(\mathcal{E}) := \bigvee_{E \in \mathrm{Obj}(\mathcal{E})} \tilde{\mathcal{E}}_{E,E}$$
$$= \bigvee_{E \in \mathrm{Obj}(\mathcal{E})} \widetilde{Hom_\mathcal{E}(E, E)}$$

in analogy with Example 2.1.6. The stabilization of this functor recovers the (spectral) topological Hochschild homology of $\mathcal{E}$, $THH(\mathcal{E})$. Indeed, by Dundas-McCarthy we have that $\underline{\mathbf{Hom}}^{\mathrm{st}}(\mathcal{E}) \simeq \underline{N}^{\mathrm{cy}}(\mathcal{E})$, the spectrally-enriched cyclic nerve of $\mathcal{E}$. Since the spectral enrichment of $\mathcal{E}$, $\tilde{\mathcal{E}}$, is pointwise cofibrant, the natural map in the stable category from $THH(\mathcal{E})$ to $\underline{N}^{\mathrm{cy}}(\mathcal{E})$ is an isomorphism by the "many-objects" version of Proposition 4.2.8 [58] (see also correction in Theorem 3.6 [50]).

Example 3.1.5 If R is a ring, and M an R-bimodule, we can generalize Example 2.1.7 to the spectral setting, defining a functor $\underline{\mathbf{Hom}}^M : \mathcal{S}_\mathcal{P} \to \mathbb{S}\text{-BiMod}$ in analogy with $\mathbf{Hom}^M$, but made with the associated FSTs as in the previous example. By work of Dundas-McCarthy

(and the aforementioned [58], [50]), we get that:

$$\left(\underline{\mathbf{Hom}}^M\right)^{\mathrm{st}}(\mathcal{P}) \simeq THH(R; M)$$

3.2 Homology and Local Coefficient Systems in Spectra

Much of the same story of local coefficient systems over categories and their associated Hochschild-Mitchell-type homologies can be translated *mutatis mutandis* to the spectral setting.

Definition 3.2.1 Let $\mathcal{C}$ be a small category and let $D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{S}\text{-BiMod}$ be a (globally stable) bi-functor (see Appendix E.3). We let $F_*(\mathcal{C}; D)$ be the simplicial FST defined by:

$$F_p(\mathcal{C}; D) = \bigvee_{\overrightarrow{C} \in N_p \mathcal{C}} D(C_0, C_p) \qquad \overrightarrow{C} := C_0 \xleftarrow{\alpha_1} C_1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_{p-1}} C_{p-1} \xleftarrow{\alpha_p} C_p$$

with face and degeneracy maps given by:

$$d_i \circ \iota_{\alpha_1,\dots,\alpha_p} = \begin{cases} \iota_{\alpha_2,\dots,\alpha_p} \circ D(\alpha_1, \mathrm{id}_{C_p}) & i = 0 \\ \iota_{\alpha_1,\dots,\alpha_i \circ \alpha_{i+1},\dots,\alpha_p} & 1 \leq i \leq p - 1 \\ \iota_{\alpha_1,\dots,\alpha_{p-1}} \circ D(\mathrm{id}_{C_0}, \alpha_p) & i = p \end{cases}$$

$$s_i \circ \iota_{\alpha_1,\dots,\alpha_p} = \iota_{\alpha_1,\dots,\alpha_i,\mathrm{id}_{C_i},\alpha_{i+1},\dots,\alpha_p} \qquad 0 \leq i \leq p$$

We denote the geometric realization of $F_*(\mathcal{C}; D)$ by $F(\mathcal{C}; D)$. The homotopy groups of $F(\mathcal{C}; D)$ are called the *spectral* (Hochschild-Mitchell) *homology of the category $\mathcal{C}$ with coefficients in the bi-functor D*, denoted $H_*(\mathcal{C}; D)$.

Remark Just as in the algebraic case, a natural transformation $\eta : D \Rightarrow D'$ of bi-functors, $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{S}\text{-BiMod}$ induces maps of simplicial FSTs, $F_*(\mathcal{C}; D) \to F_*(\mathcal{C}; D')$, and thus maps $H_*(\mathcal{C}; D) \to H_*(\mathcal{C}; D')$.

Definition 3.2.2 Let $\mathcal{E}$ be an exact category. A *spectral local coefficient system D* (at $\mathcal{E}$) associates to each $\mathcal{C} \in \mathcal{S}_{\mathcal{E}}$ a (globally stable) bi-functor $D_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{S}\text{-BiMod}$ to FSTs such that:

  i $D_{\mathcal{C}}$ is bi-reduced, that is, $D_{\mathcal{C}}(C, 0) = D_{\mathcal{C}}(0, C) = 0$ for all $C \in \mathcal{C}$.

ii $D_{(-)}$ is "natural". Given a morphism $\Lambda : \mathcal{C} \to \mathcal{C}'$ in $\mathcal{S}_{\mathcal{E}}$, let $\tilde{\Lambda} := (\Lambda^{\mathrm{op}}, \Lambda) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$ $\mathcal{C}'^{\mathrm{op}} \times \mathcal{C}'$. Then we have a natural transformation $D_{\Lambda} : D_{\mathcal{C}} \Rightarrow D_{\mathcal{C}'} \circ \tilde{\Lambda}$ of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{S}$-BiMod satisfying:

(a) $D_{id_{\mathcal{C}}} = id_{D_{\mathcal{C}}}$.

(b) If $\Lambda' : \mathcal{C}' \to \mathcal{C}''$ is another morphism in $\mathcal{S}_{\mathcal{E}}$, we have $D_{\Lambda'} \circ \tilde{\Lambda} \circ D_{\Lambda} = D_{\Lambda' \circ \Lambda}$.

Example 3.2.3 Recall that we have the "Eilenberg-Maclane" functor $Ab \xrightarrow{H} \mathbb{S}$-BiMod associating an FST to each abelian group, $A \mapsto \tilde{A}$. Let $\mathcal{E}$ be an exact category, and D a local coefficient system (of abelian groups) over $\mathcal{E}$. Post-composing with H, we get a spectral local coefficient system, $\tilde{D}$ over $\mathcal{E}$. Indeed, H is reduced, and given a morphism $\Lambda : \mathcal{C} \to \mathcal{C}'$ in $\mathcal{S}_{\mathcal{E}}$, horizontal composition of $D_{\Lambda} : D_{\mathcal{C}} \Rightarrow D_{\mathcal{C}'} \circ \tilde{\Lambda}$ with H gives the desired natural transformation $\tilde{D}_{\Lambda} : \tilde{D}_{\mathcal{C}} \Rightarrow \tilde{D}_{\mathcal{C}'} \circ \tilde{\Lambda}$. Some of the main spectral local coefficient systems we work with arise in this manner.

Example 3.2.4 A central example is the spectral analogue of 2.2.4, denoted $\widetilde{Hom}$. More generally, the notion of spectral local coefficient system makes sense for any FSP over a small pointed category $\mathcal{C}$ (see Appendix E.4). Every "spectrally-enriched" category, therefore, carries a natural local coefficient system, and furthermore, a module over such an FSP defines a spectral local coefficient system (see Appendix E.4.4).

Example 3.2.5 Let R be a ring, and let $M_1, \ldots, M_n$ be R-bimodules. Define a spectral local coefficient system $\mathfrak{D}(M_1, \ldots, M_n)$ at $\mathcal{P}$ by setting, for $\mathcal{C} \in \mathcal{S}_{\mathcal{P}}$ and $C, C' \in \mathcal{C}$,

$$\mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}}(C, C') = \bigwedge_{i=1}^{n} \widetilde{Hom}_{\mathcal{S}_I(\mathcal{C})}(\mathcal{S}_I(C), \mathcal{S}_{T_{M_i}}(C'))$$

where the wedge means tensoring over the sphere FSP $\mathbb{S}$ (see Appendix E.1). Note that $\mathfrak{D}(M_1, \ldots, M_n)$ is globally stable.

Observation We can define morphisms of spectral local coefficient systems in direct analogy with 2.2.6. Since $H : (Ab, \underset{\mathbb{Z}}{\otimes}, \mathbb{Z}) \to (\mathbb{S}-\mathrm{BiMod}, \otimes_{\mathrm{Day}}, \mathbb{S})$ is a lax symmetric monoidal functor, we get a morphism of spectral local coefficient systems at $\mathcal{P}$:

$$\mathfrak{D}(M_1, \ldots, M_n) \Rightarrow \tilde{D}(M_1, \ldots, M_n)$$

which (up to natural isomorphism) is the universal map from the extension and restriction-of-scalars adjunction from $\mathbb{S} \to \tilde{\mathbb{Z}}$ (it's the smash product of the counits).

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Example 3.2.6 Let Σ_n be the symmetric group on n letters, and let $\tau \in \Sigma_n$. Reordering of smash factors defines a morphism of spectral local coefficient systems $\tau_* : \mathfrak{D}(M_1, \ldots, M_n) \to \mathfrak{D}(M_{\tau^{-1}(1)}, \ldots, M_{\tau^{-1}(n)})$, which in the case that $M_1 = \cdots = M_n$, makes $\mathfrak{D}(M_1, \ldots, M_n)$ into a Σ_n-spectral local coefficient system, and the morphism $\mathfrak{D}(M_1, \ldots, M_n) \Rightarrow \tilde{D}(M_1, \ldots, M_n)$ Σ_n-equivariant.

Remark The main point of spectral local coefficient systems is that they interact well with spectral homology of categories. Indeed, just as before, given a spectral local coefficient system D at an exact category $\mathcal{E}$, we can form a functor $F(-; D_{(-)}) : \mathcal{S}_\mathcal{E} \to \mathbb{S}\text{-BiMod}$ sending

$$\mathcal{C} \mapsto F(\mathcal{C}; D_\mathcal{C}) := |F_*(\mathcal{C}; D_\mathcal{C})|$$

Since the Eilenberg-Maclane functor $Ab \xrightarrow{H} \mathbb{S}\text{-BiMod}$ commutes with coproducts, given an exact category $\mathcal{E}$ and a local coefficient system D (of abelian groups) at $\mathcal{E}$, we have a canonical isomorphism of functors $\mathcal{S}_\mathcal{E} \to \mathbb{S}\text{-BiMod}$:

$$F(\widetilde{-; D_{(-)}}) \cong F(-; \tilde{D}_{(-)})$$

Furthermore, if D' is another local coefficient system at $\mathcal{E}$ and $\mathcal{H} : D \to D'$ is a morphism of spectral local coefficient systems, then $\mathcal{H}$ induces a natural transformation $F(-; D_{(-)}) \Rightarrow F(-; D'_{(-)})$ of functors $\mathcal{S}_\mathcal{E} \to \mathbb{S}\text{-BiMod}$. For example, in the case of 3.2.5 we obtain a natural transformation

$$F(-; \mathfrak{D}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F(-; \mathfrak{D}(M_{\tau^{-1}(1)} \ldots, M_{\tau^{-1}(n)})_{(-)})$$

such that, in the case that $M_1 = \cdots = M_n = M$, $F(-; \mathfrak{D}(M, \ldots, M)_{(-)})$ has a (right) Σ_n-action.

Example 3.2.7 We can develop in the spectrum case many of the same constructions from the algebraic case. We can define for an exact functor $T : \mathcal{E} \to \mathcal{E}'$ and a spectral local coefficient system over $\mathcal{E}'$, D', a pullback along T, $T^*(D')$, as a spectral local coefficient system over $\mathcal{E}$. Given two D and D' over $\mathcal{E}$ we can construct a "sum" spectral local coefficient system, $D \vee D'$ as in 2.2.9, and it interacts well with both pullbacks ($T^*(D \oplus D') \cong T^*(D) \oplus T^*(D')$) and spectral homology, i.e. $F_*(-; D_{(-)}) \vee F_*(-; D'_{(-)}) \cong F_*(-; D \vee D'_{(-)})$. We also have smash products of D and D', given by

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\Delta} (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \times (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \xrightarrow{D_\mathcal{C} \times D'_\mathcal{C}} \mathbb{S}-\text{BiMod} \times \mathbb{S}-\text{BiMod} \xrightarrow{\wedge} \mathbb{S}-\text{BiMod}$$

The main computation tool is the spectrum analogue of 2.2.13, whose proof in [46] translates directly to the spectrum case.

Proposition 3.2.8 *Let D be a spectral local coefficient system at $\mathcal{E}$. The natural transformation*

$$\delta : F_0(-; D_{(-)}) \to F_*(-; D_{(-)})$$

of functors $\mathcal{S}_{\mathcal{E}} \to s\mathbb{S}\text{-BiMod}$ given by inclusion of degeneracies is a stable equivalence. Specifically,

$$\delta(S_{\bullet}^{(n)}) : \mathfrak{d}F_0\left(S_{\bullet}^{(n)}(-); D_{S_{\bullet}^{(n)}(-)}\right) \to \mathfrak{d}F_*\left(S_{\bullet}^{(n)}(-); D_{S_{\bullet}^{(n)}(-)}\right)$$

is $2n-1$-connected.

With this result, we shift focus to the 0-simplices of the $F_{\bullet}$-construction; for example, $\underline{\textbf{Hom}}$ and $\underline{\textbf{Hom}}^M$ can be similarly rewritten. The following lemmas are proved by reducing to their algebraic counterparts (2.2.14, 2.2.15 and 2.2.16) through a spectral sequence argument (see Section 4.3 [31], specifically Corollary 4.22), or by direct inspection (for 3.2.12 and 3.2.13):

Lemma 3.2.9 *Let $X_{\bullet,\dots,\bullet}$ be a n-multi-simplicial $\mathbb{S}$-bimodule. Suppose that in simplicial direction i, X is m_i-reduced (that is, $X_{k_1,\dots,k_{i-1},k_i,k_{i+1},\dots,k_n} = *$ for $k_i \leq m_i$). Then $\mathfrak{d}X$ is $(n-1+\sum_{i=1}^{n} m_i)$-connected.*

Lemma 3.2.10 *Let $\mathcal{E}, \mathcal{E}'$ be exact categories and let D, D' be spectral local coefficient systems over them, respectively. Then*

$$\left(F_0(-; \pi_{\mathcal{E}}^*(D) \wedge \pi_{\mathcal{E}'}^*(D')_{(-)})\right)^{\text{st}}(\mathcal{E} \times \mathcal{E}') \simeq 0$$

Lemma 3.2.11 *If $\eta : F \Rightarrow G$ is natural transformation of functors $\textbf{Ext} \to \mathbb{S}\text{-BiMod}$ which is a point-wise weak equivalence, then so is $\eta^{\text{st}} : F^{\text{st}} \Rightarrow G^{\text{st}}$.*

Lemma 3.2.12 *If $\eta : F \Rightarrow G$ is natural transformation of functors $\textbf{Ext} \to \mathbb{S}\text{-BiMod}$ such that for some $\mathcal{E} \in \textbf{Ext}$ $\eta_{\mathcal{E}}^{\text{st}} : F^{\text{st}}(\mathcal{E}) \to G^{\text{st}}(\mathcal{E})$ is a weak equivalence, then $\eta^{\text{st}}|_{\mathcal{S}_{\mathcal{E}}}$ is point-wise weak equivalence.*

Lemma 3.2.13 *Let $\mathcal{I}$ be a small category, and $\mathcal{F} : \mathcal{I} \to \text{Func}_*\big(\textbf{Ext}, \mathbb{S}\text{-BiMod}\big)$ a functor. Then we have an isomorphism of $\mathbb{S}$-bimodules:*

$$\operatorname*{colim}_{i \in \mathcal{I}}\big((\mathcal{F}(i,-))^{\text{st}}(\mathcal{E})\big) \cong \big(\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{F}(i,-)\big)^{\text{st}}(\mathcal{E})$$

3.3 n-fold Smash Product

We now turn to the functor $F(-; \mathfrak{D}(M_1, \ldots, M_n)_{(-)}) : \mathcal{S}_\mathcal{P} \to \mathbb{S}\text{-BiMod}$ and understanding its stabilization at $\mathcal{P}$. By 3.2.8 we need only care about $\left(F_0(-; \mathfrak{D}(M_1, \ldots, M_n)_{(-)})\right)^{\text{st}}$, up to homotopy. The majority of the lemmas and propositions for $F(-; D(M_1, \ldots, M_n)_{(-)})$ carry through.

Definition 3.3.1 Let $R, M_1, \ldots, M_n$ be as before. Define a spectral local coefficient system, denoted $\mathfrak{D}^{1_n}(M_1, \ldots, M_n)$, at $\mathcal{P}^{\times n}$ by setting $\mathfrak{D}^{1_n}(M_1, \ldots, M_n)_{\mathcal{P}^{\times n}} : (\mathcal{P}^{\times n})^{\mathrm{op}} \times (\mathcal{P}^{\times n}) \cong (\mathcal{P}^{\mathrm{op}})^{\times n} \times (\mathcal{P}^{\times n}) \to \mathbb{S}\text{-BiMod}$

$$(C_1, \ldots, C_n) \times (C_1', \ldots, C_n') \mapsto \widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C_1, C_1' \underset{R}{\otimes} M_1) \wedge \cdots \wedge \widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C_n, C_n' \underset{R}{\otimes} M_n)$$

and extending by naturality to all of $\mathcal{S}_{\mathcal{P}^{\times n}}$. For $\sigma, \sigma' \in \Sigma_n$ we set $\mathfrak{D}^{\sigma,\sigma'}(M_1, \ldots, M_n)_{\mathcal{P}^{\times n}} := \mathfrak{D}^{1_n}(M_1, \ldots, M_n)_{\mathcal{P}^{\times n}} \circ \sigma \times \sigma'$ as in the algebraic case, and define a spectral local coefficient system at $\mathcal{P}^{\times n}$ by

$$\mathfrak{D}^{\Sigma_n}(M_1, \ldots, M_n) := \bigvee_{\sigma \in \Sigma_n} \mathfrak{D}^{1_n \times \sigma}(M_1, \ldots, M_n)$$

We again abuse of notation and write $\mathfrak{D}^\sigma(M_1, \ldots, M_n)$ in place of $\mathfrak{D}^{1_n \times \sigma}(M_1, \ldots, M_n)$.

As in the algebraic case, we now compare the spectral homologies $F(-; \mathfrak{D}(M_1, \ldots, M_n)_{(-)})$ and $F(-; \mathfrak{D}^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})$. Note that $\Delta^*(\mathfrak{D}^\sigma(M_1, \ldots, M_n)) \cong \mathfrak{D}(M_1, \ldots, M_n)$ as spectral local coefficient systems at $\mathcal{P}$, and so we get a natural transformation of functors $\mathcal{S}_\mathcal{P} \to \mathbb{S}\text{-BiMod}$:

$$F(-; (\mathfrak{D}(M_1, \ldots, M_n))_{(-)}) \Rightarrow F(\mathcal{S}_\Delta(-); \mathfrak{D}^\sigma(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})$$

We construct the diamond map $\diamond$ as before, taking note that it is now a zig-zag of maps

$$
\begin{aligned}
F(-; (\mathfrak{D}(M_1, \ldots, M_n))_{(-)}) &\Rightarrow \prod_{\sigma \in \Sigma_n} F(\mathcal{S}_\Delta(-); \mathfrak{D}^\sigma(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)}) \\
&\overset{\cong}{\Leftarrow} \bigvee_{\sigma \in \Sigma_n} F(\mathcal{S}_\Delta(-); \mathfrak{D}^\sigma(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)}) \\
&\cong F(\mathcal{S}_\Delta(-); \bigvee_{\sigma \in \Sigma_n} \mathfrak{D}^\sigma(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)}) \\
&= F(\mathcal{S}_\Delta(-); \mathfrak{D}^{\Sigma_n}(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)})
\end{aligned}
$$

Each of these maps is natural in $\mathcal{C} \in \mathcal{S}_\mathcal{P}$, and we denote the generalized map by $\diamond$ as in the algebraic case. Additonally, the maps of 2.3.4 generalize to the spectral setting, and

therefore $\tau \in \Sigma_n$ induces a natural transformation of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \to \mathbb{S}\text{-BiMod}$:

$$F(-;\mathfrak{D}^{\Sigma_n}(M_1,\ldots,M_n)_{(-)}) \Rightarrow F(-;\mathfrak{D}^{\Sigma_n}(M_{\tau^{-1}(1)},\ldots,M_{\tau^{-1}(n)})_{(-)})$$

such that $\diamond$ is Σ_n-equivariant when $M_1 = \cdots = M_n$.

Example 3.3.2 We construct a map backwards, Φ, as before however care is needed in re-defining it in the spectral setting. Making a choice of exact functor $\bigoplus : \mathcal{P}^{\times n} \to \mathcal{P}$ we define a morphism of spectral local coefficient systems (at $\mathcal{P}^{\times n}$), $\phi_\sigma : \mathfrak{D}^\sigma(M_1,\ldots,M_n) \to \bigoplus^*(\mathfrak{D}(M_1,\ldots,M_n))$ by

$$\wedge_{j=1}^n (\iota_{\sigma(j)})_* \circ \pi_j^*$$

on each $(C_1,\ldots,C_n) \times (C'_1 \ldots, C'_n)$ spectral summand. Here we're using the functoriality of the Eilenberg-MacLane construction. More generally, ϕ_σ is given by using the con/covariant directions of the "spectral" enrichment of the category. Taking coproducts we get a natural transformations of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \to \mathbb{S}\text{-BiMod}$:

$$F(-;(\mathfrak{D}^{\Sigma_n}(M_1,\ldots,M_n))_{(-)}) \Rightarrow \bigvee_{\sigma \in \Sigma_n} F(-;\bigoplus^*(\mathfrak{D}(M_1,\ldots,M_n))_{(-)}$$

$$\xrightarrow{\delta} F(-;\bigoplus^*(\mathfrak{D}(M_1,\ldots,M_n))_{(-)}$$

$$\to F(\mathcal{S}_\oplus(-);\mathfrak{D}(M_1,\ldots,M_n)_{\mathcal{S}_\oplus(-)})$$

where δ is the fold map. We denote this map by Φ, as it exactly mirrors the one from the algebraic case.

Proposition 3.3.3 *The generalized natural transformation $\diamond$ is stably a zig-zag of weak equivalences.*

Proof. This is the spectral analogue of 2.3.6. Though care is taken when distinguishing between products and coproducts, the proof mimics the algebraic one closely. We begin by defining a natural transformation of functors $\mathcal{S}_\mathcal{P} \to \mathbb{S}\text{-BiMod}$:

$$\rho : F_0(-;(\mathfrak{D}(M_1,\ldots,M_n))_{(-)}) \to F_0(S_{[2]}(-);(\mathfrak{D}(M_1,\ldots,M_n))_{S_{[2]}(-)})$$

and using the face maps in the $S_\bullet$-construction. Let $\mathcal{C} \in \mathcal{S}_\mathcal{P}$ and consider the following two exact functors: $\mathcal{H} : \mathcal{C} \to \mathcal{S}_{[2]}\mathcal{C}$ given by $C \mapsto \widehat{C}$ (as defined in 2.3.6) and $s_1 : \mathcal{C} \to S_{[2]}\mathcal{C}$ the 1st-degeneracy map. Let $1 \le i \le n$ and define the following natural transformations of

functors: $\mathcal{H} \overset{\mathfrak{p}_i}{\Rrightarrow} s_1 \overset{\mathfrak{e}}{\Rightarrow} \mathcal{H}$ given by:

$$
\begin{array}{ccc}
C & =\!=\!=\!= & C \\
{\scriptstyle\Delta}\downarrow & & \| \\
C^{\oplus n} & \overset{\pi_i}{\longrightarrow} & C \\
{\scriptstyle\pm}\downarrow & & \downarrow \\
C^{\oplus(n-1)} & \overset{0}{\longrightarrow} & 0
\end{array}
\qquad\qquad
\begin{array}{ccc}
C & =\!=\!=\!= & C \\
\| & & {\scriptstyle\Delta}\downarrow \\
C & \overset{\Delta}{\longrightarrow} & C^{\oplus n} \\
\downarrow & & {\scriptstyle\pm}\downarrow \\
0 & \overset{0}{\longrightarrow} & C^{\oplus(n-1)}
\end{array}
$$

respectively. Since $\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(-, \star \underset{R}{\otimes} M)$ is a spectral local coefficient system at $\mathcal{P}$, we get, for $C \in \mathcal{C} \in \mathcal{S}_{\mathcal{P}}$, a sequence of maps:

$$
\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M) \to \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), s_1(C \underset{R}{\otimes} M))
$$

$$
\overset{(\mathfrak{e}\otimes \mathrm{id}_M)_*}{\longrightarrow} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), \widehat{C} \underset{R}{\otimes} M)
$$

$$
\overset{(\mathfrak{p}_i)^*}{\longrightarrow} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(\widehat{C}, \widehat{C} \underset{R}{\otimes} M)
$$

which in the algebraic case is precisely the map $\alpha \mapsto \widehat{\alpha}$. Smashing together the maps gives $\rho_{\mathcal{C}}$:

$$
F_0(\mathcal{C}; (\mathfrak{D}(M_1, \dots, M_n))_{\mathcal{C}}) := \bigvee_{C\in\mathrm{Obj}(\mathcal{C})} \mathfrak{D}(M_1, \dots, M_n)_{\mathcal{C}}(C, C)
$$

$$
= \bigvee_{C\in\mathrm{Obj}(\mathcal{C})} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_1) \wedge \cdots \wedge \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_n)
$$

$$
\overset{\bigvee \widehat{s_1}}{\longrightarrow} \bigvee_{C\in\mathrm{Obj}(\mathcal{C})} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), s_1(C \underset{R}{\otimes} M_1)) \wedge \cdots \wedge \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), s_1(C \underset{R}{\otimes} M_n))
$$

$$
\overset{\bigvee(\bigwedge_i (\mathfrak{e}\otimes \mathrm{id}_{M_i})_*)}{\longrightarrow} \bigvee_{C\in\mathrm{Obj}(\mathcal{C})} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), \widehat{C} \underset{R}{\otimes} M_1) \wedge \cdots \wedge \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), \widehat{C} \underset{R}{\otimes} M_n)
$$

$$
\overset{\bigvee(\bigwedge_i (\mathfrak{p}_i)^*)}{\longrightarrow} \bigvee_{C\in\mathrm{Obj}(\mathcal{C})} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(\widehat{C}, \widehat{C} \underset{R}{\otimes} M_1) \wedge \cdots \wedge \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(\widehat{C}, \widehat{C} \underset{R}{\otimes} M_n)
$$

$$
\to F_0(S_{[2]}\mathcal{C}; (\mathfrak{D}(M_1, \dots, M_n))_{S_{[2]}\mathcal{C}})
$$

Let $d_1 : F_0(S_{[2]}\mathcal{C}; \mathfrak{D}(M_1, \dots, M_n)_{S_{[2]}\mathcal{C}}) \to F_0(\mathcal{C}; \mathfrak{D}(M_1, \dots, M_n)_{\mathcal{C}})$ be the 1$^{\text{st}}$ face map of the

$S_\bullet$-construction. Composing with $\rho_{\mathcal{C}}$ gives a self-map

$$F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}}) \xrightarrow{d_1 \circ \rho_{\mathcal{C}}} F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}})$$

Because of the naturality of d_1 we can identify $\rho_{\mathcal{C}}$ on each smash factor of each summand. Indeed, on the ith smash factor we get a map

$$\widetilde{Hom}_{\textbf{Mod-}R}(C, C \underset{R}{\otimes} M_i) \to \widetilde{Hom}_{\textbf{Mod-}R}(C^{\oplus n}, (C \underset{R}{\otimes} M_i)^{\oplus n})$$

weakly equivalent to:

$$\widetilde{Hom}_{\textbf{Mod-}R}(C, C \underset{R}{\otimes} M_i) \xrightarrow{\Delta} \prod_{j=1}^n \widetilde{Hom}_{\textbf{Mod-}R}(C, C \underset{R}{\otimes} M_i)$$

$$\xleftarrow{\simeq} \bigvee_{j=1}^n \widetilde{Hom}_{\textbf{Mod-}R}(C, C \underset{R}{\otimes} M_i)$$

$$\xrightarrow{\vee(\pi_i)^*} \bigvee_{j=1}^n \widetilde{Hom}_{\textbf{Mod-}R}(C^{\oplus n}, C \underset{R}{\otimes} M_i)$$

$$\xrightarrow{\vee(\iota_j)_*} \bigvee_{j=1}^n \widetilde{Hom}_{\textbf{Mod-}R}(C^{\oplus n}, (C \underset{R}{\otimes} M_i)^{\oplus n})$$

$$\xrightarrow{\delta} \widetilde{Hom}_{\textbf{Mod-}R}(C^{\oplus n}, (C \underset{R}{\otimes} M_i)^{\oplus n})$$

where δ is the fold map. Let $\mathcal{M}_n$ be the set of all functions $\{1, \ldots, n\} \to \{1, \ldots, n\}$. By distributivity, we have:

$$\bigwedge_{i=1}^n \big(\bigvee_{j=1}^n \widetilde{Hom}_{\textbf{Mod-}R}(C, C \otimes M_i) \big) \simeq \bigvee_{\lambda \in \mathcal{M}_n} \widetilde{Hom}_{\textbf{Mod-}R}(C, C \otimes M_1) \wedge \cdots \wedge \widetilde{Hom}_{\textbf{Mod-}R}(C, C \otimes M_n)$$

and similarly:

$$\bigwedge_{i=1}^n \big(\bigvee_{j=1}^n (\pi_i)^* \circ (\iota_j)_* \big) = \bigwedge_{i=1}^n \big(\bigvee_{j=1}^n (\iota_j)_* \circ (\pi_i)^* \big)$$

$$\simeq \bigvee_{\lambda \in \mathcal{M}_n} (\iota_{\lambda(1)})_* \circ (\pi_1)^* \wedge \cdots \wedge (\iota_{\lambda(n)})_* \circ (\pi_n)^*$$

$$=: \bigvee_{\lambda \in \mathcal{M}_n} \mathfrak{l}_\lambda$$

Hence $d_1 \circ \rho_\mathcal{C}$ is weakly equivalent to the generalized map $\delta \circ \bigvee_{\lambda \in \mathcal{M}_n} \mathfrak{l}_\lambda \circ \Delta$. We now show that if $\lambda \in \mathcal{M}_n$ is not surjective, $\mathfrak{l}_\lambda$ is stably contractible. Indeed, for such a λ, let $u \in \{1, \ldots, n\}$ not be in the image. Let $\mathcal{H}_u : \mathcal{C} \to S_{[2]}\mathcal{C}$ be the exact functor given by $C \mapsto C^{\bowtie}$ where $C^{\bowtie}$ is the short exact sequence $C^{\oplus(n-1)} \rightarrowtail C^{\oplus n} \twoheadrightarrow C$ where the first map is inclusion into the non-u coordinates, and the second is projection onto the uth-coordinate. Let $1 \le i \le n$ and define the following natural transformations of functors: $\mathcal{H}_u \overset{\mathfrak{p}_i^u}{\Rightarrow} s_1 \overset{\mathfrak{e}^u}{\Rightarrow} \mathcal{H}_u$ given by:

$$
\begin{array}{ccc}
C^{\oplus(n-1)} & \xrightarrow{\pi_{\bar{i}}} & C \\
\downarrow & & \| \\
C^{\oplus n} & \xrightarrow{\pi_i} & C \\
\pi_u \downarrow & & \downarrow \\
C & \xrightarrow{\;\;0\;\;} & 0
\end{array}
\qquad\qquad
\begin{array}{ccc}
C & \xrightarrow{\iota_{\bar{\lambda}(i)}} & C^{\oplus(n-1)} \\
\| & & \downarrow \\
C & \xrightarrow{\iota_{\lambda(i)}} & C^{\oplus n} \\
\downarrow & & \pi_u \downarrow \\
0 & \xrightarrow{\;\;0\;\;} & C
\end{array}
$$

where $\bar{i}$ is i if $i < u$, $i - 1$ if $i > u$, and -1 if $i = u$, where we interpret π_{-1} as the zero map. Since $\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(-, \star \underset{R}{\otimes} M)$ is a spectral local coefficient system at $\mathcal{P}$, we get, for $C \in \mathcal{C} \in \mathcal{S}_\mathcal{P}$, a sequence of maps:

$$
\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M) \to \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), s_1(C \underset{R}{\otimes} M))
$$
$$
\xrightarrow{(\mathfrak{e}^u \otimes \mathrm{id}_M)_*} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(s_1(C), C^{\bowtie} \underset{R}{\otimes} M)
$$
$$
\xrightarrow{(\mathfrak{p}_i^u)^*} \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(C^{\bowtie}, C^{\bowtie} \underset{R}{\otimes} M)
$$

which in the algebraic case is precisely the map $\alpha \mapsto \bar{\alpha}$. Smashing together the maps for $M_1, \ldots, M_n$ gives a morphism

$$
\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_1) \wedge \cdots \wedge \widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_n)
$$
$$
\to \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(C^{\bowtie}, C^{\bowtie} \underset{R}{\otimes} M_1) \wedge \cdots \wedge \widetilde{Hom}_{S_{[2]}\mathbf{Mod}\text{-}R}(C^{\bowtie}, C^{\bowtie} \underset{R}{\otimes} M_n)
$$

and therefore a map $\rho^\lambda : F_0(\mathcal{C}; (\mathfrak{D}(M_1, \ldots, M_n))_\mathcal{C}) \to F_0(S_{[2]}\mathcal{C}; (\mathfrak{D}(M_1, \ldots, M_n))_{S_{[2]}\mathcal{C}})$. Com-

posing with d_1 we get a morphism $F_0(\mathcal{C}; (\mathfrak{D}(M_1, \ldots, M_n))_\mathcal{C}) \to F_0(\mathcal{C}; (\mathfrak{D}(M_1, \ldots, M_n))_\mathcal{C})$ which on the C-summand is weakly equivalent to $\mathfrak{l}_\lambda$. Now, consider the following diagram of natural transformations of functors $\mathcal{S}_\mathcal{P} \to \mathbb{S}\text{-BiMod}$:

$$F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_\mathcal{C}) \xrightarrow{\rho^\lambda} F_0(S_{[2]}\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{S_{[2]}\mathcal{C}}) \xrightarrow{(d_2, d_0)} F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_\mathcal{C})^{\times 2}$$

$$\Big\downarrow{\scriptstyle d_1} \qquad\qquad\qquad \Big\uparrow{\scriptstyle \simeq}$$

$$F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_\mathcal{C}) \xleftarrow{\quad\delta\quad} \bigvee_{s=1}^{2} F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_\mathcal{C})$$

Consider the composite $(d_2, d_0) \circ \rho^\lambda$. Projection onto the second factor gives a map, which on the ith-smash factor of the C-summand, is the composite $d_0 \circ (\mathfrak{p}_i^u)^* \circ (\mathfrak{e}^u \otimes \text{id}_{M_i})_* \circ \hat{s}_1$: $\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_i) \to \widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_i)$. Horizontal composition with the natural transformations

$$\mathcal{C} \quad \overset{\mathcal{H}_u}{\underset{s_1}{\mathfrak{p}_i^u \Downarrow}} \quad S_{[2]}\mathcal{C} \xrightarrow{d_0} \mathcal{C} \qquad\qquad\qquad \mathcal{C} \quad \overset{s_1}{\underset{\mathcal{H}_u}{\mathfrak{e}^u \Downarrow}} \quad S_{[2]}\mathcal{C} \xrightarrow{d_0} \mathcal{C}$$

shows that

$$
\begin{aligned}
d_0 \circ (\mathfrak{p}_i^u)^* \circ (\mathfrak{e}^u \otimes \text{id}_{M_i})_* \circ \hat{s}_1 &= (d_0 \circ_h \mathfrak{p}_i^u)^* \circ d_0 \circ (\mathfrak{e}^u \otimes \text{id}_{M_i})_* \circ \hat{s}_1 \\
&= (d_0 \circ_h \mathfrak{p}_i^u)^* \circ (d_0 \circ_h \mathfrak{e}^u \otimes \text{id}_{M_i})_* \circ d_0 \circ \hat{s}_1 \\
&= (d_0 \circ_h \mathfrak{p}_i^u)^* \circ (d_0 \circ_h \mathfrak{e}^u \otimes \text{id}_{M_i})_* \circ 0 \\
&= 0
\end{aligned}
$$

Thus $\pi_2 \circ (d_2, d_0) \circ \rho^\lambda$ is the zero map. On the other hand, $\pi_1 \circ (d_2, d_0) \circ \rho^\lambda$ is, on the uth-smash factor of the C-summand, the composite $d_2 \circ (\mathfrak{p}_u^u)^* \circ (\mathfrak{e}^u \otimes \text{id}_{M_u})_* \circ \hat{s}_1$. Now, $d_2 \circ (\mathfrak{p}_u^u)^* = (d_2 \circ_h \mathfrak{p}_u^u)^* \circ d_2$ and $d_2 \circ_h \mathfrak{p}_u^u$ is the natural transformation between $d_2 \circ \mathcal{H}_u : C \mapsto C^{\oplus(n-1)}$ and $d_2 \circ \hat{s}_1 = id_\mathcal{C}$ given by $C^{\oplus(n-1)} \xrightarrow{0} C$. Since the uth-factor is the zero map, and the n-fold smash is reduced in each variable, the induced map is on each summand is the zero map. Therefore $\pi_1 \circ (d_2, d_0) \circ \rho^\lambda$ is the zero map, and hence $(d_2, d_0) \circ \rho^\lambda$ is also. Since the above square commutes in the homotopy category after stabilization, we see by 3.2.11 that $(\mathfrak{l}_\lambda)^{\text{st}} \simeq (d_1 \circ \rho^\lambda)^{\text{st}} \simeq 0$.

Therefore,

$$(d_1 \circ \rho_{\mathcal{C}})^{\mathrm{st}} \simeq (\delta \circ \bigvee_{\lambda \in \mathcal{M}_n} \mathfrak{l}_\lambda \circ \Delta)^{\mathrm{st}} \simeq (\delta)^{\mathrm{st}} \circ \bigvee_{\lambda \in \mathcal{M}_n} (\mathfrak{l}_\lambda)^{\mathrm{st}} \circ (\Delta)^{\mathrm{st}} \simeq (\delta)^{\mathrm{st}} \circ \bigvee_{\lambda \in \Sigma_n} (\mathfrak{l}_\lambda)^{\mathrm{st}} \circ (\Delta)^{\mathrm{st}}$$

By checking on each C-summand, we see that the effect on F_0 of the zig-zag $\delta \circ \bigvee_{\lambda \in \Sigma_n} \mathfrak{l}_\lambda \circ \Delta$ is the same as the zig-zag $\Phi \circ \diamond$:

$$
\begin{aligned}
F_0(\mathcal{C}; (\mathfrak{D}(M_1, \ldots, M_n))_{\mathcal{C}}) &\Rightarrow \prod_{\sigma \in \Sigma_n} F_0(\mathcal{C}^{\times n}; \mathfrak{D}^\sigma(M_1, \ldots, M_n)_{\mathcal{C}^{\times n}}) \\
&\stackrel{\simeq}{\Leftarrow} \bigvee_{\sigma \in \Sigma_n} F_0(\mathcal{C}^{\times n}; \mathfrak{D}^\sigma(M_1, \ldots, M_n)_{\mathcal{C}^{\times n}}) \\
&\Rightarrow \bigvee_{\sigma \in \Sigma_n} F_0(\mathcal{C}^{\times n}; \bigoplus^* (\mathfrak{D}(M_1, \ldots, M_n))_{\mathcal{C}^{\times n}}) \\
&\stackrel{\delta}{\to} F_0(\mathcal{C}^{\times n}; \bigoplus^* (\mathfrak{D}(M_1, \ldots, M_n))_{\mathcal{C}^{\times n}}) \\
&\to F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}})
\end{aligned}
$$

Therefore $(d_1 \circ \rho_{\mathcal{C}})^{\mathrm{st}} \simeq (\Phi \circ \diamond)^{\mathrm{st}}$ by 3.2.11. The stabilization of of $d_1 \circ \rho_{\mathcal{C}}$ is easily analyzed. Indeed, consider the following diagram of natural transformations of functors $\mathcal{S}_{\mathcal{P}} \to \mathbb{S}$-BiMod:

$$
\begin{array}{ccccc}
F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}}) & \xrightarrow{\rho_{\mathcal{C}}} & F_0(S_{[2]}\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{S_{[2]}\mathcal{C}}) & \xrightarrow{(d_2, d_0)} & F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}})^{\times 2} \\
& & \downarrow{d_1} & & \uparrow{\simeq} \\
& & F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}}) & \xleftarrow{\delta} & \bigvee_{s=1}^2 F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}})
\end{array}
$$

Consider the composite $(d_2, d_0) \circ \rho_{\mathcal{C}}$. Projection onto the second factor gives a map, which on the ith-smash factor of the C-summand, is the composite $d_0 \circ (\mathfrak{p}_i)^* \circ (\mathfrak{e} \otimes \mathrm{id}_{M_i})_* \circ \hat{s}_1 :$ $\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_i) \to \widetilde{Hom}_{\mathbf{Mod}\text{-}R}(C, C \underset{R}{\otimes} M_i)$. Doing horizontal composition with the first pair of natural transformations

shows that this map is the zero map (similar to the case of ρ_λ). The first projection, on the

i^{th}-smash factor of the C-summand, is the composite $d_2 \circ (\mathfrak{p}_i)^* \circ (\mathfrak{e} \otimes \mathrm{id}_{M_i})_* \circ \hat{s}_1$ and we have

$$
\begin{aligned}
d_2 \circ (\mathfrak{p}_i)^* \circ (\mathfrak{e} \otimes \mathrm{id}_{M_i})_* \circ \hat{s}_1 &= (d_2 \circ_h \mathfrak{p}_i)^* \circ d_2 \circ (\mathfrak{e} \otimes \mathrm{id}_{M_i})_* \circ \hat{s}_1 \\
&= (d_2 \circ_h \mathfrak{p}_i)^* \circ (d_2 \circ_h \mathfrak{e} \otimes \mathrm{id}_{M_i})_* \circ d_2 \circ \hat{s}_1 \\
&= (d_2 \circ_h \mathfrak{p}_i)^* \circ (d_2 \circ_h \mathfrak{e} \otimes \mathrm{id}_{M_i})_* \circ id_{\mathcal{C}} \\
&= id
\end{aligned}
$$

by inspection. Therefore the map $(d_2, d_0) \circ \rho_{\mathcal{C}}$ factors through as

$$
F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}}) \xrightarrow{q} \bigvee_{s=1}^{2} F_0(\mathcal{C}; \mathfrak{D}(M_1, \ldots, M_n)_{\mathcal{C}})
$$

such that $\delta \circ q = id$. Using 3.2.11 again, since the above square commutes in the homotopy category after stabilization, we see that $(\Phi \circ \diamond)^{\text{st}} \simeq (d_1 \circ \rho_{\mathcal{C}})^{\text{st}} \simeq (id)^{\text{st}} \simeq id$. $\qquad \square$

Remark By 3.3.3, we have, when $M_1 = \cdots = M_n = M$,

$$
(F(-; (\mathfrak{D}(M_1, \ldots, M_n))_{(-)}))^{\text{st}} \simeq_{f \Sigma_n} (F(\mathcal{S}_\Delta(-); \mathfrak{D}^{\Sigma_n}(M_1, \ldots, M_n)_{\mathcal{S}_\Delta(-)}))^{\text{st}}
$$

at $\mathcal{P}$. Thus, we will now concentrate on analyzing the latter in much the same manner as was done in the algebraic case.

Definition 3.3.4 Let $R, M_1, \ldots, M_n$ and $\mathcal{S}_n \subset \Sigma_n$ be as before. We can define

$$
\mathfrak{D}^{\mathcal{S}_n}(M_1, \ldots, M_n) := \bigvee_{\sigma \in \mathcal{S}_n} \mathfrak{D}^{1_n \times \sigma}(M_1, \ldots, M_n)
$$

as a spectral local coefficient system (at $\mathcal{P}^{\times n}$) as before, with the canonical inclusion

$$
\mathfrak{D}^{\mathcal{S}_n}(M_1, \ldots, M_n) \hookrightarrow \mathfrak{D}^{\Sigma_n}(M_1, \ldots, M_n)
$$

inducing an injective natural transformation of functors $\mathcal{S}_{\mathcal{P}^{\times n}} \to \mathbb{S}\text{-BiMod}$:

$$
F(-; \mathfrak{D}^{\mathcal{S}_n}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F(-; \mathfrak{D}^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})
$$

Again we have that for each $\tau \in \Sigma_n$ the maps τ_* descend to make the aforementioned inclusion Σ_n-equivariant when $M_1 = \cdots = M_n,$.

Proposition 3.3.5 *The natural transformation of functors $\mathcal{S}_{\mathcal{P}} \to \mathbb{S}$-BiMod*

$$F(-; \mathfrak{D}^{\mathcal{S}_n}(M_1, \ldots, M_n)_{(-)}) \Rightarrow F(-; \mathfrak{D}^{\Sigma_n}(M_1, \ldots, M_n)_{(-)})$$

is a weak equivalence after stabilization.

Proof. The proof is identical to the algebraic one. First, by 3.2.8 we need only case about the stabilization of the degree 0-simplices. Second, by 3.2.13 we can equivalently show that for $\sigma \in \Sigma_n \setminus \mathcal{S}_n$, $F_0(-; \mathfrak{D}^{\sigma}(M_1, \ldots, M_n)_{(-)})$ is stably contractible. If $\sigma \in \Sigma_n \setminus \mathcal{S}_n$, then there exist distinct $\tau, \tau' \in \Sigma_n$, disjoint cycles of shorter length, such that $\sigma = \tau \cdot \tau'$. We then have an exact natural isomorphism $A : \mathcal{P}^{\times n} \cong \mathcal{P}^{\times s} \times \mathcal{P}^{\times s'}$ such that $A \circ \sigma = \mathrm{id}_{\mathcal{P}^{\times s} \times \mathcal{P}^{\times s'}}$ which extends to a natural isomorphism $\mathcal{S}_A : \mathcal{S}_{\mathcal{P}^{\times n}} \cong \mathcal{S}_{\mathcal{P}^{\times s}} \times \mathcal{S}_{\mathcal{P}^{\times s'}}$. We have a natural isomorphism of functors

$$F(-; \mathfrak{D}^{\sigma}(M_1, \ldots, M_n)_{(-)}) \cong F(-; \pi_1^*(\mathfrak{D}^{1_s}(M_{a_1}, \ldots, M_{a_s})) \wedge \pi_2^*(\mathfrak{D}^{1_{s'}}(M_{b_1}, \ldots, M_{b_{s'}}))_{(-)})$$

where $u = \{a_1, \ldots, a_s\}$ and $u' = \{b_1, \ldots, b_{s'}\}$ defined analogously, and π_i are the projections as before. By 3.2.10 and 3.2.12 we get our desired conclusion. $\square$

Remark Let $\omega = (n \ldots 2 1) \in \Sigma_n$ be as before. When $M_1 = \cdots = M_n$ the local coefficient system $\mathfrak{D}^{\omega}(M_1, \ldots, M_n)$ acquires a C_n-action, and so $F(\mathcal{C}; \mathfrak{D}^{\omega}(M_1, \ldots, M_n)_{\mathcal{C}})$ becomes a simplicial $\mathbb{C}_n$-module (recall from Appendix E.2.4 that $\mathbb{C}_n$ is the FSP with associated spectrum $\underline{\mathbb{C}_n} \cong \Sigma_+^{\infty} C_n$). The C_n-equivariant inclusion map $F(\mathcal{C}; \mathfrak{D}^{\omega}(M_1, \ldots, M_n)_{\mathcal{C}}) \hookrightarrow F(\mathcal{C}; \mathfrak{D}^{\mathcal{S}_n}(M_1, \ldots, M_n)_{\mathcal{C}})$ and the restriction/extension-of-scalars adjunction gives a map of (right) Σ_n-modules

$$F(\mathcal{C}; (\mathfrak{D}^{\omega}(M_1, \ldots, M_n))_{\mathcal{C}}) \wedge_{\mathbb{C}_n} \Sigma_n \xrightarrow{\nu} F(\mathcal{C}; \mathfrak{D}^{\mathcal{S}_n}(M_1, \ldots, M_n)_{\mathcal{C}})$$

which is natural in $\mathcal{C} \in \mathcal{S}_{\mathcal{P}^{\times n}}$. The following claim is easily checked:

Claim 3.3.6 The map of right Σ_n-modules ν is an isomorphism.

Combining this with 3.2.13 it follows that we have a Σ_n-isomorphism of the associated stabilizations:

$$\left(F(-; (\mathfrak{D}^{\omega}(M_1, \ldots, M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{C}) \wedge_{\mathbb{C}_n} \Sigma_n \simeq \left(F(-; (\mathfrak{D}^{\mathcal{S}_n}(M_1, \ldots, M_n))_{(-)})\right)^{\mathrm{st}}(\mathcal{C})$$

Altogether, in the case when $M_1 = \cdots = M_n$, we get the following chain of feeble Σ_n-weak

equivalences:

$$\left(F(-;(\mathfrak{D}(M_1,\ldots,M_n))_{(-)})\right)^{\text{st}}(\mathcal{P}) \simeq \left(F(-;(\mathfrak{D}^{\Sigma_n}(M_1,\ldots,M_n))_{(-)})\right)^{\text{st}}(\mathcal{P}^{\times n})$$
$$\simeq \left(F(-;(\mathfrak{D}^{\mathcal{S}_n}(M_1,\ldots,M_n))_{(-)})\right)^{\text{st}}(\mathcal{P}^{\times n})$$
$$\simeq \left(F(-;(\mathfrak{D}^{\omega}(M_1,\ldots,M_n))_{(-)})\right)^{\text{st}}(\mathcal{P}^{\times n}) \wedge_{\mathbb{C}_n} \Sigma_n$$

Notation The spectral homology of $\mathcal{P}^{\times n}$ with coefficients in $\mathfrak{D}^{\omega}(M_1,\ldots,M_n)$ is weakly equivalent to a known functor. Shifting momentarily to the notation of [34], let $P(i)$ for $i = 1,\ldots,n$ be the $\underline{\mathcal{P}}$-bimodules (=$\tilde{\mathcal{P}}$-bimodules in our notation) given, for $C, C' \in \mathcal{P}$, by

$$P(i)_{C,C'} := \widetilde{\widehat{Hom}}_{\mathbf{Mod}\text{-}R}(C, C' \underset{R}{\otimes} M_i)$$

We have a zig-zig of weak equivalences

$$F_0(\mathcal{P}^{\times n}; \mathfrak{D}^{\omega}(M_1,\ldots,M_n)) \simeq U_0^n(\underline{\mathcal{P}}; P(1),\ldots,P(n))$$

which is C_n-equivariant when $M_1 = \cdots = M_n$. Indeed, the zig-zag of equivalences is given by Shipley's detection functor D (see Theorem 3.1.6 [58]). We thus have:

$$\left(F(-;(\mathfrak{D}^{\omega}(M_1,\ldots,M_n))_{(-)})\right)^{\text{st}}(\mathcal{P}^{\times n}) \simeq \left(F_0(-;\mathfrak{D}^{\omega}(M_1,\ldots,M_n)_{(-)})\right)^{\text{st}}(\mathcal{P}^{\times n})$$
$$\simeq \underset{k}{\text{hocolim}}\, \Omega^{(k)}\left(\partial U_0^n\left(\underline{S_\bullet^{(k)}\mathcal{P}}; S_\bullet^{(k)}P(1),\ldots,S_\bullet^{(k)}P(n)\right)\right)$$
$$\simeq \underset{k}{\text{hocolim}}\, \Omega^{(k)}\left(\partial U^n\left(\underline{S_\bullet^{(k)}\mathcal{P}}; S_\bullet^{(k)}P(1),\ldots,S_\bullet^{(k)}P(n)\right)\right)$$
$$\simeq U^n(\underline{\mathcal{P}}; P(1),\ldots,P(n))$$

where the second and third weak equivalences come from Corollary 6.17 and Proposition 6.14 in [34], respectively. When $M_1 = \cdots = M_n = M$, we have a C_n-weak equivalence $U^n(\underline{\mathcal{P}}; P(1),\ldots,P(n)) \simeq U^n(R; M)$ by Proposition 6.13 [34].

We have shown:

Theorem 3.3.7 *Let R be an associative and unital ring, M an R-bimodule, and $n \geq 1$. There is an equivalence of Σ_n-bimodules*

$$\left(F(-;(\mathfrak{D}(M,\ldots,M))_{(-)})\right)^{\text{st}}(\mathcal{P}) \simeq U^n(R; M) \wedge_{\mathbb{C}_n} \Sigma_n$$

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CHAPTER 4

K-THEORY WITH COEFFICIENTS

In this chapter we reinterpret the F-construction used in the definition of spectral homology to incorporate "twists" by endofunctors of spectra. After reviewing the case of homogeneous functors in the sense of Goodwillie, we recast the main results of [34] as results about stabilization of towers. Finally, we discuss an approach to tackle the case of general polynomial functors.

4.1   Endofunctor Coefficients

Let $\mathcal{C}$ be a small category and let $D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbb{S}\text{-BiMod}$ be a (globally stable) bi-functor. If $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ (see Appendix C.3 for details) we may form a new (globally stable) bi-functor: $(C, C') \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \mapsto \mathfrak{F}(D(C, C'))$. We can thus form a simplicial FST, denoted $F_*(\mathcal{C}; D; \mathfrak{F})$, just as in 3.2.1 with p-simplices:

$$F_p(\mathcal{C}; D; \mathfrak{F}) = \bigvee_{\vec{C} \in N_p \mathcal{C}} \mathfrak{F}(D(C_0, C_p))$$

and face and degeneracies given just as before but with $\mathfrak{F}$ acting on the $D(\alpha_1, \mathrm{id}_{C_p})$ and $D(\mathrm{id}_{C_0}, \alpha_p)$. The geometric realization is then simply the spectral homology of $\mathcal{C}$ with coefficients in $\mathfrak{F} \circ D$. Similarly, if $\mathcal{E}$ is an exact category and D is a spectral local coefficient system at $\mathcal{E}$, the composite $\mathfrak{F} \circ D$ is as well. Some known coefficient systems arise in this manner:

Example 4.1.1 Let R, M, and $\mathcal{P}$ be as in 2.1.7. Let $\mathcal{C} \in \mathcal{S}_{\mathcal{P}}$, and consider the spectral local coefficient system at $\mathcal{P}$ from 3.2.5. Let $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ be the n-fold (derived) smash product:

$$X \mapsto X^{\wedge_{\mathbb{S}} n} = \overbrace{X \wedge \cdots \wedge X}^{n-\text{times}}$$

Then we have $\mathfrak{F} \circ \mathfrak{D}(M) \cong \mathfrak{D}(\overbrace{M, \ldots, M}^{n-\text{times}})$ as spectral local coefficient systems at $\mathcal{P}$. Further-more, reordering smash factors makes $\mathfrak{F}$ a (right) Σ_n-object in $Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$, and thus gives $\mathfrak{F} \circ \mathfrak{D}(M)$ an induced (right) Σ_n-action which agrees with 3.2.6, under the previous isomorphism.

Notation If $\mathcal{C}$ is a "spectrally-enriched" category (given by a fixed FSP $\mathfrak{C}$ over $\mathcal{C}$) we will write $F(\mathcal{C}; \mathfrak{F})$ in place of $F(\mathcal{C}; \mathfrak{C}; \mathfrak{F})$ when no confusion will arise (see 3.2.4). In particular, if $\mathcal{E}$ is an exact category with the natural Eilenberg-Maclane FSP over it ($\tilde{\mathcal{E}} = \widetilde{Hom}$), $F(\mathcal{E}; \mathfrak{F})$ will denote $F(\mathcal{E}; \widetilde{Hom}; \mathfrak{F})$. On the other hand, when using a specified bimodule $\mathfrak{D}$ over the FSP $\mathfrak{C}$, we denote that by $F(\mathcal{C}; \mathfrak{D}; \mathfrak{F})$.

Definition 4.1.2 Let $\mathcal{E}$ be an exact category and $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$. The K-theory of $\mathcal{E}$ with coefficients in $\mathfrak{F}$, denoted $K(\mathcal{E}; \mathfrak{F})$, is the $\mathbb{S}$-bimodule given by the stabi-lization:

$$K(\mathcal{E}; \mathfrak{F}) := \left(F(-; \mathfrak{F})\right)^{\mathrm{st}}(\mathcal{E})$$

Notation In accordance with the previous notation, if we wish to work with a different choice of bimodule $\mathfrak{D}$ over the FSP $\widetilde{Hom}$, we will use $K(\mathcal{E}; \mathfrak{D}; \mathfrak{F}) := \left(F(-; \mathfrak{D}; \mathfrak{F})\right)^{\mathrm{st}}(\mathcal{E})$.

Observation The definition of $K(\mathcal{E}; \mathfrak{F})$ is functorial in both arguments. Indeed, if $\eta : \mathfrak{F} \to \mathfrak{F}'$ is a natural transformation in $Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ it induces a morphism of spec-tral local coefficient systems $\mathfrak{F} \circ \tilde{\mathcal{E}} \to \mathfrak{F}' \circ \tilde{\mathcal{E}}$ and therefore a map on spectral homolo-gies, $K(\mathcal{E}; \mathfrak{F}) \to K(\mathcal{E}; \mathfrak{F}')$. Also, if $T : \mathcal{E} \to \mathcal{E}'$ is an exact functor, the spectral lo-cal coefficient system $\mathfrak{F} \circ \tilde{\mathcal{E}}$ induces a natural transformation of functors $\mathcal{S}_{\mathcal{E}} \to \mathbb{S}\text{-BiMod}$, $F(-; (\mathfrak{F} \circ \widetilde{Hom})_{(-)}) \Rightarrow F(\mathcal{S}_T(-); (\mathfrak{F} \circ \widetilde{Hom})_{\mathcal{S}_T(-)})$. Stabilizing and evaluating at $\mathcal{E}$, we get our desired map $K(\mathcal{E}; \mathfrak{F}) \to K(\mathcal{E}'; \mathfrak{F})$.

Remark The definition uses the stabilization of $F(-; \mathfrak{F})$ as a functor $\mathcal{S}_{\mathcal{E}} \to \mathbb{S}\text{-BiMod}$. There-fore by 3.2.8, $K(\mathcal{E}; \mathfrak{F})$ can just as well be defined using $\left(F_0(-; \mathfrak{F})\right)^{\mathrm{st}}(\mathcal{E})$ instead. However, we will have occasion to use *non-reduced* homotopy functors $\mathfrak{F}$ as our endofunctor coefficients. In such cases, the inclusion by degeneracies $\delta : F_0 \hookrightarrow F$ need not be an equivalence, and so we define:

$$\tilde{K}(\mathcal{E}; \mathfrak{F}) := \mathrm{hocofiber}\left(K(0; \mathfrak{F}) \to K(\mathcal{E}; \mathfrak{F})\right)$$

induced from the unique exact functor $0 \to \mathcal{E}$. Since $F_*(0; \mathfrak{F})$ is the constant simplicial $\mathbb{S}$-bimodule $F_0(0; \mathfrak{F}) = F(*)$, we see that $\tilde{K}(\mathcal{E}; \mathfrak{F}) \cong K(\mathcal{E}; \mathfrak{F})$ if $\mathfrak{F}$ is reduced.

Several results from 3.2 apply directly. For example, if $\eta : \mathfrak{F} \to \mathfrak{F}'$ is a point-wise weak equivalence, then by 3.2.11 we get a weak equivalence $K(\mathcal{E}; \mathfrak{F}) \xrightarrow{\cong} K(\mathcal{E}; \mathfrak{F}')$. Additionally, if

$\mathfrak{F}$ is given by a homotopy colimit of functors $\mathfrak{F}_i$ ($i \in \mathcal{I}$, a small category), then by 3.2.13 we have:

$$K(\mathcal{E}; \mathfrak{F}) \simeq \operatorname*{hocolim}_{i \in \mathcal{I}} K(\mathcal{E}; \mathfrak{F}_i)$$

The name is justified from the following example:

Example 4.1.3 Let $\mathcal{E}$ be an exact category, and $\mathfrak{F} = c_E : \mathbb{S}\text{-BiMod} \to \mathbb{S}\text{-BiMod}$ the (non-reduced) constant functor at a non-trivial $\mathbb{S}$-bimodule E. Then:

$$
\begin{aligned}
\tilde{K}(\mathcal{E}; c_E) &= \operatorname{hocofiber}\left(K(0; c_E) \to K(\mathcal{E}; c_E)\right) \\
&= \operatorname{hocofiber}\left(\operatorname*{hocolim}_n \Omega^{(n)}\big(\partial F\big(S_\bullet^{(n)}(0); c_E\big)\big) \to \operatorname*{hocolim}_n \Omega^{(n)}\big(\partial F\big(S_\bullet^{(n)}(\mathcal{E}); c_E\big)\big)\right) \\
&\cong \operatorname*{hocolim}_n \left(\operatorname{hocofiber}\big(\Omega^{(n)}\big(\partial F\big(S_\bullet^{(n)}(0); c_E\big)\big) \to \Omega^{(n)}\big(\partial F\big(S_\bullet^{(n)}(\mathcal{E}); c_E\big)\big)\big)\right) \\
(\star) \ &\simeq \operatorname*{hocolim}_n \left(\Omega^{(n)}\big(\operatorname{hocofiber}\big(\partial F\big(S_\bullet^{(n)}(0); c_E\big) \to \partial F\big(S_\bullet^{(n)}(\mathcal{E}); c_E\big)\big)\big)\right) \\
&= \operatorname*{hocolim}_n \left(\Omega^{(n)}\big(\operatorname{hocofiber}\big(cE \to \partial F\big(S_\bullet^{(n)}(\mathcal{E}); c_E\big)\big)\big)\right)
\end{aligned}
$$

Where $(\star)$ uses the fact that loops commute with cofiber sequences in a stable model category. The map $cE \to \partial F\big(S_\bullet^{(n)}(\mathcal{E}); c_E\big)$ is the geometric realization of a map which is given on p-simplices by the inclusion of E into the $(0 \leftarrow 0 \leftarrow \cdots \leftarrow 0)$-summand of $N_p(S_{[p]}S_{[p]} \ldots S_{[p]}\mathcal{E})$. Letting $\mathcal{C} = S_{[p]}S_{[p]} \ldots S_{[p]}\mathcal{E}$, its homotopy cofiber then is weakly equivalent to

$$\bigwedge_{\vec{\mathcal{C}} \in N_p\mathcal{C} \backslash (0 \leftarrow \cdots \leftarrow 0)} E$$

Now, using the $\underline{s\mathbf{Sets}_*}$-tensored structure of $\mathbb{S}\text{-BiMod}$ we get:

$$
\begin{aligned}
\bigwedge_{\vec{\mathcal{C}} \in N_p\mathcal{C} \backslash (0 \leftarrow \cdots \leftarrow 0)} E \ &\cong \bigwedge_{\vec{\mathcal{C}} \in N_p\mathcal{C} \backslash (0 \leftarrow \cdots \leftarrow 0)} (\mathbb{S} \wedge E) \\
&\cong \Big(\bigwedge_{\vec{\mathcal{C}} \in N_p\mathcal{C} \backslash (0 \leftarrow \cdots \leftarrow 0)} \mathbb{S}\Big) \wedge E \\
&\cong \Big(\bigwedge_{\vec{\mathcal{C}} \in N_p\mathcal{C} \backslash (0 \leftarrow \cdots \leftarrow 0)} \Sigma^\infty S^0\Big) \wedge E \\
&\cong \Sigma^\infty \left(\bigwedge_{\vec{\mathcal{C}} \in N_p\mathcal{C} \backslash (0 \leftarrow \cdots \leftarrow 0)} S^0\right) \wedge E
\end{aligned}
$$

$$\cong \Sigma^\infty(N_p\mathcal{C}) \wedge E$$
$$\cong N_p\mathcal{C} \wedge E$$

And so,
$$\tilde{K}(\mathcal{E}; c_E) \simeq \operatorname*{hocolim}_n \left(\Omega^{(n)}\big(|[p] \mapsto N_p\mathcal{C} \wedge E|\big)\right)$$

Since the functor $\mathbf{Ext} \to \mathbb{S}\text{-BiMod}$ given by $\mathcal{E} \mapsto N(\mathcal{E}) \wedge E$ is reduced, we can approximate it by 0-simplices using 3.2.8. Therefore $\tilde{K}(\mathcal{E}; c_E)$ is weakly equivalent to the stabilization of the functor $\mathcal{E} \mapsto \mathrm{Obj}(\mathcal{E}) \wedge E$. By 3.1.3, we conclude that

$$\tilde{K}(\mathcal{E}; c_E) \simeq \mathbf{K}(\mathcal{E}) \wedge E$$

that is, the E-homology of the K-theory spectrum of $\mathcal{E}$. In particular, we recover classical K-theory, $\tilde{K}(\mathcal{E}; c_{\mathbb{S}}) \simeq \mathbf{K}(\mathcal{E})$.

Example 4.1.4 Let $\mathcal{E}$ be an exact category, and $\mathfrak{F} = Id \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ the identity functor. Then:

$$K(\mathcal{E}; Id) = \big(F(-; Id)\big)^{\mathrm{st}}(\mathcal{E}) \stackrel{\simeq}{\Leftarrow} \big(F_0(-; Id)\big)^{\mathrm{st}}(\mathcal{E}) = \underline{\mathbf{Hom}}^{\mathrm{st}}(\mathcal{E})$$

using the notation of 3.1.4. Therefore, $K(\mathcal{E}; Id) \simeq THH(\mathcal{E})$.

Example 4.1.5 Consider the functor $\mathfrak{F} = l_E \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ given by smashing with a fixed $\mathbb{S}$-bimodule E, that is, $X \mapsto X \wedge E$. Then replacing E by a good $\mathbb{S}$-bimodule E', and filtering E' by its finite skeleta, we see that

$$\begin{aligned}
K(\mathcal{E}; l_E) &\simeq K(\mathcal{E}; l_{E'}) \\
&\simeq K(\mathcal{E}; l_{\operatorname*{hocolim}_n E_n}) \\
&\cong K(\mathcal{E}; \operatorname*{hocolim}_n l_{E_n}) \\
&\simeq \operatorname*{hocolim}_n K(\mathcal{E}; l_{E_n}) \\
&= \operatorname*{hocolim}_n \big(F(-; l_{E_n})\big)^{\mathrm{st}}(\mathcal{E}) \\
&\stackrel{\simeq}{\Leftarrow} \operatorname*{hocolim}_n \big(F_0(-; l_{E_n})\big)^{\mathrm{st}}(\mathcal{E}) \\
&\simeq \operatorname*{hocolim}_n \big(F_0(-; Id) \wedge E_n\big)^{\mathrm{st}}(\mathcal{E}) \\
&\simeq \operatorname*{hocolim}_n \big(F_0(-; Id)\big)^{\mathrm{st}}(\mathcal{E}) \wedge E_n
\end{aligned}$$

$$\simeq \operatorname*{hocolim}_n THH(\mathcal{E}) \wedge E_n$$

Thus, generalizing the previous example, we have $K(\mathcal{E}; l_E) \simeq THH(\mathcal{E}) \wedge E$, that is, the E-homology of the topological Hochschild homology of $\mathcal{E}$.

The previous example is characteristic of a general phenomenon whose proof is identical:

Lemma 4.1.6 *Let $\mathcal{E}$ be an exact category, $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$, and E a good $\mathbb{S}$-bimodule. Let $E \wedge \mathfrak{F}$ be the pointed reduced simplicial homotopy functor given by $X \mapsto E \wedge \mathfrak{F}(X)$. Then*

$$K(\mathcal{E}; E \wedge \mathfrak{F}) \simeq E \wedge K(\mathcal{E}; \mathfrak{F})$$

Additionally, if $\mathfrak{F}$ is a Σ_n-object in $Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$, then we can consider the associated homotopy orbits functor $\mathfrak{F}_{h\Sigma_n} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$. Recall that we construct homotopy orbits by viewing the Σ_n-object $\mathfrak{F}$ as a functor from the groupoid $\tilde{\Sigma}_n$ (see Remark following Appendix A.1.7) into $Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ and applying the homotopy colimit coend formula from Appendix B.2. Since homotopy pullbacks commute with homotopy orbits (Theorem 1 [41]), we get:

Lemma 4.1.7 *Let E be an exact category and $\mathfrak{F}$ a Σ_n-object in $Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$. Then*

$$K(\mathcal{E}; \mathfrak{F}_{h\Sigma_n}) \simeq K(\mathcal{E}; \mathfrak{F})_{h\Sigma_n}$$

Notation Let l be a positive integer, and $P^{\infty}(\mathbb{Z}/l\mathbb{Z})$ be the lth-Moore spectrum. From 4.1.3 we see that, if $\mathcal{E}$ is an exact category, $\tilde{K}(\mathcal{E}; c_{P^{\infty}(\mathbb{Z}/l\mathbb{Z})})$ is weakly equivalent to the K-theory of $\mathcal{E}$ with coefficients "mod l" as defined in Chapter IV, §2 [61].

Lastly, the cotensoring of $\mathbb{S}$-bimodules with $\underline{s\mathbf{Sets}_*}$ (last observation of Appendix E.3) interacts well with stabilization:

Lemma 4.1.8 *Let $Y \in \underline{s\mathbf{Sets}_*}^{\mathrm{fin}}$ be s-connected and $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ take s-connected values. If $\mathcal{E}$ is an exact category, then*

$$K(\mathcal{E}; \mathrm{Map}(Y, \mathfrak{F})) \simeq \mathrm{Map}\big(Y, K(\mathcal{E}; \mathfrak{F})\big)$$

Proof.

$$
\begin{aligned}
K(\mathcal{E}; \mathrm{Map}(Y, \mathfrak{F})) &= \big(F(-; \mathrm{Map}(Y, \mathfrak{F}))\big)^{\mathrm{st}}(\mathcal{E}) \\
&\overset{\cong}{\Leftarrow} \big(F_0(-; \mathrm{Map}(Y, \mathfrak{F}))\big)^{\mathrm{st}}(\mathcal{E}) \\
&= \left(\bigvee_{C \in \mathrm{Obj}(-)} \mathrm{Map}(Y, \mathfrak{F}(\widetilde{Hom}_{(-)}(C, C))) \right)^{\mathrm{st}}(\mathcal{E}) \\
(\star) \quad &\overset{\cong}{\Rightarrow} \left(\mathrm{Map}\left(Y, \bigvee_{C \in \mathrm{Obj}(-)} \mathfrak{F}(\widetilde{Hom}_{(-)}(C, C)) \right) \right)^{\mathrm{st}}(\mathcal{E}) \\
&= \underset{n}{\mathrm{hocolim}}\, \Omega^{(n)} \left(\mathfrak{d}\mathrm{Map}\left(Y, \bigvee_{C \in \mathrm{Obj}(S_\bullet^{(n)}\mathcal{E})} \mathfrak{F}(\widetilde{Hom}_{S_\bullet^{(n)}\mathcal{E}}(C, C)) \right) \right) \\
(\dagger) \quad &\simeq \underset{n}{\mathrm{hocolim}}\, \Omega^{(n)} \left(\mathrm{Map}\left(Y, \mathfrak{d} \bigvee_{C \in \mathrm{Obj}(S_\bullet^{(n)}\mathcal{E})} \mathfrak{F}(\widetilde{Hom}_{S_\bullet^{(n)}\mathcal{E}}(C, C)) \right) \right) \\
&\cong \underset{n}{\mathrm{hocolim}}\, \mathrm{Map}\left(Y, \Omega^{(n)} \left(\mathfrak{d} \bigvee_{C \in \mathrm{Obj}(S_\bullet^{(n)}\mathcal{E})} \mathfrak{F}(\widetilde{Hom}_{S_\bullet^{(n)}\mathcal{E}}(C, C)) \right) \right) \\
&\simeq \mathrm{Map}\left(Y, \underset{n}{\mathrm{hocolim}}\, \Omega^{(n)} \left(\mathfrak{d} \bigvee_{C \in \mathrm{Obj}(S_\bullet^{(n)}\mathcal{E})} \mathfrak{F}(\widetilde{Hom}_{S_\bullet^{(n)}\mathcal{E}}(C, C)) \right) \right) \\
&= \mathrm{Map}\big(Y, \big(F_0(-; \mathfrak{F})\big)^{\mathrm{st}}(\mathcal{E}) \big)
\end{aligned}
$$

Here $(\star)$ is the map induced from the canonical maps out of the coproduct, which is a weak equivalence by the standard trick of Boardman to reduce to finite wedges (see Proposition 3.11 and 3.14 [1]). On the other hand, $(\dagger)$ comes from iterating the π_*-Kan condition (see the remarks following Corollary A.6.0.4 [14]). $\qquad\square$

4.2 The Lindenstrauss-McCarthy Tower

Let $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$. Recall that the theory of calculus of homotopy functors, as developed by Goodwillie ([22], [25], [26]), constructs a sequence of reduced homotopy functors, $P_0\mathfrak{F}, P_1\mathfrak{F}, \ldots, P_n\mathfrak{F}, \ldots$ fitting into a tower under $\mathfrak{F}$:

$$\xymatrix{ & & \mathfrak{F} & & \\ \cdots \ar@{^{(}->}[r] & P_{n+1}\mathfrak{F} \ar[r] & P_n\mathfrak{F} \ar[r] & P_{n-1}\mathfrak{F} \ar[r] & \cdots }$$

such that each $P_n\mathfrak{F}$ is n-excisive, and that for $X \in \mathbb{S}\text{-BiMod}$ suitably connected, we have $\mathfrak{F}(X) \simeq \operatorname*{holim}_n P_n\mathfrak{F}(X)$. Though in general identifying $P_n\mathfrak{F}$ is difficult, McCarthy gives in [47] a model:

$$P_n\mathfrak{F}(-) \simeq \operatorname*{hocolim}_k \operatorname{Map}_{\mathbb{M}_n}((S^k)^{\wedge *}, (\hat{cr}^*\mathfrak{F})(S^k \wedge -))$$

Additionally, the layers of the tower, $D_n\mathfrak{F} := \operatorname{hofiber}(P_n\mathfrak{F} \to P_{n-1}\mathfrak{F})$, are obtained in this model from restriction to subcategories, and we have

$$D_n\mathfrak{F}(-) \simeq \left(\operatorname*{hocolim}_k \operatorname{Map}((S^k)^{\wedge n}, (\hat{cr}^n\mathfrak{F})(S^k \wedge -)) \right)_{h\Sigma_n}$$

when $\mathfrak{F}$ is *finitary*.

Remark The model we use from [47] for general $P_n\mathfrak{F}$ was inspired by Arone's work on the Goodwillie Taylor tower of $\Sigma^\infty\Omega^\infty$. The case of $\Sigma^\infty\Omega^\infty$ follows from work in [4] though it is not directly mentioned there (for a published account see Corollary 1.3 and Section 3 [2]). If $\mathfrak{F}$ is n-excissive, then $\hat{cr}^n\mathfrak{F}$ is n-multilinear (Lemma 4.1 [47]), and therefore there is an $\mathbb{S}$-bimodule, $\partial_n\mathfrak{F}$, with Σ_n-action, so that

$$\hat{cr}^n\mathfrak{F}(X) \simeq_{f\Sigma_n} \partial_n\mathfrak{F} \wedge X^{\wedge n}$$

Hence, if $\mathfrak{F}$ is degree n:

$$D_n\mathfrak{F}(-) \simeq \left(\operatorname*{hocolim}_k \operatorname{Map}((S^k)^n, (\hat{cr}^n\mathfrak{F})(S^k \wedge -)) \right)_{h\Sigma_n}$$
$$\simeq \left(\operatorname*{hocolim}_k \operatorname{Map}((S^k)^{\wedge n}, \partial_n\mathfrak{F} \wedge (S^k \wedge -)^{\wedge n}) \right)_{h\Sigma_n}$$
$$\simeq \left(\operatorname*{hocolim}_k \operatorname{Map}((S^k)^{\wedge n}, (S^k)^{\wedge n} \wedge \partial_n\mathfrak{F} \wedge (-)^{\wedge n}) \right)_{h\Sigma_n}$$

When $\mathfrak{F}$ is homogeneous of degree n, we have $\mathfrak{F} \simeq D_n\mathfrak{F}$.

Proposition 4.2.1 *Let $\mathcal{E}$ be an exact category and $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ a homogeneous degree n functor. Then*

$$K(\mathcal{E}; \mathfrak{F}) \simeq (\partial_n\mathfrak{F} \wedge K(\mathcal{E}; (-)^{\wedge n}))_{h\Sigma_n}$$

Proof. Using the lemmas from the previous section, we can reduce to:

$$K(\mathcal{E}; \mathfrak{F}) \simeq K(\mathcal{E}; D_n\mathfrak{F})$$

$$\simeq K\left(\mathcal{E}; \left(\operatorname*{hocolim}_k \operatorname{Map}((S^k)^{\wedge n}, (S^k)^{\wedge n} \wedge \partial_n\mathfrak{F} \wedge (-)^{\wedge n})\right)_{h\Sigma_n}\right)$$

$$\simeq \left(K\left(\mathcal{E}; \operatorname*{hocolim}_k \operatorname{Map}((S^k)^{\wedge n}, (S^k)^{\wedge n} \wedge \partial_n\mathfrak{F} \wedge (-)^{\wedge n})\right)\right)_{h\Sigma_n}$$

$$\simeq \left(\operatorname*{hocolim}_k K\left(\mathcal{E}; \operatorname{Map}((S^k)^{\wedge n}, (S^k)^{\wedge n} \wedge \partial_n\mathfrak{F} \wedge (-)^{\wedge n})\right)\right)_{h\Sigma_n}$$

$$\simeq \left(\operatorname*{hocolim}_k \operatorname{Map}\left((S^k)^{\wedge n}, K\left(\mathcal{E}; (S^k)^{\wedge n} \wedge \partial_n\mathfrak{F} \wedge (-)^{\wedge n}\right)\right)\right)_{h\Sigma_n}$$

$$\simeq \left(\operatorname*{hocolim}_k \operatorname{Map}\left((S^k)^{\wedge n}, (S^k)^{\wedge n} \wedge \partial_n\mathfrak{F} \wedge K\left(\mathcal{E}; (-)^{\wedge n}\right)\right)\right)_{h\Sigma_n}$$

and we conclude using the Freudenthal suspension theorem. $\qquad\square$

Remark If R is a ring and $\mathcal{P}$ the category of finitely generated projective right R-modules, $K(\mathcal{P}, (-)^{\wedge n})$ is the spectral homology of $\mathcal{P}$ with coefficients in the bi-functor $\mathfrak{D}(\overbrace{R, \dots, R}^{n-\text{times}})$ by 4.1.1. This was computed in 3.3.7. Choosing the bimodule $\mathfrak{D}(M)$ (over the FSP $\tilde{\mathcal{P}}$) over $\mathcal{P}$, we rewrite that result in the current notation:

Theorem 4.2.2 *Let R be an associative and unital ring, M an R-bimodule, and $n \geq 1$. There is a feeble equivalence of Σ_n-bimodules*

$$K(\mathcal{P}; \mathfrak{D}(M); (-)^{\wedge n}) \simeq_{f\Sigma_n} U^n(R; M) \wedge_{\mathbb{C}_n} \Sigma_n$$

The following innocuous-looking lemma is actually very powerful:

Lemma 4.2.3 *Let $\mathfrak{F} \rightarrowtail \mathfrak{F}' \twoheadrightarrow \mathfrak{F}''$ be a homotopy fiber sequence in $Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$. Then for each exact category $\mathcal{E}$ we have a natural homotopy fiber sequence in $\mathbb{S}\text{-BiMod}$*

$$K(\mathcal{E}; \mathfrak{F}) \rightarrowtail K(\mathcal{E}; \mathfrak{F}') \twoheadrightarrow K(\mathcal{E}; \mathfrak{F}'')$$

Proof. Since $\mathbb{S}\text{-BiMod}$ is stable, $\mathfrak{F} \rightarrowtail \mathfrak{F}' \twoheadrightarrow \mathfrak{F}''$ is a homotopy fiber sequence if and only if it is a homotopy cofiber sequence. We have already observed by 3.2.13 that $K(\mathcal{E}; -)$ preserves homotopy cofiber sequences, and so $K(\mathcal{E}; \mathfrak{F}) \rightarrowtail K(\mathcal{E}; \mathfrak{F}') \twoheadrightarrow K(\mathcal{E}; \mathfrak{F}'')$ is a homotopy cofiber sequence, whence it is a homotopy fiber sequence also. $\qquad\square$

Observation We have already encountered a homotopy fiber sequence of endofunctors, mainly $D_n\mathfrak{F} \rightarrowtail P_n\mathfrak{F} \twoheadrightarrow P_{n-1}\mathfrak{F}$, therefore using 4.2.1 we get homotopy fiber sequences:

$$K(\mathcal{E}; D_n\mathfrak{F}) \rightarrowtail K(\mathcal{E}; P_n\mathfrak{F}) \twoheadrightarrow K(\mathcal{E}; P_{n-1}\mathfrak{F})$$

$$(\partial_n\mathfrak{F} \wedge K\,(\mathcal{E}; (-)^{\wedge n}))_{h\Sigma_n} \rightarrowtail K(\mathcal{E}; P_n\mathfrak{F}) \twoheadrightarrow K(\mathcal{E}; P_{n-1}\mathfrak{F})$$

Applying $K(\mathcal{E}; -)$ to the Goodwillie Taylor tower, we obtain:

Notation Let R and $\mathcal{P}$ be as before, and M be an R-bimodule. We let $\mathrm{End}(R; M)$ be the category with:

- $\mathrm{Obj}(\mathrm{End}(R; M)) = \{(P, \alpha) \mid P \in \mathcal{P},\ \alpha \in \mathrm{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M)\}$

- $\mathrm{End}(R; M)\big((P, \alpha), (P', \alpha')\big) = \{f \mid f \in \mathrm{Hom}_{\mathbf{Mod\text{-}}R}(P, P')\ \text{and}\ \alpha' \circ f = (f \underset{R}{\otimes} \mathrm{id}_P) \circ \alpha\}$

This is an additive category, and inherits the structure of an exact category by the (a posteriori) exact projection functor $\pi : \mathrm{End}(R; M) \to \mathcal{P}$, given by $(P, \alpha) \mapsto P$.

Example 4.2.4 Let $\mathfrak{F} \in Fun^h(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$ be the functor $\Sigma^\infty \Omega^\infty$. The K-theory with coefficients $K(\mathcal{P}; \mathfrak{D}(M); \Sigma^\infty \Omega^\infty)$ is nothing more than the relative version of $\mathrm{End}(R; M)$. Indeed, let $P \in \mathcal{P}$. Then:

$$\Sigma^\infty \Omega^\infty(\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M)) \simeq \Sigma^\infty(Hom_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M))$$

$$\cong Hom_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M) \wedge \mathbb{S}$$

$$\cong \bigvee_{Hom_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M) \backslash 0} \mathbb{S}$$

and so

$$\bigvee_{P\in\mathrm{Obj}(\mathcal{P})} \Sigma^\infty\Omega^\infty(\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(P, P \underset{R}{\otimes} M)) \simeq \bigvee_{P\in\mathrm{Obj}(\mathcal{P})}\ \bigvee_{Hom_{\mathbf{Mod}\text{-}R}(P,P\underset{R}{\otimes}M)\backslash 0} \mathbb{S}$$

$$\cong \bigvee_{(P,\alpha)\in\mathrm{End}(R;M)\backslash(P,0)} \mathbb{S}$$

$$\cong \mathrm{hocofiber}\left(\bigvee_{(P,0)\in\mathrm{End}(R;M)} \mathbb{S} \to \bigvee_{(P,\alpha)\in\mathrm{End}(R;M)} \mathbb{S}\right)$$

$$= \mathrm{hocofiber}\left(\bigvee_{P\in\mathrm{Obj}(\mathcal{P})} \mathbb{S} \to \bigvee_{(P,\alpha)\in\mathrm{End}(R;M)} \mathbb{S}\right)$$

By analyzing the total homotopy cofiber of the diagram

$$\begin{array}{ccc} \mathbb{S} & ===== & \mathbb{S} \\ \downarrow & & \downarrow \\ \bigvee_{P\in\mathrm{Obj}(\mathcal{P})} \mathbb{S} & \longrightarrow & \bigvee_{(P,\alpha)\in\mathrm{End}(R;M)} \mathbb{S} \end{array}$$

and the fact that homotopy colimits commute, we conclude that:

$$K(\mathcal{P}; \mathfrak{D}(M); \Sigma^\infty\Omega^\infty) \simeq \mathrm{hocofiber}\left(\tilde{K}(\mathcal{P}; c_\mathbb{S}) \to \tilde{K}(\mathcal{P}; \Sigma^\infty\Omega^\infty)\right)$$

$$\simeq \mathrm{hocofiber}\left(\mathbf{K}(\mathcal{P}) \to \mathbf{K}(\mathrm{End}(R; M))\right)$$

$$\simeq \tilde{\mathbf{K}}(\mathrm{End}(R; M))$$

So, using the notation of [34], we have that $K(\mathcal{P}; \mathfrak{D}(M); \Sigma^\infty\Omega^\infty) \simeq \tilde{K}(R; M)$, the relative K-theory of endomorphisms of R parametrized by M. We now analyze the tower under $K(\mathcal{P}; \mathfrak{D}(M); \Sigma^\infty\Omega^\infty)$. Directly from the definitions in Appendix D.2 we get that

$$\hat{cr}^n(\Sigma^\infty\Omega^\infty)(X) \simeq_{f\Sigma_n} (\Sigma^\infty\Omega^\infty(X))^{\wedge n}$$

Therefore $\partial_n(\Sigma^\infty\Omega^\infty) \simeq \mathbb{S}$ for all n. The layers are completely understood. Indeed, by the

previous results we know the layers to be:

$$(\partial_n(\Sigma^\infty \Omega^\infty) \wedge K(\mathcal{P}; \mathfrak{D}(M); (-)^{\wedge n}))_{h\Sigma_n}$$

and so by Chapter 3 are:

$$(U^n(R; M) \wedge_{\mathbb{C}_n} \Sigma_n)_{h\Sigma_n}$$

The homotopy orbits of an induced representation collapse down to the homotopy orbits of the restriction, and so we conclude that the layers look like:

$$U^n(R; M)_{hC_n}$$

Now, viewing $K(\mathcal{P}; \mathfrak{D}(-); \Sigma^\infty \Omega^\infty)$ as a homotopy functor of bimodules $\mathbf{Mod}\text{-}R \to \mathbb{S}\text{-BiMod}$, we seek to compare $K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty))$ with $P_n\big(K(\mathcal{P}; \mathfrak{D}(-); \Sigma^\infty \Omega^\infty)\big)(M)$. To do so we introduce some auxiliary subcategories.

Notation Let $\underline{m}, \underline{k} \in$ Surj (see definitions in Appendix D.3) be such that $k|m$. Let $\phi_{m,k} : \underline{k} \twoheadrightarrow \underline{m}$ be the surjection given by $r = kq + t + 1 \in \underline{m} \mapsto t + 1$, where q, t are unique integers with $0 \le q < \frac{m}{k}$ and $0 \le t < k$. We let Cyc be the wide subcategory of Surj generated by the cyclic automorphisms and the $\phi_{-,\star}$-maps. That is, those evenly-covered surjections defining group homomorphisms of the associated cyclic groups (after giving $\underline{n}$ the cyclic abelian group structure of $\mathbb{Z}/n\mathbb{Z}$):

$$\mathrm{Cyc}(\underline{m}, \underline{k}) = \begin{cases} 0 & \text{if} \quad k \nmid m \\ \tau \circ \phi_{m,k} : \underline{m} \twoheadrightarrow \underline{k}, \text{ such that } \tau \in C_k & \text{if} \quad k|m \end{cases}$$

Note that $\phi_{m,m} = \mathrm{id}_{\underline{m}}$ and $\underline{1}$ is final. Denote its opposite category by $\mathbb{E} := \mathrm{Cyc}^{\mathrm{op}}$. Similarly, we set Cyc_n to be the full subcategory of Cyc with objects of size $\le n$, and denote $\mathbb{E}_n := \mathrm{Cyc}_n^{\mathrm{op}}$.

Remark For any endofunctor $\mathfrak{F} \in Fun^{\mathrm{h}}(\mathbb{S}\text{-BiMod}, \mathbb{S}\text{-BiMod})$, the McCarthy model for P_n gives a map, induced by restriction of subcategories $\mathbb{E}_n \subset \mathbb{M}_n$ (see Appendix D.5):

$$P_n\mathfrak{F}(-) \simeq \operatorname*{hocolim}_k \mathrm{Map}_{\mathbb{M}_n}((S^k)^{\wedge *}, (\hat{cr}^*\mathfrak{F})(S^k \wedge -))$$

$$\cong \operatorname*{hocolim}_k \lim_{\mathrm{tw}(\mathbb{M}_n)} \mathrm{Hom}_{\mathrm{tw}(\mathbb{M}_n)}((S^k)^{\wedge *}, (\hat{cr}^*\mathfrak{F})(S^k \wedge -))$$

$$\to \operatorname*{hocolim}_{k} \operatorname*{lim}_{\mathrm{tw}(\mathbb{E}_n)} \left(\mathrm{Hom}_{\mathrm{tw}(\mathbb{M}_n)}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -)) \right) |_{\mathrm{tw}(\mathbb{E}_n)}$$

$$= \operatorname*{hocolim}_{k} \operatorname*{lim}_{\mathrm{tw}(\mathbb{E}_n)} \mathrm{Hom}_{\mathrm{tw}(\mathbb{E}_n)}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -))$$

$$\cong \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -))$$

In particular, in the case of $\mathfrak{F} = \Sigma^\infty \Omega^\infty$, we have a comparison map, denoted τ,

$$P_n(\Sigma^\infty \Omega^\infty)(-) \to \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (\Sigma^\infty \Omega^\infty (S^k \wedge -))^{\wedge *})$$

Note also that from the commutative diagram of categories:

$$\begin{array}{ccc} \mathbb{E}_{n-1} & \lhook\joinrel\longrightarrow & \mathbb{E}_n \\ \downarrow & & \downarrow \\ \mathbb{M}_{n-1} & \lhook\joinrel\longrightarrow & \mathbb{M}_n \end{array}$$

we get a commutative diagram of $\mathbb{S}$-bimodules:

$$\begin{array}{ccc} \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{M}_n}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -)) & \longrightarrow & \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -)) \\ \downarrow & & \downarrow \\ \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{M}_{n-1}}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -)) & \longrightarrow & \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{E}_{n-1}}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -)) \end{array}$$

and so, induced maps:

$$\begin{array}{ccc} P_n \mathfrak{F}(-) & \longrightarrow & \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -)) \\ \downarrow & & \downarrow \\ P_{n-1} \mathfrak{F}(-) & \longrightarrow & \operatorname*{hocolim}_{k} \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (\hat{cr}^* \mathfrak{F})(S^k \wedge -)) \end{array}$$

Fact Let $Y \in \mathbb{S}\text{-BiMod}$. Then $S^k \wedge Y$ is a k-connected $\mathbb{S}$-bimodule. From Corollary 1.3 [2], we get that the counit map $\Sigma^\infty \Omega^\infty (S^k \wedge Y) \to S^k \wedge Y$ is $(2k+2)$-connected and so its $*$-fold smash, $(\Sigma^\infty \Omega^\infty (S^k \wedge Y))^{\wedge *} \to (S^k \wedge Y)^{\wedge *}$, is $(2k * +3 * -1)$-connected. Therefore, the associated map

$$\mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (\Sigma^\infty \Omega^\infty (S^k \wedge Y))^{\wedge *}) \to \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k \wedge Y)^{\wedge *})$$

is $(k+3)$-connected (the connectivity of the smallest smash product appearing, namely $* = 1$). Since the connectivity is linear in k, taking homotopy colimits over k we get a weak equivalence:

$$\operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (\Sigma^\infty \Omega^\infty (S^k \wedge Y))^{\wedge *}) \xrightarrow{\simeq} \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k \wedge Y)^{\wedge *})$$

From the previous fact and remark, we have a comparison map, also denoted τ:

$$P_n(\Sigma^\infty \Omega^\infty)(-) \to \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k \wedge -)^{\wedge *}) \cong \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge (-)^{\wedge *})$$

which is natural in the restrictions from n to $n-1$. For $K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty))$, taking a coproduct of τ's we get a sequence of maps:

$$\bigvee_{P \in \mathrm{Obj}(\mathcal{P})} P_n(\Sigma^\infty \Omega^\infty)(\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M))$$

$$\xrightarrow{\tau} \bigvee_{P \in \mathrm{Obj}(\mathcal{P})} \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge (\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M))^{\wedge *})$$

$$\cong \operatorname*{hocolim}_{k} \bigvee_{P \in \mathrm{Obj}(\mathcal{P})} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge (\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M))^{\wedge *})$$

$$\to \operatorname*{hocolim}_{k} \prod_{P \in \mathrm{Obj}(\mathcal{P})} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge (\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M))^{\wedge *})$$

$$\cong \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, \prod_{P \in \mathrm{Obj}(\mathcal{P})} (S^k)^{\wedge *} \wedge (\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M))^{\wedge *})$$

$$\leftarrow \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, \bigvee_{P \in \mathrm{Obj}(\mathcal{P})} (S^k)^{\wedge *} \wedge (\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M))^{\wedge *})$$

$$\cong \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge \bigvee_{P \in \mathrm{Obj}(\mathcal{P})} \overset{*}{\bigwedge}(\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M)))$$

We also have a canonical inclusion given by including "diagonal" elements:

$$\bigvee_{P \in \mathrm{Obj}(\mathcal{P})} \overset{*}{\bigwedge}(\widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P, P \underset{R}{\otimes} M)) \hookrightarrow$$

$$\bigvee_{(P_1, \ldots, P_*) \in \mathrm{Obj}(\mathcal{P}^{\times *})} \widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P_1, P_* \underset{R}{\otimes} M) \wedge \widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P_2, P_1 \underset{R}{\otimes} M) \wedge \cdots \wedge \widetilde{Hom}_{\mathbf{Mod\text{-}}R}(P_*, P_{*-1} \underset{R}{\otimes} M)$$

However, recall that the latter is *by definition* $F_0(\mathcal{P}^{\times *}; \mathfrak{D}^\omega(M, \ldots, M))$ and so is connected by a zig-zag of weak equivalences to $U_0^*(\underline{\mathcal{P}}; P(1), \ldots, P(n))$ (using the notation from the end of Chapter 3). Therefore we get a generalized map, labeled $\dashrightarrow$, from

$$\bigvee_{P \in \mathrm{Obj}(\mathcal{P})} P_n(\Sigma^\infty \Omega^\infty)(\widetilde{Hom}_{\mathbf{Mod}\text{-}R}(P, P \underset{R}{\otimes} M))$$

to

$$\mathrm{hocolim}_k \, \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge U_0^*(\underline{\mathcal{P}}; P(1), \ldots, P(n)))$$

Now, each of the morphisms involved in these generalized maps are natural in $\mathcal{S}_\mathcal{P}$, and so taking stabilizations we get a generalized map:

$$
\begin{aligned}
&K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty)) \\
&:= \big(F(-; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty))\big)^{\mathrm{st}}(\mathcal{P}) \\
&\overset{\simeq}{\leftarrow} \big(F_0(-; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty))\big)^{\mathrm{st}}(\mathcal{P}) \\
&:= \mathrm{hocolim}_t \, \Omega^{(t)} \left(\bigvee_{\overrightarrow{P} \in \mathrm{Obj}(S_\bullet^{(t)} \mathcal{P})} P_n(\Sigma^\infty \Omega^\infty)(\widetilde{Hom}_{S_\bullet^{(t)} \mathbf{Mod}\text{-}R}(\overrightarrow{P}, \overrightarrow{P} \underset{R}{\otimes} M)) \right) \\
&\dashrightarrow \mathrm{hocolim}_t \, \Omega^{(t)} \left(\mathrm{hocolim}_k \, \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge U_0^*(\underline{\mathcal{P}}; P(1), \ldots, P(n))) \right) \\
&(\star) \simeq \mathrm{hocolim}_k \, \mathrm{hocolim}_t \, \Omega^{(t)} \left(\mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge U_0^*(\underline{\mathcal{P}}; P(1), \ldots, P(n))) \right)
\end{aligned}
$$

where in $(\star)$ we have used the fact that homotopy colimits commute with homotopy colimits and that finite homotopy limits (like the ones defining $\Omega^{(t)}$) commute with directed homotopy colimits. Similarly, rewriting our mapping diagram as an inverse limit over the finite twisted arrow category, we obtain a weak equivalence:

$$
\begin{aligned}
\mathrm{hocolim}_t \, \Omega^{(t)} &\big(\mathrm{Map}_{\mathbb{E}_l}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge U_0^*(\underline{\mathcal{P}}; P(1), \ldots, P(n))) \big) \\
&\simeq \mathrm{Map}_{\mathbb{E}_l}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge \mathrm{hocolim}_t \, \Omega^{(t)} \, (U_0^*(\underline{\mathcal{P}}; P(1), \ldots, P(n))))
\end{aligned}
$$

and the latter is weakly equivalent to $U^*(R; M)$ again by the work of [34] (compare with the remarks immediately preceding Theorem 3.3.7). So our generalized map is:

$$K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty)) \dashrightarrow \mathrm{hocolim}_k \, \mathrm{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge U^*(R; M))$$

and is natural with respect to the restriction from n to $n - 1$. There is a canonical map into the homotopy colimit system

$$\operatorname{Map}_{\mathbb{E}_n}((S^0)^{\wedge *}, (S^0)^{\wedge *} \wedge U^*(R; M)) \to \operatorname*{hocolim}_{k} \operatorname{Map}_{\mathbb{E}_n}((S^k)^{\wedge *}, (S^k)^{\wedge *} \wedge U^*(R; M))$$

Let $\mathbb{N}^{\times}_{\leq n}$ be the partially ordered set $\{1, \ldots, n\}$ with $l' < l$ if and only if $l | l'$. Then a quick check shows $\operatorname{Map}_{\mathbb{E}_n}((S^0)^{\wedge *}, (S^0)^{\wedge *} \wedge U^*(R; M)) \cong \lim_{l \in \mathbb{N}^{\times}_{\leq n}} (U^l(R; M)^{C_l})$. Recall from Definition 4.6 [34] that $W_n(R; M) := \operatorname*{holim}_{l \in \mathbb{N}^{\times}_{\leq n}} (U^l(R; M)^{C_l})$. Composing with the canonical map from a limit into its homotopy limit we get a generalized map:

$$K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty)) \dashrightarrow^{\beth} W_n(R; M)$$

which we also denote by $\beth$ and which is natural with respect to restriction from n to $n - 1$. That is, we have diagrams:

$$
\begin{array}{ccc}
K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty)) & \dashrightarrow^{\beth} & W_n(R; M) \\
\downarrow & & \downarrow \\
K(\mathcal{P}; \mathfrak{D}(M); P_{n-1}(\Sigma^\infty \Omega^\infty)) & \dashrightarrow^{\beth} & W_{n-1}(R; M)
\end{array}
$$

Both homotopy fibers have been analyzed previously. Up to weak equivalence, the homotopy fiber of the vertical map on the left is

$$(\partial_n(\Sigma^\infty \Omega^\infty) \wedge K (\mathcal{P}; \mathfrak{D}(M); (-)^{\wedge n}))_{h\Sigma_n}$$

while the homotopy fiber of the vertical map on the right is $U^n(R; M)_{hC_n}$ (see Corollary 5.9 [34]). The induced map on homotopy fibers:

$$(\partial_n(\Sigma^\infty \Omega^\infty) \wedge K (\mathcal{P}; \mathfrak{D}(M); (-)^{\wedge n}))_{h\Sigma_n} \dashrightarrow^{\hat{\beth}} U^n(R; M)_{hC_n}$$

is the map induced on homotopy orbits by the stabilization of the map on 0-simplices given by the diagonal inclusion:

$$F_0(\mathcal{P}; \mathfrak{D}(M, \ldots, M)) \to F_0(\mathcal{P}^{\times n}; \mathfrak{D}^\omega(M, \ldots, M))$$

This is precisely the $\diamond$ map from Chapter 3, which by virtue of Theorem 3.3.7 is an equivalence. Therefore the square

$$K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty)) \dashrightarrow^{\beth} W_n(R; M)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$K(\mathcal{P}; \mathfrak{D}(M); P_{n-1}(\Sigma^\infty \Omega^\infty)) \dashrightarrow^{\beth} W_{n-1}(R; M)$$

is homotopy cartesian. In the case of $n = 1$, we have $P_1(\Sigma^\infty \Omega^\infty) \simeq Id \simeq \bigwedge^1(-)$ a 1-homogeneous functor, and so $\beth$ is an equivalence by our previous work. Using the long exact sequence of homotopy groups associated to a homotopy fiber sequence and the 5-lemma, we induct up the ladder to obtain:

$$K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty)) \simeq W_n(R; M)$$

Therefore, $K(\mathcal{P}; \mathfrak{D}(-); P_n(\Sigma^\infty \Omega^\infty))$ is n-excisive, and recalling how the n-truncated) topological Witt vectors appear in the Lindenstrauss-McCarthy tower:

$$P_n\big(K(\mathcal{P}; \mathfrak{D}(-); \Sigma^\infty \Omega^\infty)\big)(M) \simeq K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty))$$

We have shown:

Theorem 4.2.5 *Let R be an associative and unital ring, M an R-bimodule, and $n \geq 1$. There is an equivalence of $\mathbb{S}$-bimodules*

$$K(\mathcal{P}; \mathfrak{D}(M); P_n(\Sigma^\infty \Omega^\infty)) \simeq W_n(R; M)$$

compatible with restrictions from n to $n - 1$. Therefore the tower associated to the K-theory with coefficients in the endofunctor $\Sigma^\infty \Omega^\infty$ is weakly equivalent to the Lindenstrauss-McCarthy Taylor tower of relative K-theory:

$$\tilde{K}(R; M)$$

$$\dots \longleftarrow W_{n+1}(R; M) \longrightarrow W_n(R; M) \longrightarrow W_{n-1}(R; M) \longrightarrow \dots$$

$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$

$$U^{n+1}(R; M)_{hC_{n+1}} \qquad U^n(R; M)_{hC_n} \qquad U^{n-1}(R; M)_{hC_{n-1}}$$

APPENDIX A

SIMPLICIAL SETS

A.1 Simplex Category

Simplicial objects are the non-linear generalizations of chain complexes.

Notation For $n \in \mathbb{N}$, let $[n]$ denote the ordered set $\{0 < 1 < \cdots < n\}$. For any category $\mathcal{C}$ the set of maps between $X, Y \in \mathcal{C}$ will be denoted $\mathcal{C}(X, Y)$.

Definition A.1.1 The <u>simplex category</u>, denoted $\mathbb{\Delta}$, is the category with:

- $\mathrm{Obj}(\mathbb{\Delta}) = \{[n] \mid n \in \mathbb{N}\}$

- $\mathbb{\Delta}([n], [m]) = \{f \in \mathbf{Sets}([n], [m]) \mid f(i) \le f(j) \text{ for } i \le j\}$

that is, non-empty finite totally ordered sets and order-preserving functions.

There are morphisms in $\mathbb{\Delta}$ of particular interest.

Definition A.1.2 Fix $n \in \mathbb{N}$. The ith <u>coface</u> map, denoted $\delta_i^n (= \delta_i)$, is the morphism in $\mathbb{\Delta}([n-1], [n])$ given by

$$\delta_i(j) = \begin{cases} j & \text{for} \quad j < i \\ j+1 & \text{for} \quad j \ge i \end{cases}$$

while the ith <u>codegeneracy</u> map, denoted $\sigma_i^n (= \sigma_i)$, is the morphism in $\mathbb{\Delta}([n+1], [n])$ given by

$$\sigma_i(j) = \begin{cases} j & \text{for} \quad j \le i \\ j-1 & \text{for} \quad j > i \end{cases}$$

Remark The cofaces and codegeneracies generate all other morphisms in $\mathbb{\Delta}$. Indeed, given any $f \in \mathbb{\Delta}([n], [m])$ there is a unique epi-monic factorization

$$f = \delta_{i_1} \circ \cdots \circ \delta_{i_s} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_t}$$

where $0 \leq i_s < \cdots < i_1 \leq m$, $0 \leq j_1 < \cdots < j_t < n$, and $m = n - t + s$. Here $i_1, \ldots, i_s$ are the elements of $[m]$ not in the image of f, in reverse order, and $j_1, \ldots, j_t$ are the elements of $[n]$ such that $f(j) = f(j + 1)$, in order. Therefore to understand the morphisms and compositions in $\mathbb{\Delta}$ we need only understand how the δ_i and σ_j compose.

Observation We have the following identities in $\mathbb{\Delta}$:

$$\delta_i \circ \delta_j = \delta_{j+1} \circ \delta_i \quad \text{for} \quad i \leq j$$

$$\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1} \quad \text{for} \quad i \leq j$$

$$\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & \text{for} \quad i < j \\ \text{id} & \text{for} \quad i = j, j+1 \\ \delta_{i-1} \circ \sigma_j & \text{for} \quad i > j+1 \end{cases}$$

Definition A.1.3 Let $\mathcal{C}$ be a category. A <u>simplicial object</u> in $\mathcal{C}$ is a functor $X : \mathbb{\Delta}^{\text{op}} \to \mathcal{C}$.

Notation If X a simplicial object in $\mathcal{C}$, for ease of notation we will denote $X([n])$ by X_n, the (now) faces $X(\delta_i)$ by d_i, and the (now) degeneracies $X(\sigma_j)$ by s_j.

Remark By the unique epi-monic factorization mentioned above, a simplicial object X is determined by:

- A sequence of objects $X_n \in \mathcal{C}$, for $n \in \mathbb{N}$.

- Face and degeneracy operators, $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $i = 0, \ldots, n$, for every n.

- The "simplicial" identities:

  - $d_j \circ d_i = d_i \circ d_{j+1}$ for $i \leq j$.
  - $s_i \circ s_j = s_{j+1} \circ s_i$ for $i \leq j$.
  - $d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{for} \quad i < j \\ \text{id} & \text{for} \quad i = j, j+1 \\ s_j \circ d_{i-1} & \text{for} \quad i > j+1 \end{cases}$

Example A.1.4 Let $\mathcal{C}$ be a small category. The nerve of the category, denoted $N(\mathcal{C})$, is the simplicial set constructed as follows: The n-simplices are the collection of all diagrams of n-composable morphisms in $\mathcal{C}$,

$$C_0 \xleftarrow{f_1} C_1 \xleftarrow{f_2} \ldots \xleftarrow{f_{n-1}} C_{n-1} \xleftarrow{f_n} C_n$$

That is, as a set, $N_n(\mathcal{C}) \cong \coprod \mathcal{C}(C_1, C_0) \times \mathcal{C}(C_2, C_1) \times \cdots \times \mathcal{C}(C_n, C_{n-1})$, where the coproduct is taken over all $n+1$-tuples of objects $(C_0, \ldots, C_n) \in \mathcal{C}$. The internal faces are given by composing morphisms, while the outer ones are dropping the first/final morphism. That is, $d_i(f_1, \ldots, f_i, f_{i+1}, \ldots, f_n) = (f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n)$ for $0 < i < n$, while $d_0(f_1, f_2 \ldots, f_n) = (f_2, \ldots, f_n)$ and $d_n(f_1, \ldots, f_{n-1}, f_n) = (f_1, \ldots, f_{n-1})$. The degeneracies are given by adding identities. That is, $s_i(f_1, \ldots, f_i, f_{i+1}, \ldots, f_n) = (f_1, \ldots, f_i, \mathrm{id}_{C_i}, f_{i+1}, \ldots, f_n)$.

Example A.1.5 The standard (simplicial) n-dimensional simplex is the representable functor formed from $[n]$. That is, for $[n] \in \mathbf{\Delta}$ we can form the presheaf on $\mathbf{\Delta}$, $\Delta^n(-) := \mathbf{\Delta}(-, [n]) : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$.

Example A.1.6 A classic example of a simplicial object comes from the simplicial bar construction. Using the notation of [39] VII.1, let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category, (A, μ, e) be a monoid in $\mathcal{C}$, (M, ν) a right A-module, and (N, κ) a left A-module. The bar construction, denoted $\mathcal{B}(M, A, N)$, is the simplicial object given by:

$$[n] \mapsto \mathcal{B}_n(M, A, N) := (M \otimes \underbrace{A \otimes \cdots \otimes A}_{n-\text{times}}) \otimes N$$

where A^0 is the empty symbol and all A-parentheses are taken grouped on the left. Let $\alpha_n^{i,j}$ be the chain of associator isomorphisms $\alpha_n^{i,j} : A^{\otimes n} \to (A^{\otimes i} \otimes A^{\otimes j}) \otimes A^{\otimes(n-i-j)}$ with all (internal) parentheses grouped to the left (by the coherence theorem there's no ambiguity in choice). Let μ_n^i be the multiplication of two object after i objects, that is,

$$\mu_n^i = (\mathrm{id}_{A^{\otimes i}} \otimes \mu) \otimes \mathrm{id}_{A^{\otimes(n-i-2)}} : (A^{\otimes i} \otimes A^{\otimes 2}) \otimes A^{\otimes(n-i-2)} \to (A^{\otimes i} \otimes A) \otimes A^{\otimes(n-i-2)}$$

The face maps $d_i : \mathcal{B}_n(M, A, N) \to \mathcal{B}_{n-1}(M, A, N)$ are given by:

$$d_i = \begin{cases} ((\nu \otimes \mathrm{id}_{A^{\otimes(n-1)}}) \circ (\alpha_{M,A,A^{\otimes(n-1)}}^{-1}) \circ (\mathrm{id}_M \otimes \alpha_n^{1,n-1})) \otimes \mathrm{id}_N & \text{for} \quad i = 0 \\ (\mathrm{id}_M \otimes (\alpha_n^{i,1})^{-1} \circ \mu_n^i \circ \alpha_n^{i,2}) \otimes \mathrm{id}_N & \text{for} \quad 0 < i < n \\ \alpha_{M,A^{\otimes(n-1)},N}^{-1} \circ (\mathrm{id}_M \otimes (\mathrm{id}_{A^{\otimes(n-1)}} \otimes \kappa) \circ \alpha_{A^{\otimes(n-1)},A,N}) \circ \alpha_{M,A^{\otimes n},N} & \text{for} \quad i = n \end{cases}$$

In essence, internal faces are the monoid multiplication and the initial/final faces are the actions of right/left modules. The degeneracies, $s_i : \mathcal{B}_n(M, A, N) \to \mathcal{B}_{n+1}(M, A, N)$, involve the the left and right unitors, λ and ρ, as well as the monoid's unit, e.

$$s_i = \begin{cases} \left((\mathrm{id}_M \otimes (\alpha_{n+1}^{1,0})^{-1}) \circ \alpha_{M,A,A^{\otimes n}} \circ (\mathrm{id}_M \otimes e) \circ \rho_M^{-1} \otimes \mathrm{id}_{A^{\otimes n}}\right) \otimes \mathrm{id}_N & \text{for} \quad i = 0 \\ \left(\mathrm{id}_M \otimes ((\alpha_{n+1}^{i,2})^{-1} \circ (\mathrm{id}_{A^{\otimes i}} \otimes (e \otimes \mathrm{id}_A) \circ \lambda_A^{-1} \otimes \mathrm{id}_{A^{\otimes(n-i-1)}}) \circ \alpha_n^{i,1})\right) \otimes \mathrm{id}_N & \text{for} \quad 0 < i < n \\ (\alpha_{M,A^{\otimes n},A} \otimes \mathrm{id}_N) \circ \alpha_{M \otimes A^{\otimes n},A,N}^{-1} \circ \mathrm{id}_{M \otimes A^{\otimes n}} \otimes ((e \otimes \mathrm{id}_N) \circ \lambda_N^{-1}) & \text{for} \quad i = n \end{cases}$$

That these morphisms satisfy the simplicial identities is exactly the commutativity and coherence conditions in the definitions of $\alpha, \lambda, \rho, \mu, e, \dots$ etc. Because of the generality of this construction we can get many examples in common monoidal categories. Lastly, because of MacLane's coherence theorem (VII.2 [39]), any of diagrams constructed involving the associators will commute, and therefore subsequently we may drop the "coherence part" of the data.

Example A.1.7 Let $(\mathcal{C}, \otimes, \mathbb{1})$ be the monoidal category $(\mathbf{Sets}, \times, *)$ of sets with cartesian product, and a chosen one-element set as unit. Let (G, μ, e) be a monoid in this category, with unit $e(*) = 1_G \in G$ (i.e. a "monoid" in the traditional sense). Take $*$ as both trivial right G-module and trivial left G-module. Then $\mathcal{B}_n(*, G, *) = * \times G^{\times n} \times * \cong G^{\times n}$ (with the 0-simplex the point $*$). The face maps are given, under the previous isomorphism, by:

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{for} \quad i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{for} \quad 0 < i < n \\ (g_1, \dots, g_{n-1}) & \text{for} \quad i = n \end{cases}$$

and degeneracies by $s_i(g_1, \dots, g_n) = (g_1, \dots, g_i, 1_G, g_{i+1}, \dots, g_n)$, that is, sticking a 1_G in the ith-spot. This construction is typically referred to as the "bar construction on G", denoted $\mathcal{B}G$.

Remark The bar construction on G can also be achieved by means of the nerve construction mentioned before. A monoid can be seen as a one element category. Indeed, let G be a monoid in $(\mathbf{Sets}, \times, *)$. Denote by $\tilde{G}$ the category with object $+$, and morphisms $\tilde{G}(+, +) = G$, where the composition is the monoid multiplication (the monoid axioms on G ensure this is a category). Then we have an isomorphism of simplicial sets $N(\tilde{G}) \cong \mathcal{B}G$.

Example A.1.8 Fix a commutative ring k. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be the monoidal category (k-Mod, $\otimes_k, k$) of k-modules and tensoring over k. A monoid in this category is a (possibly

non-commutative) k-algebra A. Let M and N be right and left A-modules, respectively (they are symmetric over k since k is commutative). Then $\mathcal{B}(M, A, N)$ is the "simplicial bar resolution" of $M \underset{k}{\otimes} N$. The name comes from the fact that the nth-homology of the chain complex associated to this simplicial object in k-Mod computes $\text{Tor}_n^A(M, N)$.

A.2 Simplicial Maps and Homotopies

Definition A.2.1 Let $\mathcal{C}$ be a category, X, Y simplicial objects in $\mathcal{C}$. A <u>simplicial map</u> from X to Y, denoted $F : X \to Y$, is a natural transformation of the functors $\mathbf{\Delta}^{\text{op}} \to \mathcal{C}$. By the previous remark, this is determined by a collection of morphisms in $\mathcal{C}$, $F_n : X_n \to Y_n$ for $n \in \mathbb{N}$, commuting with the face and degeneracy operators. That is, $d_i^Y \circ F_n = F_{n-1} \circ d_i^X$ and $s_i^Y \circ F_n = F_{n+1} \circ s_i^X$ for $i = 0, \ldots, n$. The category of simplicial objects in $\mathcal{C}$ together with simplicial maps will be denoted $s\mathcal{C}$.

Observation If $X \in s\mathbf{Sets}$, then by the Yonneda Lemma we have an isomorphism of sets $s\mathbf{Sets}(\Delta^n, X) \cong X_n$, natural in n and X. So an n-simplex in X determines a unique simplicial map from Δ^n.

Example A.2.2 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. Then there is an induced simplicial map of nerves, $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ given by

$$C_0 \xleftarrow{f_1} C_1 \xleftarrow{f_2} \ldots \xleftarrow{f_{n-1}} C_{n-1} \xleftarrow{f_n} C_n \quad \mapsto$$

$$F(C_0) \xleftarrow{F(f_1)} F(C_1) \xleftarrow{F(f_2)} \ldots \xleftarrow{F(f_{n-1})} F(C_{n-1}) \xleftarrow{F(f_n)} F(C_n)$$

That is, $N(F)_n(f_1, \ldots, f_n) = (F(f_1), \ldots, F(f_n))$.

Example A.2.3 Going back to the case of a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$, suppose given *two* triples (M, A, N), (M', A', N') of monoids in $\mathcal{C}$ with right and left modules, respectively. Suppose $\phi : A \to A'$ is a morphism of monoids (so that it preserves the multiplication and the unit), and $f : M \to M'$ and $g : N \to N'$ are morphisms of right modules and left modules, respectively (so they preserve the action of the chosen monoid). The morphisms

$$\mathcal{B}_n(M, A, N) \xrightarrow{f \otimes \phi \otimes \cdots \otimes \phi \otimes g} \mathcal{B}_n(M', A', N')$$

then assemble to give a simplicial map of the simplicial bar constructions $\mathcal{B}(M, A, N) \to$

$\mathcal{B}(M', A', N')$. In particular, when working in $(\mathbf{Sets}, \times, *)$, a map of monoids $\phi : G \to H$ induces a map of bar constructions $\mathcal{B}G \to \mathcal{B}H$

Definition A.2.4 Let $\mathcal{C}$ be a small category, X, Y simplicial objects in $\mathcal{C}$, and $F, G : X \to Y$ simplicial maps. F is <u>simplicially homotopic</u> to G, denoted $F \simeq G$, if for each $n \in \mathbb{N}$ and $i = 0, \ldots, n$ there exist morphisms in $\mathcal{C}$, $h_i \in \mathcal{C}(X_n, Y_{n+1})$, such that:

- $d_0 \circ h_0 = F$ and $d_{n+1} \circ h_n = G$.

- $d_i \circ h_j = \begin{cases} h_{j-1} \circ d_i & \text{for} \quad i < j \\ d_{j+1} \circ h_{j+1} & \text{for} \quad i = j + 1 \\ h_j \circ d_{i-1} & \text{for} \quad i > j + 1 \end{cases}$

- $s_i \circ h_j = \begin{cases} h_{j+1} \circ s_i & \text{for} \quad i \leq j \\ h_j \circ s_{i-1} & \text{for} \quad i > j \end{cases}$

Example A.2.5 Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between small categories, and $\eta : F \Rightarrow G$ a natural transformation between them. We get two induced maps of simplicial sets, $N(F), N(G) : N(\mathcal{C}) \to N(\mathcal{D})$. We define a simplicial homotopy between them as follows. For every $n \in \mathbb{N}, i = 0, \ldots, n$, let $h_i \in \mathbf{Sets}(N_n(\mathcal{C}), N_{n+1}(\mathcal{D}))$ be given by:

$$h_i(f_1, \ldots, f_n) = (G(f_1), \ldots, G(f_i), \eta_{C_i}, F(f_{i+1}), \ldots, F(f_n))$$

One readily checks the identities and finds that $N(F) \simeq N(G)$.

A.3 Pointed Simplicial Sets

Let $s\mathbf{Sets}$ be the category of simplicial sets, and consider the standard 0-simplex Δ^0. Since there is only one morphism from $[n]$ to $[0]$ in $\mathbb{\Delta}$, Δ^0 is the terminal object in $s\mathbf{Sets}$.

Definition A.3.1 A <u>pointed simplicial set</u> is a simplicial set X together with a choice of simplicial map $\Delta^0 \to X$.

Observation This is equivalent to a choice of distinguished basepoint $* \in X_0$ (by the Yoneda Lemma); this determines the $*$ in each other simplicial degree, so we make no notational distinction.

A morphism of pointed simplicial sets will be a morphism in $s\mathbf{Sets}$ making the requisite triangle commute (i.e. preserving the basepoints $*_X \in X_0, *_Y \in Y_0$). We denote the category of pointed simplicial sets and pointed simplicial maps by $s\mathbf{Sets}_*$.

Remark Given an unbased simplicial set X we can canonically add a basepoint by defining $X_+ := X \amalg \Delta^0$. This functor $(-)_+ : s\mathbf{Sets} \to s\mathbf{Sets}_*$ is left adjoint to the forgetful functor $U : s\mathbf{Sets}_* \to s\mathbf{Sets}$ from pointed simplicial sets to simplicial sets (that is, that forgets the choice of morphism $\Delta^0 \to X$).

The category $s\mathbf{Sets}_*$ is symmetric monoidal. The monoidal functor is the smash product: given two pointed simplicial sets X, Y, we form the pointed simplicial set $X \wedge Y$ with n-simplices $X_n \wedge Y_n$, where $X_n \wedge Y_n$ is the quotient $(X_n \times Y_n)/(X_n \vee Y_n)$ that collapses the simplicial subset $X_n \vee Y_n = X_n \times *_Y \cup *_X \times Y_n$ of the cartesian product. The face and degeneracies are given by $d_i^{X \wedge Y} := d_i^X \wedge d_i^Y$ and $s_i^{X \wedge Y} := s_i^X \wedge s_i^Y$. This is clearly a bifunctor $s\mathbf{Sets}_* \times s\mathbf{Sets}_* \to s\mathbf{Sets}_*$. The 0-sphere, $S^0 := \Delta^0/\partial\Delta^0$ ("=" Δ^0_+, since we collapsed the empty set), is the unit with respect to the smash product. The smash product is associative, $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$ and there is a twist isomorphism $X \wedge Y \cong Y \wedge X$. All of these isomorphisms satisfy the necessary coherence conditions of VII.2 [39].

Furthermore, it is *closed* symmetric monoidal. That is, for each simplicial set Y, the functor $- \wedge Y : s\mathbf{Sets}_* \to s\mathbf{Sets}_*$ has a right adjoint, denoted $\underline{s\mathbf{Sets}}_*(Y, -)$. Explicitly, it is given by $[n] \mapsto (\underline{s\mathbf{Sets}}_*(Y, -))_n := s\mathbf{Sets}_*(Y \wedge \Delta^n_+, -)$. Note that the Δ^n assemble into a cosimplicial object in $s\mathbf{Sets}$, $\Delta^- : \mathbb{\Delta} \to s\mathbf{Sets}$, i.e. a cosimplicial simplicial set. Adding a disjoint basepoint and putting it in the contravariant entry gives a simplicial set. Closure then specializes to an adjunction:

$$s\mathbf{Sets}_*(X \wedge Y, Z) \cong s\mathbf{Sets}_*(X, \underline{s\mathbf{Sets}}_*(Y, Z))$$

Furthermore, this bijection of sets extends to an enriched adjunction, i.e. we have an isomorphism of *simplicial sets*:

$$\underline{s\mathbf{Sets}}_*(X \wedge Y, Z) \cong \underline{s\mathbf{Sets}}_*(X, \underline{s\mathbf{Sets}}_*(Y, Z))$$

Observation By pre- and post-composing with the twist isomorphism included in the symmetric monoidal structure, we also obtain an adjunction:

$$s\mathbf{Sets}_*(Y \wedge X, Z) \cong s\mathbf{Sets}_*(X, \underline{s\mathbf{Sets}}_*(Y, Z))$$

denoted by $\psi \mapsto \psi^\sharp$ (with inverse $\phi \mapsto \phi^\flat$).

Notation The (simplicial) 1-sphere will be denoted $S^1 := \Delta^1/\partial\Delta^1$. The n-sphere is defined

to be the n-fold smash of S^1, $S^n := \overbrace{S^1 \wedge \cdots \wedge S^1}^{n-\text{times}}$. Lastly, when we'll have occasion to talk about the enriched category of pointed simplicial sets (enriched over itself, as any closed symmetric monoidal category is), we will denote it by $s\mathbf{Sets}_*$.

A.4 A Convenient Category of Spaces

Definition A.4.1 A topological space X is <u>compactly generated</u> if it is a weak Hausdorff k-space (Definition 6.1.8 [54]). The category of compactly generated spaces will be denoted $\mathbf{Top}$, and the category of pointed compactly generated spaces (with non-degenerate basepoints) and pointed continuous maps will be denoted $\mathbf{Top}_*$.

Remark Limits in $\mathbf{Top}_*$ are simply the k-ification of the usual limits in the category of topological spaces (limits of weak Hausdorff spaces are weak Hausdorff), while colimits are the weak Hausdorffication of the usual colimits (colimits of k-spaces are k-spaces). Often these will reduce to the usual limits and colimits. For example, if $X, Y \in \mathbf{Top}_*$ and one happens to be locally compact, then their product in the ordinary category of topological spaces is *already* in $\mathbf{Top}_*$, so k-ification does nothing. On the other hand, if a diagram in $\mathbf{Top}_*$ is given by a sequence of closed inclusions or by a pushout along a closed inclusion, then its colimit in the ordinary category of topological spaces is *already* in $\mathbf{Top}_*$, so weak Hausdorffication does nothing (5.2 [42]).

The main purpose of working in $\mathbf{Top}_*$ rather than a more general category of topological spaces is that we have a closed symmetric monoidal structure on $\mathbf{Top}_*$ directly analogous to the one in $s\mathbf{Sets}_*$, given by the smash product of spaces. Indeed, we have an adjunction

$$\mathbf{Top}_*(Y \wedge X, Z) \cong \mathbf{Top}_*(X, \underline{\mathbf{Top}}_*(Y, Z))$$

where the "internal hom" is the k-ification of the compact-open topology of the set of continuous maps $Y \to Z$ (it is already weak Hausdorff if Y is as well).

Just as before, by pre- and post-composing with the twist isomorphism included in the symmetric monoidal structure, we again obtain an adjunction:

$$\mathbf{Top}_*(Y \wedge X, Z) \cong \mathbf{Top}_*(X, \underline{\mathbf{Top}}_*(Y, Z))$$

denoted by $\psi \mapsto \psi^\sharp$ (with inverse $\phi \mapsto \phi^\flat$). It will be clear when working with either topological spaces or simplicial sets, so that no confusion may arise. Lastly, because we're working in $\mathbf{Top}_*$, this bijection of sets extends to an enriched adjunction, i.e. we have a *homeomorphism*:

$$\underline{\mathbf{Top}}_*(Y \wedge X, Z) \cong \underline{\mathbf{Top}}_*(X, \underline{\mathbf{Top}}_*(Y, Z))$$

Notation We define the (topological) n-sphere, S^n as the one-point compactification of $\mathbb{R}^n$, and the loop spaces $\Omega^n X := \underline{\mathbf{Top}}_*(S^n, X)$. Just as with $\underline{s\mathbf{Sets}}_*$, when we'll have occasion to talk about the enriched category of pointed topological spaces (enriched over itself, as any closed symmetric monoidal category is), we will denote it by $\underline{\mathbf{Top}}_*$.

A.5   Geometric Realization

Definition A.5.1 The standard (topological) n-dimensional simplex, denoted $\underline{\Delta}^n$, is the topological space defined as the subspace

$$\underline{\Delta}^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0, \sum x_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

Let $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith-vertex of $\underline{\Delta}^n$. Then we have (closed) inclusions $\delta_i : \underline{\Delta}^{n-1} \to \underline{\Delta}^n$, called the ith-coface, given by including into the face opposite v_i, that is, setting $x_i = 0$ (there are $n + 1$ total such maps). We also have $n + 1$ surjections $\sigma_i : \underline{\Delta}^{n+1} \to \underline{\Delta}^n$, called the ith-codegeneracy, given by projecting linearly from vertices with $\sigma_i(v_{i+1}) = v_i$ and $\sigma_i(v_j) = v_j$ otherwise.

Observation $\underline{\Delta}^n$ is a (locally) compact Hausdorff space, and hence compactly generated, for all n.

One readily checks that the cofaces and codegeneracies assemble to make $\underline{\Delta}^-$ into a cosimplicial space, that is, we have a covariant functor $\underline{\Delta}^- : \mathbb{\Delta} \to \mathbf{Top}$. Now, let X be a (possibly unpointed) simplicial set. By using the inclusion $\mathbf{Sets} \hookrightarrow \mathbf{Top}$ (giving a set the discrete topology, which is clearly compactly generated) and the fact that $\mathbf{Top}$ has finite products (the $\underline{\Delta}^n$ are locally compact, so the usual product of spaces works without the k-ification), we get a functor $X_- \times \underline{\Delta}^- : \mathbb{\Delta}^{\text{op}} \times \mathbb{\Delta} \to \mathbf{Top}$.

Definition A.5.2 Let X be a simplicial set. The geometric realization of X, denoted $|X|$, is the coend $\int^{[n]:\mathbb{\Delta}^{\text{op}}} X_n \times \underline{\Delta}^n$. By the coend formula and the fact that (co)faces and

(co)degeneracies generate $\mathbb{\Delta}$, it can be explicitly computed as

$$|X| = \coprod_{n=0}^{\infty} X_n \times \underline{\Delta}^n \bigg/ \sim$$

where $(x, \delta_i(p)) \sim (d_i(x), p)$ and $(x, \sigma_i(p)) \sim (s_i(x), p)$.

Remark The coend actually exists in the target category, that is, $|X|$ *is* a compactly generated space (one sees this by noting that the filtration by partial realization gives $|X|$ the structure of a CW-complex). Also, geometric realization takes the standard simplicial n-simples to the topological one. That is, $|\Delta^n| \cong \underline{\Delta}^n$ (so our notation is in agreement). If $f : X \to Y$ is a morphism of simplicial sets, then the universal property for dinatural transformations of the coend induces a continuous map $|f| : |X| \to |Y|$, which is simply f on the representing simplex. We thus have a functor $|-| : s\mathbf{Sets} \to \mathbf{Top}$. In addition, if X was a *pointed* simplicial set, with specified map $f : \Delta^0 \to X$, then $|f| : |\Delta^0| \cong * \to |X|$ makes $|X|$ into a pointed topological space. Therefore, geometric realization gives a functor $|-| : s\mathbf{Sets}_* \to \mathbf{Top}_*$.

Example A.5.3 There is an important functor $\mathrm{Sing}(-) : \mathbf{Top} \to s\mathbf{Sets}$, called the "singular complex", whose n-simplices on X are given by $[n] \mapsto \mathbf{Top}(\underline{\Delta}^n, X)$, that is, the set of maps from the standard topological n-simplex into X. Note that this is the composition of the covariant functor $\underline{\Delta}^- : \mathbb{\Delta} \to \mathbf{Top}$ and the representable (contravariant) functor $h_X = \mathbf{Top}(-, X) : \mathbf{Top} \to \mathbf{Sets}$. Hence, for fixed X, gives a functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$, i.e. a simplicial set. If $f : X \to Y$ is a countinuous map of topological spaces, we get an induced natural transformation of representing functors $h_X \Rightarrow h_Y$, and therefore a natural transformation of functors $\mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Sets}$, i.e. a map of simplicial sets $\mathrm{Sing}(X) \xrightarrow{\mathrm{Sing}(f)} \mathrm{Sing}(Y)$. On an n-simplex, this is simply given by post composing $\underline{\Delta}^n \xrightarrow{g} X$ with $X \xrightarrow{f} Y$. The face and degeneracy operators on $\mathrm{Sing}(X)$ are pre-compositions with the coface and codegeneracy maps, δ_i and σ_i, of $\underline{\Delta}^-$.

Fact The functor $\mathrm{Sing}(-) : \mathbf{Top} \to s\mathbf{Sets}$ is left adjoint to geometric realization, $|-| : s\mathbf{Sets} \to \mathbf{Top}$. That is, for a simplicial set X and a topological space Y, we have an bi-natural isomorphism

$$\mathbf{Top}(|X|, Y) \cong s\mathbf{Sets}(X, \mathrm{Sing}(Y))$$

Immediately we get for free that geometric realization commutes with colimits (as a left adjoijnt) while the singular complex commutes with limits (as a right adjoint). On the other

hand, more is known: by Theorem 2 [48] geometric realization commutes with finite limits (here the use of compactly generated spaces is crucial).

Observation Since $\mathrm{Sing}(\underline{\Delta}^0)_n = \mathbf{Top}(\underline{\Delta}^n, \underline{\Delta}^0) = *$, we have an isomorphism of simplicial sets $\mathrm{Sing}(\underline{\Delta}^0) \cong \Delta^0$. A pointed topological space, $* \cong \underline{\Delta}^0 \to X$ is then sent to $\Delta^0 \cong \mathrm{Sing}(\underline{\Delta}^0) \to \mathrm{Sing}(X)$, a pointed simplicial set, and we get an induced functor $\mathrm{Sing}(-) : \mathbf{Top}_* \to s\mathbf{Sets}_*$. We have a similar adjunction in the pointed case. That is, for a pointed simplicial set X and a pointed topological space Y, we have an bi-natural isomorphism

$$\mathbf{Top}_*(|X|, Y) \cong s\mathbf{Sets}_*(X, \mathrm{Sing}(Y))$$

Lastly, the homeomorphisms making geometric realization finitely continuous descend to the pointed setting, making $|-| : s\mathbf{Set}_* \to \mathbf{Top}_*$ strong symmetric monoidal.

Remark The singular complex functor allows us to view $\mathbf{Top}_*$ as a *simplicially enriched category* also, by taking, for $X, Y \in \mathbf{Top}_*$, the simplicial mapping space $\mathrm{Sing}(\underline{\mathbf{Top}}_*(X, Y))$.

Example A.5.4 We define an explicit homeomorphism $f : (\underline{\Delta}^1)^\circ \to \mathbb{R}$, from the interior of the standard topological 1-simplex to $\mathbb{R}$, given by $f(x, y) := \frac{x}{y} - \frac{y}{x}$. Post-composing with the homeomorphism $|\Delta^1| \cong \underline{\Delta}^1$ we see that $|\partial\Delta^1|$ is identified with the boundary of $\underline{\Delta}^1$, and we obtain a homeomorphism $|\Delta^1|/|\partial\Delta^1| \cong S^1$. But since $|-|$ commutes with colimits we obtain a homemorphism $|S^1| \cong S^1$ (so our notation is in agreement). Similarly, using the strong symmetric monoidal structure of $|-|$, we have:

$$|S^n| := |S^1 \wedge \cdots \wedge S^1| \cong |S^1| \wedge \cdots \wedge |S^1| \cong S^1 \wedge \cdots \wedge S^1 \cong S^n$$

Notation Let X be a simplicial set. Denote by $\Omega^n X := \mathrm{Sing}\big(\underline{\mathbf{Top}}_*(|S^n|, |X|)\big)$ the (simplicial) n^{th}-loop space of X. By the previous example, we have that $\Omega^n X \cong \mathrm{Sing}(\Omega^n |X|)$.

Example A.5.5 Let G be a group and $\tilde{G}$ its one-element category. A simplicial set X with a G-action can be described as a G-object in simplicial sets, that is, a functor $\tilde{X} : \tilde{G} \to Fun(\underline{\Delta}^{\mathrm{op}}, \mathbf{Sets}) = s\mathbf{Sets}$. Composing with geometric realization we get a functor

$$\tilde{G} \xrightarrow{\tilde{X}} s\mathbf{Sets} \xrightarrow{|-|} \mathbf{Top}$$

Since geometric realization is finitely continuous, we have an isomorphism $|\lim \tilde{X}| \xrightarrow{\cong} \lim |\tilde{X}|$. Now taking the limit of any G-object (functor $\tilde{G} \to \mathcal{C}$ into a category) is the fixed-points of the action. So we have an isomorphism $|X^G| \cong |X|^G$, meaning we can take the fixed points

as a simplicial set and realize, or simply as the associated topological space. The same holds true in the pointed case.

A.6   Model Structure on Simplicial Sets

Definition A.6.1 A map of simplicial sets $f : X \to Y$ is a <u>weak equivalence</u> if $|f| : |X| \to |Y|$ is a weak equivalence of spaces, or equivalently a homotopy equivalence. It is called a <u>cofibration</u> if f is a categorical monomorphism, or equivalently, a level-wise injection of sets. It is called a <u>(Kan) fibration</u> if it has the right-lifting-property with respect to all trivial cofibrations. A simplicial set X is <u>fibrant</u> if $X \to *$ is a fibration.

The main purpose of simplicial sets is to be able to do "combinatorial" homotopy theory. That mainly comes from the following result of Quillen (Theorems 1 and 3, Chapter II [52]):

Theorem A.6.2 *The cofibrations, weak equivalences, and fibrations make $\underline{s\textbf{Sets}}_*$ into a proper model category. The model structure is monoidal with respect to the smash product. The adjoint functors:*

$$\underline{s\textbf{Sets}}_* \quad \overset{|-|}{\underset{\text{Sing}(-)}{\rightleftarrows}} \quad \bot \quad \underline{\textbf{Top}}_*$$

are a Quillen equivalence.

Fact The units and counits from the above adjunction are weak equivalences. In particular by the counit map we have $|\Omega^n X| \xrightarrow{\simeq} \Omega^n |X|$ for any simplicial set X. If $f : X \to Y$ is a weak equivalence of pointed simplicial sets then combining the previous fact with the 2-out-of-3 property for weak equivalences in $\underline{\textbf{Top}}_*$ shows that $\Omega^n X \to \Omega^n Y$ is a weak equivalence of pointed simplicial sets.

Remark If X, Y are simplicial sets, with X cofibrant (trivially) and Y fibrant (not true in general), then as a consequence of A.6.2 the induced map $|\underline{s\textbf{Sets}}_*(X,Y)| \to \underline{\textbf{Top}}_*(|X|,|Y|)$ is a weak equivalence of topological spaces. Taking the singular complex functor and precomposing with the unit map, we get a chain of weak equivalences:

$$\underline{s\textbf{Sets}}_*(X,Y) \xrightarrow{\simeq} \text{Sing}\big(|\underline{s\textbf{Sets}}_*(X,Y)|\big) \xrightarrow{\simeq} \text{Sing}\big(\underline{\textbf{Top}}_*(|X|,|Y|)\big)$$

In particular, taking the simplicial set S^n and a fibrant simplicial set X we have:

$$\underline{s\mathbf{Sets}}_*(S^n, X) \xrightarrow{\cong} \mathrm{Sing}\big(|\underline{s\mathbf{Sets}}_*(S^n, X)|\big) \xrightarrow{\cong} \Omega^n X$$

We therefore have a weak equivalence between the internal hom object of simplicial sets $\underline{s\mathbf{Sets}}_*(S^n, X)$ and the simplicial loop space functor $\Omega^n X$. However, it only holds true for X a *fibrant* simplicial set. So the realization does not in general commute with taking internal homs, and we lose out on the above fact. This is why internal hom were used for topological loop spaces, but not in the case of simplicial sets.

Because of the above constraints, it will be convenient for us moving forward to fix a functorial way to exchange a simplicial set X for a weakly equivalent fibrant one.

Definition A.6.3 The functorial fibrant replacement functor, denoted $\widehat{-}$, will be the functor $\mathrm{Sing}(|-|) : \underline{s\mathbf{Sets}}_* \to \underline{s\mathbf{Sets}}_*$. So for any simplicial set X, we have the unit weak equivalence $X \to \widehat{X}$ with $\widehat{X}$ fibrant.

APPENDIX B

HOMOTOPY COLIMITS AND LIMITS

The main references for this appendix are [10] and [28].

B.1   Slice and Coslice Categories

Definition B.1.1 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and $d \in \mathcal{D}$. The <u>slice category (over d)</u>, denoted $F \downarrow d$, is the category with:

- $\mathrm{Obj}(F \downarrow d) = \{(c, \alpha) \mid c \in \mathcal{C}, \ \alpha \in \mathcal{D}(F(c), d)\}$

- $(F \downarrow d)\big((c, \alpha), (c', \alpha')\big) = \{\beta \mid \beta \in \mathcal{C}(c, c') \text{ and } \alpha' \circ F(\beta) = \alpha\}$

That is, the morphisms $(c, \alpha) \xrightarrow{\tilde{\beta}} (c', \alpha')$ are the morphisms $\beta : c \to c'$ in $\mathcal{C}$ making the following triangle commute:

$$
\begin{array}{ccc}
F(c) & \xrightarrow{\quad F(\beta) \quad} & F(c') \\
 & \searrow^{\alpha} \quad \swarrow^{\alpha'} & \\
 & d &
\end{array}
$$

Definition B.1.2 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and $d \in \mathcal{D}$. The <u>coslice category (under d)</u>, denoted $d \downarrow F$, is the category with:

- $\mathrm{Obj}(d \downarrow F) = \{(c, \alpha) \mid c \in \mathcal{C}, \ \alpha \in \mathcal{D}(d, F(c))\}$

- $(d \downarrow F)\big((c, \alpha), (c', \alpha')\big) = \{\beta \mid \beta \in \mathcal{C}(c, c') \text{ and } F(\beta) \circ \alpha = \alpha'\}$

That is, the morphisms $(c, \alpha) \xrightarrow{\tilde{\beta}} (c', \alpha')$ are the morphisms $\beta : c \to c'$ in $\mathcal{C}$ making the following triangle commute:

$$
\begin{array}{ccc}
 & d & \\
 \swarrow^{\alpha} & & \searrow^{\alpha'} \\
F(c) & \xrightarrow{\quad F(\beta) \quad} & F(c')
\end{array}
$$

Notation In the case of $F = Id : \mathcal{C} \to \mathcal{C}$, we denote $(c \downarrow \mathcal{C}) := (c \downarrow Id)$, and similarly for the slice category.

Remark The slice and coslice categories are covariant and contravariant, respectively, in the object $d \in \mathcal{D}$. Indeed, if $d \xrightarrow{f} d'$ is a morphism in $\mathcal{D}$, then we have induced functors $f_* : (F \downarrow d) \to (F \downarrow d')$ and $f^* : (d' \downarrow F) \to (d \downarrow F)$ given by $f_*(c, \alpha) = (c, f \circ \alpha)$ and $f^*(c, \alpha) = (c, \alpha \circ f)$. Also, if $d' \xrightarrow{g} d$, then $g_* \circ f_* = (g \circ f)_*$ and $f^* \circ g^* = (g \circ f)^*$. Lastly, if $F : \mathcal{C} \to \mathcal{D}$ is a functor, and $c \in \mathcal{C}$ there are induced functors $\hat{F} : (\mathcal{C} \downarrow c) \to (\mathcal{D} \downarrow F(c))$ and $\tilde{F} : (c \downarrow \mathcal{C}) \to (F(c) \downarrow \mathcal{D})$.

On the other hand, if $\eta : F \to G$ is a natural transformation of functors $\mathcal{C} \to \mathcal{D}$, then we have induced functors of slice categories in the opposite direction, $(G \downarrow d) \to (F \downarrow d)$, given by $(c, \alpha) \mapsto (c, \alpha \circ \eta_c)$. For coslice categories the behavior is functorial; the induced functor is $\tilde{\eta} : (d \downarrow F) \to (d \downarrow G)$ given by $(c, \alpha) \mapsto (c, \eta_c \circ \alpha)$.

Observation Slice and coslice categories are "dual". Specifically, $(c \downarrow \mathcal{C}) \cong (\mathcal{C}^{\mathrm{op}} \downarrow c)^{\mathrm{op}}$.

The nerve of slice and coslice categories is going to be very important. By the previous note, we need only consider the case of coslice categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, $d \in \mathcal{D}$, and consider the simplicial nerve $N_\bullet(d \downarrow F)$. In simplicial degree n, it consists of the sets of diagrams:

$$
\begin{array}{c}
d \\
F(c_0) \xleftarrow{\beta_1} F(c_1) \xleftarrow{\beta_2} \cdots \xleftarrow{\beta_n} F(c_n)
\end{array}
$$

with $\alpha_0, \alpha_1, \alpha_n$

which by commutativity consists of an n-simplex in the nerve of $\mathcal{D}$, $(\beta_1, \ldots, \beta_n)$ and a morphism $d \xrightarrow{\alpha_n} F(c_n)$. If $d \xrightarrow{f} d'$ is a morphism in $\mathcal{D}$, the functor $f^* : (d' \downarrow F) \to (d \downarrow F)$ induces maps of simplicial sets $N(f^*) : N(d' \downarrow F) \to N(d \downarrow F)$, and assembles into a presheaf of simplicial sets $N(- \downarrow F) : \mathcal{D}^{\mathrm{op}} \to s\mathbf{Sets}$.

Also by the previous remarks, if $c \in \mathcal{C}$ then we have an induced map of nerves $\tilde{F} : N(c \downarrow \mathcal{C}) \to N(F(c) \downarrow \mathcal{D})$, which, for a morphism $c \xrightarrow{f} c'$ in $\mathcal{C}$, fits into a commutative square of simplicial sets:

$$
\begin{array}{ccc}
N(c' \downarrow \mathcal{C}) & \xrightarrow{f^*} & N(c \downarrow \mathcal{C}) \\
\downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle \tilde{F}} \\
N(F(c') \downarrow \mathcal{D}) & \xrightarrow{F(f)^*} & N(F(c) \downarrow \mathcal{D})
\end{array}
$$

That is, we have a natural transformation of functors (map of simplicial presheaves on $\mathcal{C}$) $N(- \downarrow \mathcal{C}) \Rightarrow N(F(-) \downarrow \mathcal{D})$.

B.2 Homotopy Colimits as Coends

Definition B.2.1 Let $\mathcal{C}$ be a small category and $\mathcal{X} : \mathcal{C} \to s\mathbf{Sets}_*$ a $\mathcal{C}$-diagram of pointed simplicial sets. Consider the bifunctor $\mathcal{X}(-) \wedge N(- \downarrow \mathcal{C})_+ : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to s\mathbf{Sets}_*$. The (uncorrected) homotopy colimit of $\mathcal{X}$, denoted $\mathrm{hocolim}_{\mathcal{C}}\mathcal{X}$, is the coend

$$\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} := \int^{c \in \mathcal{C}} \mathcal{X}(c) \wedge N(c \downarrow \mathcal{C})_+$$

By the coend formula this is given by:

$$\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} = \mathrm{coeq}\left(\bigvee_{c_0 \xrightarrow{f} c_1 \in \mathcal{C}} \mathcal{X}(c_0) \wedge N(c_1 \downarrow \mathcal{C})_+ \underset{\psi}{\overset{\phi}{\rightrightarrows}} \bigvee_{c \in \mathcal{C}} \mathcal{X}(c) \wedge N(c \downarrow \mathcal{C})_+ \right)$$

where on a summand $c_0 \xrightarrow{f} c_1$, ϕ is given by $\mathcal{X}(f) \wedge \mathrm{id}_{N(c_1 \downarrow \mathcal{C})_+}$ into the c_1-summand, and ψ is given by $\mathrm{id}_{\mathcal{X}(c_0)} \wedge f^*$ into the c_0-summand.

Fact Since geometric realization is a left adjoint, we have $|\mathrm{hocolim}_{\mathcal{C}}\mathcal{X}| \cong \mathrm{hocolim}_{\mathcal{C}}|\mathcal{X}|$. If $\mathcal{Y}$ is another $\mathcal{C}$-diagram in pointed simplicial sets, and $\eta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation, then by the universal property of colimits, we get an induced map $\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} \xrightarrow{\tilde{\eta}} \mathrm{hocolim}_{\mathcal{C}}\mathcal{Y}$. If η is a point-wise weak equivalence (meaning for each object c, $\eta_c : \mathcal{X}(c) \to \mathcal{Y}(c)$ is a weak equivalence of simplicial sets), then $\tilde{\eta}$ is a weak equivalence. Some additional properties of the homotopy colimit include:

- If $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ has $\mathcal{X}(c) = *$ for all c, then $\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} \cong *$.

- If $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ has $\mathcal{X}(c) = S^0$ for all c (with identity maps), then $\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} \cong N(\mathcal{C})_+$.

- More generally, if $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ has $\mathcal{X}(c) = X$ for all c for some fixed $X \in \underline{s\mathbf{Sets}_*}$, then $\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} \cong X \wedge N(\mathcal{C})_+$.

- If $\mathcal{C}$ is the trivial category with a single object $\bullet$ and identity morphism $\circlearrowleft$, $\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} \cong X(\bullet)$.

- For any $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ we have a map of pointed simplicial sets $\mathrm{hocolim}_{\mathcal{C}}\mathcal{X} \to \mathrm{colim}_{\mathcal{C}}\mathcal{X}$ such that, if $\eta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation of $\mathcal{C}$-diagrams, the following square commutes:

$$\begin{array}{ccc}
\operatorname{hocolim}_{\mathcal{C}}\mathcal{X} & \xrightarrow{\tilde{\eta}} & \operatorname{hocolim}_{\mathcal{C}}\mathcal{Y} \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\mathcal{C}}\mathcal{X} & \longrightarrow & \operatorname{colim}_{\mathcal{C}}\mathcal{Y}
\end{array}$$

- If $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ and $F : \underline{s\mathbf{Sets}_*} \to \underline{s\mathbf{Sets}_*}$ is a *simplicially enriched* functor, then we get natural maps of pointed simplicial sets $\operatorname{hocolim}_{\mathcal{C}}(F \circ \mathcal{X}) \to F(\operatorname{hocolim}_{\mathcal{C}}\mathcal{X})$ induced by the assembly maps

$$F(\mathcal{X}(c)) \wedge N(c \downarrow \mathcal{C})_+ \to F\big(\mathcal{X}(c) \wedge N(c \downarrow \mathcal{C})_+\big)$$

  that come from the tensoring of the category over simplicial sets, such that the following square commutes:

$$\begin{array}{ccc}
\operatorname{hocolim}_{\mathcal{C}}(F \circ \mathcal{X}) & \longrightarrow & F(\operatorname{hocolim}_{\mathcal{C}}\mathcal{X}) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\mathcal{C}}(F \circ \mathcal{X}) & \longrightarrow & F(\operatorname{colim}_{\mathcal{C}}\mathcal{X})
\end{array}$$

  where the bottom map is the one induced by the universal property of the colimit.

- If $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{D}}$, then there is a map of pointed simplicial sets $\operatorname{hocolim}_{\mathcal{C}}(\mathcal{X} \circ F) \xrightarrow{\tilde{F}} \operatorname{hocolim}_{\mathcal{D}}\mathcal{X}$ induced from the map of presheaves $N(- \downarrow \mathcal{C}) \Rightarrow N(F(-) \downarrow \mathcal{D})$, such that, if $\eta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation of $\mathcal{D}$-diagrams, the following square commutes:

$$\begin{array}{ccc}
\operatorname{hocolim}_{\mathcal{C}}(\mathcal{X} \circ F) & \xrightarrow{\tilde{F}} & \operatorname{hocolim}_{\mathcal{D}}\mathcal{X} \\
\downarrow & & \downarrow{\tilde{\eta}} \\
\operatorname{hocolim}_{\mathcal{C}}(\mathcal{Y} \circ F) & \xrightarrow{\tilde{F}} & \operatorname{hocolim}_{\mathcal{D}}\mathcal{Y}
\end{array}$$

  Furthermore, if F satisfies <u>homotopy cofinality</u> (that is, if $(D \downarrow F)$ is contractible for all $D \in \mathcal{D}$), then the induced map $\operatorname{hocolim}_{\mathcal{C}}(\mathcal{X} \circ F) \xrightarrow{\tilde{F}} \operatorname{hocolim}_{\mathcal{D}}\mathcal{X}$ is a weak equivalence. This occurs, for example, when $\mathcal{D}$ has a final object D, and we let $\mathcal{C} = \{D\}$ with F the inclusion functor. We conclude that the induced map $\operatorname{hocolim}_{\mathcal{C}}(\mathcal{X} \circ F) \cong \mathcal{X}(D) \xrightarrow{\tilde{F}} \operatorname{hocolim}_{\mathcal{D}}\mathcal{X}$ is a weak equivalence. Since it's a section to the natural map $\operatorname{hocolim}_{\mathcal{D}}\mathcal{X} \to \operatorname{colim}_{\mathcal{D}}\mathcal{X} \cong \mathcal{X}(D)$, we have that this map is a weak equivalence as well, and so $\mathcal{X}(D) \hookrightarrow \operatorname{hocolim}_{\mathcal{D}}\mathcal{X}$ is a deformation retraction.

- If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, $\eta : F \to G$ is a natural transformation, and $\mathcal{X} \in \underline{s\textbf{Sets}}_*^{\mathcal{D}}$, then there is a map of pointed simplicial sets $\text{hocolim}_{\mathcal{C}}(\mathcal{X} \circ F) \xrightarrow{\tilde{\eta}} \text{hocolim}_{\mathcal{C}}(\mathcal{X} \circ G)$ such that the following triangle:

$$
\begin{array}{ccc}
\text{hocolim}_{\mathcal{C}}(\mathcal{X} \circ F) & & \\
& \searrow^{\tilde{F}} & \\
\downarrow^{\tilde{\eta}} & & \text{hocolim}_{\mathcal{D}}\mathcal{X} \\
& \nearrow_{\tilde{G}} & \\
\text{hocolim}_{\mathcal{C}}(\mathcal{X} \circ G) & &
\end{array}
$$

commutes *up to homotopy.* Indeed, the simplicial homotopy h_i comes from the one induced on nerves by the natural transformation, together with the fact that we've identified the "different" images of $d_{n+1}(h_n(\mathcal{X}(F(c)); c_0 \xleftarrow{\beta_1} c_1 \xleftarrow{\beta_2} \dots \xleftarrow{\beta_n} c_n; \alpha_n))$ in the coequalizer.

Observation The coend definition of the uncorrected homotopy colimit works equally well when $\mathcal{X}$ is replaced by a diagram in a pointed simplicial model category, with the pointed smash being replaced by the tensoring over $\underline{s\textbf{Sets}}_*$. In particular we can use this to define "homotopy colimit" objects in $\textbf{Top}_*$ and $\textbf{Spec}$ (in $\textbf{Top}_*$ the tensoring with pointed simplicial sets is given by $- \wedge |K|$).

Remark Though our definition of $\text{hocolim}_{\mathcal{C}}\mathcal{X}$ works for any $X : \mathcal{C} \to \mathcal{M}$ diagram in a pointed simplicial model category, for arbitrary $\mathcal{M}$ it is only homotopy invariant for cofibrant diagrams. For example, given a map of $\mathcal{C}$-diagrams $\eta : \mathcal{X} \to \mathcal{Y}$ which is point-wise equivalence, if $\mathcal{X}(c), \mathcal{Y}(c)$ are cofibrant objects in $\mathcal{M}$ for all c, then the induced map of homotopy colimits is a weak equivalence. It is *not* true for arbitrary $\mathcal{X}, \mathcal{Y}$, hence the name *uncorrected* homotopy colimit (however, the properties in the bullets above, excepting the second, still hold). Since every object in $\underline{s\textbf{Sets}}_*$ is cofibrant in the Quillen model structure, we don't need to impose constraints to achieve homotopy invariance.

Notation When working with $\underline{s\textbf{Sets}}_*^{\mathcal{C}}$, $\text{hocolim}_{\mathcal{C}}\mathcal{X}$ will mean the above construction. When working with another pointed simplicial model category $\mathcal{M}$, care will be taken to indicate when the notation $\text{hocolim}_{\mathcal{C}}\mathcal{X}$ means the above construction applied to $\widetilde{-} \circ \mathcal{X}$, where $\widetilde{-} : \mathcal{M} \to \mathcal{M}$ is the (*functorial*) cofibrant replacement functor that is a part of the data of the simplicial model category. In this case the bulleted properties carry through with the obvious changes. For example, if $\mathcal{X}$ is the constant functor at $X \in \mathcal{M}$ then we have

$\operatorname{hocolim}_{\mathcal{C}}\mathcal{X} \cong \widetilde{X} \wedge N(\mathcal{C})_+$, or that the canonical map from the homotopy colimit to the colimit factors through the canonical map $\operatorname{colim}_{\mathcal{C}}\widetilde{\mathcal{X}} \to \operatorname{colim}_{\mathcal{C}}\mathcal{X}$.

Lastly, there is a useful enriched adjunction. For $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$, $K \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}}}$, and Z a pointed simplicial set, there is an isomorphism of simplicial sets:

$$\underline{s\mathbf{Sets}_*}\Big(\int^{c:\mathcal{C}} \mathcal{X}(c) \wedge K(c), Z\Big) \cong \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}}}(K, \underline{s\mathbf{Sets}_*}(\mathcal{X}, Z))$$

Again, this adjunction really holds in any pointed simplicial model category $\mathcal{M}$, by taking $\mathcal{X} \in \mathcal{M}^{\mathcal{C}}$, $Z \in \mathcal{M}$, replacing $\wedge$ with the tensoring over simplicial sets, and taking the simplicial mapping space in $\mathcal{M}$:

$$\underline{\mathcal{M}}\Big(\int^{c:\mathcal{C}} \mathcal{X}(c) \wedge K(c), Z\Big) \cong \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}}}(K, \underline{\mathcal{M}}(\mathcal{X}, Z))$$

The main purpose of using this alternate definition of the homotopy colimit is the following proposition (analogous to Property 1, Appendix A [5]):

Proposition B.2.2 *Let $\mathcal{C}, \mathcal{D}$ be small categories, $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C} \times \mathcal{D}}$. Then we have an isomorphism of pointed simplicial sets*

$$\operatorname{hocolim}_{\mathcal{C} \times \mathcal{D}}\mathcal{X} \cong \operatorname{hocolim}_{\mathcal{C}}(\operatorname{hocolim}_{\mathcal{D}}\mathcal{X})$$

Proof. Recall that we have an isomorphism of simplicial sets $N((c,d) \downarrow \mathcal{C} \times \mathcal{D}) \cong N(c \downarrow \mathcal{C}) \times N(d \downarrow \mathcal{D})$ natural in both c and d. Let Z be a pointed simplicial set. We have the following sequence of natural isomorphisms of pointed simplicial sets:

$$\underline{s\mathbf{Sets}_*}(\operatorname{hocolim}_{\mathcal{C} \times \mathcal{D}}\mathcal{X}, Z) = \underline{s\mathbf{Sets}_*}\Big(\int^{(c,d):\mathcal{C} \times \mathcal{D}} \mathcal{X}(c,d) \wedge N\big((c,d) \downarrow \mathcal{C} \times \mathcal{D}\big)_+, Z\Big)$$

$$\cong \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}}}\Big(N((*,-) \downarrow \mathcal{C} \times \mathcal{D})_+, \underline{s\mathbf{Sets}_*}(\mathcal{X}(*,-), Z)\Big)$$

$$\cong \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}}}\Big(\big(N(* \downarrow \mathcal{C}) \times N(- \downarrow \mathcal{D})\big)_+, \underline{s\mathbf{Sets}_*}(\mathcal{X}(*,-), Z)\Big)$$

$$\cong \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}}}\Big(N(* \downarrow \mathcal{C})_+ \wedge N(- \downarrow \mathcal{D})_+, \underline{s\mathbf{Sets}_*}(\mathcal{X}(*,-), Z)\Big)$$

$$(\star) \quad \cong \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}}}\Big(N(* \downarrow \mathcal{C})_+, \underline{s\mathbf{Sets}_*}^{\mathcal{D}^{\mathrm{op}}}\big(N(- \downarrow \mathcal{D})_+, \underline{s\mathbf{Sets}_*}(\mathcal{X}(*,-), Z)\big)\Big)$$

$$\cong \underline{s\mathbf{Sets}_*}^{\mathcal{C}^{\mathrm{op}}}\Big(N(* \downarrow \mathcal{C})_+, \underline{s\mathbf{Sets}_*}\big(\int^{d:\mathcal{D}} \mathcal{X}(*,d) \wedge N(d \downarrow \mathcal{D})_+, Z\big)\Big)$$

$$\cong \underline{s\mathbf{Sets}_*}\Big(\int^{c:\mathcal{C}} \big(\int^{d:\mathcal{D}} (\mathcal{X}(c,d) \wedge N(d \downarrow \mathcal{D})_+\big) \wedge N(c \downarrow \mathcal{C})_+, Z\Big)$$

$$\cong \underline{s\mathbf{Sets}_*}\left(\int^{c:\mathcal{C}}\left(\mathrm{hocolim}_\mathcal{D}\mathcal{X}(c,-)\right)\wedge N(c\downarrow\mathcal{C})_+, Z\right)$$

$$\cong \underline{s\mathbf{Sets}_*}(\mathrm{hocolim}_\mathcal{C}(\mathrm{hocolim}_\mathcal{D}\mathcal{X}(*,-)), Z)$$

where $(\star)$ is the exponential law adjunction induced from the evaluation map (see Proposition II.5.1 [20]). Therefore by the Yoneda Lemma, we conclude that there is an isomorphism of pointed simplicial sets $\mathrm{hocolim}_{\mathcal{C}\times\mathcal{D}}\mathcal{X} \cong \mathrm{hocolim}_\mathcal{C}(\mathrm{hocolim}_\mathcal{D}\mathcal{X})$. $\qquad\square$

Corollary B.2.3 *Let $\mathcal{C}, \mathcal{D}$ be small categories, $\mathcal{X} \in \underline{s\boldsymbol{Sets}_*}^{\mathcal{C}\times\mathcal{D}}$. Then we have an isomorphism of pointed simplicial sets*

$$\mathrm{hocolim}_\mathcal{D}(\mathrm{hocolim}_\mathcal{C}\mathcal{X}) \cong \mathrm{hocolim}_\mathcal{C}(\mathrm{hocolim}_\mathcal{D}\mathcal{X})$$

Remark In other competing definitions of the homotopy colimit of a diagram both objects would be weakly equivalent, but not necessarily isomorphic. Additionally, since the previous adjunction for coends works in any pointed simplicial model category, we have the analogous results for the *uncorrected* homotopy colimits there as well. Unfortunately, in a general pointed simplicial model category it is not true that $\widetilde{\widetilde{X}} \cong \widetilde{X}$, and so the results for the corrected homotopy colimits don't hold. Nevertheless, they are still weakly equivalent.

B.3   Homotopy Limits as Ends

Definition B.3.1 Let $\mathcal{C}$ be a small category and $\mathcal{X} : \mathcal{C} \to s\mathbf{Sets}_*$ a $\mathcal{C}$-diagram of pointed simplicial sets. Since the functor $N(\mathcal{C}\downarrow-)^{\mathrm{op}}$ is covariant, the assignment $(c, c') \mapsto \mathcal{X}(c)^{(N(\mathcal{C}\downarrow c')^{\mathrm{op}})_+}$, where K^T is the pointed simplicial set of maps from T to K, defines a bifunctor $\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to s\mathbf{Sets}_*$. The (uncorrected) homotopy limit of $\mathcal{X}$, denoted $\mathrm{holim}_\mathcal{C}\mathcal{X}$, is the end

$$\mathrm{holim}_\mathcal{C}\mathcal{X} := \int_{c\in\mathcal{C}} \mathcal{X}(c)^{(N(\mathcal{C}\downarrow c)^{\mathrm{op}})_+}$$

By the end formula this is given by:

$$\mathrm{holim}_\mathcal{C}\mathcal{X} = \mathrm{eq}\left(\prod_{c\in\mathcal{C}}\mathcal{X}(c)^{(N(\mathcal{C}\downarrow c)^{\mathrm{op}})_+} \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{c_0\overset{f}{\to}c_1\in\mathcal{C}}\mathcal{X}(c_1)^{(N(\mathcal{C}\downarrow c_0)^{\mathrm{op}})_+}\right)$$

where the projection of the map ϕ on the factor $c_0 \xrightarrow{f} c_1$ is given by $\mathcal{X}(f)^{(N(\mathcal{C}\downarrow c_0)^{\mathrm{op}})_+} \circ \pi_{c_0}$ (i.e. post-composing a map $(N(\mathcal{C} \downarrow c_0)^{\mathrm{op}})_+ \to \mathcal{X}(c_0)$ with $\mathcal{X}(f)$), and ψ is given by $(\mathrm{id}_{\mathcal{X}(c_1)})^{N(f_*)} \circ \pi_{c_1}$ (i.e. pre-composing a map $(N(\mathcal{C} \downarrow c_1)^{\mathrm{op}})_+ \to \mathcal{X}(c_1)$ with the map $N(f_*)$ defined previously).

Fact If $\mathcal{Y}$ is another $\mathcal{C}$-diagram in pointed simplicial sets, and $\eta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation, then by the universal property of limits, we get an induced map $\mathrm{holim}_{\mathcal{C}}\mathcal{X} \xrightarrow{\tilde{\eta}} \mathrm{holim}_{\mathcal{C}}\mathcal{Y}$. If η is a point-wise weak equivalence of fibrant simplicial sets, then $\tilde{\eta}$ is a weak equivalence. Some additional properties of the homotopy limit (see Lemma 2.5 [5]) include:

- If $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ has $\mathcal{X}(c) = *$ for all c, then $\mathrm{holim}_{\mathcal{C}}\mathcal{X} \cong *$.

- More generally, if $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ has $\mathcal{X}(c) = X$ for all c (with identity maps) for some fixed $X \in \underline{s\mathbf{Sets}_*}$, then $\mathrm{holim}_{\mathcal{C}}\mathcal{X} \cong X^{N(\mathcal{C}^{\mathrm{op}})_+}$.

- If $\mathcal{C}$ is the trivial category with a single object $\bullet$ and identity morphism $\circlearrowleft$, $\mathrm{holim}_{\mathcal{C}}\mathcal{X} \cong X(\bullet)$.

- For any $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ we have a map of pointed simplicial sets $\lim_{\mathcal{C}}\mathcal{X} \to \mathrm{holim}_{\mathcal{C}}\mathcal{X}$, such that, if $\eta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation of $\mathcal{C}$-diagrams, the following square commutes:

$$
\begin{array}{ccc}
\lim_{\mathcal{C}}\mathcal{X} & \longrightarrow & \lim_{\mathcal{C}}\mathcal{Y} \\
\downarrow & & \downarrow \\
\mathrm{holim}_{\mathcal{C}}\mathcal{X} & \xrightarrow{\tilde{\eta}} & \mathrm{holim}_{\mathcal{C}}\mathcal{Y}
\end{array}
$$

- If $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{C}}$ and $F : \underline{s\mathbf{Sets}_*} \to \underline{s\mathbf{Sets}_*}$ is a *simplicially enriched* functor, then we get natural maps of pointed simplicial sets $F(\mathrm{holim}_{\mathcal{C}}\mathcal{X}) \to \mathrm{hocolim}_{\mathcal{C}}(F \circ \mathcal{X})$ induced by the adjoints of the evaluation maps

$$
(N(\mathcal{C} \downarrow c)^{\mathrm{op}})_+ \wedge F\big(\underline{s\mathbf{Sets}_*}((N(\mathcal{C} \downarrow c)^{\mathrm{op}})_+, F(\mathcal{X}(c)))\big) \to F(\mathcal{X}(c))
$$

that come from the cotensoring of the category over simplicial sets, such that the following square commutes:

$$
\begin{array}{ccc}
F(\lim_{\mathcal{C}}\mathcal{X}) & \longrightarrow & \lim_{\mathcal{C}}(F \circ \mathcal{X}) \\
\downarrow & & \downarrow \\
F(\mathrm{holim}_{\mathcal{C}}\mathcal{X}) & \longrightarrow & \mathrm{holim}_{\mathcal{C}}(F \circ \mathcal{X})
\end{array}
$$

where the top map is the one induced by the universal property of the limit.

- If $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{D}}$, then there is a map of pointed simplicial sets $\mathrm{holim}_{\mathcal{D}}\mathcal{X} \xrightarrow{\hat{F}} \mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ F)$ induced from the natural transformation $N(\mathcal{C} \downarrow -)^{\mathrm{op}} \Rightarrow N(\mathcal{D} \downarrow F(-))^{\mathrm{op}}$, such that, if $\eta : \mathcal{X} \to \mathcal{Y}$ is a natural transformation of $\mathcal{D}$-diagrams, the following square commutes:

$$\begin{array}{ccc} \mathrm{holim}_{\mathcal{D}}\mathcal{X} & \xrightarrow{\hat{F}} & \mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ F) \\ \downarrow{\tilde{\eta}} & & \downarrow \\ \mathrm{holim}_{\mathcal{D}}\mathcal{Y} & \xrightarrow{\hat{F}} & \mathrm{holim}_{\mathcal{C}}(\mathcal{Y} \circ F) \end{array}$$

Furthermore, if F satisfies homotopy finality (that is, if $(F \downarrow D)$ is contractible for all $D \in \mathcal{D}$), then the induced map $\mathrm{holim}_{\mathcal{D}}\mathcal{X} \xrightarrow{\hat{F}} \mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ F)$ is a weak equivalence. This occurs, for example, when $\mathcal{D}$ has an initital object D, and we let $\mathcal{C} = \{D\}$ with F the inclusion functor. We conclude that the induced map $\mathrm{holim}_{\mathcal{D}}\mathcal{X} \xrightarrow{\hat{F}} \mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ F) \cong \mathcal{X}(D)$ is a weak equivalence. In this case the natural map $\mathcal{X}(D) \cong \lim_{\mathcal{D}}\mathcal{X} \to \mathrm{holim}_{\mathcal{D}}\mathcal{X}$ is a section of the restriction map, and so it is a weak equivalence as well. Therefore $\mathcal{X}(D) \hookrightarrow \mathrm{holim}_{\mathcal{D}}\mathcal{X}$ is a deformation retraction.

- If $F, G : \mathcal{C} \to \mathcal{D}$ are functors, $\eta : F \to G$ is a natural transformation, and $\mathcal{X} \in \underline{s\mathbf{Sets}_*}^{\mathcal{D}}$, then there is a map of pointed simplicial sets $\mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ F) \xrightarrow{\hat{\eta}} \mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ G)$ such that the following triangle:

$$\begin{array}{ccc} & & \mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ F) \\ & \nearrow^{\hat{F}} & \\ \mathrm{holim}_{\mathcal{D}}\mathcal{X} & & \downarrow{\hat{\eta}} \\ & \searrow_{\hat{G}} & \\ & & \mathrm{holim}_{\mathcal{C}}(\mathcal{X} \circ G) \end{array}$$

commutes *up to homotopy*. Indeed, the simplicial homotopy h_i comes from the one induced on nerves by the natural transformation, together with the fact that the two "different" images of $d_{n+1}(h_n((c_0 \xrightarrow{\beta_1} c_1 \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} c_n; \alpha_0) \to \mathcal{X}(c)))$ agree, since our source is in the equalizer.

Additionally, since the singular complex is a right adjoint, if $\mathcal{X}$ is a $\mathcal{C}$-diagram of *spaces*, we have $\mathrm{Sing}(\mathrm{holim}_{\mathcal{C}}\mathcal{X}) \cong \mathrm{holim}_{\mathcal{C}}\mathrm{Sing}(\mathcal{X})$.

Observation The end definition works equally well when $\mathcal{X}$ is replaced by a diagram in a pointed simplicial model category, with the simplicial mapping space being replaced by the cotensoring over $\underline{s\mathbf{Sets}}_*$. In particular, we can use this to define "homotopy limit" objects in $\mathbf{Top}_*$ and $\mathbf{Spec}$ (in $\mathbf{Top}_*$ the cotensoring with pointed simplicial sets is given by $(-)^K := \underline{\mathbf{Top}}_*(|K|, -)$).

Remark Though our definition of $\mathrm{holim}_{\mathcal{C}}\mathcal{X}$ works for any $X : \mathcal{C} \to \mathcal{M}$ diagram in a pointed simplicial model category, for arbitrary $\mathcal{M}$ it is only homotopy invariant for fibrant diagrams. For example, given a map of $\mathcal{C}$-diagrams $\eta : \mathcal{X} \to \mathcal{Y}$ which is point-wise equivalence, if $\mathcal{X}(c), \mathcal{Y}(c)$ are fibrant objects in $\mathcal{M}$ for all c, then the induced map of homotopy limits is a weak equivalence. It is *not* true for arbitrary $\mathcal{X}, \mathcal{Y}$, hence the name *uncorrected* homotopy limit (however, the properties in the bullets above still hold). Since every object in $\underline{\mathbf{Top}}_*$ is fibrant in the Quillen model structure, we don't need to impose constraints to these types of diagrams to achieve homotopy invariance.

Notation When working with $\underline{s\mathbf{Sets}}_*^{\mathcal{C}}$ (or another pointed simplicial model category $\mathcal{M}$ that isn't $\underline{\mathbf{Top}}_*$), care will be taken to indicate when the notation $\mathrm{holim}_{\mathcal{C}}\mathcal{X}$ means the above construction applied to $\widehat{-} \circ \mathcal{X}$, where $\widehat{-} : \mathcal{M} \to \mathcal{M}$ is the (*functorial*) fibrant replacement functor that is a part of the data of the simplicial model category. In this case, the bulleted properties carry through with the obvious changes. For example, if $\mathcal{X}$ is the constant functor at $X \in \mathcal{M}$ then we have $\mathrm{holim}_{\mathcal{C}}\mathcal{X} \cong \widehat{X}^{N(\mathcal{C}^{\mathrm{op}})_+}$, or that the canonical map from the limit to the homotopy limit factors through the canonical map $\lim_{\mathcal{C}}\mathcal{X} \to \lim_{\mathcal{C}}\widehat{\mathcal{X}}$.

We similarly have a useful enriched adjunction. For $\mathcal{X} \in \underline{s\mathbf{Sets}}_*^{\mathcal{C}}$, $K \in \underline{s\mathbf{Sets}}_*^{\mathcal{C}}$, and Z a pointed simplicial set, there is an isomorphism of simplicial sets:

$$\underline{s\mathbf{Sets}}_*\left(Z, \int_{c:\mathcal{C}} \mathcal{X}(c)^{K(c)}\right) \cong \underline{s\mathbf{Sets}}_*^{\mathcal{C}}(K, \underline{s\mathbf{Sets}}_*(Z, \mathcal{X}))$$

Again, this adjunction really holds in any pointed simplicial model category $\mathcal{M}$, by taking $\mathcal{X} \in \mathcal{M}^{\mathcal{C}}$, $Z \in \mathcal{M}$, replacing the simplicial mapping space $\mathcal{X}(c)^{K(c)}$ with the cotensoring over simplicial sets, and taking the simplicial mapping spaces in $\mathcal{M}$ for the natural transformations:

$$\underline{\mathcal{M}}\left(Z, \int_{c:\mathcal{C}} \mathcal{X}(c)^{K(c)}\right) \cong \underline{s\mathbf{Sets}}_*^{\mathcal{C}}(K, \underline{\mathcal{M}}(Z, \mathcal{X}))$$

The analogous Fubini theorems hold for uncorrected homotopy limits:

Proposition B.3.2 *Let $\mathcal{C}, \mathcal{D}$ be small categories, $\mathcal{X} \in \underline{s\boldsymbol{Sets}}_*^{\mathcal{C} \times \mathcal{D}}$. Then we have an iso-*

morphism of pointed simplicial sets

$$\mathrm{holim}_{\mathcal{C}\times\mathcal{D}}\mathcal{X} \cong \mathrm{holim}_{\mathcal{C}}(\mathrm{holim}_{\mathcal{D}}\mathcal{X})$$

Corollary B.3.3 *Let* $\mathcal{C},\mathcal{D}$ *be small categories,* $\mathcal{X} \in \underline{s\boldsymbol{Sets}_*}^{\mathcal{C}\times\mathcal{D}}$. *Then we have an isomorphism of pointed simplicial sets*

$$\mathrm{holim}_{\mathcal{D}}(\mathrm{holim}_{\mathcal{C}}\mathcal{X}) \cong \mathrm{holim}_{\mathcal{C}}(\mathrm{holim}_{\mathcal{D}}\mathcal{X})$$

Remark In other competing definitions of the homotopy limit of a diagram both objects would be weakly equivalent, but not necessarily isomorphic. Additionally, since the previous adjunction for ends works in any pointed simplicial model category, we have the analogous results for the *uncorrected* homotopy limits there as well. Unfortunately, in a general pointed simplicial model category it is not true that $\widehat{\widehat{X}} \cong \widehat{X}$, and so the results for the corrected homotopy limits don't hold. Nevertheless, they are still weakly equivalent.

B.4   Adjunction between Both

These models of the homotopy colimit and homotopy limits allow us to switch between each by means of an adjunction.

Proposition B.4.1 *Let* $\mathcal{C}$ *be a small category,* $\mathcal{X} \in \underline{s\boldsymbol{Sets}_*}^{\mathcal{C}}$, *and* $Y \in \underline{s\boldsymbol{Sets}_*}$. *Then there is a natural isomorphism of pointed simplicial sets*

$$\underline{s\boldsymbol{Sets}_*}(\mathrm{hocolim}_{\mathcal{C}}\mathcal{X}, Y) \cong \mathrm{holim}_{\mathcal{C}^{\mathrm{op}}}\underline{s\boldsymbol{Sets}_*}(\mathcal{X}, Y)$$

Proof. In simplicial degree n, $\underline{s\boldsymbol{Sets}_*}(\mathrm{hocolim}_{\mathcal{C}}\mathcal{X}, Y)_n$, we have a chain of natural isomorphisms:

$$= s\boldsymbol{Sets}_*((\mathrm{hocolim}_{\mathcal{C}}\mathcal{X}) \wedge \Delta^n_+, Y)$$

$$\cong s\boldsymbol{Sets}_*(\mathrm{hocolim}_{\mathcal{C}}\mathcal{X}, \underline{s\boldsymbol{Sets}_*}(\Delta^n_+, Y))$$

$$= s\boldsymbol{Sets}_*\Big(\mathrm{coeq}\big(\bigvee_{c_0\xrightarrow{f}c_1\in\mathcal{C}} \mathcal{X}(c_0) \wedge N(c_1 \downarrow \mathcal{C})_+ \underset{\psi}{\overset{\phi}{\rightrightarrows}} \bigvee_{c\in\mathcal{C}} \mathcal{X}(c) \wedge N(c \downarrow \mathcal{C})_+\big), \underline{s\boldsymbol{Sets}_*}(\Delta^n_+, Y)\Big)$$

$$\cong \mathrm{eq}\Big(s\boldsymbol{Sets}_*\big(\bigvee_{c\in\mathcal{C}} \mathcal{X}(c) \wedge N(c \downarrow \mathcal{C})_+, \underline{s\boldsymbol{Sets}_*}(\Delta^n_+, Y)\big)$$

$$\overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} s\mathbf{Sets}_*\Big(\bigvee_{c_0 \xrightarrow{f} c_1 \in \mathcal{C}} \mathcal{X}(c_0) \wedge N(c_1 \downarrow \mathcal{C})_+, \underline{s\mathbf{Sets}}_*(\Delta_+^n, Y)\Big)\Big)$$

$$\cong \mathrm{eq}\Big(\prod_{c \in \mathcal{C}} s\mathbf{Sets}_*\Big(\mathcal{X}(c) \wedge N(c \downarrow \mathcal{C})_+, \underline{s\mathbf{Sets}}_*(\Delta_+^n, Y)\Big)$$

$$\overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} \prod_{c_0 \xrightarrow{f} c_1 \in \mathcal{C}} s\mathbf{Sets}_*\Big(\mathcal{X}(c_0) \wedge N(c_1 \downarrow \mathcal{C})_+, \underline{s\mathbf{Sets}}_*(\Delta_+^n, Y)\Big)\Big)$$

$$\cong \mathrm{eq}\Big(\prod_{c \in \mathcal{C}} s\mathbf{Sets}_*\big(\mathcal{X}(c) \wedge N(c \downarrow \mathcal{C})_+ \wedge \Delta_+^n, Y\big)$$

$$\overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} \prod_{c_0 \xrightarrow{f} c_1 \in \mathcal{C}} s\mathbf{Sets}_*\big(\mathcal{X}(c_0) \wedge N(c_1 \downarrow \mathcal{C})_+ \wedge \Delta_+^n, Y\big)\Big)$$

$$\cong \mathrm{eq}\Big(\prod_{c \in \mathcal{C}} s\mathbf{Sets}_*\Big(N(c \downarrow \mathcal{C})_+ \wedge \Delta_+^n, \underline{s\mathbf{Sets}}_*(\mathcal{X}(c), Y)\Big)$$

$$\overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} \prod_{c_0 \xrightarrow{f} c_1 \in \mathcal{C}} s\mathbf{Sets}_*\Big(N(c_1 \downarrow \mathcal{C})_+ \wedge \Delta_+^n, \underline{s\mathbf{Sets}}_*(\mathcal{X}(c_0), Y)\Big)\Big)$$

$$\cong \mathrm{eq}\Big(\prod_{c \in \mathcal{C}^{\mathrm{op}}} s\mathbf{Sets}_*\Big(N(\mathcal{C}^{\mathrm{op}} \downarrow c)_+^{\mathrm{op}} \wedge \Delta_+^n, \underline{s\mathbf{Sets}}_*(\mathcal{X}(c), Y)\Big)$$

$$\overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} \prod_{c_1 \xrightarrow{f} c_0 \in \mathcal{C}^{\mathrm{op}}} s\mathbf{Sets}_*\Big(N(\mathcal{C}^{\mathrm{op}} \downarrow c_1)_+^{\mathrm{op}} \wedge \Delta_+^n, \underline{s\mathbf{Sets}}_*(\mathcal{X}(c_0), Y)\Big)\Big)$$

$$\cong \mathrm{eq}\left(\prod_{c \in \mathcal{C}^{\mathrm{op}}} \Big(\underline{s\mathbf{Sets}}_*(\mathcal{X}(c), Y)^{N(\mathcal{C}^{\mathrm{op}} \downarrow c)_+^{\mathrm{op}}}\Big)_n \overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} \prod_{c_1 \xrightarrow{f} c_0 \in \mathcal{C}^{\mathrm{op}}} \Big(\underline{s\mathbf{Sets}}_*(\mathcal{X}(c_0), Y)^{N(\mathcal{C}^{\mathrm{op}} \downarrow c_1)_+^{\mathrm{op}}}\Big)_n \right)$$

$$\cong \Big(\mathrm{eq}\Big(\prod_{c \in \mathcal{C}^{\mathrm{op}}} \underline{s\mathbf{Sets}}_*(\mathcal{X}(c), Y)^{N(\mathcal{C}^{\mathrm{op}} \downarrow c)_+^{\mathrm{op}}} \overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} \prod_{c_1 \xrightarrow{f} c_0 \in \mathcal{C}^{\mathrm{op}}} \underline{s\mathbf{Sets}}_*(\mathcal{X}(c_0), Y)^{N(\mathcal{C}^{\mathrm{op}} \downarrow c_1)_+^{\mathrm{op}}}\Big)\Big)_n$$

$$\cong \big(\mathrm{holim}_{\mathcal{C}^{\mathrm{op}}} \underline{s\mathbf{Sets}}_*(\mathcal{X}, Y)\big)_n$$

$$\square$$

Remark We emphasize here that we are working with the uncorrected homotopy limit and colimit above. With these, this property also holds in an arbitrary pointed simplicial model category $\mathcal{M}$. That is, for $\mathcal{X} \in \mathcal{M}^{\mathcal{C}}$ and $Y \in \mathcal{M}$ we have a natural isomorphism of pointed simplicial sets:

$$\underline{\mathcal{M}}(\mathrm{hocolim}_{\mathcal{C}} \mathcal{X}, Y) \cong \mathrm{holim}_{\mathcal{C}^{\mathrm{op}}} \underline{\mathcal{M}}(\mathcal{X}, Y)$$

by taking the induced $\mathcal{C}^{\mathrm{op}}$-diagram given by pointed simplicial mapping spaces. When working with the *corrected* homotopy limits and colimits we hit a snag; it is not true in an arbitrary pointed simplicial model category $\mathcal{M}$ that if $\widetilde{X} \xrightarrow{\simeq} X$ is a functorial cofibrant replacement of X, then given $Y \in \mathcal{M}$, the induced map $\underline{\mathcal{M}}(X,Y) \to \underline{\mathcal{M}}(\widetilde{X},Y)$ is the fibrant replacement of $\underline{\mathcal{M}}(X,Y)$.

APPENDIX C

NAIVE SPECTRA

C.1 Definition and Enrichment

Definition C.1.1 A spectrum, $\underline{X}$, is a sequence of pointed simplicial sets, $X_0, X_1, \ldots, X_n, \ldots$, together with structure maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ for each $n \geq 0$. A map of spectra $f : \underline{X} \to \underline{Y}$ is a collection of maps $f_n : X_n \to Y_n$ making the following diagram commute for each $n \geq 0$:

$$
\begin{array}{ccc}
S^1 \wedge X_n & \xrightarrow{\sigma_n^X} & X_{n+1} \\
\downarrow{\scriptstyle \mathrm{id} \wedge f_n} & & \downarrow{\scriptstyle f_{n+1}} \\
S^1 \wedge Y_n & \xrightarrow{\sigma_n^Y} & Y_{n+1}
\end{array}
$$

Notation The category of spectra with maps of spectra will be denoted $\mathbf{Spec}_0$.

Observation The trivial spectrum, denoted $*$, is the spectrum that has the trivial simplicial set in each spectrum degree. That is, $(*)_n := *$, and the structure maps are the trivial maps $S^1 \wedge * \cong *$. For any spectrum $\underline{X}$ there is a unique map of spectra $* \to \underline{X}$ (since each X_n is pointed), and a unique map $\underline{X} \to *$, making $\mathbf{Spec}_0$ a pointed category.

Example C.1.2 Let $\mathbb{S}$ be the spectrum given by $\mathbb{S}_n := S^n$ for each $n \geq 0$, with structure maps $\sigma_n : S^1 \wedge S^n \cong S^{n+1}$ (recall the definition of S^n in pointed simplicial sets). Then $\mathbb{S}$ is a spectrum called the "sphere spectrum."

Example C.1.3 Let $X \in \underline{s\mathbf{Sets}}_*$. We construct a spectrum, denoted $\underline{\Sigma^\infty X}$, by setting $(\underline{\Sigma^\infty X})_n := S^n \wedge X$ with structure maps $\sigma_n : S^1 \wedge (S^n \wedge X) \cong S^{n+1} \wedge X$ (the associator). This spectrum is the "suspension spectrum" of X. Note that the previous example is just an instance of this, with $\mathbb{S} \cong \underline{\Sigma^\infty S^0}$.

The category of spectra is not just a category but carries an enrichment with it. Indeed, let Δ^n be the standard simplicial n-simplex, and let $\underline{X}$ be a spectrum. We form a new spectrum,

$\Delta^n_+ \wedge \underline{X}$, defined as $(\Delta^n_+ \wedge \underline{X})_m := \Delta^n_+ \wedge X_m$ and structure maps $\sigma_m^{\Delta^n_+ \wedge \underline{X}} : S^1 \wedge (\Delta^n_+ \wedge \underline{X})_m \to (\Delta^n_+ \wedge \underline{X})_{m+1}$ given by:

$$S^1 \wedge (\Delta^n_+ \wedge X_m) \cong (S^1 \wedge \Delta^n_+) \wedge X_m \cong (\Delta^n_+ \wedge S^1) \wedge X_m \cong \Delta^n_+ \wedge (S^1 \wedge X_m) \xrightarrow{\mathrm{id} \wedge \sigma^X_m} \Delta^n_+ \wedge X_{m+1}$$

where the first three isomorphisms are the associator, twist, and associator (respectively) of the symmetric monoidal structure on pointed simplicial sets. Note that we have an isomorphism of spectra $\Delta^0_+ \wedge \underline{X} \cong \underline{X}$ (given by level-wise unit isomorphisms). For a fixed spectrum $\underline{X}$, $\Delta^-_+ \wedge \underline{X}$ assembles into a cosimplicial object in $\mathbf{Spec}_0$. Therefore, for spectra $\underline{X}, \underline{Y}$, we define the *simplicial set* of maps from $\underline{X}$ to $\underline{Y}$, denoted $\mathbf{Spec}(\underline{X}, \underline{Y})$, by $\mathbf{Spec}(\underline{X}, \underline{Y})_n := \mathbf{Spec}_0(\Delta^n_+ \wedge \underline{X}, \underline{Y})$. This simplicial set is canonically pointed, by taking $* \in \mathbf{Spec}_0(\underline{X}, \underline{Y})$ to be the constant map of spectra (factoring through the zero object $*$ in $\mathbf{Spec}_0$).

There is a composition $\mathbf{Spec}(\underline{Y}, \underline{Z}) \wedge \mathbf{Spec}(\underline{X}, \underline{Y}) \to \mathbf{Spec}(\underline{X}, \underline{Z})$ (induced by the diagonal $\Delta^n \to \Delta^n \times \Delta^n$) which at simplicial degree 0 recovers the composition in $\mathbf{Spec}_0$. A quick check shows that this gives an enrichment of $\mathbf{Spec}_0$ over $\underline{s\mathbf{Sets}_*}$, which we'll denote by $\mathbf{Spec}$.

Remark The simplicially enriched category $\mathbf{Spec}$ is also tensored and cotensored over $\underline{s\mathbf{Sets}_*}$. Indeed, for $K \in \underline{s\mathbf{Sets}_*}$ and $\underline{X} \in \mathbf{Spec}$ we may form $K \wedge \underline{X}$ in exactly the same manner as we did for $\Delta^n_+ \wedge \underline{X}$ obtaining an isomorphism of simplicial sets:

$$\mathbf{Spec}(K \wedge \underline{X}, \underline{Y}) \cong \underline{s\mathbf{Sets}_*}(K, \mathbf{Spec}(\underline{X}, \underline{Y}))$$

defining the tensoring.

Observation By using the n-fold twisting map in $\underline{s\mathbf{Sets}_*}$, $K \wedge S^n \cong S^n \wedge K$, we get an isomorphism of spectra $K \wedge \mathbb{S} \cong \underline{\Sigma^\infty K}$, so that the suspension spectrum is simply the tensored $\underline{s\mathbf{Sets}_*}$-structure on $\mathbf{Spec}$ (and therefore makes $\Sigma^\infty(-) : \underline{s\mathbf{Sets}_*} \to \mathbf{Spec}$ into a functor). On the other hand, $\mathbf{Spec}(\mathbb{S}, \underline{Y})$ is a simplicial set whose n-simplices are $\mathbf{Spec}(\mathbb{S}, \underline{Y})_n = \mathbf{Spec}_0(\Delta^n_+ \wedge \mathbb{S}, \underline{Y}) \cong \mathbf{Spec}_0(\underline{\Sigma^\infty \Delta^n_+}, \underline{Y})$. We denote this simplicial set by $\Omega'^\infty(\underline{Y}) := \mathbf{Spec}(\mathbb{S}, \underline{Y})$. By the functorality of the simplicial enrichment internal mapping space, we get that $\Omega'^\infty(-) : \mathbf{Spec} \to \underline{s\mathbf{Sets}_*}$ is a functor Explicitly, the definition is isomorphic to the 0th-space of the spectrum, $\underline{Y} \mapsto Y_0$. The tensored structure then reads:

$$\mathbf{Spec}(\Sigma^\infty(K), \underline{Y}) \cong \underline{s\mathbf{Sets}_*}(K, \Omega'^\infty(\underline{Y}))$$

That is, we have a simplicially enriched adjunction between the suspension spectrum functor

and the 0th-space functor.

Remark It is also cotensored. Let $\mathbf{Spec}(K, \underline{X})$ be the spectrum whose m^{th}-pointed simplicial set is defined as $\mathbf{Spec}(K, \underline{X})_m := \underline{s\mathbf{Sets}_*}(K, X_m)$ with structure maps given by:

$$S^1 \wedge \underline{s\mathbf{Sets}_*}(K, X_m) \to \underline{s\mathbf{Sets}_*}(K, S^1 \wedge X_m) \to \underline{s\mathbf{Sets}_*}(K, X_{m+1})$$

where the first map is the assembly map and the second the image of σ_m^X under $\underline{s\mathbf{Sets}_*}(K, -)$. Here we're using the fact that the internal hom in a closed symmetric monoidal category (enriched over itself) is an enriched endofunctor, e.g. that $\underline{s\mathbf{Sets}_*}(K, -)$ is a *simplicial* functor. The cotensoring comes from the natural isomorphism of simplicial sets:

$$\mathbf{Spec}(\underline{X}, \mathbf{Spec}(K, \underline{Y})) \cong \underline{s\mathbf{Sets}_*}(K, \mathbf{Spec}(\underline{X}, \underline{Y}))$$

Fact Since $\mathbf{Spec}$ is tensored and cotensored over $\underline{s\mathbf{Sets}_*}$ we have some natural isomorphisms (true for any enriched tensored and cotensored category). For $K, K' \in \underline{s\mathbf{Sets}_*}$ and $\underline{X}, \underline{Y} \in \mathbf{Spec}$ we have:

$$(K \wedge K') \wedge \underline{X} \cong K \wedge (K' \wedge \underline{X}) \quad \text{and} \quad \mathbf{Spec}(K \wedge K', \underline{X}) \cong \mathbf{Spec}(K, \mathbf{Spec}(K', \underline{X}))$$

Remark When the notion of *symmetric spectrum* is introduced together with its associative, symmetric, and unital smash product, we will see that for $K \in \underline{s\mathbf{Sets}_*}$ and $\underline{X} \in \mathbf{Spec}$, we have $\Sigma^\infty(K) \wedge \underline{X} \cong K \wedge \underline{X}$, where the left-hand side is the smash product of symmetric spectra, and the right-hand side is the tensoring of $\mathbf{Spec}$ over $\underline{s\mathbf{Sets}_*}$.

C.2 Naive Equivalences

Definition C.2.1 Let $\underline{X}$ be a spectrum. Let $\sigma_n^\sharp : X_n \to \Omega X_{n+1}$ be the maps obtained from the adjoints of the structure map:

$$X_n \to \underline{s\mathbf{Sets}_*}(S^1, X_{n+1}) \xrightarrow{\cong} \mathrm{Sing}(|\underline{s\mathbf{Sets}_*}(S^1, X_{n+1})|) \to \mathrm{Sing}(\underline{\mathbf{Top}_*}(|S^1|, |X_{n+1}|)) = \Omega X_{n+1}$$

Consider the directed system of maps

$$X_0 \xrightarrow{\sigma_0^\sharp} \Omega X_1 \xrightarrow{\Omega(\sigma_1^\sharp)} \Omega^2 X_2 \xrightarrow{\Omega^2(\sigma_2^\sharp)} \Omega^3 X_3 \longrightarrow \dots$$

Taking homotopy groups, and using the (adjunction) isomorphisms $\pi_i(\Omega^s X) \cong \pi_{i+s}(X)$, we get a directed system of (abelian) groups

$$\pi_i(X_0) \xrightarrow{\pi_i(\sigma_0^\sharp)} \pi_{i+1}(X_1) \xrightarrow{\pi_i(\Omega(\sigma_1^\sharp))} \pi_{i+2}(X_2) \xrightarrow{\pi_i(\Omega^2(\sigma_2^\sharp))} \pi_{i+3}(X_3) \longrightarrow \ldots$$

The ith-homotopy group of the spectrum, denoted $\pi_i(\underline{X})$, is the colimit of the previous system, i.e. $\pi_i(\underline{X}) := \text{colim}_j \pi_{i+j}(X_j)$. A map of spectra $f : \underline{X} \to \underline{Y}$ is a stable weak equivalence if the induced map of homotopy groups $\pi_i(f) : \pi_i(\underline{X}) \to \pi_i(\underline{Y})$ is an isomorphism for all i. It is a levelwise equivalence if $f_n : X_n \to Y_n$ is a weak equivalence of pointed simplicial sets for all n.

Observation A levelwise equivalence of spectra is automatically a stable weak equivalence.

Definition C.2.2 Let $\underline{X}$ be a spectrum. We say it is n-connected if $\pi_*(\underline{X}) = 0$ for $* \leq n$. It is connective if it is -1-connected. It is bounded below if it is n-connected for some n.

Example C.2.3 If a spectrum $\underline{X}$ has the property that X_n is $(n+k)$-connected for large n, then $\underline{X}$ is k-connected.

Definition C.2.4 Let $f : \underline{X} \to \underline{Y}$ be a map of spectra. The homotopy fiber of f is the homotopy limit of the cospan:

$$\begin{array}{ccc} & & \underline{X} \\ & & \downarrow f \\ * & \longrightarrow & \underline{Y} \end{array}$$

while the homotopy cofiber of f is the homotopy colimit of the span:

$$\begin{array}{ccc} \underline{X} & \xrightarrow{f} & \underline{Y} \\ \downarrow & & \\ * & & \end{array}$$

Notation A map of spectra $f : \underline{X} \to \underline{Y}$ is said to be n-connected if the homotopy fiber is $(n-1)$-connected.

Definition C.2.5 Let $\sigma_n^\sharp : X_n \to \Omega X_{n+1}$ be the adjoints of the structure maps in the spectrum $\underline{X}$ as defined in C.2.1. An Ω-spectrum is a spectrum such that all $\sigma_n^\sharp$ are weak equivalences of simplicial sets.

Consider the directed system of maps

$$X_n \xrightarrow{\sigma_n^\sharp} \Omega X_{n+1} \xrightarrow{\Omega(\sigma_{n+1}^\sharp)} \Omega^2 X_{n+2} \xrightarrow{\Omega^2(\sigma_{n+2}^\sharp)} \Omega^3 X_{n+3} \longrightarrow \dots$$

and define $(\underline{\Omega^\infty X})_n := \operatorname{hocolim}_j \Omega^j X_{n+j}$. We have weak equivalences ϕ_n:

$$\operatorname{hocolim}_j \Omega^j X_{n+j} \xrightarrow{\simeq} \operatorname{hocolim}_j \Omega^{j+1} X_{n+1+j} \xrightarrow{\simeq} \Omega(\operatorname{hocolim}_j \Omega^j X_{n+1+j})$$

were both maps are the canonical maps induced by the homotopy colimits, and are weak equivalences by homotopy cofinality and the fact that the loops functor commutes with directed homotopy colimits, respectively. Then the adjoint maps $\phi_n^\flat : S^1 \wedge (\underline{\Omega^\infty X})_n \to (\underline{\Omega^\infty X})_{n+1}$ define an Ω-spectrum, $\underline{\Omega^\infty X}$. There are natural maps $X_n \to (\underline{\Omega^\infty X})_n$ for each n, and they assemble into commuting squares:

$$
\begin{array}{ccc}
X_n & \xrightarrow{\sigma_n^\sharp} & \Omega X_{n+1} \\
\downarrow & & \downarrow \\
(\underline{\Omega^\infty X})_n & \xrightarrow{\phi_n} & \Omega((\underline{\Omega^\infty X})_{n+1})
\end{array}
$$

Taking the adjoint of these squares, we get a map of spectra $\underline{X} \to \underline{\Omega^\infty X}$, which represents our "fibrant replacement functor". Indeed, by comparing homotopy groups, we see that $\underline{X} \to \underline{\Omega^\infty X}$ is a weak equivalence. This replacement is functorial; if $f : \underline{X} \to \underline{Y}$ is a map of spectra, we can construct commuting ladder diagrams as above, to obtain induced maps of Ω-spectra, and a commuting square:

$$
\begin{array}{ccc}
\underline{X} & \longrightarrow & \underline{\Omega^\infty X} \\
\downarrow f & & \downarrow \Omega^\infty(f) \\
\underline{Y} & \longrightarrow & \underline{\Omega^\infty Y}
\end{array}
$$

Definition C.2.6 A spectrum $\underline{X}$ is called $\underline{\text{good}}$ if, for all $n \geq 0$, the structure map $S^1 \wedge X_n \xrightarrow{\sigma_n^X} X_{n+1}$ is an inclusion, or equivalently, if the same is true for $\sigma_n^\sharp$.

Remark Good spectra are homotopically well-behaved. For example, if $\underline{X}$ is good, the "fibrant replacement functor" $\underline{\Omega^\infty X}$ can be defined using the colimit instead of the homotopy colimit.

Definition C.2.7 Let $\Omega^\infty(-) : \mathbf{Spec} \to \underline{s\mathbf{Sets}_*}$ be the composite functor $\underline{X} \mapsto \Omega'^\infty(\underline{\Omega^\infty X})$, i.e. taking the 0th-space of the Ω-spectrum associated to $\underline{X}$.

Remark We again obtain a simplicially-enriched adjunction,

$$\mathbf{Spec}(\Sigma^\infty(K), \underline{Y}) \cong \underline{s\mathbf{Sets}_*}(K, \Omega^\infty(\underline{Y}))$$

C.3 Functor Category

Definition C.3.1 Let $F : \mathbf{Spec} \to \mathbf{Spec}$ be a pointed simplicial functor. We say it is a <u>homotopy functor</u> if it preserves weak equivalences of spectra.

Remark Recall that we have defined **Spec** as the pointed simplicial category of spectra of pointed simplicial sets. In particular, since its enrichment is pointed simplicial, an enriched functor $F : \mathbf{Spec} \to \mathbf{Spec}$ must send the basepoint in $\mathbf{Spec}(\underline{X}, \underline{Y})$ to the one in $\mathbf{Spec}(F(\underline{X}), F(\underline{Y}))$. That is, we must have $F(*) = *$. In the general literature, when working in the *unpointed* setting (so that our categories are simplicial, but not necessarily *pointed simplicial*), simplicial functors F need not satisfy the condition $F(*) = *$. If it holds, F is said to be *reduced*, while F is said to be *weakly reduced* if the weaker condition $F(*) \simeq *$ is true.

Notation We will denote by $Fun^{\mathrm{h}}(\mathbf{Spec}, \mathbf{Spec})$ the category of pointed simplicial homotopy functors of spectra taking good values. To avoid set-theoretic difficulties, what we will mean here (by abuse of notation) are the functors defined on the category of *finite* spectra; since $\mathbf{Spec}^{\mathrm{fin}}$ is skeletally small, this is a well-defined functor category. The morphisms between functors $F, G : \mathbf{Spec} \to \mathbf{Spec}$ are natural transformations $\eta : F \Rightarrow G$. Note that since **Spec** is $\underline{s\mathbf{Sets}_*}$-enriched, one can give $Fun^{\mathrm{h}}(\mathbf{Spec}, \mathbf{Spec})$ a pointed simplicial enrichment as well, which denote by $\underline{Fun}^{\mathrm{h}}(\mathbf{Spec}, \mathbf{Spec})$.

Example C.3.2 Let F be the composite of the $(\Sigma^\infty, \Omega^\infty)$-adjunction, given by $\Sigma^\infty \Omega^\infty : \mathbf{Spec} \to \mathbf{Spec}$. Then $F \in Fun^{\mathrm{h}}(\mathbf{Spec}, \mathbf{Spec})$.

APPENDIX D

CUBICAL HOMOTOPY THEORY

D.1   Total Fibers and Cofibers

Definition D.1.1 Let S be a finite set. The cubical category of S, denoted $\mathcal{P}(S)$ is the category with objects all subsets of S and morphisms inclusions of sets (it's the category associated to the poset). We let $\mathcal{P}_0(S) \subset \mathcal{P}(S)$ be the full subcategory consisting of all non-empty subsets, and $\mathcal{P}^1(S) \subset \mathcal{P}(S)$ the full subcategory of proper subsets. An S-cube in a category $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{P}(S) \to \mathcal{C}$.

Example D.1.2 Let $S = \{1,2,3\}$ be the three-element set. The category $\mathcal{P}(S)$ is represented by the poset:
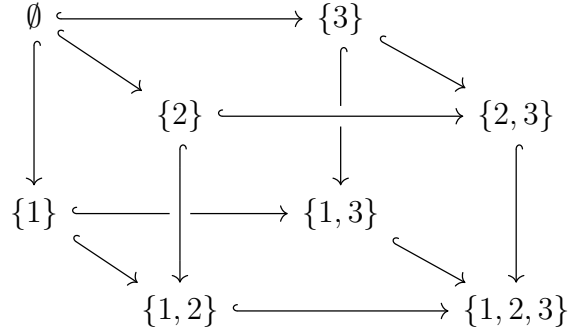
while $\mathcal{P}_0(S)$ and $\mathcal{P}^1(S)$, respectively, are represented by:
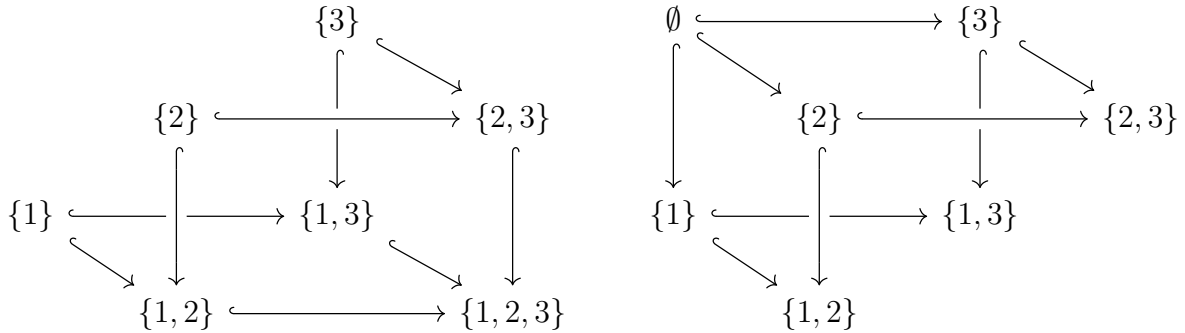
Observation Let $\mathcal{X} : \mathcal{P}(S) \to \mathcal{C}$ be an S-cube in a category. Then $\lim_{\mathcal{P}(S)}\mathcal{X} \cong \mathcal{X}(\emptyset)$ while $\mathrm{colim}_{\mathcal{P}(S)}\mathcal{X} \cong \mathcal{X}(S)$. Suppose $\mathcal{C}$ is a simplicially enriched category (like $\underline{s\mathbf{Sets}_*}$ or $\mathbf{Spec}$) so we can talk about uncorrected homotopy limits and colimits. Then the inclusion of the subcategory $\mathcal{P}_0(S) \subset \mathcal{P}(S)$ induces a map $\mathrm{holim}_{\mathcal{P}(S)}\mathcal{X} \to \mathrm{holim}_{\mathcal{P}_0(S)}(\mathcal{X}|_{\mathcal{P}_0(S)})$, and therefore maps:

$$\mathcal{X}(\emptyset) \cong \lim_{\mathcal{P}(S)}\mathcal{X} \to \mathrm{holim}_{\mathcal{P}(S)}\mathcal{X} \to \mathrm{holim}_{\mathcal{P}_0(S)}(\mathcal{X}|_{\mathcal{P}_0(S)})$$

Similarly, the inclusion of the subcategory $\mathcal{P}^1(S) \subset \mathcal{P}(S)$ induces a map $\mathrm{hocolim}_{\mathcal{P}^1(S)}(\mathcal{X}|_{\mathcal{P}^1(S)}) \to \mathrm{hocolim}_{\mathcal{P}(S)}\mathcal{X}$, and therefore maps:

$$\mathrm{hocolim}_{\mathcal{P}^1(S)}(\mathcal{X}|_{\mathcal{P}^1(S)}) \to \mathrm{hocolim}_{\mathcal{P}(S)}\mathcal{X} \to \mathrm{colim}_{\mathcal{P}(S)}\mathcal{X} \cong \mathcal{X}(S)$$

Definition D.1.3 Let $\mathcal{X}$ be an S-cube in $\underline{s\mathbf{Sets}_*}$ or $\mathbf{Spec}$. If the natural map $\mathcal{X}(\emptyset) \to \mathrm{holim}_{\mathcal{P}_0(S)}(\mathcal{X}|_{\mathcal{P}_0(S)})$ is n-connected, we say $\mathcal{X}$ is $\underline{n\text{-Cartesian}}$; it is $\underline{\text{Cartesian}}$ if the map is an equivalence. If the natural map $\mathrm{hocolim}_{\mathcal{P}^1(S)}(\mathcal{X}|_{\mathcal{P}^1(S)}) \to \mathcal{X}(S)$ is n-connected, we say $\mathcal{X}$ is $\underline{n\text{-coCartesian}}$; it is $\underline{\text{coCartesian}}$ if the map is an equivalence. We say $\mathcal{X}$ is $\underline{\text{strongly coCartesian}}$ if every subcube is coCartesian.

Fact S-cubes in spectra are much better behaved than in simplicial sets. Indeed, a cube in spectra is Cartesian if and only if it is coCartesian (see Lemma 2.6 [37]).

In the case that $\mathcal{X}$ is not Cartesian or coCartesian the fibers and cofibers of the natural maps described above will be of importance:

Definition D.1.4 Let $\mathcal{X}$ be an S-cube in $\underline{s\mathbf{Sets}_*}$ or $\mathbf{Spec}$. The $\underline{\text{total fiber}}$ of $\mathcal{X}$, denoted $\mathrm{tfib}\,\mathcal{X}$, is defined as $\mathrm{tfib}\,\mathcal{X} := \mathrm{hofib}\big(\mathcal{X}(\emptyset) \to \mathrm{holim}_{\mathcal{P}_0(S)}(\mathcal{X}|_{\mathcal{P}_0(S)})\big)$, while the $\underline{\text{total cofiber}}$ of $\mathcal{X}$, denoted $\mathrm{tcofib}\,\mathcal{X}$, is defined as $\mathrm{tcofib}\,\mathcal{X} := \mathrm{hocofib}\big(\mathrm{hocolim}_{\mathcal{P}^1(S)}(\mathcal{X}|_{\mathcal{P}^1(S)}) \to \mathcal{X}(S)\big)$.

Fact An S-cube $\mathcal{X}$ is n-Cartesian if and only if $\mathrm{tfib}\,\mathcal{X}$ is $(n-1)$-connected. It is n-coCartesian if and only if $\mathrm{tcofib}\,\mathcal{X}$ is $(n-1)$-connected.

Remark Recall our convention for taking homotopy limits and colimits. If $\mathcal{X}$ is valued in pointed simplicial sets and is not object-wise fibrant, then the total fiber is applied *only after* replacing $\mathcal{X}$ by the equivalent, object-wise fibrant, S-cube $\{U \subset S \mapsto \widehat{\mathcal{X}}(U))\}$. The total cofiber in this setting requires no change. Similarly, if $\mathcal{X}$ is valued in $\mathbf{Spec}$ and is not object-wise fibrant, $\mathrm{tfib}\,\mathcal{X}$ refers to the total fiber of the S-cube $\{U \subset S \mapsto \underline{\Omega^\infty(\mathcal{X}(U))}\}$. If $\mathcal{X}$ is valued in $\mathbf{Spec}$ and is not object-wise cofibrant, $\mathrm{tcofib}\,\mathcal{X}$ will refer to the total cofiber of the the S-cube $\{U \subset S \mapsto \widetilde{\mathcal{X}(U)})\}$

Observation Let $\Theta : S \to T$ be a bijection of sets. We get an induced functor $\mathcal{P}(S) \to \mathcal{P}(T)$ with a commutative square of categories:

$$\begin{array}{ccc} \mathcal{P}_0(S) & \longhookrightarrow & \mathcal{P}(S) \\ \downarrow & & \downarrow \\ \mathcal{P}_0(T) & \longhookrightarrow & \mathcal{P}(T) \end{array}$$

Let $\mathcal{X}$ be a T-cube in $\underline{s\mathbf{Sets}}_*$ or $\mathbf{Spec}$. By considering the restriction of $\mathcal{X}$ to both $\mathcal{P}_0(T)$ and $\mathcal{P}_0(S)$, we get a commutative square:

$$\begin{array}{ccc} \mathcal{X}(\emptyset) & \longrightarrow & \mathrm{holim}_{\mathcal{P}_0(T)}(\mathcal{X}|_{\mathcal{P}_0(T)}) \\ \downarrow & & \downarrow \\ \mathcal{X}(\emptyset) & \longrightarrow & \mathrm{holim}_{\mathcal{P}_0(S)}(\mathcal{X}|_{\mathcal{P}_0(S)}) \end{array}$$

and therefore maps on homotopy fibers, $\mathrm{tfib}\,\mathcal{X} \to \mathrm{tfib}\,\Theta^*\mathcal{X}$. It is clear that this is functorial in S, and so the induced map is a homeomorphism. In particular, if $S = T = \underline{n}$, $\Theta = \sigma \in \Sigma_n$, and $\mathcal{X}$ has the property that $\mathcal{X} \circ \Theta = \mathcal{X}$, then the induced maps $\mathrm{tfib}\,\mathcal{X} \xrightarrow{\sigma} \mathrm{tfib}\,\mathcal{X}$ make $\mathrm{tfib}\,\mathcal{X}$ into a (right) Σ_n-object. Similarly, the commutative square of categories:

$$\begin{array}{ccc} \mathcal{P}^1(S) & \longhookrightarrow & \mathcal{P}(S) \\ \downarrow & & \downarrow \\ \mathcal{P}^1(T) & \longhookrightarrow & \mathcal{P}(T) \end{array}$$

gives a commutative square:

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathcal{P}^1(S)}(\mathcal{X}|_{\mathcal{P}_0(S)}) & \longrightarrow & \mathcal{X}(\emptyset) \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{\mathcal{P}^1(T)}(\mathcal{X}|_{\mathcal{P}_0(T)}) & \longrightarrow & \mathcal{X}(\emptyset) \end{array}$$

and therefore maps on homotopy cofibers, $\mathrm{tcofib}\,\Theta^*\mathcal{X} \to \mathrm{tcofib}\,\mathcal{X}$. In the aforementioned case of $S = T = \underline{n}$, $\Theta = \sigma \in \Sigma_n$, and $\mathcal{X} \circ \Theta = \mathcal{X}$, the induced maps $\mathrm{tcofib}\,\mathcal{X} \xrightarrow{\sigma} \mathrm{tcofib}\,\mathcal{X}$ make $\mathrm{tcofib}\,\mathcal{X}$ into a (right) Σ_n-object.

Remark There is an alternate formulation of the total fibers and total cofibers of a cube which is remarkably useful. Let $\mathcal{X}$ be an S-cube in a category $\mathcal{C}$, and pick a distinguished element $s \in S$. Then we can construct two $(S - s)$-cubes in $\mathcal{C}$, denoted $\mathcal{X}_{source}$ and $\mathcal{X}_{target}$, as follows: for $U \subset S - s$, $\mathcal{X}_{source}(U) := \mathcal{X}(U)$ and $\mathcal{X}_{target}(U) := \mathcal{X}(U \amalg \{s\})$ with the maps induced from those in $\mathcal{X}$. Note also that the inclusion $U \subset U \amalg \{s\}$ for $U \subset S - s$ induces a

map of $(S-s)$-cubes $\mathcal{X}_{source} \to \mathcal{X}_{target}$. On the other hand, given a map of S-cubes $\mathcal{X} \to \mathcal{Y}$, we can form the finite set $T := S \amalg *$ by adding a disjoint element. Then we can view $\mathcal{X} \to \mathcal{Y}$ as a single cube, $\mathcal{Z} : \mathcal{P}(T) \to \mathcal{C}$ by sending, for $U \subset T$

$$
\mathcal{Z}(U) = \begin{cases} \mathcal{X}(U) & \text{if} \quad * \notin U \\ \mathcal{Y}(U - *) & \text{if} \quad * \in U \end{cases}
$$

In particular, if $S = \{1, \ldots, n-1\}$, then a map of $(n-1)$-cubes is equivalent to an n-cube, by taking $* = n$ or $s = n$. This is used extensively and successfully to induct on the size of cubes (see p. 300 [25]). In particular, we have the following result:

Proposition D.1.5 *Let $\mathcal{X}$ be an $\underline{n}$-cube in $s\boldsymbol{Sets_*}$ or $\boldsymbol{Spec}$, and view it as a map of $(n-1)$-cubes $\mathcal{X}_{source} \to \mathcal{X}_{target}$. Then we have natural homeomorphisms*

$$
\mathrm{tfib}\, \mathcal{X} \cong \mathrm{hofib}\big(\mathrm{tfib}\,(\mathcal{X}_{source}) \to \mathrm{tfib}\,(\mathcal{X}_{target})\big)
$$

and

$$
\mathrm{tcofib}\, \mathcal{X} \cong \mathrm{hocofib}\big(\mathrm{tcofib}\,(\mathcal{X}_{source}) \to \mathrm{tcofib}\,(\mathcal{X}_{target})\big)
$$

Proof. These homeomorphisms come directly from the end and coend formulas for the homotopy limits and colimits, much in the same manner that Goodwillie shows Definitions 1.1 and 1.1b in [25] are equivalent. □

D.2 Co-cross Effects

Notation Throughout this section, let $\mathcal{T}$ denote either $s\boldsymbol{Sets_*}$ or $\boldsymbol{Spec}$.

Definition D.2.1 Let S be a finite non-empty set and $f : S \to \mathcal{T}$ be a function. Define an S-cube in $\mathcal{T}$, denoted S^f, by:

$$
U \subset S \mapsto \prod_{u \in U} f(u) \quad \text{and} \quad V \subset U \subset S \mapsto \prod_{v \in V} f(v) \to \prod_{u \in U} f(u)
$$

is given by inclusion into the basepoint ($\emptyset$ is sent to $*$).

Example D.2.2 Let $S = \{1, 2\}$, $X, Y \in s\boldsymbol{Sets_*}$, and $f : S \to s\boldsymbol{Sets_*}$ be given by $f(1) = X$, $f(2) = Y$. Then S^f is the 2-cube:

$$\begin{array}{ccc}
* & \longrightarrow & Y \\
\downarrow & & \downarrow {\scriptstyle *\times \mathrm{id}_Y} \\
X & \xrightarrow{\ \mathrm{id}_X \times *\ } & X \times Y
\end{array}$$

If $S' \xrightarrow{\alpha} S$ is a function, we get an induced S'-cube, $S'^{f\circ\alpha}$

Example D.2.3 Continuing with the previous example, if $S' = \{1,2,3\}$ and $\alpha : \{1,2,3\} \to \{1,2\}$ is given by $\alpha(1) = \alpha(3) = 1, \alpha(2) = 2$, then $S'^{f\circ\alpha}$ is the 3-cube:

Observation We also get an induced functor $\mathcal{P}(S) \to \mathcal{P}(S')$ given by taking the pre-image of a subset under α. If $S' \xrightarrow{\alpha} S$ is surjective, then we define an S-subcube of the S'-cube, $S'^{f\circ\alpha}$, as the composite $\mathcal{P}(S) \to \mathcal{P}(S') \xrightarrow{S'^{f\circ\alpha}} \mathcal{C}$. That is,

$$U \subset S \mapsto \prod_{u' \in \alpha^{-1}(U)} f(\alpha(u'))$$

We denote this S-subcube by $S^f(\alpha)$

Example D.2.4 If $S = \{1,2\}$ $S' = \{1,2,3\}$ and $\alpha : S' \to S$ is given by $\alpha(1) = \alpha(3) = 1, \alpha(2) = 2$, then $S^f(\alpha)$ is the 2-subcube given by (in blue):

We let $\tilde{\alpha} : S^f \to S^f(\alpha)$ be the map of S-cubes given by:

$$U \subset S \mapsto \prod_{u \in U} f(u) \xrightarrow{\tilde{\alpha}_U} \prod_{u' \in \alpha^{-1}(U)} f(\alpha(u'))$$

where $\tilde{\alpha}_U(x_1, \ldots, x_t) = (x_{\alpha(1)}, \ldots, x_{\alpha(s)})$, and $t = |U|, s = |\alpha^{-1}(U)|$.

Example D.2.5 In our previous example, the induced map of 2-cubes $\tilde{\alpha} : S^f \to S^f(\alpha)$ is:
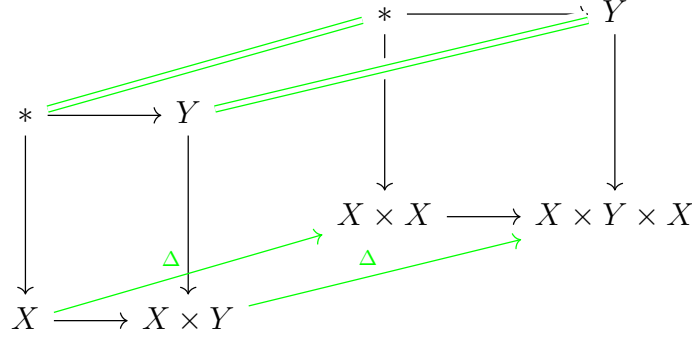
This is *not* the map induced from two subcubes sitting inside the 3-cube $S'^{f \circ \alpha}$.

Definition D.2.6 Let $F : \mathcal{T} \to \mathcal{T}'$ be a reduced homotopy functor. Let S be a finite set, and $f : S \to \mathcal{T}$ a function. We define:

$$\hat{cr}^f F := \operatorname{tcofib} F(S^f)$$

Example D.2.7 Let $S = \{1, 2\}$, $X, Y \in \underline{s\mathbf{Sets}_*}$, and $f : S \to \underline{s\mathbf{Sets}_*}$ be given by $f(1) = X, f(2) = Y$. Let F be a reduced functor. Then $\hat{cr}^f F$ is the the iterated cofiber of the 2-cube:

$$
\begin{array}{ccc}
* & \longleftarrow & F(Y) \\
\downarrow & & \downarrow{\scriptstyle * \times \mathrm{id}_Y} \\
F(X) & \xrightarrow{\mathrm{id}_X \times *} & F(X \times Y)
\end{array}
$$

In the case that $F = Id_{\underline{s\mathbf{Sets}_*}}$, then $\hat{cr}^f Id \simeq X \wedge Y$.

Let $\alpha : S' \to S$ be a surjective map of sets. We have constructed thus far three cubes from this data: S^f, $S'^{f \circ \alpha}$, and the S-cube sitting inside it, $S^f(\alpha)$. Taking F we obtain three cubes in the $\mathcal{T}'$, mainly, $F(S^f)$, $F(S'^{f \circ \alpha})$, and $F(S^f(\alpha))$. Note that we have a morphism of S-cubes with the first and last, that is, $F(\tilde{\alpha}) : F(S^f) \to F(S^f(\alpha))$, and therefore we get induced maps of total cofibers:

$$\operatorname{tcofib}(F(S^f)) \xrightarrow{\tilde{F}(\tilde{\alpha})} \operatorname{tcofib}(F(S^f(\alpha)))$$

On the other hand, since $S^f(\alpha)$ is defined as the composite $\mathcal{P}(S) \xrightarrow{\alpha^{-1}(-)} \mathcal{P}(S') \xrightarrow{S'^{f \circ \alpha}} \mathcal{C}$, by the properties of the homotopy colimit with respect to change of indexing category, we get

an induced map:

$$\mathrm{tcofib}(F(S^f(\alpha))) \to \mathrm{tcofib}(F(S'^{f\circ\alpha}))$$

Combining both maps, we get a map in $\mathcal{T}'$:

$$\hat{\alpha} : \hat{cr}^f F \longrightarrow \hat{cr}^{f\circ\alpha} F$$

Example D.2.8 Let $S = \{1, 2\}$, $X, Y \in \underline{s\mathbf{Sets}_*}$, and $f : S \to \underline{s\mathbf{Sets}_*}$ be given by $f(1) = X, f(2) = Y$. Let $S' = \{1, 2, 3\}$ and $\alpha : S' \to S$ is given by $\alpha(1) = \alpha(3) = 1, \alpha(2) = 2$. Let $F = Id_{s\mathbf{Sets}_*}$. Then, $\hat{cr}^f Id \simeq X \wedge Y$, and, by taking the iterated cofibers of the 3-cube $S'^{f\circ\alpha}$, we see that $\hat{cr}^{f\circ\alpha} Id \simeq (X \wedge Y) \wedge X$. The map $\hat{\alpha}$ is then the composite of:

$$\hat{cr}^f Id \simeq X \wedge Y \xrightarrow{\Delta \times \mathrm{id}_Y} (X \times X) \wedge Y \xrightarrow{\pi \times \mathrm{id}_Y} (X \wedge X) \wedge Y \cong (X \wedge Y) \wedge X \simeq \hat{cr}^{f\circ\alpha} Id$$

Fact Let $\alpha : S' \to S$ and $\beta : S'' \to S'$ be surjections of finite sets. Then for any $f : S \to \mathcal{T}$, $\widehat{\alpha \circ \beta} = \hat{\beta} \circ \hat{\alpha}$.

Notation For $X \in \mathcal{T}$, denote also by X the function from $\{1\}$ to $\mathcal{T}$ given by $1 \mapsto X$. For S any finite non-empty set, there is a unique surjective map $\tau_S : S \to \{1\}$.

Definition D.2.9 Let $F : \mathcal{T} \to \mathcal{T}'$ be a reduced homotopy functor, and $X \in \mathcal{T}$. For S a finite non-empty set, define

$$\hat{cr}^S F(X) := \hat{cr}^{X\circ\tau_S} F$$

Example D.2.10 Let $F = Id_{s\mathbf{Sets}_*}$. Then $\hat{cr}^S Id(X) \simeq \bigwedge_S X$, the smash product of X indexed over the elements of S.

Observation If $\alpha : S' \to S$ is a surjective map, we get an induced map $\hat{cr}^S F(X) \to \hat{cr}^{S'} F(X)$ by the map $\hat{\alpha} : \hat{cr}^{X\circ\tau_S} F \to \hat{cr}^{X\circ\tau_S\circ\alpha} F$ (since $\tau_S \circ \alpha = \tau_{S'}$), which was induced from the diagonals, in the map of S-cubes $F(S^{X\circ\tau_S}) \xrightarrow{F(\hat{\alpha})} F(S^{X\circ\tau_S}(\alpha))$.

Example D.2.11 Let $S = \{1, 2\}, S' = \{1, 2, 3\}$ and $\alpha : S' \to S$ is given by $\alpha(1) = \alpha(3) = 1, \alpha(2) = 2$. Fix $X \in \underline{s\mathbf{Sets}_*}$. Then the induced map $\hat{\alpha} : \hat{cr}^S Id(X) \to \hat{cr}^{S'} Id(X)$ is the map:

$$X \wedge X \to X \wedge X \wedge X$$
$$(x_1, x_2) \mapsto (x_1, x_2, x_1)$$

D.3   As $\mathbb{M}$-diagrams

Notation Let Surj be the category with:

- $\mathrm{Obj}(\mathrm{Surj}) = \{\underline{n} \mid \underline{n} := \{1, \ldots, n\}, n \in \mathbb{N}\}$

- $\mathrm{Surj}(\underline{n}, \underline{m}) = \{\alpha \mid \alpha : \underline{n} \to \underline{m} \text{ surjective}\}$

Denote its opposite category by $\mathbb{M} := \mathrm{Surj}^{\mathrm{op}}$. Similarly, we set Surj_n to be the full subcategory of Surj with objects of size $\leq n$, and denote $\mathbb{M}_n := \mathrm{Surj}_n{}^{\mathrm{op}}$.

Claim D.3.1 *For a fixed non-empty finite set S, and $F \in Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}')$, the assignment $X \mapsto \hat{cr}^S F(X)$ is in $Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}')$ again.*

Proof. Let $f : X \to Y$ be a map in $\mathcal{T}$. There is a map of S-cubes in $\mathcal{T}$, $S^{X \circ \tau_S} \to S^{Y \circ \tau_S}$ given by, for $U \subset S$, $\prod_{u \in U} X_u \xrightarrow{f \times \cdots \times f} \prod_{u \in U} Y_u$, and therefore maps of S-cubes in $\mathcal{T}'$, $F(S^{X \circ \tau_S}) \to F(S^{Y \circ \tau_S})$. The induced map on total cofibers

$$\hat{cr}^S F(X) = \mathrm{tcofib}\, F(S^{X \circ \tau_S}) \to \mathrm{tcofib}\, F(S^{Y \circ \tau_S}) = \hat{cr}^S F(Y)$$

makes $\hat{cr}^S F(-) : \mathcal{T} \to \mathcal{T}'$ into a functor. Also, note that $\hat{cr}^S F(*) = \mathrm{tcofib}\, F(S^{* \circ \tau_S})$. Since F is reduced, then $F(S^{* \circ \tau_S})$ is the constant S-cube of $*$, so by the properties of our model of the homotopy colimit, $\mathrm{tcofib}\, F(S^{* \circ \tau_S}) \cong *$, and therefore $\hat{cr}^S F(-)$ is a reduced functor. Lastly, suppose that $X \xrightarrow{\simeq} Y$ is a weak equivalence in $\mathcal{T}$. Then since F is a homotopy functor, the map of S-cubes in $\mathcal{T}'$, $F(S^{X \circ \tau_S}) \to F(S^{Y \circ \tau_S})$, is a weak equivalence in the diagram category $Func(\mathcal{P}(S), \mathcal{T}')$. Since homotopy colimits preserve weak equivalences we get that:

$$\hat{cr}^S F(X) = \mathrm{tcofib}\, F(S^{X \circ \tau_S}) \xrightarrow{\simeq} \mathrm{tcofib}\, F(S^{Y \circ \tau_S}) = \hat{cr}^S F(Y)$$

$\square$

Claim D.3.2 *For a fixed $F \in Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}')$, the assignment $S \in \mathbb{M} \mapsto \hat{cr}^S F(-)$ defines a functor $\mathbb{M} \to Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}')$.*

Proof. Let $\alpha^{\mathrm{op}} : S \to S'$ be a morphism in $\mathbb{M}$ (so $\alpha : S' \to S$ is a surjective map of sets). Then for every $X \in \mathcal{T}$ we have an induced map $\hat{\alpha} : \hat{cr}^S F(X) \to \hat{cr}^{S'} F(X)$ defined earlier as $\hat{cr}^{X \circ \tau_S} F \to \hat{cr}^{X \circ \tau_{S'}} F$. If $\beta^{\mathrm{op}} : S' \to S''$ is another morphism in $\mathbb{M}$ (so $\beta : S'' \to S'$ is a surjective maps of sets), and $\hat{\beta} : \hat{cr}^{X \circ \tau_{S'}} F \to \hat{cr}^{X \circ \tau_{S''}} F$ its induced map, we've already seen

that $\widehat{\alpha \circ \beta} = \hat{\beta} \circ \hat{\alpha}$, and therefore,

$$\hat{cr}^S F(X) \to \hat{cr}^{S'} F(X) \to \hat{cr}^{S''} F(X)$$

has $\hat{\beta}^{\mathrm{op}} \circ \hat{\alpha}^{\mathrm{op}} = \widehat{\beta^{\mathrm{op}} \circ \alpha^{\mathrm{op}}}$. Now, we need to check that for $\alpha^{\mathrm{op}} : S \to S'$ in $\mathbb{M}$ we have a *morphism* in $Fun^{\mathrm{red}}(\mathcal{T}, \mathcal{T}')$, that is, a natural transformation of the functors $\hat{cr}^S F(-) \to \hat{cr}^{S'} F(-)$. For each $X \in \mathcal{T}$, we'll denote the previous map $\hat{\alpha}$ by the object X, by $\hat{\alpha}_X : \hat{cr}^S F(X) \to \hat{cr}^{S'} F(X)$. Recall that that map was induced from the map of S-cubes induced from diagonals $\tilde{\alpha}_X : S^{X \circ \tau_S} \to S^{X \circ \tau}(\alpha)$. Let $f : X \to Y$ be a map in $\mathcal{T}$. Since the "diagonals of the f's are the f's of the diagonals" we have a commutative square of S-cubes:

$$\begin{array}{ccc} S^{X \circ \tau_S} & \xrightarrow{\tilde{\alpha}_X} & S^{X \circ \tau_S}(\alpha) \\ \downarrow & & \downarrow \\ S^{Y \circ \tau_S} & \xrightarrow{\tilde{\alpha}_Y} & S^{Y \circ \tau_S}(\alpha) \end{array}$$

and therefore a commutative square of S-cubes in $\mathcal{T}'$

$$\begin{array}{ccc} F(S^{X \circ \tau_S}) & \xrightarrow{F(\tilde{\alpha}_X)} & F(S^{X \circ \tau_S}(\alpha)) \\ \downarrow & & \downarrow \\ F(S^{Y \circ \tau_S}) & \xrightarrow{F(\tilde{\alpha}_Y)} & F(S^{Y \circ \tau_S}(\alpha)) \end{array}$$

Taking total cofibers, we get a commutative square in $\mathcal{T}'$:

$$\begin{array}{ccc} \mathrm{tcofib}\, F(S^{X \circ \tau_S}) & \longrightarrow & \mathrm{tcofib}\, F(S^{X \circ \tau_S}(\alpha)) \\ \downarrow & & \downarrow \\ \mathrm{tcofib}\, F(S^{Y \circ \tau_S}) & \longrightarrow & \mathrm{tcofib}\, F(S^{Y \circ \tau_S}(\alpha)) \end{array}$$

On the other hand, the map $f : X \to Y$ induces a map of S'-cubes in $\mathcal{T}$, $S'^{X \circ \tau_{S'}} \to S'^{Y \circ \tau_{S'}}$ as in the previous claim. Since the homotopy colimit of the pre-composition is natural with respect to natural transformations of diagrams before composition, we have a commutative square:

$$\begin{array}{ccc} \mathrm{tcofib}\, F(S^{X \circ \tau_S}(\alpha)) & \longrightarrow & \mathrm{tcofib}\, F(S^{X \circ \tau_{S'}}) \\ \downarrow & & \downarrow \\ \mathrm{tcofib}\, F(S^{Y \circ \tau_S}(\alpha)) & \longrightarrow & \mathrm{tcofib}\, F(S^{Y \circ \tau_{S'}}) \end{array}$$

Pasting this square next to the previous one, we get that:

$$\begin{CD} \hat{cr}^S F(X) @>\hat{\alpha}_X>> \hat{cr}^{S'} F(X) \\ @VVV @VVV \\ \hat{cr}^S F(Y) @>\hat{\alpha}_Y>> \hat{cr}^{S'} F(Y) \end{CD}$$

Therefore $\hat{\alpha}_- : \hat{cr}^S F(-) \to \hat{cr}^{S'} F(-)$ is a natural transformation of functors $\mathcal{T} \to \mathcal{T}'$, and this concludes the claim. $\square$

Claim D.3.3 *The assignment $F \in Fun^h(\mathcal{T}, \mathcal{T}') \mapsto \hat{cr}^\star F(-) \in Fun\big(\mathbb{M}, Fun^h(\mathcal{T}, \mathcal{T}')\big)$ defines a functor.*

Proof. Let $F, G \in Fun^{red}(\mathcal{T}, \mathcal{T}')$, and $\eta : F \to G$ be a natural transformation. We want a natural transformation $\hat{cr}^\star F(-) \to \hat{cr}^\star G(-)$ of functors $\mathbb{M} \to Fun^{red}(\mathcal{T}, \mathcal{T}')$. For $S \in \mathbb{M}$ and $X \in \mathcal{T}$, we have a map of S-cubes $F(S^{X \circ \tau_S}) \xrightarrow{\tilde{\eta}_S} G(S^{X \circ \tau_S})$ given, for $U \subset S$ by:

$$F\big(\prod_{u \in U} X_u\big) \xrightarrow{\eta_{\prod_{u \in U} X_u}} G\big(\prod_{u \in U} X_u\big)$$

The naturality of η ensures that this is a map of S-cubes. Again by the naturality of η, if $f : X \to Y$ is a map in $\mathcal{T}$, then we have a commutative square of S-cubes:

$$\begin{CD} F(S^{X \circ \tau_S}) @>\tilde{\eta}_S>> G(S^{X \circ \tau_S}) \\ @VVV @VVV \\ F(S^{Y \circ \tau_S}) @>\tilde{\eta}_S>> G(S^{Y \circ \tau_S}) \end{CD}$$

Taking total cofibers, we get a commuting square in $\mathcal{T}'$:

$$\begin{CD} \hat{cr}^S F(X) @>\tilde{\eta}_{S,X}>> \hat{cr}^S G(X) \\ @VVV @VVV \\ \hat{cr}^S F(Y) @>\tilde{\eta}_{S,Y}>> \hat{cr}^S G(Y) \end{CD}$$

So $\tilde{\eta}_S$ is a natural transformation between $\hat{cr}^S F(-) \to \hat{cr}^S G(-)$ as functors from $\mathcal{T} \to \mathcal{T}'$. If $\alpha : S' \to S$ is a surjective map of sets, by the naturality of η, we have a commutative square of S-cubes:

$$\begin{CD} F(S^{X \circ \tau_S}) @>F(\tilde{\alpha}_X)>> F(S^{X \circ \tau_S}(\alpha)) \\ @V\tilde{\eta}_{S,X}VV @VV\tilde{\eta}_{S,X}V \\ G(S^{X \circ \tau_S}) @>G(\tilde{\alpha}_X)>> G(S^{X \circ \tau_S}(\alpha)) \end{CD}$$

Taking total cofibers, we have the commutative diagram in $\mathcal{T}'$:

$$\begin{array}{ccc}
\text{tcofib } F(S^{X \circ \tau_S}) & \xrightarrow{F(\tilde{\alpha}_X)} & \text{tcofib } F(S^{X \circ \tau_S}(\alpha)) \\
\downarrow{\scriptstyle \tilde{\eta}_{S,X}} & & \downarrow{\scriptstyle \tilde{\eta}_{S,X}} \\
\text{tcofib } G(S^{X \circ \tau_S}) & \xrightarrow{G(\tilde{\alpha}_X)} & \text{tcofib } G(S^{X \circ \tau_S}(\alpha))
\end{array}$$

On the other hand, the natural transformation η induces a map of S'-cubes in $\mathcal{T}'$, $F(S'^{X \circ \tau_{S'}}) \xrightarrow{\tilde{\eta}_{S',X}} G(S'^{X \circ \tau_{S'}})$. Since the homotopy colimit of the pre-composition is natural with respect to natural transformations of diagrams before composition, we have a commutative square:

$$\begin{array}{ccc}
\text{tcofib } F(S^{X \circ \tau_S}(\alpha)) & \longrightarrow & \text{tcofib } F(S^{X \circ \tau_{S'}}) \\
\downarrow & & \downarrow \\
\text{tcofib } G(S^{X \circ \tau_S}(\alpha)) & \longrightarrow & \text{tcofib } G(S^{X \circ \tau_{S'}})
\end{array}$$

Pasting this square next to the previous one, we get that:

$$\begin{array}{ccc}
\hat{cr}^S F(X) & \xrightarrow{\hat{\alpha}_X} & \hat{cr}^{S'} F(X) \\
\downarrow{\scriptstyle \tilde{\eta}_{S,X}} & & \downarrow{\scriptstyle \tilde{\eta}_{S',X}} \\
\hat{cr}^S G(X) & \xrightarrow{\hat{\alpha}_X} & \hat{cr}^{S'} G(X)
\end{array}$$

So $\tilde{\eta}_{\star,-}$ is a natural transformation between $\hat{cr}^\star F(-) \to \hat{cr}^\star G(-)$ as functors from $\mathbb{M} \to Fun^h(\mathcal{T}, \mathcal{T}')$. If $\theta : G \to H$ is another natural transformation of functors $Fun^h(\mathcal{T}, \mathcal{T}')$, then we have $\tilde{\theta}_{\star,-} \circ \tilde{\eta}_{\star,-} = \widetilde{\theta \circ \eta}_{\star,-}$ by inspection. $\qquad\square$

Altogether we have that "taking co-cross effects" defines a functor

$$\hat{cr} := \hat{cr}^\star(\dagger)(-) : Fun^h(\mathcal{T}, \mathcal{T}') \to Fun\big(\mathbb{M}, Fun^h(\mathcal{T}, \mathcal{T}')\big)$$

We can also define another functor between these categories which behaves like the trivial functor, denoted tr,

$$tr := tr^\star(\dagger)(-) : Fun^h(\mathcal{T}, \mathcal{T}') \to Fun\big(\mathbb{M}, Fun^h(\mathcal{T}, \mathcal{T}')\big)$$

which is much easier to define. For $F \in Fun^h(\mathcal{T}, \mathcal{T}')$, $S \in \mathbb{M}$, and $X \in \mathcal{T}$, define $tr^S F(X) := F(X)$ and for $f : X \to Y$ in $\mathcal{T}$, we set $tr^S F(X) \to tr^S F(Y)$ to be $F(f)$. That is, for any finite set S, we set $tr^S F$ to be the functor F. For $\alpha : S' \to S$ a surjective map of sets, we set $tr^S F(X) \to tr^{S'} F(X)$ to be the identity map on $F(X)$; this clearly defines the identity natural transformation. Lastly, for $F, G \in Fun^h(\mathcal{T}, \mathcal{T}')$ and $\eta : F \to G$ a natural

transformation, we set $\tilde{\eta}_{S,X} := \eta_X : F(X) \to G(X)$, thus defining a natural transformation $tr(F) \to tr(G)$ as functors $\mathbb{M} \to Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}')$.

There is a natural transformation $tr \to \hat{c}r$ as functors $Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}') \to Fun(\mathbb{M}, Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}'))$. We define it via an auxiliary diagram; for $S \in \mathbb{M}$ and $X \in \mathcal{T}$, let $tr_X(S)$ be the S-cube in $\mathcal{T}$ given by, for $U \subset S$:

$$tr_X(S)(U) = \begin{cases} * & \text{if} \quad U \neq S \\ X & \text{if} \quad U = S \end{cases}$$

This diagram has the zero object in $\mathcal{T}$ everywhere except at the terminal point of the S-cube, where it equals X. Let $\Delta : tr_X(S) \to S^{X \circ \tau_S}$ be the map of S-cubes determined by the diagonal map $X \xrightarrow{\Delta} \prod_{s \in S} X_s$. If $F \in Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}')$, we get an induced map of S-cubes in $\mathcal{T}'$, $F(tr_X(S)) \xrightarrow{F(\Delta)} F(S^{X \circ \tau_S})$, and then taking total cofibers:

$$tr^S F(X) = F(X) \cong \mathrm{tcofib}\, F(tr_X(S)) \xrightarrow{F(\Delta)} \mathrm{tcofib}\, F(S^{X \circ \tau_S}) = \hat{c}r^S F(X)$$

A quick check shows that this defines a natural transformation of functors (all the maps are induced by diagonals, which are compatible with $F(\Delta)$).

Example D.3.4 Let $\mathcal{T} = \mathcal{T}' = \underline{s\mathbf{Sets}_*}$ and let $F = Id_{\underline{s\mathbf{Sets}_*}}$. Then $\hat{c}r^\star Id(-)$ is the functor that sends a finites set $S \mapsto \hat{c}r^S Id(X) \simeq \bigwedge_S X$, the smash product of X indexed over the elements of S. The maps on $f : X \to Y$ and $\alpha : S' \to S$ are induced by diagonals. Furthermore, the natural transformation $tr^\star Id(-) \to \hat{c}r^\star Id(-)$ *is the diagonal,*

$$tr^S Id(X) = X \xrightarrow{\Delta} \bigwedge_S X \simeq \hat{c}r^S Id(X)$$

D.4   As Multi-functors

Definition D.4.1 Let $n \in \mathbb{N}_{>0}$, and $F \in Fun^{\mathrm{h}}(\mathcal{T}, \mathcal{T}')$. Define an n-multi-functor, denoted $\hat{c}r^n F : \mathcal{T}^{\times n} \to \mathcal{T}'$, by sending $(X_1, \ldots, X_n) \mapsto \hat{c}r^f F$, where $f : \underline{n} \to \mathcal{T}$ is such that $f(i) = X_i$.

Observation Let $(X_1, \ldots, X_n) \xrightarrow{(f_1, \ldots, f_n)} (X_1', \ldots, X_n')$ be a morphism in $\mathcal{T}^{\times n}$. It induces a map of n-cubes in $\mathcal{T}$, $\underline{n}^f \to \underline{n}^{f'}$: on $U \subset \underline{n}$, $\prod_{u \in U} X_u \xrightarrow{\Pi f_u} \prod_{u \in U} X_u'$. Applying F and taking

total cofibers we get the map

$$\hat{cr}^n F(X_1, \ldots, X_n) \to \hat{cr}^n F(X_1', \ldots, X_n')$$

giving the structure of a multi-functor. The multi-functor $\hat{cr}^n F$ is multi-reduced, that is, reduced in each entry. Indeed, if $X_i = *$ for some i, then we can partition the n-cube $F(\underline{n}^f)$ as a map of $(n-1)$-cubes by taking $\underline{n} - i$, and defining $F(\underline{n}^f)_{source} \to F(\underline{n}^f)_{target}$ as in the section on cubical homotopy theory. For each $U \subset (\underline{n} - i)$ the map $\underline{n}^f{}_{source}(U) \to \underline{n}^f{}_{target}(U)$ is an equivalence, since it is the inclusion into the basepoint on $X_i \cong *$ and identity on the remaining $X_j, j \neq i$. Since F is a homotopy functor, we see that for each $U \subset (\underline{n} - i)$, $F(\underline{n}^f)_{source}(U) \to F(\underline{n}^f)_{target}(U)$ is an equivalence. Taking total cofibers, we get that $\hat{cr}^f F \cong *$. Furthermore, $\hat{cr}^n F$ is a homotopy functor in each direction. That is, if $X_i \xrightarrow{\simeq} X_i'$ is a weak equivalence, then the induced map $\hat{cr}^n F(X_1, \ldots, X_i, \ldots, X_n) \to \hat{cr}^n F(X_1, \ldots, X_i', \ldots, X_n)$ is a weak equivalence. Indeed, if $g : X_i \xrightarrow{\simeq} X_i'$ is a weak equivalence, then for any other $Z \in \mathcal{T}$, $Z \times X_i \xrightarrow{id_Z \times g} Z \times X_i'$ is a weak equivalence also. The map of cubes $\underline{n}^f \to \underline{n}^{f'}$ is a weak equivalence on each vertex, and therefore so is $F(\underline{n}^f) \to F(\underline{n}^{f'})$. Taking total cofibers gives the result.

Fact The nth-co-cross effect construction is compatible with natural transformations. That is, taking n^{th}-co-cross effects defines a functor:

$$\hat{cr}^n(-) : Fun^{\text{h}}(\mathcal{T}, \mathcal{T}') \to Fun^{\text{h}}(\mathcal{T}^{\times n}, \mathcal{T}')$$

Example D.4.2 Let $F = Id_{\underline{s\mathbf{Sets}}_*}$, then $\hat{cr}^n Id : \underline{s\mathbf{Sets}}_*{}^{\times n} \to \underline{s\mathbf{Sets}}_*$ is the n-multi-functor:

$$(X_1, \ldots, X_n) \mapsto X_1 \wedge \cdots \wedge X_n$$

Remark Let $\sigma \in \mathbb{M}(\underline{n}, \underline{n}) = \Sigma_n$. For $F \in Fun^{\text{h}}(\mathcal{T}, \mathcal{T}')$, define $\hat{cr}^n F \cdot \sigma : \mathcal{T}^{\times n} \to \mathcal{T}'$ by:

$$(\hat{cr}^n F \cdot \sigma)(X_1, \ldots, X_n) := \hat{cr}^n F(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)})$$

Since our iterated cofibers are isomorphic regardless of the directions taken for the homotopy cofibers, we have that

$$\hat{cr}^n F(X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(n)}) \cong \hat{cr}^n F(X_1, \ldots, X_n)$$

We call such a multi-functor symmetric, that is, invariant under permutations of the variables

(up to isomorphism).

Observation Composing $\hat{cr}^n F$ with the diagonal $\mathcal{T} \xrightarrow{\Delta} \mathcal{T}^{\times n}$ gives a (right) Σ_n-object in $Fun^{\mathrm{h}}(\mathcal{T}^{\times n}, \mathcal{T}')$.

Fact Any n^{th}-co-cross effect can be realized as a second co-cross effect. Indeed, we have a weak equivalence:

$$\hat{cr}^n F(X_1, \ldots, X_n) \simeq \hat{cr}^2 [\hat{cr}^{n-1} F(X_1, \ldots, X_{n-2}, \star)](X_{n-1}, X_n)$$

by D.1.5 or more directly by Lemma 5.7.6 [49].

D.5   Twisted Arrow Category

Notation Let $\mathcal{C}$ be a category. The category of arrows in $\mathcal{C}$ is denoted $\mathrm{Ar}(\mathcal{C}) := Fun([1], \mathcal{C})$.

Definition D.5.1 Let $\mathcal{C}$ be a category. The twisted arrow category on $\mathcal{C}$, denoted $\mathrm{tw}(\mathcal{C})$, is the category with:

- $\mathrm{Obj}(\mathrm{tw}(\mathcal{C})) = \{f \mid f \in \mathrm{Ar}(\mathcal{C})\}$

- $\mathrm{tw}(\mathcal{C})(f, g) = \{(\alpha, \beta) \mid \alpha \in \mathcal{C}(d_o, c_0) \; g \in \mathcal{C}(c_1, d_1) \text{ with } \beta \circ f \circ \alpha = g\}$

That is, the objects are arrows $c_0 \xrightarrow{f} c_1$ in $\mathcal{C}$, and the morphisms between $c_0 \xrightarrow{f} c_1$ and $d_0 \xrightarrow{g} d_1$ are pairs (α, β) making the following diagram commute:

$$
\begin{array}{ccc}
c_0 & \xleftarrow{\;\alpha\;} & d_0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
c_1 & \xrightarrow{\;\beta\;} & d_1
\end{array}
$$

Composition is defined as $(\alpha', \beta') \circ (\alpha, \beta) := (\alpha \circ \alpha', \beta' \circ \beta)$, i.e. from the commuting square:

$$
\begin{array}{ccccc}
c_0 & \xleftarrow{\;\alpha\;} & d_0 & \xleftarrow{\;\alpha'\;} & e_0 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
c_1 & \xrightarrow{\;\beta\;} & d_1 & \xrightarrow{\;\beta'\;} & e_1
\end{array}
$$

Observation There is an obvious "forgetful" functor $\pi : \mathrm{tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ given by $(c_0 \xrightarrow{f} c_1) \mapsto (c_0, c_1)$ and $(\alpha, \beta) \mapsto (\alpha^{\mathrm{op}}, \beta)$. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, we get an induced functor $\mathrm{tw}(\mathcal{C}) \to \mathrm{tw}(\mathcal{D})$ by sending $(c_0 \xrightarrow{f} c_1) \mapsto (F(c_0) \xrightarrow{F(f)} F(c_1))$ and $(\alpha, \beta) \mapsto (F(\alpha), F(\beta))$.

Remark Let V be a closed symmetric monoidal category, and let $\mathcal{D}$ be a V-enriched category, with V-mapping spaces denoted by $\underline{\mathcal{D}}(d, d')$ for objects $d, d' \in \mathcal{D}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be $\mathcal{C}$-diagrams in $\mathcal{D}$. We can form an $\mathrm{tw}(\mathcal{C})$-diagram in V, denoted $\mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(F, G)$, as follows: $(c_0 \xrightarrow{f} c_1) \mapsto \underline{\mathcal{D}}(F(c_0), G(c_1))$ and for the morphism $(c_0 \xrightarrow{f} c_1) \xrightarrow{(\alpha, \beta)} (d_0 \xrightarrow{g} d_1)$ in $\mathrm{tw}(\mathcal{C})$, we get the morphism in V:

$$\mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(F, G)(c_0 \xrightarrow{f} c_1) = \underline{\mathcal{D}}(F(c_0), G(c_1)) \to \underline{\mathcal{D}}(F(d_0), G(d_1)) = \mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(F, G)(d_0 \xrightarrow{g} d_1)$$

given by $h \mapsto G(\beta) \circ h \circ F(\alpha)$.

Example D.5.2 Let $V = \mathcal{D} = \underline{s\mathbf{Sets}_*}$ and $X, Y : \mathcal{C} \to \underline{s\mathbf{Sets}_*}$. The above construction gives us a $\mathrm{tw}(\mathcal{C})$-diagram of pointed simplicial sets, denoted $\mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(X, Y)$. On the other hand, the category of $\mathcal{C}$-diagrams in $\underline{s\mathbf{Sets}_*}$ is $\underline{s\mathbf{Sets}_*}$-enriched, tensored and cotensored. Given $X, Y : \mathcal{C} \to \underline{s\mathbf{Sets}_*}$, the simplicial mapping space, denoted $\mathrm{Map}_{\mathcal{C}}(X, Y)$, is given in simplicial degree n by $(\mathrm{Map}_{\mathcal{C}}(X, Y))_n := s\mathbf{Sets}_*^{\mathcal{C}}(X \wedge \Delta_+^n, Y)$. These two mapping spaces are related:

Claim D.5.3 *There is an isomorphism* $\mathrm{Map}_{\mathcal{C}}(X, Y) \cong \lim_{\mathrm{tw}(\mathcal{C})} \mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(X, Y)$ *(as pointed simplicial sets).*

Proof. By definition:

$$\lim_{\mathrm{tw}(\mathcal{C})} \mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(X, Y) = \mathrm{eq}\Big(\prod_{(c_0 \xrightarrow{f} c_1) \in \mathrm{tw}(\mathcal{C})} \mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(X, Y)(c_0 \xrightarrow{f} c_1)$$

$$\underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} \prod_{(\alpha, \beta) \in \mathrm{tw}(\mathcal{C})\left(c_0 \xrightarrow{f} c_1, d_0 \xrightarrow{g} d_1\right)} \mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(d_0 \xrightarrow{g} d_1) \Big)$$

$$= \mathrm{eq}\Big(\prod_{c_0 \xrightarrow{f} c_1} s\mathbf{Sets}_*(X(c_0), Y(c_1)) \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} \prod_{g = \beta \circ f \circ \alpha} s\mathbf{Sets}_*(X(d_0), Y(d_1)) \Big)$$

where $\pi_{(\alpha, \beta)} \circ \delta^0 = \pi_{d_0 \xrightarrow{g} d_1}$ and $\pi_{(\alpha, \beta)} \circ \delta^1 = \mathrm{Hom}_{\mathrm{tw}(\mathcal{C})}(\alpha, \beta) \circ \pi_{c_0 \xrightarrow{f} c_1}$. The map $\phi : \mathrm{Map}_{\mathcal{C}}(X, Y) \to s\mathbf{Sets}_*(X(c_0), Y(c_1))$ is given, in simplicial degree n by $\eta \in s\mathbf{Sets}_*^{\mathcal{C}}(X \wedge \Delta_+^n, Y) \mapsto Y(f) \circ \eta_{c_0}$. Assembling the maps for all such $(c_0 \xrightarrow{f} c_1)$ gives a simplicial map that clearly factors through the equalizer, obtaining the desired isomorphism. $\square$

APPENDIX E

FUNCTORS WITH SMASH PRODUCT

E.1 Enrichment and Morphisms

Notation Recall from A that $s\mathbf{Sets}_*$ forms a complete and cocomplete closed symmetric monoidal category under the smash product of pointed simplicial sets. A $s\mathbf{Sets}_*$-enriched category $\mathcal{C}$ will be called pointed simplicial. Let $\mathcal{C}, \mathcal{D}$ be two pointed simplicial categories. A $s\mathbf{Sets}_*$-enriched functor $F : \mathcal{C} \to \mathcal{D}$ will be called pointed simplicial. See [14] Appendix 10 or [32] Chapters 1 and 2 for details on enriched category theory.

Let $F : \mathcal{C} \to \mathcal{D}$ be a pointed simplicial functor between pointed simplicial categories. Assume further that both $\mathcal{C}, \mathcal{D}$ are tensored over $s\mathbf{Sets}_*$. For $X \in s\mathbf{Sets}_*, C \in \mathcal{C}$ we get assembly maps $\lambda_{X,C} : X \otimes F(C) \to F(X \otimes C)$ defined via

$$
\begin{array}{ccc}
S^0 \xrightarrow{\mathrm{id}_{X \otimes C}} \underline{\mathcal{C}}(X \otimes C, X \otimes C) & \xrightarrow{\cong} & s\mathbf{Sets}_*(X, \underline{\mathcal{C}}(C, X \otimes C)) \\
& & \downarrow{\scriptstyle F \circ (-)} \\
\underline{\mathcal{D}}(X \otimes F(C), F(X \otimes Y)) & \xleftarrow{\cong} & s\mathbf{Sets}_*(X, \underline{\mathcal{D}}(F(C), F(X \otimes C)))
\end{array}
$$

in the underlying category of $\mathcal{D}$, $U_0\mathcal{D}$. By the naturality of the isomorphisms defining the closed monoidal structure on $s\mathbf{Sets}_*$ and from the fact that F is enriched we get that $F(\ell_C) \circ \lambda_{S^0,C} \circ \ell_{F(C)}^{-1} = \mathrm{id}_{F(C)}$, where ℓ is the left unitor of the monoidal category $s\mathbf{Sets}_*$. In a tensored category we have for $X, Y \in s\mathbf{Sets}_*, C \in \mathcal{C}$ natural isomorphisms $\tilde{\alpha}_{X,Y,C} : (X \wedge Y) \otimes C \cong X \otimes (Y \otimes C)$. In particular, with $(X \wedge Y) \otimes F(C) \cong X \otimes (Y \otimes F(C))$ in $\mathcal{D}$, we get that $F(\tilde{\alpha}_{X,Y,C}^{-1}) \circ \lambda_{X,Y \otimes C} \circ (\mathrm{id}_X \otimes \lambda_{Y,C}) \circ \alpha_{X,Y,F(C)} = \lambda_{X \wedge Y,C}$. Furthermore, in the special case that $\mathcal{C} = \mathcal{D} = s\mathbf{Sets}_*$, if $\tau_{X,Y} : X \wedge Y \xrightarrow{\cong} Y \wedge X$ is the switch isomorphism defining the *symmetric* monoidal structure, we have that $F(\tau_{Z,X \wedge Y}) \circ \lambda_{Z,X \wedge Y} \circ \tau_{F(X \wedge Y),Z} \circ (\lambda_{X,Y} \wedge \mathrm{id}_Z) = F(\alpha_{X,Y,Z}^{-1}) \circ \lambda_{X,Y \wedge Z} \circ (\mathrm{id}_X \wedge (F(\tau_{Z,Y}) \circ \lambda_{Z,Y} \circ \tau_{F(Y),Z})) \circ \alpha_{X,F(Y),Z}$. In this last equality it is crucial that we use the fact that a symmetric monoidal category is closed if and only if it is biclosed, with the right adjoint to $X \wedge -$ given by the internal hom $[X, -]$.

Notation Let $\underline{s\mathbf{Sets}_*}^{\text{fin}}$ be the full subcategory of $\underline{s\mathbf{Sets}_*}$ which consists of finite pointed simplicial sets (pointed simplicial sets with finitely many non-degenerate simplices).

Fact The category $\underline{s\mathbf{Sets}_*}^{\text{fin}}$ is small, and inherits the structure of a (*non-closed*) symmetric monoidal $\underline{s\mathbf{Sets}_*}$-category (see [14] A.10.3).

Remark Even though $\underline{s\mathbf{Sets}_*}^{\text{fin}}$ is $\underline{s\mathbf{Sets}_*}$-enriched it is *not* tensored or cotensored. However, it admits "products with (finite) pointed simplicial sets" (see [17] 1.0.3), which is enough to give assembly maps $\lambda_{X,C} : X \otimes F(C) \to F(X \otimes C)$ for $X \in \underline{s\mathbf{Sets}_*}^{\text{fin}}, C \in \mathcal{C}$ given a pointed simplicial functor $F : \underline{s\mathbf{Sets}_*}^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$.

Definition E.1.1 A <u>functor with stabilization</u> (FST) is a pointed simplicial functor $F : s\mathbf{Sets}_*^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$ such that

- If X is n-connected, then $F(X)$ is also n-connected.

- If X is n-connected, the assembly map $\lambda_{S^1,X} : S^1 \wedge F(X) \to F(S^1 \wedge X)$ is $(2n - c)$-connected for some number c not dependent on X.

Remark In the literature, the conditions on the assembly maps mentioned previously, $F(\ell_C) \circ \lambda_{S^0,C} \circ \ell_{F(C)}^{-1} = \text{id}_{F(C)}$ and $F(\tilde{\alpha}_{X,Y,C}^{-1}) \circ \lambda_{X,Y \otimes C} \circ (\text{id}_X \otimes \lambda_{Y,C}) \circ \alpha_{X,Y,F(C)} = \lambda_{X \wedge Y,C}$, are included in the definition of an FST when the language of enriched category theory is not used.

Definition E.1.2 A <u>functor with smash product</u> (FSP) is a functor with stabilization which is lax monoidal. A <u>commutative</u> FSP is an FST which is symmetric lax monoidal.

Unpacking the definition, an FSP F has a natural transformation of functors $\underline{s\mathbf{Sets}_*}^{\text{fin}} \times s\mathbf{Sets}_*^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$, the "product"

$$\mu_{-,\star}^F = \mu_{-,\star} : F(-) \wedge F(\star) \Rightarrow F(- \wedge \star)$$

and a "unit" morphism $\mathbb{1}_{S^0} : S^0 \to F(S^0)$ which satisfy (up to an associator):

$$\mu_{X \wedge Y, Z} \circ \left(\mu_{X,Y} \wedge \text{id}_{F(Z)} \right) = \mu_{X, Y \wedge Z} \circ \left(\text{id}_{F(X)} \wedge \mu_{Y,Z} \right)$$

$$F(\ell_X) \circ \mu_{S^0, X} \circ (\mathbb{1}_{S^0} \wedge \text{id}_{F(X)}) \circ \ell_{F(X)}^{-1} = \text{id}_{F(X)}$$

$$F(r_X) \circ \mu_{X, S^0} \circ (\text{id}_{F(X)} \wedge \mathbb{1}_{S^0}) \circ r_{F(X)}^{-1} = \text{id}_{F(X)}$$

However, since F is an FST (in particular, pointed simplicial) we also get assembly maps. The assembly maps and the lax monoidal structure are not that far off; we see from the

second equation that $\lambda_{S^0,X} = \mu_{S^0,X} \circ (\mathbb{1}_{S^0} \wedge \mathrm{id}_{F(X)})$. For any X we can define a morphism $\mathbb{1}_X^F : X \to F(X)$ by $F(r_X) \circ \lambda_{X,S^0} \circ (\mathrm{id}_X \wedge \mathbb{1}_{S^0}) \circ r_X{}^{-1}$. Since all the morphisms are natural, we thus obtain a natural transformation of functors $\underline{s\mathbf{Sets}_*}^{\mathrm{fin}} \to \underline{s\mathbf{Sets}_*}$

$$\mathbb{1}_-^F = \mathbb{1}_- : Id_{\underline{s\mathbf{Sets}_*}}(-) \Rightarrow F(-)$$

Combining the last two equations then gives $\mu_{X,Y} \circ (\mathbb{1}_X \wedge \mathrm{id}_{F(Y)}) = \lambda_{X,Y}$ (in particular, $\lambda_{S^1,X}$ is given by the "product" with the unit map $\mathbb{1}_{S^1}^F : S^1 \to F(S^1)$). Furthermore, the relationship between assembly maps and the "twist" isomorphism from the previous subsection gets translated to (up to associators) $F(\tau_{Y,X}) \circ \mu_{Y,X} \circ (\mathrm{id}_{F(Y)} \wedge \mathbb{1}_X) \circ \tau_{X,F(Y)} = \lambda_{X,Y}$. Lastly, the associativity of μ and the simplicial structure of F give $\mu_{X,Y} \circ (\mathbb{1}_X \wedge \mathbb{1}_Y) = \mathbb{1}_{X \wedge Y}$.

Remark The conditions mentioned in these previous paragraphs regarding the two natural transformations $\mathbb{1}_X^F = \mathbb{1}_X : X \to F(X)$ and $\mu_{X,Y}^F = \mu_{X,Y} : F(X) \wedge F(Y) \to F(X \wedge Y)$ for the FST F are included in the definition of an FSP when the language of enriched category theory is not used.

Observation Under this dictionary *commutative* FSP's satisfy:

$$F(\tau_{X,Y}) \circ \mu_{X,Y} = \mu_{Y,X} \circ \tau_{F(X),F(Y)}$$
$$\tau_{F(X),F(Y)} \circ (\mathbb{1}_X \wedge \mathrm{id}_{F(Y)}) = (\mathrm{id}_{F(Y)} \wedge \mathbb{1}_X) \circ \tau_{X,F(Y)}$$

and these again are included in older definitions in the literature.

Notation Let $\mathcal{C}, \mathcal{D}$ be two pointed simplicial categories. The functor category between them is $Fun(\mathcal{C}, \mathcal{D})$, whereas the $\underline{s\mathbf{Sets}_*}$-enriched functor category is denoted $\underline{Fun}(\mathcal{C}, \mathcal{D})$, with objects pointed simplicial functors $\mathcal{C} \to \mathcal{D}$ and hom $\underline{s\mathbf{Sets}_*}$-objects given by the $\underline{s\mathbf{Sets}_*}$-enriched ends:

$$\underline{Fun}(\mathcal{C}, \mathcal{D})(F, G) := \int_{c \in \mathcal{C}} \underline{\mathcal{D}}(F(c), G(c))$$

over the functor $\underline{\mathcal{D}}(F(-), G(-)) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \underline{s\mathbf{Sets}_*}$.

In the case that $\mathcal{C}$ is small $\underline{s\mathbf{Sets}_*}$-enriched monoidal and $\mathcal{D} = \underline{s\mathbf{Sets}_*}$ (or more generally, the base closed symmetric monoidal category we are enriched over), the enriched functor category $\underline{Fun}(\mathcal{C}, \mathcal{D})$ becomes a closed monoidal category in its own right (see Theorem 3.3 [12]), under the monoidal product given by Day convolution

$$\underline{Fun}(\mathcal{C}, \mathcal{D}) \times \underline{Fun}(\mathcal{C}, \mathcal{D}) \to \underline{Fun}(\mathcal{C}, \mathcal{D})$$

$$(F, G) \mapsto \left\{ F \otimes_{\text{Day}} G : \ c \mapsto \int^{(c_1, c_2) \in \mathcal{C} \times \mathcal{C}} \underline{\mathcal{C}}(c_1 \otimes_{\mathcal{C}} c_2, c) \wedge F(c_1) \wedge G(c_2) \right\}$$

that is, the left Kan extension of the external product $\bar{\otimes} : \underline{Fun}(\mathcal{C}, \mathcal{D}) \times \underline{Fun}(\mathcal{C}, \mathcal{D}) \to$ $\underline{Fun}(\mathcal{C}, \mathcal{D})$ through the monoidal functor on $\mathcal{C}$, $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. In particular, from the definition of left Kan extension, we know precisely how to map out of a Day product. Indeed, we have a $\underline{s\mathbf{Sets}_*}$-natural isomorphism:

$$\underline{Fun}(\mathcal{C}, \mathcal{D})(F \otimes_{\text{Day}} G, H) \cong \underline{Fun}(\mathcal{C} \times \mathcal{C}, \mathcal{D})(F\bar{\otimes}G, H \circ \otimes_{\mathcal{C}})$$

Remark With the Day monoidal structure, lax monoidal functors $\underline{s\mathbf{Sets}_*}^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$ are identified with monoids in $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\text{fin}}, \underline{s\mathbf{Sets}_*})$. Similarly, *symmetric* lax monoidal functors are identified with *commutative* monoids in $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\text{fin}}, \underline{s\mathbf{Sets}_*})$. That is, thanks to the Day convolution, we can identify FSTs and FSPs as the monoids and commutative monoids, respectively, in $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\text{fin}}, \underline{s\mathbf{Sets}_*})$ satisfying stabilization.

Definition E.1.3 The <u>category of FSPs</u>, denoted $\mathcal{FSP}$, is the (pointed simplicial) subcategory of $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\text{fin}}, \underline{s\mathbf{Sets}_*})$ of monoids satisfying stabilization and monoid maps. The <u>category of commutative FSPs</u>, $\mathcal{FSP}^{\text{com}}$, is the full subcategory of $\mathcal{FSP}$ consisting of commutative monoids satisfying stabilization.

Observation Concretely, a morphism between FSPs, $\eta : F \to G$, is an $\underline{s\mathbf{Sets}_*}$-enriched natural transformation of functors $\underline{s\mathbf{Sets}_*}^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$ strictly compatible with μ and $\mathbb{1}$. That is, $\eta_X \circ \mathbb{1}_X^F = \mathbb{1}_X^G$ and $\eta_{X \wedge Y} \circ \mu_{X,Y}^F = \mu_{X,Y}^G \circ (\eta_X \wedge \eta_Y)$. Also, since the forgetful functor from the simplicially-enriched category $\underline{s\mathbf{Sets}_*}$ down to its underlying category $s\mathbf{Sets}_*$ (taking the 0-simplices of the hom-sets) is faithful, the notions of $\underline{s\mathbf{Sets}_*}$-enriched natural transformation and (ordinary) natural transformation coincide.

E.2   Connection to Symmetric Spectra

Notation We take $S^m := S^1 \wedge \cdots \wedge S^1$ (m-times) as our model of the simplicial sphere.

Definition E.2.1 The spectrum associated to an FSP F, denoted $\underline{F}$, is the sequence of spaces $\{F(S^m)\}_{m \in \mathbb{N}}$, with structure maps $\lambda_{S^1, S^m} : S^1 \wedge F(S^m) \to F(S^{m+1})$. From the stability conditions, this is a connective spectrum.

Remark Let Σ_m be the symmetric group on m-letters, and $X \in \mathcal{S}_*$. The iterated smash $X^{\wedge m}$ has a natural Σ_m-action by setting, for $\sigma \in \Sigma_m$, $\sigma_X : X^{\wedge m} \to X^{\wedge m}$ to be coordinate permu-

tation, that is, $\sigma_X(x_1 \wedge \cdots \wedge x_m) := x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(m)}$. Then $F(X^{\wedge m})$ acquires a Σ_m-action by means of functorality. In particular, $F(S^m)$ has a Σ_m-action. By playing around with (enriched) adjunctions, it is easy to see that $F(\mathrm{id}_{S^1} \wedge \sigma_{S^m}) \circ \lambda_{S^1,S^m} = \lambda_{S^1,S^m} \circ (\mathrm{id}_{S^1} \wedge F(\sigma_{S^m}))$. In other words, $\lambda_{S^1 \wedge S^m}$ is $\Sigma_1 \times \Sigma_m$-equivariant. In fact, a similar argument shows that, for $\theta \in \Sigma_p$, we have $F(\theta_{S^p} \wedge \sigma_{S^m}) \circ \lambda_{S^p,S^m} = \lambda_{S^p,S^m} \circ (\theta_{S^p} \wedge F(\sigma_{S^m}))$; that is λ_{S^p,S^m} is $\Sigma_p \times \Sigma_m$-equivariant.

Note that by uniqueness of adjunctions, the iterated map:

$$\underbrace{S^1 \wedge \cdots \wedge S^1}_{p-\text{times}} \wedge F(S^m) \xrightarrow{\mathrm{id}_{S^{p-1}} \wedge \lambda_{S^1,S^m}} \underbrace{S^1 \wedge \cdots \wedge S^1}_{p-1-\text{times}} \wedge F(S^{m+1}) \to \cdots$$

$$\cdots \to S^1 \wedge F(S^{m+p-1}) \xrightarrow{\lambda_{S^1,S^{m+p-1}}} F(S^{m+p})$$

is simply the assembly map λ_{S^p,S^m}. By the previous remark, the iterated map is $\Sigma_p \times \Sigma_m$-equivariant. In sum, given an FSP F, the associated spectrum $\underline{F}$ is a *symmetric* spectrum in the sense of [30], and morphisms of FSP's give morphisms of symmetric spectra (the naturality of η guarantees the equivariance of the level-wise maps). So far, only the pointed simplicial structure of the FSP's and the naturality of their morphisms has been used; no μ or $\mathbb{1}$.

Observation We actually have more structure. For each $m \in \mathbb{N}$, we have a map $\mathbb{1}_{S^m} : S^m \to F(S^m)$ that is, by definition, Σ_m-equivariant. We also have for each (p,q) a map $\mu_{S^p,S^q} : F(S^p) \wedge F(S^q) \to F(S^{p+q})$ which is, again by naturality, $\Sigma_p \times \Sigma_q$-equivariant. The remaining conditions for F to be an FSP make $\underline{F}$ into a symmetric ring spectrum. For example, the last condition taken with $X = S^m, Y = S^n$ gives the (in)famous shuffle/centraility condition:

$$
\begin{array}{ccc}
S^m \wedge F(S^n) & \xrightarrow{\tau_{S^m,F(S^n)}} & F(S^n) \wedge S^m \\
\downarrow{\scriptstyle \mathbb{1}_{S^m} \wedge \mathrm{id}_{F(S^n)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(S^n)} \wedge \mathbb{1}_m} \\
F(S^m) \wedge F(S^n) & & F(S^n) \wedge F(S^m) \\
\downarrow{\scriptstyle \mu_{S^m,S^n}} & & \downarrow{\scriptstyle \mu_{S^n,S^m}} \\
F(S^m \wedge S^n) & \xrightarrow{F(\tau_{S^m,S^n})} & F(S^n \wedge S^m)
\end{array}
$$

were $F(\tau_{S^m,S^n})$ is playing the role of $\chi_{m,n}$, the (m,n)-shuffle that moves the first m elements past the last n elements keeping the two blocks in order. Also, the conditions $\eta_X \circ \mathbb{1}_X^F = \mathbb{1}_X^G$ and $\eta_{X \wedge Y} \circ \mu_{X,Y}^F = \mu_{X,Y}^G \circ (\eta_X \wedge \eta_Y)$ guarantee that $\underline{\eta} : \underline{F} \to \underline{G}$ is a morphism of symmetric

ring spectra. If we are working with a commutative FSP, then $\underline{F}$ will be a commutative symmetric ring spectrum.

The (naive) homotopy groups of an FSP F are defined to be the (naive) homotopy groups of the associated symmetric spectrum:

$$\pi_i(F) := \pi_i(\underline{F}) = \operatorname{colim}_m \pi_{i+m} F(S^m)$$

Definition E.2.2 A morphism $\eta : F \to G$ of FSPs is a <u>stable weak equivalence</u> if it induces an isomorphism of groups $\pi_i(F) \to \pi_i(G)$ for all i.

Since $\underline{F}$ is connective, our FSP F has no negative homotopy groups. And since the associated spectrum $\underline{F}$ is a symmetric ring spectrum, the multiplication $\mu_{\underline{F}} : \underline{F} \wedge \underline{F} \to \underline{F}$ makes $\pi_*(F)$ into a graded ring.

Fact Let F, G be two FSPs. There is a canonical weak equivalence of symmetric spectra $\underline{F} \otimes_{\text{Day}} G \xrightarrow{\simeq} \underline{F} \wedge \underline{G}$.

E.2.1   Examples

Example E.2.3 "The universal FSP" is the sphere spectrum. It is attained by $\mathbb{S}(X) = X$, with identity maps for μ and $\mathbb{1}$ (the connectivity conditions are tautological). It is initial among all FSP's. Indeed, $\mathbb{S}$ is the inclusion functor from $\underline{s\mathbf{Sets}_*}^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$, and given any FSP G, the natural transformation $\mathbb{1}^G$ is really a morphism of FSP's from $\mathbb{S} \to G$. Note that the (commutative symmetric ring) spectrum associated to $\mathbb{S}$ is simply the sphere spectrum, $\underline{\mathbb{S}} = \mathbb{S}$. Also, as a silly example, there is the analogue of the zero ring, where $0(X) = *$, and it is final, with associated spectrum $*$.

Example E.2.4 "The group ring" FSP associated to a simplicial group G (or monoid for that matter), denoted $\mathbb{G}$. is defined by $\mathbb{G}(X) := G_+ \wedge X$. Here $(-)_+$ means adding a disjoint basepoint to G. It is clearly pointed, and it is simplicial because of the simplicial nature of the smash product. To see the FSP structure we need to define the multiplication and unit maps. The unit map, $\mathbb{1}_X : X \to G_+ \wedge X$ is given by $x \mapsto 1 \wedge x$, while the product map is

given by:

$$\mathbb{G}(X) \wedge \mathbb{G}(Y) := (G_+ \wedge X) \wedge (G_+ \wedge Y)$$

$$\xrightarrow{\operatorname{id}_{G_+} \wedge \tau_{X,G_+} \wedge \operatorname{id}_Y} (G_+ \wedge G_+) \wedge (X \wedge Y)$$

$$= (G \times G)_+ \wedge (X \wedge Y)$$

$$\to (G)_+ \wedge (X \wedge Y) = \mathbb{G}(X \wedge Y)$$

Here we've used the fact that for any two spaces, $A_+ \wedge B_+ \cong (A \times B)_+$, and the (simplicial) multiplication map of the group $G \times G \to G$. The defining properties of a FSP are easy to check (notice, for example, that the assembly map here $\lambda_{X,Y} : X \wedge \mathbb{G}(Y) \to \mathbb{G}(X \wedge Y)$ is simply the "switch" map $x \wedge (g \wedge y) \mapsto g \wedge (x \wedge y)$). Also, the connectivity conditions follow from the higher excision estimates on homotopy groups, and that, as in the previous example, product with $\mathbb{1}^{\mathbb{G}}_{S^1}$ is an isomorphism. Note also that if G is commutative, then $\mathbb{G}$ is a commutative FSP.

Clearly, the associated spectrum is isomorphic to the suspension spectrum of the space G_+, $\underline{\mathbb{G}} \cong \underline{\Sigma^\infty_+ G}$. Because of the map of FSP's $\mathbb{S} \to \mathbb{G}$ (and consequently, the map of symmetric ring spectra $\underline{\mathbb{S}} = \underline{\mathbb{S}} \to \underline{\mathbb{G}} = \underline{\Sigma^\infty_+ G}$), we think of this as the "group ring" of G over the sphere spectrum.

Going further, if we are given a (simplicial) group homomorphism $\phi : G \to H$, we get a natural transformation $\mathbb{G} \xrightarrow{\tilde{\phi}} \mathbb{H}$ by $G_+ \wedge X \xrightarrow{\phi_+ \wedge \operatorname{id}_X} H_+ \wedge X$. Inspection shows that this is in fact a map of FSPs.

Example E.2.5 "Ordinary rings" have FSP's associated to them. If R is a ring (associative and unital), then define $\tilde{R}(X) := R[X]/R[*]$ to be the free (reduced) simplicial R-module with basis the simplicial set X. That is, in simplicial degree j we have the free R-module $R[X_j]$ with $R[*]$ modded out. This functor is clearly pointed and simplicial because of the freeness of $R[-]$. Let's describe its structure as an FSP. We have a unit map $\mathbb{1}_X : X \to \tilde{R}(X)$ given by $x \mapsto 1 \cdot x$, while the product map is given by:

$$\tilde{R}(X) \wedge \tilde{R}(Y) := (R[X]/R[*]) \wedge (R[Y]/R[*])$$

$$\to (R[X]/R[*]) \otimes (R[Y]/R[*])$$

$$\cong (R \otimes R)[X \wedge Y]/(R \otimes R)[*]$$

$$\to R[X \wedge Y]/R[*] = \tilde{R}(X \wedge Y)$$

Here we've used the fact that $\tilde{R}(X) \otimes \tilde{R}(Y) \cong \widetilde{R \otimes R}(X \wedge Y)$, and the multiplication map for the ring $R \otimes R \to R$. Just as with group rings, the defining properties of a FSP are easy to check (e.g. that the assembly map here $\lambda_{X,Y} : X \wedge \tilde{R}(Y) \to \tilde{R}(X \wedge Y)$ is simply the "switch" map $x \wedge (r \cdot y) \mapsto r \cdot (x \wedge y)$). The connectivity conditions come from the Dold-Kan theorem. Indeed, the homotopy groups of $\tilde{R}(X)$ compute the reduced homology of the space X with coefficients in R, and so if X is n-connected, tautologically so is $\tilde{R}(X)$. Note also that if R is commutative, then $\tilde{R}$ is a commutative FSP. Lastly, the associated spectrum, $\underline{\tilde{R}}$ is the Eilenberg-Maclane (commutative symmetric ring) spectrum $H(R)$ for the (abelian) group R.

Remark The previous construction can be carried out with simplicial rings as well. Suppose $R_* = \{R_j\}_{j \in \mathbb{N}}$ is a simplicial ring, so that we have ring homomorphisms $d_i^{R,j} = d_i^R = d_i : R_j \to R_{j-1}$, $0 \leq i \leq j$ ("face maps"), and $s_i^{R,j} = s_i^R = s_i : R_j \to R_{j+1}$, $0 \leq i \leq j$ ("degeneracy maps"), satisfying the simplicial identities. Similarly, let $X = \{X_j\}_{j \in \mathbb{N}}$ be a (pointed finite) simplicial set, with face and degeneracy maps $d_i^{X,j}$ and $s_i^{X,j}$, respectively. Then we can also make sense of $\tilde{R}_*(X)$. Indeed, there is a first quadrant diagram of abelian groups:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & \\
\dots \quad R_{n+1}[X_{m-1}] & & R_{n+1}[X_m] & & R_{n+1}[X_{m+1}] & \dots \\
& & & & & \\
\dots \quad R_n[X_{m-1}] & \rightleftarrows & R_n[X_m] & \rightleftarrows & R_n[X_{m+1}] & \dots \\
& & & & & \\
\dots \quad R_{n-1}[X_{m-1}] & & R_{n-1}[X_m] & & R_{n-1}[X_{m+1}] & \dots \\
& & & & & \\
& & \vdots & & \vdots & \\
\end{array}
$$

where in row n we have $R_n[X_*]$. Note that the horizontal maps are homomorphisms of R_n-modules, and are determined by the simplicial maps of X_*, e.g. face maps are given by $r_n \cdot x_m \mapsto r_n \cdot d_i^{X,m}(x_m)$. On the other hand, the vertical maps are determined by the simplicial ring structure of R_*, e.g. face maps are given by $r_n \cdot x_m \mapsto d_i^{R,n}(r_n) \cdot x_m$. Since this is, in particular, a bisimplicial abelian group, we can take its realization component-wise or (equivalently) diagonally, and the same is true of the reduced bicomplex. So, we define $\tilde{R}_*(X) := \{R_j[X_j]/R_j[*]\}_{j \in \mathbb{N}}$. There are no issues with the unit map, and note that the

product map at simplicial degree j simply involves the multiplication map $R_j \otimes R_j \to R_j$ only.

In terms of functorality, if $\phi : R \to S$ is a homomorphism of rings, then we get an induced map $\tilde{R}(X) \to \tilde{S}(X)$ which assembles into a morphism of FSPs $\tilde{R} \xrightarrow{\tilde{\phi}} \tilde{S}$.

Example E.2.6 We can construct "products" of FSP's. Indeed, let F and G be FSP's. Define $(F \times G)(X) := F(X) \times G(X)$. This (pointed and simplicial) functor trivially satisfies the connectivity estimates, and has an FSP structure as follows: $\mathbb{1}_X^{F \times G} : X \xrightarrow{\mathbb{1}_X^F \times \mathbb{1}_X^G} F(X) \times G(X)$ and product map:

$$
\begin{aligned}
\mu_{X,Y}^{F \times G} : (F \times G)(X) \wedge (F \times G)(Y) &:= (F(X) \times G(X)) \wedge (F(Y) \times G(Y)) \\
&\cong (F(X) \wedge F(Y)) \times (G(X) \wedge G(Y)) \\
\xrightarrow{\mu_{X,Y}^F \times \mu_{X,Y}^G} F(X \wedge Y) &\times G(X \wedge Y) = (F \times G)(X \wedge Y)
\end{aligned}
$$

One can readily check the properties of an FSP (following directly from those of F and G), and that the projection maps $\pi_F : F \times G \to F$ and $\pi_G : F \times G \to G$ are morphisms of FSP's (so $F \times G$ is the categorical product).

Example E.2.7 We can construct "Matrix FSP's", that is, the "$r \times r$ matrix ring" of an FSP F, denoted $M_r(F)$. For $r \in \mathbb{N}_{>0}$, let $\underline{r}$ be the finite set $\{1, \ldots, r\}$. Then $\underline{r}$ can be considered as a constant simplicial set. For $X \in \underline{s\mathbf{Sets}_*}^{\text{fin}}$, define

$$
M_r(F)(X) := \underline{s\mathbf{Sets}_*}(\underline{r}_+, \underline{r}_+ \wedge F(X))
$$

That is, the simplicial set of simplicial maps $\underline{r}_+ \to \underline{r}_+ \wedge F(X)$. This is clearly a functor by post-composition. Since F is pointed and simplicial, so is $M_r(F)$. The structure of an FSP is given as follows: the unit map, $\mathbb{1}_X^{M_r(F)}$ is given as the adjoint of $\text{id}_{\underline{r}_+} \wedge \mathbb{1}_X^F : \underline{r}_+ \wedge X \to \underline{r}_+ \wedge F(X)$. Combining the naturality of $\mathbb{1}_X^F$ with the adjunction shows $\mathbb{1}_-^{M_r(F)}$ is natural. For the product, $\mu_{X,Y}^{M_r(F)} : M_r(F)(X) \wedge M_r(F)(Y) \to M_r(F)(X \wedge Y)$, we use $\mu_{X,Y}^F$ as follows: $(A, B) \mapsto C$ where C is the composite

$$
\underline{r}_+ \xrightarrow{B} \underline{r}_+ \wedge F(Y) \xrightarrow{A \wedge \text{id}_{F(Y)}} (\underline{r}_+ \wedge F(X)) \wedge F(Y) \cong \underline{r}_+ \wedge (F(X) \wedge F(Y)) \xrightarrow{\text{id}_+ \wedge \mu_{X,Y}^F} \underline{r}_+ \wedge F(X \wedge Y)
$$

Here we've indicated the map at 0-simplices, with higher simplices induced from the diagonals on $\Delta^n \to \Delta^n \times \Delta^n$, and its naturality comes directly from that of F. Verification that this satisfies the identities of the FSP is a (very) tedious calculation, and it reduces solely to the

fact that F was an FSP. We think of this as a matrix as follows: Let $A \in M_r(F)(X)$ and for $1 \leq i, j \leq r$ set $A_{i,j}$ by writing $A(j) = i \wedge A_{i,j}$, and setting all other $A_{i,j} = *$. So A is an $r \times r$ collection of points in $F(X)$, with at most one non-zero entry in each column. The multiplication law can then be written

$$\mu_{X,Y}^{M_r(F)}(A \wedge B) = C, \quad C_{i,j} = \sum \mu_{X,Y}^F(A_{i,k} \wedge B_{k,j})$$

where the sum is well-defined since only one element of $B_{k,j}$, $k = 1, \ldots, r$ is $\neq *$ (the jth-column of B). Therefore the multiplication is analogous to matrix multiplication. Additionally, from the adjoint formula we have that $\mathbb{1}_X^{M_r(F)}$ is given by $x \mapsto \{i \mapsto i \wedge \mathbb{1}_X^F(x)\}$. Therefore, in the matrix re-interpretation, $\mathbb{1}_X^{M_r(F)}(x)$ is the $r \times r$ matrix with $\mathbb{1}_X^F(x)$ along the diagonal and $*$ elsewhere.

Observation Even if F is a commutative FSP, $M_r(F)$ need not be. For, if $A \wedge B \in M_r(F)(X) \wedge M_r(F)(Y)$, then the two maps:

$$\underline{r}_+ \xrightarrow{B} \underline{r}_+ \wedge F(Y) \xrightarrow{A \wedge \mathrm{id}_{F(Y)}} (\underline{r}_+ \wedge F(X)) \wedge F(Y) \cong \underline{r}_+ \wedge (F(Y) \wedge F(X))$$

$$\underline{r}_+ \xrightarrow{A} \underline{r}_+ \wedge F(X) \xrightarrow{B \wedge \mathrm{id}_{F(X)}} (\underline{r}_+ \wedge F(Y)) \wedge F(X) \cong \underline{r}_+ \wedge (F(Y) \wedge F(X))$$

need not agree.

Remark The matrix FSP $M_r(F)$ has the correct stable homotopy type; the product of r copies of the coproduct of r copies of F. If $\underline{F}$ is the spectrum associated to F, then the spectrum associated to $M_r(F)$ is $\underline{M_r(F)} = \underline{s\mathbf{Sets}_*}(\underline{r}_+, \underline{r}_+ \wedge \underline{F})$, where we are using the tensoring and cotensoring of the category of spectra by that of pointed simplicial sets. The latter is the spectrum $\underline{F}$ tensored by the (constant) pointed simplicial set $\underline{r}_+$ and cotensored against $\underline{r}_+$. Taking homotopy groups, we find that the ring of homotopy groups $\pi_*(M_r(F))$ is isomorphic to the matrix ring over the graded ring $\pi_*(F)$, that is: $\pi_*(M_r(F)) \cong M_r(\pi_*(F))$ as graded rings.

E.3   Modules over FSPs

Definition E.3.1 Let F be an FSP, thought of as a monoid in $\underline{Fun}(s\mathbf{Sets}_*^{\text{fin}}, s\mathbf{Sets}_*)$ satisfying stabilization. We say $M \in \underline{Fun}(s\mathbf{Sets}_*^{\text{fin}}, s\mathbf{Sets}_*)$ is a <u>left module over F</u>, if it satisfies stabilization and is a left module over the monoid.

Unpacking this definition in steps, first note that M is a pointed simplicial functor $\underline{s\mathbf{Sets}_*}^{\text{fin}} \to$ $\underline{s\mathbf{Sets}_*}$, and as such carries assembly maps, which in order to distinguish from those of F we will denote by $\lambda^M_{X,Y}$. In particular, these assembly maps satisfy all of the properties listed in E.1:

$$\text{id}_{M(X)} = M(\ell_X) \circ \lambda^M_{S^0,X} \circ \ell_{M(X)}^{-1}$$

$$\lambda^M_{X \wedge Y, Z} = M(\alpha^{-1}_{X,Y,Z}) \circ \lambda^M_{X,Y \wedge Z} \circ (\text{id}_X \wedge \lambda^M_{Y,Z}) \circ \alpha_{X,Y,M(Z)}$$

$$M(\tau_{Z,X \wedge Y}) \circ \lambda^M_{Z,X \wedge Y} \circ \tau_{M(X \wedge Y),Z} \circ (\lambda^M_{X,Y} \wedge \text{id}_Z) =$$

$$M(\alpha^{-1}_{X,Y,Z}) \circ \lambda^M_{X,Y \wedge Z} \circ (\text{id}_X \wedge (M(\tau_{Z,Y}) \circ \lambda^M_{Z,Y} \circ \tau_{M(Y),Z})) \circ \alpha_{X,M(Y),Z}$$

Second, since it is a left module over the monoid F in $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\text{fin}}, \underline{s\mathbf{Sets}_*})$, we have a morphism in $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\text{fin}}, \underline{s\mathbf{Sets}_*})$, $l : F \otimes_{\text{Day}} M \to M$ satisfying (up to an associator): $l \circ (Id_F \otimes_{\text{Day}} l) = l \circ (\mu^F \otimes_{\text{Day}} Id_M)$ and $l \circ (\mathbb{1}^F \otimes_{\text{Day}} Id_M) = \ell^{\text{Day}}_M$, where ℓ^{Day} is the left unitor for the Day convolution. Recall that the unit under Day convolution is the functor corepresenting the unit of the source monoidal category, in this case $\underline{s\mathbf{Sets}_*}$, so is $\underline{s\mathbf{Sets}_*}(S^0, -) \cong Id_{\underline{s\mathbf{Sets}_*}}(-)$. Using the universal property of mapping out of a Day convolution, we can write l as a natural transformation of functors $\underline{s\mathbf{Sets}_*}^{\text{fin}} \times \underline{s\mathbf{Sets}_*}^{\text{fin}} \to$ $\underline{s\mathbf{Sets}_*}$,

$$l_{-,\star} : F(-) \wedge M(\star) \Rightarrow M(- \wedge \star)$$

satisfying:

$$l_{X \wedge Y, Z} \circ (\mu^F_{X,Y} \wedge \text{id}_{M(Z)}) = M(\alpha_{X,Y,Z}^{-1}) \circ l_{X,Y \wedge Z} \circ (\text{id}_{F(X)} \wedge l_{Y,Z}) \circ \alpha_{F(X),F(Y),M(Z)}$$

$$l_{X,Y} \circ (\mathbb{1}^F_X \wedge \text{id}_{M(Y)}) = \lambda^M_{X,Y}$$

Remark Similarly, we define a right module over F to be an FST $N \in \underline{Fun}(\underline{s\mathbf{Sets}_*}^{\text{fin}}, \underline{s\mathbf{Sets}_*})$ which is a *right* module over F thought of as a monoid. Working through the definitions, we get a natural transformation of functors $\underline{s\mathbf{Sets}_*}^{\text{fin}} \times \underline{s\mathbf{Sets}_*}^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$,

$$\rho_{-,\star} : N(-) \wedge F(\star) \Rightarrow N(- \wedge \star)$$

satisfying:

$$\rho_{X \wedge Y, Z} \circ (\rho_{X,Y} \wedge \text{id}_{F(Z)}) = N(\alpha_{X,Y,Z}^{-1}) \circ \rho_{X,Y \wedge Z} \circ (\text{id}_{N(X)} \wedge \mu^F_{Y,Z}) \circ \alpha_{N(X),F(Y),F(Z)}$$

$$N(\tau_{Y,X}) \circ \rho_{Y,X} \circ (\mathrm{id}_{N(Y)} \wedge \mathbb{1}_X^F) \circ \tau_{X,Y} = \lambda_{X,Y}^N$$

Definition E.3.2 Let F be an FSP. We say $M \in \underline{Fun}(s\mathbf{Sets}_*^{\mathrm{fin}}, s\mathbf{Sets}_*)$ is a <u>bimodule over F</u> if it satisfies stabilization and is a bimodule under the monoid given by F.

M is therefore both a left module and a right module as defined previously. The bimodule compatibility (up to an associator) $\rho \circ (l \otimes_{\mathrm{Day}} Id_F) = l \circ (Id_F \otimes_{\mathrm{Day}} \rho)$ means that:

$$\rho_{X \wedge Y, Z} \circ (l_{X,Y} \wedge \mathrm{id}_{F(Z)}) = M(\alpha_{X,Y,Z}{}^{-1}) \circ l_{X, Y \wedge Z} \circ (\mathrm{id}_{F(X)} \wedge \rho_{Y,Z}) \circ \alpha_{F(X), M(Y), F(Z)}$$

Example E.3.3 Let $\mathbb{S}$ be the sphere spectrum FSP, where $\mathbb{S}(X) = X$ with μ and $\mathbb{1}$ identity natural transformations. Then as mentioned above $\mathbb{S}$ is a unit for the Day convolution monoidal structure on $\underline{Fun}(s\mathbf{Sets}_*^{\mathrm{fin}}, s\mathbf{Sets}_*)$. Recall that in any monoidal category *any* object is a bimodule over the unit monoid, by means of the left and right unitors. In particular, any functor $G \in \underline{Fun}(s\mathbf{Sets}_*^{\mathrm{fin}}, s\mathbf{Sets}_*)$ (satisfying stabilization) is a bimodule over $\mathbb{S}$. In other words, all FSTs are $\mathbb{S}$-bimodules. Also, there is a strong symmetric monoidal functor $(s\mathbf{Sets}_*, \wedge, S^0) \to (\mathbb{S} - \mathrm{BiMod}, \otimes_{\mathrm{Day}}, \mathbb{S})$ sending Y to the functor $X \mapsto X \wedge Y$. So we may associate to each pointed simplicial set Y an FST $\mathbb{Y}$. In terms of associated spectra, we have a natural isomorphism between $\underline{\mathbb{Y}}$ and the suspension spectrum of Y.

Example E.3.4 Let $\eta : F \to G$ be a morphism of FSPs. Then $l_{X,Y} := \mu_{X,Y}^G \circ (\eta_X \wedge \mathrm{id}_{G(Y)})$ gives G the structure of a left F-module, while $\rho_{X,Y} := \mu_{X,Y}^G \circ (\mathrm{id}_{G(X)} \wedge \eta_Y)$ makes G into a right F-module. Furthermore, the left and right actions are compatible (directly by the definition of η as a map of FSPs and the multiplication associativity on G). In short, given any morphism of FSPs $\eta : F \to G$, G naturally is an F-bimodule.

Example E.3.5 Let R be a ring, and M a left R-module. Define $\tilde{M} : s\mathbf{Sets}_*^{\mathrm{fin}} \to s\mathbf{Sets}_*$ by $X \mapsto \tilde{M}(X) := M[X]/M[*]$, with the simplicial structure inherited from X. Then $\tilde{M}$ is clearly pointed simplicial. The connectivity conditions come from the Dold-Kan theorem. Indeed, the homotopy groups of $\tilde{M}(X)$ compute the reduced homology of the space X with coefficients in (the additive abelian group) M, and so if X is n-connected, tautologically so is $\tilde{M}(X)$. So $\tilde{M}$ is an FST. As such it is an $\mathbb{S}$-bimodule. More is true, though. Since M is a left R-module, we have a function $R \times M \to M$ defining the action. For $X, Y \in s\mathbf{Sets}_*$ we have a map $\tilde{R}(X) \wedge \tilde{M}(Y) \cong (\widetilde{R \times M})(X \wedge Y) \to \tilde{M}(X \wedge Y)$. A quick check shows that $\tilde{M}$ acquires the structure of a left $\tilde{R}$-module. If M had been a right R-module, we similarly find that $\tilde{M}$ is a right $\tilde{R}$-module. Lastly, if M were an R-bimodule, then we get that $\tilde{M}$ is an $\tilde{R}$-bimodule.

Definition E.3.6 Let F be an FSP, M and N left F-modules. A <u>morphism of left F-mod-</u><u>ules</u>, $\phi : M \to N$, is a morphism of left modules over $F \in \underline{Fun}(s\mathbf{Sets_*}^{\mathrm{fin}}, s\mathbf{Sets_*})$ (thought of as a monoid).

Unwrapping this definition a bit, first and foremost ϕ needs to be an enriched natural transformation. Indeed, M, N are pointed simplicial functors $\underline{s\mathbf{Sets_*}}^{\mathrm{fin}} \to \underline{s\mathbf{Sets_*}}$, so a morphism between them is a $\underline{s\mathbf{Sets_*}}$-enriched natural transformation:

$$\phi_X : M(X) \to N(X)$$

for $X \in \underline{s\mathbf{Sets_*}}$. Second, it must be a morphism of left modules over the module structures defined on M and N. Using the universal property of the Day convolution, we have that for each $X, Y \in \underline{s\mathbf{Sets_*}}$ we have a commuting diagram:

$$
\begin{array}{ccc}
F(X) \wedge M(Y) & \xrightarrow{l^M_{X,Y}} & M(X \wedge Y) \\
\downarrow{\scriptstyle \mathrm{id}_{F(X)} \wedge \phi_Y} & & \downarrow{\scriptstyle \phi_{X \wedge Y}} \\
F(X) \wedge N(Y) & \xrightarrow[l^N_{X,Y}]{} & N(X \wedge Y)
\end{array}
$$

We can similarly defined notions of morphism of right modules and bimodules.

Definition E.3.7 Let F be an FSP, M and N right F-modules. A <u>morphism of right</u> <u>F-modules</u>, $\phi : M \to N$, is a morphism of right modules over $F \in \underline{Fun}(s\mathbf{Sets_*}^{\mathrm{fin}}, s\mathbf{Sets_*})$ (thought of as a monoid).

Again, we have a natural transformation ϕ, but now the compatibility condition is a commuting diagram:

$$
\begin{array}{ccc}
M(X) \wedge F(Y) & \xrightarrow{\rho^M_{X,Y}} & M(X \wedge Y) \\
\downarrow{\scriptstyle \phi_X \wedge \mathrm{id}_{F(Y)}} & & \downarrow{\scriptstyle \phi_{X \wedge Y}} \\
N(X) \wedge F(Y) & \xrightarrow[\rho^N_{X,Y}]{} & N(X \wedge Y)
\end{array}
$$

Definition E.3.8 Let F be an FSP, M and N F-bimodules. A <u>morphism of F-bimodules,</u> $\phi : M \to N$, is a morphism of bimodules over $F \in \underline{Fun}(s\mathbf{Sets_*}^{\mathrm{fin}}, s\mathbf{Sets_*})$ (thought of as a monoid).

That is, we require both previous squares to commute. We can of course define the categories of such objects.

Definition E.3.9 Let F be an FSP. The category of left F-modules, with objects left F-modules and morphisms the morphisms of left F-modules, will be denoted F-Mod. Similarly, the categories of right F-modules and F-bimodules will be denoted Mod-F and F-BiMod, respectively.

Notation In our definition of left/right modules over an FSP F we have already included the condition on stabilization. Therefore, F-Mod/Mod-F should not be confused with the category of left/right modules over F when viewed simply as a monoid in $\underline{Fun}(s\mathbf{Sets}_*{}^{\text{fin}}, s\mathbf{Sets}_*)$ under Day convolution. Rather, it is the full subcategory of those modules that are *also* FSTs. This distinction will not be mentioned moving forward.

Fact Since $\mathbb{S}$-BiMod is equivalently the category of all FSTs, it is closed symmetric monoidal by E.1. Its monoidal tensor is simply the Day convolution, and its internal hom is defined as follows: If S and T are FSTs we define an FST $Hom_{\mathbb{S}}(S, T) \in \mathbb{S}$-BiMod by sending

$$X \mapsto Hom_{\mathbb{S}}(S, T)(X) := \int_{Y \in \underline{s\mathbf{Sets}_*}{}^{\text{fin}}} \underline{s\mathbf{Sets}_*}(S(Y), T(X \wedge Y))$$

This is a pointed simplicial functor, and one readily checks the connectivity conditions. Indeed, working through the universal properties of mapping out of a coend and into an end, we have a natural isomorphism

$$\mathbb{S}-\text{BiMod}(F \otimes_{\text{Day}} G, H) \cong \mathbb{S}-\text{BiMod}(F, Hom_{\mathbb{S}}(G, H))$$

which extends to an enriched natural isomorphism:

$$Hom_{\mathbb{S}}(F \otimes_{\text{Day}} G, H) \cong Hom_{\mathbb{S}}(F, Hom_{\mathbb{S}}(G, H))$$

For details see 5.1, Proposition 5.2 and Corollary 5.3 in [38].

Observation The functor $(\mathbb{S} - \text{BiMod}, \otimes_{\text{Day}}, \mathbb{S}) \to (\underline{s\mathbf{Sets}_*}, \wedge, S^0)$ sending $T \mapsto T(S^0)$ is strong symmetric monoidal. The induced $\underline{s\mathbf{Sets}_*}$-enrichment on $\mathbb{S} - \text{BiMod}$ agrees with the one introduced in E.1. In addition to being $\underline{s\mathbf{Sets}_*}$-enriched, the category $\mathbb{S}$-BiMod is tensored and cotensored over $\underline{s\mathbf{Sets}_*}$. We work the details of the tensored case, leaving the cotensoring for the interested reader. For notational purposes we record here: the cotensoring of $T \in \mathbb{S}$-BiMod and $Y \in \underline{s\mathbf{Sets}_*}$. is denoted $\text{Map}(Y, T)$. As for the tensoring, define $T \otimes Y \in \underline{Fun}(s\mathbf{Sets}_*{}^{\text{fin}}, s\mathbf{Sets}_*)$ by $X \mapsto T(X) \wedge Y$. This is clearly a pointed simplicial functor. Furthermore, if X is n-connected, so is $T(X) \wedge Y$. Similarly, if $S^1 \wedge T(X) \to T(S^1 \wedge X)$

is $(2n - c)$-connected, then so is $S^1 \wedge T(X) \wedge Y \to T(S^1 \wedge X) \wedge Y$. So $T \otimes Y \in \mathbb{S}$-BiMod. Lastly, if T' is another $\mathbb{S}$-bimodule, we have natural isomorphisms:

$$
\begin{aligned}
\int_{W \in \underline{s\mathbf{Sets}_*}^{\mathrm{fin}}} \underline{s\mathbf{Sets}_*}((T \otimes Y)(W), T'(W)) &:= \int_{W \in \underline{s\mathbf{Sets}_*}^{\mathrm{fin}}} \underline{s\mathbf{Sets}_*}(T(W) \wedge Y, T'(W)) \\
&\cong \int_{W \in \underline{s\mathbf{Sets}_*}^{\mathrm{fin}}} \underline{s\mathbf{Sets}_*}(Y, \underline{s\mathbf{Sets}_*}(T(W), T'(W))) \\
&\cong \underline{s\mathbf{Sets}_*}\left(Y, \int_{W \in \underline{s\mathbf{Sets}_*}^{\mathrm{fin}}} \underline{s\mathbf{Sets}_*}(T(W), T'(W))\right)
\end{aligned}
$$

In fact, the tensoring of $\mathbb{S} - \mathrm{BiMod}$ over $\underline{s\mathbf{Sets}_*}$ is the one induced from the strong symmetric monoidal functor in E.3.3, as given $T \in \mathbb{S}$-BiMod and $Y \in \underline{s\mathbf{Sets}_*}$ we have $T \otimes Y \cong T \otimes_{\mathrm{Day}} \mathbb{Y}$.

Remark More generally, the category F-Mod is pointed simplicial. Indeed, it is the category of left-modules over a monoid in an enriched category with all equalizers, $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\mathrm{fin}}, \underline{s\mathbf{Sets}_*})$, and so we have a pointed simplicial set of maps between two left F-modules, denoted $\underline{F\text{-}\mathrm{Mod}}(M, N)$. The same is true of Mod-F and F-BiMod.

Remark Let $\mathcal{I}$ be a small category. The category F-Mod is closed under finite products, coproducts, and homotopy colimits. However, in general it will *not* be closed under arbitrary diagrams. This is an unfortunate consequence of the generality of our definition of functor with stabilization. Specifically, for a pointed simplicial functor $F : \underline{s\mathbf{Sets}_*}^{\mathrm{fin}} \to \underline{s\mathbf{Sets}_*}$ we only suppose that the assembly maps $\lambda_{S^1, X} : S^1 \wedge F(X) \to F(S^1 \wedge X)$ (for X an n-connected simplicial set) are $(2n - c)$-connected for some constant c not dependent on X. If we have a diagram of FSTs $A \in (F - \mathrm{Mod})^{\mathcal{I}}$, the constants c_i of each A_i can tend to infinity and so products, coproducts, and homotopy colimits may not stabilize (other treatments in the literature require FSTs to have $2n$-connected assembly maps, in which case this is a moot point). If $A \in (F - \mathrm{Mod})^{\mathcal{I}}$ is *globally stable* (Section 5.4 [47]), then $\lim_{\mathcal{I}} A$, $\mathrm{colim}_{\mathcal{I}} A$, and $\mathrm{hocolim}_{\mathcal{I}} A$ satisfy the stabilization criteria and therefore we get an FST again (in fact, and F-module).

Example E.3.10 Let G be a (simplicial) group or monoid. Then one can form the the "group ring over the sphere spectrum", $\mathbb{G}$, as defined in E.2.4. However, associated to G is also a (simplicial) integral group ring $\mathbb{Z}(G)$, so one can form the Eilenberg-Maclane FSP, $\widetilde{\mathbb{Z}(G)}$, as defined in E.2.5. There is a natural transformation of functors $\mathbb{G} \xrightarrow{\eta} \widetilde{\mathbb{Z}(G)}$ given, for $X \in \underline{s\mathbf{Sets}_*}^{\mathrm{fin}}$, by: $\mathbb{G}(X) := G_+ \wedge X \xrightarrow{\eta_X} \mathbb{Z}(G)[X]/\mathbb{Z}(G)[*] =: \widetilde{\mathbb{Z}(G)}(X)$, $(g \wedge x) \mapsto (1 \cdot g) \cdot x$.

One easily checks that this is well-defined, natural in X, enriched over $\underline{s\mathbf{Sets}_*}$, and preserves the multiplications and identities. So we have a morphism of FSP's $\mathbb{G} \xrightarrow{\eta} \widetilde{\mathbb{Z}(G)}$ for each (simplicial) group G. Note that the map on the associated spectra, $\underline{\mathbb{G}} \cong \Sigma_+^\infty G \to H(\mathbb{Z}(G)) = \widetilde{\mathbb{Z}(G)}$, is the "base change of rings." Indeed, we have morphisms of FSPs $\mathbb{S} \to \mathbb{G}$ and $\mathbb{S} \to \tilde{\mathbb{Z}}$. Viewing $\mathbb{G}$ as an $\mathbb{S}$-module (via the first morphism) and base changing along the second morphism, we get a $\tilde{\mathbb{Z}}$-module, $\tilde{\mathbb{Z}} \otimes_{\mathrm{Day}} \mathbb{G} = \tilde{\mathbb{Z}} \wedge_\mathbb{S} \mathbb{G}$ (in fact, it is a $\tilde{\mathbb{Z}}$-algebra).

We claim that $\tilde{\mathbb{Z}} \wedge_\mathbb{S} \mathbb{G} \cong \widetilde{\mathbb{Z}(G)}$ as FSPs. By the universal property of the Day convolution, to give a map $\phi : \tilde{\mathbb{Z}} \wedge_\mathbb{S} \mathbb{G} \to \widetilde{\mathbb{Z}(G)}$ we need a natural transformation of functors $\underline{s\mathbf{Sets}_*}^{\mathrm{fin}} \times \underline{s\mathbf{Sets}_*}^{\mathrm{fin}} \to \underline{s\mathbf{Sets}_*}$

$$\tilde{\mathbb{Z}}(X) \wedge \mathbb{G}(Y) \to \widetilde{\mathbb{Z}(G)}(X \wedge Y)$$

It is given by $(n \cdot x) \wedge (g \wedge y) \mapsto (n \cdot g) \cdot (x \wedge y)$. One readily checks that this is an isomorphism of FSTs (i.e. $\mathbb{S}$-bimodules), and furthermore that it preserves the multiplication and identities.

Remark In any symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$, the monoidal product of two monoids, M and N, is again a monoid, with the obvious multiplication. This induces a symmetric monoidal structure on the category of monoids $\mathrm{Mon}(\mathcal{C})$ in $\mathcal{C}$. However, unless both monoids are *commutative*, the monoidal product $M \otimes N$ will not be the coproduct in $\mathrm{Mon}(\mathcal{C})$. That is, given monoid maps $M \xrightarrow{f} X$ and $N \xrightarrow{g} X$, the unique map that exists $M \otimes N \xrightarrow{f \otimes g} X$ in $\mathcal{C}$, is *not* going to be a morphism of monoids (and therefore a map in $\mathrm{Mon}(\mathcal{C})$).

In our case, $\tilde{\mathbb{Z}}$ is a commutative FSP. If in addition, G is commutative (so that $\mathbb{G}$ is a commutative FSP), then they are both commutative monoids with regard to the Day convolution monoidal structure on $\underline{Fun}(\underline{s\mathbf{Sets}_*}^{\mathrm{fin}}, \underline{s\mathbf{Sets}_*})$. In this case, $\tilde{\mathbb{Z}} \wedge_\mathbb{S} \mathbb{G}$ is the coproduct in $\mathcal{FSP}$:

$$
\begin{array}{ccc}
\mathbb{S} & \longrightarrow & \mathbb{G} \\
\downarrow & \ulcorner & \downarrow \\
\tilde{\mathbb{Z}} & \longrightarrow & \tilde{\mathbb{Z}} \wedge_\mathbb{S} \mathbb{G}
\end{array}
$$

After composing the right-most map with the isomorphisms of FSPs $\phi : \tilde{\mathbb{Z}} \wedge_\mathbb{S} \mathbb{G} \xrightarrow{\cong} \widetilde{\mathbb{Z}(G)}$ defined above, we get a morphism of FSPs $\mathbb{G} \to \widetilde{\mathbb{Z}(G)}$, which agrees with η! This justifies our earlier comment that the map of associated spectra $\underline{\mathbb{G}} \cong \Sigma_+^\infty G \to H(\mathbb{Z}(G)) = \widetilde{\mathbb{Z}(G)}$, is the "base change of rings." We can thus think of $\widetilde{\mathbb{Z}(G)}$ as the "linearization" of $\mathbb{G}$.

E.4   FSPs over Categories

E.4.1   Enrichment

Definition E.4.1 Let $\mathcal{C}$ be a small category. A <u>functor with smash product over $\mathcal{C}$</u> is a choice of enrichment of $\mathcal{C}$ over the closed symmetric monoidal category $\mathbb{S}$-BiMod.

Notation If no confusion arises, we will denote such a choice of enrichment by $F^{\mathcal{C}} = F$ and call it an "FSP over $\mathcal{C}$."

Unpacking this definition in steps, we have for each pair of objects in $\mathcal{C}$, (A, B), an object $F_{A,B} := F(A, B)$ of $\mathbb{S}$-BiMod. That is, we have an FST $F_{A,B}$ for each pair (A, B). Furthermore, for each triple (A, B, C) in $\mathcal{C}$ we have a morphism of FSTs $F_{B,C} \otimes_{\text{Day}} F_{A,B} \to F_{A,C}$ which are the "composition" maps, assembling into a natural transformation of functors. We also have unit maps $\mathbb{S} \to F_{A,A}$ for every object A, which together with the composition maps satisfy the obvious commuting diagrams (see [14] A.10). As is the case with any enriched category, the endomorphism objects in $\mathcal{C}$ are monoid objects in $\mathbb{S}$-BiMod; that is, each $F_{A,A}$ is an FSP. Because of this, we will denote the composition maps for a triple (A, B, C), by $\mu_{A,B,C} : F_{B,C} \otimes_{\text{Day}} F_{A,B} \to F_{A,C}$. By the universal property of Day convolution, we get natural transformation of functors $\underline{s\mathbf{Sets}_*}^{\text{fin}} \times \underline{s\mathbf{Sets}_*}^{\text{fin}} \to \underline{s\mathbf{Sets}_*}$:

$$\mu_{A,B,C;-,\star} = \mu_{-,\star} : F_{B,C}(-) \wedge F_{A,B}(\star) \Rightarrow F_{A,C}(- \wedge \star)$$

as well as a unit map $\mathbb{1}_{A;X} : X \to F_{A,A}(X)$ satisfying identities similar to the ones of an FSP (see [34] 1.3).

Remark In older treatments in the literature, where the language of enriched category was not used, the definition of an FSP over a category $\mathcal{C}$ was given by means of the aforementioned properties and identities (see the diagrams in [17] 1.1). Because each $F_{A,A}$ is an FSP, functors with smash product over categories are sometimes referred to as "rings with several objects."

Example E.4.2 Let $\mathcal{C}$ be the trivial category. Then an FSP over $\mathcal{C}$ is precisely the same data as an FSP in the earlier sense of definition E.1.2. More generally, if F is an FSP over a general category $\mathcal{C}$ and we have a functor $\mathcal{B} \xrightarrow{\eta} \mathcal{C}$, then $\mathcal{B}$ naturally acquires the structure of an $(\mathbb{S}$-BiMod$)$-enriched category. The induced FSP over $\mathcal{B}$ is denoted $\eta^* F$. In particular, if F is an FSP over $\mathcal{C}$ and $C \in \mathcal{C}$ is an object, we have an inclusion $C^{\text{trv}} \hookrightarrow \mathcal{C}$, where C^{trv} is the trivial category on C. The induced FSP over C^{trv} is simply $F_{C,C}$, an FSP in the earlier definition. More generally, if we let $\tilde{C}$ be the *full* subcategory of $\mathcal{C}$ on the single object

C, the induced FSP $F_{C,C}$ is an FSP with extra structure, mainly, natural transformations $F_{C,C} \Rightarrow F_{C,C}$ for each element of $\mathrm{End}_{\mathcal{C}}(C)$ satisfying certain commutativity squares.

Remark We saw in E.2 that to an FST M we may associate a symmetric spectrum $\underline{M}$ by evaluating on spheres. The story translates naturally to the relative setting. Indeed, taking associated spectra gives a strong symmetric monoidal functor $(\mathbb{S}-\mathrm{BiMod}, \otimes_{\mathrm{Day}}, \mathbb{S}) \to (\mathrm{Spec}^{\Sigma}, \wedge, \mathbb{S})$ between the category of FSTs and the closed symmetric monoidal category of symmetric spectra (see [30]). Therefore, a category enriched in $(\mathbb{S} - \mathrm{BiMod}, \otimes_{\mathrm{Day}}, \mathbb{S})$ naturally acquires an enrichment over $(\mathrm{Spec}^{\Sigma}, \wedge, \mathbb{S})$. So an FSP over $\mathcal{C}$ gives $\mathcal{C}$ the structure of a spectrally-enriched category.

E.4.2   Examples

Example E.4.3 Let $\mathcal{C}$ be a simplicial category. We define the "half-smash FSP over $\mathcal{C}$" as follows: for $(A, B) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ define the FST $X \mapsto \underline{\mathcal{C}}(A, B)_+ \wedge X$ where $\underline{\mathcal{C}}(-, -)$ is the simplicial enrichment of $\mathcal{C}$. The multiplication is induced via composition:

$$
\begin{aligned}
(\underline{\mathcal{C}}(B,C)_+ \wedge X) \wedge (\underline{\mathcal{C}}(A,B)_+ \wedge Y) &\cong (\underline{\mathcal{C}}(B,C)_+ \wedge \underline{\mathcal{C}}(A,B)_+) \wedge (X \wedge Y) \\
&= (\underline{\mathcal{C}}(B,C) \times \underline{\mathcal{C}}(A,B))_+ \wedge (X \wedge Y) \\
&\to \underline{\mathcal{C}}(A,C)_+ \wedge (X \wedge Y)
\end{aligned}
$$

which by assumption is a simplicial map. The unit map $\mathbb{1}_{A;X} : X \to \underline{\mathcal{C}}(A, A)_+ \wedge X$ is given by the unit simplicial map in the enrichment $(S^0 \to \underline{\mathcal{C}}(A, A))$. Note that over an object, $\underline{\mathcal{C}}(A, A)$ is a (simplicial) monoid, and the FSP associated here is the one from Example E.2.4.

Example E.4.4 Let sAb be the closed symmetric monoidal category of simplicial abelian groups and let $\mathcal{C}$ be an sAb-enriched category. Recalling the Eilenberg-Maclane FSP $\tilde{\mathbb{Z}}$ from Example E.2.5, we define an FSP over $\mathcal{C}$ as follows: for $(A, B) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ construct the FST $X \mapsto \underline{\mathcal{C}}(A, B) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(X)$. The multiplication is given by sending smash to tensor (using the FSP structure of $\tilde{\mathbb{Z}}$) followed by composition in $\mathcal{C}$:

$$
\begin{aligned}
(\underline{\mathcal{C}}(B,C) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(X)) \wedge (\underline{\mathcal{C}}(A,B) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(Y)) &\to (\underline{\mathcal{C}}(B,C) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(X)) \otimes_{\mathbb{Z}} (\underline{\mathcal{C}}(A,B) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(Y)) \\
&\to (\underline{\mathcal{C}}(B,C) \otimes_{\mathbb{Z}} \underline{\mathcal{C}}(A,B)) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(X \wedge Y) \\
&\to \underline{\mathcal{C}}(A,C) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(X \wedge Y)
\end{aligned}
$$

while the unit map $\mathbb{1}_{A,X}$ is given by the composition of the (simplicial) inclusion $\mathbb{Z} \to \underline{\mathcal{C}}(A,A)$ and the unit map $\mathbb{1}_X : X \to \tilde{\mathbb{Z}}(X)$. The $(\mathbb{S} - \mathrm{BiMod})$-enrichment of $\mathcal{C}$ will be denoted $\tilde{\mathcal{C}}$.

Observation The previous example is simply the fact that we have a lax symmetric monoidal functor $(sAb, \underset{\mathbb{Z}}{\otimes}, \mathbb{Z}) \xrightarrow{H} (\mathbb{S} - \mathrm{BiMod}, \otimes_{\mathrm{Day}}, \mathbb{S})$ from the closed symmetric monoidal category of simplicial abelian groups to the closed symmetric monoidal category of FSTs, sending $M \mapsto \tilde{M}$. Therefore, a category enriched over $(sAb, \underset{\mathbb{Z}}{\otimes}, \mathbb{Z})$ naturally acquires an enrichment over $\mathbb{S}$-BiMod. Objectwise, is is given by sending $A, B \in \mathcal{C} \mapsto \widetilde{\underline{\mathcal{C}}(A,B)}$, the FST associated to the $\mathbb{Z}$-module $\underline{\mathcal{C}}(A,B)$ from E.3.5.

Remark Given a (simplicial) ring R we can view it as an *Ab*-enriched (or even *sAb*-enriched) category with a single object. In this case, the above construction exactly recovers $\tilde{R}$, the Eilenberg-Maclane FSP from Example E.2.5.

Example E.4.5 There is a strong symmetric monoidal functor $(\mathbf{Sets}, \times, *) \to (\mathbb{S}-\mathrm{BiMod}, \otimes_{\mathrm{Day}}, \mathbb{S})$ sending a set D to the FST given by $X \mapsto D_+ \wedge X$. Let $\mathcal{C}$ be a category. Since *any* category can be considered as $(\mathbf{Sets})$-enriched, we get a canonical FSP over $\mathcal{C}$ in a similar vein to the previous note. Explicitly, for $A, B \in \mathcal{C}$ the associated FST is $X \mapsto \mathcal{C}(A,B)_+ \wedge X$ with the multiplication given by composition in the category.

Example E.4.6 Let F, F' be FSPs over $\mathcal{C}, \mathcal{C}'$, respectively. We define a new FSP $F \times F'$ over the product category $\mathcal{C} \times \mathcal{C}'$ by setting for $(A,B) \times (A',B')$ the FST $F_{A,B} \times F_{A',B'}$, with the multiplication defined by $\mu \times \mu'$ (that is, componentwise). If $\mathcal{C} = \mathcal{C}' = \mathcal{D}$, then we can can pullback the FSP $F \times F'$ along the diagonal functor $\mathcal{D} \xrightarrow{\Delta} \mathcal{D} \times \mathcal{D}$ (as in E.4.2) to obtain the "internal product" $F \times F'$, which is an FSP over $\mathcal{D}$. In the case that $\mathcal{D}$ is the trivial category, we recover the product of FSPs in E.2.6.

Similarly, given F, F' FSPs over $\mathcal{C}, \mathcal{C}'$ we can form a *non-unital* "FSP" $F \vee F'$ over $\mathcal{C} \times \mathcal{C}'$. First, let us briefly recall how we form the coproduct of two FSTs. If F and G are two FSTs, we define $F \vee G$ as the functor given by $X \mapsto F(X) \vee G(X)$. This is pointed simplicial as each F and G is. Note that if X is n-connected then $F(X) \vee G(X)$ is also n-connected. Also, since $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$, tracing through the adjunctions one finds that the assembly map $\lambda_{S^1,X}^{F \vee G}$ is given by:

$$\lambda_{S^1,X}^{F \vee G} : S^1 \wedge ((F \vee G)(X)) := S^1 \wedge (F(X) \vee G(X))$$
$$\cong (S^1 \wedge F(X)) \vee (S^1 \wedge G(X))$$

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$$\xrightarrow{\lambda^F_{S^1,X} \vee \lambda^G_{S^1,X}} F(S^1 \wedge X) \vee G(S^1 \wedge X) = (F \vee G)(S^1 \wedge X)$$

Now, by the Blakers-Massey theorem, if $\lambda^F_{S^1,X}$ is $(2n-c)$-connected and $\lambda^G_{S^1,X}$ is $(2n-d)$-connected, then $\lambda^F_{S^1,X} \vee \lambda^G_{S^1,X}$ is $\min(2n+3, 2n-c, 2n-d)$-connected. In any case, $F \vee G$ defines an FST. Now, in our setting, we define the FSP $F \vee F'$ over $\mathcal{C} \times \mathcal{C}'$ by taking, for $(A, A'), (B, B') \in \mathcal{C} \times \mathcal{C}'$, the FST $F_{A,B} \vee F'_{A',B'}$. If we had a third object $(C, C') \in \mathcal{C} \times \mathcal{C}'$, then the composition maps $\left((F \vee F')_{(B,B'),(C,C')}\right) \otimes_{\text{Day}} \left((F \vee F')_{(A,A'),(B,B')}\right) \to (F \vee F')_{(A,A'),(C,C')}$ are defined (using the universal property of mapping out of a Day convolution product) by distributing the smash over the wedges, collapsing the "non-composable" pairs $F_{B,C} \wedge F'_{A',B'}$ and $F'_{B',C'} \wedge F_{A,B}$, and using the multiplications $\mu_{A,B,C}$ and $\mu'_{A',B',C'}$ coming from the enrichments in $\mathcal{C}$ and $\mathcal{C}'$ individually. These are associative up to natural isomorphisms. However, the $F_{A,B} \vee F'_{A',B'}$ don't quite determine an enrichment of $\mathcal{C} \times \mathcal{C}'$ in $\mathbb{S}$-BiMod. Indeed, the desired unit map $\mathbb{S} \to (F \vee F')_{(A,A'),(A,A')}$ is, for $X \in \underline{s\mathbf{Sets}_*}^{\text{fin}}$, supposed to be a map of simplicial sets $X \to F_{A,A}(X) \vee F'_{A',A'}(X)$ that composes to the identity natural transformation *in both variables*. Such a map in general does not exist (unless one of the two FSTs is trivial). So $F \vee F'$ is *not* an FSP in our setting. In [17], there is a distinction between "ring functors" and "unital ring functors", that precisely addresses this issue. Specifically, their axioms for a ring functor are exactly that of an enrichment in $(\mathbb{S} - \text{BiMod}, \otimes_{\text{Day}}, \mathbb{S})$ that's missing the identity structure, i.e. when viewed as a "semigroup with many objects" instead of a symmetric monoidal category.

In any case, for $(A, A'), (B, B') \in \mathcal{C} \times \mathcal{C}'$, there is a natural map of FSTs $F_{A,B} \vee F'_{A',B'} \to F_{A,B} \times F'_{A',B'}$ which is stable weak equivalence by the Blakers-Massey theorem. Furthermore, the map is compatible with the multiplications previously defined.

Example E.4.7 Let F, F' be FSPs over $\mathcal{C}, \mathcal{C}'$, respectively. We can form the disjoint union category $\mathcal{C} \amalg \mathcal{C}'$ as in Example 2.2.5 [44]. Then $\mathcal{C} \amalg \mathcal{C}'$ acquires an $(\mathbb{S} - \text{BiMod})$-enrichment, denoted $F \amalg F'$, taking the associated FSTs from F and F' if both objects are in $\mathcal{C}$ or $\mathcal{C}'$, respectively. If $(C, C') \in \mathcal{C} \amalg \mathcal{C}'$ has $C \in \mathcal{C}$ and $C' \in \mathcal{C}'$, then $(F \amalg G')_{C,C'} = *$, that is, the FST taking constant value at the point. The multiplication is then well-defined (since for any FST G, $* \otimes_{\text{Day}} G \cong *$).

E.4.3   Morphisms in the Relative Setting

Definition E.4.8 Let F, F' be FSPs over $\mathcal{C}, \mathcal{C}'$, respectively. A <u>unital morphism of FSPs</u> from F to F' is an ($\mathbb{S}$-BiMod)-enriched functor $\mathcal{C} \xrightarrow{\eta} \mathcal{C}'$.

Remark Since $\mathcal{C} \xrightarrow{\eta} \mathcal{C}'$ is an enriched functor, it preserves the unit. We will need to make use at times of *non-unital* morphisms between FSPs. These will be functors $\mathcal{C} \xrightarrow{\eta} \mathcal{C}'$ that are *almost* enriched, in the sense that they commute with μ and μ', but do not necessarily preserve the unit maps.

Example E.4.9 Let F be an FSP over $\mathcal{C}$. Consider an *unenriched* functor $\mathcal{B} \xrightarrow{\eta} \mathcal{C}$. Then, if we give $\mathcal{B}$ the FSP $\eta^* F$ as in E.4.2, the functor η becomes ($\mathbb{S}$-BiMod)-enriched, and therefore a unital morphism of FSPs.

Fact Let $\eta, \eta' : \mathcal{B} \to \mathcal{C}$ be two functors between categories, and F an FSP over $\mathcal{C}$. If $\epsilon : \eta \Rightarrow \eta'$ is a natural transformation of functors, then ϵ induces a morphism of FSPs $\eta^* F \to \eta'^* F$ over $\mathcal{C}$ who's underlying functor on objects is the identity functor. Is η is a natural isomorphism, then the induced unital morphism of FSPs is an isomorphism.

Example E.4.10 Let F, F' be FSPs over $\mathcal{C}, \mathcal{C}'$, respectively. We can form the external product FSP $F \times F'$ over $\mathcal{C} \times \mathcal{C}'$ as in E.4.6. Then the categorical projection functors $p : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}$ and $p' : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'$ are ($\mathbb{S}$-BiMod)-enriched by construction. Therefore, the projection functors $F \times F'$ to F and F' are unital morphisms of FSPs. Furthermore, $F \times F'$ then *is* the categorical product in the category of (relative) FSPs.

Example E.4.11 Let $\mathcal{C}, \mathcal{D}$ be simplicial categories, and $\eta : \mathcal{C} \to \mathcal{D}$ a pointed simplicial functor. Then the simplicial maps $\underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(F(A), F(B))$ for each $A, B \in \mathcal{C}$ assemble to give maps of the associated FSTs commuting with the multiplications and unit. That is, the enrichment of η over pointed simplicial sets induces an ($\mathbb{S}$-BiMod)-enriched functor between the induced enrichments.

Example E.4.12 Let $\mathcal{C}, \mathcal{C}'$ be *sAb*-categories and $\eta : \mathcal{C} \to \mathcal{C}'$ be a *sAb*-enriched functor (Under the symmetric monoidal Dold-Kan correspondence, this might appear in the literature under dg-categories and a dg-functor). Then from the note following E.4.4 we see that η induces a morphism of FSPs between the associated FSPs over $\mathcal{C}$ and $\mathcal{C}'$, $\eta : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}'$.

E.4.4   Modules in the Relative Setting

Just as in the case of any enriched category, we have notions of left module, right module and bi-modules.

Definition E.4.13 Let F be a functor with smash product over $\mathcal{C}$. A <u>left F-module over $\mathcal{C}$</u>, M, is an $(\mathbb{S}\text{-BiMod})$-enriched functor $\mathcal{C} \xrightarrow{M} \mathbb{S}\text{-BiMod}$. A <u>right F-module over $\mathcal{C}$</u>, N, is an $(\mathbb{S}\text{-BiMod})$-enriched functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{N} \mathbb{S}\text{-BiMod}$. An <u>$F$-bimodule over $\mathcal{C}$</u>, P, is an $(\mathbb{S}\text{-BiMod})$-enriched functor $\mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Day}} \mathcal{C} \xrightarrow{P} \mathbb{S}\text{-BiMod}$.

Remark For unpacked versions of these definitions see sections 1 in both [17] and [34].

Example E.4.14 If $\mathcal{C}$ is the trivial category, and F an FSP over $\mathcal{C}$, the notions of left/right/bi-F-module over $\mathcal{C}$ coincides with the earlier notions in section E.3.

Example E.4.15 Let $\mathcal{C}$ be an sAb-category, and let $\tilde{\mathcal{C}}$ be the $(\mathbb{S} - \mathrm{BiMod})$-enrichment obtained by composing the lax symmetric monoidal functor $(sAb, \underset{\mathbb{Z}}{\otimes}, \mathbb{Z}) \xrightarrow{H} (\mathbb{S} - \mathrm{BiMod}, \otimes_{\mathrm{Day}}, \mathbb{S})$ as in E.4.4. Let M be a left sAb-module over $\mathcal{C}$. That is, M is a sAb-enriched functor $\mathcal{C} \to sAb$. Composing with H, we get a functor $\mathcal{C} \to \mathbb{S} - \mathrm{BiMod}$ which respects the enrichment of $\tilde{\mathcal{C}}$. We denote the enriched functor by $\tilde{M}$. Thus we get a left $\tilde{\mathcal{C}}$-module $\tilde{M}$. This similarly works for right and bimodules. For example, if $\mathcal{A}$ is an Ab-enriched category (a "linear category" in the language of Example 1.4 [34]) and T an Ab-enriched functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to Ab$ (i.e. a bilinear or additive functor), the associated $\tilde{\mathcal{A}}$-bimodule $\tilde{T}$ is given, for $A, B \in \mathcal{A}$, by: $X \mapsto T(A, B) \underset{\mathbb{Z}}{\otimes} \tilde{\mathbb{Z}}(X)$.

Fact A natural way to get a bimodule over an Ab-enriched category $\mathcal{A}$ is to have a pair of Ab-enriched functors $G_1, G_2 : \mathcal{A} \to \mathcal{B}$ to another Ab-enriched category $\mathcal{B}$. Then the abelian group of morphisms $\underline{\mathcal{B}}(G_1(-), G_2(\star)) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to Ab$ forms an $\mathcal{A}$-bimodule.

Definition E.4.16 Let F be an FSP over $\mathcal{C}$, M, M' two left F-modules. A morphism of left F-modules $\phi : M \to M'$ is an $(\mathbb{S}\text{-BiMod})$-enriched natural transformation of functors $\mathcal{C} \xrightarrow{M} \mathbb{S}\text{-BiMod}$. Morphisms of right F-modules and F-bimodules are defined analogously as $(\mathbb{S}\text{-BiMod})$-enriched natural transformations of the appropriate functors. The categories of left/right/bimodules over F will be denoted exactly as in the case of E.3.

Example E.4.17 If $\mathcal{C}$ is the trivial category, and F an FSP over $\mathcal{C}$, the notions of morphism of left/right/bi-F-modules over $\mathcal{C}$ coincide with the earlier notions in section E.3.

Remark Let F, F' be FSPs over $\mathcal{C}, \mathcal{C}'$ and let $\eta : \mathcal{C} \to \mathcal{C}'$ be a unital morphism of FSPs. Then it induces unital morphisms of FSPs $\eta^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'^{\mathrm{op}}$, and $\eta^{\mathrm{op}} \otimes_{\mathrm{Day}} \eta : \mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Day}} \mathcal{C} \to \mathcal{C}'^{\mathrm{op}} \otimes_{\mathrm{Day}} \mathcal{C}'$. If P' is a left/right/bi-F'-module, then precomposing with η, η^{op}, or $\eta^{\mathrm{op}} \otimes_{\mathrm{Day}} \eta$ restricts P' to a left/right/bi-F-module, denoted $\eta^* P'$, since a composition of enriched functors is enriched.

Fact For a fixed FSP F over $\mathcal{C}$, the category of left F-modules is enriched over $\mathbb{S}$-BiMod. Indeed, it is the functor category of enriched functors between enriched categories, $\underline{Fun}(\mathcal{C}, \mathbb{S}-$ BiMod). The $(\mathbb{S}$-BiMod)-natural transformation objects are given by the end formula as in E.1. This $(\mathbb{S}$-BiMod)-category is also $(\mathbb{S}$-BiMod)-tensored and cotensored, as the target category of the enriched functor category is tensored and cotensored. Combining with the "evaluate at S^0" functor from before, we get an induced $\underline{s\mathbf{Sets}_*}$-enrichment, tensoring and cotensoring. For example, the tensoring is defined as follows: Let F be an FSP over $\mathcal{C}$, $P \in F$- BiMod, and $Y \in \underline{s\mathbf{Sets}_*}$. Define $P \otimes Y$ as the F-bimodule sending $C \in \mathcal{C} \mapsto P_C(-) \otimes Y$. That is, we utilize the tensoring of $\mathbb{S}$-BiMod over $Y \in \underline{s\mathbf{Sets}_*}$ mentioned in E.3.

The following definition will let us compare modules over different categories:

Definition E.4.18 Let F, F' be FSPs over $\mathcal{C}, \mathcal{C}'$, respectively. Let P, P' be left modules over F, F', respectively. A morphism of left modules $(F; P) \to (F'; P')$ is a pair (η, ϕ), where $\eta : \mathcal{C} \to \mathcal{C}'$ is a unital morphism of FSPs, and $\phi : P \to \eta^* P'$ a morphism of left F-modules. We define similarly pairs (η, ϕ) for right modules and bimodules.

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