NON COMMUTATIVE VERSION OF ARITHMETIC GEOMETRIC MEAN INEQUALITY AND CROSSED PRODUCT OF TERNARY RING OF OPERATORS

BY

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DISSERTATION

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Abstract

This thesis is structured into two parts. In the first two chapters, we prove the noncommutative version of the Arithmetic Geometric Mean (AGM) inequality (this is a joint work with Mingyue Zhao and Maruis Junge). We start Chapter 2 by giving some background about the partition and Möbius function. We then prove the two main theorems: The AGM inequality for the norm and for the order. In Chapter 3, we provide some applications from random matrices such as Wishart random matrices, vector-valued moments of convex bodies, and freely independent operators.

The second part is about a ternary ring of operators (TRO). After giving a quick survey for the work of Todorov on the operator space version of Zettl's decomposition theorem, we introduce crossed products of ternary ring of operators (the full crossed product and the reduced crossed product). We also prove that $V \rtimes_{\alpha^V} G$ as the off-diagonal corner of the C^*-algebra $A(V) \rtimes_{\alpha^{A(V)}} G$. Equivalently, we have the $*$-isomorphism between the two linking C^*-algebras, i.e. $A(V \rtimes_{\alpha^V} G) = A(V) \rtimes_{\alpha^{A(V)}} G$. By using this identity, we obtain that if the group G is amenable, some local properties for TRO's preserve with the crossed product. We also provide a counter example which shows that if the linking C^*-algebras $A(V)$ and $A(W)$ are $*$-isomorphic or if their diagonal components are $*$-isomorphic, then their TRO's are not isomorphic. Similar example will be applied for W^*-TRO's.

To my lovely mother, brothers and sisters.

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Table of Contents

List of Symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathcal{K}(\mathcal{H})$	the space of compact operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}(\mathcal{H})$	the space of bounded operators on a Hilbert space $\mathcal{H}$
$P_d(A_1, ..., A_n)$	the average product of noncommutative operators of length d
λ	the left regular representation of a group G
M_n	the algebra of $n \times n$ matrices
$H \otimes \ell^2(G)$	Hilbert space crossed product
$A \rtimes_{\alpha,r} G$	the reduced crossed product of a C^*-algebra A
$A \rtimes_{\alpha,f} G$	the full crossed product of a C^*-algebra A
$A \rtimes_{\alpha,c} G$	the conditional crossed product of a C^*-algebra A
$V \rtimes_{\alpha,r} G$	the reduced crossed product of a TRO V
$V \rtimes_{\alpha,f} G$	the full crossed product of a TRO V

Chapter 1

Introduction

The focus of the first part of this exposition is to study the arithmetic geometric mean inequality in a noncommutative context. As pointed out by Ré and Recht in [RR12], noncommutative versions of the AGM inequalities are relevant to machine learning. In particular, their proof, which employed the classical MacLaurin inequalities, led to improved convergence rate of the algorithms in machine learning.

Let us recall the famous MacLaurin inequalities for positive real numbers $x_1, ..., x_n$ and the normalized d-th symmetric sums as

$$S_d(x_{i_1}, ..., x_{i_n}) = \binom{n}{d}^{-1} \sum_{\substack{\tau \subset \{1,...,n\} \\ |\tau| = d}} \prod_{i \in \tau} x_i.$$

where $1 \leq d \leq n$ and $|\tau| :=$ the cardinality of τ. According to the MacLaurin inequalities, we have

$$S_1 \geq \sqrt[2]{S_2} \geq \sqrt[3]{S_3} \geq ... \geq \sqrt[n]{S_n}.$$

In particular, $S_1 \geq \sqrt[n]{S_n}$ is the standard AGM inequality. For more details about the classical AGM inequality see [HLP52]. In this project, we will discuss noncommutative versions of MacLaurin's inequalities. Indeed, we will consider a generalized AGM inequality for the norm and the order. It may come as a surprise to the operator algebra community that these inequalities are motivated by problems in machine learning, stochastic gradient method (see Buttou [Bot98] and the reference there is in [RR12]), and randomized coordinates descent (see Nesterov [Nes12]). This interesting connection along with an overview of known results on this topic can be found in [RR11] and [RR12]. In fact, these methods contain an iteration procedure which can be performed with or without replacement samples. Recht and Ré, in [RR11], study the performance of both. They show that the expected convergence rate without replacement is faster than that with replacement. They proved this result by using a particular AGM inequality.

In the effort to generalize the classical AGM inequality to the noncommutative setting, a standard but naive procedure in noncommutative analysis is to replace scalars by operators. Famous examples of this strategy are Cauchy-Schwarz type inequalities for C^*-modules, Khintchine, and martingale inequalities.(See e.g. [Lus86], [LPP91], [PX03], [Ran02], [Jun02] , [JX03], [JX03]. For a general survey see [PX03].) Proving these noncommutative extensions often employs a combination of functional analytic and combinatorial methods. In fact, the key results of this project heavily rely on Pisier's interpretation of Rota's Möbius formulae for partitions in the setting of martingales in non-commutative L_p-spaces. A NC-AGM inequality

would ask whether

$$A_1 \cdots A_n \overset{?}{\leq} (\frac{1}{n}\sum_{j=1}^{n} A_j)^n \tag{1.0.1}$$

holds for positive operators $A_1, ..., A_n$ on a Hilbert space. (In this context we shall interpret $x \leq y$ as requiring that $y - x$ is positive semi-definite.) However, for positive operators A and B, the product AB may not be positive or even self-adjoint. Thus, the inequality (1.1) may not make sense. Inspired by Recht and Ré, we modify (1.0.1) by replacing the left hand side with the average of all the products of the operators A_i, which turns out to be self-adjoint. Following the MacLaurin approach, we may now ask whether the AGM inequality holds on average, i.e.

$$\frac{1}{n!}\sum_{\sigma\in S_n} A_{\sigma(1)} \cdots A_{\sigma(n)} \overset{?}{\leq} (\frac{1}{n}\sum_{j=1}^{n} A_j)^n. \tag{1.0.2}$$

$$\frac{1}{n!}\sum_{\sigma\in S_n} A_{\sigma(1)} \cdots A_{\sigma(n)} \overset{?}{\leq} (\frac{1}{n}\sum_{j=1}^{n} A_j)^n. \tag{1.0.3}$$

Unfortunately, we can not prove (1.0.3) in general. A milder version of (1.0.3) is to ask for

$$\|\frac{1}{n!}\sum_{\sigma\in S_n} A_{\sigma(1)} \cdots A_{\sigma(n)}\| \overset{?}{\leq} \|(\frac{1}{n}\sum_{j=1}^{n} A_j)^n\|, \tag{1.0.4}$$

where $\|x\| = \|x\|_{B(H)}$ refers to the standard operator norm of bounded operators on a Hilbert space H. The inequality (1.0.4) is a particular case of the noncommutative MacLaurin inequalities discussed in [RR11]. Indeed, for fixed d we may consider the following average product of noncommutative operators of length d:

$$P_d(A_1, ..., A_n) = \frac{1}{n \cdots (n-d+1)} \sum_{1\leq j_1,...,j_d\leq n \text{ all different}} A_{j_1} \cdots A_{j_d}.$$

We refer to the example in [RR12] for the fact that the symmetrization for the operators in the AGM inequality is required. In [RR11], Ré and Recht posed the following question: Is it true that for positive bounded operators $A_1, ..., A_n$ on a Hilbert space one has

$$\|P_d(A_1, ..., A_n)\|^{1/d} \leq \|P_1(A_1, ..., A_n)\|. \tag{1.0.5}$$

They proved that (1.0.5) holds when $A_1, ..., A_n$ are matrices that mutually commute. Moreover, they observed that for operators $A_1, ..., A_n$ on an m-dimensional Hilbert space one has

$$\|P_d(A_1, ..., A_n)\|_{B(\ell_2^m)}^{1/d} \leq m \ \|P_1(A_1, ..., A_n)\|.$$

We have two goals in this project. First, is to prove the AGM inequality for the norm for more general operators with a constant independent of the dimension m. Second, is to prove the AGM for the order with constant equal one.

The AGM inequalities for noncommutative operators will be covered in Chapter 2 and Chapter 3. In Chapter 2, we collect the important definitions for partitions and identities that we will use throughout these two chapters. Then at the last section, we prove the AGM inequality for the norm and for the order. In Chapter 3, we provide some interesting applications for the AGM inequality. We prove a version of the NC-AGM inequality for random matrices. For these examples, we prove first a deviation inequality. Then with additional assumptions, we have the AGM inequality for the norm.

The second part of this exposition focuses on the study of a ternary ring of operators (or simply TRO's). A concrete definition for the ternary ring of operators V is a norm closed subspace V of $B(H, K)$ where both H, K are complex Hilbert spaces, which is closed under the triple product

$$(x, y, z) \in V \times V^\sharp \times V \mapsto xy^*z \in V$$

for all $x, y, z \in V$. Note that $V^\sharp$ is the conjugate space of V that is contained in $B(K, H)$. In general, a TRO is defined as the off-diagonal corner of its linking C*-algebra which is

$$A(V) = \begin{bmatrix} C(V) & V \\ V^\sharp & D(V) \end{bmatrix}$$

where $C(V)$ and $D(V)$ are both C*-algebras generated by $VV^\sharp$ and $V^\sharp V$ respectively. The readers are referred to Hestenes [Hes62], Harris [Har81] , Zettl [Zet83], Hamana [Ham99] [Ham11], Exel [Exe97], Kirchberg [Kir95], and Effros, Ozawa and Ruan [EOR01] for more details. It is important to know that every TRO has a natural operator space structure, i.e. if we define a TRO $V \subset B(H, K)$, then $M_n(V) \subset M_n(B(H, K)) = B(H^n, K^n)$ is also a TRO. This imply that V has a natural canonical operator space structure $(V, \|.\|_n)$. Then $V^\sharp = \{x^* \in B(K, H) : x \in V\}$ is a TRO where its canonical TRO matrix norm satisfies $\|[x^*_{ij}]\| = \|[x_{ji}]\|$ for all $[x^*_{ij}] \in M_n(V^\sharp)$. It was proved by Ruan [Rua89] that an injective operator space is a TRO. Therefore, TRO's are considered as a special class of operator spaces.

Given V and W are two TRO's and a linear map $\theta : V \to W$, then θ from V to W is called a TRO-homomorphism if it preserves the triple product as follows:

$$\theta(xy^*z) = \theta(x)\theta(y)^*\theta(z)$$

for all $x,\ y,\ z \in V$. Moreover, if θ is bijection then we call θ a TRO-isomorphism from V onto W. Also, from [Zet83] a map $\theta : V \to W$ is called an anti-TRO homomorphism if it satisfies

$$\theta(xy^*z) = -\theta(x)\theta(y)^*\theta(z).$$

If, in addition, $\theta : V \to W$ is a TRO-homomorphism, then $\theta_n : M_n(V) \to M_n(W)$ is a TRO-homomorphism for each n. Thus, it is contraction for each n (by Harries [Har81]), i.e. θ is a complete contraction. This shows that every TRO-homomorphism is a complete contraction (see appendix A for the proof). We can obtain the following important result of Hamana and Ruan for TRO-isomorphism [Ham99]. If V and W are two TRO's and $\theta : V \to W$ is an onto linear map, then the following are equivalent:

$$\theta \text{ is a 2-isometry } \Leftrightarrow\ \theta \text{ is a triple isomorphism } \Leftrightarrow\ \theta \text{ is a complete isometry.}$$

TRO's have many common properties with its linking C^*-algebras. For instance, we know that every C^*-homomorphisms are completely contraction maps and every TRO homomorphisms are a completely contraction maps. Also, C^*-homomorphisms which are onto maps are quotient maps and TRO-homomorphisms which are onto maps are completely quotient maps (see appendix A for the proof).

An Abstract characterization for TRO's is given first by Zettl [Zet83]. Zettl introduced the concept of C^*-ternary ring of operators which is defined as follows:
A C^*-ternary ring is a Banach space X with ternary product

$$\langle ., ., . \rangle : X \times X^* \times X \to X$$

which is linear on the first and third variables and conjugate linear on the second such that it is associative

$$\langle\langle a, b^*, c\rangle, d^*, e\rangle = \langle a\langle d, c^*, b\rangle^* e\rangle = \langle a, b^*, \langle c, d^*, e\rangle\rangle$$

and satisfies the following conditions

$$\|\langle x, x^*, x\rangle\| \leq \|x\|\|y\|\|z\| \text{ and } \|\langle x, x^*, x\rangle\| = \|x\|^3.$$

Then he proved a decomposition theorem for C^*-ternary ring. This theorem states that every C^*-ternary ring can be written as a decomposition of two sub-ternary rings where the first part is isometrically isomorphic to a TRO and the second part is isometrically anti-isomorphic to a TRO. In 2002, Todorov proved the cb-version of Zettl's decomposition theorem. Todorov first introduced a ternary operator system which is an operator space X equipped with a triple product $X \times X^* \times X \to X$ which is linear on the first and third variables and conjugate linear on the second such that

1. $\|\langle [x_{ij}], [y^*_{jk}], [z_{kl}]\rangle\| \leq \|[x_{ij}]\|\|[y^*_{jk}]\|\|[z_{kl}]\|$ for all $[x_{ij}], [y_{kj}], [z_{kl}] \in M_n(X)$
2. $\langle\langle [x_{ij}], [y^*_{jk}], [z_{kl}]\rangle, [d^*_{ls}], [e_{st}]\rangle = \langle [x_{ij}], \langle [d_{sl}], [z^*_{lk}], [y_{kj}]\rangle^*, [e_{st}]\rangle = \langle [x_{ij}], [y^*_{jk}], \langle [z_{kl}], [d^*_{ls}], [e_{st}]\rangle\rangle$ for all $[x_{ij}], [y_{kj}], [z_{kl}], [d_{ls}], [e_{st}] \in M_n(X)$
3. $\|[x_{ij}] \odot [x_{ij}] \odot [x_{ij}]\| = \|[x_{ij}]\|^3$ for all $[x_{ij}] \in M_n(X)$, and $n \in \mathbb{N}$.

Here $\odot$ denotes the formal matrix product. For $n = 1$, the above definition is for C^*-ternary ring. It turns out that with these conditions we will have the cb-version of Zettl's decomposition theorem. Moreover, Todorov proved that any ternary operator system is completely isometric to a TRO. This means these conditions for ternary operator system are not enough to obtain the isomorphism as we expected. More details about C^*-ternary ring and its decomposition theorem will be covered in section 4.2.

Motivated by C^*-algebras theory, we define the crossed product of TRO's. There is an increasing interest to study the crossed product for operator spaces and TRO's. The crossed product of W*-TRO's has been studied recently by Salmi and Skalski [SS17]. Since TRO is a special class of operator space, it is also considered as an approach to study the crossed product of operator spaces. We were interested to study this topic to answer a question related to Morita equivalent theory. Let us first recall the definition of Morita equivalent between two C^*-algebras A and B in the sense of Rieffel [Rie82]. Let A, B be a C^*-algebras. These algebras are called strong Morita equivalent in the sense of Rieffel, if there exists injective $*$-homomorphisms $\pi : A \to B(H)$ and $\rho : B \to B(K)$ where H, K are two different Hilbert spaces and there exists a TRO

$V \subset B(H,K)$ such that

$$\pi(A) = \overline{[V^\sharp V]}^{\|\cdot\|} \text{ and } \rho(B) = \overline{[VV^\sharp]}^{\|\cdot\|}.$$

It is important to know that this TRO V is not necessarily unique and we proved that in Theorem 4.3.5 for TRO's and Theorem 4.4.6 for W^*-TRO's. Back to our question that we stated as follows:

If we have two C^*-algebras $C(V)$ and $D(V)$, denoted as the diagonal components of the linking C^*-algebra of the TRO V, which are Morita equivalent (M.E) via V, i.e.

$$C(V) \overset{M.E}{\cong} D(V),$$

does this imply that their crossed product are Morita equivalent via $V \rtimes_\alpha G$, i.e.

$$C(V) \rtimes G \overset{M.E}{\cong} D(V) \rtimes G$$

for certain action α on a TRO V. It turns out that the answer of this question is connected to the existence of this identity

$$A(V \rtimes_\alpha G) = A(V) \rtimes_\alpha G,$$

which means that the TRO $V \rtimes_\alpha G$ is defined to be the off diagonal corner of the linking C^*-algebra $A(V) \rtimes_\alpha G$. We prove this identity for the conditional crossed product and the reduced crossed product of C^*-algebras at Chapter 5 (see section 5.2 and 5.4 for more details). By using this identity, we prove that some local properties preserve with the crossed product when G is amenable.

In Chapter 4, we start by preliminary parts for TRO's theory. It is known that if two TRO's V and W are TRO-isomorphic, then their linking C^*-algebras are also $*$-isomorphic:

$$V \cong W \Rightarrow A(V) \cong A(W).$$

On the other hand, it's a natural question to ask if the other direction is also true. Quiet surprisingly, this direction is not true in general. We prove this by giving a counter example for TRO's and corresponding result works for W*-TRO's. The first example of TRO's is based on the CAR algebra theory (see [Dav96] for more details) and the second example of W*-TRO's is based on one of the important Ruan's result in [Rua04].

We start Chapter 5 by introducing the definition of TRO's crossed product. We discuss the reduced crossed product of a TRO (which is denoted as $V \rtimes_{\alpha,r} G$) and the full crossed product of a TRO (which is denoted as $V \rtimes_{\alpha,f} G$). Also we obtain the definition of a conditional crossed product for the linking C^*-algebra which is characterized by its representation.

Since local properties like nuclearity and exactness behave nicely in the level of C^*-algebras crossed product. Then, one of our goals of this project is to see how this local properties behave in the level of TRO's crossed product. Using an important result for Ruan and Kaur [KR02], which proves that local properties of TRO's have strong connections with the local properties of their linking C^*-algebras, we prove the strong connections between the local properties of the crossed product of TRO's and its linking C^*-algebras. In order to prove that, we use the following an identity,

$$A(V \rtimes_{\alpha} G) = A(V) \rtimes_{\alpha} G.$$

Based on this result, it is easy to show the connection between local properties for crossed product of TRO's and its linking C^*-algebras. We end up this section of this chapter discussing about the conditional crossed product for the linking C^*-algebras. The important things about this class that we can relate the covariant representation of TRO's with the covariant representation of its linking C^*-algebras.

Chapter 2

Non commutative Arithmetic Geometric Mean Inequality

In this section, we are proving the following two main theorems: The first theorem is the norm version of the AGM inequality.

Theorem 2.0.1. *For operators $A_1, ..., A_n \geq 0$ on a Hilbert space H,*

$$\|P_d(A_1, ..., A_n)\|^{1/d} \leq d\ \|P_1(A_1, ..., A_n)\|.$$

and the second is the order version of the AGM inequality

Theorem 2.0.2. *Fix n and d. Suppose $A_1, ..., A_n$ and a_i are as above, $\sum\limits_i A_i = n$, $a_i = A_i - 1$ and*

i) $P_1(A_1, ..., A_n) = \frac{\sum_i A_i}{n} = 1$,

ii) $\|(\sum a_j^2)^{\frac{1}{2}}\| \leq \frac{n}{3d}$.

Then the AGM inequality holds in the order sense:

$$P_d(A_1, ..., A_n) \leq P_1(A_1, ..., A_n)^d = 1.$$

Let us now consider an example for the order version of the AGM inequality. In the second theorem we added the additional assumption $\sum A_i = n$. In order to illustrate the technique we use generally, it is good to start with $d = 3$.

Theorem 2.0.3. *Let $n \geq 6$. If $A_1, ..., A_n$ are self-adjoint operators such that $\sum_i A_i = n$. Then*

$$P_3(A_1, ..., A_n)^{1/3} \leq 1.$$

For the proof we consider the mean-zero operators $a_i := A_i - 1$. Observe the operators a_i are self-adjoint and $\sum_{i=1}^{n} a_i = 0$. It follows easily that

$$P_3(A_1, ..., A_n) = 1 + \binom{3}{1} P_1(a_1, ..., a_n) + \binom{3}{2} P_2(a_1, ..., a_n) + \binom{3}{3} P_3(a_1, ..., a_n).$$

Straightforward computations using $\sum a_i = 0$ reveal that
$P_1(a_1, ..., a_n) = \frac{(n-1)!}{n!} \sum_i a_i = 0$
$P_2(a_1, ..., a_n) = \frac{(n-2)!}{n!} \sum_{i \neq j} a_i a_j = \frac{(n-2)!}{n!} \Big((\sum a_i)^2 - \sum a_i^2 \Big) = -\frac{(n-2)!}{n!} \sum a_i^2$

$$\begin{aligned} P_3(a_1, ..., a_n) &= \frac{(n-3)!}{n!} (\sum_{i \neq j \neq k} a_i a_j a_k) \\ &= \frac{(n-3)!}{n!} \Big((\sum_{i \neq j \neq k} a_i)^3 - (\sum_{i=j} a_i^2)(\sum_k a_k) - (\sum_i a_i)(\sum_{j=k} a_j^2) - \sum_j \sum_{i=k \neq j} a_i a_j a_i + 2(\sum_{i=j=k} a_i^3) \Big) \\ &= 2 \frac{(n-3)!}{n!} \Big(\sum_i a_i^3 \Big). \end{aligned}$$

This leads to the form $P_3(A_1, ..., A_n) = 1 - \frac{3}{n(n-1)} \sum a_i^2 + \frac{2}{n(n-1)(n-2)} \sum a_i^3$. Together with

$$\sum a_i^3 \leq \|a_i\| \sum a_i^2 \leq n \sum a_i^2,$$

this yields

$$P_3(A_1, ..., A_n) \leq 1 - \frac{3}{n(n-1)} \sum a_i^2 + \frac{2n}{n(n-1)(n-2)} \sum a_i^2. \tag{2.0.1}$$

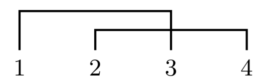
Since $\frac{2n}{n(n-1)(n-2)} \leq \frac{3}{n(n-1)}$ holds for all $n \geq 6$, the right side of (2.0.1) is at most 1 and we are done. A far-reaching generalization of this idea leads to the following result. Note that these techniques work efficiently when d is very large. This chapter is organized as the following: we first give an introduction for Partition and Möbius Formula

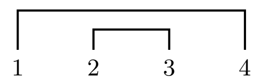
2.1 Partition and Möbius Formula

In this section, we review the analytic and combinatorial tools needed to prove Theorem 2.0.1, especially Pisier's interpretation of Rota's results on Möbius transforms for partitions. We need some definitions from the combinatorial theory of partitions. Let $\mathbb{P}_d$ be the lattice of all the partitions of $\{1, ..., d\}$. For two partitions σ and π, we write $\sigma \leq \pi$ if every block of the partition σ is contained in some block of π (i.e., any block of the partition of π can be written as a union of blocks of σ). In other words, π is a refinement of σ. There are two trivial partitions, $\dot{0}$ and $\dot{1}$, where $\dot{0}$ is the partition into n singletons and $\dot{1}$ is the partition of a single block. For a partition π, $\nu(\pi)$ is the number of the blocks of the partition π and $r_i(\pi)$ is the number of blocks of π with cardinality i such that $\sum_{i=1}^{d} ir_i(\pi) = d$; and $\sum_{i=1}^{d} r_i(\pi) = \nu(\pi)$. Also, we need to recall the definition of crossing and non-crossing partition.

Definition 2.1.1. A partition $\pi \in \mathbb{P}_d$ is called a crossing partition if there exist four numbers $1 \leq i < k < j < l \leq d$ such that i and j are in the same block, k and l are in the same block but i, j and k, l belong to two different blocks. If this situation does not happen, then we call π non-crossing.

Example 2.1.2.

- $\dot{0}$ and $\dot{1}$ are the smallest and largest non-crossing partition.
- The partition $\sigma = \{\{1, 3\}, \{2, 4\}\} =$ 

is crossing.
- The partition $\sigma = \{\{1, 4\}, \{23\}\} =$ 

is non-crossing.

For more information on partitions, see [Spe97] [And98] and [Rot64].

Let us recall two results on the Möbius function μ in [Pis00] which are crucial for our paper.

Proposition 2.1.3. [[Pis00], Proposition 1.1] For any $d \in \mathbb{N}$ there exists a function $\mu : \mathbb{P}_d \times \mathbb{P}_d \longrightarrow \mathbb{Z}$ such that for every vector space V and functions $\phi : \mathbb{P}_d \longrightarrow V$ and $\psi : \mathbb{P}_d \longrightarrow V$, we have the following properties:

1. If $\psi(\sigma) = \sum\limits_{\pi \leq \sigma} \phi(\pi)$, then $\phi(\sigma) = \sum\limits_{\pi \leq \sigma} \mu(\pi, \sigma)\psi(\pi)$;
2. If $\psi(\sigma) = \sum\limits_{\pi \geq \sigma} \phi(\pi)$, then $\phi(\sigma) = \sum\limits_{\pi \geq \sigma} \mu(\sigma, \pi)\psi(\pi)$;
3. Moreover, $\forall \sigma \neq \dot{0}$, $\sum\limits_{\dot{0} \leq \pi \leq \sigma} \mu(\pi, \sigma) = 0$.

Theorem 2.1.4. [[Pis00], Proposition 1.2]

The Möbius function satisfies the following properties:

1. $\mu(\dot{0}, \dot{1}) = (-1)^{d-1}(d-1)!$.
2. $\mu(\dot{0}, \pi) = \prod_{i=1}^{d} [(-1)^{i-1}(i-1)!]^{r_i(\pi)}$, and consequently,
3. $\sum_{\pi \in P_d} |\mu(\dot{0}, \pi)| = d!$.

If σ is a partition of $\{1, ..., d\}$, then there exists a coordinate function $f : \{1, ..., d\} \to \{1, ..., \nu(\sigma)\}$ such that $f^{-1}(t) = A_t$ where each A_t represents a block in our partition. Note that this coordinate function isn't unique. For every partition σ we can fix an enumeration of the blocks $f : \{1, 2, ..., d\} \longrightarrow \{1, 2, ..., |\sigma|\}$ where $\sigma{:=}\langle j_1, j_2, ..., j_d \rangle$. This means $j_r = j_s$ if and only if $r, s \in A_{r,s}$ where $A_{r,s}$ is a block in $\sigma = \langle j_1, j_2, ..., j_d \rangle$.

Using this notation for partitions, we are able to define the restricted and full partition for operators $x^i_{j_i}$ from an algebra $\mathcal{A}$ where the operators are written according to the blocks A_k in the partition σ. Note that the upper indices for the operator $x^i_{j_i}$ refer to the position of the operator and the lower indices refer to a block A_k in the partition σ where $j_i \in A_k$.

Definition 2.1.5. Let $\mathcal{A}$ be an algebra and $x^i_{j_i} \in \mathcal{A}$. The restricted partition is defined by:

$$\langle \sigma \rangle = \sum_{\substack{\langle j_1, j_2, \ldots, j_d \rangle = \sigma \\ j_i \neq j_k \text{ if } j_i \in A_i, j_k \in A_k \text{ and } i \neq k}} x^1_{j_1} \ldots x^d_{j_d},$$

where the sum run over all the partition σ where indices in the same block are equal and indices in different blocks are not equal. This is considered as a restricted condition. Unlike the restricted partition, the full partition run over all the partitions π such that $\pi \geq \sigma$. consider all the cases of partitions. This can be written in the form

$$[\sigma] = \sum_{\pi \geq \sigma} \langle \pi \rangle.$$

The restricted and full partitions, which are denoted as $\langle \sigma \rangle$ and $[\sigma]$ respectively, give expressions for the operators in the given $B(H)$ according to the algebraic combinatorial partition σ. In order to understand the difference between the definition of restricted partition and full partition, consider the following example.

Example 2.1.6. Let both the numbers of total samples and chosen samples be 3 ($n = d = 3$). For the full partition $[1\ 2, 3]$ with the assumption that $x^i_j = x_j$ we have

$$[1\ 2, 3] \quad = \sum_{\pi \geq [1\ 2,3]} \langle \pi \rangle = \langle 1\ 2, 3 \rangle + \langle 1\ 2\ 3 \rangle.$$

This can be written in terms of operators as

$$(\sum x_i^2)(\sum x_i) = \sum_{i_1 = i_2 \neq i_3} x^2_{i_1} x_{i_3} + \sum_{i_1 = i_2 = i_3} x^3_{i_1},$$

where the restricted partition $\langle 1\ 2, 3 \rangle$ is defined as $\langle 1\ 2, 3 \rangle = \sum_{i_1 = i_2 \neq i_3} x^2_{i_1} x_{i_3}$. Also, we can write the restricted partition in term of the full partition as follows

$$\langle 1\ 2, 3 \rangle = [1\ 2, 3] - \langle 1\ 2\ 3 \rangle.$$

We reformulate Proposition 2.1.3 in our context.

Proposition 2.1.7. *Let $x_j^k \in \mathcal{A}$ as above. Then we have*

$$\langle \pi \rangle = \sum_{\nu \geq \pi} \mu(\pi,\nu)[\nu], \text{ where } [\pi] = \sum_{\nu \geq \pi} \langle \nu \rangle,$$

$$\langle \pi \rangle = \sum_{\nu \leq \pi} \mu(\pi,\nu)[\nu], \text{ where } [\pi] = \sum_{\nu \leq \pi} \langle \nu \rangle.$$

Moreover, we have

$$\langle \dot{0} \rangle = [\dot{0}] + \sum_{\dot{0} \lneqq \nu \leq \dot{1}} \mu(\dot{0},\nu)[\nu]. \tag{2.1.1}$$

Let's give another example, if we have the restricted partition $\langle 1,2,3 \rangle \cong \{\{1\},\{2\},\{3\}\}$, then by using the previous formula (2.1.1):

$$\langle 1,2,3 \rangle = [1,2,3] - [12,3] - [1,23] - [13,2] + 2[123]$$

$$\begin{aligned} \sum_{i_1,i_2,i_3 \text{ all distinct}} x_{i_1}x_{i_2}x_{i_3} &= (\sum x_i)^3 - (\sum x_i^2)(\sum x_i) - (\sum x_i)(\sum x_i^2) \\ &\quad - \sum_{i,\ j} x_i x_j x_i + 2(\sum x_i^3). \end{aligned}$$

The coefficients (-1,-1,-1,+2) are computed by using the the Möbius function formula (2). For instance, the Möbius function for the partition $[1\ 2,3]$ is computed as follows

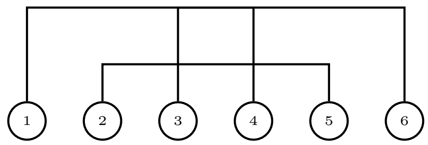
$$\mu(\dot{0},[1\ 2,3]) = \prod_{i=1}^{3} [(-1)^{i-1}(i-1)!]^{r_i(\pi)} = -1.$$

In [Pis00], in order to separate different partition blocks into disjoint subspaces, Pisier uses a trick to embed operators $x_{ik} \in B(H)$ into $B(K \otimes H)$ (for another Hilbert space K). Our first goal is to modify Pisier's trick by using matrix units.

Consider first the trivial partition that has only one block $[1\ 2\ \cdots\ d]$. We can write

$$\begin{aligned} \dot{1} = [1\ 2\ \cdots\ d] &= \sum x_{i_1}^1 x_{i_2}^2 \cdots x_{i_d}^d \\ &= (\sum e_{1i_1} \otimes x_{i_1}^1) \times (\sum e_{i_2 i_2} \otimes x_{i_2}^2) \times \cdots \\ &\times (\sum e_{i_{d-1}i_{d-1}} \otimes x_{i_{d-1}}^{d-1}) \times (\sum e_{i_d 1} \otimes x_{i_d}^d). \end{aligned}$$

Now if we have 6 elements and our partition σ has two crossing blocks, one containing $\{1,3,4,6\}$ and the other containing $\{2,5\}$ as seen in the following graph:



then the full partition of σ will be of the form:

$$[\sigma] = \sum_{\substack{i_1=i_3=i_4=i_6 \\ i_2=i_5}} x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6}.$$

We rewrite these elements into a tensor form, as follows:

$$\begin{aligned} Z_{i_1} &= e_{1i_1} \otimes 1 \otimes x_{i_1}\ ,\ Z_{i_2} = 1 \otimes e_{1i_2} \otimes x_{i_2} \\ Z_{i_3} &= e_{i_3i_3} \otimes 1 \otimes x_{i_3},\ Z_{i_4} = e_{i_4i_4} \otimes 1 \otimes x_{i_4} \\ Z_{i_5} &= 1 \otimes e_{i_51} \otimes x_{i_5}\ ,\ Z_{i_6} = e_{i_61} \otimes 1 \otimes x_{i_6}. \end{aligned}$$

With this new notation, we get

$$\begin{aligned} [\sigma] &= \sum_{\substack{i_1=i_3=i_4=i_6 \\ i_2=i_5}} x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}x_{i_6} \\ &= \sum_{i_1,i_2,i_3,i_4,i_5,i_6} Z_{i_1}Z_{i_2}Z_{i_3}Z_{i_4}Z_{i_5}Z_{i_6} \\ &= \prod_{j=1}^{6}(\sum_{i_j} Z_{i_j}) = \prod_{j=1}^{6} Z_j,\ (Z_j := \sum_{i_j} Z_{i_j}). \end{aligned}$$

In a more general setting, assume σ has more than one block. Denote $A_1,\ldots,A_{|\sigma|}$ as the blocks of the partition σ with cardinality larger than one.

Then we define

$$Z_{j_k}^k \in B(H)^{\otimes^{|\sigma|}} \otimes B(H)$$

as follows:

$$\begin{aligned} &\forall\ k \in A_1,\ Z_{j_k}^k = t_{A_1}(j_k) \otimes 1 \otimes \cdots \otimes x_{j_k}^k \\ &\forall\ k \in A_2,\ Z_{j_k}^k = 1 \otimes t_{A_2}(j_k) \otimes 1 \cdots \otimes x_{j_k}^k \end{aligned}$$

$$\forall\, k \in A_{|\sigma|},\; Z^k_{j_k} = 1 \otimes \cdots \otimes t_{A_{|\sigma|}}(j_k) \otimes x^k_{j_k},$$

$$t_{A_m(j_k)} = \begin{cases} e_{1j_k} & j_k = \min A_m \\ e_{j_k j_k} & \text{otherwise} \\ e_{j_k 1} & j_k = \max A_m. \end{cases}$$

Here, $\min A_m$ means the smallest index number and $\max A_m$ means the largest index number in the partition A_m. Finally, if k belongs to singleton block of the partition σ, then we set

$$Z^k_{j_k} = 1 \otimes \cdots \otimes 1 \otimes x^k_{j_k}.$$

To sum up, the method places each element into larger spaces, which will allow us to interchange the summation and multiplication as in the above example and the following lemma.

Lemma 2.1.8. *For an arbitrary partition σ for d elements, we have*

$$[\sigma] = \sum_{i_1,\ldots,i_d} Z^1_{i_1} \ldots Z^d_{i_d}.$$

Indeed, this immediately follows from

$$Z^i_{i_j} \cdot Z^k_{i_k} = 0, \text{if } i_j \neq i_k.$$

Follow Pisier's approach in [Pis00], we deduce the following norm estimate.

Theorem 2.1.9. *For an arbitrary partition σ for d elements, we have*

$$\|[\sigma]\|_{B(H)} \leq \prod_{k=1}^{d} \Big(\| \sum_{j_k} Z^k_{j_k} \| \cdot 1_{\sigma_s}(k) + \| \sum_{j_k} Z^k_{j_k} \| \cdot 1_{\sigma_{ns}}(k) \Big).$$

Moreover,

$$\|[\sigma]\|_{B(H)} \leq \prod_{k \in \sigma_s} \| \sum_{j_k} Z_{j_k} \| \times \prod_{k \in \sigma_{ns}} |||(Z_{j_k})|||,$$

where $|||(Z_{j_k})||| = \max\{\| \sum Z_{j_{k_1}} Z^*_{j_{k_1}} \|^{\frac{1}{2}},\ \| \sum Z^*_{j_{k_p}} Z_{j_{k_p}} \|^{\frac{1}{2}}, \sup_{j_k} \|Z_{j_k}\|\}$. *Here σ_s means the set of singletons in the partition σ, and σ_{ns} means the set of non-singleton elements in the partition σ. The functions $1_{\sigma_{ns}}(k), 1_{\sigma_s}(k)$ represent the characteristic functions, i.e.*

$$1_{\sigma_{ns}}(k) = \begin{cases} 1 & k \in \sigma_{ns} \\ 0 & otherwise \end{cases}, \ 1_{\sigma_s}(k) = \begin{cases} 1 & k \in \sigma_s \\ 0 & otherwise. \end{cases}$$

Proof. Taking the norm for the full partition, we have

$$\begin{aligned} \|[\sigma]\| &= \|\sum_{\pi \geq \sigma} \langle \pi \rangle\| = \| \sum_{\langle j_1, \cdots, j_d \rangle \geq \sigma} x^1_{j_1} \cdots x^d_{j_d} \| \\ &= \| \sum_{j_1, j_2, \ldots, j_d} Z^1_{j_1} \cdots Z^d_{j_d} \| \qquad (2.1.2) \\ &= \| \prod_{k \in \sigma_s} \sum_{j_k} Z^k_{j_k} \cdot \prod_{k \in \sigma_{ns}} \sum_{j_k} Z^k_{j_k} \| \qquad (2.1.3) \\ &\leq \| \prod_{k \in \sigma_s} \sum_{j_k} Z^k_{j_k} \| \cdot \| \prod_{k \in \sigma_{ns}} \sum_{j_k} Z^k_{j_k} \| \\ &\leq \prod_{k \in \sigma_s} \| \sum_{j_k} Z^k_{j_k} \| \cdot \prod_{k \in \sigma_{ns}} \| \sum_{j_k} Z^k_{j_k} \|. \end{aligned}$$

The equality (2.1.2) comes from Lemma 2.1.8. The equality (2.1.3) follows from the definition of $Z^k_{j_k}$, which means it allows us to perform summation first and then multiplication. Next,

$$\begin{aligned} &\|[\sigma]\|_{B(H)} \leq \prod_{k \in \sigma_s} \| \sum_{j_k} Z_{j_k} \| \cdot \prod_{k \in \sigma_{ns}} \| \sum_{j_k} Z_{j_k} \| \\ &\leq \prod_{k \in \sigma_s} \| \sum_{j_k} Z_{j_k} \| \times \prod_{k \in A_m \subset \sigma_{ns}} \| \sum_{j_k} Z_{j_k} \| \cdot (1_{\min A_m} + 1_{\max A_m} + 1_{\mathrm{mid} \ A_m}) \\ &\leq \prod_{k \in \sigma_s} \| \sum_{j_k} Z_{j_k} \| \times \\ &\prod_{k \in A_m \subset \sigma_{ns}} \Big(\| \sum_{j_k} Z_{j_k} \| \cdot 1_{\min A_m} + \| \sum_{j_k} Z_{j_k} \| \cdot 1_{\max A_m} + \| \sum_{j_k} Z_{j_k} \| \cdot 1_{\mathrm{mid} \ A_m} \Big) \\ &\leq \prod_{k \in \sigma_s} \| \sum_{j_k} Z_{j_k} \| \times \\ &\prod_{k \in A_m \subset \sigma_{ns}} \Big(\| \sum_{j_k} Z_{j_k} Z^*_{j_k} \|^{\frac{1}{2}} \cdot 1_{\min A_m} + \| \sum_{j_k} Z^*_{j_k} Z_{j_k} \|^{\frac{1}{2}} \cdot 1_{\max A_m} + \sup_{j_k} \|Z_{j_k}\| \cdot 1_{\mathrm{mid} \ A_m} \Big) \\ &\leq \prod_{k \in \sigma_s} \| \sum_{j_k} Z_{j_k} \| \times \prod_{k \in \sigma_{ns}} |||(Z_{j_k})|||, \end{aligned}$$

where $|||(Z_{j_k})||| = \max\{\| \sum Z_{j_{k_1}} Z^*_{j_{k_1}} \|^{\frac{1}{2}}, \ \| \sum Z^*_{j_{k_p}} Z_{j_{k_p}} \|^{\frac{1}{2}}, \sup_{j_k} \|Z_{j_k}\|\}$. ☐

The next corollary states the norm estimate in $B(H)$ rather than in $B(K \otimes H)$. For simplicity we replace $x^k_{i_k}$ by x_{i_k}.

Corollary 2.1.10. *If σ is a partition and x_{j_k} is a self-adjoint operator for arbitrary $k \in \{1, ..., d\}$, then*

$$\|[\sigma]\|_{B(H)} \leq \prod_{k \in \sigma_s} \|\sum x_{j_k}\| \cdot \prod_{k \in \sigma_{ns}} \|\sum x_{j_k}^2\|^{\frac{1}{2}}.$$

Proof. We need to discuss two cases:

(i) For $k \in \sigma_s$, $\|\sum_j Z_{j_k}\| = \|\sum 1 \otimes \cdots \otimes x_{j_k}\| = \|1 \otimes \cdots \otimes \sum x_{j_k}\| = \|\sum x_{j_k}\|$.

(ii) For $A_m \in \sigma_{ns}$,

$$\begin{aligned}
\|\sum Z_{j_{k_1}} Z_{j_{k_1}}^*\|^{\frac{1}{2}} &= \|\sum [1 \otimes \cdots \otimes e_{1 j_{k_1}} \otimes \cdots \otimes x_{j_{k_1}}] \cdot [1 \otimes \cdots \otimes e_{j_{k_1} 1} \otimes \cdots \otimes x_{j_{k_1}}^*]\|^{\frac{1}{2}} \\
&= \|\sum 1 \otimes \cdots \otimes e_{11} \otimes \cdots \otimes x_{j_{k_1}} x_{j_{k_1}}^*\|^{\frac{1}{2}} \\
&= \|1 \otimes \cdots \otimes \sum x_{j_{k_1}} x_{j_{k_1}}^*\|^{\frac{1}{2}} = \|\sum x_{j_{k_1}} x_{j_{k_1}}^*\|^{\frac{1}{2}} = \|\sum x_{j_{k_1}}^2\|^{\frac{1}{2}}. \qquad (2.1.4)
\end{aligned}$$

and

$$\begin{aligned}
\|\sum Z_{j_{k_p}}^* Z_{j_{k_p}}\|^{\frac{1}{2}} &= \|\sum [1 \otimes \cdots \otimes e_{1 j_{k_p}} \otimes \cdots \otimes x_{j_{k_p}}^*] \cdot [1 \otimes \cdots \otimes e_{j_{k_p} 1} \otimes \cdots \otimes x_{j_{k_p}}]\|^{\frac{1}{2}} \\
&= \|\sum 1 \otimes \cdots \otimes e_{11} \otimes \cdots \otimes x_{j_{k_p}}^* x_{j_{k_p}}\|^{\frac{1}{2}} = \|1 \otimes \cdots \otimes \sum x_{j_{k_p}}^* x_{j_{k_p}}\|^{\frac{1}{2}} \\
&= \|\sum x_{j_{k_p}}^* x_{j_{k_p}}\|^{\frac{1}{2}} = \|\sum x_{j_{k_p}}^2\|^{\frac{1}{2}}.
\end{aligned}$$

For the middle term, we have

$$\begin{aligned}
\sup_{k \in \{k_2, \ldots, k_{p-1}\}} \sup_{j_k} \|Z_{j_k}\| &= \sup_{k \in \{k_2, \ldots, k_{p-1}\}} \sup_{j_k} \|Z_{j_k}^* Z_{j_k}\|^{\frac{1}{2}} = \sup_{k \in \{k_2, \ldots, k_{p-1}\}} \sup_{j_k} \|x_{j_k}^* x_{j_k}\|^{\frac{1}{2}} \\
&\leq \sup_{k \in A_m} \|\sum x_{j_k}^2\|^{\frac{1}{2}}.
\end{aligned}$$

Combining (i) and (ii) finishes the proof. □

2.2 AGM inequality for the norm

In this section we prove the AGM inequality for the norm and for the order. We need the following lemma which handles positive or self-adjoint operators $\{x_{i_k}\}$ in a C*-algebra $\mathcal{A}$.

Lemma 2.2.1. *(i) If $x_{j_k} \geq 0$, then $\|\sum x_{j_k}^2\|^{\frac{1}{2}} \leq \|\sum x_{j_k}\|$.*

(ii) If x_{j_k} are self-adjoint, then $\|\sum x_{j_k}^2\|^{\frac{1}{2}} = \|(\sum x_{j_k}^2)^{\frac{1}{2}}\|$.

Proof. (i) Indeed, we have

$$\begin{aligned}
\|\sum x_{j_k}^2\|^{\frac{1}{2}} &= \|\sum x_{j_k}^{\frac{1}{2}} x_{j_k} x_{j_k}^{\frac{1}{2}}\|^{\frac{1}{2}} \\
&\leq (\|\sum x_{j_k}\|^{\frac{1}{2}} \cdot \|\sum x_{j_k}\| \cdot \|\sum x_{j_k}\|^{\frac{1}{2}})^{\frac{1}{2}} \\
&= \|\sum x_{j_k}\|.
\end{aligned}$$

(ii) Holds trivially using $\|x^2\| = \|x\|^2$, for $x = (\sum x_{j_k}^2)^{\frac{1}{2}}$. □

Now we have done all the preparation to prove the NC-AGM inequality for the norm.

Theorem 2.2.2. *Suppose $x_1, \ldots, x_n$ are positive operators in $B(H)$. Then*

$$\|P_d(x_1, ..., x_n)\|_{B(H)}^{1/d} \leq d\ \|P_1(x_1, ..., x_n)\|_{B(H)}.$$

Proof. From Corollary 2.1.10 and Lemma 2.2.1, we deduce that for a given arbitrary partition σ and positive elements $x_{j_k} = x_j$, we have

$$\|[\sigma]\|_{B(H)} \leq \|\sum x_j\|^d.$$

Recall identity 2.1.1 from Proposition 2.1.7:

$$\langle 1, \cdots, d\rangle = [1, \cdots, d] + \sum_{\upsilon \gneqq \dot{0}} \mu(\dot{0}, \nu)[\nu], \text{ where} \sum_{\upsilon \gneqq \dot{0}} |\mu(\dot{0}, \nu)| = d! - 1. \tag{2.2.1}$$

Taking the norm of both sides of the equality (2.2.1) we get

$$\begin{aligned}
\|\langle 1, \cdots, d\rangle\|_{B(H)} &= \|[1, \cdots, d] + \sum_{\upsilon \gneqq \dot{0}} \mu(\dot{0}, \nu)[\nu]\|_{B(H)} \\
&\leq \|[1, \cdots, d]\|_{B(H)} + \sum_{\upsilon \gneqq \dot{0}} |\mu(\dot{0}, \nu)| \|[\nu]\|_{B(H)} \\
&\leq \|\sum x_j\|_{B(H)}^d + (d! - 1)\|\sum x_j\|_{B(H)}^d \\
&\leq d!\|\sum x_j\|_{B(H)}^d \\
&= d! n^d \|\frac{1}{n}\sum x_j\|_{B(H)}^d \\
&= d! n^d \|P_1(x_1, ..., x_n)\|_{B(H)}^d.
\end{aligned}$$

Thus,$\|P_d(x_1,...,x_n)\|_{B(H)} \le \frac{d!n^d(n-d)!}{n!}\|P_1(x_1,...,x_n)\|_{B(H)}$. Denote $C(n,d) := \frac{d!n^d(n-d)!}{n!}$, and for fixed d define $f(n) := \sum\limits_{i=0}^{d-1} \log \frac{n}{n-i}$. Then

$$\begin{aligned} C(n,d) &= \frac{d!n^d(n-d)!}{n!} = \frac{d!n^d}{n(n-1)(n-2)\cdots(n-d+1)} \\ &= d! \cdot \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdots \frac{n}{n-d+1} \\ &= d! \cdot \exp(f(n)). \end{aligned}$$

Since $f(n)$ is a decreasing function in n, $C(n,d)$ is also a decreasing function with respect to the variable n. From the definition of d, we know $n \ge d$, so $\max\limits_{n \ge d} C(n,d) = C(d,d) = d^d$. □

2.3 AGM inequality for the order

Recall that the average product is defined by:

$$P_d(x_1,x_2,...,x_n) = \frac{(n-d)!}{n!} \sum_{\langle\sigma\rangle=\dot{0}} x_{i_1}...x_{i_d}.$$

Lemma 2.3.1. *Let $\{x_i\}$ be a finite family of positive operators in $B(H)$ which satisfy the condition $\sum\limits_{i=1}^{n} x_i = n$. If $a_i := x_i - 1$ then*

$$P_d(x_1,x_2,...,x_n) = 1 + \sum_{k=1}^{d} \binom{d}{k} P_k(a_1,a_2,...,a_n). \tag{2.3.1}$$

Proof. This lemma can be proved by two methods. The first method is by induction. For $d = 1$, the equation is trivial. Since $\sum\limits_i x_i = \sum\limits_i a_i + 1$, it follows that

$$\begin{aligned} P_1(x_1,x_2,...,x_n) = \frac{(n-1)!}{n!} \sum_{i=1}^{n} x_i = \frac{1}{n} \sum_{i=1}^{n} x_i \\ = 1 + \frac{1}{n} \sum_{i=1}^{n} a_i = 1 + \sum_{k=1}^{1} \binom{1}{k} P_k(a_1,...,a_n). \end{aligned}$$

Assume now this is true for $d = m$,

$$P_m(x_1,x_2,...,x_n) = 1 + \sum_{k=1}^{m} \binom{m}{k} P_k(a_1,a_2,...,a_n),$$

where $P_k(a_1, a_2, ..., a_n) = \frac{(n-k)!}{n!} \sum_{\langle\sigma\rangle = \dot{0}} a_{i_1}...a_{i_k}$. Then we need to check it is also true when $d = m+1$. Indeed,

$$\begin{aligned}
\frac{n!}{(n-m-1)!} P_{m+1}(x_1, x_2, ..., x_n) &= \sum_{i=1}^{n} x_i \sum_{\substack{\langle\sigma\rangle = \dot{0} \\ x_i \notin \sigma}} x_1 ... \check{x}_i ... x_{m+1} \\
&= \frac{(n-1)!}{(n-1-m))!} \sum_{i=1}^{n} (a_i + 1) P_m(x_1, x_2, ...\check{x}_i ..., x_n),
\end{aligned}$$

and

$$\begin{aligned}
& nP_{m+1}(x_1, x_2, ..., x_n) \\
&= \sum_{i=1}^{n} (a_i + 1) P_m(x_1, x_2, ...\check{x}_i ..., x_n) \\
&= \sum_{i=1}^{n} (a_i + 1) \big[1 + \sum_{k=1}^{m} \binom{m}{k} P_k(a, ...\check{a}_i, ..., a_n)\big] \\
&= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \Big[\sum_{k=1}^{m} \binom{m}{k} a_i P_k(a, ..., \check{a}_i, ..., a_n) + 1 + \sum_{k=1}^{m} \binom{m}{k} P_k(a_1, a_2, ..., \check{a}_i, ..., a_n) \Big] \\
&= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \sum_{k=1}^{m} \binom{m}{k} a_i P_k(a_1, ..., \check{a}_i, ..., a_n) + n + \sum_{i=1}^{n} \sum_{k=1}^{m} \binom{m}{k} P_k(a_1, a_2, ..., \check{a}_i, ..., a_n) \\
&= n + \sum_{i=1}^{n} a_i + \sum_{k=1}^{m} \binom{m}{k} P_{k+1}(a_1, ..., a_n).n + \sum_{k=1}^{m} \binom{m}{k} P_k(a_1, ..., a_n).n \\
&= n + \sum_{i=1}^{n} a_i + n \Big[\sum_{k=2}^{m+1} \binom{m}{k-1} P_k(a_1, ..., a_n) + \sum_{k=1}^{m} \binom{m}{k} P_k(a_1, ..., a_n) \Big].
\end{aligned}$$

Here $P_k(a_1, ..., \check{a}_i, ..., a_n)$ means we consider all the elements except a_i. Dividing both sides by n yields

$$\begin{aligned}
P_{m+1}(x_1, x_2, ..., x_n) &= 1 + \frac{1}{n} \sum_{i=1}^{n} a_i + \sum_{k=2}^{m+1} \binom{m}{k-1} P_k(a_1, ..., a_n) + \sum_{k=1}^{m} \binom{m}{k} P_k(a_1, ..., a_n) \\
&= 1 + \sum_{k=1}^{m+1} \binom{m}{k-1} P_k(a_1, ..., a_n) + \sum_{k=1}^{m} \binom{m}{k} P_k(a_1, ..., a_n) \\
&= 1 + \sum_{k=1}^{m} \Big[\binom{m}{k-1} + \binom{m}{k}\Big] P_k(a_1, ..., a_n) + P_{m+1}(a_1, ..., a_n) \\
&= 1 + \sum_{k=1}^{m} \binom{m+1}{k} P_k(a_1, ..., a_n) + P_{m+1}(a_1, ..., a_n) \\
&= 1 + \sum_{k=1}^{m+1} \binom{m+1}{k} P_k(a_1, ..., a_n).
\end{aligned}$$

For the second proof, we use the binomial identity. Then we have

$$\begin{aligned} P_d(x_1, ..., x_n) &= \frac{(n-d)!}{n!} \sum_{\langle\sigma\rangle=\dot{0}} x_{i_1}...x_{i_d} \\ &= \frac{(n-d)!}{n!} \sum_{\langle\sigma\rangle=\dot{0}} (a_{i_1}+1)(a_{i_2}+1)...(a_{i_d}+1) = 1 + \sum_{k=1}^{d} \lambda_k P_k(a_1, ..., a_n). \end{aligned}$$

Let $x_1 = x_2 = ... = x_n = t$, where $t = a + 1$. Then

$$P_d(x_1, ..., x_n) = t^d = (1+a)^d = 1 + \sum_{k=1}^{d} \binom{d}{k} a^k,$$

which implies that $\lambda_k = \binom{d}{k}$, so $P_d(x_1, ..., x_n) = 1 + \sum_{k=1}^{d} \binom{d}{k} P_k(a_1, ..., a_n)$. $\square$

In Theorem 2.0.3 for d=3, we deduce that each term in $P_3(x_1, ..., x_n)$ has an upper bound of some scalar multiple of $\sum a_i^2$. For $d > 3$, we need the following lemma.

Lemma 2.3.2. *If $\{x_i\}, \{a_i\}$ are defined as above, then*

$$\max_i \|a_i\| \leq \|\sum a_i^2\|^{\frac{1}{2}} \leq \|\sum_i x_i^2\|^{\frac{1}{2}}.$$

In particular, $\|a_i\|^k \leq n^k \|\frac{1}{n^2} \sum_i x_i^2\|^{\frac{k}{2}}$.

Proof. Since we have $a_j^2 \leq \sum a_i^2$,

$$\|a_i\| = \|a_i^2\|^{\frac{1}{2}} \leq \|\sum a_i^2\|^{\frac{1}{2}}.$$

Moreover, for each a_i, we have $x_i = a_i + 1$. Thus, we have

$$\sum x_i^2 = \sum a_i^2 + n \geq \sum a_i^2.$$

This finishes the proof. $\square$

Note that for a partition with $d = 3$, the proof of the AGM inequality in the order sense was easily done in the introduction. However, the proof is much more complicated for $d \geq 4$. The complication comes from crossing partitions, so we need the following useful known lemma [Pau02].

Lemma 2.3.3. *Assume $a, b \in B(H)$ and $t \geq 0$. Then*

$$(1) \quad -(a^*a + b^*b) \leq a^*b + b^*a \leq a^*a + b^*b.$$

$$(2) \quad ab + b^*a^* \leq t^2 aa^* + t^{-2} b^* b.$$

To prove (1), we start by observing $(a+b)^*(a+b), (a-b)^*(a-b) \geq 0$. This directly gives $-(a^*a+b^*b) \leq a^*b + b^*a$ and $a^*b + b^*a \leq a^*a + b^*b$. It is clear that (2) is a special case of (1), using the assumptions that $a = ta^*$ and $b = t^{-1}b$ for the upper bound of (1).

The two previous lemmas will help in establishing our result for general case of the AGM inequality for the order. For convenience, we will write $A_i := \sum_i Z_i$ where Z_i is defined as at the beginning of Section 2.1. We now provide upper and lower bounds for $P_d(a_{i_1}, ..., a_{i_n})$.

Lemma 2.3.4. *If $\{a_i\}$ and $\{x_i\}$ are defined as above, then for $S = \| \sum x_i^2 \|^{1/2}$*

$$-\frac{(n-d)!}{n!} d!\ S^{d-2} \sum a_i^2 \leq P_d(a_1, a_2, \cdots, a_n) \leq \frac{(n-d)!}{n!} d!\ S^{d-2} \sum a_i^2.$$

Proof. From Proposition 2.1.7, we know that

$$\frac{n!}{(n-d)!} P_d(a_1, a_2, \cdots, a_n) = \langle \dot{0} \rangle_d = [\dot{0}]_d + \sum_{\dot{0} \lneq \nu \leq \dot{1}} \mu(\dot{0}, \nu)[\nu]_d.$$

We will prove first the case when $\mu(\dot{0}, \nu) \geq 0$. We will obtain an upper bound for the sum $[\nu]_d$ by introducing $[\bar{\nu}]_d$ as the following:

$$[\nu]_d = \sum_{\langle i_1, i_2, \cdots, i_d \rangle \geq \nu} a_{i_1} a_{i_2} \cdots a_{i_d},$$

$$[\bar{\nu}]_d := \sum_{\langle i_1, i_2, \cdots, i_d \rangle \geq \nu} a_{i_d} a_{i_{d-1}} \cdots a_{i_1}.$$

Here the $\bar{\nu}$ can be viewed as the transposition of the partition ν. By Theorem 2.1.4, we have $\mu(\dot{0}, \pi) = \prod_{i=1}^{d} [(-1)^{i-1}(i-1)!]^{r_i(\pi)}$. So $\mu(\dot{0}, \nu) = \mu(\dot{0}, \bar{\nu})$. Thus, we can sum these two items together.

Claim: For every partition ν and $S = \| \sum x_i^2 \|^{1/2}$ we have

$$-2\ S^{d-2} \sum a_i^2 \leq [\nu]_d + [\bar{\nu}_d] \leq 2\ S^{d-2} \sum a_i^2.$$

The idea here is to use our modification of Pisier's trick for these two partitions. Recall that $Z_{i_1} = e_{1i_1} \otimes a_{i_1}$ is for the first component in the partition, $Z_{i_j} = e_{jj} \otimes a_{i_j}$ is for the elements in the middle of the partition, and $Z_{i_d} = e_{i_d 1} \otimes a_{i_d}$ is for the last element in the partition. Then we have

$$\begin{aligned}
[\nu]_d + [\bar{\nu}]_d &= \sum_{\langle i_1, i_2, \cdots, i_d \rangle \geq \nu} a_{i_1} a_{i_2} \cdots a_{i_d} + a_{i_d} a_{i_{d-1}} \cdots a_{i_1} \\
&= \sum_{i_1} Z_{i_1} \ldots \sum_{i_d} Z_{i_d} + \sum_{i_d} Z^*_{i_d} \ldots \sum_{i_1} Z^*_{i_1} \\
&= A_1 \ldots A_d + A^*_d \ldots A^*_1.
\end{aligned}$$

By applying Lemma 2.3.3 with $S = \| \sum x_i^2 \|^{1/2}$, we obtain

$$\begin{aligned}
[\nu]_d + [\bar{\nu}]_d &= A_1 \cdots A_d + A^*_d \cdots A^*_1 && (2.3.2) \\
&\leq t^2 A_1 A^*_1 + t^{-2} A^*_d \cdots A^*_2 A_2 \cdots A_d \\
&\leq t^2 A_1 A^*_1 + t^{-2} \prod_{j=2}^{d-1} \|A^*_j A_j\| A^*_d A_d \\
&\leq t^2 A_1 A^*_1 + t^{-2} \prod_{j=2}^{d-1} \|A_j\|^2 A^*_d A_d \\
&\leq t^2 A_1 A^*_1 + t^{-2} \prod_{j=2}^{d-1} \| \sum a_j^2 \| A^*_d A_d && (2.3.3) \\
&\leq t^2 A_1 A^*_1 + t^{-2} \| \sum a_i^2 \|^{d-2} A^*_d A_d && (2.3.4) \\
&\leq \| \sum a_i^2 \|^{d/2-1} (A_1 A^*_1 + A^*_d A_d) \leq 2 \sum a_i^2 \; S^{d-2}.
\end{aligned}$$

Indeed, if our partition contains the singleton then $[\nu]_d + [\bar{\nu}]_d$ is already zero. Hence we may assume there are no singletons in our partition as it also can be noticed in inequality (2.3.4). Indeed, if the index is a singleton in partition ν, then it is controlled by the summation norm $\| \sum a_i \|$ which is zero by our construction. On the other hand, if the index is in a non-singleton block, then by Theorem 2.1.10 it is controlled by the square norm $\| \sum a_i^2 \|$. Therefore, in both cases, $\|A_i\|$ is controlled by the square norm of a_i. To get inequality (2.3.3), we may apply the norm equality as in equality (2.1.4) from section 2. For the inequality (2.3.4), we use Lemma (2.3.3) by choosing $t^2 = S^{d/2-1}$. Then we have

$$\begin{aligned}
\frac{n!}{(n-d)!} P_d(a_1, a_2, \cdots, a_n) &= [\dot{0}]_d + \sum_{\dot{0} \lneq \nu \leq \dot{1}} \mu(\dot{0}, \nu) [\nu]_d \\
&= \sum_{\dot{0} \lneq \nu \leq \dot{1}} \mu(\dot{0}, \nu) [\nu]_d \\
&= \sum_{\mu(\dot{0}, \nu) \geq 0} \mu(\dot{0}, \nu) [\nu]_d + \sum_{\mu(\dot{0}, \nu) \leq 0} \mu(\dot{0}, \nu) [\nu]_d
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\sum_{\mu(\dot{0},\nu)\geq 0} \mu(\dot{0},\nu)[\nu]_d + \sum_{\mu(\dot{0},\bar{\nu})\geq 0} \mu(\dot{0},\bar{\nu})[\bar{\nu}]_d\Big) \\
&+ \frac{1}{2}\Big(\sum_{\mu(\dot{0},\nu)\leq 0} \mu(\dot{0},\nu)[\nu]_d + \sum_{\mu(\dot{0},\bar{\nu})\leq 0} \mu(\dot{0},\bar{\nu})[\bar{\nu}]_d\Big) \\
&\leq \sum_{\mu(\dot{0},\nu)\geq 0} \mu(\dot{0},\nu) S^{d-2} \sum a_i^2 - \sum_{\mu(\dot{0},\nu)\leq 0} \mu(\dot{0},\nu) S^{d-2} \sum a_i^2 \\
&= \sum |\mu(\dot{0},\nu)|\ S^{d-2}(\sum a_i^2) = d!\ S^{d-2} \sum a_i^2.
\end{aligned}
$$

For the lower bound, the proof is similar to the one above replacing A_1 by $-A_1$. $\square$

Theorem 2.3.5. *(AGM inequality for the order) Fix n and d. Let $x_1, ..., x_n$ be self-adjoint operators such that $\sum_i x_i = n$ and $a_i = x_i - 1$ as above. Assume the following conditions hold:*

i) $P_1(x_1, ..., x_n) = \frac{\sum_1^n x_i}{n} = 1,$

ii) $\|(\sum x_i^2)^{\frac{1}{2}}\| \leq \frac{n}{3d}.$

Then the AGM inequality holds in the order sense:

$$
P_d(x_1, x_2, \cdots, x_n) \leq \left(\frac{\sum_1^n x_i}{n}\right)^d = 1.
$$

Proof. According to Lemma 2.3.2, we have $\|\sum a_i^2\|^{1/2} \leq \frac{n}{3d}$. Using this upper bound for the average of noncommutative operators a_i with the identity (2.3.1) where $S = \|\sum x_i^2\|^{1/2} \leq \Delta n$ and let $\Delta := \frac{1}{3d}$, we have

$$
\begin{aligned}
P_d(x_1, x_2, ..., x_n) &= 1 + \sum_{k=1}^{d} \binom{d}{k} P_k(a_1, a_2, ..., a_n) \\
&= 1 - \binom{d}{2} \frac{(n-2)!}{n!} (\sum a_i^2) + \sum_{k=3}^{d} \binom{d}{k} P_k(a_1, a_2, ..., a_n) \\
&\leq 1 - \binom{d}{2} \frac{(n-2)!}{n!} (\sum a_i^2) + \sum_{k=3}^{d} \binom{d}{k} \frac{(n-k)!}{n!} k! \Delta^{k-2} n^{k-2} (\sum a_i^2).
\end{aligned}
$$

Now we need the following condition:

$$
\binom{d}{2} \frac{(n-2)!}{n!} \overset{?}{\geq} \sum_{k=3}^{d} \binom{d}{k} \frac{(n-k)!}{n!} k! \Delta^{k-2} n^{k-2}.
$$

Simplifying the right hand side gives

$$\begin{aligned}
\sum_{k=3}^{d}\binom{d}{k}\frac{(n-k)!}{n!}k!\Delta^{k-2}n^{k-2} &= \sum_{k=3}^{d}\frac{d!}{(d-k)!k!}\frac{(n-k)!k!}{n!}\Delta^{k-2}n^{k-2} \\
&= \sum_{k=3}^{d}\frac{d!}{(d-k)!}\frac{(n-k)!}{n!}\Delta^{k-2}n^{k-2} \\
&= \frac{1}{n(n-1)}\sum_{k=3}^{d}\frac{d!}{(d-k)!}\frac{n^{k-2}}{(n-2)\cdots(n-k+1)}\Delta^{k-2}. \qquad (2.3.5)
\end{aligned}$$

Fix k, and denote $f(n) := \frac{n^{k-2}}{(n-2)\cdots(n-k+1)}$. Then, by taking the logarithm, we have $g(n) := \log f(n) = \sum_{i=2}^{k-1}\log\frac{n}{n-i}$. Observe that $g(n)$ is a decreasing function and thus $f(n)$ is a decreasing function as well. Therefore, we get the inequality:

$$\frac{n^{k-2}}{(n-2)\cdots(n-k+1)} \leq \frac{d^{k-2}}{(d-2)\cdots(d-k+1)}. \qquad (2.3.6)$$

We continue the calculation in (2.3.5) with the help of inequality (2.3.6), we have

$$\begin{aligned}
\sum_{k=3}^{d}\binom{d}{k}\frac{(n-k)!}{n!}k!\Delta^{k-2}n^{k-2} &= \frac{1}{n(n-1)}\sum_{k=3}^{d}\frac{d!}{(d-k)!}\frac{n^{k-2}}{(n-2)\cdots(n-k+1)}\Delta^{k-2} \\
&\leq \frac{1}{n(n-1)}\sum_{k=3}^{d}\frac{d!}{(d-k)!}\frac{d^{k-2}}{(d-2)\cdots(d-k+1)}\Delta^{k-2} \\
&\leq \frac{1}{n(n-1)}\sum_{k=3}^{d}d(d-1)d^{k-2}\Delta^{k-2} \\
&= \frac{d(d-1)}{n(n-1)}\frac{d\Delta(1-(d\Delta)^{d-2})}{1-d\Delta} \leq \frac{d(d-1)}{n(n-1)}\frac{d\Delta}{1-d\Delta}.
\end{aligned}$$

With our choice of $\Delta = \frac{1}{3d}$ we deduce indeed $\frac{d(d-1)}{n(n-1)}\frac{d\Delta}{1-d\Delta} \leq \binom{d}{2}\frac{(n-2)!}{n!}$ and this completes the proof. $\square$

Chapter 3

AGM inequalities in application

In this section we show that a combination of Pisier's partition method and probabilistic results allow AGM inequalities hold in many different scenarios. We confirm the AGM inequality up to ε for many random matrices, in particular for Wishart random matrices, more general vector-valued moments of convex bodies, and freely independent operators. We should point out that in contrast to results on averages of random matrices in Ré and Recht in [RR12], our estimates hold with high probabilities. In this section, we prove a version of the non-commutative AGM inequality for random matrices. We start with a deviation inequality. Let us use the norm $|||X|||_p = (E\|X\|^p_{B(H)})^{1/p}$ defined for a random variable $X : \Omega \to B(H)$.

3.1 AGM inequality for random matrices

Proposition 3.1.1. *Let $\{a_i\}$ be a family of self-adjoint random operators. Let $\varepsilon > 0$, $p \geq 2$, $p_d = \frac{p}{d}$ and $x_i = a_i + 1$. Define*

(i) $\varepsilon_p := \left|\left|\left| \frac{1}{n} \sum a_i - E\frac{1}{n}\sum a_i \right|\right|\right|_p$,

(ii) $\delta_p := \frac{1}{n} \left|\left|\left| (\sum a_i^2)^{1/2} \right|\right|\right|_p$,

(iii) $\gamma_p := \max(\varepsilon_p, \delta_p)$.

Assume $\sum\limits_i Ea_i = 0$, $\gamma_p \leq \frac{1}{3d}$ *and* $\varepsilon = 3d\gamma_p$.
Then, $|||P_d(x_1, ..., x_n) - EP_d(x_1, ..., x_n)|||_{p_d} \leq \varepsilon$.

Proof. From the assumption above, we get that $\left|\left|\left| (\frac{1}{n}\sum a_i) \right|\right|\right|_p = \varepsilon_p$. Fix a partition ν. According to Corollary 2.1.10 and by using Hölder's inequality we have that

$$\begin{aligned} E\|[\nu]\|_\infty^{p_d} &\leq E\left(\|(\sum a_i^2)^{1/2}\|_\infty^{(d-|\nu_s|)p_d} \|(\sum a_i)\|_\infty^{|\nu_s|p_d} \right) \\ &= E\left(\|(\sum a_i^2)^{1/2}\|_\infty^{\frac{(d-|\nu_s|)p_d d}{d}} \cdot \|(\sum a_i)\|_\infty^{\frac{|\nu_s|p_d d}{d}} \right) \end{aligned}$$

$$
\begin{aligned}
&\leq \left(E(\|(\sum a_i^2)^{1/2}\|_\infty^{p_d d}\right)^{\frac{p_d(d-|\nu_s|)}{p_d d}} \left(E\|(\sum a_i)\|_\infty^{p_d d}\right)^{\frac{p_d|\nu_s|}{p_d d}} \\
&= \left(E(\|(\sum a_i^2)^{1/2}\|_\infty^{p}\right)^{\frac{p_d(d-|\nu_s|)}{p}} \left(E\|(\sum a_i)\|_\infty^{p}\right)^{\frac{p_d|\nu_s|}{p}} \\
&= \left(\left\|\left\|(\sum a_i^2)^{1/2}\right\|\right\|_p\right)^{p_d(d-|\nu_s|)} \left(\left\|\left\|\sum a_i\right\|\right\|_p\right)^{p_d|\nu_s|} \\
&= (\delta_p \cdot n)^{p_d(d-|\nu_s|)}.(\varepsilon_p \cdot n)^{p_d|\nu_s|} \\
&= \delta_p^{p_d(d-|\nu_s|)}\varepsilon_p^{p_d|\nu_s|}n^p = \delta_p^{p_d(d-|\nu_s|)}\varepsilon_p^{p_d|\nu_s|}n^{p_d d}.
\end{aligned}
$$

Since $\gamma_p = \max(\delta_p, \varepsilon_p)$,

$$
\|\|[\nu]\|\|_{p_d} = (E\|[\nu]\|_\infty^{p_d})^{\frac{1}{p_d}} \leq \gamma_{p_d d}^d \cdot n^d \tag{3.1.1}
$$

By using our definition of γ_p and the upper bound for inequality (3.1.1) we obtain

$$
\begin{aligned}
&(E\|P_k(a_1,...,a_n) - EP_k(a_1,...,a_n)\|_\infty^{p_d})^{1/p_d} \\
&\leq \frac{(n-k)!}{n!}\sum |\mu(0,\nu)|\Big(E(\|[\nu] - E[\nu]\|_\infty)^{p_d}\Big)^{1/p_d} \\
&\leq \frac{(n-k)!}{n!}\sum |\mu(0,\nu)| \cdot 2(E\|[\nu]\|_\infty^{p_d})^{1/p_d} \\
&\leq 2\frac{(n-k)!}{n!}k!\gamma_{p_d k}^k n^k.
\end{aligned}
$$

From the above we will have

$$
\begin{aligned}
&\|\|P_d(x_1,...,x_n) - EP_d(x_1,...,x_n)\|\|_{p_d} \\
&= \left\|\left\|\sum_{k=1}^d \binom{d}{k}(P_k(a_1,...,a_n) - EP_k(a_1,...,a_n))\right\|\right\|_{p_d} \\
&= (E\|\sum_{k=1}^d \binom{d}{k}(P_k(a_1,...,a_n) - EP_k(a_1,...,a_n))\|_\infty^{p_d})^{1/p_d} \\
&\leq \sum_{k=1}^d \binom{d}{k}(E\|(P_k(a_1,...,a_n) - EP_k(a_1,...,a_n))\|_\infty^{p_d})^{1/p_d} \\
&\leq 2\sum_{k=1}^d \binom{d}{k}\frac{(n-k)!}{n!}k!\gamma_{p_d k}^k \cdot n^k = 2\sum_{k=1}^d \frac{d!}{k!(d-k)!}\frac{(n-k)!}{n!}k!\gamma_{p_d k}^k \cdot n^k \\
&\leq 2\sum_{k=1}^d \frac{d!(n-k)!n^k}{(d-k)!n!}\gamma_p^k.
\end{aligned} \tag{3.1.2}
$$

Recall the definition $\gamma_{p_d k} = \max(\delta_{p_d k}, \varepsilon_{p_d k})$. Each $\delta_{p_d k}, \varepsilon_{p_d k}$ is increasing since $L_{p_d k}$ is defined as probability space which is norm increasing in probability measure. Thus $\gamma_{p_d k} \leq \gamma_{p_d d} = \gamma_p, \forall\ 1 \leq k \leq d$, which justifies the last inequality (3.1.2). Let $f(n) = \frac{d!(n-k)!n^k}{(d-k)!n!}$. This function is a decreasing function in n, so $f(d) = \max f(n) = d^k$. Then we have

$$\begin{aligned}
&|||P_d(x_1,...,x_n) - EP_d(x_1,...,x_n)|||_{p_d} \\
&\leq 2\sum_{k=1}^{d}(d\cdot\gamma_p)^k = 2\cdot d\cdot\gamma_p\frac{(1-(d\cdot\gamma_p)^d)}{1-d\cdot\gamma_p} \leq 2\cdot\frac{d\cdot\gamma_p}{1-d\cdot\gamma_p} \leq \varepsilon.
\end{aligned}$$

The last inequality follows from $d\cdot\gamma_p \leq \frac{\varepsilon}{1-\varepsilon/2}$. $\square$

We now present conditions for positive random operators $\{x_i\}$ where $a_i = x_i - 1$. Note that for $A := \sum_{i=1}^n \frac{a_i}{n}$, we have $E\|A - EA\|_p = E\|(\frac{\sum x_j}{n}) - E(\frac{\sum x_j}{n})\|_p$. Therefore, whenever we control the x_i's, we control the a_i's.

Lemma 3.1.2. *Let $\{x_i\}$ be a family of self-adjoint random operators. Then*

$$\left|\left|\left|(\sum_1^n (x_i-1)^2)^{1/2}\right|\right|\right|_p \leq 6\left|\left|\left|(\sum_1^n x_i^2)^{1/2}\right|\right|\right|_p.$$

Proof. Observe that $\left|\left|\left|(\sum x_i^2)^{1/2}\right|\right|\right|_p = |||\sum e_{i,1}\otimes x_i|||_p$ is given by the column norm. Define operators $\phi : C_n(B(H)) \to C_n(B(H))$ and $\Phi : C_n \to C_n$ such that $\Phi(\alpha_i) = (\frac{1}{n}\sum_i \alpha_i)_j$ where $\phi = \Phi \otimes Id$. Then it is easy to check that $\|\Phi\|_{cb} = \|\phi\|_{cb} \leq 1$. Indeed

$$\|\sum_{j=1}^n e_{j,1}\otimes\Phi(y_i)_j\| = \|\sum_{j=1}^n e_{j,1}\otimes(\frac{1}{n}\sum_{i=1}^n y_i)\| = \frac{1}{\sqrt{n}}\|\sum_{j=1}^n y_j\| \leq \|\sum_i e_{1,i}\otimes y_i\|.$$

Denote $z_i := x_i - Ex_i$, so $\|\sum_i e_{1,i}\otimes(Id+\phi)(z_i)\| \leq 2\|\sum_i e_{1,i}\otimes x_i\|$. Also,

$$\begin{aligned}
(Id+\phi)(z_i) &= x_i - Ex_i + \frac{1}{n}\sum x_i - \frac{1}{n}\sum Ex_i = x_i - 1 - Ex_i + \frac{1}{n}\sum x_i, \\
(x_i - 1) &= (Id+\phi)(z_i) + Ex_i - \frac{1}{n}\sum x_i.
\end{aligned}$$

By the triangle inequality, we can get

$$\begin{aligned}
&\left|\left|\left|\sum(x_i-1)\otimes e_{i,1}\right|\right|\right| \\
&\leq \left|\left|\left|\sum(Id+\phi)(z_i)\otimes e_{i,1}\right|\right|\right| + \left|\left|\left|\sum Ex_i\otimes e_{i,1}\right|\right|\right| + \left|\left|\left|\sum_j(\frac{1}{n}\sum x_i)\otimes e_{j,1}\right|\right|\right|
\end{aligned}$$

$$\leq 2\left\|\left|\sum z_i \otimes e_{i,1}\right|\right\| + \left\|\left|\sum Ex_i \otimes e_{i,1}\right|\right\| + \left\|\left|\sum_j (\frac{1}{n}\sum x_i) \otimes e_{j,1}\right|\right\|$$

$$= 2\left\|\left|\sum (x_i - Ex_i) \otimes e_{i,1}\right|\right\| + \left\|\left|\sum Ex_i \otimes e_{i,1}\right|\right\| + \left\|\left|\sum_j (\frac{1}{n}\sum x_i) \otimes e_{j,1}\right|\right\|$$

$$\leq 2\left\|\left|\sum x_i \otimes e_{i,1}\right|\right\| + 3\left\|\left|\sum Ex_i \otimes e_{i,1}\right|\right\| + \left\|\left|(\sum x_i^2)^{\frac{1}{2}}\right|\right\|$$

$$\leq 2\left\|\left|(\sum x_i^2)^{\frac{1}{2}}\right|\right\| + 3\left\|\left|\sum x_i \otimes e_{i,1}\right|\right\| + \left\|\left|(\sum x_i^2)^{\frac{1}{2}}\right|\right\| = 6\left\|\left|(\sum x_i^2)^{\frac{1}{2}}\right|\right\|.$$

The second-to-last inequality $|||\sum Ex_i \otimes e_{i,1}||| \leq |||\sum x_i \otimes e_{i,1}|||$ follows from the fact that the conditional expectation from $E : L_\infty(\Omega, B(H)) \to B(H)$ is a complete contraction. The inequality $\left|\left|\left|\frac{1}{n}\sum x_i\right|\right|\right| \leq \left|\left|\left|(\sum x_i^2)^{\frac{1}{2}}\right|\right|\right|$ is true by the Cauchy-Schwarz inequality. □

Thanks to Theorem 3.1.1 and Lemma 3.1.2 we obtain the following deviation result.

Theorem 3.1.3. *Let $p \geq 2$, $p_d := \frac{p}{d}$, and $\{x_i\}$ be a random family of positive operators such that $Ex_i = 1$. Define*

(i) $\varepsilon_p := \left|\left|\left|\frac{1}{n}\sum x_i - E\frac{1}{n}\sum x_i\right|\right|\right|_p$,

(ii) $\delta_p := \frac{1}{n}\left|\left|\left|(\sum x_i^2)^{1/2}\right|\right|\right|_p$,

(iii) $\gamma_p := \max(\varepsilon_p, 4\delta_p)$.

If $3d \cdot \gamma_p \leq 1$ then

$$|||P_d(x_1, ..., x_n) - EP_d(x_1, ..., x_n)|||_{p_d} \leq 3d \cdot \gamma_p.$$

Corollary 3.1.4. *If in addition $\{x_i\}$ are matrix-valued i.i.d. Then*

$$|||P_d(x_1, ..., x_n)|||_{p_d} \leq 1 + 3d \cdot \gamma_p.$$

Proof. Since x_i's are matrix-valued i.i.d, then $E(P_d(x_1, ..., x_n)) = P_d(Ex_1, ..., Ex_n)$. Moreover, for $\varepsilon := 3d\cdot\gamma_p$ by (ii) in the above Theorem 3.1.3, we have

$$\left\|\left|(\sum_1^n Ex_i^2)^{1/2}\right|\right\|_p = \left\|\left|\sum_1^n E(x_i) \otimes e_{i,1}\right|\right\|_{C_n \otimes B(H)} \leq \left\|\left|\sum_1^n x_i \otimes e_{i,1}\right|\right\|_{C_n \otimes B(H)} \leq \delta_p \cdot n.$$

Then we can use Theorem 2.3.5 for $E(x_i)$'s and the classical AGM inequality (here $\delta_p \leq \frac{1}{4}\gamma_p \leq \frac{1}{4d} < \frac{1}{3d}$).

$$E(P_d(x_1, ..., x_n)) \leq P_1(Ex_1, ..., Ex_n) = \sum_1^n \frac{Ex_i}{n} = 1.$$

Using the upper bound above and Theorem 3.1.3, we have the required inequality.

$$\left\|\left\|P_d(x_1,...,x_n)\right\|\right\|_{p_d} \le \left\|\left\|P_1(Ex_1,...,Ex_n)\right\|\right\|_{p_d} + \epsilon \le 1+\epsilon.$$

3.2 Applications for log concave measures

In this section we want to study random AGM inequalities for log-concave measures.

Definition 3.2.1. A Borel measure μ on n-dimensional Euclidean space $\mathbb{R}^n$ is called logarithmically concave (or log-concave) if for any compact subsets A and B of $\mathbb{R}^n$ and $0 \le \lambda \le 1$ we have

$$\mu\big(\lambda A + (1-\lambda)B\big) \ge \mu\big(A\big)^{\lambda}\mu\big(B\big)^{(1-\lambda)}.$$

Let us recall the isotropic measure μ in $\mathbb{R}^n$.

Definition 3.2.2. The isotropic measure μ is the measure which satisfies

$$\int_{\mathbb{R}^n} |\langle\theta, x\rangle|^2 d\mu(x) = L_\mu \|\theta\|^2,$$

for all $\theta \in \mathbb{R}^n$ where L_μ is denoted as isotropic constant.

Also let us recall Rosenthal's inequality, which will be used frequently in this section.

Theorem 3.2.3. [JZ13] *Let A_i be a fully independent sub-algebra over N where $N \subset M$ and M is a von Neumann algebra, and $1 \le p < \infty$. Let $x_i \in L_p(A_i)$ with $E_N(x_i) = 0$. Then*

$$\|\sum_{i=1}^n x_i\|_p \le C \max\{\sqrt{p}\|\sum_{i=1}^n E_N(x_i^* x_i + x_i x_i^*)^{1/2}\|_p, p(\sum_{i=1}^n \|x_i\|_p^p)^{1/p}\}.$$

We can prove the following result.

Theorem 3.2.4. *Let $n,\ d \in \mathbb{N}$, $p \ge 2$. Let $(\mathbb{R}^d, \mu)$ be log-concave Borel measure μ in isotropic position on $\mathbb{R}^d$ with constant L. Define random variable $y : \mathbb{R}^d \to \mathbb{R}^d$ by $y(\omega) = \frac{\omega}{\sqrt{L}}$ where $\omega \in \mathbb{R}^d$. Let y_i be independent copies of y. Then $x_i(\omega) := |y_i(\omega)\rangle\langle y_i(\omega)|$ is a $d \times d$ random matrix satisfying*

(i) $\forall i,\ Ex_i = 1,$

(ii) $\left\|\left\|\sum_1^n (x_i - Ex_i)\right\|\right\|_p \le \gamma_p \cdot n,$

(iii) $\left\|\left|\sum x_i^2\right|\right\|_p^{1/2} \leq \gamma_p \cdot n,$

where

$$\gamma_p = \begin{cases} p^{1/2}\sqrt{\frac{d}{n}} + p^{5/2}\frac{d}{n} & p \geq \ln n \ \textit{or} \\ 2C\sqrt{\ln d}\delta^{1/2} & d \leq \frac{n}{\ln n^5} \\ 2C(\ln n)^3\delta & d \geq \frac{n}{\ln n^5}. \end{cases}$$

(iv) Moreover, assume $\gamma_p \leq (1 - \frac{2}{2+\varepsilon})\frac{1}{d}$, $\varepsilon \geq 0$, *and* $p_k := \frac{p}{k}$. *Then the following hold.*

(a) $\|\|P_k(x_1, ..., x_n) - EP_k(x_1, ..., x_n)\|\|_{p_k} \leq \varepsilon.$

(b) The AGM inequality holds $\|\|P_k(x_1, ..., x_n)\|\|_{p_k} \leq (1 + 2\varepsilon).$

Proof. We apply Rosenthal's inequality for $q \geq p$ to $x_i - 1$ instead of x_i. Let us introduce the norm in the space $L_q(S_q)$ where S_q is the Schatten class, $|x|_q := (E\|x_i\|_{S_q}^q)^{\frac{1}{q}} = (\int \|x(\omega)\|_{S_q}^q d\mu)^{\frac{1}{q}}$, where $\|x(\omega)\|_{S_q}^q = \mathrm{tr}(|x|^q)$. So, we have

$$\begin{aligned}
&\left\|\left|\sum_{i=1}^n (x_i - Ex_i)\right|\right\|_q \\
&\leq c\max\{\sqrt{q}|\sum_{i=1}^n E((x_i-1)^*(x_i-1) + (x_i-1)(x_i-1)^*)|_{\frac{q}{2}}^{\frac{1}{2}}, q(\sum_{i=1}^n |x_i - Ex_i|_q^q)^{\frac{1}{q}}\} \\
&\leq c\sqrt{q}|\sum_{i=1}^n E((x_i-1)^*(x_i-1) + (x_i-1)(x_i-1)^*)|_{\frac{q}{2}}^{\frac{1}{2}} + cq(\sum_{i=1}^n |x_i - Ex_i|_q^q)^{\frac{1}{q}} \\
&\leq 2c\sqrt{q}|\sum_{i=1}^n E(x_i-1)^2|_{\frac{q}{2}}^{\frac{1}{2}} + cq(\sum_{i=1}^n |x_i - Ex_i|_q^q)^{\frac{1}{q}} \\
&\leq 2c\sqrt{q}|\sum_{i=1}^n Ex_i^2|_{\frac{q}{2}}^{\frac{1}{2}} + 2cqn^{1/q}|x_1|_q
\end{aligned}$$

By Rosenthal's inequality, we need to separately estimate the two terms of the right side. We denote

$$\mathrm{I} = |(\sum_1^n Ex_i^2)^{1/2})|_q \text{ and } \mathrm{II} = |x_1|_q.$$

We claim that (ii) holds for γ_q and $Ex_i^2 \leq dEx_i \leq cd \cdot \mathbf{1}_{M_d}$. Using Borel inequality (see [MS86] where $\|.\|$ is seminorm), we have

$$(E\|y\|_X^q)^{\frac{1}{q}} \leq C_q E\|y\|_X \leq C_q(E\|y\|_X^2)^{\frac{1}{2}}.$$

Recall that $E\|y\|^2 = \sum_{i=1}^d E|\langle e_i, \frac{\omega}{\sqrt{L}}\rangle|^2 = d$. So, we have for $x_i := x_1 = |y\rangle\langle y|$

$$\langle\theta, Ex_1^2\theta\rangle = E\langle\theta, y\rangle\langle y, y\rangle\langle\theta, y\rangle = E\|y\|^2|\langle\theta, y\rangle|^2$$

$$\text{(by Cauchy Shwarz inequality)} \leq (E\|y\|^4)^{\frac{1}{2}}(E|\langle\theta, y\rangle|^4)^{\frac{1}{2}}$$

$$\text{(by Borel inequality)} \leq C_4^4\ E\|y\|^2 E(|\langle\theta, y\rangle|^2) = C_4^4 \cdot d\ \|\theta\|^2.$$

i.e. $Ex_i^2 \leq dEx_i \leq cd \cdot \mathbf{1}_{M_d}$. This implies

$$|(\sum_1^n Ex_i^2)^{1/2})|_q \leq C \cdot d^{1/2+1/q}$$

which proves our claim for (I). For (II), note that the q-norm is defined to be $|x|_q = (E\mathrm{tr}|x|^q)^{\frac{1}{q}}$. We use a Bra-ket notation where $|y_i\rangle$ and $\langle y_i|$ represent a row and column operators such that $\langle y_i, y_i\rangle \in \mathbb{C}$ and $|y_i\rangle\langle y_i| \in M_n$. Let's first take $q = m$ be an integer. We have

$$\begin{aligned} x_i^m =& (|y_i\rangle\langle y_i|)^m = |y_i\rangle\langle y_i, y_i\rangle \cdots \langle y_i, y_i\rangle\langle y_i| \\ =& |y_i\rangle\|y_i\|^{2(m-1)}\langle y_i|. \end{aligned}$$

Then, by using the Borel inequality (where $C_{2m} := C \cdot 2m$)(see [MS86] for details about C_{2m}), we have

$$\begin{aligned} E\mathrm{tr}(x_i^m) =& E\mathrm{tr}(y_i\rangle\|y_i\|^{2(m-1)}\langle y_i) = E(\|y_i\|_2^{2m}) \\ &\leq (C \cdot 2m)^{2m}((E\|y\|_2^2)^{1/2})^{2m} \\ &\leq (C \cdot 2m)^{2m} d^m. \end{aligned}$$

So we get the inequality $|x|_m \leq (C \cdot 2m)^2 d$ for arbitrary integer m. Then for any real number q, we can find an integer m, such that $m \leq q \leq m+1$, and by interpolation between m and $m+1$, we get

$$|x|_q \leq (C \cdot 2q)^2\ d. \tag{3.2.1}$$

Thanks to (3.2.1), we can now prove condition (iii).

$$\begin{aligned} \left|\left|\left|(\sum_1^n x_i^2)^{1/2}\right|\right|\right|_q &\leq \left|\left|\left|\sum_1^n x_i^2\right|\right|\right|_{\frac{q}{2}}^{1/2} \leq (\sum_1^n |||x_i^2|||_{\frac{q}{2}})^{1/2} \\ &\leq (\sum_1^n |||x_i|||_q)^{1/2} = \sqrt{n}|||x_1|||_q \\ &\leq \sqrt{n}|x_1|_q \leq \sqrt{n} d\ (C \cdot 2q)^2. \end{aligned}$$

Combining (I) and (II) we obtain

$$\begin{aligned}\left\| \left\| \sum (x_i - 1) \right\| \right\|_q \leq & \tilde{c}(qnd)^{1/2} d^{1/q} + cqCn^{1/q} q^2 d \\ = & \tilde{C}(qn)^{1/2} d^{\frac{1}{2}+\frac{1}{q}} + C' n^{1/q} q^3 d.\end{aligned}$$

And then divide each term by n, we have

$$\begin{aligned}\frac{\|\|\sum (x_i - 1)\|\|_q}{n} = \frac{\|\|\sum (x_i - Ex_i)\|\|_q}{n} &\leq C(q,d,n) := \left(\frac{q}{n}\right)^{1/2} d^{\frac{1}{2}+\frac{1}{q}} + n^{\frac{1}{q}-1} q^3 d \\ &= \left(\frac{q}{n}\right)^{1/2} d^{\frac{1}{2}+\frac{1}{q}} + \frac{qd}{n} q^2 n^{\frac{1}{q}} \\ &= d^{1/q}\left(q^{1/2}\left(\frac{d}{n}\right)^{1/2} + q^3 \frac{d}{n}\,\frac{n^{1/q}}{d^{1/q}}\right) = d^{1/q}\left(q^{1/2}\left(\frac{d}{n}\right)^{1/2} + q^3\left(\frac{d}{n}\right)^{1-1/q}\right).\end{aligned}$$

If we denote $\frac{d}{n} = \delta$, then

$$\begin{aligned}\frac{\|\|\sum (x_i - Ex_i)\|\|_q}{n} &\leq d^{\frac{1}{q}} (q^{\frac{1}{2}} \delta^{\frac{1}{2}} + q^3 \delta^{1-\frac{1}{q}}) \\ &= d^{1/q} q^{1/2} \delta^{1/2} (1 + q^{5/2} \delta^{1/2-1/q}).\end{aligned}$$

Now our goal is to find $\hat{\gamma}_q = \inf\limits_{q \geq q_0} d^{1/q} q^{1/2} \delta^{1/2} (1 + q^{5/2} \delta^{1/2-1/q})$ by optimization over q where $q_0 \geq 2$. Define $f(q,\delta) := q^{5/2} \delta^{1/2-1/q}$ and consider $g := \ln f(q,\delta) = \frac{5}{2} \ln q + (\frac{1}{2} - \frac{1}{q}) \ln \delta$, with derivative $g' = \frac{5}{2}\frac{1}{q} + \frac{1}{q^2} \ln \delta$. The critical point for $f(q,\delta)$ is $q(\delta) = \frac{2}{5} \ln \frac{1}{\delta}$. Since $f(q,\delta)$ is a convex function then it has no more than one minimum point which is $q(\delta)$. Then we have to consider the following cases for the choices of q,

1. $q_0 \leq \ln d \leq q_1$ where $q_1 = (\frac{1}{\delta})^{1/5}$
2. $\ln d \leq q_0 \leq \ln n$
3. $\ln d \leq \ln n \leq q_0$.

This can be done by using optimization over q for the term $d^{1/q} q^{1/2} \delta^{1/2}$. For the first case, we choose $q = \ln d$ and $C(q,\delta) = 2C\sqrt{\ln d}\,\delta^{1/2}$ where $f(q,\delta) \leq 1$. We also calculate q_1 which represents the upper bound for our choice of q from $q^{5/2}\delta^{1/2} = 1$. For the second case, if $(\frac{n}{d})^{1/5} \geq \ln \frac{n}{d}$, then we simply choose $q = \ln n$. This leads to $\frac{d}{n} \backsimeq \frac{1}{\ln n^4} \leq \frac{1}{\ln n^5}$. We can summarize the cases in the following

$$\hat{\gamma}_q = \begin{cases} q^{1/2}\sqrt{\frac{d}{n}} + q^{5/2}\frac{d}{n} & q \geq \ln n \text{ or} \\ 2C\sqrt{\ln d}\,\delta^{1/2} & d \leq \frac{n}{\ln n^5} \\ 2C(\ln n)^3 \delta & d \geq \frac{n}{\ln n^5}\ . \end{cases}$$

We apply the estimate for $q \geq p$ and appeal to Theorem 3.1.3 and Corollary 3.1.4 to deduce the AGM inequality. $\square$

3.3 Wishart random variable matrices

Let us recall the definition of Wishart random matrices. Let $[g^i_{r,s}]$ is a family of $d \times m$ Gaussian random matrices such that $i \in [1, n]$, $r \in [1, d]$ and $s \in [1, m]$. Define $G_i = \frac{1}{\sqrt{m}}[g^i_{rs}]$ and $x_i = G_i G_i^*$. We call the $d \times d$ matrices x_i Wishart random matrices. Then we have $Ex_i = EG_iG_i^* = 1$, which implies that $\sum_{i=1}^n Ex_i = n$. In this section we assume that $m \geq n$. Let us start with some useful lemmas which will be used in the main theorem. Each of these lemmas proves one of the conditions of Theorem 3.1.3 separately.

Lemma 3.3.1. *Let* $\varepsilon_{q,m,n,d} = \left(\frac{\sqrt{d}+\sqrt{m}}{\sqrt{m}}\right)^2 \frac{q}{\sqrt{n}}$. *Those $d \times d$ Wishart random matrices $\{x_i\}$ from above satisfy*

$$\frac{1}{n}\left|\!\left|\!\left|(\sum x_i^2)^{1/2}\right|\!\right|\!\right|_q \leq \varepsilon_{q,m,n,d}.$$

Proof. Denote $A = \frac{1}{\sqrt{m}}\sum\limits_{r,s} g_{r,s}e_{r,s}$. Then for all $h \in H$, and $x = AA^*$

$$\begin{aligned} E(h, x^2h) =& E(h, |AA^*|^2h) = E(h, AA^*AA^*h) = E(AA^*h, AA^*h) \\ =& E\|AA^*h\|^2 \leq E(\|A\|^2_{op} \cdot \|A^*h\|^2) \leq E\|A\|^2_{op} \cdot E\|A^*h\|^2. \end{aligned}$$

Note that $E\|A^*h\|^2 = E(h, A^*Ah) = \|h\|^2$. Using Chevet's inequality [Gor85],

$$E\|A\| = E\|\sum_{r=1}^{d}\sum_{s=1}^{m} g_{r,s}e_r \otimes e_s\|_{X\check{\otimes}Y} \leq E(\|\sum_{s=1}^{m} g_{r,s}e_s\|) + E(\|\sum_{r=1}^{d} g_{r,s}e_r\|),$$

where $X = l_2^m$ and $Y = l_2^d$. We deduce that if $A = \frac{1}{\sqrt{m}}\sum g^i_{rs}e_r \otimes e_s$ then by using Kahane's inequality (see proposition 3.3.1 and proposition 3.4.1 in [KW92]) we have that

$$(E\|A\|^2_{op})^{1/2} \leq \sqrt{2}\Big(\frac{\sqrt{d}+\sqrt{m}}{\sqrt{m}}\Big) =: C(d, m). \tag{3.3.1}$$

Therefore $|\!|\!|x_i|\!|\!|_2 = (E\|x_i\|^2_{op})^{1/2} \leq C(d, m)$. For $q \geq 2$

$$\begin{aligned} \left|\!\left|\!\left|(\sum_1^n x_i^2)^{1/2}\right|\!\right|\!\right|_q &= \left|\!\left|\!\left|\sum_1^n x_i^2\right|\!\right|\!\right|_{q/2}^{1/2} \leq (\sum_1^n |\!|\!|x_i^2|\!|\!|_{q/2})^{1/2} \\ &\leq (\sum_1^n |\!|\!|x_i|\!|\!|_q^2)^{1/2} \leq \sqrt{n}|\!|\!|x_i|\!|\!|_q = \sqrt{n}[(E\|A\|^{2q})^{1/2q}]^2 \end{aligned}$$

$$\leq \sqrt{n}(\sqrt{q})^2[(E\|A\|^2)^{1/2}]^2 = 2q\sqrt{n}\Big(\frac{\sqrt{d}+\sqrt{m}}{\sqrt{m}}\Big)^2. \tag{3.3.2}$$

The last inequality comes from Kahane's inequality and inequality (3.3.1). Thus, taking $\varepsilon_{q,m,n,d} = \big(\frac{\sqrt{d}+\sqrt{m}}{\sqrt{m}}\big)^2 \frac{2q}{\sqrt{n}}$, we have

$$\frac{1}{n}\Big\|\Big|(\sum x_i^2)^{1/2}\Big\|\Big|_q \leq \varepsilon_{q,m,n,d}.$$

The following lemma is used to prove the first condition in Theorem 3.1.3.

Lemma 3.3.2. *For $d \times d$ Wishart random variables x_i, the following is satisfied*

$$\frac{1}{n}\Big\|\Big|\sum(x_i - Ex_i)\Big\|\Big|_q \leq \gamma_q',$$

where $\gamma_q' = \begin{cases} C' \ln d \, \sqrt{\frac{\ln d}{n}} & q \leq \ln d \leq n \\ C' d^{\frac{1}{q}} q \max\{\sqrt{\frac{q}{n}}, \frac{q}{n}\} & q \geq \ln d \,. \end{cases}$

Proof. By Rosenthal's inequality, we have

$$\begin{aligned}
&\Big\|\Big|\sum(x_i - Ex_i)\Big\|\Big|_q \\
&\leq \Big(E\|\sum_i(x_i - Ex_i)\|_q^q\Big)^{\frac{1}{q}} \\
&\leq_c \sqrt{q}\Big(E|(\sum_i Ex_i^2)^{1/2}|_q^q\Big)^{1/q} + q\Big(\sum |x_i - Ex_i|_q^q\Big)^{1/q} \\
&\leq_c \sqrt{q}\Big(E|(\sum_i Ex_i^2)^{1/2}|_q^q\Big)^{1/q} + qn^{\frac{1}{q}} \cdot \max_i(E|x_i|_q^q)^{\frac{1}{q}} \\
&\leq \sqrt{q}d^{\frac{1}{q}}\Big(E|(\sum x_i^2)^{\frac{1}{2}}|_\infty^q\Big)^{1/q} + qn^{\frac{1}{q}}d^{\frac{1}{q}}q\Big[\frac{\sqrt{d}+\sqrt{m}}{\sqrt{m}}\Big]^2 \\
&\leq \sqrt{qn}d^{\frac{1}{q}}q\Big[1+\sqrt{\frac{d}{m}}\Big]^2 + qn^{\frac{1}{q}}d^{\frac{1}{q}}q\Big[1+\sqrt{\frac{d}{m}}\Big]^2 \\
&\leq d^{\frac{1}{q}}\Big[1+\sqrt{\frac{d}{m}}\Big]^2(\sqrt{qn}q + q^2n^{\frac{1}{q}}).
\end{aligned}$$

The second-to-last inequality uses Kahane's inequality [KW92] and inequality (3.3.2). Dividing the inequality by n, we obtain

$$\frac{\||\sum(x_i - Ex_i)\||_q}{n} \leq d^{\frac{1}{q}}\Big[1+\sqrt{\frac{d}{m}}\Big]^2 q\Big(\sqrt{\frac{q}{n}} + \frac{\sqrt{q}}{n}\sqrt{q}n^{\frac{1}{q}}\Big) \,. \tag{3.3.3}$$

Let $2 \leq q_0 \leq q$. We have two cases to estimate the upper bound:

1. $q_0 \leq \ln d \leq n$

2. $\ln d \le q_0 \le q$.

We follow the optimization for q from the proof of Theorem 3.2.4. Define $f(q) = \sqrt{q} n^{\frac{1}{q}}$, and consider $g(q) = \ln f(q) = \frac{1}{2}\ln q + \frac{1}{q}\ln n$, then $g'(q) = \frac{1}{2q} - \frac{\ln n}{q^2} = 0$ at $q = 2\ln n$. Then

$$\frac{\sqrt{q}}{n} f(q) \le \begin{cases} C\sqrt{\frac{q}{n}} & 2 \le q < n \\ C\frac{q}{n} & q \ge n. \end{cases}$$

Moreover, by (3.3.3), when $d \le m$, we obtain

$$\begin{aligned} d^{\frac{1}{q}}\Big[1+\sqrt{\frac{d}{m}}\Big]^2 q\Big(\sqrt{\frac{q}{n}}+\sqrt{\frac{q}{n}}\sqrt{q}n^{\frac{1}{q}-\frac{1}{2}}\Big) &\le 2Cd^{\frac{1}{q}}\Big[1+\sqrt{\frac{d}{m}}\Big]^2 q\max\Big\{\sqrt{\frac{q}{n}},\frac{q}{n}\Big\} \\ &\le 8Cd^{\frac{1}{q}} q\max\Big\{\sqrt{\frac{q}{n}},\frac{q}{n}\Big\}. \end{aligned}$$

Denote $F(d,n) = 8Cd^{\frac{1}{q}} q\max\{\sqrt{\frac{q}{n}},\frac{q}{n}\}$. We choose $q = \ln d$ and we get that

$$F(d,n) = C'\ln d\sqrt{\frac{\ln d}{n}}$$

if we have $q_0 \le \ln d \le n$. Otherwise we choose $q \ge q_0$, and we get

$$F(d,n) = C'd^{\frac{1}{q}} q\max\Big\{\sqrt{\frac{q}{n}},\frac{q}{n}\Big\}.$$

Moreover,

$$\hat{\gamma}_q = \begin{cases} C'\ln d\sqrt{\frac{\ln d}{n}} & q \le \ln d \le n \\ C'd^{\frac{1}{q}} q\max\{\sqrt{\frac{q}{n}},\frac{q}{n}\} & q \ge \ln d\,. \end{cases}$$

We apply the estimate for $q \ge p$ and appeal to Theorem 3.1.3 and Corollary 3.1.4. $\square$

Now, we can prove the AGM inequality for random matrices, which holds up to $(1+\varepsilon)$.

Theorem 3.3.3. *Let $\{x_i\}$ be a family of self-adjoint family of $d \times d$ Wishart random matrices. For $2 \le p \le \ln d \le n$, we have*

(i) $\Big|\Big|\Big|\sum_i (x_i - E(x_i))\Big|\Big|\Big|_p \le \gamma_p n$;

(ii) $\frac{1}{n}\sum_{i=1}^{n} E(x_i) = 1$;

(iii) $\Big|\Big|\Big|(\sum_i x_i^2)^{\frac{1}{2}}\Big|\Big|\Big|_p \le \gamma_p n$, *where* $\gamma_p = C'\ln d\sqrt{\frac{\ln d}{n}}$, $p_0 \le \ln d \le n$;

(iv) Moreover, for $\varepsilon \geq 0$ if $\gamma_p \leq \frac{\varepsilon}{3k}$, $p_k := \frac{p}{k}$ then the following hold.

- $\left\|\left|P_k(x_1, ..., x_n) - EP_k(x_1, ..., x_n)\right|\right\|_{p_k} \leq \varepsilon.$
- *The random AGM inequality holds,*

$$\left\|\left|P_k(x_1, ..., x_n)\right|\right\|_{p_k} \leq (1 + 2\varepsilon).$$

Proof. Condition (ii) comes from definition of the Wishart random matrices. For condition (i) we directly use Lemma 3.3.2 for the case when $p_k \leq \ln d \leq n$. For condition (iii), we use Lemma 3.3.1. This implies that all the conditions of Theorem 3.1.4 are satisfied, since $p_k \leq \ln d \leq n$. Thus, we get the random AGM inequality. □

3.4 Application of Pisier's construction for freely independent random variables

Let (M, τ) be a von Neumann algebra where τ is a faithful normal and normalized trace. An example of a finite von Neumann algebra is given by the group von Neumann algebra $L(G)$ associated to the left regular representation $\lambda(G)$ of a discrete group G. It is defined as the strong operator closure of the linear span of $\lambda(G)$. Recall that $L_p(M, \tau)$ where $1 \leq p < \infty$ is defined as the completion of M with respect to the norm $\|x\|_p = (\tau(|x|^p))^{1/p}$ (see [PX03] for more details). Note that $L(G) = L_\infty(L(G))$ and $L(G) \subset L_p(L(G))$. We want to prove a version of the AGM inequality with respect to the norm $\|.\|_p$. For this version of the AGM inequality, we need the following key lemma.

Lemma 3.4.1. *Let M be a von Neumann algebra. Let ν be a partition. Then there exists a group G and $b_i(j) \in L(G)$ such that for $x_i(j) \in L_p(M)$, the elements $X_i(j) = b_i(j) \otimes x_i(j) \in L_p(L(G) \otimes M)$ satisfy*

$$[\nu] = E_M \sum_{i_1, i_2, i_3, \ldots, i_d} X_{i_1}(1) X_{i_2}(2) \ldots X_{i_d}(d).$$

Moreover,

$$\left\|\sum_i X_i(j)\right\|_p \leq \begin{cases} C \max\left\{\|(\sum x_i(j)^* x_i(j))^{1/2}\|_p, \|(\sum x_i(j) x_i(j)^*)^{1/2}\|_p\right\} & j \in A_{n.s} \in \sigma_{n.s} \\ \|\sum\ x_i(j)\ \|_p & \{j\} \in \sigma_s, \end{cases} \tag{3.4.1}$$

where C is a universal constant. Note that $b_i(j) = 1$ if $\{i\} \in \sigma_s$.

Remark 3.4.2. The norm inequality (3.4.1) was proved by Pisier and Xu for even integers $p \geq 2$ in [PX03]. The general case follows from [JPX07].

Now we can state the AGM inequality for $L_p(M)$ where $p \geq d$.

Theorem 3.4.3. *Let M be a von Neumann algebra and $x_i \in L_p(M,\tau)_{sa}$ satisfy the following condition for some $\delta \geq 0$,*

$$\Big\|(\sum_1^n x_i^2)^{1/2}\Big\|_p \leq \delta \Big\|\sum_1^n x_i\Big\|_p.$$

Then we have

$$\|P_d(x_1,...,x_n)\|_{\frac{p}{d}} \leq \Big(1+(\delta C)(d!-1)\Big)\frac{n^d(n-d)!}{n!}\Big\|\frac{1}{n}\sum_1^n x_i\Big\|_p^d.$$

We will only give the sketch of the proof of this theorem since it is similar to the proof of Theorem 2.2.2 for $p_d = \frac{p}{d} \geq 1$.

Proof. By using Lemma 3.4.1, Hölder's inequality and the contractivity of conditional expectations we have

$$\begin{aligned}\|\langle\sigma\rangle\|_{p_d} &\leq \Big\|\sum x_i\Big\|_p^d + \sum_{\upsilon \gneq \dot{0}} |\mu(\dot{0},\nu)| C^{|v_{n.s}|} \Big\|(\sum x_i^2)^{1/2}\Big\|_p^{|v_{n.s}|} \Big\|\sum x_i\Big\|_p^{|v_s|} \\ &\leq \Big\|\sum x_i\Big\|_p^d + \sum_{\upsilon \gneq \dot{0}} |\mu(\dot{0},\nu)| (\delta C)^{|v_{n.s}|} \Big\|\sum x_i\Big\|_p^d.\end{aligned}$$

Thus for $\delta C \leq 1$

$$\|P_d(x_1,...,x_n)\|_{p_d} \leq (1+(\delta C)(d!-1))\frac{n^d(n-d)!}{n!}\Big\|\frac{1}{n}\sum x_i\Big\|_p^d.$$

Remark 3.4.4. If $\delta \leq 1$, we get the AGM inequality with a constant $C(d,n) = C^d d^d$.

As a matter of completeness, we want to include the limit case of the Wishart random matrices as an application for the AGM inequality. Let's first give the definition of freely independent von Neumann algebra (for more details see [VDN92]).

Definition 3.4.5. The sequence of algebras $\{A_i\}$ is called copies over a von Neumann algebra M if the following conditions hold

1. $M \subset A_j$ for every $j \in \mathbb{N}$
2. There exists a trace preserving and $*$-homomorphism such that $\pi_{1,j} : A_1 \to A_j$ where $\pi_{1,j}|_M = I_M$.

Definition 3.4.6. Let $\{A_i\}$ be a family of unital von Neumann subalgebras of A.

- Then $\{A_i\}$ is called a freely independent algebra (with respect to a unital linear functional ϕ) if $\phi(x_1...x_n) = 0$ whenever $\phi(x_j) = 0$ for all $x_j \in A_{i_j}$ and $i_1 \neq i_2, i_2 \neq i_3, ..., i_{k-1} \neq i_k$.

- We say that operators $x_i \in A_i$ are freely independent if their algebra $\{A_i\}$ are freely independent.

We also need to recall Voiculescu's inequality in the following proposition, which is considered as an operator-valued free analogue of Rosenthal's inequality for homogeneous free polynomials of degree 1 and $p = \infty$.

Proposition 3.4.7. *Let $a_k \in A_k$ where $A_1, ..., A_n$ are freely independent algebras over M. Then*

$$\|\sum_{k=1}^{n} a_k\| \leq \sup_{k=1,\ldots,n} \|a_k\| + \|\sum_{k=1}^{n} E_M(a_k^* a_k)\|^{\frac{1}{2}} + \|\sum_{k=1}^{n} E_M(a_k a_k^*)\|^{\frac{1}{2}}.$$

In the following theorem we prove the deviation inequality up to ε and apply this to the AGM inequality.

Theorem 3.4.8. *Fix $n, d \in \mathbb{N}$ such that $n \geq d$. If $\{x_i\}$ are freely independent over M such that*

1. $E_M(x_i) = 1$

2. $x_i^* = x_i$

3. $2 + (4\sqrt{n}) \sup \|x_i\| \leq \frac{\varepsilon n}{3d}$,

then

1. $\|P_d(x_1, ..., x_n) - E_M P_d(x_1, ..., x_n)\|_\infty \leq \varepsilon$

2. $\|P_d(x_1, ..., x_n)\|_\infty \leq 1 + \varepsilon$.

Proof. Let $a_i = x_i - 1$. By assumption we have $E_M(a_i) = 0$. Define $C = \sup \|x_i\|$. By a simple modification of Voiculescu's inequality [Jun05] , we get that

$$\begin{aligned}\Big\|(\sum_1^n a_i^2)^{1/2}\Big\| = \Big\|\sum_1^n e_{i1} \otimes a_i\Big\| &\leq \sup \|a_i\| + 2\Big\|(\sum_1^n E_M(a_i^2))^{1/2}\Big\| \\ &\leq 2(1 + C) + 2\sqrt{n}C \\ &\leq 2 + (4\sqrt{n}C) \leq \frac{\varepsilon n}{3d}.\end{aligned}$$

Indeed, $\|a_i\| = \|x_i - 1\| \leq 1 + \|x_i\| \leq 1 + C$ and

$$\begin{aligned}E_M(a_i^2) &= E_M(x_i - E_M(x_i))^2 = E_M(x_i^2) - E_M(x_i)^2 \\ &\leq E_M(x_i^2) = E_M(x_i^{1/2}|x_i|x_i^{1/2}) \\ &\leq \|x_i\| E_M(x_i) = \|x_i\| \leq C.\end{aligned}$$

Again, using Voiculescu's inequality we have,

$$\Big\|\sum a_i\Big\| \le \sup\|a_i\| + 2\Big\|\Big(\sum E_M(a_i^2)\Big)^{1/2}\Big\| \le 2(1+C) + 2\sqrt{n}C \le \frac{\varepsilon n}{3d}.$$

Following the proof of Proposition 3.1.1, we get

$$\|[\nu]\| \le \Big\|(\sum a_i^2)^{1/2}\Big\|^{d-|v_s|}\Big\|\sum a_i\Big\|^{|v_s|} \le \Big(\frac{\varepsilon n}{3d}\Big)^d.$$

Applying the techniques of Proposition 3.1.1 to the case $p = \infty$, we have

$$\|P_k(a_1, ..., a_n) - EP_k(a_1, ..., a_n)\| \le 2\frac{(n-k)!}{n!}k!\Big(\frac{\varepsilon n}{3d}\Big)^k.$$

Then we have

$$\begin{aligned}\|P_d(x_1, ..., x_n) - EP_d(x_1, ..., x_n)\| &= \|\sum_{k=1}^{d}\binom{d}{k}(P_k(a_1, ..., a_n) - EP_k(a_1, ..., a_n))\| \\ &\le 2\sum_{k=1}^{d}\underbrace{\frac{d!(n-k)!}{(d-k)!n!}n^k}_{f(n)\text{ is a decreasing function}}(\frac{\varepsilon}{3d})^k \\ &\le 2\sum_{k=1}^{d}(d)^k(\frac{\varepsilon}{3d})^k \le \varepsilon.\end{aligned}$$

We apply Theorem 2.3.5 for $y_i = Ex_i$ instead of x_i, where $\frac{\sum y_i}{n} = 1$. Note that by free independence, we have $EP_d(x_1, ..., x_n) = P_d(Ex_1, ..., Ex_n)$ using the fact that $\{x_n\}$ in $P_d(x_1, ..., x_n)$ has no repetition. Therefore, we get

$$\|P_d(x_1, ..., x_n)\| \le \|P_1(Ex_1, ..., Ex_n)\| + \epsilon \le 1 + \epsilon.$$

□

Remark 3.4.9. The norm version of the AGM inequality also holds for the family of freely independent $\{x_i\}$. Indeed, we have that

$$\|P_d(x_1, ..., x_n)\| \le (1+\tilde{\varepsilon})\|\frac{1}{n}\sum_1^n x_i\|^d.$$

In this case we use again the Voiculescu inequality and deduce that $\|\frac{1}{n}\sum\limits_1^n x_i - \frac{1}{n}\sum\limits_1^n Ex_i\| \le \frac{\varepsilon}{3d}$. This implies

$\|\frac{1}{n}\sum\limits_{1}^{n} x_i\| \geq 1 - \frac{\varepsilon}{3d}$. Hence,

$$\|P_d(x_1, ..., x_n)\| \leq \frac{(1+\varepsilon)}{(1-\frac{\varepsilon}{3d})^d}\|\frac{1}{n}\sum_{1}^{n} x_i\|^d.$$

Since we have $(1-t)^n \geq (1-nt)$ for $t \in [0,1]$ and $n \geq 1$, this implies that for $t = \frac{\varepsilon}{3d}$, we have

$$(1-\frac{\varepsilon}{3d})^d \geq (1-\frac{\varepsilon}{3d}d) = (1-\frac{\varepsilon}{3}).$$

Thus, we have

$$\|P_d(x_1, ..., x_n)\| \leq \frac{(1+\varepsilon)}{(1-\frac{\varepsilon}{3})}\|\frac{1}{n}\sum_{1}^{n} x_i\|^d.$$

Note that $\frac{(1+\varepsilon)}{(1-\frac{\varepsilon}{3})} = 1 + \frac{4\varepsilon}{3-\varepsilon} = 1 + \tilde{\varepsilon}$ where $\tilde{\varepsilon} = \frac{4\varepsilon}{3-\varepsilon}$ and $\varepsilon \leq \frac{3}{5}$. This implies the AGM inequality up to the constant $1 + \tilde{\varepsilon}$. ■

Another interesting application for freely independent copies $\{x_i\}$ is given as follows:

Corollary 3.4.10. *Let $\{x_i\}$ be a sequence of freely independent copies over an algebra M such that*

1. $E_M(x_1) = 1_M$
2. $x_i^* = x_i$
3. $\|x_1\| \leq C$.

Then the AGM inequality holds up to $(1+\varepsilon)$.

Proof. Using the free independence for the $\{x_i\}$'s, where $d \leq p \leq \infty$ we get

1. $E_M(x_i) = 1_M$;
2. $\|x_i\|_p = \|x_1\|_p \leq C$;
3. $\|(\sum x_i^2)^{1/2}\|_p \leq \tilde{c}\|x_i\|_p n^{1/p} + \sqrt{n}\|(E_M x_i^2)^{1/2}\|_p$.

Indeed, for the property (3) we just apply a version of Voiculescu's inequality for free variables [JPX07],

$$\begin{aligned}\|\sum_{1}^{n} x_i \otimes e_{i1}\|_p &\leq c\Big(\sum_{1}^{n}\|x_i\|^p\Big)^{1/p} + \Big\|\Big(\sum_{1}^{n} E_M(x_i^* x_i)\Big)^{1/2}\Big\|_p \\ &\leq \tilde{C}n^{1/p} + \sqrt{n}\|(E_M x_1^2)^{1/2}\|_p.\end{aligned}$$

Note that

$$
\begin{aligned}
\|\sum_1^n x_i\|_p &\geq \|\sum_1^n E_M(x_i)\|_p - \|\sum_1^n \Big(x_i - E_M(x_i)\Big)\|_p \\
&\geq n\|E_M(x_1)\|_p - \underbrace{\Big(\tilde{C}n^{1/p} + \sqrt{n}\|E_M(x_i^2)^{1/2}\|_p\Big)}_{\text{A}}.
\end{aligned}
$$

Now, if $A \leq \frac{n}{2}\|E_M(x_1)\|$, then we have

$$
\begin{aligned}
\|(\sum_1^n x_i^2)^{1/2}\|_p &\leq 2\frac{\tilde{C}n^{1/p} + \sqrt{n}\|E_M(x_1^2)^{1/2}\|_p}{n\|E_M(x_1)\|_p}\|\sum_1^n x_i\|_p \\
&= n^{-1/2}\underbrace{\left(\frac{2\tilde{C}n^{1/p-1/2} + 2\|E_M(x_1^2)^{1/2}\|_p}{\|E_M(x_1)\|_p}\right)}_{C_n}\|\sum_1^n x_i\|_p.
\end{aligned}
$$

Then we get

$$
\|(\sum_1^n x_i^2)^{1/2}\|_p \leq \delta_n\|\sum_1^n x_i\|_p,
$$

where $\delta_n = \frac{C_n}{\sqrt{n}}$. Then for $\sqrt{n} \gg d!$ we have $\delta_n \to 0$. This implies that when n is large enough, we get the following AGM inequality:

$$
\|P_d(x_1, ..., x_n)\|_{\frac{p}{d}} \leq (1+\varepsilon)\|\sum_1^n x_i\|_p^d.
$$

□

In conclusion, we see that the AGM inequality almost holds for the two extreme situations, namely, for commutating or free independent random variables. Indeed, there should be some balance between the degree of the polynomial and the size of the matrices. This can be seen from the restriction of the parameters required to prove the AGM inequality for Wishart random matrices, log-concave measures and freely independent (see Theorem 3.3.3, Theorem 3.2.4, Theorem 3.4.8). Therefore, without this balance, it seems the AGM inequality is hard to prove especially in the absent of the central limit theorem (CLT) because we have to control the norm of polynomials in non-commuting variables of high degree.

Chapter 4

Ternary ring of operators

4.1 Definitions and properties

In this section we recall some basic facts about C^*-ternary ring of operators. We start with the algebraic concept of C^*-ternary ring of operators (see [Hes62],[Zet83] for more details).

A complex ternary ring is a linear space X over complex number $\mathbb{C}$ such that equipped with a map

$$\langle .,.,. \rangle : X \times X^\sharp \times X \longrightarrow X$$
$$(x, y^*, z) \mapsto \langle x, y^*, z \rangle \in X,$$

which is linear on the first and third variables and conjugate linear on the second and it's satisfying the following associativity condition:

$$\langle \langle x, y^*, z \rangle, e^*, f \rangle = \langle x, \langle e, z^*, y \rangle^*, f \rangle = \langle x, y^*, \langle z, e^*, f \rangle \rangle$$

for all $x,\ y,\ z,\ e,\ f \in X$.

A Banach ternary ring X is a complex ternary ring and also a Banach space such that the map $\langle .,.,. \rangle$: $X \times X^\sharp \times X \to X$ is a contraction with respect to the Banach space norm, i.e.

$$\|\langle x, y^*, z \rangle\| \leq \|x\| \|y^*\| \|z\|$$

for all $x, y, z \in X$.

This leads to the following important definitions.

Definition 4.1.1. A C^*-ternary ring is a Banach ternary ring such that for all $x \in X$

$$\|\langle x, x^*, x \rangle\| = \|x\|^3.$$

The concrete definition of a ternary ring of operators (TRO) is given as follows:

Definition 4.1.2. A ternary ring of operators (TRO) between Hilbert spaces H and K is a norm closed operator subspace V of $B(H, K)$, which is closed under the triple product

$$\begin{aligned} V \times V^{\sharp} \times V \to V \\ (x, y^*, z) \to \langle x, y^*, z \rangle = xy^*z \in V \subseteq B(H, K). \end{aligned}$$

It is important to know that every TRO has a natural operator space structure, i.e. if we define a TRO $V \subset B(H, K)$, then $M_n(V) \subset M_n(B(H, K)) = B(H^n, K^n)$ is also a TRO. This imply that V has a natural canonical operator space structure $(V, \|.\|_n)$.

It is easy to see that every C^*-algebra is a TRO. In fact, if p, q are projections in a C^*-algebra A, then pAq is a TRO. This is an equivalent definition of a TRO V which is defined as the off-diagonal corner of its linking C^*-algebra $A(V)$ such that

$$A(V) = \begin{bmatrix} C(V) & V \\ V^{\sharp} & D(V) \end{bmatrix}$$

where $C(V)$ and $D(V)$ are both C^*-algebras generated by $VV^{\sharp}$ and $V^{\sharp}V$ respectively (see [KR02] for more details).

Remark 4.1.3. It is clear from the definition that

$$\text{TRO } \subseteq C^*\text{-ternary ring} \subseteq \textit{Banah ternary ring}.$$

But the inverse inclusions are not necessarily true. We provide the following known examples.

Example 4.1.4. Let $\ell_1(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} : \|f\|_1 = \sum |f(n)| < +\infty\}$ where the multiplication is given by the convolution,

$$f * g := \sum_{m \in \mathbb{Z}} f(m)g(n-m).$$

It is well-known that $\ell_1(\mathbb{Z})$ is an involutive Banach algebra and hence a natural ternary ring by

$$\begin{aligned} \langle ., ., . \rangle : \ell_1(\mathbb{Z}) \times \ell_1(\mathbb{Z})^{\sharp} \times \ell_1(\mathbb{Z}) \longrightarrow \ell_1(\mathbb{Z}) \\ (f, g^*, h) \mapsto f * g^* * h), \end{aligned}$$

for all $f, g, h \in \ell_1(\mathbb{Z})$. It is clear that

$$\|f * g^* * h\|_1 \leq \|f\|_1 \|g^* * h\|_1 \leq \|f\|_1 \|g^*\|_1 \|h\|_1 = \|f\|_1 \|g\|_1 \|h\|_1.$$

It is a Banach ternary ring but not a C^*-ternary ring. Indeed, consider the element $x = \delta_0 + i\delta_1 + \delta_2 \in \ell_1(\mathbb{Z})$ where δ_n is the characteristic function on n and $\delta_n * \delta_m = \delta_{n+m}$. Then, $x^* = \delta_0 - i\delta_{-1} + \delta_{-2}$ and

$$\|x^*\|_1 = \|x\|_1 = 3 \text{ and } \|x^* * x\|_1 = 5.$$

This implies that $\|x * x^* * x\|_1 \leq \|x\|_1 \|x^* * x\|_1 = 3 \cdot 5 = 15 < \|x\|_1^3 = 27$. ■

Next, for the second example, we recall the following definitions.

Definition 4.1.5. Let V and W are two TRO's. A linear map $\theta : V \to W$ between two TRO's V and W is called a TRO-homomorphism if it preserves the triple product as follows:

$$\theta(xy^*z) = \theta(x)\theta(y)^*\theta(z)$$

for all $x,\ y,\ z \in V$. Moreover, θ is a TRO-isomorphism if it is bijection. A linear map $\theta : V \to W$ is called an anti-TRO-homomorphism if for all $x, y, z \in V$

$$\theta(xy^*z) = -\theta(x)\theta(y)^*\theta(z).$$

The following decomposition theorem for C^*-ternary ring is due to Zettl [Zet83].

Theorem 4.1.6. *Let $(X, (.,.,.), \|.\|)$ be a C^*-ternary ring. Then X is the direct sum of two C^*-ternary subrings X_+ and X_- where X_+ is a TRO-isomorphic to a TRO V and X_- is anti-TRO-isomorphic to a TRO W. Moreover, V and W are unique up to TRO-isomorphism.*

In particular, the above theorem shows that every C^*-ternary ring consists of a TRO part and an anti-TRO part. In general, a C^*-ternary ring is not isomorphic to a TRO. Zettl provides an example for the decomposition of a C^*-ternary ring of operators.

Example 4.1.7. Suppose Ω is a compact Hausdorff space and let $\Omega_1 \neq \Omega$ be a nonempty set which is open and closed. Define $\chi : \Omega \to \{0,\ 1\}$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } t \in \Omega_1 \\ & \\ 0 & \text{if } t \in \Omega_2 = \Omega/\Omega_1 \end{cases}.$$

Define the triple product map for $C(\Omega)$ as the following

$$\langle .,.,. \rangle : C(\Omega) \times C(\Omega)^\sharp \times C(\Omega) \longrightarrow C(\Omega),$$

$$\langle f, \bar{g}, h \rangle \mapsto f\bar{g}h(2\chi - 1)(t) = \begin{cases} f(t)\overline{g(t)}h(t) & \text{if } t \in \Omega_1 \\ & \\ -f(t)\overline{g(t)}h(t) & \text{if } t \in \Omega_2. \end{cases}$$

Then $(C(\Omega), \langle .,.,. \rangle, \|.\|_{\sup})$ is a C^*-ternary ring which has the following decomposition: $C(\Omega) = C(\Omega)_+ + C(\Omega)_-$ where $C(\Omega)_+ = C(\Omega_1)$ and $C(\Omega)_- = C(\Omega/\Omega_1)$. ■

From the definition we can see that every TRO is a C^*-ternary ring with anti-TRO part $X_- = 0$, but not every C^*-ternary ring is a TRO. However, we can make it a TRO by appropriate modification that is given by Zettl which corrects the anti-TRO part to the TRO-part.

Theorem 4.1.8. *For every C^*-ternary ring $(X, \langle .,.,. \rangle, \|.\|)$, there exists a unique map operator $T : X \to X$ satisfying*

1. $T^2 = Id_X$;
2. $T(\langle x, y^*, z \rangle) = \langle Tx, y^*, z \rangle = \langle x, Ty^*, z \rangle = \langle x, y^*, Tz \rangle$ *for all* $x,\ y,\ z \in X$;
3. $(X, T \circ \langle .,.,. \rangle, \|.\|)$ *is a C^*-ternary ring which is isomorphic to a TRO.*

Example 4.1.9. Back to the Example 4.1.7 of $C(\Omega)$, it is clear that

$$X_+ = \Omega_1 = \{t \in \Omega : \langle f, \bar{g}, h \rangle(t) \geq 0\},\ X_- = \Omega_2 = \{t \in \Omega : \langle f, \bar{g}, h \rangle(t) = -f\bar{g}h(t) \leq 0\}.$$

From Theorem 4.1.8, the map $T = 2\chi - 1$ corrects the anti-TRO part of a C^*-ternary ring such that $T \circ \langle f, \bar{g}, h \rangle = f\bar{g}h$. ■

4.2 Cb-version of Zettl's decomposition theorem

We have seen from the last section Zettl's decomposition theorem for C^*-ternary ring where X is a Banach space. A natural question is to ask what happened if we are given X as operator space. Todorov [Tod02] has proved the cb-version of Zettl's decomposition theorem. In this section, we recall some of Todorov's results. Let's start first by the following definitions.

Definition 4.2.1. A completely contractive ternary ring (c.c ternary ring) is a ternary ring with an operator space structure, such that the map $\langle ., ., . \rangle : X \times X^{\sharp} \times X \to X$ is completely contractive with respect to the norm.

Definition 4.2.2. A ternary operator system X is a completely contractive ternary ring such that for all $x = [x_{ij}] \in M_n(X)$

$$\|x \odot x^{\sharp} \odot x\| = \|x\|^3.$$

Here $\odot$ denotes the formal matrix product.

Remark 4.2.3. Every TRO is a ternary operator system, i.e.

$$TRO \subseteq \text{Ternary operator system}$$

but the inverse inclusion is not true in general. In fact, a ternary operator system is a matricial structure of a C^*-ternary ring.

Example 4.2.4. From Example 4.1.7, we find that $(C(\Omega), \langle ., ., . \rangle, \|.\|_n)$ is a ternary operator system. Indeed, for $M_n(C(\Omega, \mathbb{C})) \cong C(\Omega, M_n)$ as a C^*-algebras. The map

$$\langle ., ., . \rangle : C(\Omega) \times C(\Omega)^{\sharp} \times C(\Omega) \to C(\Omega)$$

is completely contractive by the minimal operator space structure of $C(\Omega)$ [ER00]. To check the last condition of a ternary operator system we use the functional calculus. Let $x = [f_{ij}]$ and since $xx^* \geq 0$, we have

$$\begin{aligned}
\|x \odot x^* \odot x\|^2 &= \|[f_{ij}] \odot [f_{kj}]^* \odot [f_{kl}]\|^2 \\
&= \|[\sum_{j=1,k=1}^{n} f_{ij} f_{jk}^* f_{kl}]_{i,l}\|^2 \\
&= \|[\sum_{j=1,k=1}^{n} f_{ij} f_{jk}^* f_{kl} f_{lk}^* f_{kj} f_{ji}^*]\| \\
&= \|x^* x x^* x x^* x\| = \|x\|^6.
\end{aligned}$$

So this is an example of a C^*-ternary ring which is also a ternary operator system. ■

The following cb-version of Zettl's decomposition theorem is due to Todorov [Tod02].

Corollary 4.2.5. *Let X be a ternary operator system. Then there exists ternary operator sub-systems X_+ and X_- such that $X = X_+ \oplus X_-$ and X_+ is completely isometrically isomorphic to a TRO, while X_- is completely isometrically anti-isomorphic to a TRO.*

This result tells us that the triple operation is not preserved by the representation $T : X \to B(H, K)$, i.e.

$$T(\langle x, y^*, z\rangle) \neq T(x)T(y)^*T(z).$$

Corollary 4.2.6. *Let X be a ternary operator system. Then X is completely linear isometric to a TRO.*

4.3 Equivalence between TRO's

Recall the equivalent definition of a TRO V as the off-diagonal corner of its linking C*-algebra

$$A(V) = \begin{bmatrix} C(V) & V \\ V^\sharp & D(V) \end{bmatrix}$$

where $C(V)$ and $D(V)$ are both C*-algebra generated by $VV^\sharp$ and $V^\sharp V$ respectively [KR02]. It is known that if two TRO's V and W are TRO-isomorphic, then their linking C^*-algebras are $*$-isomorphic, i.e.

$$V \cong W \Rightarrow A(V) \cong A(W).$$

Questions to ask:

1. If the converse of this statement is also true,
2. If the diagonal components between two linking C^*-algebra of two TRO's V and W are $*$-isomorphic then is this imply that V and W are TRO-isomorphic.

Surprisingly, these questions are not true in general. Our main goals in this section are to prove the above statements are not true, i.e. If we have two TRO's V and W such that

$$C(V) \cong C(W),\ D(V) \cong D(W) \text{ are } *\text{-isomorphic as a } C^*\text{-algebras but } V \ncong W \text{ as TRO-isomorphic.} \quad (4.3.1)$$

Also we prove

$$A(V) \cong A(W) \text{but} V \ncong W. \quad (4.3.2)$$

Note that this relation in (4.3.2) also tells us that TRO is not unique in the definition of Morita equivalent between two C^*-algebras. We begin by recalling the definitions of the UHF algebras, CAR algebras and other related results [Dav96, Bla06].

Definition 4.3.1. A C^*-algebra A is called uniformly hyperfinite (or UHF) if $A = \overline{\cup_{n=1}^{\infty} M_{k_n}}^{\|\cdot\|}$ is an increasing union of unital subalgebras which are isomorphic to matrix algebras M_{k_n}.

Example 4.3.2. Let A be algebra obtained as the union of subalgebras $A_n = M_{2^n}$ where the embedding $\phi_{n,n+1} : A_n \to A_{n+1}$ is defined by

$$\phi_{n,n+1}(a) = diag(a, a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

This is an embedding of multiplicity 2 and A is called the CAR algebra.

Definition 4.3.3. The supernatural number associated to the sequence A_n is defined as a formal product of the form

$$\delta(A) := \prod_{p \text{ prime}} p^{\epsilon_p},$$

where for each prime integer p there is a unique $\epsilon_p \in \mathbb{N} \bigcup \{\infty\}$ which is the supremum of the exponents of power of p which divide k_n (k_n is the dimension of the matrices M_{kn}) as n tends to infinity.

For instance, the CAR algebra A has the supernatural number $\delta(A) = 2^\infty$, i.e. $\epsilon_2 = \infty$ and for all other $p \neq 2$ we have $\epsilon_p = 0$ (see [Dav96] for details about the supernatural number). We have the following diagram for A and $M_2(A)$ as follows,

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook] \arrow[d] & M_2 \arrow[r, hook] \arrow[d] & M_2 \otimes M_2 \ldots\ldots \arrow[r, hook] \arrow[d] & \overline{\cup_{k\in\mathbb{Z}} M_{2^k}}^{\|\cdot\|} \arrow[d] \\
M_2 \arrow[r, hook, "\phi"'] \arrow[ur, "id"] & M_2 \otimes M_2 \arrow[r, hook] \arrow[ur, "id"] & M_2 \otimes M_2 \otimes M_2 \ldots\ldots \arrow[r, hook] & \overline{\cup_{k\in\mathbb{Z}} M_{2^k}}^{\|\cdot\|}
\end{tikzcd}$$

It is easy to check that this diagram commutes and hence $\delta(A) = \delta(M_2(A)) = 2^\infty$. This implies that A and $M_2(A)$ are isomorphic (Note that UHF is uniquely determined by its supernatural number, see Theorem III.5.2 in [Dav96]). Using this fact, we are able to prove the following result.

Proposition 4.3.4. *Let A be the CAR algebra. Then $M_{1,2}(A)$ is not TRO isomorphic to A.*

Proof. Let $V := M_{1,2}(A)$ and $W := A$. Suppose that V and W are TRO-isomorphic, i.e. there exists a map $\phi : M_{1,2}(A) \to A$ which is TRO-isomorphism from $M_{1,2}(A)$ onto A. Since the row vector $v = [0\ , 1]$ is a partial isometry in $M_{1,2}(A)$ such that $vv^*x = x$ for any $x \in M_{1,2}(A)$, its image $w = \phi(v)$ must be partial isometry in A such that $ww^*y = y$ for all $y \in A$ where $y = \phi(x)$ for some $x \in M_{1,2}(A)$. If $y = 1$ then we have $ww^* = 1$ and because the CAR algebra A is finite then $w^*w = 1$. Furthermore, there exists a non-zero element $x_1 = [1\ ,0]$ in $M_{1,2}(A)$ such that

$$\phi(x_1)w^*w = \phi(x_1v^*v) = 0 \neq \phi(x_1),$$

which leads to a contradiction. ☐

The above proposition leads us to the following theorem.

Theorem 4.3.5. *There exist two TRO's V and W such that*

1. *If $C(V) \cong C(W)$ and $D(V) \cong D(W)$ but $V \ncong W$*
2. *If $A(V) \cong A(W)$but $V \ncong W$*

Proof. Consider the TRO's $W := M_{1,2}(A)$ and $V := A$, where A is the CAR algebra. Then from Proposition 4.3.4 we know that $W := M_{1,2}(A) \ncong V := A$ as TRO-isomorphism. However, $C(V) = \overline{VV^*}^{\|\cdot\|} = A$ and $D(V) = \overline{V^*V}^{\|\cdot\|} = A$. Therefore, $C(W) = \overline{WW^*}^{\|\cdot\|} = A$ and $D(W) = \overline{W^*W}^{\|\cdot\|} = M_2(A)$. Since A and $M_2(A)$ are $*$-isomorphic, then $C(V) \cong C(W)$ and $D(V) \cong D(W)$ are both $*$-isomorphic. But we have that $V \ncong W$ as TRO isomorphism.
For (2) we obtain two different UHF algebras B_1 and B_2 with the following diagrams:

$$\mathbb{C} \hookrightarrow M_2 \hookrightarrow M_2 \otimes M_3 \hookrightarrow M_2 \otimes M_3 \otimes M_2 \ldots\ldots \hookrightarrow \overline{\cup_{n,m\in\mathbb{Z}} M_{2^n \times 3^m}}^{\|\cdot\|} := B_1$$

and

$$\mathbb{C} \hookrightarrow M_3 \hookrightarrow M_3 \otimes M_2 \hookrightarrow M_3 \otimes M_2 \otimes M_3 \ldots\ldots \hookrightarrow \overline{\cup_{n,m\in\mathbb{Z}} M_{2^n \times 3^m}}^{\|\cdot\|} := B_2$$

It turns out that B_1 and B_2 are isomorphic since they have the same supernatural number, i.e. $\delta(B_1) = \delta(B_2) = 2^\infty \times 3^\infty$ as n and m tend to infinity. Also, for the same reason $M_3(B_1)$ and $M_2(B_2)$ are isomorphic. Now consider the TRO's V and W such that $V := B_1$ and $W := M_{1,2}(B_1)$. Then by simple modification of the result of Proposition 4.3.4 we know that V is not isomorphic to W but we know that $A(V) = M_2(B_1)$ is isomorphic to $A(W) = A(M_{1,2}(B_1)) = M_3(B_1)$. ☐

4.4 Equivalence between W^*-TRO

A concrete W^*-TRO is defined as a weak*-closed subspace $V \subseteq B(H,K)$ such that $xy^*z \in V$ for all $x,y,z \in V$. It is a corner of the von Neumann algebras $R(V)$, defined as follows:

$$R(V) = \begin{bmatrix} M(C) & V \\ V^* & N(D) \end{bmatrix},$$

where $M(C)$ and $N(D)$ are von Neumann algebras. Our main result in this section is to prove the W^*-version of Theorem 4.3.5. We first recall a number of definitions for von Neumann algebras (see[Rua04, Bla06] for more details.)

Let $M \in B(H)$ is a von Neumann algebra and p and q are projections in M, we say p is dominated by q, denoted as ($p \lesssim q$) if there is an operator $u \in M$ with $uu^* = p$ and $u^*uq = u^*u$. We say that p and q are equivalent, denoted as ($p \sim q$), if there is $u \in M$ with $uu^* = p$, $u^*u = q$. It is true that $p \sim q$ if $p \lesssim q$ and $q \lesssim q$. A projection q is finite if $p \lesssim q$, $p \sim q$ implies $p = q$ and it is infinite if there is a p such that $p \sim q$ and $p \lnsim q$. A projection $p \neq 0$ is called minimal if it dominates no other projection in M other than 0.

Definition 4.4.1. A factor is a von Neumann algebra R with trivial center, i.e. $R \cap R' = \mathbb{C}$.

Definition 4.4.2. A von Neumann algebra is finite if every isometry $v \in M$ is a unitary. i.e $v^*v = 1 \Rightarrow vv^* = 1$ for all $v \in M$, i.e. (1 is finite).

Definition 4.4.3. A factor M is of type II_1 if M has no minimal projections and every projection is finite.

Definition 4.4.4. A separable von Neumann algebra R is said to be approximately finite dimensional (AFD) (or hyperfinite) if there exists an increasing sequence of finite dimensional C^*-subalgebra N_n such that $R = (\cup_{n=1}^{\infty} N_n)'' = \overline{\cup_{n=1}^{\infty} N_n}^{s.o.t}$

It is known that all AFD factors of type II_1 are $*$-isomorphic. The following proposition proved by Ruan [Rua04], will help us to construct an example we need for (4.3.1).

Proposition 4.4.5. *Let R be a finite von Neumann algebra. Then $M_{1,2}(R)$ is not TRO-isomorphic to R.*

Theorem 4.4.6. *There exist two W^*-TRO's V and W such that*

If $N(V) \cong N(W)$, $M(V) \cong M(W)$ and $R(V) \cong R(W)$ but $V \ncong W$

Proof. Consider the TRO's $W := M_{1,2}(R)$ and $V := R$, where R is the AFD factor of type II_1. Then from Proposition 4.4.5 we know that $W := M_{1,2}(R) \ncong V := R$ as TRO's. However we have that $M(V) = \overline{VV^*}^{w*} = R$ and $N(V) = \overline{V^*V}^{w*} = R$. Therefore, $M(W) = \overline{WW^*}^{w*} = R$ and $N(W) = \overline{W^*W}^{w*} = M_2(R)$.

Since all von Neumann algebra which are AFD factor of type II_1 are $*$-isomorphic, then $N(V) \cong N(W)$ and $M(V) \cong M(W)$ but $V \ncong W$ as W^*-TRO's. Moreover, their linking von Neumann algebras are $*$-isomorphic since they are hyperfinite II_1 factor. This conclude our example. $\square$

Note that the above result is stronger than the following theorem which can be proved by using the same example.

Theorem 4.4.7. *There exist two W^*-TRO's V and W such that*

1. *If $N(V) \cong N(W)$ and $M(V) \cong M(W)$ but $V \ncong W$*
2. *If $R(V) \cong R(W)$ but $V \ncong W$.*

Chapter 5

Crossed Product of TRO's

5.1 Crossed product of C^*-algebras

In this section, we present some known results for crossed product of C^*-algebras which will be used throughout this chapter (see [BO08] for more details).

Let G be a discrete group and A be a C^*-algebra. An action of G on a C^*-algebra A is a group homomorphism $\alpha : G \to Aut(A)$; $s \mapsto \alpha_s$ of G into the group of $Aut(A)$ of $*$-automorphisms of A. The $*$-algebra $C_c(G, A)$ is defined as the linear space of the finitely supported function on G with values in A. Let the action is defined as the inner action, i.e. $\alpha_g(a) = gag^{-1}$ for all $a \in A$ and $g \in G$. Then the α-twisted convolution and the $*$-operation of $C_c(G, A)$ are given as follows:
for $S = \sum\limits_{s \in G} a_s s$ and $T = \sum\limits_{t \in G} b_t t \in C_c(G, A)$,

$$S *_\alpha T = \sum_{s,t \in G} a_s \alpha_s(b_t) st, \;\; S^* = \sum_{s \in G} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

In order to understand the crossed product of C^*-algebras better, it is useful to start with the reduced crossed product. If G acts on a Hilbert space $\ell^2(G)$ and we start with a faithful representation $A \subset B(H)$ which takes value in a Hilbert space H, then we can define a new faithful representation from A into $B(H \otimes \ell^2(G))$ where $H \otimes \ell^2(G)$ is the new Hilbert space such that

$$\pi(a)(v \otimes \delta_g) = (\alpha_{g^{-1}}(a)(v)) \otimes \delta_g$$

where $\{\delta_g\}_{g \in G}$ is the orthonormal basis for the $\ell^2(G)$. Namely,

$$\pi(a) = \oplus_{g \in G} \alpha_g^{-1}(a) \in B(H \otimes \ell^2(G)) = B(\oplus_{g \in G} H).$$

Let $\lambda : G \to U(\ell^2(G))$ be the left regular representation of G on $\ell^2(G)$ such that $\lambda_s \delta_t = \delta_{st}$. Then the regular covariant representation for the reduced crossed product is a pair $(\pi, \mathbb{I}_H \otimes \lambda)$ where π is the above $*$-representation of A into $B(H \otimes \ell^2(G))$ and $\mathbb{I}_H \otimes \lambda$ is a unitary representation of G into $B(H \otimes \ell^2(G))$ satisfying

$$\pi(\alpha_s(a)) = (\mathbb{I}_H \otimes \lambda_s)\pi(a)(\mathbb{I}_H \otimes \lambda_s)^*.$$

Now, we are ready to recall the definition of the reduced crossed product of a C^*-algebras.

Definition 5.1.1. The reduced crossed product, denoted $A \rtimes_{\alpha,r} G$, is defined to be the norm closure of the image of the regular representation $C_c(G, A) \to B(H \otimes \ell^2(G))$.

We recall the general definition for the covariant representation for the full crossed product of a C^*-algebras.

Definition 5.1.2. A covariant representation of $C_c(G, A)$ is a pair (π, U) which consists of a unitary representation U of G into $H \otimes \ell^2(G)$ and a $*$-representation π of A into $H \otimes \ell^2(G)$ such that

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^* \tag{5.1.1}$$

for every $g \in G$ and $a \in A$.

The definition of the full crossed product of a C^*-algebra is given as follows.

Definition 5.1.3. The full crossed product, denoted $A \rtimes_{\alpha,f} G$, is the completion of $C_c(G, A)$ with respect to the norm

$$\|x\|_f = \sup \|\sigma(x)\|,$$

where the supremum is taken over all the $*$-homomorphisms $C_c(G, A) \to B(H)$. From the above construction, we know that the family of $*$-representation of $C_c(G, A)$ into $B(H)$ is not empty.

The following universal property of the full crossed product of C^*-algebras highlights that every covariant representation (π, U) of $C_c(G, A)$ yields a $*$-representation of the $A \rtimes_\alpha G$..

Proposition 5.1.4. *For every covariant representation (U, π, H) of $C_c(G, A)$, there is a $*$-homomorphism $\sigma : A \rtimes_\alpha G \to B(H)$ such that*

$$\sigma(\sum_{s \in G} a_s s) = \sum_{s \in G} \pi(a_s) U_s \tag{5.1.2}$$

for all $\sum_{s \in G} a_s s \in C_c(G, A)$.

Remark 5.1.5. We have to distinguish between the covariant condition in (5.1.1), which multiplied by the same unitary from both sides, and the universal property (5.1.2), which multiplied by a unitary from one side.

Example 5.1.6. If $A = \mathbb{C}$, then the full crossed product $C^*(G) := \mathbb{C} \rtimes_f G$ is called the full group C^*-algebra of G ($\mathbb{C}$ has only the trivial $*$-automorphism and trivial representation) and the reduced crossed product $C^*_r(G) := \mathbb{C} \rtimes_r G$ is called the reduced group C^*-algebra which is defined as the norm closure of $\lambda(\overline{C_c(G)}) \subset B(\ell^2(G))$ where λ is the left regular representation.

5.2 Crossed product of C^*-algebras and its local properties

The local properties for C^*-algebras preserve with the crossed product when G is amenable or the action of G is amenable. Throughout this section, we will present some important known results for crossed product of C^*-algebras and its local properties that we will use them in the next section when we are proving the same results for TRO's. The definitions of the local properties and the following theorems can be found in [BO08].

Theorem 5.2.1. *For an amenable group G and an action $\alpha : G \to Aut(A)$, the following statements hold.*

1. $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$
2. *A is nuclear if and only if $A \rtimes_\alpha G$ is nuclear.*
3. *A is exact if and only if $A \rtimes_\alpha G$ is exact.*

When the group G is not amenable then the result remains valid for the amenable action which is defined as follows (see [BO08] for more details).

Definition 5.2.2. Let A be a unital C^*-algebra. An action $\alpha : G \to Aut(A)$ is amenable if there exist finitely supported functions $T_i : G \to A$ with the following properties:

1. $T_i(g) \geq 0$ and $T_i(g) \in Z(A)$ (the center of A) for all $i \in \mathbb{N}$ and $g \in G$.
2. $\langle T_i, T_i \rangle = \sum_{g \in G} T_i(g)^2 = 1_A$.
3. $\|s *_\alpha T_i - T_i\| \longrightarrow 0$, for all $s \in G$ where $s *_\alpha T_i(p) = \alpha_s(T_i(s^{-1}p))$ for all $p \in G$.

Theorem 5.2.3. *For any amenable action of α on A, the following statements hold.*

1. $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$.
2. *A is nuclear if only if $A \rtimes_\alpha G$ is nuclear.*
3. *A is exact if and only if $A \rtimes_\alpha G$ exact.*

5.3 Reduced and full crossed product of TRO's

We have seen in the previous section the definitions of the full crossed product and the reduced crossed product of C^*-algebras. In this section, our goal is to define the crossed products of TRO's.

Definition 5.3.1. Let G be a discrete group and V be a TRO. An action of G on V is defined to be a group homomorphism $\alpha : G \to Aut(V)$; $s \mapsto \alpha_s$ of G into $Aut(V)$ where $Aut(V)$ is the group of the TRO-isomorphism on V. In this case we say that V is equipped with a G action α on V.

Suppose G is a discreet group and V is a TRO equipped with a G-action α on V. Let $C_c(G, V)$ be the linear space of finitely supported functions on G with values in V. An element $S \in C_c(G, V)$ is written as a finite sum

$$S = \sum_{s \in G} a_s s$$

where $a_s \in V$ and $s \in G$ and its conjugate is written as

$$S^* = (\sum_{s \in G} a_s s)^* = \sum_{s \in G} (s^{-1} a_s^* s) s^{-1} = \sum_{s \in G} \alpha_{s^{-1}}(a_s)^* s^{-1}$$

where $\alpha_s(a) = sas^{-1}$ for all $a \in V$ and $s \in G$. For $S, T, R \in C_c(G, V)$, where $S = \sum_{s \in G} a_s s$, $T^* = \sum_{t \in G} \alpha_{t^{-1}}(b_t)^* t^{-1}$ and $R = \sum_{r \in G} c_r r$, we define the triple convolution product as follows:

$$\begin{aligned} S * T^* * R &= \sum a_s s * \sum \alpha_{t^{-1}}(b_t)^* t^{-1} * \sum c_r r \\ &= \sum a_s (s \alpha_{t^{-1}}(b_t)^* s^{-1}) s t^{-1} c_r r \\ &= \sum a_s \alpha_s(\alpha_{t^{-1}}(b_t)^*) s t^{-1} c_r r \\ &= \sum \underbrace{a_s \alpha_{st^{-1}}(b_t)^* \alpha_{st^{-1}}(c_r)}_{\in V} \underbrace{st^{-1} r}_{\in G}, \end{aligned}$$

where the first part belongs to V and the second part belong to G. This implies that $S * T^* * R \in C_c(G, V)$, which means that $C_c(G, V)$ is the algebraic ternary ring, i.e. it is closed under the ternary product.

There are two different ways to take the completion of this ternary ring $C_c(G, V)$. The completion with respect to the reduced crossed product norm or the full crossed product norm. Motivated by the crossed product of C^*-algebras, we first construct the reduced crossed product of TRO's. We define the regular representation of TRO's as follows:

let's start with a faithful representation $V \subset B(H, K)$. Then we define a new representation π of V into

$B(H \otimes \ell^2(G), K \otimes \ell^2(G))$ by

$$\pi(a)(v \otimes \delta_g) = \alpha_{g^{-1}}(a)v \otimes \delta_g,$$

Under the identification $H \otimes \ell^2(G) \cong \oplus_{g \in G} H$, this is the direct sum representation

$$\pi(a) = \oplus_{g \in G} \alpha_g^{-1}(a) \in B(\oplus_{g \in G} H, \oplus_{k \in K} K).$$

Let λ be the left regular representation of G on $\ell^2(G)$ and let $\tilde{\lambda}_s^H = \mathbb{I}_H \otimes \lambda_s$ be the amplification of the left regular representation of G on $H \otimes \ell^2(G)$ and $\tilde{\lambda}_s^K = \mathbb{I}_K \otimes \lambda_s$ on $K \otimes \ell^2(G)$. Then the representation which consists of $(\pi, \tilde{\lambda}_s^H, \tilde{\lambda}_s^K)$ is called a regular covariant representation such that

$$\sigma(\sum_{s \in G} a_s s) = \sum_{s \in G} \pi(a_s)(\mathbb{I}_H \otimes \lambda_s)$$

as it is satisfied the following condition for every $a \in V$ and $s \in G$,

$$\pi(\alpha_s(a)) = \tilde{\lambda}_s^K \pi(a) \tilde{\lambda}_s^H = (\mathbb{I}_K \otimes \lambda_s)\pi(a)(\mathbb{I}_H \otimes \lambda_s)^*. \tag{5.3.1}$$

Remark 5.3.2. In the covariant condition of C^*-algebras (5.1.1), we see that it is multiplied by the same unitary from both sides since $H = K$. But for TRO's (5.3.1), we multiply each side by a different unitary since $H \neq K$.

Using the covariant condition, we check that

$$\mathrm{span}\{\pi(a)(\mathbb{I}_H \otimes \lambda_s) : a \in V, s \in G\} \subset B(H \otimes \ell^2(G), K \otimes \ell^2(G))$$

is closed under the triple product. Indeed, for $x = \pi(a)(\mathbb{I}_H \otimes \lambda_s)$, $y = \pi(b)(\mathbb{I}_H \otimes \lambda_t)$ and $z = \pi(c)(\mathbb{I}_H \otimes \lambda_u)$,

$$\begin{aligned}
xy^*z &= \pi(a)(\mathbb{I}_H \otimes \lambda_s)(\pi(b)(\mathbb{I}_H \otimes \lambda_t))^* \pi(c)(\mathbb{I}_H \otimes \lambda_u) \\
&= \pi(a)(\mathbb{I}_H \otimes \lambda_s)(\mathbb{I}_H \otimes \lambda_t))^* \pi(b)^* \pi(c)(\mathbb{I}_H \otimes \lambda_u) \\
&= \pi(a)(\mathbb{I}_H \otimes \lambda_{st^{-1}})\pi(b)^* \pi(c)(\mathbb{I}_H \otimes \lambda_u) \\
&= (\mathbb{I}_K \otimes \lambda_{st^{-1}})\pi(\alpha_{ts^{-1}}(a))\pi(b)^* \pi(c)(\mathbb{I}_H \otimes \lambda_u) \\
&= \pi(\alpha_{st^{-1}}(\alpha_{ts^{-1}}(a)b^*c))(\mathbb{I}_H \otimes \lambda_{ts^{-1}u}).
\end{aligned}$$

Thus, $xy^*z \in span\{\pi(a)(\mathbb{I}_H \otimes \lambda_t) : a \in V, t \in G\}$. Then by taking the completion of this linear span we obtain the reduced cross products of TRO's.

Definition 5.3.3. The reduced crossed product of a TRO, denoted as $V \rtimes_{\alpha,r} G$, is defined to be the norm closure of the image of a regular representation $C_c(G, V) \to B(H \otimes \ell^2(G), K \otimes \ell^2(G))$, i.e.

$$V \rtimes_{\alpha,r} G = \overline{span\{\pi(a)(\mathbb{I}_H \otimes \lambda_t) : a \in V, t \in G\}}^{\|\cdot\|}.$$

Before proving the main result in this section, we recall the following result by Hamana [Ham99], which shows us how we relate the C^*-homomorphisms of $C(V)$ and $D(V)$ with the TRO-homomorphism of a TRO V on $B(H, K)$.

Theorem 5.3.4. *Let V and W be two TRO's and $\pi : V \to W$ be a TRO-homomorphism. Then*

$$\pi_{A(V)} = \begin{bmatrix} \pi_C & \pi \\ \pi^* & \pi_D \end{bmatrix} : A(V) \to A(W)$$

is a well-defined C^*-homomorphism, where $\pi_C : C(V) \to C(W)$ and $\pi_D : D(V) \to D(W)$ are C^*-homomorphism which are defined as follows:

$$\pi_C(\sum_i x_i y_i^*) = \sum_i \pi(x_i)\pi(y_i)^* \tag{5.3.2}$$

$$\pi_D(\sum_i y_i^* z_i) = \sum_i \pi(y_i)^*\pi(x_i).$$

Remark 5.3.5. For $S = \sum_{s\in G} a_s s$, $T = \sum_{t\in G} a_t t$ and $R = \sum_{r\in G} c_r r \in C_c(G, V)$, we have

- $T^* = \sum_{t\in G} \alpha_{t^{-1}}(b_t)^* t^{-1} \in C_c(G, V^\sharp)$

- $S * T^* = \sum\limits_{s,t\in G} \underbrace{a_s \alpha_{st^{-1}}(b_t)^*}_{\in C(V)} st^{-1} \in C_c(G, C(V))$

- $T^* * R = \sum\limits_{t,r\in G} \underbrace{\alpha_{t^{-1}}(b_t)^* \alpha_{t^{-1}}(c_r)}_{\in D(V)} t^{-1} r \in C_c(G, D(V))$

Recall from the introduction that $V^\sharp$ is the conjugate space of V that is contained in $B(K, H)$ and $C(V)$ and $D(V)$ are both C^*-algebras generated by $VV^\sharp$ and $V^\sharp V$ respectively.

1. For an element $S * T^* \in C_c(G, C(V))$ we claim that

$$\pi_C \times (\mathbb{I}_K \otimes \lambda)\ (S * T^*) = \pi \times (\mathbb{I}_H \otimes \lambda)(S)\ \pi \times (\mathbb{I}_H \otimes \lambda)(T)^*.$$

Indeed,

$$
\begin{aligned}
\pi_C \times (\mathbb{I}_K \otimes \lambda)\ (S * T^*) &= \pi_C \times (\mathbb{I}_K \otimes \lambda)\Big(\sum_{s,t\in G} a_s \alpha_{st^{-1}}(b_t)^* st^{-1}\Big) \\
&= \sum_{s,t\in G} \pi_C(a_s \alpha_{st^{-1}}(b_t)^*)(\mathbb{I}_K \otimes \lambda_{st^{-1}}) \\
&= \sum_{s,t\in G} \pi(a_s)\pi(\alpha_{st^{-1}}(b_t))^*(\mathbb{I}_K \otimes \lambda_{st^{-1}}) \qquad (5.3.3) \\
&= \sum_{s,t\in G} \pi(a_s)(\mathbb{I}_H \otimes \lambda_{st^{-1}})[(\mathbb{I}_K \otimes \lambda_{ts^{-1}})\pi(\alpha_{st^{-1}}(b_t))(\mathbb{I}_H \otimes \lambda_{st^{-1}})]^* \qquad (5.3.4) \\
&= \sum_{s,t\in G} \pi(a_s)(\mathbb{I}_H \otimes \lambda_{st^{-1}})\pi(b_t)^* \\
&= \sum_{s,t\in G} \pi(a_s)(\mathbb{I}_H \otimes \lambda_s)[\pi(b_t)(\mathbb{I}_H \otimes \lambda_t)]^* \\
&= \pi \times (\mathbb{I}_H \otimes \lambda)(S)\ \pi \times (\mathbb{I}_H \otimes \lambda)(T)^*.
\end{aligned}
$$

The equality (5.3.3) follows from the equality 5.3.2 in Hamana's result. The covariant condition 5.3.1 is applied in the equality 5.3.4 to conclude the result. The argument for an element belong to $D(V)$ follows similarly i.e. $\pi_D \times (\mathbb{I}_H \otimes \lambda)\ (T^* * D) = \pi \times (\mathbb{I}_H \otimes \lambda)(T)^*\ \pi \times (\mathbb{I}_H \otimes \lambda)(D)$.

2. For $\pi \times (\mathbb{I}_H \otimes \lambda)\ (S * T^* * R)$ where S, T, R are defined as before we have

$$
\begin{aligned}
\pi \times (\mathbb{I}_H \otimes \lambda)\ (S * T^* * R) &= \sum_{s,t,r\in G} \pi(\alpha_{st^{-1}}(\alpha_{ts^{-1}}(a_s) b_t^* c_r))(\mathbb{I}_H \otimes \lambda_{ts^{-1}r}) \\
&= \sum_{s,t,r\in G} \pi(a_s)\pi(\alpha_{st^{-1}}(b_t))^*\pi(\alpha_{st^{-1}}(c_r))(\mathbb{I}_H \otimes \lambda_{ts^{-1}r}) \\
&= \sum_{s,t,r\in G} \pi(a_s)(\mathbb{I}_H \otimes \lambda_t)(\pi(b_t)(\mathbb{I}_H \otimes \lambda_t))^*\pi(c_r) \otimes (\mathbb{I}_H \otimes \lambda_r) \\
&= \pi \times (\mathbb{I}_H \otimes \lambda)(S)\ \pi \times (\mathbb{I}_H \otimes \lambda)(T)^*\ \pi \times (\mathbb{I}_H \otimes \lambda)(R) \in B(H, K).
\end{aligned}
$$

Let α^V be an action of G into the group of $Aut(V)$ and $\alpha^{A(V)}$ be its natural extension. A natural question to ask if we can identify $V \rtimes_{\alpha^V} G$ as the off-diagonal corner of the C^*-algebra $A(V) \rtimes_{\alpha^{A(V)}} G$. Equivalently, if we can obtain the $*$-isomorphism between the following linking C^*-algebras, i.e.

$$A(V \rtimes_{\alpha^V} G) = A(V) \rtimes_{\alpha^{A(V)}} G.$$

In the following proposition, we prove this identity for the reduced crossed product of $A(V)$.

Proposition 5.3.6. *Let G be a discrete group and let V be a TRO such that an action $\alpha^V : G \to Aut(V)$ can naturally be extended to $\alpha^{A(V)} : G \to Aut(A(V))$. Then we have the following C*-isomorphism,*

$$A(V \rtimes_{\alpha^V,r} G) = A(V) \rtimes_{\alpha^{A(V)},r} G. \tag{5.3.5}$$

More precisely,

$$A(V \rtimes_{\alpha^V,r} G) = \begin{bmatrix} C(V \rtimes_{\alpha^V,r} G) & V \rtimes_{\alpha^V,r} G \\ (V \rtimes_{\alpha^V,r} G)^\sharp & D(V \rtimes_{\alpha^V,r} G) \end{bmatrix} = \begin{bmatrix} C(V) \rtimes_{\alpha^{C(V)},r} G & V \rtimes_{\alpha^V} G \\ V^\sharp \rtimes_{\alpha^V,r} G & D(V) \rtimes_{\alpha^{D(V)},r} G \end{bmatrix} = A(V) \rtimes_{\alpha^{A(V)},r} G.$$

Proof. It is sufficient to check the C^*-isomorphism $C(V) \rtimes_{\alpha^{C(V)},r} G = C(V \rtimes_{\alpha^V,r} G)$. The proof for the other components will be similar. Note that

$$C(V) \rtimes_{\alpha^{C(V)},r} G = \overline{\{\pi_C(ab^*)(\mathbb{I}_K \otimes \lambda_s) : a, b \in V, s \in G\}}^{\|\cdot\|}$$

and

$$C(V \rtimes_{\alpha^V,r} G) = \overline{(V \rtimes_{\alpha^V,r} G)(V \rtimes_{\alpha^V,r} G)^\sharp}^{\|\cdot\|}.$$

Let $xy^* \in C(V \rtimes_{\alpha^V,r} G)$ where $\pi : V \to B(H,K)$ is a TRO-homomorphism such that $\pi|_C = \pi_C$ and $\pi|_D = \pi_D$ are $*$-homomorphisms for C*-algebras $C(V)$ and $D(V)$. Using the covariant condition, we have

$$\begin{aligned} xy^* &= \pi(a)(\mathbb{I}_H \otimes \lambda_s)[\pi(b)(\mathbb{I}_H \otimes \lambda_t)]^* \\ &= \pi(a)\ [\mathbb{I}_K \otimes \lambda_{ts^{-1}} \pi(\alpha_{st^{-1}}(b))]^* \\ &= \pi(a)\pi(\alpha_{st^{-1}}(b))^*(\mathbb{I}_K \otimes \lambda_{st^{-1}}) \\ &= \pi_C(a\alpha_{st^{-1}}(b)^*)(\mathbb{I}_K \otimes \lambda_{st^{-1}}) \in C(V) \rtimes_{\alpha^{C(V)},r} G. \end{aligned}$$

Note that in the last equation we use the identity (5.3.2) in Hamana's result. Now for the other direction let $\pi_C(ab^*)(\mathbb{I}_K \otimes \lambda_s) \in C(V) \rtimes_{\alpha^{C(V)},r} G$, then

$$\pi_C(ab^*)(\mathbb{I}_K \otimes \lambda_s) = \pi(a)\pi(b)^*(\mathbb{I}_K \otimes \lambda_s) = \pi(a)(\mathbb{I}_H \otimes \lambda_s)[\pi(\alpha_{s^{-1}}(b))(\mathbb{I}_H \otimes \lambda_{s^2})]^* \in C(V \rtimes_{\alpha^V,r} G).$$

The proof of the other three corners will be similar to this one. □

Now we consider for the full crossed product of TRO's. We replace the unitaries $\tilde{\lambda_s}^H$, $\tilde{\lambda_s}^K$ in the regular covariant representation by general unitaries u_s and v_s on Hilbert spaces H and K respectively. The general covariant representation (π, u_s, v_s) is defined to be the representation consists of TRO-homomorphism π and the unitaries u_s and v_s on Hilbert spaces H and K such that

$$\pi(\alpha_s(a)) = v_s \pi(a) u_s^*.$$

Definition 5.3.7. The full crossed product of a TRO is defined to be the completion of the space $C_c(G, V)$ with respect to the norm

$$\|S\|_f = \sup\{\|\pi \times u_s(S)\| : \pi \times u_s \text{ is a non-degenerate covariant representation of } C_c(G, V)\}$$

where $S \in C_c(G, V)$, denoted by $V \rtimes_{\alpha,f} G$.

Note that the covariant representation of a TRO has also the universal property.

5.4 Local properties

Since a TRO is defined as the off-diagonal component of its linking C^*-algebra $A(V)$, then there is a strong connection between some local properties of TRO's and their linking C^*-algebras [KR02]. In this section we prove that the local properties for TRO's preserve with the crossed product when the group G is amenable. The following theorem was proved by Kaur and Ruan in [KR02].

Theorem 5.4.1. *Let V be a TRO. Then the following are true*

1. *V is 1-exact (or equivalently, λ-exact) if and only if $A(V)$ is 1-exact (or equivalently, λ-exact).*

2. *V is nuclear if and only if $A(V)$ is nuclear.*

3. *V is local reflexive (or equivalently, λ-local reflexive) if and only if $A(V)$ is local reflexive (or equivalently, λ-local reflexive).*

Now, we use the identity (5.3.5) and Theorem 5.4.1 to prove the following theorem.

Theorem 5.4.2. *For any amenable group and any action $\alpha : G \to Aut(V)$, the following statements are all true.*

1. *$V \rtimes_{\alpha,f} G = V \rtimes_{\alpha,r} G$;*

2. *V is nuclear if and only if $V \rtimes_\alpha G$ is nuclear;*

3. *V is exact if and only if $V \rtimes_\alpha G$ is exact.*

Proof. We know that $A(V) \rtimes_{\alpha,f} G = A(V) \rtimes_{\alpha,r} G$ by Theorem 5.2.1. For (1), by using Proposition 5.3.6 we have $A(V \rtimes_{\alpha,f} G) = A(V \rtimes_{\alpha,r} G)$. Since the two maps $V \rtimes_{\alpha,f} G \hookrightarrow A(V \rtimes_{\alpha,f} G)$ and $V \rtimes_{\alpha,r} G \hookrightarrow A(V \rtimes_{\alpha,r} G)$ are completely isometric as upper-right corner, we have $V \rtimes_{\alpha,f} G = V \rtimes_{\alpha,r} G$. For (2), if V is nuclear, then by Theorem 5.4.1 $A(V)$ is nuclear. By Theorem 5.2.1, the linking C^*-algebra $A(V \rtimes_\alpha G) = A(V) \rtimes_\alpha G$ is also nuclear. So by Theorem 5.4.1, $V \rtimes_\alpha G$ is nuclear. Similarly if we assume that $V \rtimes_\alpha G$ is nuclear, then its linking C^*-algebra is also nuclear. Again, using the equality that $A(V \rtimes_\alpha G) = A(V) \rtimes_\alpha G$, $A(V)$ and hence V is nuclear. We leave (3) to the reader since the proof is identical to (2). □

5.5 Conditional crossed product

In this section, we consider a special type of the crossed product of linking C^*-algebras of TRO's where we have a one to one correspondence between the covariant representation of TRO's and its linking C^*-algebras.

Let $\alpha_s \in Aut(A(V))$ is defined as an inner action, $\alpha_s(a) = U_s a U_s^*$ for all $s \in G$ and U_s is a unitary in $B(H \oplus K)$, such that it satisfies one of the conditions in the following proposition.

Proposition 5.5.1. *Suppose α_s is an action on $A(V)$. Then the following are equivalent:*

1. $\alpha_s\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *and* $\alpha_s\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *for all* $s \in G$
2. $\alpha_s|_V \in Aut(V)$ *for all* $s \in G$.

Proof. By assuming (1) is true and using TRO-homomorphism we have

$$\begin{aligned}\alpha_s\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) &= \alpha_s\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\alpha_s\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V.\end{aligned}$$

This implies that $\alpha_s|_V \in Aut(V)$. For (2), the first equality implies (1). □

Definition 5.5.2. Let $A(V)$ be the linking C^*-algebra of a TRO V and let α be an action of G into $Aut(A(V))$. If α satisfies one of the condition of Proposition 5.5.1, then we call this crossed product as a conditional crossed product of $A(V)$, denoted as $A(V) \rtimes_{\alpha,c} G$.

Note that Proposition 5.5.1 defines a corner preserving action α_s which helps us to obtain the unitary representations u_s and v_s in the covariant representations for both $C(V)$ and $D(V)$.

Corollary 5.5.3. *Let $(\tilde{\pi}, U, L)$ be a covariant representation for the linking C^*-algebra $A(V)$ such that U is a unitary on the Hilbert space $L = H \oplus K$. Let's define an action α_s on $A(V)$ satisfies one of the conditions of Proposition 5.5.1. Then there exist orthogonal projections p and q, such that $pL = H$ and $qL = K$, which split U on $B(L) = B(H \oplus K)$ into two unitaries u_s and v_s for both $C(V)$ and $D(V)$ such that*

$$u_s = pU_sp \text{ is a unitary on } H$$

$$v_s = qU_sq \text{ is a unitary on } K.$$

Proof. From given, we have the covariant representation $(\tilde{\pi}, U)$ for $A(V) \rtimes_\alpha G$ where $\tilde{\pi} : A(V) \to B(H \oplus K)$ and

$$\tilde{\pi}(\alpha_s(a)) = U_s\tilde{\pi}(a)U_s^*$$

We claim that there exists a unitary u_s such that $u_s^*u_s = 1$ for the covariant representation of $C(V)$, where $u_s = pU_sp$, and similarly for $D(V)$.

The orthogonal projections p and q are defined as follows:

$$p = \tilde{\pi}\Big(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\Big),\ q = \tilde{\pi}\Big(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\Big),$$

where $p + q = \tilde{\pi}\Big(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Big) = I_L$ and $pq = 0$. Now, consider $a \in C(V)$ and let $u_s = pU_sp$. Then we have

$$\begin{aligned}
u_s^*u_s\tilde{\pi}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)p &= pU_s^*p \cdot pU_sp \cdot \tilde{\pi}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)p \\
&= pU_s^*p\left(U_s\tilde{\pi}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)U_s^*\right)U_sp \\
&= pU_s^*p \cdot \tilde{\pi}(\alpha_s\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right))U_sp \\
&= pU_s^*p \cdot \tilde{\pi}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\alpha_s\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\right)U_sp
\end{aligned}$$

$$= p \underbrace{U_s^* \tilde{\pi}\left(\alpha_s\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\right)U_s}_{\text{use the covariant condition again}} p$$

$$= p\ \tilde{\pi}\left(\alpha_s^{-1}\alpha_s\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\right)p = \tilde{\pi}\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)p.$$

This imply that $u_s^* u_s = I_K$. Now, using a similar argument we prove that $u_s u_s^* = I_K$. The argument for $D(V)$ follows similarly. ☐

Remark 5.5.4. For the converse, we know that if we have two covariant representations for $C(V)$ and $D(V)$ with unitaries u_s and v_s on $B(H)$ and $B(K)$ where H and K are two different Hilbert spaces then we can define a unitary for $A(V)$, $U_s = \begin{bmatrix} u_s & 0 \\ 0 & v_s \end{bmatrix}$ on $B(H \oplus K)$, and an action $\alpha_s(a) = U_s a U_s^*$ which satisfies the condition of Proposition 5.5.1 that $\alpha_s|_V(x) = u_s x v_s^* \in V$ for all $x \in V$. This is also called the conditional crossed product of $A(V)$, denoted as $A(V) \rtimes_{\alpha,c} G$. Of course, not all actions for $A(V)$ satisfy the condition of Proposition 5.5.1.

Example 5.5.5. Let $G = \mathbb{Z}_2$, $V = \mathbb{C}$. Then $A(V) = M_2(\mathbb{C})$. Define the following two unitaries in $A(V)$:

$$U_0 = I_{M_2(\mathbb{C})} \text{ and } U_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Define α_s as the associated inner action $\alpha_s(a) = U_s a U_s^*$. It is clear that $U_1 \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} U_1^* \notin V$.

Motivated by Proposition 5.3.6, we present the same result for the conditional crossed product of $A(V)$.

Theorem 5.5.6. *Let G be a discrete group and let V be a TRO such that the conditional crossed product exist for $A(V)$. Then we have the C^*-isomorphisms,*

$$A(V \rtimes_\alpha G)_c = A(V) \rtimes_c G \tag{5.5.1}$$

$$C(V \rtimes_\alpha G)_c = C(V) \rtimes_{\alpha,c} G \text{ and } D(V \rtimes_\alpha G)_c = D(V) \rtimes_{\alpha,c} G.$$

The proof is similar to Proposition 5.3.6 for the reduced crossed product of $A(V)$.

Theorem 5.5.7. *Let V be a TRO and G be a discrete group. Then*

$$V \rtimes_{\alpha,r} G = V \rtimes_{\alpha} G \text{ if and only if } A(V) \rtimes_{\alpha,r} G = A(V) \rtimes_{\alpha,c} G.$$

Proof. If $A(V) \rtimes_{\alpha,r} G = A(V) \rtimes_{\alpha,c} G$ then by using the identity (5.5.1), (5.3.5) and the orthogonal projection $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have that

$$V \rtimes_{\alpha,r} G = pA(V \rtimes_{\alpha,r} G)q = pA(V \rtimes_{\alpha,c} G)q = V \rtimes_{\alpha} G.$$

For the other direction, if we have $V \rtimes_{\alpha,r} G = V \rtimes_{\alpha} G$, then the linking C^*-algebras are the same and again by using the identity (5.5.1), (5.3.5), we get that $A(V) \rtimes_{\alpha,r} G = A(V \rtimes_{\alpha,r} G) = A(V \rtimes_{\alpha} G) = A(V) \rtimes_{\alpha,c} G$. □

This class of crossed product sits in the middle between the full and reduced crossed product of $A(V)$.

Theorem 5.5.8. *The following maps h and g*

$$A(V) \rtimes_{f} G \xrightarrow{h} A(V) \rtimes_{\alpha,c} G \xrightarrow{g} A(V) \rtimes_{\alpha,r} G$$

are quotient maps. In particular $V \rtimes_{\alpha,c} G \xrightarrow{g} V \rtimes_{\alpha,r} G$ is a quotient map.

Proof. It's straightforward to check that these maps are completely quotient maps by extending the identity map between its dense algebra $C_c(G, A(V))$. Then by taking different completion for the algebra $C_c(G, A(V))$ (the full, the conditional and the reduced one), we get contractive maps which are onto $*$-homomorphism maps which are quotient maps. Similarly, we prove it for TRO's. We just need to recall (3) from Proposition A.0.11 in the appendix A that was proved in [EOR01], that every surjective TRO-homomorphism is a contraction map and a completely quotient map. This completes the proof. □

Remark 5.5.9. In this Chapter, we proved this identity $A(V \rtimes_{\alpha^V} G) = A(V) \rtimes_{\alpha^{A(V)}} G$ for the reduced and conditional crossed products of $A(V)$. This answered our question about Morita equivalent between two C^*-algebras. We conclude that if we have two C^*-algebras $C(V)$ and $D(V)$, denoted as the diagonal components of the linking C^*-algebra $A(V)$, which are Morita equivalent then their reduced and conditional crossed product are also Morita equivalent, i.e. $C(V) \overset{M.E}{\cong} D(V) \Rightarrow C(V) \rtimes G \overset{M.E}{\cong} D(V) \rtimes G$.

Appendix A

TRO-homomorphism

Throughout this appendix, we state and recall some known results for TRO-homomorphism. We state their proofs for the convenience of the reader (see [EOR01] for more details).

Definition A.0.10. A norm-closed subspace J in a TRO V is called a TRO ideal in V if $JV^\sharp V \subset J$ and if $VV^\sharp J \subset J$.

Theorem A.0.11. *Let V and W be two TRO's, and let J be a TRO ideal for V.*

1. *Every TRO-homomorphism $\phi : V \to W$ is completely contractive.*
2. *Every injective TRO-homomorphism is completely isometric.*
3. *V/J is a TRO with the induced ternary product and operator space structure.*
4. *If the map between two TRO's V and W is a surjective TRO-homomorphism, then it must be a completely quotient map.*

Proof.

For (1) and (2), the argument is based on some spectral arguments that are proved by Harries (see [Har81]for more details). For $a \in V$ and a TRO-homomorphism $\phi : V \to W$, we have $\sigma(\phi(a)\phi(a)^*) \subset \sigma(aa^*) \cup 0$ by (Lemma 3.5 [Har81]). This implies $\|\phi(a)\|^2 \leq \|a\|^2$ and a simple modification for this argument where $\phi : V \to W$ is injective implies that $\|\phi(a)\|^2 = \|a\|^2$. Similarly, for each $n \in \mathbb{N}$, $\pi_n : M_n(V) \to M_n(W)$ is also TRO-homomorphism and the result follows as above.

For (3) and (4), they are proved by Ruan, Ozawa and Effroce in [EOR01]. Let us start by recalling the proof of (3). Let J be a TRO ideal in V. Then the quotient operator space $V/J = \{x_j = x + J : x \in V\}$ has the triple product which is defined as follows:

$$x_j y_j^* z_j = (xy^* z)_j.$$

Let

$$A(V) = \begin{bmatrix} \mathbb{C}1 + C(V) & V \\ V^\sharp & \mathbb{C}1 + D(V) \end{bmatrix}$$

be the linking C^*-algebra of V and

$$I(V) = \begin{bmatrix} \overline{JV^\sharp + VJ^\sharp} & J \\ J^\sharp & \overline{J^\sharp V + V^\sharp J}. \end{bmatrix}$$

is a closed ideal in $A(V)$, and we have the completely isometric TRO-isomorphisms

$$V \cong eA(V)(1-e) \text{ and } J \cong eI(V)(1-e)$$

where $e \in A(V)$ is the projection such that $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $1 - e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then there is a natural complete contraction

$$\pi : V/J \to A(V)/I(V)$$
$$x_j \mapsto x_I$$

where π preserves the ternary product $\pi(x_j y_j^* z_j) = \pi(x_j)\pi(y_j)^*\pi(z_j)$ for all $x_j, y_j, z_j \in V/J$ which maps the off-diagonal corner $e_I A/I(1-e_I)$ of the C^*-algebra, where $e_I = e + I(V)$. Claim: π is complete isometry. Indeed, given $x_j \in V/J$, we have

$$\begin{aligned} \|\pi(x_j)\| = \inf\{\|x+y\| : y \in I\} &\geq \inf\{\|e(x+y)(1-e) : y \in I\|\} \\ &= \inf\{\|x+y\| : y \in J\} = \|x_j\|. \end{aligned}$$

This shows that π is an isometry. The similar argument is used to prove it for π_n for each $n \in \mathbb{N}$. This implies that V/J is completely isometrically TRO-isomorphic to the off diagonal corner $e_I A(V)/I(V)(1-e_I)$ of the C^*-algebra $A(V)/I(V)$.

For (4), if $\phi : V \to W$ is a TRO-homomorphism, then $J = \mathrm{Ker}(\phi)$ is a TRO-ideal in V by (3). The map ϕ induces an injective completely isometric TRO-homomorphism from V/J into W. So we have the following diagram:

$$\begin{tikzcd}
V \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & W = \phi(V) \\
& V/ker(\phi) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

It follows that $\phi(V)$ is a sub-TRO of W. Since ϕ is a TRO-homomorphism which is surjective then this implies that the map $\phi : V \to W$ is a completely quotient map. Recall that the map is completely quotient if for each $n \in \mathbb{N}$, ϕ_n maps the closed unit ball of $M_n(V)$ onto the closed unit ball of $M_n(W)$. This result is an elementary result for the C^*-algebras. From (3), we have $V/J \cong e_I(A(V)/I(V))(1 - e_I)$. Given $w \in V/J$ with $\|w\| \leq 1$ we may choose $a \in A$ such that $\phi(a) = w$. Then there exists $v = ea(1 - e) \in V$ satisfies $\|v\| \leq 1$ and $\phi(v) = w$. We use the same argument to proof the map ϕ_n is a quotient map for every n. $\square$

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