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COMPUTING THE GOODWILLIE-TAYLOR TOWER FOR DISCRETE MODULES

BY

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DISSERTATION

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Abstract

A functor from finite sets to chain complexes is called atomic if it is completely determined by its value on a particular set. We present a new resolution for these atomic functors, which allows us to easily compute their Goodwillie polynomial approximations. By a rank filtration, any functor from finite sets to chain complexes is built from atomic functors. Computing the linear approximation of an atomic functor is a classic result involving partition complexes. Robinson constructed a bicomplex, which can be used to compute the linear approximation of any functor. We hope to use our new resolution to similarly construct bicomplexes that allow us to compute polynomial approximations for any functor from finite sets to chain complexes.

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Chapter 1

Introduction

As a way to better understand the behavior of a large class of functors, Goodwillie developed the calculus of homotopy functors. Analogous to calculus of real-valued functions, in which a crucial tool is approximation by polynomials, we have a notion of polynomial functors, whose behavior is easier to understand, and we use these to approximate other functors. A functor F can be approximated by a degree n functor, denoted P_nF, in a way that is analogous to the degree n Taylor polynomial, p_nf, of a real-valued function f. Taylor polynomials of functions assemble into the Taylor series. The analogous construction for functors is the Taylor tower, for which the approximation P_nF is called the n-th level of the tower.

The difference of the degree n and degree $n-1$ Taylor polynomials of a function, $p_nf - p_{n-1}f$, is a homogeneous degree n term. In order to define the degree n homogeneous layer D_nF, we use the homotopy fiber

$$D_nF = hofib(P_nF \to P_{n-1}F).$$

Unlike adding together the homogeneous terms of a function to reconstruct the Taylor series, reassembling the Taylor tower from the homogeneous layers is a nontrivial problem.

It can be shown that a large class of functors can be reduced down to the study of *discrete modules*, which are functors from pointed finite sets to chain complexes. This thesis studies such functors.

We consider two properties of discrete modules: rank and degree. In order to better understand the behavior of a real valued function, we might approximate it with a sum of easier to understand functions. The intuition here is the same. We break down a discrete module into elemental functors based on rank and degree. We do this in an orderly way, preserving structure so that the original functor can be reassembled from these pieces.

To motivate the definition of rank, consider a method of approximating real-valued functions called Lagrangian approximation, or polynomial interpolation. For a real valued function f, the n-th Lagrangian approximation, l_nf, is the degree n polynomial that agrees with f at $n+1$ points. There is also an analog for Lagrangian approximation in Goodwillie calculus. For a functor F from finite pointed sets to chain complexes, there is a unique functor L_nF that agrees with F on sets of size at most $n+1$ and its behavior on larger sets is completely determined by this information. To be more explicit, L_nF is the left Kan extension of the restriction of F to the subcategory of sets of size at most $n+1$. If a functor is completely determined by its behavior on sets of size at most $n+1$, we say it has rank n.

The rank n approximations of a functor F fit into a filtration of F:

$$L_1 F \to L_2 F \to \cdots \to \lim_n L_n F \cong F.$$

The quotients of successive terms, $C_n F = \frac{L_n F}{L_{n-1} F}$, in this rank filtration are called atomic functors, that is, they are completely determined by their value on one set and are zero on all smaller sets. When we talk about breaking down a discrete module F in terms of rank, we mean computing the atomic pieces $C_0 F$, $C_1 F$, $C_2 F$, and so on. Atomic functors can be shown to have the following form, [IJM08]

$$C_n F(X) = \frac{L_n F(X)}{L_{n-1} F(X)} \cong \mathcal{I}^n(X) \otimes_{\Sigma_n} \mathrm{cr}_n F([1]),$$

where $\mathcal{I}^n(X) = \widetilde{R}\,[Inj_+([n], X)]$, $Inj_+([n], X)$ denotes the set of basepoint preserving injections from $[n] = \{0, 1, \ldots, n\}$ to X, $\mathrm{cr}_n F([1])$ is the n-th cross effect of F composed with the diagonal map evaluated at $[1] = \{0, 1\}$, and $\widetilde{R}\,[X]$ is the reduced free R-module on the finite pointed set X.

Now that we have dissected the discrete modules by rank, we further dismantle them in terms of degree. More precisely, we wish to compute the Taylor Tower of the atomic functors.

We start by computing the homogeneous layers of the tower. From the classification of atomic functors, we have

$$D_k C_n F(X) \simeq D_k \mathcal{I}^n(X) \otimes_{h\Sigma_n} \mathrm{cr}_n F[1].$$

Thus, in order to compute the k-homogeneous layer of an atomic functor, we first compute $D_k \mathcal{I}^n$ and its Σ_n action.

Based on a combinatorial result on partially ordered sets, it was previously known that

$$D_1 C_n F(X) = \Sigma^{n-1} \varepsilon \mathrm{Lie}_n^* \otimes \widetilde{R}\,[X] \otimes_{\Sigma_n} \mathrm{cr}_n F([1]),$$

where ε indicates that the standard Σ_n action is twisted by the sign representation and Lie_n^* is the dual of the n-multilinear part of the free Lie algebra on n letters [Rob03].

Multilinearization, a tool from Goodwillie calculus, enables us to compute general $D_k \mathcal{I}^n$ given $D_1 \mathcal{I}^n$ by taking the k-th cross effect, applying D_1(i.e. linearizing) in each variable, and taking homotopy orbits. The general statement for multilinearization is

$$D_k F \simeq (D_1^{(k)} \mathrm{cr}_k F)_{h\Sigma_k}.$$

In order to state the result, we will need to introduce some more notation. We define $\mathrm{Ord}(n, k)$ as the set of ordered surjections $\mathbf{n} \to \mathbf{k}$, where $\mathbf{n} = \{1, 2, \ldots, n\}$. Alternatively, it is the set of partitions of $\{1, 2, \ldots, n\}$ into k nonempty blocks, where we describe the partitions by surjections $\mathbf{n} \to \mathbf{k}$ with a particular order fixed on the blocks.

Let

$$A_{n-k}(n)F(X) = \bigoplus_{\alpha \in \mathrm{Ord}(n,k)} \varepsilon\mathrm{Lie}(\alpha^{-1}(1))^* \otimes \cdots \otimes \varepsilon\mathrm{Lie}(\alpha^{-1}(k))^* \otimes \widetilde{R}\left[X^k\right] \otimes_{h\Sigma_n} \mathrm{cr}_n F([1]).$$

The chain complex $D_1C_nF(X)$ above is equivalent to $\Sigma^{n-1}A_1(n)F(X)$.

Theorem 1.0.1. *For a functor F from finite pointed sets to chain complexes,*

$$D_kC_nF(X) \simeq \Sigma^{n-k}A_{n-k}(n)F(X).$$

We further show the existence of a resolution of C_nF, where the terms of the resolution are the layers D_kC_nF and such that truncations of the resolution allow us to compute P_kC_nF for any k.

Theorem 1.0.2. *There exists a unique resolution as functors*

$$0 \to A_{n-1}(n)F \to A_{n-2}(n)F \to \cdots \to A_0(n)F \to C_nF \to 0$$

and the chain complex

$$0 \to A_{n-1}(n)F \to A_{n-2}(n)F \to \cdots \to A_{n-k}(n)F \to 0 \to \cdots \to 0$$

is quasi-isomorphic to P_kC_nF for all $k \geq 0$.

In order to use the resolution effectively, we need to know that it is suitably natural with respect to n. To that end, we give an explicit description of both the groups and the maps in the resolution.

Our resolution captures the layers and levels of the Goodwillie-Taylor tower for any atomic functor. However, we are interested in discrete modules in general. Through the rank filtration, such functors can be broken down into atomic functors. We conjecture that the layers D_kC_iF, for $i = 1, \ldots, n$, can be assembled as the columns of a bicomplex that has the same homology as D_kF. Inspiration for this conjecture comes from a bicomplex defined by Robinson in [Rob03], and modified by Intermont, Johnson, and McCarthy in [IJM08], which is just such a construction for D_1F. For any k and any discrete module F, we use multilinearization to construct a multi-complex from the Robinson bicomplex that is quasi-isomorphic to D_kF. Totalizing this multi-complex yields a complex with entries quasi-isomorphic to the entries of our degree resolutions of atomic functors. Future work will be to describe the maps in the bicomplexes for D_kF and to check if they can be assembled, using the boundary maps of the resolution we have constructed, into a tricomplex that computes P_nF up to homology.

Chapter 2

Functor Calculus

2.1 Intro

Just like real calculus was developed to help us understand a large class of functions, functor calculus helps us better understand a large class of functors, namely homotopy functors.

The analog to the Taylor series of a function is the Taylor Tower for a functor F, which is a tower of functors and natural transformations:

$$\begin{tikzcd}
& {} \arrow[d] \\
& P_kF \arrow[d] \\
& P_{k-1}F \arrow[d] \\
F \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \vdots \arrow[d] \\
& P_1F \arrow[d] \\
& P_0F
\end{tikzcd}$$

Each P_kF, called a *level* of the tower, is the "degree k" approximation of F, where our notion of degree will vary depending on the setting we are working in. Each P_kF is universal for degree k functors with natural transformations from F. The level P_kF is the analog of the degree k partial expansion, p_kf, of a the Taylor series of a function f. For functions, one could take the difference of partial expansions $p_kf - p_{k-1}f$ and arrive at a single degree k homogeneous term. The analog for functors is the k *layer* of the tower, D_kF, defined by

$$D_kF = hofib(P_kF \to P_{k-1}F).$$

Unlike Taylor series, which can be easily recovered by summing the homogeneous terms, reassembling the Taylor tower from its layers is difficult.

The notion of degree k that we will here comes from [JM04]. In order to define degree, we must first define the cross effect of a functor.

2.2 Cross effects and degree

For a functor $F : \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ a category with finite coproducts and $\mathcal{B}$ an abelian category, the k-th cross effect $\mathrm{cr}_k F : \mathcal{A}^{\times k} \to \mathcal{B}$ is defined recursively in the following manner:

$$\mathrm{cr}_0 F = F(+),$$

$$\mathrm{cr}_1 F(X) \oplus F(+) = F(X),$$

$$\mathrm{cr}_2 F(X, Y) \oplus \mathrm{cr}_1 F(X) \oplus \mathrm{cr}_1 F(Y) = \mathrm{cr}_1 F(X \vee Y),$$

and in general

$$\mathrm{cr}_k F(X_1, \ldots, X_k) \oplus \mathrm{cr}_{k-1} F(X_1, X_3, \ldots, X_k) \oplus \mathrm{cr}_{k-1} F(X_2, X_3, \ldots, X_k)$$

$$= \mathrm{cr}_{k-1} F(X_1 \vee X_2, X_3, \ldots, X_k).$$

If $\mathrm{cr}_0 F \cong 0$, then we say that F is *reduced*, and it follows that $\mathrm{cr}_1 F(X) \cong F(X)$. Let $\mathrm{cr}_k F(X)$ denote $\mathrm{cr}_k F(X, \ldots, X)$, where all k inputs are the same object X. The following example demonstrates a calculation of cross effects.

Example 2.2.1. Consider the functor of R-modules, $T : \mathrm{Mod}_R \to \mathrm{Mod}_R$, defined by

$$T(M) = M \otimes_R M.$$

The 0-th cross effect is given by the what the functor does on the basepoint, in this case

$$\mathrm{cr}_0 T \cong 0.$$

Since T is reduced,

$$\mathrm{cr}_1 T(M) \cong T(M) \cong M \otimes M.$$

Considering the second cross effect on two R-modules M and N,

$$\mathrm{cr}_2 T(M, N) \oplus (M \otimes M) \oplus (N \otimes N) = \otimes^2 (M \oplus N).$$

Expanding the right side, we get a direct sum of several modules, including $M \otimes M$ and $N \otimes N$. The remaining terms are the second cross effect,

$$\mathrm{cr}_2 T(M,N) \cong (M \otimes N) \oplus (N \otimes M).$$

It can be shown that $\mathrm{cr}_3 T \cong 0$, and it follows all higher cross effects are as well.

It will sometimes be helpful to have another way of calculating cross effects.

Lemma 2.2.2.

$$cr_k F(X_1, \ldots, X_k) \simeq ker\big(F(X_1 \vee \cdots \vee X_k) \to \bigvee_{i=1}^{k} F(X_1 \vee \cdots \vee \widehat{X_i} \vee \cdots \vee X_k)\big).$$

We say that F is *degree* k if $\mathrm{cr}_{k+1} F(X_1, \ldots, X_{k+1}) \cong 0$, for all choices of X_i. The tensor functor in Example 2.2.1 is degree 2. Since $T(M) = M \otimes M$ is reminiscent of the degree two function $f(x) = x * x$, this fits with analogy to real valued calculus.

2.3 The Taylor tower for functors to abelian categories

The definition of Taylor tower for abelian functors is given in [JM04]. An equivalent construction is given in [BJO$^+$], which we will state here. For $F : \mathcal{A} \to \mathcal{B}$ a functor of abelian categories, let

$$\perp_{k+1} F(X) = \Delta^* \circ \mathrm{cr}_{k+1} F(X),$$

where Δ^* denotes precomposition with the diagonal functor. The k-th polynomial approximation $P_k F : \mathcal{A} \to Ch\mathcal{B}$ is

$$\cdots \to \perp_{k+1}^{\times 3} F(X) \to \perp_{k+1}^{\times 2} F(X) \to \perp_{k+1} F(X) \to F(X).$$

There maps here are given by alternating sums involving the counit map of the cotriple defined by $\perp_{k+1}$. Details about the cotriple construction can be found in [JM04, BJO$^+$].

For the purposes of this thesis, we will rely heavily on multilinearization, which is stated in the following proposition from [JM04].

Proposition 2.3.1.

$$D_k F \simeq (D_1^{(k)} cr_k F)_{h\Sigma_k}.$$

Chapter 3

Discrete Modules, Rank Filtration, and Atomic functors

3.1 Discrete Modules

Let $\mathbb{F}_+$ denote the category of finite pointed sets with basepoint preserving maps and let CMod_R denote the category of chain complexes of R-modules. A *discrete module* is a functor $F : \mathbb{F}_+ \to \mathrm{CMod}_R$

We use the notation $[n]$ for the set $\{0, 1, \ldots, n\}$ in $\mathbb{F}_+$, where 0 is regarded as the basepoint, and $\mathbf{n} = \{1, 2, \ldots, n\}$. For $X \in \mathbb{F}_+$, $\widetilde{R}[X]$ is the reduced free R module on X.

The discrete module $\mathcal{I}^n$ is essential to our constructions.

$$\begin{aligned} \mathcal{I}^n(-) &= \widetilde{R}\left[Inj([n], -)_+\right] \\ &= \widetilde{R}\left[\frac{\bigwedge^n X}{\Delta^n X}\right], \end{aligned}$$

where Δ^n is the fat diagonal $\Delta^n = \{(x_1, \ldots, x_n) \in \wedge^n X : x_i = x_j \text{ for some } i \neq j\}$. As shown in the next section, discrete modules can be broken down into *atomic functors* by means of a filtration.

3.2 Rank Filtration

Returning briefly to calculus of real valued functions, there is a method of approximating functions by polynomials called Lagrangian approximation, or polynomial interpolation. For a function

$$f : \mathbb{R} \to \mathbb{R},$$

let $l_n f$ be the degree n polynomial such that

$$l_n f(k) = f(k)$$

for $k = 0, 1, \ldots, n$. In other words, $l_n f$ the unique polynomial that depends only on the value of f at these points, $0, 1, \ldots, n$.

From Lagrangian approximation comes an analogous construction for functors, the discrete module $L_n F$

that depends only on the behavior of F at $[0],[1],\ldots,[n]$. Following [IJM08], we construct the approximation L_nF by restricting F to the subcategory $\mathbb{F}_+^n$ of sets of size at most $n+1$ and then taking the left Kan extension back to the category $\mathbb{F}$.

$$\begin{tikzcd}
\mathbb{F}_+^{\leq n} \arrow[d, "\iota"'] \arrow[dr, "F|_{\leq n}"] & \\
\mathbb{F}_+ \arrow[r, dashed, "L_nF"'] & \mathrm{CMod}_R
\end{tikzcd}$$

If a discrete module depends only on its values for the sets $[0],\ldots,[n]$ in the proceeding way, then we say it is *rank* n.

From $\iota_n : \mathbb{F}^{\leq n} \to \mathbb{F}^{\leq n+1}$, we have natural transformations $L_nF \to L_{n+1}F$. We assemble these into a filtration

$$L_0F \to L_1F \to \cdots \to L_nF \to L_{n+1}F \to \cdots \to \mathrm{colim} L_nF \cong F$$

This is called the *rank filtration* of F.

We say a discrete module F is *atomic* if its behavior is completely determined by its value at a particular set and is 0 for smaller sets. That is, if there is some n such that F is rank n and $F([k]) = 0$ for all $k < n$.

Atomic functors are classified by quotients of successive terms of the rank filtration [IJM08].

$$\begin{aligned} C_nF(X) &= \frac{L_nF(X)}{L_{n-1}F(X)} \\ &\cong \mathcal{I}^n(X) \otimes_{\Sigma_n} \mathrm{cr}_nF([1]). \end{aligned}$$

By Lemma 3.2.1, we could instead consider the derived tensor.

Lemma 3.2.1. *If M and N are R modules, and the action of R on N is free, then*

$$M \otimes_R N \simeq M \otimes_{hR} N.$$

Proof. If N is a free $R[\Sigma_n]$-module, then it is flat and thus $\otimes_{R[\Sigma_n]}N$ is an exact functor. Therefore, $Tor_\bullet^{R[\Sigma_n]}(M,N) = 0$ for $\bullet \geq 1$ and $M \otimes_R N \simeq M \otimes_{hR} N$.

□

So, we have

$$\begin{aligned} C_nF(X) &\simeq \mathcal{I}^n(X) \otimes_{h\Sigma_n} \mathrm{cr}_nF([1]) \\ &\simeq \mathcal{I}^n(X) \widehat{\otimes}_{\Sigma_n} \mathrm{cr}_nF([1]), \end{aligned} \tag{3.1}$$

where $\widehat{\otimes}_{\Sigma_n}$ denotes the two-sided bar construction. That is, the simplicial module

$$m \mapsto \mathcal{I}^n(X) \otimes R[\Sigma_n^m] \otimes \mathrm{cr}_n F([1])$$

with face maps

$$d_0(f, \sigma_1, \ldots, \sigma_m, c) = (\sigma_1 \cdot f, \sigma_2, \ldots, \sigma_m, c),$$

for $1 \leq i \leq m-1$

$$d_i(f, \sigma_1, \ldots, \sigma_m, c) = (f, \sigma_1, \ldots, \sigma_i \circ \sigma_{i+1}, \ldots, \sigma_m, c),$$

$$d_m = (f, \sigma_1, \ldots, \sigma_m, c) = (f, \sigma_1, \ldots, \sigma_{m-1}, \sigma_m \cdot c),$$

and the face maps correspond to inserting a new coordinate with the identity permutation.

For a filtration

$$F_0 C \to F_1 C \to F_2 C \to \cdots$$

of chain complex C, there is a natural way of constructing a spectral sequence with

$$E^0_{pq} = \frac{F_p C_{p+q}}{F_{p-1} C_{pq}},$$

as described in [Wei94, Theorem 5.4.1]. By [Wei94, Theorem 5.5.1], if the filtration is bounded below and $\mathrm{colim} F_p C \cong C$, then the spectral sequence converges naturally to $H_*(C)$. For a discrete module F, the quotients of successive terms of the rank filtration are atomic functors. So, as we seek to describe the Taylor tower of any discrete module F, we will do so by assembling the towers for the atomic pieces, $C_n F$, of F.

3.3 Lie algebras

3.3.1 Lie_n

Recall that a Lie bracket is a binary operation satisfying three axioms:

1. Bilinearity:

$$[ax + by, z] = [ax, z] + [by, z] = a[x, z] + b[yz],$$

$$[x, ay + bz] = [x, ay] + [x, bz] = a[x, y] + b[x, z].$$

2. Alternating:

$$[x, x] = 0.$$

3. Jacobi Relation:

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

Note, axioms 1 and 2 imply antisymmetry:

$$[x,y]=-[y,x].$$

Let R be a commutative ring with unit. A *Lie algebra* over R is a R-module $\mathfrak{L}$ together with a R-bilinear Lie bracket

$$[-,-]:\mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}.$$

For a set A, recursively generate a collection of expressions, $M(A)$ such that $A\subset M(A)$ and if $a,b\in M(A)$, then $[a,b]\in M(A)$. The *free Lie algebra* $\mathfrak{L}(A)$ over A can be constructed by taking the free R-module M generated by $M(A)$ and applying the three Lie bracket relations above.

We are concerned with a particular Lie algebra, Lie_A, generated by the expressions $a\in M(A)$, such that each letter of A appears in a exactly once.

It can be shown that Lie_n is a free R module of rank $(n-1)!$ and one can take as a basis the right justified brackets on A, i.e. terms of the form

$$[\sigma(1),[\sigma(2),[\ldots,[\sigma(n-1),n]\ldots]]],$$

where $A=\mathbf{n}$ and $\sigma\in\Sigma_{n-1}$.

We denote the dual of Lie_n with its Σ_n action twisted by ε by $\varepsilon\mathrm{Lie}^*_n$. This is well known to be the cohomology of the partition poset, where the twist is tied to the fact that the blocks in a partition are not ordered.

For a surjection of sets, $\varphi:A\to\mathbf{k}$, we will sometimes use the notation $\varepsilon\mathrm{Lie}^*_\varphi$ as shorthand for

$$\varepsilon\mathrm{Lie}^*_{\varphi^{-1}(1)}\otimes\cdots\otimes\varepsilon\mathrm{Lie}^*_{\varphi^{-1}(k)}.$$

For $p_1,\ldots p_k\in\mathbb{N}$ and $\vec{p}=(p_1,\ldots,p_k)$,

$$\varepsilon\mathrm{Lie}^*_{\vec{p}}=\varepsilon\mathrm{Lie}^*_{p_1}\otimes\cdots\otimes\varepsilon\mathrm{Lie}^*_{p_k}.$$

3.3.2 Alternative vocabulary for Lie

A Lie monomial in $\mathrm{Lie}(\mathcal{A})$ is a bracketing, in some order, of the symbols in $\mathcal{A}$. For example, if $\mathcal{A} = \{a, b, c\}$ some monomials in $\mathrm{Lie}(\mathcal{A})$ would be

$$[a,[b,c]],\ [[a,b]c],\ [b,[c,a]],\ \text{etc.}$$

Rather than as a word consisting of letters and brackets, we can think of a monomial as a bracketing of slots defined for n inputs and evaluated on an n-tuple of letters. For example

$$[a,[b,c]] = [-_1,[-_2,-_3]](a,b,c) = [-_2,[-_1,-_3]]](b,a,c) = \cdots$$

For an arbitrary finite set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, we define a Σ_n action on $\mathrm{Lie}(\mathcal{A})$ by

$$\rho \cdot f(a_1, \ldots, a_n) = f(a_{\rho(1)}, \ldots, a_{\rho(n)}),$$

for every $\rho \in \Sigma_n$ and monomial $f(a_1, a_2, \ldots, a_n)$ in $\mathrm{Lie}(\mathcal{A})$.

The Σ_n module $\varepsilon\mathrm{Lie}(\mathcal{A})$ is the same underlying R-module as $\mathrm{Lie}(\mathcal{A})$, with the Σ_n action twisted by the sign representation, i.e.

$$\rho \cdot f(a_1, \ldots, a_n) = (-1)^{|\rho|} f(a_{\rho(1)}, \ldots, a_{\rho(n)}),$$

for every $\rho \in \Sigma_n$ and monomial $f(a_1, a_2, \ldots, a_n)$ in $\varepsilon\mathrm{Lie}(\mathcal{A})$.

3.4 Taylor Tower for Discrete Modules

We will build up a description of the Taylor Tower for discrete modules starting with the n-homogeneous terms of atomic functors.

$$D_k C_n F(X) \simeq D_k \mathcal{I}^n(X) \widehat{\otimes}_{\Sigma_n} \mathrm{cr}_n F[1]$$

For $k = 1$, it is shown in [Rob03] that

$$D_1 \mathcal{I}^n(X) \simeq \Sigma^{n-1} \varepsilon\mathrm{Lie}^*_{\mathbf{n}} \otimes_R \widetilde{R}[X],$$

where $\varepsilon\mathrm{Lie}^*_{\mathbf{n}}$ is the n-multilinear free Lie algebra discussed in Section 3.3.

In general, by multilinearizing we can compute $D_k \mathcal{I}^n(-)$.

Proposition 3.4.1.

$$D_k\mathcal{I}^n(X) \simeq \Sigma^{n-k} \bigoplus_{\varphi\in Ord(n,k)} \varepsilon Lie^*_{\varphi^{-1}(1)} \otimes \cdots \otimes \varepsilon Lie^*_{\varphi^{-1}(k)} \otimes \widetilde{R}\left[\wedge^k X\right]$$

The case where $k = 2$ gives a good illustration of the proof.

Example 3.4.2. Let $X_1 = X_2 = X$. We start by noticing that an injection from $[n]$ into a wedge $X_1 \vee X_2$ can be defined by first choosing which X_i each element of $[n]$ is mapped to, then choosing an injection into X_1 and an injection into X_2. This defines an isomorphism:

$$\mathcal{I}^n(X_1 \vee X_2) \cong \bigoplus_{\varphi:\mathbf{n}\to\mathbf{2}} \widetilde{R}\left[Inj(\varphi^{-1}(1), X_1)_+\right] \otimes \widetilde{R}\left[Inj(\varphi^{-1}(2), X_2)_+\right].$$

The second cross effect kills the elements corresponding to sending all elements of $[n]$ to X_1 or all to X_2, so

$$\mathrm{cr}_2\mathcal{I}^n(X_1, X_2) \cong \bigoplus_{\varphi:\mathbf{n}\twoheadrightarrow\mathbf{2}} \widetilde{R}\left[Inj(\varphi^{-1}(1), X_1)_+\right] \otimes \widetilde{R}\left[Inj(\varphi^{-1}(2), X_2)_+\right].$$

We already know how to calculate $D_1\mathcal{I}^n$, so we can linearize in each variable:

$$D_1^{(2)}\mathrm{cr}_2\mathcal{I}^n(X_1, X_2) \simeq$$

$$\bigoplus_{\varphi:\mathbf{n}\twoheadrightarrow\mathbf{2}} \left(\Sigma^{|\varphi^{-1}(1)|-1}\varepsilon\mathrm{Lie}^*_{\varphi^{-1}(1)} \otimes \widetilde{R}\left[X_1\right]\right) \otimes \left(\Sigma^{|\varphi^{-1}(2)|-1}\varepsilon\mathrm{Lie}^*_{\varphi^{-1}(2)} \otimes \widetilde{R}\left[X_2\right]\right).$$

Simplify by recalling $X_1 = X_2$.

$$\simeq \Sigma^{n-2} \bigoplus_{\varphi:\mathbf{n}\twoheadrightarrow\mathbf{2}} \varepsilon\mathrm{Lie}^*_{\varphi^{-1}(1)} \otimes \varepsilon\mathrm{Lie}^*_{\varphi^{-1}(2)} \otimes \widetilde{R}\left[\wedge^2 X\right]$$

Finally, taking homotopy orbits computes $D_2\mathcal{I}^n$ by multilinearization.

$$D_2\mathcal{I}^n(X) \simeq \Sigma^{n-2} \bigoplus_{\varphi\in\mathrm{Ord}(n,2)} \varepsilon\mathrm{Lie}^*_{\varphi^{-1}(1)} \otimes \varepsilon\mathrm{Lie}^*_{\varphi^{-1}(2)} \otimes \widetilde{R}\left[\wedge^2 X\right],$$

where $\mathrm{Ord}(n, 2) = Surj(n, 2)/\Sigma_2$.

Proof of Proposition 3.4.1. Let $X = X_1 = \cdots = X_k$. Notice that a basepoint preserving injection from $[n]$ to $X_1 \vee \cdots \vee X_k$ can be defined by deciding which X_i each element of $[n]$ should be sent to, and then picking an injection from those elements into X_i. This defines an equivalence of sets

$$\left(Inj_+([n], \bigvee_{i=1}^k X_i)\right)_+ \cong \bigvee_{\varphi:\mathbf{n}\to\mathbf{k}} \bigwedge_{i=1}^k \left(Inj_+(\varphi^{-1}(i), X_i)\right)_+,$$

which induces an isomorphism

$$\begin{aligned}
\mathcal{I}^n(\bigvee_{i=1}^k X_i) &= \widetilde{R}\left[\left(Inj_+([n], \bigvee_{i=1}^k X_i)\right)_+\right] \\
&\cong \widetilde{R}\left[\bigvee_{\varphi:\mathbf{n}\to\mathbf{k}} \bigwedge_{i=1}^k \left(Inj_+(\varphi^{-1}(i), X_i)\right)_+\right] \\
&\cong \bigoplus_{\varphi:\mathbf{n}\to\mathbf{k}} \bigotimes_{i=1}^k \widetilde{R}\left[\left(Inj_+(\varphi^{-1}(i), X_i)\right)_+\right].
\end{aligned}$$

By Lemma 2.2.2,

$$\mathrm{cr}_k\mathcal{I}^n(X_1,\ldots,X_k) \simeq \bigoplus_{\varphi:\mathbf{n}\twoheadrightarrow\mathbf{k}} \bigotimes_{i=1}^k \widetilde{R}\left[\left(Inj_+(\varphi^{-1}(i), X_i)\right)_+\right].$$

Linearize in each variable, making use of the calculation $D_1\mathcal{I}^n(X) \simeq \Sigma^{n-1}\varepsilon\mathrm{Lie}_n^* \otimes \widetilde{R}\,[X]$, and simplify by recalling that $X_1 = \cdots = X_k$:

$$\begin{aligned}
D_1^{(k)}\mathrm{cr}_k\mathcal{I}^n(X_1,\ldots,X_k) &\simeq \bigoplus_{\varphi:\mathbf{n}\twoheadrightarrow\mathbf{k}} \bigotimes_{i=1}^k \Sigma^{|\varphi^{-1}(i)|-1}\varepsilon\mathrm{Lie}^*_{|\varphi^{-1}(i)|} \otimes \widetilde{R}\,[X_i] \\
&\cong \Sigma^{|\varphi^{-1}(1)|-1+\cdots+|\varphi^{-1}(k)|-1} \bigoplus_{\varphi:\mathbf{n}\twoheadrightarrow\mathbf{k}} \bigotimes_{i=1}^k \varepsilon\mathrm{Lie}^*_{|\varphi^{-1}(i)|} \otimes \widetilde{R}\,[X] \\
&\cong \Sigma^{n-k} \bigoplus_{\varphi\mathbf{n}\twoheadrightarrow\mathbf{k}} \varepsilon\mathrm{Lie}^*_\varphi \otimes \widetilde{R}\left[\wedge^k X\right].
\end{aligned}$$

Take homotopy orbits Σ_k and apply Proposition 2.3.1,

$$D_k\mathcal{I}^n(X) \simeq \Sigma^{n-k} \bigoplus_{\varphi\in\mathrm{Ord}(n,k)} \varepsilon\mathrm{Lie}^*_\varphi \otimes \widetilde{R}\left[\wedge^k X\right]$$

□

Let

$$A_{n-k}(n)(X) = \bigoplus_{\varphi\in\mathrm{Ord}(n,k)} \varepsilon\mathrm{Lie}^*_{\varphi^{-1}(1)} \otimes \cdots \otimes \varepsilon\mathrm{Lie}^*_{\varphi^{-1}(k)} \otimes \widetilde{R}\left[\wedge^k X\right].$$

In order to describe the Taylor tower for any atomic functor, we first construct a resolution of $\mathcal{I}^n$.

Theorem 3.4.3. *There exists a resolution*

$$\cdots \to 0 \to A_{n-1}(n) \to A_{n-2}(n) \to \cdots \to A_0(n) \to \mathcal{I}^n \to 0,$$

such that the truncation at $A_{n-k}(n)$ is equivalent as a complex of Σ_n-modules to $P_k\mathcal{I}^n$, for all k.

By truncations we mean

$$A_{n-1} \to 0 \to \cdots \to 0 \ \simeq D_1\mathcal{I}^n = P_1\mathcal{I}^n,$$

$$A_{n-1} \to A_{n-2} \to 0 \to \cdots \to 0 \ \simeq P_2\mathcal{I}^n,$$

and so on.

Combining Theorem 3.4.3 with the characterization of atomic functors in (3.1), we also have a resolution for any atomic functor.

Corollary 3.4.4. *For atomic functor G, there exists a resolution*

$$\cdots \to 0 \to A_{n-1}(n)\widehat{\otimes}_{\Sigma_n} cr_nG([1]) \to \cdots \to A_0(n)\widehat{\otimes}_{\Sigma_n} cr_nG([1]) \to \mathcal{I}^n(-)\widehat{\otimes}_{\Sigma_n} cr_nG([1]),$$

such that the truncation at $A_{n-k}(n)G$ is equivalent as a complex of Σ_n-modules to P_kG, for all k.

Chapter 4

The Construction

In this chapter, we give a construction of the *degree resolution* of $\mathcal{I}^n$ of Theorem 3.4.3. We show that it is a chain complex, that the boundary maps are Σ_n-equivariant, and finally that it is indeed a resolution of $\mathcal{I}^n$.

4.1 The Chain Complex

To construct the complex

$$0 \to A_{n-1}(n) \xrightarrow{\partial^*_{n-1}} \cdots \to A_0(n) \xrightarrow{\partial^*_0} \mathcal{I}^n,$$

define

$$A_{n-k}(n)(X) = \bigoplus_{\alpha \in \mathrm{Ord}(\mathbf{n},\mathbf{k})} \varepsilon \mathrm{Lie}^*_\alpha \otimes \widetilde{R}\left[\wedge^k X\right].$$

Note, $A_0(n)(X) \cong \widetilde{R}\left[\wedge^n X\right]$. A basis element of $\mathcal{I}^n(X)$ can be written as an n-tuple of distinct elements of X. Define $\partial^*_0 : A_0(n) \to \mathcal{I}^n$ by

$$\partial^*_0(x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_n) & \text{if } x_i \text{ are all distinct} \\ 0 & \text{else} \end{cases}$$

To define $\partial^*_k : A_k(n) \to A_{k-1}(n)$ for $k > 0$, we will consider the dual complex and define ∂_k. In the dual complex,

$$A_{n-k}(n)^*(X) = \bigoplus_{\alpha \in \mathrm{Ord}(\mathbf{n},\mathbf{k})} \varepsilon \mathrm{Lie}_\alpha \otimes \widetilde{R}\left[\wedge^k X\right]^*.$$

Writing $\widetilde{R}\left[X^k\right]$ is perhaps more natural when thinking of these modules as coming from the multilinearization. However, we will introduce some notation that will make it easier to keep track of the symmetric group actions later.

For $\alpha \in \mathrm{Ord}(n,k)$, we write $\widetilde{R}\left[X^\alpha\right]$ for the submodule of $\widetilde{R}\left[\wedge^n X\right]$ such that, for $(x_1, \ldots x_n) \in \widetilde{R}\left[\wedge^n X\right]$, if $\alpha(i) = \alpha(j)$ then $x_i = x_j$. Note that this is isomorphic to $\widetilde{R}\left[\wedge^k X\right]$.

Example 4.1.1. For $\varphi : \mathbf{3} \twoheadrightarrow \mathbf{2}$ defined by

$$\varphi : \begin{array}{r} 1,3 \mapsto 2 \\ 2 \mapsto 1 \end{array}$$

and $X = \{x, y, z\}$, some examples of elements of $\widetilde{R}\,[X^{\varphi}]$ are (x, x, x) and (x, y, x), but not (x, y, z) or (x, x, z).

To construct ∂_k, let $\alpha \in \mathrm{Ord}(n, k+1)$, $\beta \in \mathrm{Ord}(k+1, k)$, and $g < h$ the unique pair in $k+1$ such that $\beta(g) = \beta(h)$. For $w = w_1 \otimes w_2 \otimes \cdots \otimes w_{k+1}$ an element of $\varepsilon\mathrm{Lie}(\alpha^{-1}(1)) \otimes \cdots \otimes \varepsilon\mathrm{Lie}(\alpha^{-1}(k+1))$, define

$$\partial_\beta(w) = (-1)^{g+1} w_1 \otimes w_2 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes w_{h-1} \otimes [w_g, w_h] \otimes \cdots \otimes w_{k+1}.$$

Note, $\beta \circ \alpha \in \mathrm{Ord}(n, k)$. For $x = (x_1, x_2, \ldots, x_n)^*$ a basis element of $\widetilde{R}\,[X^{\alpha}]^*$, define

$$t_\beta(\alpha; x) = \begin{cases} 0 & \text{if } x \notin \widetilde{R}\left[X^{\beta\circ\alpha}\right]^* \\ x & \text{else} \end{cases}$$

We define the boundary map for the chain complex by

$$\partial_{n-k}(w \otimes x) = \sum_{\beta \in \mathrm{Ord}(k+1,k)} \partial_\beta(w) \otimes t_\beta(\alpha; x).$$

Example 4.1.2. Consider

$$w \otimes \vec{x} = 1 \otimes 2 \otimes 3 \otimes 4 \otimes 5 \otimes (x, x, y, x, y)^*$$

$$\begin{aligned} \partial_4(w \otimes \vec{x}) = \big(&[1,2] \otimes 3 \otimes 4 \otimes 5 + 2 \otimes 3 \otimes [1,4] \otimes 5 \\ &- 1 \otimes 3 \otimes [2,4] \otimes 5 + 1 \otimes 2 \otimes 4 \otimes [3,5]\big) \otimes (x, x, y, x, y)^* \end{aligned}$$

Proposition 4.1.3. *$A_*(n)(X)$ is a chain complex.*

Proposition 4.1.3 is proved by observing that the terms of $\partial \circ \partial(w \otimes x)$ can be grouped based on surjections and then showing cancellation either due to signs or Lie bracket relations, as shown in the following example.

Example 4.1.4. Continuing from Example 4.1.2,

$$w \otimes \vec{x} = 1 \otimes 2 \otimes 3 \otimes 4 \otimes 5 \otimes (x, x, y, x, y)^*.$$

Apply the boundary map a second time:

$$\begin{aligned}
\partial_3 \circ \partial_4(w \otimes \vec{x}) = {} & 3 \otimes [[1,2],4] \otimes 5 \otimes (y,x,y)^* - [1,2] \otimes 4 \otimes [3,5] \otimes (x,x,y)^* \\
& + 3 \otimes [2,[1,4]] \otimes 5 \otimes (y,x,y)^* - 2 \otimes [1,4] \otimes [3,5] \otimes (x,x,y)^* \\
& - 3 \otimes [1,[2,4]] \otimes 5 \otimes (y,x,y)^* + 1 \otimes [2,4] \otimes [3,5] \otimes (x,x,y)^* \\
& + [1,2] \otimes 4 \otimes [3,5] \otimes (x,x,y)^* + 2 \otimes [14] \otimes [3,5] \otimes (x,x,y)^* \\
& - 1 \otimes [2,4] \otimes [3,5] \otimes (x,x,y)^*.
\end{aligned}$$

Group the terms by the composition β of indexing surjections and see that they all cancel.

$$\begin{aligned}
& \beta = 3|1,2,4|5 : \\
& \qquad + 3 \otimes [[1,2],4] \otimes 5 + 3 \otimes [2,[1,4]] \otimes 5 - 3 \otimes [1,[2,4]] \otimes 5 \\
& \beta = 1,2|4|3,5 : \\
& \qquad - [1,2] \otimes 4 \otimes [3,5] + [1,2] \otimes 4 \otimes [3,5] \\
& \beta = 2|1,4|3,5 : \\
& \qquad - 2 \otimes [1,4] \otimes [3,5] + 2 \otimes [14] \otimes [3,5] \\
& \beta = 1|2,4|3,5 : \\
& \qquad + 1 \otimes [2,4] \otimes [3,5] - 1 \otimes [2,4] \otimes [3,5]
\end{aligned}$$

Proof of Proposition 4.1.3. Fix an ordered surjection $\alpha : n \twoheadrightarrow k+1$ and let

$$w \in \varepsilon\mathrm{Lie}(\alpha^{-1}(1)) \otimes \cdots \otimes \varepsilon\mathrm{Lie}(\alpha^{-1}(k+1))$$

be a basis element, i.e. a pure tensor where $w = w_1 \otimes \cdots \otimes w_{k+1}$ and each w_i is a Lie bracket in $\varepsilon\mathrm{Lie}(\alpha^{-1}(i))$. Let $x = (x_1, x_2, \ldots, x_n)^*$ be a basis element of $\widetilde{R}\,[X^\alpha]^*$.

$$\partial \circ \partial(w \otimes x) = \sum_{\varphi' \in \mathrm{Ord}(k,k-1)} \sum_{\varphi \in \mathrm{Ord}(k+1,k)} \partial_{\varphi'} \circ \partial_\varphi(w) \otimes t_{\varphi'} \circ t_\varphi(\alpha; x).$$

The terms $\partial_{\varphi'}\partial_\varphi$ of $\partial \circ \partial$ can be partitioned based on the composition of the indexing surjections,

$$\beta = \varphi' \circ \varphi : k+1 \twoheadrightarrow k-1.$$

We will show that the terms of each block of this partition cancel.

Consider $\beta : k+1 \twoheadrightarrow k-1$. We can factor β as a composition of ordered surjections in either two ways or three ways, depending on which of the following two types of surjection it is.

We will say β is Type 1 if there exists $1 \leq a \leq k-1$ such that $|\beta^{-1}(a)| = 3$. The first β in Example 4.1.4 was Type 1.

Let $\{g, h, i\} = \beta^{-1}(a)$ and $g < h < i$. There are three ways of factoring β, corresponding to each choice of a pair of elements that map to the same element in k, i.e. a choice of φ. Let these factorizations be $\varphi' \circ \varphi$ where $\varphi(g) = \varphi(h)$, $\psi' \circ \psi$ where $\psi(g) = \psi(i)$, and $\gamma' \circ \gamma$ where $\gamma(h) = \gamma(i)$. We can describe all of these pairs of surjections explicitly.

$$\varphi(\ell) = \begin{cases} \ell & \text{if } \ell < g \\ h-1 & \text{if } \ell = g \\ \ell - 1 & \text{if } g < \ell \end{cases}, \varphi'(m) = \begin{cases} m & \text{if } m < \varphi(h) = h-1 \\ \varphi(i) - 1 = i-2 & \text{if } m = h-1 \\ m-1 & \text{if } h-1 < m \end{cases}$$

$$\psi(\ell) = \begin{cases} \ell & \text{if } \ell < g \\ i-1 & \text{if } \ell = g \\ \ell - 1 & \text{if } g < \ell \end{cases}, \psi'(m) = \begin{cases} m & \text{if } m < \psi(h) = h-1 \\ \psi(i) - 1 = i-2 & \text{if } m = h-1 \\ m-1 & \text{if } h-1 < m \end{cases}$$

$$\gamma(\ell) = \begin{cases} \ell & \text{if } \ell < h \\ i-1 & \text{if } \ell = h \\ \ell - 1 & \text{if } h < \ell \end{cases}, \gamma'(m) = \begin{cases} m & \text{if } m < \gamma(g) = g \\ \gamma(i) - 1 = i-2 & \text{if } m = g \\ m-1 & \text{if } g < m \end{cases}$$

We can describe the corresponding boundary map terms. Note that if $x \in \widetilde{R}\left[X^{\beta\circ\alpha}\right]^*$, then it is also in $\widetilde{R}\left[X^{\varphi\circ\alpha}\right]^*$, $\widetilde{R}\left[X^{\psi\circ\alpha}\right]^*$, and $\widetilde{R}\left[X^{\gamma\circ\alpha}\right]^*$. If $x \notin \widetilde{R}\left[X^{\beta\circ\alpha}\right]^*$, then $t_{\varphi'}t_{\varphi}(\alpha; x) = t_{\psi'}t_{\psi}(\alpha; x) = t_{\gamma'}t_{\gamma}(\alpha; x) = 0$, and trivially the corresponding boundary map terms are 0. Assume $x \in \widetilde{R}\left[X^{\beta\circ\alpha}\right]^*$.

$$\begin{aligned} &\partial_{\varphi'} \circ \partial_{\varphi}(w) \otimes t_{\varphi'}t_{\varphi}(x) \\ &= \partial_{\varphi'}\left((-1)^{g+1} w_1 \otimes \cdots \otimes w_{g-1} \otimes \widehat{w_g} \otimes \cdots \otimes w_{h-1} \otimes [w_g, w_h] \otimes \cdots \otimes w_i \otimes \cdots \otimes w_{k+1}\right) \otimes x \\ &= (-1)^{(h-1)+1+g+1} w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes \widehat{w_h} \otimes \cdots \otimes w_{i-1} \otimes [[w_g, w_h], w_i] \otimes \cdots \otimes w_{k+1} \otimes x \end{aligned}$$

$$\begin{aligned} &\partial_{\psi'} \circ \partial_{\psi}(w) \otimes t_{\psi'}t_{\psi}(x) \\ &= \partial_{\psi'}\left((-1)^{g+1} w_1 \otimes \cdots \otimes w_{g-1} \otimes \widehat{w_g} \otimes \cdots \otimes \otimes w_h \otimes \cdots \otimes w_{i-1} \otimes [w_g, w_i] \otimes \cdots \otimes w_{k+1}\right) \otimes x \\ &= (-1)^{(h-1)+1+g+1} w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes \widehat{w_h} \otimes \cdots \otimes w_{i-1} \otimes [w_h, [w_g, w_i]] \otimes \cdots \otimes w_{k+1} \otimes x \end{aligned}$$

$$\begin{aligned} &\partial_{\gamma'} \circ \partial_{\gamma}(w) \otimes t_{\gamma'}t_{\gamma}(x) \\ &= \partial_{\gamma'}\left((-1)^{h+1} w_1 \otimes \cdots \otimes w_g \otimes \cdots \otimes w_{h-1} \otimes \widehat{w_h} \otimes \cdots \otimes w_{i-1} \otimes [w_h, w_i] \otimes \cdots \otimes w_{k+1}\right) \otimes x \\ &= (-1)^{g+1+h+1} w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes \widehat{w_h} \otimes \cdots \otimes w_{i-1} \otimes [w_g, [w_h, w_i]] \otimes \cdots \otimes w_{k+1} \otimes x \end{aligned}$$

Adding the three together, we have

$$w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes \widehat{w_h} \otimes \cdots \otimes w_{i-1} \otimes u \otimes \cdots \otimes w_{k+1} \otimes x,$$

where

$$u = (-1)^{h+g+2} \left(-[[w_g, w_h], w_i] - [w_h, [w_g, w_i]] + [w_g, [w_h, w_i]]\right).$$

From the Jacobi relation, we get

$$0 = [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = -[[b, c], a] + [b, [c, a]] - [c, [b, a]].$$

Thus, $u = 0$ and it follows that

$$\partial_{\varphi'} \circ \partial_\varphi(w) + \partial_{\psi'} \circ \partial_\psi(w) + \partial_{\gamma'} \circ \partial_\gamma(w) = 0.$$

This completes the proof for the Type 1 case. We now consider the second case.

We say β is Type 2 if there exist $1 \le a < b \le k-1$ such that $|\beta^{-1}(a)| = |\beta^{-1}(b)| = 2$. The last three β in Example 4.1.4 were Type 2.

Let $\{g, h\} = \beta^{-1}(a)$, with $g < h$, and $\{i, j\} = \beta^{-1}(b)$, with $i < j$.

There are two ways of factoring β, namely, $\varphi' \circ \varphi$ where $\varphi(g) = \varphi(h)$, and $\psi' \circ \psi$ where $\psi(i) = \psi(j)$. We can describe these surjections explicitly.

$$\varphi(\ell) = \begin{cases} \ell & \text{if } \ell < g \\ h-1 & \text{if } \ell = g \\ \ell - 1 & \text{if } g < \ell \end{cases}, \; \varphi'(m) = \begin{cases} m & \text{if } m < \varphi(i) \\ \varphi(j) - 1 & \text{if } m = \varphi(i) \\ m-1 & \text{if } \varphi(i) < m \end{cases}$$

and

$$\psi(\ell) = \begin{cases} \ell & \text{if } \ell < i \\ j-1 & \text{if } \ell = i \\ \ell - 1 & \text{if } i < \ell \end{cases}, \text{ so } \psi'(m) = \begin{cases} m & \text{if } m < \psi(g) \\ \psi(h) - 1 & \text{if } m = \psi(g) \\ m-1 & \text{if } \psi(g) < m \end{cases}$$

Because of the antisymmetry relationship for Lie brackets and the pairs of surjections came from factoring β, both $\partial_{\varphi'}\partial_\varphi(w)$ and $\partial_{\psi'}\partial_\psi(w)$ will result in the same element of $\varepsilon\mathrm{Lie}_{(\beta\alpha)^{-1}}$, up to a sign. For example, when $g < h < i < j$ we have

$$\pm w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes [w_g, w_h] \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [w_i, w_j] \otimes \cdots \otimes w_{k+1},$$

when $g < i < h < j$ we have

$$\pm w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [w_g, w_h] \otimes \cdots \otimes [w_i, w_j] \otimes \cdots \otimes w_{k+1},$$

etc.

Similarly to Part 1, we only have non-trivial terms if $x \in \widetilde{R}\left[X^{\beta\circ\alpha}\right]$. What remains is to check that the two factorizations result in different signs, and thus in canceling terms.

The signs can be calculated as follows

$$sgn(\partial_{\varphi'} \circ \partial_{\varphi}) = sgn(\partial_{\varphi'})sgn(\partial_{\varphi}) = (-1)^{\varphi(i)+1}(-1)^{g+1} = (-1)^{\varphi(i)+g+2}$$

and

$$sgn(\partial_{\psi'} \circ \partial_{\psi}) = sgn(\partial_{\psi'})sgn(\partial_{\psi}) = (-1)^{\psi(g)+1}(-1)^{i+1} = (-1)^{i+\psi(g)+2}.$$

If $g < i$,

$$\varphi(i) + g + 2 = i - 1 + g + 2 = i + g + 1 \text{ and } \psi(g) + i + 2 = g + i + 2.$$

If $i < g$

$$\varphi(i) + g + 2 = i + g + 2 \text{ and } \psi(g) + i + 2 = g - 1 + i + 2 = g + i + 1.$$

So, in both cases

$$sgn(\partial_{\varphi'} \circ \partial_{\varphi}) = -sgn(\partial_{\psi'} \circ \partial_{\psi})$$

as desired.

□

4.2 Σ_n equivariance

4.2.1 Alternative notation for the complex

Any basis element $w \otimes x$ of

$$A_{n-k}(n)^*(X) = \bigoplus_{\alpha \in \mathrm{Ord}(\mathbf{n},\mathbf{k})} \varepsilon\mathrm{Lie}_{\alpha} \otimes \widetilde{R}\left[\wedge^k X\right]^*,$$

corresponds to some ordered surjection α. In this section, we will write $(\alpha; w \otimes x)$ for $w \otimes x$ to improve the bookkeeping.

Recall that for an arbitrary surjection $\varphi : \mathbf{n} \twoheadrightarrow \mathbf{k}$, there is a unique factorization of $\varphi = \sigma \circ \alpha$ where $\sigma \in \Sigma_k$

and $\alpha \in \mathrm{Ord}(n,k)$. These unique factorizations give the isomorphism

$$A_{n-k}(n)^n(X) \cong \bigoplus_{\varphi \in Surj(n,k)} \varepsilon\mathrm{Lie}_\varphi \otimes \widetilde{R}\,[X^\varphi]^* \,/\sim,$$

where

$$(\varphi; w_1 \otimes \cdots \otimes w_k \otimes x) \sim (-1)^{|\sigma|}(\alpha; w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)} \otimes x)$$

for all $\sigma \in \Sigma_k$. So, we can consider representatives, $(\varphi; w_1 \otimes \cdots \otimes w_k \otimes x)$, of these equivalence classes such that φ is not ordered.

Example 4.2.1. For $n = 3$.

$$1 \otimes 2 \otimes 3 \otimes (x,y,z) \sim -2 \otimes 1 \otimes 3 \otimes (x,y,z) \sim 2 \otimes 3 \otimes 1 \otimes (x,y,z)$$

$$[1,2] \otimes 3 \otimes (x,x,y) \sim -3 \otimes [1,2] \otimes (x,x,y)$$

4.2.2 Symmetric group action on $A_k(n)(X)$

There is an action of Σ_n on $A_{n-k}(n)(X)$ by permuting letters, which will be denoted with $\cdot$, and an action of Σ_k on $A_{n-k}(n)(X)$ permuting blocks (w_i), which will be denoted with $\star$.

Fix $\varphi \in Surj(n,k)$, a basis element $w \in \varepsilon\mathrm{Lie}_\varphi$, and a basis element $x \in \widetilde{R}\,[X^\varphi]^*$. Recall, $w = w_1 \otimes \cdots \otimes w_k$, where $w_i \in \varepsilon\mathrm{Lie}(\varphi^{-1}(i))$ and $x = (x_1, \ldots, x_n)^*$. Adapting notation from Lie monomials in Section 3.3.2,

$$\begin{aligned} w &= (w_1 \otimes \cdots \otimes w_k)(a_{\tau(1)}, \ldots, a_{\tau(n)}) \\ &= w_1(a_{\tau(1)}, \ldots, a_{\tau(d_1)}) \otimes \cdots \otimes w_k(a_{\tau(d_{k-1}+1)}, \ldots, a_{\tau(n)}) \end{aligned}$$

where τ is some ordering of the alphabet $\mathbf{n}$, $d_i = |\varphi^{-1}\{1,2,\ldots,i\}|$ and $\{a_{\tau(d_{i-1}+1)}, \ldots, a_{\tau(d_i)}\} = \varphi^{-1}(i)$.

The Σ_n action is defined by

$$\begin{aligned} \rho \cdot (w \otimes x) &= (\rho \cdot w) \otimes \rho^*(x) \\ &= (-1)^{|\rho|} w\left(a_{\rho\circ\tau(1)}, \ldots, a_{\rho\circ\tau(n)}\right) \otimes \left(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)}\right)^* \end{aligned}$$

for $\rho \in \Sigma_n$. Note, $\rho \cdot w \in \mathrm{Lie}_{\varphi\circ\rho^{-1}}$.

Example 4.2.2. Let

$$w \otimes x = [1,3] \otimes 4 \otimes [2,5] \otimes (x,z,x,y,z)^*.$$

For $\rho_1 = (23)$,

$$\rho_1 \cdot (w \otimes x) = -[1,2] \otimes 4 \otimes [3,5] \otimes (x,x,z,y,z)^*.$$

For $\rho_2 = (35)$,

$$\begin{aligned}\rho_2 \cdot (w \otimes x) &= -[1,5] \otimes 4 \otimes [2,3] \otimes (x,z,z,y,x)^* \\ &\sim [2,3] \otimes 4 \otimes [1,5] \otimes (x,z,z,y,x)^*.\end{aligned}$$

For $\rho_3 = (35)(34)$,

$$\begin{aligned}\rho_3 \cdot (w \otimes x) &= [1,4] \otimes 5 \otimes [2,3] \otimes (x,z,z,x,y)^* \\ &\sim [2,3] \otimes [1,4] \otimes 5 \otimes (x,z,z,x,y)^*.\end{aligned}$$

Suppose $1 \leq i < j \leq n$ such that $\varphi \circ \rho^{-1}(i) = \varphi \circ \rho^{-1}(j)$. If $x \in \widetilde{R}\,[X^{\varphi}]^*$, then by definition $x_{\rho^{-1}(i)} = x_{\rho^{-1}(j)}$. So, whenever $x \in \widetilde{R}\,[X^{\varphi}]^*$, $\rho^*(x) \in \widetilde{R}\left[X^{\varphi \circ \rho^{-1}}\right]^*$.

The Σ_k action is defined by

$$\sigma \star (\varphi; w \otimes x) = (-1)^{sgn(\sigma)}(\sigma \circ \varphi; w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k)} \otimes x).$$

The equivalence relation in Section 4.2.1 can be written as

$$\sigma \star (\varphi; w \otimes x) \sim (\varphi; w \otimes x).$$

Observation 4.2.3. Because $\sigma \in \Sigma_k$ permutes the various Lie blocks and $\rho \in \Sigma_n$ permutes the letters that are plugged into the blocks,

$$\rho \cdot (\sigma \star w \otimes x) = \sigma \star (\rho \cdot w \otimes x).$$

Alternatively, this could be argued by the associativity of composition, $\sigma \circ (\varphi \circ \rho) = (\sigma \circ \varphi) \circ \rho$.

4.2.3 Alternative Notation for ∂

To work with the symmetric group action, it will be helpful to write the map ∂ in terms of indexes, rather than surjections.

For $1 \leq g < h \leq k$, let $s_{g,h} : \mathbf{k} \twoheadrightarrow \mathbf{k-1}$ be the ordered surjection such that $g \mapsto h-1$ and $h \mapsto h-1$. Let $s_{g,h} = s_{h,g}$.

Let $\varphi : \mathbf{n} \twoheadrightarrow \mathbf{k}$ be a surjection, possibly not ordered. Define

$$\mathfrak{Br}_{g,h} : \varepsilon\mathrm{Lie}_{\varphi} \to \varepsilon\mathrm{Lie}_{s_{g,h} \circ \varphi}$$

by

$$\mathfrak{Br}_{g,h}(\varphi; w) = (s_{g,h} \circ \varphi; w_1 \otimes w_2 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes w_{h-1} \otimes [w_g, w_h] \otimes \cdots \otimes w_k),$$

where $w = w_1 \otimes w_2 \otimes \cdots \otimes w_k \in \varepsilon\mathrm{Lie}_\varphi$ is a basis element. Let $\mathfrak{Br}_{h,g} = \mathfrak{Br}_{g,h}$.

This $\mathfrak{Br}_{g,h}$ notation can be used to write an equivalent definition of ∂. Let $\alpha \in \mathrm{Ord}(n,k)$, and $1 \leq g < h \leq k$. For $w = w_1 \otimes w_2 \otimes \cdots \otimes w_k$ an element of $\varepsilon\mathrm{Lie}_\alpha$,

$$\partial_{s_{g,h}}(w) = (-1)^{g+1}\mathfrak{Br}_{g,h}(w)$$

and so

$$\partial_k(\alpha; w \otimes x) = \sum_{1 \leq g < h \leq k+1} (-1)^{g+1}(s_{g,h} \circ \alpha; \mathfrak{Br}_{g,h}(w) \otimes t_{g,h}(\alpha; x)),$$

where

$$t_{g,h}(\alpha; x) = \begin{cases} x & \text{if } x \in \widetilde{R}\,[X^{s_{g,h}\circ\alpha}]^* \\ 0 & \text{else} \end{cases}$$

for $x = (x_1, x_2, \ldots, x_n)^*$ a basis element of $\widetilde{R}\,[X^\alpha]^*$

This definition is only for $w \in \varepsilon\mathrm{Lie}_\alpha$ where α is ordered. If $\sigma \in \Sigma_k$ and $w \in \mathrm{Lie}_{\sigma\circ\alpha}$, then

$$\begin{aligned}(\sigma \circ \alpha; w \otimes x) &= (\sigma \circ \alpha; w_1 \otimes \cdots \otimes w_k \otimes x) \\ &\sim (-1)^{|\sigma|}(\alpha; w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)} \otimes x) \\ &= \sigma^{-1} \star (\sigma \circ \alpha; w \otimes x)\end{aligned}$$

so we use

$$\begin{aligned}\partial_k(\sigma \circ \alpha; w \otimes x) &= \partial_k(\sigma^{-1} \star (\sigma \circ \alpha; w \otimes x)) \\ &= (-1)^{|\sigma|}\partial_k(\alpha; w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)} \otimes x).\end{aligned}$$

4.2.4 Naturality

For an injection $\iota : [m] \to [n]$ and for $\alpha \in \mathcal{I}^n(X)$, $\alpha \circ \iota \in \mathcal{I}^m(X)$. So, ι induces a map $\iota^* : \mathcal{I}^n(X) \to \mathcal{I}^m(X)$ and dualizing gives us a map $\iota_* : \mathcal{I}^m(X)^* \to \mathcal{I}^n(X)^*$. We want to show $\partial \circ \iota_* = \iota_* \circ \partial$, i.e. naturality with respect to all injections.

The injection $\iota : m \to n$ can be factored as a composition $\sigma_\iota \circ \iota_{n-m} \circ \cdots \circ \iota_1$, where $\sigma_\iota \in \Sigma_n$ and $\iota_i : [m+i-1] \to [m+i]$ is defined by $\iota_i(j) = j$ for all $j \in [m+i-1]$. The choice of permutation σ_ι may not be unique, but we will define it to be the unique permutation so that $\sigma_\iota \circ \iota_{n-m} \circ \cdots \circ \iota_1 = \iota$ and σ_ι is strictly order preserving when its domain is restricted to $\{m+1, m+2, \ldots, n\}$. Using this factorization, we need only check that $\partial \circ (\iota_i)_* = (\iota_i)_* \circ \partial$ for all i and that $\partial \circ \sigma_\iota = \sigma_\iota \circ \partial$.

Proposition 4.2.4. *For $\rho \in \Sigma_n$,*

$$\partial(\rho \cdot (w \otimes x)) = \rho \cdot \partial(w \otimes x).$$

In order to prove Proposition 4.2.4, we first prove several supporting lemmas.

Let $\varphi : \mathbf{n} \twoheadrightarrow \mathbf{k}$, $w \in \varepsilon\mathrm{Lie}_\varphi$, $x \in \widetilde{R}\,[X^\varphi]^*$, and $\rho \in \Sigma_n$. For now, φ does not have to be ordered. There exists a unique $\sigma \in \Sigma_k$ such that $\sigma \circ \varphi$ is ordered.

Recall,

$$\begin{aligned} \sigma \star (\varphi; w) &= \sigma \star (\varphi; w_1 \otimes \cdots \otimes w_k) \\ &= (-1)^{|\sigma|}(\sigma \circ \varphi; w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k)}). \end{aligned}$$

Combining this with the definition of ∂,

$$\begin{aligned} \partial(\varphi; w \otimes x) &= \partial\left(\sigma \star (\varphi; w \otimes x)\right) \\ &= \sum_{1 \le g < h \le k+1} (-1)^{g+1} \mathfrak{Br}_{g,h}\left(\sigma \star (\varphi; w)\right) \otimes t_{g,h}(\sigma \circ \varphi; x) \\ &= \sum_{1 \le g < h \le k+1} (-1)^{g+1+|\sigma|} \mathfrak{Br}_{g,h}(\sigma \circ \varphi; w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k)}) \otimes t_{g,h}(\sigma \circ \varphi; x) \end{aligned} \tag{4.1}$$

Lemma 4.2.5.

$$t_{g,h}(\sigma \circ \varphi; x) = t_{\sigma^{-1}(g), \sigma^{-1}(h)}(\varphi; x)$$

Proof. The surjection φ determines a partition of $\mathbf{n}$ into $\mathbf{k}$ ordered blocks, $B_i = \varphi^{-1}(i)$. The composition $\sigma \circ \varphi$ determines the same partition with a different order on the blocks. If $x \in \widetilde{R}\,[X^\varphi]^*$, then x associates the same element of X to every element in a block of the partition. The map $t_{g_h}(\varphi; -)$ checks whether the g-th block has the same element of X associated to it as the h-th block.

If $B_1, \ldots, B_k$ are the blocks from φ and $B'_1, \ldots, B'_k$ are the blocks from $\sigma \circ \varphi$, then $B'_i = B_{\sigma^{-1}(i)}$. Therefore, checking that B'_g and B'_h are associated with the same element of X is equivalent to checking $B_{\sigma^{-1}(g)}$ and $B_{\sigma^{-1}(h)}$. $\square$

Lemma 4.2.6.

$$t_{gh}(\rho \cdot x; \varphi \circ \rho^{-1}) = \rho \cdot (t_{gh}(x; \varphi))$$

Proof. Recall, $\rho \cdot x = (x_{\rho^1(1)}, \ldots, x_{\rho^{-1}(n)})^*$ We consider $x \in \widetilde{R}\,[X^\varphi]^*$. There is an ordered partition $B_1, \ldots, B_{k+1}$ of $\mathbf{n}$ defined by φ. So $x = (x_1, \ldots, x_n)^*$ has an element of X for each element of $\mathbf{n}$. If $x \in \widetilde{R}\,[X^\varphi]$, then x assigns a single element of X to each block of the partition. Then $t_{g,h}$ tests whether B_g and B_h are assigned the same element of X. The surjection $\varphi \circ \rho^{-1}$ defines another partition $B'_1, \ldots, B'_{k+1}$. $x \in \widetilde{R}\,[X^\varphi]$ if and only if $\rho(x) \in \widetilde{R}\left[X^{\varphi \circ \rho^{-1}}\right]$. In fact, x assigns the same element of X to B_i as $\rho \cdot x$ does to B'_i. So, B_g and B_h have same corresponding element of X exactly when B'_g and B'_h do. So, $t_{gh}(\rho \cdot x) = 0 = \rho \cdot t_{g,h}(x)$ or $t_{g,h}(\rho \cdot x) = \rho \cdot x = \rho \cdot t_{g,h}(x)$. $\square$

Example 4.2.7.

$$\sigma: \begin{array}{c} 2 \mapsto 3 \\ 1 \mapsto 2 \\ 3 \mapsto 1 \end{array}$$

$$(-1)^{|\sigma|+g+1}\mathfrak{Br}_{g,h}\left(w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k)}\right) = (-1)^{2+g+1}\mathfrak{Br}_{g,h}(w_3 \otimes w_1 \otimes w_2)$$

$g = 1, h = 2$:

$$\begin{aligned}(-1)^{2+1+1}\mathfrak{Br}_{1,2}(w_3 \otimes w_1 \otimes w_2) &= [w_3, w_1] \otimes w_2 \\ &\sim -w_2 \otimes [w_3, w_1] \\ &\sim w_2 \otimes [w_1, w_3] \\ &= \mathfrak{Br}_{1,3}(w_1 \otimes w_2 \otimes w_3)\end{aligned}$$

$g = 1, h = 3$:

$$\begin{aligned}(-1)^{2+1+1}\mathfrak{Br}_{1,3}(w_3 \otimes w_1 \otimes w_2) &= w_1 \otimes [w_3, w_2] \\ &\sim -w_1 \otimes [w_2, w_3] \\ &= (-1)^{2+1}\mathfrak{Br}_{2,3}(w_1 \otimes w_2 \otimes w_3)\end{aligned}$$

$g = 2, h = 3$:

$$\begin{aligned}(-1)^{2+2+1}\mathfrak{Br}_{2,3}(w_3 \otimes w_1 \otimes w_2) &= -w_3 \otimes [w_1, w_2] \\ &\sim [w_1, w_2] \otimes w_3 \\ &= \mathfrak{Br}_{1,2}(w_1 \otimes w_2 \otimes w_3)\end{aligned}$$

Lemma 4.2.8. *$w = w_1 \otimes w_2 \otimes \cdots \otimes w_k \in \varepsilon Lie_\varphi$, with w possibly not ordered.*

$$(-1)^{|\sigma|+g+1}\mathfrak{Br}_{g,h}\left(w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k)}\right) = (-1)^{min(\sigma^{-1}(g,h))+1}\mathfrak{Br}_{\sigma^{-1}(g,h)}(w)$$

Proof. Consider a transposition $\tau = (\ a\ b\ ) \in \Sigma_k$ and suppose that $a < b$. Applying the transposition to w, we have

$$\tau \star w = -w_1 \otimes \cdots \otimes w_{a-1} \otimes w_b \otimes w_{a+1} \otimes \cdots \otimes w_{b-1} \otimes w_a \otimes w_{b+1} \otimes \cdots \otimes w_k$$

Let $1 \leq g < h \leq k$. We want to show

$$\mathfrak{Br}_{g,h}(\tau \star w) = (-1)^{min(\tau(g),\tau(h))+g}\mathfrak{Br}_{\tau(g),\tau(h)}(w). \tag{4.2}$$

If $\{g, h\}$ and $\{a, b\}$ are disjoint, then $\mathfrak{Br}_{g,h}(\tau \star w) \sim \mathfrak{Br}_{g,h}(w)$.

Suppose $b = g$.

$$\begin{aligned}\mathfrak{Br}_{g,h}(\tau \star w) &= \mathfrak{Br}_{g,h}(\tau \star w_1 \otimes \cdots \otimes w_{a-1} \otimes w_a \otimes w_{a+1} \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_h \otimes \cdots \otimes w_k) \\ &= -\mathfrak{Br}_{g,h}(w_1 \otimes \cdots \otimes w_{a-1} \otimes w_g \otimes w_{a+1} \otimes \cdots \otimes w_{g-1} \otimes w_a \otimes w_{g+1} \otimes \cdots \otimes w_h \otimes \cdots \otimes w_k) \\ &= -w_1 \otimes \cdots \otimes w_{a-1} \otimes w_g \otimes w_{a+1} \otimes \cdots \otimes w_{g-1} \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes [w_a, w_h] \otimes \cdots \otimes w_k\end{aligned}$$

In order to make this look more like $\mathfrak{Br}$ applied to w, move w_g to be between w_{g-1} and w_{g+1}. To do this, apply $g - 1 - a$ transpositions to switch w_g with w_{a+1}, then w_g with w_{a+2} and so on.

$$\begin{aligned}&= (-1)^{g-1-a+1} w_1 \otimes \cdots \otimes w_{a-1} \otimes w_{a+1} \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes [w_a, w_h] \otimes \cdots \otimes w_k \\ &= (-1)^{a+g} \mathfrak{Br}_{a,h}(w) \\ &= (-1^{\tau(g)+g} \mathfrak{Br}_{\tau(g),h}(w).\end{aligned}$$

Suppose $a = h$.

$$\begin{aligned}\mathfrak{Br}_{g,h}(\tau \star w) &= \mathfrak{Br}_{g,h}(\tau \star w_1 \otimes \cdots \otimes w_g \otimes \cdots \otimes w_{h-1} \otimes w_h \otimes w_{h+1} \otimes \cdots \otimes w_{b-1} \otimes w_b \otimes w_{b+1} \otimes \cdots \otimes w_k) \\ &= -\mathfrak{Br}_{g,h}(w_1 \otimes \cdots \otimes w_g \otimes \cdots \otimes w_{h-1} \otimes w_b \otimes w_{h+1} \otimes \cdots \otimes w_{b-1} \otimes w_h \otimes w_{b+1} \otimes \cdots \otimes w_k) \\ &= -w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes w_{h-1} \otimes [w_g, w_b] \otimes w_{h+1} \otimes \cdots \otimes w_{b-1} \otimes w_h \otimes \cdots \otimes w_k \\ &\sim w_1 \otimes \cdots \otimes \widehat{w_g} \otimes \cdots \otimes w_{h-1} \otimes w_h \otimes \cdots \otimes w_{b-1} \otimes [w_g, w_b] \otimes \cdots \otimes w_k \\ &= \mathfrak{Br}_{g,b}(w)\end{aligned}$$

Suppose $a = g$. We have

$$\begin{aligned}\mathfrak{Br}_{g,h}(\tau \star w) &= \mathfrak{Br}_{g,h}(\tau \star w_1 \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_{b-1} \otimes w_b \otimes w_{b+1} \otimes \cdots \otimes w_k) \\ &= -\mathfrak{Br}_{g,h}(w_1 \otimes \cdots \otimes w_{g-1} \otimes w_b \otimes w_{g+1} \otimes \cdots \otimes w_{b-1} \otimes w_g \otimes w_{b+1} \otimes \cdots \otimes w_k).\end{aligned}$$

Case $b < h$:

$$\begin{aligned}&\mathfrak{Br}_{g,h}(\tau \star w) \\ &= \mathfrak{Br}_{g,h}(\tau \star w_1 \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_{b-1} \otimes w_b \otimes w_{b+1} \otimes \cdots \otimes w_h \otimes \cdots \otimes w_k) \\ &= -\mathfrak{Br}_{g,h}(w_1 \otimes \cdots \otimes w_{g-1} \otimes w_b \otimes w_{g+1} \otimes \cdots \otimes w_{b-1} \otimes w_g \otimes w_{b+1} \otimes \cdots \otimes w_h \otimes \cdots \otimes w_k) \\ &= -w_1 \otimes \cdots \otimes w_{g-1} \otimes w_{g+1} \otimes \cdots \otimes w_{b-1} \otimes w_g \otimes w_{b+1} \otimes \cdots \otimes w_{h-1} \otimes [w_b, w_h] \otimes w_{h+1} \otimes \cdots \otimes w_k.\end{aligned}$$

We want to move w_g to be in between w_{g-1} and w_{g+1} in order to make this look more like a bracketing of w. To do this, we apply $b-1-g$ transpositions to switch w_g with w_{b-1}, then switch w_g with w_{b-2} and so on. We arrive at

$$\begin{aligned}
&\mathfrak{Br}_{g,h}(\tau \star w) \\
&= (-1)^{b-g} w_1 \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_{b-1} \otimes w_{b+1} \otimes \cdots \otimes w_{h-1} \otimes [w_b, w_h] \otimes w_{h+1} \otimes \cdots \otimes w_k \\
&= (-1)^{b-g} \mathfrak{Br}_{b,h}(w) \\
&= (-1)^{\tau(g)+g} \mathfrak{Br}_{\tau(g),h}(w).
\end{aligned}$$

Case $b = h$**:**

$$\begin{aligned}
\mathfrak{Br}_{g,h}(\tau \star w) &= \mathfrak{Br}_{g,h}(\tau \star w_1 \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes w_h \otimes w_{h+1} \otimes \cdots \otimes w_k) \\
&= -\mathfrak{Br}_{g,h}(w_1 \otimes \cdots \otimes w_{g-1} \otimes w_h \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes w_g \otimes w_{h+1} \otimes \cdots \otimes w_k) \\
&= -w_1 \otimes \cdots \otimes w_{g-1} \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes [w_h, w_g] \otimes w_{h+1} \otimes \cdots \otimes w_k
\end{aligned}$$

Using Lie bracket properties, $[w_h, w_g] = -[w_g, w_h]$.

$$\begin{aligned}
&= w_1 \otimes \cdots \otimes w_{g-1} \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes [w_g, w_h] \otimes w_{h+1} \otimes \cdots \otimes w_k \\
&= \mathfrak{Br}_{g,h}(w).
\end{aligned}$$

Case $h < b$**:**

$$\begin{aligned}
\mathfrak{Br}_{g,h}(\tau \star w) &= \mathfrak{Br}_{g,h}(w_1 \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_h \otimes \cdots \otimes w_{b-1} \otimes w_b \otimes w_{b+1} \otimes \cdots \otimes w_k) \\
&= -\mathfrak{Br}_{g,h}(w_1 \otimes \cdots \otimes w_{g-1} \otimes w_b \otimes w_{g+1} \otimes \cdots \otimes w_h \otimes \cdots \otimes w_{b-1} \otimes w_g \otimes w_{b+1} \otimes \cdots \otimes w_k) \\
&= -w_1 \otimes \cdots \otimes w_{g-1} \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes [w_b, w_h] \otimes w_{h+1} \otimes \cdots \otimes w_{b-1} \otimes w_g \otimes w_{b+1} \otimes \cdots \otimes w_k
\end{aligned}$$

Using Lie bracket properties, $[w_h, w_b] = -[w_b, w_h]$

$$= w_1 \otimes \cdots \otimes w_{g-1} \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes [w_h, w_b] \otimes w_{h+1} \otimes \cdots \otimes w_{b-1} \otimes w_g \otimes w_{b+1} \otimes \cdots \otimes w_k$$

In order to make this look more like $\mathfrak{Br}$ applied to w, w_g needs to be between w_{g-1} and w_{g+1}.

$$= (-1)^{b-1-g} w_1 \otimes \cdots \otimes w_{g-1} \otimes w_g \otimes w_{g+1} \otimes \cdots \otimes w_{h-1} \otimes [w_h, w_b] \otimes w_{h+1} \otimes \cdots \otimes w_{b-1} \otimes w_{b+1} \otimes \cdots \otimes w_k$$

Since $b > h$, use transpositions to move $[w_h, w_b]$ to be between w_{b-1} and w_{b+1}

$$\begin{aligned}
&= (-1)^{b-1-g+b-1-h} w_1 \otimes \cdots \otimes w_g \otimes \cdots \otimes w_{h-1} \otimes w_{h+1} \otimes \cdots \otimes w_{b-1} \otimes [w_h, w_b] \otimes w_{b+1} \otimes \cdots \otimes w_k \\
&= (-1)^{h+g} \mathfrak{Br}_{h,b}(w) \\
&= (-1)^{\tau(h)+g} \mathfrak{Br}_{\tau(h),\tau(g)}(w).
\end{aligned}$$

□

Proof of Proposition 4.2.4. Let $\alpha \in \mathrm{Ord}(\mathbf{n}, \mathbf{k})$, $w \in \mathrm{Lie}_\alpha$, $x \in \widetilde{R}[X^\alpha]^*$, and $\rho \in \Sigma_n$. Let $\sigma \in \Sigma_k$ be the unique permutation such that $\sigma \circ \alpha \circ \rho^{-1}$ is ordered.

By applying (4.1)

$$\begin{aligned}
\partial_{g,h}(\rho \cdot w \otimes \rho(x)) &= (-1)^{|\sigma|} \partial_{gh}(\rho \cdot w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k+1)} \otimes \rho \cdot x) \\
&= (-1)^{|\sigma|+g+1} \mathfrak{Br}_{g,h}\left(\rho \cdot w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k+1)}\right) \otimes t_{gh}(\rho \cdot x; \sigma \circ \alpha \circ \rho^{-1})
\end{aligned}$$

ρ commutes with $\mathfrak{Br}$:

$$= (-1)^{|\sigma|+g+1} \rho \cdot \mathfrak{Br}_{g,h}\left(w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k+1)}\right) \otimes t_{gh}(\rho(x); \sigma \circ \alpha \circ \rho^{-1})$$

By Lemma 4.2.8, we can write this as a bracketing of w instead of $\sigma \star w$:

$$= (-1)^{min(\sigma^{-1}(g,h))+1} \rho \cdot \mathfrak{Br}_{\sigma^{-1}(g,h)}(w) \otimes t_{gh}(\rho(x); \sigma \circ \alpha \circ \rho^{-1})$$

By Lemma 4.2.6, ρ commutes with $t_{g,h}$:

$$= \rho \cdot \left((-1)^{min(\sigma^{-1}(g,h))+1} \mathfrak{Br}_{\sigma^{-1}(g,h)}(w) \otimes t_{g,h}(x; \sigma \circ \alpha)\right)$$

By Lemma 4.2.5, we can move σ to the index on t so that the surjection used in t is the same as the one determined by w

$$= \rho \cdot \left((-1)^{min(\sigma^{-1}(g,h))+1} \mathfrak{Br}_{\sigma^{-1}(g,h)}(w) \otimes t_{\sigma^{-1}(g,h)}(x; \alpha)\right)$$

Now, this looks like the definition:

$$= \rho \cdot \partial_{\sigma^{-1}(g,h)}(w \otimes x)$$

□

4.3 Proof of Resolution

Theorem 4.3.1.

$$0 \to A_{n-1}(n) \xrightarrow{\partial^*_{n-1}} \cdots \to A_0(n) \xrightarrow{\partial^*_0} \mathcal{I}^n \to 0$$

is a long exact sequence.

To prove Theorem 4.3.1, we will show there is a natural quasi-isomorphism $\eta : A_\bullet(n) \to \mathcal{I}^n$. We will apply several results from [IJM08] to reduce the quasi-isomorphism problem to the easier problem of showing $\mathrm{cr}_t A_\bullet(n)([1]) \overset{\simeq}{\to} \mathrm{cr}_t \mathcal{I}^n[1]$ for all $1 \leq t \leq n$. By the calculations in Lemmas 4.3.2 and 4.3.3, this can be further reduced to showing $A_\bullet(\ell)([1]) \overset{\simeq}{\to} \mathcal{I}^\ell([1])$ for all $1 \leq \ell \leq n$. For the case $\ell > 1$, since $\mathcal{I}^\ell([1]) \cong 0$, we can further reduce this to showing that $A_\bullet(\ell)([1]) \simeq 0$, which is done in Lemma 4.3.4.

Lemma 4.3.2.

$$cr_t \mathcal{I}^n(X_1, \ldots, X_t) \simeq \bigoplus_{\alpha \in Surj(\mathbf{n},\mathbf{t})} \bigotimes_{i=1}^{t} \mathcal{I}^{|\alpha^{-1}(i)|}(X_i)$$

Proof. Consider a wedge $\bigvee_{i=1}^t X_i$, where $X_1 = \cdots X_t = X$ for some finite set X.

$$\mathcal{I}^n(\bigvee_{i=1}^t X_i) \cong \bigoplus_{\alpha \in Hom(\mathbf{n},\mathbf{t})} \bigotimes_{i=1}^{t} \mathcal{I}^{|\alpha^{-1}(i)|}(X_i)$$

The isomorphism can be seen in the following way. In order to write an injection $\mathbf{n} \to \bigvee X_i$, we can first decide which elements of $\mathbf{n}$ map to each X_i, thus producing an ordered partition $\alpha \in Hom(\mathbf{n}, \mathbf{t})$, and then pick an injection on each X_i.

From the cokernel formulation of cross effect, we see that the only basis elements of $\mathcal{I}^n(\bigvee X_i)$ that survive taking the cokernel are the ones that map an element of $\mathbf{n}$ to each X_i, i.e. such that α is a surjection. □

Lemma 4.3.3.

$$cr_t A_\bullet(n)[1] \simeq \bigoplus_{\alpha \in Surj(\mathbf{n},\mathbf{t})} \bigotimes_{i=1}^{t} A_\bullet(|\alpha^{-1}(i)|)[1]$$

Proof. Consider $A_k(n)$ on a wedge $\vee^t[1]$.

$$\begin{aligned} A_k(n)(\vee^t[1]) &\cong \bigoplus_{\varphi \in \mathrm{Ord}(n,n-k)} \varepsilon\mathrm{Lie}^*_\varphi \otimes \widetilde{R}\left[\wedge^{n-k}(\vee^t[1])\right] \\ &\cong \bigoplus_{\varphi \in \mathrm{Ord}(n,n-k)} \varepsilon\mathrm{Lie}^*_\varphi \otimes (\otimes^{n-k}(\oplus^t \widetilde{R}[1])) \\ &\cong \bigoplus_{\varphi \in \mathrm{Ord}(n,n-k)} \varepsilon\mathrm{Lie}^*_\varphi \otimes \bigoplus_{\beta \in \mathrm{hom}(n-k,t)} \widetilde{R}[1] \end{aligned}$$

Taking the cokernel

$$\mathrm{cr}_t A_k(n)([1],\ldots,[1]) \simeq \bigoplus_{\varphi\in \mathrm{Ord}(n,n-k)} \varepsilon\mathrm{Lie}^*_\varphi \otimes \bigoplus_{\beta\in \mathrm{Surj}(n-k,t)} \widetilde{R}\,[1]$$

$$\simeq \bigoplus_{\varphi\in \mathrm{Ord}(n,n-k)} \bigoplus_{\beta\in \mathrm{Surj}(n-k,t)} \varepsilon\mathrm{Lie}^*_\varphi$$

Let

$$\mathcal{A} = \bigoplus_{\varphi\in \mathrm{Ord}(n,n-k)} \bigoplus_{\beta\in \mathrm{Surj}(n-k,t)} \varepsilon\mathrm{Lie}^*_\varphi$$

and

$$\mathcal{B} = \bigoplus_{\alpha\in \mathrm{Surj}(n,t)} Tot_k(A_\bullet(\alpha^{-1}(1))\otimes\cdots\otimes A_\bullet(\alpha^{-1}(t)))$$

$$= \bigoplus_{\alpha\in \mathrm{Surj}(n,t)} \bigoplus_{k_1+\cdots+k_t=k} A_{k_1}(\alpha^{-1}(1))\otimes\cdots\otimes A_{k_t}(\alpha^{-1}(t)))$$

We will show $\mathcal{A}\cong\mathcal{B}$, motivating the isomorphism with examples.

Example $\mathcal{B}\to\mathcal{A}$: Consider $n=5, k=2, t=2$. Let $(\alpha,\vec{k};u\otimes v)$ be element of

$$\mathcal{B} = \bigoplus_{\alpha\in \mathrm{Surj}(5,2)} \bigoplus_{k_1+k_2=2} A_{k_1}(\alpha^{-1}(1))\otimes A_{k_2}(\alpha^{-1}(2)))$$

with α defined by

$$\alpha = \begin{cases} 1,3,4 \mapsto 2 \\ 2,5\mapsto 1 \end{cases},$$

$\vec{k}=(1,1)$, $u=[2,5]\in A_1(\{2,5\})$, and $v=[1,3]\otimes 4\in A_1(\{1,3,4\})$. Suppose $u\in\varepsilon\mathrm{Lie}*_{\varphi_1}$ and $v\in\varepsilon\mathrm{Lie}^*_{\varphi_2}$. We have

$$\varphi_1\coprod\varphi_2 = \begin{cases} 4\mapsto 3 \\ 1,3\mapsto 2 \\ 2,5\mapsto 1 \end{cases}.$$

Then $\alpha=\gamma\circ\varphi_1\coprod\varphi_2$, where $\gamma\in\mathrm{Ord}(3,2)$ defined by

$$\gamma = \begin{cases} 2,3\mapsto 2 \\ 1\mapsto 1 \end{cases}.$$

Can factor $\varphi_1\coprod\varphi_2=\sigma\circ\varphi$, where $\sigma\in\Sigma_3$ is defined by

$$\sigma = \begin{cases} 2\mapsto 3 \\ 1\mapsto 2 \\ 3\mapsto 1 \end{cases}$$

and $\varphi \in \mathrm{Ord}(5,3)$ is defined by

$$\varphi = \begin{cases} 2,5 \mapsto 3 \\ 4 \mapsto 2 \\ 1,3 \mapsto 1 \end{cases}.$$

Let $\beta = \gamma \circ \sigma$. Explicitly, $\beta \in \mathrm{Surj}(3,2)$ is defined by

$$\beta = \begin{cases} 1,2 \mapsto 2 \\ 3 \mapsto 1 \end{cases}.$$

Reorder subwords in $u \otimes v$:

$$\begin{array}{ccccccccccc} [2,5] & \otimes & [1,3] & \otimes & 4 & \mapsto & [1,3] & \otimes & 4 & \otimes & [2,5] \\ v_1 & & v_2 & & v_3 & & v_2 & & v_3 & & v_1 \end{array}$$

Then $w = [1,3] \otimes 4 \otimes [2,5] \in \varepsilon\mathrm{Lie}^*_{\varphi}$.

The element$(\alpha, \vec{k}; u \otimes v) \in \mathcal{B}$ goes to $(\varphi, \beta; w)$ in $\mathcal{A}$.

Proof for $\mathcal{B} \to \mathcal{A}$: Consider $(\alpha, \vec{k}; u) \in \mathcal{B}$, where

$$u = (u_1^1 \otimes \cdots \otimes u^1_{k'_1}) \otimes (u_1^2 \otimes \cdots \otimes u^2_{k'_2}) \otimes \cdots \otimes (u_1^t \otimes \cdots \otimes u^t_{k'_t})$$

and $k'_i = |\alpha^{-1}(i)| - k_i$ for all i.

For all i, we have $u_1^i \otimes \cdots \otimes u^i_{k'_i} \in \varepsilon\mathrm{Lie}^*_{\varphi_i}$, for some $\varphi_i : \alpha^{-1}(i) \mathrel{\circ\!\!\twoheadrightarrow} k'_i$. Let $\ell_i = k'_1 + \cdots k'_i$. The subwords of u can be re-indexed as $u = v_1 \otimes \cdots \otimes v_{n-k}$, where $v_{\ell_{i-1}+1} \otimes \cdots \otimes v_{\ell_i} = u^i_1 \otimes \cdots \otimes u^i_{k_i}$. Following this re-indexing, we can think of φ_i equivalently as an ordered surjection $\alpha^{-1}(i) \mathrel{\circ\!\!\twoheadrightarrow} \{\ell_{i-1}+1, \ldots, \ell_i\}$. Since $\sum_i k'_i = \sum_i (|\alpha^{-1}(i)| - k_i) = n-k$, the coproduct $\varphi_1 \coprod \cdots \coprod \varphi_t$ is a surjection $n \to n-k$. Given $\varphi_1 \coprod \cdots \coprod \varphi_t$, there is a unique strictly order preserving $\gamma : \mathbf{n} - \mathbf{k} \twoheadrightarrow \mathbf{t}$ such that $\alpha = \gamma \circ (\varphi_1 \coprod \cdots \coprod \varphi_t)$. More explicitly,

$$\gamma : j \mapsto i \text{ if } \ell_{i-1} < j \leq \ell_i.$$

Although each φ_i is an ordered surjection, $\varphi_1 \coprod \cdots \coprod \varphi_t$ may not be ordered. There exists a unique $\sigma \in \Sigma_{n-k}$ and $\varphi \in \mathrm{Ord}(n, n-k)$ such that $\sigma \circ \varphi = \varphi_1 \coprod \cdots \coprod \varphi_t$. Let $\beta = \gamma \circ \sigma$ and $w = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n-k)} \in \varepsilon\mathrm{Lie}^*_{\varphi}$. We have defined an element $(\varphi, \beta; w) \in \mathcal{A}$.

Proof $\mathcal{A} \to \mathcal{B}$: Consider $(\varphi, \beta; w) \in \mathcal{A}$. Let $\alpha = \beta \circ \varphi$. There exists $\rho \in \Sigma_{n-k}$ and a unique strictly order preserving $\eta \in \mathrm{Ord}(n-k, t)$ such that $\beta = \eta \circ \rho$. Letting $\ell_i = \sum_{j=1}^i |\beta^{-1}(i)|$, we can explicitly define η by

$$\eta : j \mapsto i \text{ if } \ell_{i-1} < j \leq \ell_i.$$

If we further require ρ to be order preserving when restricted to each of $\beta^{-1}(1), \ldots, \beta^{-1}(t)$, then the choice of ρ is unique. If we let $k_i(\beta,\varphi) = |\alpha^{-1}(i)| - |\beta^{-1}(i)|$ for all i, then

$$\sum_i k_i(\beta,\varphi) = \sum_i (|\alpha^{-1}(i)| - |\beta^{-1}(i)|) = n - (n-k) = k.$$

Let $u = w_{\rho^{-1}(1)} \otimes \cdots \otimes w_{\rho^{-1}(n-k)}$. We have defined an element $(\alpha, \vec{k}(\beta,\varphi); u) \in \mathcal{B}$

Check $\mathcal{B} \to \mathcal{A} \to \mathcal{B} = id$: Start with $(\alpha, \vec{k}, u) \in \mathcal{B}$. Following the description above,

$$(\alpha, \vec{k}, u) \mapsto (\sigma^{-1} \circ \coprod \varphi_i, \gamma \circ \sigma, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n-k)}).$$

Mapping back to $\mathcal{B}$, we get

$$(\sigma^{-1} \circ \coprod \varphi_i, \gamma \circ \sigma, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n-k)}) \mapsto (\beta \circ \varphi, \vec{k}(\beta,\varphi), w_{\rho^{-1}(1)} \otimes \cdots \otimes w_{\rho^{-1}(n-k)}),$$

where

$$\beta \circ \varphi = (\gamma \circ \sigma) \circ \left(\sigma^{-1} \circ \coprod \varphi\right) = \alpha.$$

By the definition of γ, $|\gamma^{-1}(i)| = k'_i$. Since

$$\begin{aligned} k_i(\beta,\varphi) &= |(\beta \circ \varphi)^{-1}(i)| - |\beta^{-1}(i)| \\ &= |\alpha^{-1}(i)| - |(\gamma \circ \sigma)^{-1}(i)| \\ &= |\alpha^{-1}(i)| - k'_i \\ &= k_i, \end{aligned}$$

we conclude $\vec{k}(\beta,\varphi) = \vec{k}$. Notice that if we factor β as $\beta = \eta \circ \rho$ such that $\rho \in \Sigma_{n-k}$ is strictly order preserving on each $\beta^{-1}(i)$ and $\eta : \mathbf{n-k} \twoheadrightarrow \mathbf{t}$, then $\eta = \gamma$ and $\rho = \sigma$. Since $w_{\rho^{-1}(i)} = v_{\sigma(\rho^{-1}(i))} = v_i$, we have

$$(\beta \circ \varphi, \vec{k}(\beta,\varphi), w_{\rho^{-1}(1)} \otimes \cdots \otimes w_{\rho^{-1}(n-k)}) = (\alpha, \vec{k}, u)$$

as desired.

Check $\mathcal{A} \to \mathcal{B} \to \mathcal{A} = id$: Consider $(\varphi, \beta, w) \in \mathcal{A}$. Following the description above,

$$(\varphi, \beta, w) \mapsto (\beta \circ \varphi, \vec{k}(\beta,\varphi), w_{\rho^{-1}(1)} \otimes \cdots \otimes w_{\rho^{-1}(n-k)}).$$

Mapping back to $\mathcal{A}$, we get

$$(\beta \circ \varphi, \vec{k}(\beta,\varphi), w_{\rho^{-1}(1)} \otimes \cdots \otimes w_{\rho^{-1}(n-k)}) \mapsto (\sigma^{-1} \circ \coprod \varphi_i, \gamma \circ \sigma, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n-k)}).$$

The tensor $w_{\rho^{-1}(1)} \otimes \cdots \otimes w_{\rho^{-1}(n-k}$ determines a partition of $\mathbf{n}$ into $n-k$ blocks, which determines a unique ordered surjection, $\sigma^{-1} \circ (\coprod \varphi_i)$. Since $w_1 \otimes \cdots \otimes w_{n-k}$ is just a reordering of blocks, it describes the same ordered surjection. So $\varphi = \sigma^{-1} \circ (\coprod \varphi_i)$ and it follows that $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n-k)} = w_1 \otimes \cdots \otimes w_{n-k}$. The strictly ordered surjection γ is defined by $\gamma \circ (\coprod \varphi_i) = \alpha$, so

$$\begin{aligned}
\gamma \circ (\coprod \varphi_i) &= \alpha \\
&= \beta \circ \varphi \\
&= \beta \circ \sigma^{-1} \circ (\coprod \varphi_i)
\end{aligned}$$

and $\gamma = \beta \circ \sigma^{-1}$. We have

$$(\sigma^{-1} \circ \coprod \varphi_i, \gamma \circ \sigma, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n-k)}) = (\varphi, \beta, w)$$

as desired.

$\square$

Lemma 4.3.4. *For $n > 1$, the chain complex*

$$0 \to A_{n-1}(n)[1] \to \cdots \to A_0(n)[1] \to 0$$

is a long exact sequence.

Proof. Fix $n > 1$. Let $A_k = A_k(n)^*[1]$. Because $X = [1]$, $\widetilde{R}[X] = R$, and we can drop the $\widetilde{R}[X]$ coordinates when we write elements of A_k. In other words, $A_k \cong \bigoplus_{\varphi \in \mathrm{Ord}(\mathbf{n}, \mathbf{n}\text{-}\mathbf{k})} \varepsilon\mathrm{Lie}_\varphi$.

We will use the dual complex

$$0 \leftarrow A_{n-1} \leftarrow A_{n-2} \leftarrow \cdots \leftarrow A_0 \leftarrow 0$$

and construct a contracting homotopy.

Define $s : A_k \to A_{k-1}$ recursively in the following way. Let $w \in A_k$ be a basis element, i.e. $w = w_1 \otimes \cdots \otimes w_{n-k} \in \varepsilon\mathrm{Lie}_\varphi$ for some $\varphi \in \mathrm{Ord}(n, n-k)$ such that each w_i is a bracket in $\varepsilon\mathrm{Lie}_{\varphi^{-1}(i)}$. If $w_1 = [a]$, where $a \in \varphi^{-1}(1)$, then

$$s(w) = 0.$$

If $|w_1| > 1$, then we can write $w_1 = [a, w_1']$ and let

$$s(w) = a \otimes w_1' \otimes w_2 \otimes \cdots \otimes w_{n-k} - \sum_{h=3}^{n-k+1} s(\partial_{1,h}(a \otimes w_1' \otimes w_2 \otimes \cdots \otimes w_{n-k})).$$

For indexing reasons, sometimes it is convenient to use the equivalent definition

$$s(w) = a \otimes w_1' \otimes w_2 \otimes \cdots \otimes w_{n-k} - \sum_{h=2}^{n-k} s(\partial_{1,h+1}(a \otimes w_1' \otimes w_2 \otimes \cdots \otimes w_{n-k})).$$

We will show $\partial s + s\partial = Id$.

First, consider the case where $w_1 = [a]$. By definition,

$$\partial s(w) = \partial(0) = 0.$$

We also have

$$\begin{aligned}
s\partial(w) &= \sum_{1 \leq g < h \leq n-k} s\left(\partial_{g,h}(w)\right) && (4.3)\\
&= \sum_{2 \leq h \leq n-k} s\left(\partial_{1,h}(w)\right) + \sum_{2 \leq g < h \leq n-k} s\left((-1)^{g+1} a \otimes w_2 \otimes \cdots \otimes [w_g, w_h] \otimes \cdots \otimes w_{n-k}\right) && (4.4)\\
&= \sum_{2 \leq h \leq n-k} s\left(\partial_{1,h}(w)\right) && (4.5)\\
&= s\left([a, w_2] \otimes w_3 \otimes \cdots \otimes w_{n-k}\right) + \sum_{3 \leq h \leq n-k} s\left(\partial_{1,h}(w)\right) && (4.6)\\
&= [a] \otimes w_2 \otimes \cdots \otimes w_{n-k} - \sum_{h=3}^{n-k} s(\partial_{1,h}(a \otimes w_2 \otimes w_3 \otimes \cdots \otimes w_{n-k})) + \sum_{3 \leq h \leq n-k} s\left(\partial_{1,h}(w)\right) && (4.7)\\
&= w && (4.8)
\end{aligned}$$

where (4.3) and (4.6) follow from definition of ∂ and $\partial_{g,h}$, (4.5) follows from the first part of the definition of s, and (4.7) from the second part of the definition of s. So, in the case where $w_1 = [a]$, we have $\partial s + s\partial = Id$.

We will induct on the number of letters in w_1. Suppose we have shown $\partial s(w) + s\partial(w) = w$ for all w with $|w_1| \leq m$. Consider $w = w_1 \otimes w_2 \otimes \cdots \otimes w_{n-k}$, where $|w_1| = m + 1$. Let $w_1 = [a, w_1']$. To simplify some of the writing, let $w' = a \otimes w_1 \otimes \cdots \otimes w_{n-k}$.

Consider $\partial s(w)$. Applying s,

$$s(w) = w' - \sum_{h=3}^{n-k+1} s\partial_{1,h}(w').$$

Applying ∂,

$$\partial s(w) = \partial(w') - \sum_{h=3}^{n-k+1} \partial s\partial_{1,h}(w').$$

Since $|w_1'| = m$ and w_1' will be the first factor in $\partial_{1,h}(w')$ for all $3 \leq h \leq n - k + 1$, by assumption

$$\partial s(\partial_{1,h}(w')) = \partial_{1,h}(w') - s\partial(\partial_{1,h}(w')). \tag{4.9}$$

Then we have

$$\partial s(w) = \partial(w') - \sum_{h=3}^{n-k+1} \partial s \partial_{1,h}(w') \tag{4.10}$$

$$= \sum_{1\le i<j\le n-k+1} \partial_{i,j}(w') - \sum_{h=3}^{n-k+1} \partial s \partial_{1,h}(w') \tag{4.11}$$

$$= \sum_{1\le i<j\le n-k+1} \partial_{i,j}(w') - \sum_{h=3}^{n-k+1} \partial_{1,h}(w') + \sum_{h=3}^{n-k+1} s\partial\partial_{1,h}(w') \tag{4.12}$$

$$= w + \sum_{2\le i<j\le n-k+1} \partial_{i,j}(w') + \sum_{h=3}^{n-k+1} s\partial\partial_{1,h}(w') \tag{4.13}$$

$$= w + \sum_{2\le i<j\le n-k+1} \partial_{i,j}(w') + \sum_{h=2}^{n-k} s\partial\partial_{1,h+1}(w') \tag{4.14}$$

where (4.11) comes from definition of ∂, (4.12) follows from (4.9), and (4.14) shifts the index on the second sum in (4.13) down by 1 and the indexes on to make the terms easier to match up later.

Expand the second term in (4.14) by cases for possible values for the index i.

$$\sum_{2\le i<j\le n-k+1} \partial_{i,j}(w') = \sum_{1\le i<j\le n-k} \partial_{i+1,j+1}(w') \tag{4.15}$$

$$= -a \otimes [w'_1, w_2] \otimes \cdots \otimes w_{n-k} \tag{4.16}$$

$$- \sum_{j=3}^{n-k} a \otimes w_2 \otimes \cdots \otimes [w'_1, w_j] \otimes \cdots \otimes w_{n-k} \tag{4.17}$$

$$- \sum_{j=3}^{n-k} \sum_{i=2}^{j-1} a \otimes w'_1 \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [w_i, w_j] \otimes \cdots \otimes w_{n-k} \tag{4.18}$$

Expand $\sum \partial\partial_{1,h+1}(w')$:

$$\sum_{h=2}^{n-k} s\partial\partial_{1,h+1}(w') = \sum_{h=2}^{n-k} s\partial(w'_1 \otimes w_2 \otimes \cdots \otimes w_{h-1} \otimes [a, w_h] \otimes \cdots \otimes w_{n-k})$$

$$= \sum_{h=2}^{n-k} \sum_{1\le i<j\le n-k} s\partial_{i,j}(w'_1 \otimes w_2 \otimes \cdots \otimes w_{h-1} \otimes [a, w_h] \otimes \cdots \otimes w_{n-k})$$

$$= \sum_{h=3}^{n-k} s([w'_1, w_2] \otimes w_3 \otimes \cdots \otimes w_{h-1} \otimes [a, w_h] \otimes \cdots \otimes w_{n-k}) \tag{4.19}$$

$$+ \sum_{h=2}^{n-k} \sum_{j=3}^{h-1} s(w_2 \otimes \cdots \otimes [w'_1, w_j] \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes w_{n-k}) \tag{4.20}$$

$$+ \sum_{h=2}^{n-k} \sum_{j=2}^{h-1} \sum_{i=2}^{j-2} s(w'_1 \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [w_i, w_j] \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes w_{n-k}) \tag{4.21}$$

$$+ s([w'_1, [a, w_2]] \otimes w_3 \otimes \cdots \otimes w_{n-k}) \tag{4.22}$$

$$+ \sum_{h=3}^{n-k} s(w_2 \otimes \cdots \otimes w_{h-1} \otimes [w'_1, [a, w_h]] \otimes \cdots \otimes w_{n-k}) \tag{4.23}$$

$$+ \sum_{h=3}^{n-k} \sum_{i=2}^{h-1} (-1)^{i+1} s(w'_1 \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [w_i, [a, w_h]] \otimes \cdots \otimes w_{n-k}) \tag{4.24}$$

$$+ \sum_{j=3}^{n-k} s([a, w_2] \otimes w_3 \otimes \cdots \otimes [w'_1, w_j] \otimes \cdots \otimes w_{n-k}) \tag{4.25}$$

$$+ \sum_{h=3}^{n-k} \sum_{j=h+1}^{n-k} s(w_2 \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes w_{j-1} \otimes [w'_1, w_j] \otimes \cdots \otimes w_{n-k}) \tag{4.26}$$

$$+ \sum_{h=3}^{n-k} \sum_{j=h+1}^{n-k} (-1)^{i+1} s(w'_1 \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes [w_i, w_j] \otimes \cdots \otimes w_{n-k}) \tag{4.27}$$

$$+ \sum_{h=2}^{n-k} \sum_{j=h+1}^{n-k} (-1)^{h+1} s(w'_1 \otimes \cdots \otimes \widehat{[a, w_h]} \otimes \cdots \otimes [[a, w_h], w_j] \otimes \cdots \otimes w_{n-k}) \tag{4.28}$$

$$+ \sum_{h=2}^{n-k} \sum_{j=h+1}^{n-k} \sum_{i=h+1}^{j-1} (-1)^{i+1} s(w'_1 \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [w_i, w_j] \otimes \cdots \otimes w_{n-k}) \tag{4.29}$$

We want a contracting homotopy with $s\partial$, so we will expand that also in order to compare terms.

$$s\partial(w) = \sum_{j=2}^{n-k} s(w_2 \otimes \cdots \otimes [[a, w'_1], w_j] \otimes \cdots \otimes w_{n-k})$$

$$+ \sum_{j=3}^{n-k} \sum_{i=2}^{j-1} (-1)^{i+1} s([a, w'_1] \otimes \cdots \otimes \widehat{w_i} \otimes \cdots \otimes [w_i, w_j] \otimes \cdots \otimes w_{n-k}. \tag{4.30}$$

Apply the definition of s to (4.30).

$$s\partial(w) = s([[a,w_1'],w_2]\otimes\cdots\otimes w_{n-k}) \tag{4.31}$$

$$+\sum_{j=3}^{n-k} s(w_2\otimes\cdots\otimes[[a,w_1'],w_j]\otimes\cdots\otimes w_{n-k}) \tag{4.32}$$

$$+\sum_{j=3}^{n-k}\sum_{i=2}^{j-1}(-1)^{i+1}a\otimes w_1'\otimes\cdots\otimes\widehat{w_i}\otimes\cdots\otimes[w_i,w_j]\otimes\cdots\otimes w_{n-k} \tag{4.33}$$

$$-\sum_{j=2}^{n-k}\sum_{i=2}^{j-1}\sum_{g=j+1}^{n-k}(-1)^{i+1}s(w_1'\otimes\cdots\otimes[w_i,w_j]\otimes\cdots\otimes[a,w_g]\otimes\cdots\otimes w_{n-k}) \tag{4.34}$$

$$-\sum_{j=3}^{n-k}\sum_{i=2}^{j-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes w_{j-1}\otimes[a,[w_i,w_j]]\otimes\cdots\otimes w_{n-k}) \tag{4.35}$$

$$-\sum_{j=3}^{n-k}\sum_{i=2}^{j-1}\sum_{g=i+1}^{j-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes\widehat{w_i}\otimes\cdots\otimes[a,w_g]\otimes\cdots\otimes[w_i,w_j]\otimes\cdots\otimes w_{n-k}) \tag{4.36}$$

$$-\sum_{j=3}^{n-k}\sum_{i=2}^{j-1}\sum_{g=2}^{i-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes[a,w_g]\otimes\cdots\otimes\widehat{w_i}\otimes\cdots\otimes[w_i,w_j]\otimes\cdots\otimes w_{n-k}) \tag{4.37}$$

Our goal is to show $\partial s(w)+s\partial(w)=w$. As desired, w shows up in (4.14), but we need to show that everything else will cancel. It is immediate that the following pairs cancel: (4.21) and (4.34), (4.27) and (4.36), (4.18) and (4.33), and (4.29) and (4.37). With some re-indexing of the sums,

$$\begin{aligned}
(4.24)+(4.28)+(4.35) &= \sum_{h=3}^{n-k}\sum_{i=2}^{h-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes[w_i,[a,w_h]]\otimes\cdots\otimes w_{n-k})\\
&\quad+\sum_{h=2}^{n-k}\sum_{j=h+1}^{n-k}(-1)^{h+1}s(w_1'\otimes\cdots\otimes[[a,w_h],w_j]\otimes\cdots\otimes w_{n-k})\\
&\quad-\sum_{j=3}^{n-k}\sum_{i=2}^{j-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes[a,[w_i,w_j]]\otimes\cdots\otimes w_{n-k})\\
&= \sum_{j=3}^{n-k}\sum_{i=2}^{j-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes[w_i,[a,w_j]]\otimes\cdots\otimes w_{n-k})\\
&\quad+\sum_{i=2}^{n-k}\sum_{j=i+1}^{n-k}(-1)^{i+1}s(w_1'\otimes\cdots\otimes[[a,w_i],w_j]\otimes\cdots\otimes w_{n-k})\\
&\quad-\sum_{j=3}^{n-k}\sum_{i=2}^{j-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes[a,[w_i,w_j]]\otimes\cdots\otimes w_{n-k})\\
&= \sum_{j=3}^{n-k}\sum_{i=2}^{j-1}(-1)^{i+1}s(w_1'\otimes\cdots\otimes([w_i,[a,w_j]]+[[a,w_i],w_j]-[a,[w_i,w_j]])\otimes\cdots\otimes w_{n-k}).
\end{aligned}$$

By the Jacobi relation,

$$(4.24)+(4.28)+(4.35)=0.$$

Apply the definition of s to (4.25).

$$
\begin{aligned}
(4.25) = & \sum_{j=3}^{n-k} s([a, w_2] \otimes w_3 \otimes \cdots \otimes [w_1', w_j] \otimes \cdots \otimes w_{n-k}) \\
= & \sum_{j=3}^{n-k} a \otimes w_2 \otimes \cdots \otimes [w_1', w_j] \otimes \cdots \otimes w_{n-k} && (4.38) \\
- & \sum_{j=3}^{n-k} \sum_{h=3}^{j-1} s(w_2 \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes [w_1', w_j] \otimes \cdots \otimes w_{n-k}) && (4.39) \\
- & \sum_{h=3}^{n-k} s(w_2 \otimes \cdots \otimes [a, [w_1', w_h]] \otimes \cdots \otimes w_{n-k}) && (4.40) \\
- & \sum_{j=3}^{n-k} \sum_{h=j+1}^{n-k} s(w_2 \otimes \cdots \otimes [w_1', w_j] \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes w_{n-k}) && (4.41)
\end{aligned}
$$

The following pairs cancel: (4.38) and (4.17), (4.26) and (4.39), and (4.20) and (4.41). By a similar argument to $(4.24) + (4.28) + (4.35) = 0$, by the Jacobi relation,

$$(4.23) + (4.32) + (4.40) = 0.$$

Applying the Jacobi relation to (4.31) and (4.22), and then applying the definition of s,

$$
\begin{aligned}
(4.22) + (4.31) &= s([w_1', [a, w_2]] \otimes w_3 \otimes \cdots \otimes w_{n-k}) + s([[a, w_1'], w_2] \otimes \cdots \otimes w_{n-k}) \\
&= s([a, [w_1', w_2]] \otimes \cdots \otimes w_{n-k}) \\
&= a \otimes [w_1', w_2] \otimes \cdots \otimes w_{n-k} && (4.42) \\
&- \sum_{h=3}^{n-k} s([w_1', w_2] \otimes \cdots \otimes [a, w_h] \otimes \cdots \otimes w_{n-k} && (4.43)
\end{aligned}
$$

Finally, $(4.42) + (4.16) = 0$ and $(4.43) + (4.19) = 0$. We have shown $\partial s(w) + s\partial(w) = w$ for all w.

$\square$

With the special case of $X = [1]$ complete, we can prove Theorem 4.3.1.

Proof of Theorem 4.3.1. We wish to show there is a quasi-isomorphism $\eta : A_\bullet(n) \to \mathcal{I}^n$.

Since $\mathcal{I}^n$ is a chain complex concentrated in degree 0, $(\eta)_k : A_k(n) \to (\mathcal{I}^n)_k$ must be trivial for $k > 0$. Then $(\eta)_0$ should be ∂_0^*.

This problem can be reduced to consider only a finite number of sets. Both $A_\bullet(n)$ and $\mathcal{I}^n$ are degree n discrete modules. By [IJM08, Corollary 2.4], in order to show this is a quasi-isomorphism, it suffices to show

$$\eta_{[s]} : A_\bullet(n)[s] \xrightarrow{\simeq} \mathcal{I}^n[s]$$

for all $s \leq n$.

We can further reduce to a question on the cross effects of $A_\bullet(n)$ and $\mathcal{I}^n$ evaluated at the set [1]. There is a result [JM04, Proposition 1.2] that lets us write a functor in terms of its cross effects.

$$F(\bigvee_{i=1}^{s} X_i) \cong F(0) \oplus \left(\bigoplus_{\{s_1,\ldots,s_t\}\subset \mathbf{s}} \mathrm{cr}_t F(X_{s_1},\ldots,X_{s_t}) \right).$$

In this case,

$$[s] = \bigvee_{i=1}^{s} [1]$$

and

$$A_\bullet(n)([0]) \simeq 0 \simeq \mathcal{I}^n[0],$$

so we can rewrite both $A_\bullet(n)[s]$ and $\mathcal{I}^n[s]$ as

$$A_\bullet(n)(\bigvee_{i=1}^{s}[1]) \cong \left(\bigoplus_{\{s_1,\ldots,s_t\}\subset \mathbf{s}} \mathrm{cr}_t A_\bullet(n)([1],\ldots,[1]) \right)$$

and

$$\mathcal{I}^n(\bigvee_{i=1}^{s}[1]) \cong \left(\bigoplus_{\{s_1,\ldots,s_t\}\subset \mathbf{s}} \mathrm{cr}_t \mathcal{I}^n([1],\ldots,[1]) \right).$$

Therefore, we only need to show that the restriction of η to $\mathrm{cr}_t A_\bullet(n)[1] \xrightarrow{\simeq} \mathrm{cr}_t \mathcal{I}^n[1]$ for $1 \leq t \leq n$. We show this quasi-isomorphism by the following cases.

Case $t = 1$ and $n = 1$: This is automatic because $\mathrm{cr}_1 A_\bullet([1])$ is a copy of R concentrated at 0, $\mathrm{cr}_1\mathcal{I}^1([1]) \cong R$, and η in this case sends 1 to 1.

Case $t = 1$ and $n > 1$: Reduced functors are equivalent to their first cross effects. For $n > 1$, $\mathcal{I}^n([1]) = 0$. In Lemma 4.3.4, we show $A_\bullet(n)[1] \xrightarrow{\simeq} 0$ via contracting homotopy.

Case $t > 1$:

From the lemmas we have

$$\mathrm{cr}_t A_\bullet(n)[1] \simeq \bigoplus_{\alpha\in \mathrm{Surj}(\mathbf{n},\mathbf{t})} \bigotimes_{i=1}^{t} A_\bullet(|\alpha^{-1}(i)|)([1])$$

and

$$\mathrm{cr}_t\mathcal{I}^n([1],\ldots,[1]) \simeq \bigoplus_{\alpha\in \mathrm{Surj}(\mathbf{n},\mathbf{t})} \bigotimes_{i=1}^{t} \mathcal{I}^{|\alpha^{-1}(i)|}([1]).$$

From the previous cases, we have quasi-isomorphisms $A_\bullet(|\alpha^{-1}(i)|)([1]) \xrightarrow{\simeq} \mathcal{I}^{|\alpha^{-1}(i)|}([1])$. So, we have a quasi-isomorphism on the whole thing.

□

Chapter 5

Tricomplex

5.1 Generalized Robinson Complex

Let Ξ denote the (reduced) Robinson bicomplex. We know from [IJM08] that there is a quasi-isomorphism

$$\Xi F \simeq D_1 F.$$

We will use this fact and multilinearization to compute $D_k F$ for any k.

Proposition 5.1.1. *[JM04]*

$$D_k F(X) \simeq (D_1^{(k)} cr_k F([1]))\widehat{\otimes}_{\Sigma_k} \widetilde{R}\left[\wedge^k X\right]$$

Given the proposition and $\Xi F \simeq D_1 F$, we have the following corollary.

Corollary 5.1.2. *For a discrete module F,*

$$D_k F(X) \simeq Tot(\Xi^{(k)} cr_k F([1]))\widehat{\otimes}_{\Sigma_k} \widetilde{R}\left[\wedge^k X\right],$$

where the exponent on $\Xi^{(k)}$ indicates that the Robinson complex should be applied to each of the k inputs of $cr_k F$.

The Robinson bicomplex, ΞF, is the following complex of chain complexes:

$$\cdots \to \varepsilon\mathrm{Lie}_3^*\widehat{\otimes}_{\Sigma_3}\mathrm{cr}_3 F(X) \to \varepsilon\mathrm{Lie}_2^*\widehat{\otimes}_{\Sigma_2}\mathrm{cr}_2 F(X) \to \varepsilon\mathrm{Lie}_1^*\widehat{\otimes}_{\Sigma_1}\mathrm{cr}_1 F(X).$$

If instead of F we consider $\mathrm{cr}_2 F(-,\star)$ and apply Ξ in one variable and then the other, we get the following bicomplex of bicomplexes:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \ldots\ \varepsilon\mathrm{Lie}_3^*\widehat{\otimes}_{\Sigma_3}\mathrm{cr}_3^1\left(\varepsilon\mathrm{Lie}_3^*\widehat{\otimes}_{\Sigma_3}\mathrm{cr}_3^2\mathrm{cr}_2F\right) & \to & \varepsilon\mathrm{Lie}_3^*\widehat{\otimes}_{\Sigma_3}\mathrm{cr}_3^1\left(\varepsilon\mathrm{Lie}_2^*\widehat{\otimes}_{\Sigma_2}\mathrm{cr}_2^2\mathrm{cr}_2F\right) & \to & \varepsilon\mathrm{Lie}_3^*\widehat{\otimes}_{\Sigma_3}\mathrm{cr}_3^1\left(\varepsilon\mathrm{Lie}_1^*\widehat{\otimes}_{\Sigma_1}\mathrm{cr}_1^2\mathrm{cr}_2F\right) \\ \downarrow & & \downarrow & & \downarrow \\ \ldots\ \varepsilon\mathrm{Lie}_2^*\widehat{\otimes}_{\Sigma_2}\mathrm{cr}_2^1\left(\varepsilon\mathrm{Lie}_3^*\widehat{\otimes}_{\Sigma_3}\mathrm{cr}_3^2\mathrm{cr}_2F\right) & \to & \varepsilon\mathrm{Lie}_2^*\widehat{\otimes}_{\Sigma_2}\mathrm{cr}_2^1\left(\varepsilon\mathrm{Lie}_2^*\widehat{\otimes}_{\Sigma_2}\mathrm{cr}_2^2\mathrm{cr}_2F\right) & \to & \varepsilon\mathrm{Lie}_2^*\widehat{\otimes}_{\Sigma_2}\mathrm{cr}_2^1\left(\varepsilon\mathrm{Lie}_1^*\widehat{\otimes}_{\Sigma_1}\mathrm{cr}_1^2\mathrm{cr}_2F\right) \\ \downarrow & & \downarrow & & \downarrow \\ \ldots\ \varepsilon\mathrm{Lie}_1^*\widehat{\otimes}_{\Sigma_1}\mathrm{cr}_1^1\left(\varepsilon\mathrm{Lie}_3^*\widehat{\otimes}_{\Sigma_3}\mathrm{cr}_3^2\mathrm{cr}_2F\right) & \to & \varepsilon\mathrm{Lie}_1^*\widehat{\otimes}_{\Sigma_1}\mathrm{cr}_1^1\left(\varepsilon\mathrm{Lie}_2^*\widehat{\otimes}_{\Sigma_2}\mathrm{cr}_2^2\mathrm{cr}_2F\right) & \to & \varepsilon\mathrm{Lie}_1^*\widehat{\otimes}_{\Sigma_1}\mathrm{cr}_1^1\left(\varepsilon\mathrm{Lie}_1^*\widehat{\otimes}_{\Sigma_1}\mathrm{cr}_1^2\mathrm{cr}_2F\right) \end{array}$$

given by

$$(\Xi^1\Xi^2\mathrm{cr}_2F(-,\star)_{p-1,q-1} \simeq \varepsilon\mathrm{Lie}_p^*\widehat{\otimes}_{\Sigma_p}\mathrm{cr}_p^1\left(\varepsilon\mathrm{Lie}_q^*\widehat{\otimes}_{\Sigma_q}\mathrm{cr}_q^2\mathrm{cr}_2F(-,\star)\right).$$

Let $-=\star=[1]$. To get a chain complex of bicomplexes, we take the total complex:

$$\begin{aligned} T_n(2)F :&= Tot\left(\Xi^1\Xi^2\mathrm{cr}_2F([1],[1])\right)_n \\ &\simeq \bigoplus_{\substack{p+q=n+2 \\ p,q,\geq 1}} \varepsilon\mathrm{Lie}_p^*\widehat{\otimes}_{\Sigma_p}\mathrm{cr}_p^1\left(\varepsilon\mathrm{Lie}_q^*\widehat{\otimes}_{\Sigma_q}\mathrm{cr}_q^2\mathrm{cr}_2F([1])\right)([1]). \end{aligned}$$

In general, we can consider cr_kF, apply Ξ in each variable to get a k-complex of k-complexes, and then take the total complex to get a chain complex of k-complexes:

$$\begin{aligned} T_n(k)F :&= Tot\left(\Xi^1\cdots\Xi^k\mathrm{cr}_kF([1],\ldots,[1])\right)_n \\ &\simeq \bigoplus_{\substack{p_1+\cdots+p_k=n+k \\ p_1,\ldots,p_k\geq 1}} \varepsilon\mathrm{Lie}_{p_1}^*\widehat{\otimes}_{\Sigma_{p_1}}\mathrm{cr}_{p_1}^1\left(\cdots\left(\varepsilon\mathrm{Lie}_{p_k}^*\widehat{\otimes}_{\Sigma_{p_k}}\mathrm{cr}_{p_k}^k\mathrm{cr}_kF([1])\right)\cdots\right)([1]). \end{aligned}$$

We will show that $(T_\bullet(k))_{h\Sigma_k}$ is quasi-isomorphic to a chain complex of k-complexes whose entries match up with our resolution.

We start with a lemma on cross effects.

Lemma 5.1.3.

$$cr_{k_1}^1\cdots cr_{k_n}^n\, cr_nF \cong cr_{k_1+\cdots k_n}F.$$

Proof. In [JM04, Example 1.8], they show there is an adjunction between precomposition with the diagonal functor Δ_n^* and cr_n. In other words,

$$\mathrm{Hom}_{functors}(F\circ\Delta_n, G) \cong \mathrm{Hom}_{n-reduced}(F, \mathrm{cr}_nG). \tag{5.1}$$

Note $F : \mathcal{A}^n \to \mathcal{B}$, so $F = (-_1, -_2, \ldots, -_n)$. Applying (5.1) to $\mathrm{cr}^1_{n_1}$,

$$\mathrm{Hom}(F, \mathrm{cr}^1_{n_1} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k G) \cong \mathrm{Hom}(F(\Delta_{n_1}, -_{n_1+1}, \ldots, -_n), \mathrm{cr}^2_{n_2} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k G).$$

We can repeat this for $\mathrm{cr}^2_{n_2}$ and so on:

$$\begin{aligned}
\mathrm{Hom}(F, \mathrm{cr}^1_{n_1} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k G) &\cong \mathrm{Hom}(F(\Delta_{n_1}, -_{n_1+1}, \ldots, -_n), \mathrm{cr}^2_{n_2} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k G) \\
&\cong \mathrm{Hom}(F(\Delta_{n_1}, \Delta_{n_2}, -_{n_1+n_2+1}, \ldots, -_n), \mathrm{cr}^3_{n_3} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k G) \\
&\vdots \\
&\cong \mathrm{Hom}(F(\Delta_{n_1}, \ldots, \Delta_{n_k}), \mathrm{cr}_k G) \\
&\cong \mathrm{Hom}(F \circ \Delta_n, G).
\end{aligned}$$

From these two natural(in F and G) isomorphisms with $\mathrm{Hom}(F \circ \Delta_n, G)$, we get a natural isomorphism

$$\mathrm{Hom}_{n-red.}(F, \mathrm{cr}^1_{n_1} \mathrm{cr}^2_{n_2} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k G) \cong \mathrm{Hom}_{n-red.}(F, \mathrm{cr}_n G).$$

That is an isomorphism in

$$Nat(\mathrm{Hom}(-, \mathrm{cr}^1_{n_1} \mathrm{cr}^2_{n_2} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k), \mathrm{Hom}(-, \mathrm{cr}_n)),$$

which corresponds to a natural isomorphism $\mathrm{Hom}(\mathrm{cr}^1_{n_1} \mathrm{cr}^2_{n_2} \cdots \mathrm{cr}^k_{n_k} \mathrm{cr}_k, \mathrm{cr}_n)$.

□

In order to make use of Lemma 5.1.3, we re-write the entries of $T_\bullet(2)$ with the following lemma.

Lemma 5.1.4.

$$\begin{aligned}
&\bigoplus_{p_1+\cdots+p_k=n+k} cr^1_{p_k} \left(\cdots \left(cr^k_{p_k}\, cr_k F \widehat{\otimes}_{\Sigma_{p_k}} \varepsilon Lie^*_{p_k} \right) \cdots \right) \widehat{\otimes}_{\Sigma_{p_1}} \varepsilon Lie^*_{p_1} \\
\simeq &\bigoplus_{p_1+\cdots+p_k=n+k} \left(cr^1_{p_1} \cdots cr^k_{p_k}\, cr_k F \right) \widehat{\otimes}_{\Sigma_{p_k}} \varepsilon Lie^*_{p_k} \cdots \widehat{\otimes}_{\Sigma_{p_1}} \varepsilon Lie^*_{p_1}
\end{aligned}$$

Proof. Since cr_{p_i} is an exact functor, it preserves derived functors.[Wei94, Ex 2.4.2]

□

We need another lemma to take this bisimplicial complex to a complex.

Lemma 5.1.5.

$$F \widehat{\otimes}_S M \widehat{\otimes}_T N \simeq F \widehat{\otimes}_{S \times T} (M \otimes N).$$

Proof. Let F be a $S \times T$-module, M be a S-module, and N be an T-module. Consider

$$(F \widehat{\otimes}_S M) \widehat{\otimes}_T N.$$

This is a bisimplicial complex where

$$(m, n) \mapsto F \otimes R[S]^{\otimes m} \otimes M \otimes R[T]^{\otimes n} \otimes N.$$

We get a simplicial complex by diagonalizing:

$$n \mapsto F \otimes R[S]^{\otimes n} \otimes M \otimes R[T]^{\otimes n} \otimes N$$

and

$$\begin{aligned} F \otimes R[S]^{\otimes n} \otimes M \otimes R[T]^{\otimes n} \otimes N &\cong F \otimes R[S] \otimes \overset{\times n}{\cdots} \otimes R[S] \otimes M \otimes R[T] \otimes \overset{\times n}{\cdots} \otimes R[T] \otimes N \\ &\cong F \otimes (R[S] \otimes R[T])^{\otimes n} (M \otimes N) \\ &\cong F \otimes R[S \times T]^{\otimes n} \otimes (M \otimes N). \end{aligned}$$

So, the diagonalization of $(F \widehat{\otimes}_S M) \widehat{\otimes}_T N$ is isomorphic as a simplicial module to $F \widehat{\otimes}_{S \times T} (M \otimes N)$. The fact that the diagonalization of $(F \widehat{\otimes}_S M) \widehat{\otimes}_T N$ is equivalent to the total complex of the associated bicomplex is precisely the Eilenberg-Zilber Theorem [Wei94, Theorem 8.5.1]. Thus,

$$Tot((F \widehat{\otimes}_S M) \widehat{\otimes}_T N) \simeq F \widehat{\otimes}_{S \times T} M \otimes N.$$

□

By repeatedly apply Lemma 5.1.5, we have the following corollary.

Corollary 5.1.6.

$$M \widehat{\otimes}_{\Sigma_{p_k}} \varepsilon Lie^*_{p_k} \cdots \widehat{\otimes}_{\Sigma_{p_1}} \varepsilon Lie^*_{p_1} \simeq M \widehat{\otimes}_{\Sigma_{p_1} \times \cdots \times \Sigma_{p_k}} (\varepsilon Lie^*_{p_1} \otimes \cdots \otimes \varepsilon Lie^*_{p_k})$$

Lemma 5.1.7 follows from Corollary 5.1.6, Lemma 5.1.3, and Lemma 5.1.4.

Lemma 5.1.7.

$$T_n(k)F[1] \simeq \bigoplus_{p_1 + \cdots + p_k = n+k} \varepsilon Lie^*_{p_1} \otimes \cdots \otimes \varepsilon Lie^*_{p_k} \widehat{\otimes}_{\Sigma_{p_1} \times \cdots \times \Sigma_{p_k}} cr_{n+k} F([1])$$

5.2 Isomorphism between $T_n(k)$ and $A_k(n)$

In this section, we show that the entries of the total complex of the generalized Robinson complex are quasi-isomorphic to the entries from the degree resolution.

Theorem 5.2.1.

$$(T_{n-k}(k)F)\widehat{\otimes}_{\Sigma_k}\widetilde{R}\left[X^k\right] \cong A_{n-k}(n)F(X)$$

as $R[\Sigma_n]$-modules.

In order to prove Theorem 5.2.1, we will use the following more general propositions for R-modules.

Suppose we have an R-module, M_i, for every positive integer i such that M_i has a Σ_i action. Fix n. Consider

$$A = \bigoplus_{\alpha\in \mathrm{Surj}(\mathbf{n},\mathbf{k})} M_{|\alpha^{-1}(1)|} \otimes \cdots \otimes M_{|\alpha^{-1}(k)|}$$

and

$$B = \bigoplus_{\ell_1+\ell_2+\cdots+\ell_k=n} M_{\ell_1} \otimes \cdots \otimes M_{\ell_k} \otimes_{\Sigma_{\ell_1}\times\cdots\times\Sigma_{\ell_k}} \Sigma_n,$$

where all $\ell_i \geq 1$.

Proposition 5.2.2. *There is an isomorphism $f : A \to B$ that commutes with the Σ_n actions.*

Proof. Consider $(\alpha; m_1 \otimes \cdots \otimes m_k)$ be an element of the summand of A corresponding to $\alpha \in \mathrm{Surj}(\mathbf{n},\mathbf{k})$. If we let $\ell_j = |\alpha^{-1}(j)|$ for $1 \leq j \leq k$, then $\ell_1 + \cdots + \ell_k = n$. Let $\ell = (\ell_1, \ldots, \ell_k)$. Define a strictly order preserving surjection $b_\ell : \mathbf{n} \to \mathbf{k}$ corresponding to ℓ by

$$b_\ell(i) = \begin{cases} 1 & \text{if } i \leq \ell_1 \\ 2 & \text{if } \ell_1 < i \leq \ell_1 + \ell_2 \\ & \vdots \\ k & \text{if } \ell_1 + \ell_2 + \cdots + \ell_{k-1} < i \leq n \end{cases}$$

The surjection α defines a partition of $\mathbf{n}$ into k blocks, where the i-th block is $\alpha^{-1}(i)$. We can factor $\alpha = b_\ell \circ \tau_\alpha$, where $\tau_\alpha \in \Sigma_n$. If τ and τ' differ only by transpositions within blocks of α, then $b_\ell \circ \tau = b_\ell \circ \tau'$. So, the factorization may not be unique. We pick τ_α such that the restriction of τ_α to $\alpha^{-1}(i)$ is order preserving for each i.

Define $f : A \to B$ by

$$f(\alpha; m_1 \otimes \cdots \otimes m_k) = (\ell; m_1 \otimes \cdots \otimes m_k \otimes \tau_\alpha).$$

To find an inverse for f, consider $(\ell; m_1 \otimes \cdots \otimes m_k \otimes \sigma)$, an element of the summand of B corresponding to $\ell = (\ell_1, \ldots, \ell_k)$. By the same construction as above, we have a strictly order preserving surjection $b_\ell : \mathbf{n} \to \mathbf{k}$. The composition $b_\ell \circ \sigma$ is another surjection $n \to k$. The restriction of σ to $(b_\ell \circ \sigma)^{-1}(i)$ may not be order preserving for all i. However, we can write σ as a composition $\tau\rho$, where τ is order preserving on blocks of $b_\ell \circ \sigma$ and $\rho \in \Sigma_n$ permutes elements within blocks. We can think of ρ as $(\rho_1, \ldots, \rho_k) \in \Sigma_{\ell_1} \times \cdots \times \Sigma_{\ell_k}$.

Because we are tensoring over $\Sigma_{\ell_1} \times \cdots \times \Sigma_{\ell_k}$, we have

$$(\ell; m_1 \otimes \cdots \otimes m_k \otimes \sigma) \sim (\ell; m_1 \cdot \rho_1^{-1} \otimes \cdots \otimes m_k \cdot \rho_k^{-1} \otimes \tau).$$

Define $g : B \to A$ by

$$g(\ell; m_1 \otimes \cdots \otimes m_k \otimes \sigma) = g(\ell; m_1 \cdot \rho_1^{-1} \otimes \cdots \otimes m_k \cdot \rho_k^{-1} \otimes \tau) = (b_\ell \circ \tau; m_1 \cdot \rho_1^{-1} \otimes \cdots \otimes m_k \cdot \rho_k^{-1}).$$

Note g is the inverse of f, so f is an isomorphism. We also have that f commutes with the Σ_n action.

$$\begin{aligned}
(\ell; m_1 \otimes \cdots \otimes m_k \otimes \sigma) \cdot \mu &= (\ell; m_1 \otimes \cdots \otimes m_k \otimes \sigma \circ \mu) \\
&\sim (\ell; m_1\rho_1^{-1} \otimes \cdots \otimes m_k\rho_k^{-1} \otimes \tau_{\sigma\mu}) \\
f(\ell; m_1\rho_1^{-1} \otimes \cdots \otimes m_k\rho_k^{-1} \otimes \tau_{\sigma\mu}) &= (b_\ell\tau_{\sigma\mu}; m_1\rho_1^{-1} \otimes \cdots \otimes m_k\rho_k^{-1}) \\
&= (b_\ell\tau_\sigma; m_1 \otimes \cdots \otimes m_k) \cdot \mu \\
&\sim f(\ell; m_1 \otimes \cdots \otimes m_k \otimes \sigma) \cdot \mu.
\end{aligned}$$

□

Proposition 5.2.3. *For a group G with a subgroup H, if M is an H module, and N is a G module, then*

$$(M \otimes_H G)\widehat{\otimes}_G N \simeq M\widehat{\otimes}_H N.$$

Proof. Since H is a subgroup of G, the action of H on G is free. By Lemma 3.2.1,

$$M \otimes_H G \simeq M\widehat{\otimes}_H G.$$

So,

$$
\begin{aligned}
(M \otimes_H G)\widehat{\otimes}_G N &\simeq (M\widehat{\otimes}_H G)\widehat{\otimes}_G N \\
&\cong M\widehat{\otimes}_H (G\widehat{\otimes}_G N) \\
&\simeq M\widehat{\otimes}_H (G \otimes_G N) \\
&\cong M\widehat{\otimes}_H N
\end{aligned}
$$

□

We now proceed with the proof of the theorem.

Proof. (of Theorem 5.2.1)

Recall

$$A_{n-k}(n)F(X) = \mathrm{cr}_n F([1])\widehat{\otimes}_{\Sigma_n} \left(\bigoplus_{\varphi \in \mathrm{Surj}(n,k)} \varepsilon\mathrm{Lie}^*_\varphi \otimes \widetilde{R}\left[X^k\right] \right)_{h\Sigma_k}.$$

By Lemma 5.1.7,

$$(T_{n-k}(k)F)\widehat{\otimes}_{\Sigma_k}\widetilde{R}\left[X^k\right] \simeq \left(\bigoplus_{|\vec{p}|=n} \varepsilon\mathrm{Lie}^*_{\vec{p}} \widehat{\otimes}_{\Sigma_{\vec{p}}} \mathrm{cr}_n F([1]) \right) \widehat{\otimes}_{\Sigma_k}\widetilde{R}\left[X^k\right],$$

where $\vec{p} = (p_1, \ldots, p_k)$ such that $p_1 + \cdots + p_k = n$, $\varepsilon\mathrm{Lie}^*_{\vec{p}} = \varepsilon\mathrm{Lie}^*_{p_1} \otimes \cdots \otimes \varepsilon\mathrm{Lie}^*_{p_k}$, and $\Sigma_{\vec{p}} = \Sigma_{p_1} \times \cdots \times \Sigma_{p_k}$.

Applying Proposition 5.2.3,

$$\bigoplus_{|\vec{p}|=n} \varepsilon\mathrm{Lie}^*_{\vec{p}} \widehat{\otimes}_{\Sigma_{\vec{p}}} \mathrm{cr}_n F([1]) \simeq \mathrm{cr}_n F([1])\widehat{\otimes}_{\Sigma_n} \left(\bigoplus_{|\vec{p}|=n} \varepsilon\mathrm{Lie}^*_{\vec{p}} \otimes_{\Sigma_{\vec{p}}} R[\Sigma_n] \right).$$

So,

$$(T_{n-k}(k)F)\widehat{\otimes}_{\Sigma_k}\widetilde{R}\left[X^k\right] \simeq \mathrm{cr}_n F([1])\widehat{\otimes}_{\Sigma_n} \left(\bigoplus_{|\vec{p}|=n} \varepsilon\mathrm{Lie}^*_{\vec{p}} \otimes_{\Sigma_{\vec{p}}} \Sigma_n \right) \widehat{\otimes}_{\Sigma_k}\widetilde{R}\left[X^k\right].$$

The final step is to show

$$\bigoplus_{|\vec{p}|=n} \varepsilon\mathrm{Lie}^*_{\vec{p}} \otimes_{\Sigma_{\vec{p}}} \Sigma_n \simeq \bigoplus_{\varphi \in \mathrm{Surj}(n,k)} \varepsilon\mathrm{Lie}^*_\varphi,$$

which follows from Proposition 5.2.2.

□

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