COLORING AND COVERING PROBLEMS ON GRAPHS

BY

SARAH LOEB

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

Doctoral Committee:

Professor Alexandr Kostochka, Chair
Emeritus Professor Douglas B. West, Director of Research
Professor Alexander Yong
Dr. Theodore Molla

Abstract

The *separation dimension* of a graph G, written $\pi(G)$, is the minimum number of linear orderings of $V(G)$ such that every two nonincident edges are "separated" in some ordering, meaning that both endpoints of one edge appear before both endpoints of the other. We introduce the *fractional separation dimension* $\pi_f(G)$, which is the minimum of a/b such that some a linear orderings (repetition allowed) separate every two nonincident edges at least b times.

In contrast to separation dimension, we show fractional separation dimension is bounded: always $\pi_f(G) \leq 3$, with equality if and only if G contains K_4. There is no stronger bound even for bipartite graphs, since $\pi_f(K_{m,m}) = \pi_f(K_{m+1,m}) = \frac{3m}{m+1}$. We also compute $\pi_f(G)$ for cycles and some complete tripartite graphs. We show that $\pi_f(G) < \sqrt{2}$ when G is a tree and present a sequence of trees on which the value tends to 4/3. We conjecture that when $n = 3m$ the K_4-free n-vertex graph maximizing $\pi_f(G)$ is $K_{m,m,m}$.

We also consider analogous problems for circular orderings, where pairs of nonincident edges are separated unless their endpoints alternate. Let $\pi^\circ(G)$ be the number of circular orderings needed to separate all pairs, and let $\pi_f^\circ(G)$ be the fractional version. Among our results: (1) $\pi^\circ(G) = 1$ if and only G is outerplanar. (2) $\pi^\circ(G) \leq 2$ when G is bipartite. (3) $\pi^\circ(K_n) \geq \log_2 \log_3(n-1)$. (4) $\pi_f^\circ(G) \leq \frac{3}{2}$, with equality if and only if $K_4 \subseteq G$. (5) $\pi_f^\circ(K_{m,m}) = \frac{3m-3}{2m-1}$.

A *star k-coloring* is a proper k-coloring where the union of any two color classes induces a star forest. While every planar graph is 4-colorable, not every planar graph is star 4-colorable. One method to produce a star 4-coloring is to partition the vertex set into a 2-independent set and a forest; such a partition is called an I,F-*partition*. We use discharging to prove that every graph with maximum average degree less than $\frac{5}{2}$ has an I,F-partition, which is sharp and improves the result of Bu, Cranston, Montassier, Raspaud, and Wang (2009). As a corollary, we gain that every planar graph with girth at least 10 has a star 4-coloring.

A proper vertex coloring of a graph G is *r-dynamic* if for each $v \in V(G)$, at least $\min\{r, d(v)\}$ colors appear in $N_G(v)$. We investigate 3-dynamic versions of coloring and list coloring. We prove that planar and toroidal graphs are 3-dynamically 10-choosable, and this bound is sharp for toroidal graphs.

Given a proper total k-coloring c of a graph G, we define the *sum value* of a vertex v to be $c(v)$ +

$\sum_{uv \in E(G)} c(uv)$. The smallest integer k such that G has a proper total k-coloring whose sum values form a proper coloring is the *neighbor sum distinguishing total chromatic number* $\chi''_{\Sigma}(G)$. Pilśniak and Woźniak (2013) conjectured that $\chi''_{\Sigma}(G) \leq \Delta(G) + 3$ for any simple graph with maximum degree $\Delta(G)$. We prove this bound to be asymptotically correct by showing that $\chi''_{\Sigma}(G) \leq \Delta(G)(1 + o(1))$. The main idea of our argument relies on Przybyło's proof (2014) for neighbor sum distinguishing edge-coloring.

Acknowledgments

I would like to begin my acknowledgments by thanking my advisor, Douglas West. I have enjoyed doing mathematics with him and the time I have spent socially with him and Ching. I have learned a great deal about mathematical writing from Doug and appreciated his patient corrections. As this section is the only part of the thesis he did not get to read, any grammatical errors in it are mine alone.

I would like to thank my committee members, Sasha Kostochka, Theo Molla, and Alex Yong. Over the years they have given me excellent advise about mathematics and my career. Thank you also to Mike Ferrara for being a superb mentor and collaborator. In addition, thanks to Rick Laugesen, Randy McCarthy, and Jennifer McNeilly for their advise and support during my PhD.

From a young age, my family has been extremely supportive of my mathematical interests. I couldn't have made it to UIUC without their support. In particular, I want to mention my parents, their spouses, my brother, and my grandparents.

My collaborators made this thesis possible and deserve acknowledgment here in addition to the relevant chapters of this thesis. Many of these collaborators have become friends with whom I look forward to continuing collaboration.

Lest I forget any, I omit the names of the many friends who lent me support along the way though they all have my thanks. These friends and supporters include friends from high school, Harvey Mudd College, the University of Illinois, and friends made at conferences and workshops. Within the category of friends, I owe particular thanks to the romantic partners who supported me through my PhD.

Finally, my research had several funding sources. Research in Chapter 2 was supported in part by a gift from Gene H. Golub to the Mathematics Department of the University of Illinois. Research in Chapter 3 was supported in part by NSF grant DMS-1500662 "The 2015 Rocky Mountain–Great Plains Graduate Research Workshop in Combinatorics". Research in Chapter 4 was supported in part by NSF grant DMS 08-38434, "EMSW21-MCTP: Research Experience for Graduate Students".

Table of Contents

Chapter 1

Introduction

In this thesis, we study fractional separation dimension and several types of graph colorings.

Both separation dimension and traditional graph coloring can be viewed as hypergraph covering problems, as can graph domination and poset dimension. Given a hypergraph H, the *covering number* $\tau(H)$ is the minimum number of edges in H whose union is the full vertex set.

A k-coloring $c\colon V(G) \to \{1, \ldots, k\}$ of a graph G is *proper* if c assigns distinct colors to adjacent vertices. The *chromatic number* of G is the minimum k such that G has a proper k-coloring. Framing coloring as a covering problem, the vertex set of H is $V(G)$ and $e \in E(H)$ if and only if e does not contain both vertices of any edge of G.

List coloring is a variation on coloring introduced independently by Vizing [42] and by Erdős, Rubin, and Taylor [19]. A *list assignment* L for G assigns to each vertex v a list $L(v)$ of permissible colors. Given a list assignment L for a graph G, if a proper coloring ϕ can be chosen so that $\phi(v) \in L(v)$ for all $v \in V(G)$, then G is *L-colorable*. The *choosability* of G is the least k such that G is L-colorable for any list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$.

Subsequent sections of this chapter give an overview of results in each chapter. In section 1.5, we present notation used in this thesis and relevant graph theoretic definitions.

1.1 Fractional separation dimension

A pair of nonincident edges in a graph G is *separated* by a linear ordering of $V(G)$ if both vertices of one edge precede both vertices of the other. The *separation dimension* $\pi(G)$ of a graph G is the minimum number of vertex orderings that together separate every pair of nonincident edges of G. Graphs with at most three vertices have no such pairs, so their separation dimension is 0. We therefore consider only graphs with at least four vertices.

Introduced by Basavaraju, Chandran, Golumbic, Mathew, and Rajendraprasad [9] (full version in [10]), separation dimension is motivated by a geometric interpretation. By viewing the orderings as giving coordinates for each vertex, the separation dimension is the least k such that the vertices of G can be embedded in $\mathbb{R}^k$ so that any two nonincident edges of G are separated by a hyperplane perpendicular to some coordinate axis (ties in a coordinate may be broken arbitrarily.)

Framing separation dimension as a covering problem, the vertex set of H is the set of pairs of nonincident edges in G, and the edges of H are the sets of pairs separated by a single ordering of $V(G)$.

Given a hypergraph covering problem, the corresponding fractional problem considers the difficulty of covering each vertex multiple times and measures the average number of edges needed. In particular, the *t-fold covering number* $\tau_t(H)$ is the least number of edges in a list of edges (repetition allowed) that covers each vertex at least t times, and the *fractional covering dimension* is $\liminf_t \tau_t(H)/t$. In the special case that H is the hypergraph associated with separation dimension, we obtain the *t-fold separation dimension* $\pi_t(G)$ and the *fractional separation dimension* $\pi_f(G)$.

Every list of s edges in a hypergraph H provides an upper bound on $\tau_f(H)$; if it covers each vertex at least t times, then it is called an *$(s : t)$-covering*, and $\tau_f(H) \le s/t$. This observation will enable us to obtain the maximum value of the fractional separation dimension. It is bounded, even though the separation dimension is not. In Section 2.3, we show:

Theorem 1.1.1. *$\pi_f(G) \le 3$ for any graph G, with equality if and only if $K_4 \subseteq G$.*

No smaller bound can be given even for bipartite graphs; we prove $\pi_f(K_{m,m}) = \frac{3m}{m+1}$.

In Sections 2.4 and 2.5 we consider sparser graphs. The *girth* of a graph is the minimum length of its cycles (infinite if it has no cycles). In Section 2.4 we show $\pi_f(C_n) = \frac{n}{n-2}$. Also, the value is $\frac{30}{17}$ for the Petersen graph and $\frac{28}{17}$ for the Heawood graph. Although these results suggested asking whether graphs with fixed girth could admit better bounds on separation number, Alon [3] pointed out by using expander graphs that large girth does not permit bounding $\pi_f(G)$ by any constant less than 3 (see Section 2.4). Nevertheless, we can still ask the question for planar graphs.

Question 1.1.2. *How large can $\pi_f(G)$ be when G is a planar graph with girth at least g?*

In Section 2.5, we consider graphs without cycles.

Theorem 1.1.3. *$\pi_f(G) < \sqrt{2}$ when G is a tree.*

The bound in Theorem 1.1.3 improves to $\pi_f(T) \le \frac{4}{3}$ for trees obtained from a subdivision of a star by adding any number of pendant edges at each leaf. This is sharp; the tree with $4m+1$ vertices obtained by subdividing every edge of $K_{1,2m}$ has diameter 4 and fractional separation dimension $\frac{4m-2}{3m-1}$, which tends to $\frac{4}{3}$. We believe that the optimal bound for trees is strictly between $\frac{4}{3}$ and $\sqrt{2}$.

Question 1.1.4. *What is the supremum of $\pi_f(G)$ when G is a tree?*

In Section 2.6, we return to the realm of dense graphs with values of π_f near 3. We first compute $\pi_f(K_{m+1,qm})$. The formula yields $\pi_f(K_{m,r}) < 3(1-\frac{1}{2m-1})$ for all r, so both parts of a bipartite graph must grow to obtain a sequence of values approaching 3. In the special case $q=1$, we obtain $\pi_f(K_{m+1,m}) = \frac{3m}{m+1}$. In addition, $\pi_f(K_{m,m}) = \frac{3m}{m+1}$, for $m \geq 2$. We conjecture that, among bipartite n-vertex graphs, π_f is maximized by $K_{n,n}$.

We also studied complete balanced tripartite graphs.

Theorem 1.1.5. $\pi_f(K_{m,m,m}) = \frac{6m}{2m+1}$ *for* $m \geq 2$.

When $n = 6r$, we thus have $\pi_f(K_{2r,2r,2r}) > \pi_f(K_{3r,3r})$. Surprisingly, the value is larger for a quite different complete tripartite graph.

Theorem 1.1.6. $\pi_f(K_{1,m,m}) = \frac{24m}{8m+5+3/(2\lceil m/2\rceil-1)}$ *for* $m \geq 1$.

Computer search verifies the extreme among tripartite graphs up to 14 vertices. Using Theorems 2.6.2 and 2.6.4, we compare $\pi_f(K_{2r+1,2r+1,2r+1}) = \frac{6(2r+1)}{4r+3}$ and $\pi_f(K_{1,3r+1,3r+1}) \geq \frac{24(3r+1)}{24r+13+1/r}$. Each graph has $6r+3$ vertices. When $r > 1$, the value of $\pi_f(K_{1,3r+1,3r+1})$ is larger. On the other hand, for $n = 9$, there is an anomaly, with $\pi_f(K_{3,3,3}) > \pi_f(K_{1,4,4})$.

Conjecture 1.1.7. *For $n \geq 10$, the graph $K_{1,\lfloor (n-1)/2\rfloor,\lceil (n-1)/2\rceil}$ achieves the maximum value of π_f among n-vertex graphs not containing K_4.*

Since $\pi_f(G)$ is always rational, we ask

Question 1.1.8. *Which rational numbers (between* 1 *and* 3*) occur as the fractional separation dimension of some graph?*

Finally, in Section 2.7, we consider the analogues of π and π_f defined by using circular orderings of the vertices rather than linear ones; we use the notation π° and π_f°. We show first that $\pi^\circ(G) = 1$ if and only if G is outerplanar. Surprisingly, $\pi^\circ(K_{m,n}) = 2$ when $m, n \geq 2$ and $mn > 4$, but π° is unbounded.

Theorem 1.1.9. $\pi^\circ(K_n) > \log_2 \log_3 (n-1)$.

For the fractional context, we prove $\pi_f^\circ(G) \leq \frac{3}{2}$ for all G, with equality if and only if $K_4 \subseteq G$. Again no better bound holds for bipartite graphs; we prove $\pi_f^\circ(K_{m,qm}) = \frac{6(qm-1)}{4mq+q-3}$, which tends to $\frac{3}{2}$ as $m \to \infty$ when $q = 1$. It tends to $\frac{6m}{4m+1}$ when $q \to \infty$, so again both parts must grow to obtain a sequence on which π_f° tends to $\frac{3}{2}$. The proof is different from the linear case. The questions remaining are analogous to those for π_f.

Question 1.1.10. *How large can π_f° be when G is a planar graph with girth at least g? Which are the n-vertex graphs maximizing π_f° among bipartite graphs and among those not containing K_4? Which rational numbers between 1 and $\frac{3}{2}$ occur?*

This chapter contains joint work with Douglas West.

1.2 I,F-partitions

Acyclic coloring was first introduced by Grünbaum [22]. A proper vertex coloring is *acyclic* if the union of any two color classes induces a forest. The least k such that G has an acyclic k-coloring is the *acyclic chromatic number* of G, denoted $\chi_a(G)$. An acyclic k-coloring of G is a *star k-coloring* if the components of the forest induced by the union of two color classes are stars. The least k such that G has a star k-coloring is the star chromatic number of G, denoted $\chi_s(G)$. It follows immediately that $\chi(G) \leq \chi_a(G) \leq \chi_s(G)$ for any graph G, although is not difficult to see that $\chi \neq \chi_a$ in general by considering, for instance, any bipartite graph containing a cycle. We refer the reader to the thorough survey of Borodin [13] for additional results on acyclic and star colorings beyond what we present next.

In this chapter, we are interested in the problem of star-coloring planar graphs. The well-known Four Color Theorem of Appel and Haken [7, 8] states that $\chi(G) \leq 4$ if G is planar, while Grünbaum [22] constructed a planar graph with no acyclic 4-coloring (and so, in particular, no star 4-coloring). Subsequently, Borodin [12] showed $\chi_a(G) \leq 5$ for all planar G. Albertson, Chappell, Kierstead, Kündgen, and Ramamurthi [2] showed that every planar graph G satisfies $\chi_s(G) \leq 20$ and also constructed a planar graph with star chromatic number at least 10. Kündgen and Timmons [29] proved that every planar graph of girth 6 (respectively 7 and 8) can be star-colored with 8 (respectively 7 and 6) colors. Kierstead, Kündgen and Timmons [25] showed that every bipartite planar graph can be star 14-colored, and they constructed a bipartite planar graph with star chromatic number 8. It is worthwhile to note that, while not our focus here, the results in [29] and [25] hold for the natural extension of star-colorings to a list coloring framework.

Given the Four Color Theorem, it is natural to search for conditions that ensure a planar graph can be star 4-colored. Albertson *et al.* [2] also showed that for every girth g, there exists a graph G_g with girth at least g and $\chi_s(G_g) = 4$, and further that there is some girth g such that every planar graph of girth at least g is star 4-colorable. Timmons [41] showed that $g = 14$ is sufficient and also gave a planar graph with girth 7 and star chromatic number 5. Bu, Cranston, Montassier, Raspaud, and Wang [15] improved upon Timmons' result by showing that every planar graph with girth $g \geq 13$ has a star 4-coloring.

The *maximum average degree* $\mathrm{Mad}(G)$ of a graph G is the $\max_{H \subseteq G} \overline{d}(H)$. Our main result shows:

Theorem 1.2.1. *If G is a graph with* $\mathrm{Mad}(G) < \frac{5}{2}$*, then* $\chi_s(G) \leq 4$.

A straightforward application of Euler's formula shows that if G is a planar graph with girth at least g, then $\mathrm{Mad}(G) < \frac{2g}{g-2}$. Thus, as a corollary to Theorem 1.2.1 we have the following improvement on [15].

Corollary 1.2.2. *If G is a planar graph with girth at least* 10*, then* $\chi_s(G) \leq 4$.

To prove Theorem 1.2.1 we will use I,F-partitions, which were first introduced in [2]. A *2-independent set* in G is a set of vertices that have pairwise distance greater than 2. An I,F*-partition* of a graph G is a partition of $V(G)$ as $(\mathcal{I}, \mathcal{F})$, where $\mathcal{I}$ is a 2-independent set in G and $G[\mathcal{F}]$ is a forest. Albertson *et al.* [2] observed that if G has an I,F-partition $(\mathcal{I}, \mathcal{F})$, then $\chi_s(G) \leq 4$; because $\chi_s(T) \leq 3$ for any tree T there is a 3-coloring of $G[\mathcal{F}]$ which can be extended to all of G by assigning the vertices in $\mathcal{I}$ a new color. Note that the converse does not hold; for example, $\chi_s(K_{3,3}) = 4$, but $K_{3,3}$ has no I,F-partition. Timmons [41] and Bu *et al.* [15] showed that maximum average degree less than $\frac{7}{3}$ and $\frac{26}{11}$, respectively, imply the existence of an I,F-partition, which in turn imply that the above mentioned girth bounds sufficient for a planar graph to be star 4-colorable. Along the same lines, Theorem 1.2.1 is a consequence of the following theorem.

Theorem 1.2.3. *If G is a graph with* $\mathrm{Mad}(G) < \frac{5}{2}$*, then G has an* I,F*-partition.*

Theorem 1.2.3 is sharp in the sense that there are graphs with maximum average degree $\frac{5}{2}$ that do not have an I,F-partition. Indeed, given a cycle C, for each vertex v in the cycle add a 3-cycle $a_v b_v c_v$ and the edge va_v. To see that such a graph, which has maximum average degree $\frac{5}{2}$, does not have an I,F-partition, simply note that no vertex v on the cycle C can be in the 2-independent set, as then $a_v b_v c_v$ would necessarily have to be in the forest $\mathcal{F}$, an impossibility. However, this then implies that every vertex on C must be in $\mathcal{F}$, which is also impossible.

This chapter contains joint work with Axel Brandt, Michael Ferrara, Mohit Kumbhat, Derrick Stolee, and Matthew Yancey.

1.3 3-Dynamic coloring of planar and toroidal graphs

For a graph G and a positive integer r, an *r-dynamic coloring* of G is a proper vertex coloring such that for each $v \in V(G)$, at least $\min\{r, d(v)\}$ distinct colors appear in $N_G(v)$. The *r-dynamic chromatic number*, denoted $\chi_r(G)$, is the minimum k such that G admits an r-dynamic k-coloring. Montgomery [35] introduced 2-dynamic coloring and the generalization to r-dynamic coloring.

We consider the r-dynamic version of list coloring. For further work, see [1, 23, 24]. A graph G is *r-dynamically L-colorable* when an r-dynamic coloring can be chosen from the list assignment L. The r-

dynamic choosability of G, denoted $\mathrm{ch}_r(G)$, is the least k such that G is *r-dynamically L-colorable* for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$.

The *square* of a graph G, denoted G^2, is the graph resulting from adding an edge between every pair of vertices of distance 2 in G. For any graph G,

$$\begin{aligned} \chi(G) = \ \chi_1(G) &\leq \ \chi_2(G) \leq \cdots \leq \ \chi_{\Delta(G)}(G) = \cdots = \ \chi(G^2), \\ \mathrm{ch}(G) = \mathrm{ch}_1(G) &\leq \mathrm{ch}_2(G) \leq \cdots \leq \mathrm{ch}_{\Delta(G)}(G) = \cdots = \mathrm{ch}(G^2), \end{aligned} \tag{1}$$

and that $\chi_r(G) \leq \mathrm{ch}_r(G)$ for all r. Thus we can think of r-dynamic coloring as bridging the gap between coloring a graph and coloring its square.

Wegner [45] conjectured bounds for the chromatic number of squares of planar graphs in terms of their maximum degree. For a graph G with $\Delta(G) \leq 3$, proper colorings of G^2 and 3-dynamic colorings of G are equivalent. Thomassen [40] proved Wegner's conjecture for maximum degree 3, showing that $\chi_3(G) \leq 7$ for any planar subcubic graph G. Cranston and Kim [17] studied the list coloring version and proved that when G is a planar subcubic graph, $\mathrm{ch}_3(G) \leq 7$ if the girth is at least 7 and $\mathrm{ch}_3(G) \leq 6$ if the girth is at least 9.

Thomassen [39] proved that planar graphs are 5-choosable, and Voigt [43] proved sharpness.

Our main results are on the 3-dynamic chromatic number and choice number for toroidal graphs. A graph is *toroidal* if it can be drawn on the torus without crossing edges. In particular, planar graphs are also toroidal.

Theorem 1.3.1. *If G is a toroidal graph, then $\chi_3(G) \leq \mathrm{ch}_3(G) \leq 10$.*

Theorem 1.3.1 is sharp: the Petersen graph P is embeddable on the torus. It has maximum degree 3 and diameter 2, so $\chi_3(P) = \chi(P^2) = \chi(K_{10}) = 10$.

As an immediate corollary of Theorem 1.3.1, we have:

Corollary 1.3.2. *If G is a planar graph, then $\chi_3(G) \leq \mathrm{ch}_3(G) \leq 10$.*

We do not believe that Corollary 1.3.2 is sharp. An example of a planar graph G with $\chi_3(G) = 7$ is the graph obtained from K_4 by subdividing the three edges incident to one vertex. Note that G has maximum degree 3 and diameter 2, so $\chi_3(G) = \chi(G^2) = \chi(K_7)$.

This chapter contains joint work with Thomas Mahoney, Benjamin Reiniger, and Jennifer Wise.

1.4 Neighbor sum distinguishing total colorings

For an edge-coloring c, define the *sum value* $s_c(v)$ of a vertex v by $\sum_{u \in N(v)} c(uv)$. An edge-coloring is a *proper edge-weighting* if s_c forms a proper coloring. The least k such that G has a proper k-edge-coloring that is a proper edge-weighting is the *neighbor sum distinguishing edge-chromatic number* of a graph, denoted $\chi'_\Sigma(G)$. This parameter is well defined only for graphs with no isolated edges. Clearly, $\chi'_\Sigma(G) \geq \chi'(G) \geq \Delta(G)$. Flandrin, Marczyk, Przybyło, Saclé, and Woźniak [20] conjectured that:

Conjecture 1.4.1 ([20]). *If G is a connected graph with at least three vertices other than C_5, then $\chi'_\Sigma(G) \leq \Delta(G) + 2$.*

Przybyło [37] proved an asymptotically optimal upper bound for graphs with large maximum degree. Specifically, he showed:

Theorem 1.4.2 ([37]). *If G is a connected graph with $\Delta(G)$ sufficiently large, then $\chi'_\Sigma(G) \leq \Delta(G) + 50\Delta(G)^{5/6} \ln^{1/6} \Delta(G)$.*

A *proper total k-coloring* of G is a function $c : V(G) \cup E(G) \to [k]$ such that c restricted to $V(G)$ is a proper coloring, c restricted to $E(G)$ is a proper edge-coloring, and the color on each vertex is different from the color on its incident edges. For a total coloring c, define the *sum value* $s_c(v)$ of a vertex v by $c(v) + \sum_{uv \in E(G)} c(uv)$. A total coloring is a *proper total weighting* if s_c is a proper coloring. The least k such that G has a proper total k-coloring that is a proper total weighting is the *neighbor sum distinguishing total chromatic number* of G, denoted $\chi''_\Sigma(G)$. Clearly, $\chi''_\Sigma(G) \geq \chi''(G) \geq \Delta(G) + 1$. Pilśniak and Woźniak [36] conjectured that

Conjecture 1.4.3 ([36]). *If G is a connected graph with maximum degree $\Delta(G)$, then $\chi''_\Sigma(G) \leq \Delta(G) + 3$.*

Pilśniak and Woźniak [36] proved that Conjecture 1.4.3 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. Using the Combinatorial Nullstellensatz, Wang, Ma, and Han [44] proved that the conjecture holds for triangle-free planar graphs with maximum degree at least 7. Dong and Wang [18] showed that Conjecture 1.4.3 holds for graphs with $\mathrm{Mad}(G) < 3$ and $\Delta(G) \geq 4$, and Li, Liu, and Wang [31] proved that the conjecture holds for K_4-minor-free graphs. Li, Ding, Liu, and Wang [30] also confirmed Conjecture 1.4.3 for planar graphs with maximum degree at least 13. Finally, Xu, Wu, and Xu [46] proved $\chi''_\Sigma(G) \leq \Delta(G) + 2$ for graphs G with $\Delta(G) \geq 14$ that can be embedded in a surface of nonnegative Euler characteristic.

By modifying Przybyło's proof that Conjecture 1.4.1 is asymptotically correct for graphs with large maximum degree, we confirm that Conjecture 1.4.3 is also asymptotically correct.

Theorem 1.4.4. *If G is a connected graph with $\Delta(G)$ sufficiently large, then*

$$\chi''_{\Sigma}(G) \leq \Delta(G) + 50\Delta(G)^{5/6} \ln^{1/6} \Delta(G).$$

This chapter contains joint work with Yunfang Tang. The results were independently obtained by Jakub Przybyło and appear in a joint paper with him.

1.5 Definitions and notation

The set $\{1, \dots, n\}$ is abbreviated $[n]$. For a set S and natural number k, the set $\binom{S}{k}$ is the family of all k-element subsets of S.

A hypergraph H consists of a set $V(H)$ of *vertices* and a set $E(H)$, of *edges* where each element in $E(H)$ is a subset of $V(H)$. A hypergraph is a *graph* if each edge is a 2-element subset of $V(G)$. For an edge in a graph, we write uv instead of $\{u, v\}$. Unless noted otherwise, the remaining definitions in this section are given for graphs.

Two vertices are *adjacent* if there is an edge containing both of them. Two edges are *incident* if their intersection is nonempty. A *walk* is an alternating list $v_0, e_1, v_1, \dots, v_k$ of vertices and edges with $e_i = v_{i-1}v_i$ for $1 \leq i \leq k$. The *length* of a walk is the number of edges in it. The *endpoints* of an edge are the vertices in the edge. The *endpoints* of a walk are the first and last vertex in the walk. An n-vertex *path* is a walk with n vertices, none repeated. An n-vertex *cycle* is a walk with length n, the same initial and terminal vertex, and no other repeated vertices.

Two vertices are *connected* if there is a path having them as endpoints. If u and v are connected, then the *distance* $d(u, v)$ between them is the length of a shortest path with endpoints u and v. The *components* of a graph are its maximal connected subgraphs. If G has exactly one component, then it is *connected*, otherwise *disconnected.*

The *diameter* of a graph is the maximum distance between two vertices.

A *clique* is a set of pairwise adjacent vertices. The *clique number* $\omega(G)$ of a graph G is the number of vertices in a largest clique in G. A *independent set* is a set of vertices that are pairwise nonadjacent. The *independence number* $\alpha(G)$ of G is the maximum size of an independent set in G. A *k-independent set* is a set of vertices in a graph with pairwise distance greater than k. Note that 1-independent sets are just independent sets.

The *k-neighborhood* $N_k(v)$ of a vertex v is the set of vertices at distance k from v. When $k = 1$, we omit the subscript and refer to $N(v)$ as the *neighborhood* of v. For a set $S \subseteq V(G)$, the *neighborhood* $N(S)$ of S is

$\cup_{v\in S}N(v)$. The *degree* $d(v)$ of a vertex v is the size of $N(v)$. A *k-vertex* is a vertex of degree k, a *k^+-vertex* is a vertex of degree at least k, and a *k^--vertex* is a vertex of degree at most k.

The *maximum degree* $\Delta(G)$ of a graph G is $\max_{v\in V(G} d(v)$. The *minimum degree* $\delta(G)$ is $\min_{v\in V(G)} d(v)$. The *average degree* $\overline{d}(G)$ of a graph G is $\frac{1}{|V(G)|}\sum_{v\in V(G)} d(v)$, which equals $\frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* $\mathrm{Mad}(G)$ of a graph G is $\max\limits_{H\subseteq G}\frac{2|E(H)|}{|V(H)|}$.

An *isomorphism* from a graph G to a graph H is a map $f\colon V(G)\to V(H)$ such that $f(u)f(v)\in E(H)$ if and only if $uv\in E(G)$. Two graphs are *isomorphic* if there is an isomorphism from one to the other. The class of graphs isomorphic to a given graph G is the *isomorphism class of* G. An *automorphism* of G is an isomorphism from G to G.

The isomorphism classes of n-vertex paths and cycles are denoted P_n and C_n. A *forest* is a graph containing no cycle. A *tree* is a forest with a single component. The isomorphism class containing the graph G with $V(G)=[n]$ and $E(G)=\binom{[n]}{2}$ is K_n, and we refer to such graphs as *complete graphs*.

A graph G is *bipartite* if $V(G)$ can be partitioned as (X,Y) such that X and Y are independent sets; the sets X and Y are the *parts*. A *complete bipartite graph* is a bipartite graph such that every vertex in one part is adjacent to every vertex in the other. The automorphism class of complete bipartite graphs with one part of size m and one part of size n is denoted $K_{m,n}$. A graph G is *tripartite* if $V(G)$ can be partitioned as (X,Y,Z) such that X, Y, and Z are independent sets; the sets X, Y, and Z are the *parts*. A *complete tripartite graph* is a tripartite such that every vertex in each part is adjacent to every vertex in the other two parts. The automorphism class of complete tripartite graphs with parts of sizes i, j and k is denoted $K_{i,j,k}$.

The *Kneser graph*, denoted $K(n,k)$, is the graph with $V(K(n,k))=\binom{n}{k}$ and $uv\in E(K(n,k))$ if and only if $u\cap v=\emptyset$. The *Petersen graph* is $K(5,2)$.

The *Fano plane* is the hypergraph with vertex set [7] and edge set $\{124,457,561,346,235,672,713\}$. Given a hypergraph H, the *incidence graph* of H is a graph with vertex set $V(H)\cup E(H)$ and with edge set $\{ve\colon v\in V(H), e\in E(H), v\in e\}$. The *Heawood graph* is the incidence graph of the Fano plane.

A *subgraph* of a graph G is a graph H such that $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. When H is a subgraph of G, we write $H\subseteq G$. We also use "H is a subgraph of G" in the context of isomorphism classes, to mean that G has a subgraph isomorphic to H. A subgraph H is an *induced subgraph* of G if $u,v\in V(H)$ and $uv\in E(G)$ together imply $uv\in E(H)$. For a subset S of $V(G)$, we use $G[S]$ to denote the induced subgraph of G with vertex set S and say that it is the subgraph *induced* by S.

The *complement* of a graph G is the graph $\overline{G}$ with $V(\overline{G})=V(G)$ and $E(\overline{G})=\binom{V(G)}{2}-E(G)$. The *square* of a graph G, denoted G^2, is the graph obtained by adding an edge joining every two vertices at distance 2 that are not adjacent in G.

Given $uv \in E(G)$, we *subdivide* uv by adding a new vertex w to $V(G)$ and replacing uv in $E(G)$ by uw and vw. Any graph that can by obtained from G by a succession of edge subdivisions is a *subdivision* of G.

For $S \subset V(G)$, we write $G - S$ for the subgraph obtained from G by deleting all vertices of S and all edges incident to vertices in S. In the case that $S = \{v\}$ for some $v \in V(G)$, we write $G - v$. For $S \subseteq E(G)$, we write $G - S$ for the subgraph obtained from G by deleting all edges of S. For $S \subseteq \binom{V(G)}{2}$, we write $G \cup S$ for the graph obtained from G by adding all edges of $S - E(G)$. In the case that $S = \{e\}$ for $e \in \binom{V(G)}{2}$, we write $G - e$ or $G \cup e$.

If G is connected and there exists $S \subseteq V(G)$ such that $G - S$ is disconnected, then S is a *cut-set*. When $S \subseteq V(G)$ and $V_1, \ldots, V_k$ are the vertex sets of the components of $G - S$, then the *S-lobes* of G are the subgraphs induced by $S \cup V_1, \ldots, S \cup V_k$. (Note that S-lobe is well defined even when S is not a cut set.)

A *coloring* of a graph is an assignment of labels, called *colors*, to the vertices. When the labels are from a k-element set, a coloring is a *k-coloring*. A coloring is *proper* if adjacent vertices receive distinct colors. A graph is *k-colorable* if it can be properly k-colored. The *chromatic number* $\chi(G)$ of G is the least k such that G is k-colorable.

A *list assignment* for G is an assignment of a list $L(v)$ to each vertex v. For a list assignment L, a graph is L-colorable if G has a proper coloring in which each vertex v receives an element of $L(v)$. A graph G is *k-choosable* if G is L-colorable for every list assignment L such that $|L(v)| \geq k$ for all $v \in V(G)$. The *choice number* $\mathrm{ch}(G)$ for G is the least k such that G is k-choosable.

A *list assignment* L for $E(G)$ assigns to each vertex e a list $L(e)$ of permissible colors. Given a list assignment L for the edges of G, if a proper edge-coloring c can be chosen so that $c(e) \in L(v)$ for all $e \in E(G)$, then we say that G is *L-edge-colorable*. The *list edge-chromatic number* $\chi'_\ell(G)$ of G is the least k such that G is L-edge-colorable for any list assignment L satisfying $|L(e)| \geq k$ for all $e \in E(G)$.

A *drawing* of a graph on a surface is a mapping of the vertices into distinct points and the edges into continuous curves on the surface that preserves the incidence relations. Since the incidence relation is preserved, we may view these points and curves *as* the vertices and edges. By moving edges slightly, we may restrict drawings by requiring that no three edges share a single internal point, that no edge has a vertex as an internal point, that no two edges are tangent, and that no two incident edges cross. In a drawing, a *crossing* of two edges is a common internal point. An *embedding* of a graph is a drawing with no crossings. A graph is *planar* if it has an embedding in the plane. A particular embedding in the plane is a *plane graph*. In a plane graph, the *faces* are the regions of the complement of the drawing.

Given an ordering $v_1, \ldots, v_n$ of $V(G)$, the *adjacency matrix* $A(G)$ is the $0,1$-matrix defined by $A_{i,j} = 1$

if and only if $v_iv_j \in E(G)$. The *eigenvalues* of a graph G are the eigenvalues of its adjacency matrix $A(G)$.

Chapter 2

Fractional separation dimension

2.1 Introduction

This chapter contains joint work with Douglas B. West.

A pair of nonincident edges in a graph G is *separated* by a linear ordering of $V(G)$ if both vertices of one edge precede both vertices of the other. The *separation dimension* $\pi(G)$ of a graph G is the minimum number of vertex orderings that together separate every pair of nonincident edges of G. Graphs with at most three vertices have no such pairs, so their separation dimension is 0. We therefore consider only graphs with at least four vertices.

Introduced by Basavaraju, Chandran, Golumbic, Mathew, and Rajendraprasad [9] (full version in [10]), separation dimension is motivated by a geometric interpretation. By viewing the orderings as giving coordinates for each vertex, the separation dimension is the least k such that the vertices of G can be embedded in $\mathbb{R}^k$ so that any two nonincident edges of G are separated by a hyperplane perpendicular to some coordinate axis (ties in a coordinate may be broken arbitrarily.)

The upper bounds on $\pi(G)$ proved by Basavaraju *et al.* [9, 10] include $\pi(G) \le 3$ when G is planar (sharp for K_4) and $\pi(G) \le 4\log_{3/2} n$ when G has n vertices. Since all pairs needing separation continue to need separation when other edges are added, $\pi(G) \le \pi(H)$ when $G \subseteq H$; we call this fact *monotonicity.* By monotonicity, the complete graph K_n achieves the maximum among n-vertex graphs. In general, $\pi(G) \ge \log_2\lfloor \frac{1}{2}\omega(G)\rfloor$, where $\omega(G) = \max\{t\colon\ K_t \subseteq G\}$. This follows from the lower bound $\pi(K_{m,n}) \ge \log_2 \min\{m, n\}$ [9, 10] and monotonicity. Hence the growth rate of $\pi(K_n)$ is logarithmic. (For the *induced separation dimension*, introduced in Golumbic, Mathew, and Rajendraprasad [21], the only pairs needing separation are those whose vertex sets induce exactly two edges, and monotonicity does not hold.)

Basavaraju, Chandran, Mathew, and Rajendraprasad [11] proved $\pi(G) \in O(k \log\log n)$ for the n-vertex graphs G in which every subgraph has a vertex of degree at most k. Letting K_n' denote the graph produced from K_n by subdividing every edge, they also showed $\pi(K_n') \in \Theta(\log\log n)$. Thus separation dimension is unbounded already on the family of graphs with average degree less than 4. We ask, what is the largest value of b such that $\pi(G)$ is bounded (by a constant) when $\mathrm{Mad}(G) < b$? Since planar graphs have separation dimension at most 3 and trees are planar, the value is at least 2.

In terms of the maximum vertex degree $\Delta(G)$, Alon, Basavaraju, Chandran, Mathew, and Rajendraprasad [6] proved $\pi(G) \leq 2^{9\log_2^* \Delta(G)}\Delta(G)$. They also proved that almost all d-regular graphs G satisfy $\pi(G) \geq \lceil d/2 \rceil$.

Separation dimension is equivalently the restriction of another parameter to the special case of line graphs. The *boxicity* of a graph G, written $\mathrm{box}(G)$, is the least k such that G can be represented by assigning each vertex an axis-parallel box in $\mathbb{R}^k$ (that is, a cartesian product of k intervals) so that vertices are adjacent in G if and only if their assigned boxes intersect. The initial paper [9] observed that $\pi(G) = \mathrm{box}(L(G))$, where $L(G)$ denotes the line graph of G (including when G is a hypergraph).

We study a fractional version of separation dimension, using techniques that apply for hypergraph covering problems in general. Given a hypergraph H, the *covering number* $\tau(H)$ is the minimum number of edges in H whose union is the full vertex set. For separation dimension $\pi(G)$, the vertex set of H is the set of pairs of nonincident edges in G, and the edges of H are the sets of pairs separated by a single ordering of $V(G)$. Many minimization problems, including chromatic number, domination, poset dimension, and so on, can be expressed in this way.

Given a hypergraph covering problem, the corresponding fractional problem considers the difficulty of covering each vertex multiple times and measures the average number of edges needed. In particular, the *t-fold covering number* $\tau_t(H)$ is the least number of edges in a list of edges (repetition allowed) that covers each vertex at least t times, and the *fractional covering dimension* is $\liminf_t \tau_t(H)/t$. Note that $\tau_f(H) \leq \tau_1(H) = \tau(H)$. In the special case that H is the hypergraph associated with separation dimension, we obtain the *t-fold separation dimension* $\pi_t(G)$ and the *fractional separation dimension* $\pi_f(G)$.

Every list of s edges in a hypergraph H provides an upper bound on $\tau_f(H)$; if it covers each vertex at least t times, then it is called an *$(s:t)$-covering*, and $\tau_f(H) \leq s/t$. This observation will enable us to obtain the maximum value of the fractional separation dimension. It is bounded, even though the separation dimension is not (recall $\pi(K_n) \geq \log\lfloor n/2 \rfloor$). In Section 2.3, we show:

Theorem 2.1.1. *$\pi_f(G) \leq 3$ for any graph G, with equality if and only if $K_4 \subseteq G$.*

Proof. We may assume $|V(G)| \geq 4$, since otherwise there are no separations to be established and $\pi_f(G) \leq \pi(G) = 0$. Now consider the set of all linear orderings of $V(G)$. For any two nonincident edges ab and cd, consider fixed positions of the other $n-4$ vertices in a linear ordering. There are 24 such orderings, and eight of them separate ab and cd. Grouping the orderings into such sets shows that ab and cd are separated $n!/3$ times. Hence $\pi_f(G) \leq 3$.

Now suppose $K_4 \subseteq G$. In a copy of K_4 there are three pairs of nonincident edges, and every linear ordering separates exactly one of them. Hence to separate each at least t times, $3t$ orderings must be used. We obtain $\pi_t(G) \geq 3t$ for all t, so $\pi(G) \geq 3$. □

Theorem 2.1.1 gives a sharp bound, even for bipartite graphs; we prove $\pi_f(K_{m,m}) = \frac{3m}{m+1}$.

When G is disconnected, the value on G of π_t for any t (and hence also the value of π_f) is just its maximum over the components of G. We therefore focus on connected graphs. Also monotonicity holds for π_f just as for π.

Fractional versions of hypergraph covering problems are discussed in the book of Scheinerman and Ullman [38]. For every hypergraph covering problem, the fractional covering number is the solution to the linear programming relaxation of the integer linear program specifying $\tau(H)$. One can use this to express $\tau_f(G)$ in terms of a matrix game; we review this transformation in Section 2.2 to make our presentation self-contained. The resulting game yields a strategy for proving results about $\tau_f(H)$ and in particular about $\pi_f(G)$. To prove results about π_f via this matrix game expression, we probability distributions on the vertex orders and on the pairs of nonincident edges.

In Section 2.3, we characterize the extremal graphs for fractional separation dimension, proving that $\pi_f(G) = 3$ only when $K_4 \subseteq G$. No smaller bound can be given even for bipartite graphs; we prove $\pi_f(K_{m,m}) = \frac{3m}{m+1}$. For $K_{m,m}$, the pairs of nonincident edges fall into a single orbit, thus our proof consists of giving a family of orders that separates a maximum number of pairs.

In Sections 2.4 and 2.5 we consider sparser graphs. The *girth* of a graph is the minimum length of its cycles (infinite if it has no cycles). In Section 2.4 we show $\pi_f(C_n) = \frac{n}{n-2}$. Also, the value is $\frac{30}{17}$ for the Petersen graph and $\frac{28}{17}$ for the Heawood graph. Although these results suggested asking whether graphs with fixed girth could admit better bounds on separation number, Alon [3] pointed out by using expander graphs that large girth does not permit bounding $\pi_f(G)$ by any constant less than 3 (see Section 2.4). Nevertheless, we can still ask the question for planar graphs.

Question 2.1.2. *How large can $\pi_f(G)$ be when G is a planar graph with girth at least g?*

In Section 2.5, we consider graphs without cycles.

Theorem 2.5.1. $\pi_f(G) < \sqrt{2}$ *when G is a tree.*

The proof involves building a probability distribution on the orders of and calculating the probability of separation for the nonincident edges. The bound in Theorem 2.5.1 improves to $\pi_f(T) \leq \frac{4}{3}$ for trees obtained from a subdivision of a star by adding any number of pendant edges at each leaf. This is sharp; the tree with $4m+1$ vertices obtained by subdividing every edge of $K_{1,2m}$ has diameter 4 and fractional separation dimension $\frac{4m-2}{3m-1}$, which tends to $\frac{4}{3}$. We believe that the optimal bound for trees is strictly between $\frac{4}{3}$ and $\sqrt{2}$.

Question 2.1.3. *What is the supremum of $\pi_f(G)$ when G is a tree?*

In Section 2.6, we return to the realm of dense graphs with values of π_f near 3. We first compute $\pi_f(K_{m+1,qm})$. The formula yields $\pi_f(K_{m,r}) < 3(1-\frac{1}{2m-1})$ for all r, so both parts of a bipartite graph must grow to obtain a sequence of values approaching 3. In the special case $q = 1$, we obtain $\pi_f(K_{m+1,m}) = \frac{3m}{m+1}$. In addition, $\pi_f(K_{m,m}) = \frac{3m}{m+1}$, for $m \geq 2$. We conjecture that, among bipartite n-vertex graphs, π_f is maximized by $K_{n,n}$. We also studied complete balanced tripartite graphs.

Theorem 2.6.2. $\pi_f(K_{m,m,m}) = \frac{6m}{2m+1}$ *for* $m \geq 2$.

When $n = 6r$, we thus have $\pi_f(K_{2r,2r,2r}) > \pi_f(K_{3r,3r})$. Surprisingly, the value is larger for a quite different complete tripartite graph.

Theorem 2.6.4. $\pi_f(K_{1,m,m}) = \frac{24m}{8m+5+3/(2\lceil m/2\rceil-1)}$ *for* $m \geq 1$.

Computer search verifies the extreme among tripartite graphs up to 14 vertices. For $n = 9$, there is an anomaly, with $\pi_f(K_{3,3,3}) > \pi_f(K_{1,4,4})$.

Conjecture 2.1.4. *For $n \geq 10$, the graph $K_{1,\lfloor(n-1)/2\rfloor,\lceil(n-1)/2\rceil}$ achieves the maximum value of π_f among n-vertex graphs not containing K_4.*

Since $\pi_f(G)$ is always rational, we ask

Question 2.1.5. *Which rational numbers (between 1 and 3) occur as the fractional separation dimension of some graph?*

Finally, in Section 2.7, we consider the analogues of π and π_f defined by using circular orderings of the vertices rather than linear ones; we use the notation π° and π_f°. We show first that $\pi^\circ(G) = 1$ if and only if G is outerplanar. Surprisingly, $\pi^\circ(K_{m,n}) = 2$ when $m, n \geq 2$ and $mn > 4$, but π° is unbounded.

Theorem 2.7.3. $\pi^\circ(K_n) > \log_2 \log_3(n-1)$.

For the fractional context, we prove $\pi_f^\circ(G) \leq \frac{3}{2}$ for all G, with equality if and only if $K_4 \subseteq G$. Again no better bound holds for bipartite graphs; we prove $\pi_f^\circ(K_{m,qm}) = \frac{6(qm-1)}{4mq+q-3}$, which tends to $\frac{3}{2}$ as $m \to \infty$ when $q = 1$. It tends to $\frac{6m}{4m+1}$ when $q \to \infty$, so again both parts must grow to obtain a sequence on which π_f° tends to $\frac{3}{2}$. The proof is different from the linear case. The questions remaining are analogous to those for π_f.

Question 2.1.6. *How large can π_f° be when G is a planar graph with girth at least g? Which are the n-vertex graphs maximizing π_f° among bipartite graphs and among those not containing K_4? Which rational numbers between 1 and $\frac{3}{2}$ occur?*

2.2 Fractional covering and matrix games

Given a hypergaph H with vertex set $V(H)$ and edge set $E(H)$, let $E_v = \{e \in E(H)\colon\ v \in e\}$ for $v \in V(H)$. The covering number $\tau(H)$ is the solution to the integer linear program "minimize $\sum_{e\in E(H)} x_e$ such that $x_e \in \{0,1\}$ for $e \in E(H)$ and $\sum_{e\in E_v} x_e \geq 1$ for $v \in V(H)$." The linear programming relaxation replaces the constraint $x_e \in \{0,1\}$ with $0 \leq x_e \leq 1$.

It is well known (see Theorem 1.2.1 of [38]) that the resulting solution τ^* equals $\tau_f(H)$. Multiplying the values in that solution by their least common multiple t yields a list of edges covering each vertex at least t times, and hence $\tau_f(H) \leq \tau^* t/t = \tau^*$. Similarly, normalizing an $(s : t)$-covering yields $\tau^* \leq s/t$. Note that since the solution to a linear program with integer constraints is always rational, always $\tau_f(H)$ is rational (when H is finite).

A subsequent transformation to a matrix game yields a technique for proving bounds on $\tau_f(H)$. The constraint matrix M for the linear program has rows indexed by $E(H)$ and columns indexed by $V(H)$, with $M_{e,v} = 1$ when $v \in e$ and otherwise $M_{e,v} = 0$. In the resulting matrix game, the edge player chooses a row e and the vertex player chooses a column v, and the outcome is $M_{e,v}$. In playing the game repeatedly, each player uses a strategy that is a probability distribution over the options, and then the expected outcome is the probability that the chosen vertex is covered by the chosen edge. The edge or "covering" player wants to maximize this probability; the vertex player wants to minimize it.

Using the probability distribution x over the rows guarantees outcome at least the smallest entry in $x^T M$, no matter what the vertex player does. Hence the edge player seeks a probability distribution x to maximize t such that $\sum_{e\in E_v} x_e \geq t$ for all $v \in V(H)$. Dividing by t turns this into the linear programming formulation for $\tau_f(H)$, with the resulting optimum being $1/t$. This yields the following relationship.

Proposition 2.2.1. *(Theorem 1.4.1 of [38]) If M is the covering matrix for a hypergraph H, then $\tau_f(H) = 1/t$, where t is the value of the matrix game given by M.*

Just as any strategy x for the edge player establishes $\min x^T M$ as a lower bound on the value, so any strategy y for the vertex player establishes $\max My$ as an upper bound. The value is established by providing strategies x and y so that these bounds are equal. As noted in [38], such strategies always exist.

For fractional separation dimension, we thus obtain the *separation game*. The rows correspond to vertex orderings and the columns to pairs of nonincident edges. The players are the *ordering player* and the *pair player*, respectively. To prove $\pi_f(G) \leq 1/t$, it suffices to find a distribution for the ordering player such that each nonincident pair is separated with probability at least t. To prove $\pi_f(G) \geq 1/t$, it suffices to find a distribution for the pair player such that for each ordering the probability that the chosen pair is separated is at most t.

The proof of Theorem 2.1.1 can be phrased in this language. By making all vertex orderings equally likely, the ordering player achieves separation probability exactly $\frac{1}{3}$ for each pair, yielding $\pi_f(G) \leq 3$. By playing the three nonincident pairs in a single copy of K_4 with equal probability and ignoring all other pairs, the pair player achieves separation probability exactly $1/3$ against any ordering, yielding $\pi_f(G) \geq 3$.

Another standard result about these games will be useful to us. Let $\mathcal{P}$ denote the set of pairs of nonincident edges in a graph G. Symmetry in G greatly simplifies the task of finding an optimal strategy for the pair player.

Proposition 2.2.2. *(follows from Exercise 1.7.3 of [38]) If, for any two pairs of nonincident edges in a graph G, some automorphism of G maps one to the other, then there is an optimal strategy for the pair player in which all pairs in $\mathcal{P}$ are made equally likely. In general, there is an optimal strategy that is constant on orbits of the pairs under the automorphism group of G.*

Proof. Consider an optimal strategy y, yielding $\max My = t$. Automorphisms of G induce permutations of the coordinates of y. The entries in My' for any resulting strategy y' are the same as in My. Summing these vectors over all permutations and dividing by the number of permutations yields a strategy y^* that is constant over orbits and satisfies $\max My^* \leq t$. $\square$

When there is an optimal strategy in which the pair player plays all pairs in $\mathcal{P}$ equally, the value of the separation game is just the largest fraction of $\mathcal{P}$ separated by any ordering. For $\tau_f(H)$ in general, Proposition 1.3.4 in [38] states this by saying that for a vertex-transitive hypergraph H, always $\tau_f(H) = |V(H)|/r$, where r is the maximum size of an edge. For separation dimension, this yields the following:

Corollary 2.2.3. *Let G be a graph. If for any two pairs of nonincident edges in G, there is an automorphism of G mapping one pair of edges to the other, then $\tau_f(G) = q/r$, where q is the number of nonincident pairs of edges in G and r is the maximum number of pairs separated by any vertex ordering.*

2.3 Characterizing the extremal graphs

When $K_4 \not\subseteq G$, we can separate $\pi_f(G)$ from 3 by a function of n.

Theorem 2.3.1. *If G is an n-vertex graph and $K_4 \not\subseteq G$, then $\pi_f(G) \le 3\left(1 - \frac{12}{n^4} + O(\frac{1}{n^5})\right)$.*

Proof. Let $p = \frac{1}{3} + \frac{4(n-4)!}{n!}$; note that $1/p$ has the form $3\left(1 - \frac{12}{n^4} + O(\frac{1}{n^5})\right)$. It suffices to give a probability distribution on the orderings of $V(G)$ such that each nonincident pair of edges is separated with probability at least p. We do this by modifying the list of all orderings.

Choose any four vertices $a, b, c, d \in V(G)$. For each ordering ρ of the remaining $n-4$ vertices, 24 orderings begin with $\{a, b, c, d\}$ and end with σ'. By symmetry, we may assume $ac \notin E(G)$. Thus the possible pairs of nonincident edges induced by $\{a, b, c, d\}$ are $\{ab, cd\}$ and $\{ad, bc\}$. We increase the separation probability for these pairs, even though these four edges need not all exist.

The pairs $\{ab, cd\}$ and $\{ad, bc\}$ are each separated eight times in the list of 24 orderings. We replace these 24 with another list of 24 (that is, the same total weight) that separate $\{ab, cd\}$ and $\{ad, bc\}$ each at least twelve times, while any other pair of disjoint vertex pairs not involving $\{a, c\}$ is separated at least eight times. Since $\{a, b, c, d\}$ is arbitrary and we do this for each 4-set, the pairs $\{ab, cd\}$ and $\{ad, bc\}$ remain separated at least eight times in all other groups of 24 orderings. Thus the separation probability increases from $\frac{1}{3}$ to at least p for all pairs of nonincident edges.

Use four orderings each that start with *abcd* or *bcad* and eight each that start with *cdba* or *adbc*, always followed by σ'. By inspection, each of $\{ab, cd\}$ and $\{ad, bc\}$ is separated twelve times in the list. The number of orderings that separate any pair of nonincident edges having at most two vertices in $\{a, b, c, d\}$ does not change.

It remains only to check pairs with three vertices in this set, consisting of one edge induced by this set and another edge with one endpoint in the set. The induced edge is one of $\{ab, cd, bc, ad, bd\}$ (never ac), and the other edge uses one of the remaining two vertices in $\{a, b, c, d\}$. In each case, the endpoints of the induced edge appear before the third vertex in at least eight of the orderings in the new list of 24; this completes the proof. □

For n-vertex graphs not containing K_4, Theorem 2.3.1 separates $\pi_f(G)$ from 3 by a small amount. We believe that a much larger separation also holds (Conjecture 2.1.4). Nevertheless, we show next that even when G is bipartite there is no upper bound less than 3.

Theorem 2.3.2. $\pi_f(K_{m,m}) = \frac{3m}{m+1}$ *for $m \ge 2$.*

Proof. The pairs in $\mathcal{P}$ all lie in the same orbit under automorphisms of $K_{m,m}$, so Corollary 2.2.3 applies. There are $2\binom{m}{2}^2$ pairs in $\mathcal{P}$ (played equally by the pair player). It suffices to show that the maximum number of pairs separated by any ordering is $\frac{m+1}{3m} 2\binom{m}{2}^2$.

Let the parts of $K_{m,m}$ be X and Y. Let σ be an ordering $v_1, \ldots, v_{2m}$ such that each pair $\{v_{2i-1}, v_{2i}\}$ consists of one vertex of X and one vertex of Y. The ordering player will in fact make all such orderings equally likely. It suffices to show that σ separates $\frac{m+1}{3m} 2\binom{m}{2}^2$ pairs and that no ordering separates more.

By symmetry, we may index X as $x_1, \ldots, x_m$ and Y as $y_1, \ldots, y_m$ in order in σ, so that $\{v_{2i-1}, v_{2i}\} = \{x_i, y_i\}$ for $1 \le i \le m$, though x_i and y_i may appear in either order. Consider an element of $\mathcal{P}$ separated by σ. The vertices involved in the separation may use two, three, or four indices among 1 through m.

Pairs hitting i, j, k, l with $i < j < k < l$ must be separating $x_i y_j$ or $y_i x_j$ from $x_k y_l$ or $y_k x_l$. Hence there are $4\binom{m}{4}$ such pairs.

Pairs hitting only i, j, k with $i < j < k$ involve two vertices with the same index. If that index is i or k, then there are two ways to complete the edge pair. However, if x_j and y_j are both used, then there is only one way to choose from $\{x_i, y_i\}$ and from $\{x_k, y_k\}$ to complete a separated pair, determined by the order of x_j and y_j. Hence there are $5\binom{m}{3}$ such pairs.

A separated pair hitting only i and j must be $\{x_i y_i, x_j y_j\}$. Hence in total σ separates $4\binom{m}{4} + 5\binom{m}{3} + \binom{m}{2}$ pairs in $\mathcal{P}$. In fact, this sum equals $\frac{m+1}{3m} 2\binom{m}{2}^2$.

Now let σ be an ordering not of the specified form. By symmetry we may again index X as $x_1, \ldots, x_m$ and Y as $y_1, \ldots, y_m$ in order in σ. However, now some vertex precedes another vertex with a lesser index. That is, by symmetry we may assume that y_j appears immediately before x_i for some i and j with $j > i$.

Form σ' from σ by interchanging the positions of y_j and x_i. Any pair separated by exactly one of σ and σ' has x_i and y_j as endpoints of the two distinct edges. There are $(i-1)(m-j)$ such pairs in σ and $(j-1)(m-i)$ such pairs in σ'. Since $m \ge 2$ and $j > i$, comparing these quantities shows that σ' separates strictly more pairs than σ. $\square$

To prove that always $\pi_f(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) = \frac{3m}{m+1}$, where $m = \lfloor n/2 \rfloor$, we need also to compute $\pi_f(K_{m+1,m})$. We postpone this to Section 2.6. Note that the simple final expression arises when we cancel common factors in the numerator and denominator. We would hope that such a simple formula has a simple direct proof, but we have not found one.

2.4 Graphs with larger girth

Among sparser graphs, it is natural to think first about cycles.

Proposition 2.4.1. *$\pi_f(C_n) = \frac{n}{n-2}$, for $n \ge 4$.*

Proof. The ordering player uses the n rotations of an n-vertex path along the cycle, equally likely. Nonincident edges e and e' are separated unless e or e' consists of the first and last vertex. Hence any pair in $\mathcal{P}$ is

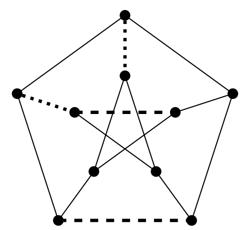


Figure 2.1: The Petersen graph.

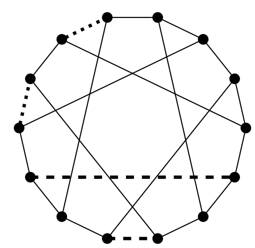


Figure 2.2: The Heawood graph.

separated with probability $\frac{n-2}{n}$.

Let the vertices be $v_1, \ldots, v_n$ in order along the cycle. The pair player makes the pairs $\{v_{i-1}v_i, v_{i+1}v_{i+2}\}$ (modulo n) equally likely. It suffices to show that any ordering separates at most $n-2$ of these pairs. Otherwise, by symmetry some ordering σ separates the $n-1$ such pairs satisfying $2 \leq i \leq n$. By symmetry we may assume that $\{v_1, v_2\}$ precedes $\{v_3, v_4\}$ in σ. If v_i precedes v_{i+2}, then separating v_iv_{i+1} from $v_{i+2}v_{i+3}$ requires v_{i+1} to precede v_{i+3}. Iterating this argument yields v_{n-2} before v_n and v_{n-1} before v_1 in σ. Since v_1 precedes both v_3 and v_4, choosing the right one by parity leads to v_1 preceding v_1, a contradiction. $\square$

Proposition 2.4.1 suggests that π_f decreases as girth increases. We compute the values for the smallest 3-regular graphs of girth 5 and girth 6. These are the Petersen graph and the Heawood graph. The Petersen graph is the Kneser graph $K(5,2)$, shown in Figure 2.1. The *Fano plane* is the hypergraph with vertex set $[7]$ and edge set $\{124, 457, 561, 346, 235, 672, 713\}$. The Heawood graph is the incidence graph of the Fano plane, and is shown in Figure 2.2. The pairs of nonincident edges in both the Petersen graph and Heawood graph have two orbits under automorphisms. In the figures, the two pairs of dashed edges represent the two orbits.

Proposition 2.4.2. *The fractional separation dimension of the Petersen graph is* $\frac{30}{17}$.

Proof. The 75 pairs of nonincident edges in the Petersen graph G fall into two orbits: the 15 pairs that occur as opposite edges on a 6-cycle (Type 1) and the 60 pairs that do not (Type 2). Thus some optimal strategy for the pair player will make Type 1 pairs equally likely and make Type 2 pairs equally likely. There exist orderings that separate nine Type 1 pairs and 34 Type 2 pairs. With the graph expressed as the disjointness graph of the 2-element subsets of $\{1,2,3,4,5\}$, such an ordering σ is

$$12, 34, 51, 23, 45, 13, 42, 35, 41, 25$$

Since $\frac{9}{15} > \frac{34}{60}$, making the 120 orderings generated from σ by permuting $\{1,2,3,4,5\}$ equally likely yields $\pi_f(G) \leq \frac{60}{34} = \frac{30}{17}$.

Since $\frac{9}{15} > \frac{34}{60}$, the pair player establishes a matching lower bound by playing only Type 2 pairs, equally likely, if no ordering separates more than 34 Type 2 pairs. Computer search shows that this is true. $\square$

Proposition 2.4.3. *The fractional separation dimension of the Heawood graph is* $\frac{28}{17}$.

Proof. The 168 pairs of nonincident edges in the Heawood graph H fall into two orbits: 84 pairs that are opposite on a a 6-cycle (Type 1) and 84 pairs that have a common incident edge (Type 2). Some optimal strategy for the pair player will make Type 1 pairs equally likely and make Type 2 pairs equally likely. There exist orderings that separate 51 Type 2 pairs and 54 Type 1 pairs. With the graph expressed as the incidence graph of the Fano plane, such an ordering is

$$1, 124, 4, 457, 5, 561, 6, 346, 3, 235, 2, 672, 7, 713$$

The automorphism group of H is isomorphic to the projective linear gropu, $\mathrm{PGL}_2(7)$, which has size 336. Making each ordering generated by the automorphisms equally likely yields $\pi_f(H) \leq \frac{84}{54} = \frac{28}{17}$.

The pair player establishes a matching lower bound by playing only Type 2 pairs, equally likely, if no ordering separates more than 51 Type 2 pairs. Computer search (reduced by symmetries) shows that this is true. $\square$

These small graphs suggest that perhaps $\pi_f(G) < 2$ when G has girth at least 5. However, Alon [3] observed using the Expander Mixing Lemma that expander graphs with large girth (such as Ramanujan graphs) still have π_f arbitrarily close to 3. We sketch the argument.

Lubotzky, Phillips, and Sarnak [32] introduced *Ramanujan graphs* as d-regular graphs in which every eigenvalue with magnitude less than d has magnitude at most $2\sqrt{d-1}$. For $d-1$ being prime, they further introduced an infinite family of such graphs whose girth is at least $\frac{2}{3}\log_{d-1} n$, where n is the number of vertices.

Let G be a d-regular n-vertex graph whose eigenvalues other than d have magnitude at most λ. The Expander Mixing Lemma of Alon and Chung [4] states that whenever A and B are two vertex sets in G, the number of edges of G joining A and B differs from $|A|\,|B|(d/n)$ by at most $\lambda\sqrt{|A|\,|B|}$ (edges with both endpoints in $A \cap B$ are counted twice).

Alon [3] applied this lemma to an arbitrary vertex ordering σ of G, breaking σ into k blocks of consecutive vertices, each with length at most $\lceil n/k \rceil$. Intuitively, by the Expander Mixing Lemma the vast majority of the edges can be viewed as forming a blowup of a complete graph with k vertices. With k chosen to be about $d^{1/3}$, Alon showed that asymptotically only $\frac{d^2n^2}{24}$ pairs of nonincident edges can be separated by σ . However, there are asymptotically $\frac{d^2n^2}{8}$ pairs of nonincident edges. Thus every ordering can separate only about a third of the pairs. Lubotzky, Phillips, and Sarnak [32] showed this graph G can be chosen to have arbitrarily large girth.

Alon extended the question in our Conjecture 2.1.4 by asking how small ϵ can be made so that there is an n-vertex graph G with girth at least g such that $\pi_f(G) \geq 3 - \epsilon$. His detailed computations [3] with the error terms yield $\epsilon < n^{-c/g}$ for some positive constant c.

Graphs with good expansion properties are not planar. The original paper [9] proved $\pi(G) \leq 2$ for every outerplanar graph G, and hence also $\pi_f(G) \leq 2$. Equality holds for outerplanar graphs with 4-cycles. We suggest seeking sharp upper bounds for the family of outerplanar graphs with girth at least g, and similarly for planar graphs with girth at least g (Question 2.1.2 states this for planar graphs).

2.5 Trees

Although $\lim_{g\to\infty} \frac{g}{g-2} = 1$, it is not true that $\pi_f(G) = 1$ whenever G is a tree. The graphs G with $\pi_f(G) = 1$ are just the graphs with $\pi(G) = 1$, as holds for every hypergraph covering parameter. These graphs were characterized in BCGMR [9]. Each component is obtained from a path P by adding independent vertices that have one neighbor or two consecutive neighbors on P, but for any two consecutive vertices on P at most one common neighbor can be added.

This implies that the trees with fractional separation dimension 1 are the caterpillars. We seek the sharpest general upper bound for trees.

Theorem 2.5.1. *$\pi_f(G) < \sqrt{2}$ when G is a tree.*

Proof. We construct a strategy for the ordering player to show that the separation game has value at least $\frac{1}{\sqrt{2}}$. Since $\pi_f(G)$ is rational, the inequality is strict.

Root T at a vertex v. For a vertex u other than v, let u' be the parent of u. We describe the strategy for the ordering player by an iterative probabilistic algorithm that generates an ordering. Starting with v, we iteratively add the children of previously placed vertices according to the following rules, where β is a probability to be specified later.

(R1) The children of v are placed before or after v with probability $\frac{1}{2}$, independently.

(R2) The children of a non-root vertex u are put between u and its parent u' with probability $1 - \beta$; they are placed on the side of u away from u' with probability β.

(R3) The children placed on each side of a vertex are placed immediately next to it by a random permutation.

The resulting ordering has the following property:

$(*)$ Any vertex between a vertex u and a child of u is a descendant of u.

We must prove that the separation probability is at least $\frac{1}{\sqrt{2}}$ for each pair of nonincident edges. Given nonincident edges ab and cd, let w denote the common ancestor of these vertices that is farthest from the root. We may assume $a = b'$ and $c = d'$. Without loss of generality, there are three types of pairs, as shown in Figure 2.3.

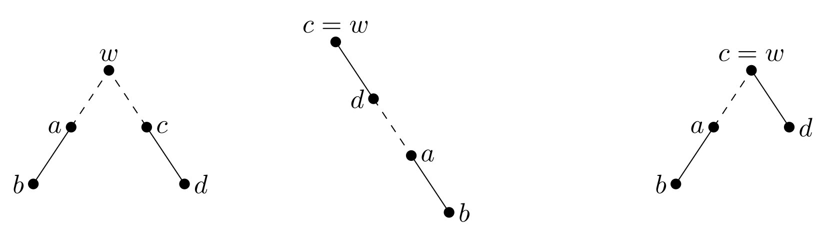


Figure 2.3: The three possible types of pairs.

In Type 1, neither edge contains an ancestor of a vertex in the other edge. Hence $(*)$ implies that no vertex of one edge can lie between the vertices of the other edge. Thus Type 1 pairs are separated with probability 1.

In Type 2, both vertices in one edge are descendants of the vertices in the other edge, say a and b are descendents of d. By $(*)$, the pair fails to be separated if and only if a is between c and d. This occurs if and only if the child of d on the path from d to a is placed between d and its parent, c. That event has probability $1 - \beta$, so the separation probability is β.

In Type 3, the vertices in ab are descendents of c but not d. Again separation fails if and only if a is between c and d. This requires that d and the child of c on the path to a are placed on the same side of c, after which the probability of having a between c and d is $\frac{1}{2}$. The probability of having two specified children of c on the same side of c is $(1 - \beta)^2 + \beta^2$ if $c \neq v$; it is $\frac{1}{2}$ if $c = v$. If $c = v$, then the separation probability is $\frac{3}{4}$, greater than $\frac{1}{\sqrt{2}}$. If $c \neq v$, then the separation probability is $1 - \frac{1}{2}(1 - \beta)^2 - \frac{1}{2}\beta^2$.

We optimize by solving $\beta = 1 - \frac{1}{2}(1-\beta)^2 - \frac{1}{2}\beta^2$ and setting $\beta = \frac{1}{\sqrt{2}}$. Now each pair of nonincident edges is separated with probability at least $\frac{1}{\sqrt{2}}$. $\square$

If a root v can be chosen in a tree G so that the all pairs of Type 3 involve v, then in the proof of Theorem 2.5.1 setting $\beta = \frac{3}{4}$ yields $\pi_f(G) \leq \frac{4}{3}$. This proves the following corollary.

Corollary 2.5.2. *$\pi_f(G) \leq \frac{4}{3}$ for any tree G produced from a subdivision of a star by adding any number of pendent vertices to each leaf.*

The bound in Corollary 2.5.2 cannot be improved.

Proposition 2.5.3. *$\pi_f(K'_{1,n}) = \frac{4m-2}{3m-1}$, where $m = \lceil n/2 \rceil$ and $K'_{1,n}$ is the graph obtained from $K_{1,n}$ by subdividing every edge once.*

Proof. Form $K'_{1,n}$ from the star with center v and leaves $y_1, \ldots, y_n$ by introducing x_i to subdivide vy_i, for $1 \leq i \leq n$. Let $X = x_1, \ldots, x_n$.

If in some ordering a vertex of degree 1 does not appear next to its neighbor, then moving it next to its neighbor does not make any separated pair unseparated. Hence the ordering player can optimally play only orderings in which every vertex of degree 1 appears next to its neighbor; it does not matter on which side of its neighbor the vertex is placed.

Nonincident edges of the form x_iy_i and x_jy_j are always separated by any ordering that puts y_i next to x_i for all i; the pair player will not play these. The remaining $n(n-1)$ pairs of nonincident edges have the form $\{vx_i, x_jy_j\}$ and lie in a single orbit. By Corollary 2.2.3, some optimal strategy for the pair player makes them equally likely.

An optimal strategy for the ordering player will thus make equally likely all orderings obtained by permuting the positions of the pairs x_ry_r within an ordering that maximizes the number of separated pairs of the nontrivial form $\{vx_i, x_jy_j\}$. Such a pair is separated when x_i and x_j lie on opposite sides of v and when x_i is between v and x_j.

To count such pairs, it matters only how many vertices of X appear to the left of v, since y_i appears next to x_i for all i. If k vertices of X appear to the left of v, then the count of separated nontrivial pairs is $2k(n-k) + \binom{k}{2} + \binom{n-k}{2}$. This formula simplifies to $\binom{n}{2} + k(n-k)$, which is maximized only when $k \in \{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor\}$.

Thus the ordering player puts v in the middle, $\lfloor \frac{n}{2} \rfloor$ vertices of X on one side, and $\lceil \frac{n}{2} \rceil$ vertices of X on the other side. Whether n is $2m$ or $2m-1$, the ratio of $\binom{n}{2} + \lfloor \frac{n^2}{4} \rfloor$ to $n(n-1)$ simplifies to $\frac{3m-1}{4m-2}$, as desired. $\square$

It remains to find trees with fractional separation dimension between $\frac{4}{3}$ and $\sqrt{2}$. In the proof of Theorem 2.5.1, it is the balance of separating the Type 2 and Type 3 pairs (see Figure 2.3) that leads to our bound of $\sqrt{2}$. Let $T(i,j,k)$ be the tree with root v, $d(v) = i$, $d(w) = j+1$ for each $w \in N(v)$, and $d(w) = k+1$ for each $w \in N_2(v)$. We suggest studying the fractional separation dimension of the class of trees $T(i,j,k)$ with $i \geq 2$, $j \geq 2$ and $k = 1$.

2.6 Complete multipartite graphs

To prove that always $\pi_f(K_{\lceil n/2\rceil,\lfloor n/2\rfloor}) = \frac{3m}{m+1}$, where $m = \lfloor n/2 \rfloor$, we need also to compute $\pi_f(K_{m+1,m})$. This is the special case $q = 1$ of our next theorem. We postponed it because the counting argument for the generalization is more technical than our earlier arguments.

Theorem 2.6.1. $\pi_f(K_{m+1,qm}) = 3\left(1 - \frac{(q+1)m-2}{(2m+1)mq-m-2}\right)$ *for* $m, q \in \mathbb{N}$ *with* $mq > 1$.

Proof. Note that $3\left(1 - \frac{(q+1)m-2}{(2m+1)mq-m-2}\right) = \frac{6m(mq-1)}{(2m+1)mq-m-2}$. Let $p = \frac{(2m+1)mq-m-2}{6m(mq-1)}$. The pairs in $\mathcal{P}$ all lie in the same orbit, so Corollary 2.2.3 applies, and the pair player can make all $2\binom{m+1}{2}\binom{mq}{2}$ pairs in $\mathcal{P}$ equally likely. It suffices to show that the maximum number of pairs separated by any ordering is $2p\binom{m+1}{2}\binom{mq}{2}$. The proof is similar to that of Theorem 2.3.2.

Let the parts of $K_{m+1,qm}$ be X and Y, with $|X| = m+1$ and $|Y| = qm$. Let σ be an ordering $v_0, \ldots, v_{(q+1)m}$ such that $v_i \in X$ if and only if $i \equiv 0 \mod (q+1)$. The ordering player will in fact make all such orderings equally likely. We show that σ separates $2p\binom{m+1}{2}\binom{mq}{2}$ pairs and that no ordering separates more.

Let $X = \{x_0, \ldots, x_m\}$, indexed in order of appearance in σ, and similarly let $Y = \{y_1, \ldots, y_{qm}\}$. Let $B_0 = \{x_0\}$, and for $1 \leq i \leq m$ let B_i consist of $\{y_{q(i-1)+1}, \ldots, y_{qi}, x_i\}$. To count pairs in $\mathcal{P}$ separated by σ, we consider which blocks contain the vertices used.

If the indices are i,j,k,l with $1 \leq i < j < k < l \leq m$, then one edge consists of x_i or x_j and a Y-vertex from the other block among $\{B_i, B_j\}$, and similarly for $\{B_k, B_l\}$. Hence there are $4q^2\binom{m}{4}$ such pairs. If $i = 0$, then we must use x_0, and there are $2q^2\binom{m}{3}$ such pairs.

If the indices are i,j,k with $1 \leq i < j < k \leq m$, then we use two vertices from one block. If we use two in B_i, then the other edge uses x_j or x_k and a Y-vertex from the remaining block, yielding $2q^2$ separated pairs. Similarly, $2q^2$ separated pairs use two vertices in B_k. Two vertices used from B_j may both be from Y or may include x_j. In the first case x_i and x_k are used, while in the second case x_j and x_i are used; thus the vertices from Y can be chosen in $\binom{q}{2} + q^2$ ways. Hence for such index choices a total of $(\binom{q}{2} + 5q^2)\binom{m}{3}$ pairs are separated.

If the indices are $0,j,k$ with $1 \le j < k \le m$, then x_0 is used. If x_j is used, then there are q^2 ways to complete the pair of separated edges coming from selecting one vertex from Y in each of B_j and B_k. If x_k is used, then there are $\binom{q}{2} + q^2$ ways to complete a pair in $\mathcal{P}$ since both vertices from Y may be chosen one from B_j and one from B_k, or both from B_j. Hence this case contributes $(\binom{q}{2} + 2q^2)\binom{m}{2}$ pairs.

If the indices are i and j with $1 \le i < j \le m$, then either we use two vertices from each of B_i and B_j (with one edge within each block) or we use three vertices from B_j and only the vertex x_i from B_i. This yields $(q^2 + \binom{q}{2})\binom{m}{2}$ separated pairs. If $i = 0$, then we must use x_0 and three vertices from B_j, for a total of $\binom{q}{2}m$ pairs.

Thus σ separates $[4q^2]\binom{m}{4} + [7q^2 + \binom{q}{2}]\binom{m}{3} + [3q^2 + 2\binom{q}{2}]\binom{m}{2} + \binom{q}{2}m$ pairs. Direct computation shows that this equals $2p\binom{m+1}{2}\binom{mq}{2}$. In particular, since $p = \frac{(2m+1)mq-m-2}{6m(mq-1)}$, the formula $2p\binom{m+1}{2}\binom{mq}{2}$ simplifies to $\frac{1}{12}[(2m+1)mq - m - 2](m+1)mq$.

It remains to show that no ordering separates more pairs than the orderings of this type. Let σ be an ordering not of this type. Index X and Y as before. If σ does not start with x_0, then let y be the vertex immediately preceeding x_0. Form σ' from σ by exchanging the positions of y and x_0. Since no pair of the form $x'y, x_0y'$ is separated by σ, every pair separated by σ is also separated by σ'.

Hence we may assume by symmetry that σ starts with x_0 and ends with x_m. If σ does not have the desired form, then by symmetry there is a least index j such that more than qj vertices of Y precede x_j, while fewer than $q(m-j)$ follow x_j. Form σ' by exchanging the positions of x_j and the vertex y immediately preceding it in σ. Let r be the number of vertices of Y preceding x_j. The number of pairs separated by σ but not σ' is $j(mq-r)$, while the number separated by σ' but not σ is $(r-1)(m-j)$. The difference is $m(r-jq)-(m-j)$. Since $r > qj$ and $j < m$, the difference is positive, and σ' separates more pairs than σ. $\square$

Since $\frac{1}{p} = \frac{6(mq-1)}{(2m+1)q-m-2/m} \le \frac{6m}{2m+1} = 3(1-\frac{1}{2m+1})$, with equality only when $m = 1$, always $\pi_f(K_{m,r})$ is bounded away from 3 by a function of m. In particular, having π_f tend to 3 on a sequence of bipartite graphs requires the sizes of both parts to grow.

We expect that $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ maximizes π_f among n-vertex bipartite graphs. By monotonicity, the maximum occurs at $K_{k,n-k}$ for some k. We have $\pi_f(K_{m,m}) = \pi_f(K_{m+1,m}) = \frac{3m}{m+1}$. For unbalanced instances with $2m+1$ vertices (assuming integrality of ratios for simplicity), Theorem 2.6.1 yields $\pi_f(K_{\frac{2m}{q+1}+1,\frac{2qm}{q+1}})$. The value is highest for the balanced case.

It would be desirable to have a direct argument showing that moving a vertex from the larger part to the smaller part in $K_{k,n-k}$ increases π_f when $k \le n/2-1$. This would prove that balanced bipartite graphs maximize π_f. However, the statement surprisingly is not true in general for complete tripartite graphs.

Computation has shown $\pi_f(K_{m+2,m,m}) > \pi_f(K_{m+1,m+1,m})$ when $2 \le m \le 4$. Even more surprising, by computing the values of π_f for $K_{m,m,m}$ and $K_{1,m,m}$, we obtain $\pi_f(K_{1,(n-1)/2,(n-1)/2}) > \pi_f(K_{n/3,n/3,n/3})$ when n is an odd multiple of 3. This follows from the remaining results in this section and motivates our Conjecture 2.1.4.

Theorem 2.6.2. *$\pi_f(K_{m,m,m}) = \frac{6m}{2m+1}$ for $m \ge 2$.*

Proof. There are two types of pairs in $\mathcal{P}$: those with endpoints in two parts, called *double-pairs* or *D-pairs*, and those with endpoints in all three parts, called *triple-pairs* or *T-pairs*. Within these two types of pairs in $\mathcal{P}$, any pair can be mapped to any other pair via an automorphism, so by Corollary 2.2.3 some optimal strategy for the pair player makes D-pairs equally likely and makes T-pairs equally likely.

Let the parts of $K_{m,m,m}$ be X, Y, and Z. Let σ be an ordering $v_1, \ldots, v_{3m}$ such that each triple $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$ consists of one vertex from each part, for $1 \le i \le m$. The ordering player will make all such orderings equally likely. The restrictions of such orderings to two parts are the orderings used in Theorem 2.3.2, which separate the fraction $\frac{m+1}{3m}$ of the D-pairs.

We show that each such ordering separates the fraction $\frac{2m+1}{6m}$ of the T-pairs. This fraction is smaller than $\frac{m+1}{3m}$. Hence this strategy shows that the separation game has value at least $\frac{2m+1}{6m}$. By making the T-pairs equally likely, the pair player establishes equality if also no other ordering separates more T-pairs.

For use in the next theorem, we distinguish each T-pair as a W-pair, for $W \in \{X, Y, Z\}$, when W is the part contributing two vertices to the pair. Furthermore, with $w \in \{x, y, z\}$, we index W as $w_1, \ldots, w_m$ in order of appearance in σ. Let the *block* B_i be $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$, so $B_i = \{x_i, y_i, z_i\}$ for $1 \le i \le m$, though $\{x_i, y_i, z_i\}$ may appear in any order in σ. The vertices of a T-pair separated by σ may use two, three, or four indices in $\{1, \ldots, m\}$. In each case, let W be the part contributing a vertex to each edge of the pair.

T-pairs hitting B_i, B_j, B_k, B_l with $i < j < k < l$ consist of one edge in $B_i \cup B_j$ and the other in $B_k \cup B_l$. We can choose the blocks for the two vertices of W in four ways (B_i or B_j, and B_j or B_k), and then we just choose which of the other two parts finishes the first edge. Hence there are $8\binom{m}{4}$ such W-pairs for each W, together $24\binom{m}{4}$ such T-pairs.

Pairs hitting only B_i, B_j, B_k with $i < j < k$ consist of one edge in $B_i \cup B_j$ and the other in $B_j \cup B_k$, with care needed in making a separated pair when B_j contributes two vertices. If the repeated part W contributes w_i and w_k, then there are five ways to complete the pair, one with two vertices from B_j and two each having an edge within B_i or B_k. If w_i and w_j are used, then there are two W-pairs having an edge within B_i. The number of W-pairs using a second vertex from B_j is $t-1$ when w_j is the tth vertex of B_j in σ. Finally, the number of W-pairs using two vertices from B_k is zero. By symmetry, assuming w_j is in the tth position of B_j in σ, when w_j and w_k are used, there are 2 W-pairs using two vertices from B_k, $3-t$ W-pairs using two vertices from B_j and zero using two vertices from B_i. Since $5 + [2 + (t-1)] + [2 + (3-t)] = 11$ for $t \in [3]$, in each case 11 W-pairs are separated, for a total of $33\binom{m}{3}$ T-pairs.

A separated T-pair hitting only B_i and B_j uses w_i and w_j. Picking one additional vertex from each block yields a separated W-pair in two ways. There is also one separated W-pair with three vertices in B_i if w_i is not the last vertex of B_i, and one with three vertices in B_j if w_j is not the first vertex of B_j. Summing over W yields $10\binom{m}{2}$ T-pairs of this type; six with two vertices in each of B_i and B_j, two with three vertices in B_i and two with three vertices in B_j.

In total σ separates $24\binom{m}{4} + 33\binom{m}{3} + 10\binom{m}{2}$ T-pairs. This sum equals $\frac{2m+1}{m}\binom{m}{2}$. Altogether there are $6m^2\binom{m}{2}$ T-pairs, so the fraction of them separated is $\frac{2m+1}{6m}$, as desired.

For an ordering σ not of the specified form, index the vertices of each part in increasing order in σ. Avoiding the specified form means that some vertex precedes another vertex with a lesser index. By symmetry, we may assume that y_j appears immediately before x_i in σ for some i and j with $j > i$. Let k be the number of vertices of Z appearing before y_j.

Form σ' from σ by interchanging the positions of y_j and x_i. Any T-pair separated by exactly one of σ and σ' has x_i and y_j as endpoints of the two distinct edges. Considering whether a vertex of Z is used to complete the first, second, or both edges, there are $k(m-j) + (i-1)(m-k) + k(m-k)$ such T-pairs in σ and $k(m-i) + (j-1)(m-k) + k(m-k)$ such T-pairs in σ'. The difference is $m(j-i)$. Since $m \geq 2$ and $j > i$, the comparison shows that σ' separates strictly more T-pairs than σ. $\square$

We also compute the fractional separation dimension of $K_{m+1,m,m}$. As with $K_{m+1,m}$, the extra vertex imposes no extra cost.

Theorem 2.6.3. *$\pi_f(K_{m+1,m,m}) = \frac{6m}{2m+1}$ for $m \geq 2$.*

Proof. Let the parts of $K_{m+1,m,m}$ be X, Y, and Z with $|X| = m+1$. By monotonicity, $\pi_f(K_{m+1,m,m}) \geq \pi_f(K_{m,m,m}) = \frac{6m}{2m+1}$. To prove equality, it suffices to give a strategy for the ordering player that separates any pair in $\mathcal{P}$ with probability at least $\frac{2m+1}{6m}$. Given an ordering σ as $v_1, \ldots, v_{3m+1}$, let $B_i = \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ for $1 \leq i \leq m$ as in Theorem 2.6.2. Use $W \in \{X, Y, Z\}$ and $W = \{w_1, \ldots, w_t\}$ as before, indexed as ordered in σ. The ordering player makes equally likely all orderings such that $(v_{3i-2}, v_{3i-1}, v_{3i}) = (x_i, y_i, z_i)$ in order or $(v_{3i-2}, v_{3i-1}, v_{3i}) = (x_i, z_i, y_i)$ in order and with $v_{3m+1} = x_{m+1}$. By Corollary 2.2.3, it suffices to show that σ separates at least the fraction $\frac{2m+1}{6m}$ of the pairs in each orbit.

For the pairs in $\mathcal{P}$ with endpoints in only two parts, the number of pairs separated by σ depends only on the restriction of σ to those parts. The restriction is precisely an ordering used in Theorem 2.3.2 or Theorem 2.6.1. There we showed that the fraction of such pairs separated is $\frac{m+1}{3m}$, which is larger than

$\frac{2m+1}{6m}$.

In remains to consider the T-pairs. As in Theorem 2.6.2, classify these as W-pairs for $W \in \{X, Y, Z\}$. The Y-pairs and Z-pairs are in one orbit, the X-pairs in another.

Deleting x_{3m+1} (the last vertex) leaves an ordering considered in Theorem 2.6.2. There we counted W-pairs within that ordering. There were the same number of separated T-pairs of each type, except for those hitting only two blocks. Since each block B_k appears in the order (x_k, y_k, z_k), each pair of blocks yields three separated X-pairs, four Y-pairs, and three Z-pairs among the 10 T-pairs counted earlier.

We conclude that the ordering separates $8\binom{m}{4}+11\binom{m}{3}+3\binom{m}{2}$ X-pairs and a total of $16\binom{m}{4}+22\binom{m}{3}+7\binom{m}{2}$ Y-pairs and Z-pairs not involving x_{3m+1}.

Separated T-pairs involving x_{3m+1} hit at most three earlier blocks. Using one vertex each from B_i, B_j, and B_k with $i < j < k$, we obtain $4\binom{m}{3}$ X-pairs and a total of $4\binom{m}{3}$ Y-pairs and Z-pairs. Using x_{3m+1} and vertices from B_i and B_j, there are $5\binom{m}{2}$ X-pairs, $2\binom{m}{2}$ Y-pairs and $\binom{m}{2}$ Z-pairs. Using x_{3m+1} and all three vertices of B_i, we obtain one X-pair, since x_i comes first, and no Y-pairs or Z-pairs.

Summing these possibilities, we find that σ separates $8\binom{m}{4}+15\binom{m}{3}+8\binom{m}{2}+m$ of the $m^3(m+1)$ X-pairs and $16\binom{m}{4}+26\binom{m}{3}+10\binom{m}{2}$ of the $2m^2(m^2-1)$ Y-pairs and Z-pairs. Remarkably, each ratio is exactly $\frac{2m+1}{6m}$. □

In Theorem 2.6.2 we used more general orderings to simplify the optimality argument. Since the use of monotonicity eliminated the need to prove optimality, in Theorem 2.6.3 we used more restricted orderings to simplify counting T-pairs.

The most surprising aspect of fractional separation dimension of dense n-vertex graphs is that it is not generally maximized by the balanced complete tripartite graph.

Theorem 2.6.4. *$\pi_f(K_{1,m,m}) = \frac{24m}{8m+5+3/(2\lceil m/2\rceil-1)}$ for $m \geq 1$.*

Proof. Let the parts be X, Y, and Z with $X = \{x\}$. Again we have D-pairs and T-pairs, but the vertices of the D-pairs all lie in $Y \cup Z$, and the T-pairs all use x and are Y-pairs or Z-pairs, designated by the part contributing a vertex to each edge. The D-pairs lie in one orbit, as do the T-pairs, so by Corollary 2.2.3 some optimal strategy for the pair player makes D-pairs equally likely and makes T-pairs equally likely.

Let σ be a vertex ordering of the form $v_1, \ldots, v_{2k}, x, v_{2k+1}, \ldots, v_{2m}$ such that each pair of the form $\{v_{2i-1}, v_{2i}\}$ consists of one vertex from each of Y and Z, for $1 \leq i \leq m$. We count the pairs separated by σ. After optimizing over k, the ordering player will make all orderings with that k equally likely.

For all k, the restrictions of such orderings to $Y \cup Z$ are the orderings used in Theorem 2.3.2, which separate the fraction $\frac{m+1}{3m}$ of the D-pairs, and no ordering separates more such pairs.

Index Y as $y_1, \ldots, y_m$ and Z as $z_1, \ldots, z_m$ in order in σ, so that $\{v_{2i-1}, v_{2i}\} = \{y_i, z_i\}$ for $1 \le i \le m$. Each T-pair separated by σ involves x. For the edge xw, an edge separated from xw by the ordering is obtained by picking one vertex each from Y and Z that are both on the opposite side of x from w or both on the opposite side of w from x. When $w \in \{y_j, z_j\}$ with $1 \le j \le k$, taking the two cases of y_j and z_j together yields $(j-1)(j-1+j) + 2(m-k)^2$ pairs. Summing over j yields $2k(m-k)^2 + \sum_{j=1}^{k} \binom{2j-1}{2}$ pairs. Similarly, summing over j with $k+1 \le j \le m$ yields $2(m-k)k^2 + \sum_{i=1}^{m-k} \binom{2i-1}{2}$ pairs.

Let $f(k)$ be the sum of these two quantities, the total number of T-pairs separated. Note that

$$f(k) = 2mk(m-k) + \sum_{i=1}^{k} \binom{2i-1}{2} + \sum_{i=1}^{m-k} \binom{2i-1}{2}.$$

Letting $g(k) = f(k) - f(k-1)$, we have

$$g(k) = 2m(m-2k+1) + \binom{2k-1}{2} - \binom{2(m-k+1)-1}{2},$$

which simplifies to $m - 2k + 1$. Thus $g(k)$ is a decreasing function of k. Also, $g(\frac{m}{2}) > 0$ and $g(\frac{m+1}{2}) = 0$. Hence the number of T-pairs is maximized by choosing k as the integer closest to $m/2$.

By induction on k, it is easily verified that $\sum_{i=1}^{k} \binom{2i-1}{2} = \frac{1}{6}(4k+1)k(k-1)$. Hence when m is even and $k = m/2$, our orderings separate $\frac{m}{12}(8m^2 - 3m - 2)$ pairs. When m is odd, they separate $\frac{m-1}{12}(8m^2+5m+3)$. With altogether $2m^2(m-1)$ T-pairs, the ratio is $\frac{8m^2-3m-2}{24m(m-1)}$ when m is even and $\frac{8m^2+5m+3}{24m^2}$ when m is odd. Dividing numerator and denominator by $m-1$ or m yields the unified formula $\frac{8m+5+3/(2\lceil m/2\rceil-1)}{24m}$ for the fraction separated.

Note that the fraction of T-pairs separated is smaller than the fraction of D-pairs separated. It suffices to show that no ordering that does not pair vertices of Y and Z and place x between two pairs separates the maximum number of T-pairs. The pair player then achieves equality in the game by making the T-pairs equally likely.

Since we have considered all k, avoiding the specified form means that some vertex in $Y \cup Z$ precedes another vertex with a lesser index or that x occurs between y_i and z_i for some i. In the first case, we may assume that y_j appears before z_i with $j > i$ and no vertex of $Y \cup Z$ between y_j and z_i. In the second case, we may assume by symmetry that $i < m$ and y_i is before z_i. In either case, form σ' from σ by moving z_i one position earlier; this exchanges z_i with y_j or with x, see Figure 2.4.

If x appears before y_j in σ, then $m-j$ T-pairs are separated in σ but not σ', and $m-i$ T-pairs are separated in σ' but not σ. If x appears after z_i, then $i-1$ T-pairs are separated by σ but not σ', and $j-1$ T-pairs are separated by σ' but not σ. Since $j > i$, in each case σ' separates more T-pairs.

$$\ldots, y_j, z_i, \ldots \quad \rightarrow \quad \ldots, z_i, y_j, \ldots$$

$$\ldots, y_i, x, z_i, \ldots \quad \rightarrow \quad \ldots, y_i, z_i, x, \ldots$$

Figure 2.4: Two suboptimal orders and local exchanges considered in the proof of Theorem 2.6.4

In the remaining case, x appears between y_j and z_i with $j \geq i$ and $i < m$. Now $(i+j-1)(m-j)$ T-pairs are separated by σ but not σ', and $(2m-i-j)j$ T-pairs separated by σ' but not σ. We have $(2m-i-j)j > (i+j-1)(m-j)$ when $j < m(j-i+1)$, which is true when $i < j \leq m$ and $i < m$. $\square$

Using Theorems 2.6.2 and 2.6.4, we compare $\pi_f(K_{2r+1,2r+1,2r+1}) = \frac{6(2r+1)}{4r+3}$ and $\pi_f(K_{1,3r+1,3r+1}) \geq \frac{24(3r+1)}{24r+13+1/r}$. Each graph has $6r+3$ vertices. When $r > 1$, the value of $\pi_f(K_{1,3r+1,3r+1})$ is larger. Similarly, using Theorems 2.6.3 and 2.6.4, we compare $\pi_f(K_{2r+1,2r,2r}) = \frac{12r}{4r+1}$ and $\pi_f(K_{1,3r,3r}) \geq \frac{24(3r)}{24r+5+3/(3r-1)}$. Each graph has $6r+1$ vertices. When $r > 1$, the value of $\pi_f(K_{1,3r,3r})$ is larger.

2.7 Circular separation dimension

Instead of considering linear orderings of $V(G)$, we may consider circular orderings of $V(G)$. A pair of nonincident edges $\{xy, zw\}$ is *separated* by a circular ordering σ if the endpoints of the two edges do not alternate. The *circular separation dimension* is the minimum number of circular orderings needed to separate all pairs of nonincident edges in this way. The *circular t-separation dimension* $\pi_t^\circ(G)$ is the minimum size of a multiset of circular orderings needed to separate all the pairs at least t times. The *fractional circular separation dimension* $\pi_f^\circ(G)$ is $\liminf_{t\to\infty} \pi_f^\circ(G)/t$.

Like $\pi(G)$, also $\pi^\circ(G)$ is a hypergraph covering problem. The vertex set $\mathcal{P}$ of the hypergraph H is the same, but the edges corresponding to vertex orderings of G are larger. Thus $\pi^\circ(G) \leq \pi(G)$ and $\pi_f^\circ(G) \leq \pi_f(G)$.

Before discussing the fractional problem, one should first determine the graphs G such that $\pi^\circ(G)$ (and hence also $\pi_f^\circ(G)$) equals 1. Surprisingly, this characterization is quite easy. Unfortunately, it does not generalize to geometrically characterize graphs with $\pi^\circ(G) = t$ like the boxicity result in [9, 10].

Proposition 2.7.1. *$\pi^\circ(G) = 1$ if and only if G is outerplanar.*

Proof. When $\pi^\circ(G) = 1$, the ordering provides an outerplanar embedding of G by drawing all edges as chords. Chords cross if and only if their endpoints alternate in the ordering.

For sufficiency, it suffices to consider a maximal outerplanar graph, since the parameter is monotone. The outer boundary in an embedding is a spanning cycle; use that as the vertex order. All pairs in $\mathcal{P}$ are separated, since alternating endpoints yield crossing chords. $\square$

The lower bound $\pi(K_{m,n}) \geq \log_2(\min\{m, n\})$ relies on the fact that when two vertices of one part precede two vertices of the other, both nonincident pairs induced by these four vertices fail to be separated. In an circular ordering, always at least one of the two pairs is separated. This leads to the surprising result that $\pi^\circ(G) \in \{1, 2\}$ when G is bipartite.

Proposition 2.7.2. *$\pi^\circ(K_{m,n}) = 2$ when $m, n \geq 2$ with $mn > 4$.*

Proof. The exceptions are the cases where $K_{m,n}$ is outerplanar and Proposition 2.7.1 applies. Let σ be an circular ordering in which each partite set occurs as a consecutive segment of vertices. Obtain σ' from σ by reversing one of the partite sets. A nonincident pair of edges alternates endpoints in σ if and only if it does not alternate endpoints in σ'. Hence it is separated in exactly one of the two orderings. $\square$

Nevertheless, π° is unbounded. It suffices to consider K_n, where a classical result provides the lower bound. A list of d-tuples is *monotone* if in each coordinate the list is strictly increasing or weakly decreasing. The multidimensional generalization of the Erdős–Szekeres Theorem by de Bruijn states that any list of more than l^{2^d} vectors in $\mathbb{R}^d$ contains a monotone sublist of more than l vectors. The result is sharp, but does not necessarily yield sharpness in our proof of the lower bound on $\pi^\circ(K_n)$. Our best upper bound is logarithmic, from $\pi^\circ(K_n) \leq \pi(K_n) \leq 4 \log_{3/2} n$ [10].

Theorem 2.7.3. $\pi^\circ(G) > \log_2 \log_3(\omega(G) - 1)$.

Proof. By monotonicity, it suffices to prove this for $G = K_n$. Note first that a set of circular orderings separates all pairs of nonincident edges in K_n if and only if every 4-set appears cyclically ordered in more than one way (not counting reversal). This follows because each cyclic ordering of K_4 alternates endpoints of exactly one pair of nonincident edges, and for the three cyclic orderings (unchanged under reversal) the pairs that alternate are distinct.

Consider d circular orderings of $\{v_1, \ldots, v_n\}$. Write them linearly by starting with v_1. Associate with each v_i a vector w_i in $\mathbb{R}^d$ whose jth coordinate is the position of v_i in the jth linear ordering. If $n > 3^{2^d}$, then by the multidimensional generalization of the Erdős–Szekeres Theorem $w_1, \ldots, w_n$ has a monotone sublist of four elements. The four corresponding vertices x_1, x_2, x_3, x_4 appear in increasing order or in decreasing order in each linear order. Hence they appear in the same cyclic order or its reverse in each of the original circular orderings. In particular, x_1x_3 and x_2x_4 are not separated by these circular orderings. Since we considered any d circular orderings, $\pi^\circ(K_n) > d$ when $n = 3^{2^d} + 1$. $\square$

We turn now to the fractional context. Since $\pi^\circ(G)$ is a hypergraph covering problem, again π°_f is computed from a matrix game, with each row being the incidence vector for the set of pairs in $\mathcal{P}$ separated by a circular ordering of $V(G)$.

Our earlier results have analogues in the circular context. A circular ordering of four vertices separates two of the three pairs instead of one, which improves some bounds by a factor of 2. The characterization of the extremal graphs then mirrors the proof of Theorem 2.3.1.

Theorem 2.7.4. *$\pi^\circ_f(G) \le \frac{3}{2}$, with equality if and only if $K_4 \subseteq G$. Furthermore, if G has n vertices and $K_4 \subseteq G$, then $\pi^\circ_f(G) \le \frac{3}{2}\left(1 - \frac{6}{n^4} + O(\frac{1}{n^5})\right)$.*

Proof. A circular ordering separates two of the three pairs in each set of four vertices, so making all circular orderings of n vertices equally likely yields $\pi^\circ_f(G) \le \frac{3}{2}$. Equality holds when $K_4 \subseteq G$, since the pair player can give probability $\frac{1}{3}$ to each pair of nonincident edges in a copy of K_4.

Now suppose $K_4 \not\subseteq G$. Let $p = \frac{2}{3} + \frac{4(n-4)!}{n!}$. We provide a distribution on the circular orderings of $V(G)$ such that each nonincident pair of edges is separated with probability at least p. We create a list of $n!$ linear orderings of $V(G)$, which we view as $n!$ circular orderings.

Consider $S = \{a, b, c, d\} \subseteq V(G)$. For each ordering σ' of the remaining $n-4$ vertices, 24 orderings begin with S and end with σ'. By symmetry, we may assume $ac \notin E(G)$. Thus the possible pairs of nonincident edges induced by S are $\{ab, cd\}$ and $\{ad, bc\}$. We increase the separation probability for these vertex pairs.

Circular separation includes nesting when written linearly; only alternation of endpoints fails. The pairs $\{ab, cd\}$ and $\{ad, bc\}$ are each separated 16 times in the 24 orderings of S followed by σ'. The new 24 orderings will separate $\{ab, cd\}$ and $\{ad, bc\}$ each at least 20 times and any other pair, not involving the nonadjacent pair consising of a and c, at least 16 times.

The 24 new orderings are two copies each where the first four vertices are (in order) $abdc$, $badc$, $dcba$, $cbad$, $adbc$, $adcb$, $acbd$, or $dbac$, and four copies each using $cdab$ or $bcda$, always followed by σ'. By inspection, each of $\{ab, cd\}$ and $\{ad, bc\}$ is separated 20 times in the list.

The number of orderings that separate any pair of nonincident edges having at most two vertices in S is the same as before. Hence we need only check pairs with three vertices in S, consisting of one edge in $\{ab, cd, bc, ad, bd\}$ (never ac) and another edge with one endpoint among the remaining two vertices in S. In each case, the endpoints of the induced edge appear before or after the third vertex in at least 16 of the orderings in the new list of 24.

Since $\{a, b, c, d\}$ is arbitrary and we do this for each 4-set, the pairs $\{ab, cd\}$ and $\{ad, bc\}$ are separated with probability at least $\frac{5}{6}$ by the 24 orderings that start with $\{a, b, c, d\}$ and then are made circular, and with probability at least $\frac{2}{3}$ among the remaining orderings. Thus the separation probability increases from

$\frac{2}{3}$ to at least p for each pair. $\square$

Again there is no sharper bound for bipartite graphs or graphs with girth 4: $\pi_f^\circ(K_{m,m}) \to \frac{3}{2}$. The orderings used to give the optimal upper bound for $\pi_f^\circ(K_{m,qm})$ are in some sense the farthest possible from those giving the optimal upper bound for $\pi^\circ(K_{m,qm})$ in Proposition 2.7.2.

Theorem 2.7.5. $\pi_f^\circ(K_{m,qm}) = \frac{6(qm-1)}{4mq+q-3}$. *In particular,* $\pi_f^\circ(K_{m,m}) = \frac{3m-3}{2m-1}$.

Proof. Again Corollary 2.2.3 (for the circular separation game) applies. The $2\binom{m}{2}\binom{qm}{2}$ pairs of nonincident edges lie in one orbit, so it suffices to make circular orderings that separate $\frac{4mq+q-3}{6(qm-1)}2\binom{m}{2}\binom{qm}{2}$ pairs equally likely and show that no ordering separates more.

Let X and Y be the parts of the bipartition, with $|X| = m$. Let σ be a circular ordering in which the vertices of X are equally spaced, with q vertices of Y between any two successive vertices of X.

There are two types of pairs separated by σ. In one, the parts for the four vertices alternate as $XYXY$; in the other, they occur as $XYYX$, cyclically. Choose the first member of X in m ways. Let k be the number of steps within X taken to get from there to the other member of X used. In the first case, there are $kq(m-k)q$ ways to choose the vertices from Y and two ways to group the chosen vertices to form a separated nonincident pair, but either of the vertices of X could have been called the first vertex. In the second case, there are $\binom{kq}{2}$ ways to choose from Y, one way to group, and only one choice for the first vertex of X.

Thus, to count the separated pairs we sum over k and use $\sum_{k=-n}^{m}\binom{n+k}{r}\binom{m-k}{s} = \binom{n+m+1}{r+s+1}$ and $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ to compute

$$\begin{aligned} m\sum_{k=1}^{m-1} kq(m-k)q + m\sum_{k=1}^{m-1}\binom{kq}{2} &= m\sum_{k=0}^{m} q^2\binom{0+k}{1}\binom{m-k}{1} + \frac{mq}{2}\sum_{k=1}^{m-1}(k^2q-k) \\ &= mq^2\binom{m+1}{3} + \frac{mq^2}{2}\frac{(m-1)m(2m-1)}{6} - \frac{mq}{2}\binom{m}{2}. \end{aligned}$$

Factoring out $2\binom{m}{2}\frac{qm}{2}$ leaves $\frac{1}{6}(4mq+q-3)$, as desired.

It remains to show that no other circular ordering separates as many pairs of nonincident edges. We do this by finding, for every circular ordering σ other than those discussed above, an ordering $\hat{\sigma}$ that separates more pairs.

With $X = \{x_1, \ldots, x_m\}$ in cyclic order, the ordering σ is described by a list $q_1, \ldots, q_m$ of nonnegative integers summing to qm, where q_i is the number of vertices of Y between x_{i-1} and x_i (indexed modulo m). Index so that $q_1 = \max_i q_i$; we may assume $q_1 \geq q+1$.

Let σ' be the ordering obtained by interchanging x_m with the vertex y immediately following it (note that $y \in Y$, since $q_1 > q$). The pairs in $\mathcal{P}$ separated by σ or σ' but not both are those consisting of an edge yx_k for some k with $1 \le k \le m-1$ and an edge $x_m y'$. For those separated by σ but not σ' there are $\sum_{j=k+1}^{m} q_j$ choices for y'. For those separated by σ' but not σ there are $(\sum_{i=1}^{k} q_i) - 1$ choices for y'.

After isolating the terms involving q_1, the net gain in switching from σ to σ' is thus

$$\sum_{k=1}^{m-1} \left(q_1 - 1 + \sum_{i=2}^{k} q_i - \sum_{j=k+1}^{m} q_j \right).$$

Consider instead the ordering σ'' obtained from σ by interchanging x_1 with the vertex y immediately preceding it (again $y \in Y$, since $q_1 > q$). The net change in the number of separated pairs follows the same computation, except that $q_2, \ldots, q_m$ are indexed in the reverse order. More precisely, the change in moving from σ to σ'' is

$$\sum_{k=2}^{m} \left(q_1 - 1 + \sum_{j=k+1}^{m} q_j - \sum_{i=2}^{k} q_i \right).$$

In summing the two net changes, the summations in the terms for $2 \le k \le m-1$ cancel. The sum is thus

$$2(q_1 - 1)(m-1) - \sum_{j=2}^{m} q_j - \sum_{i=2}^{m} q_i.$$

Since $\sum_{j=2}^{m} q_j = qm - q_1$, the net sum simplifies to $2q_1 m - 2qm - 2(m-1)$. Since $q_1 \ge q+1$, the value is at least 2. Since the sum of the two net changes is positive, at least one of them is positive, and σ does not separate the most pairs. $\square$

Note that $K_{2,r}$ is planar with girth 4, for $r \ge 2$. Theorem 2.7.5 yields $\pi_f^\circ(K_{2,2q}) = \frac{4q-4}{3q-1} \to \frac{4}{3}$. It remains open how large π_f° can be for planar graphs with girth 4, and for graphs (planar or not) with larger girth. For girth 5, computer search shows that the fractional circular separation dimension of the Petersen graph is $\frac{8}{7}$.

Chapter 3

I,F-partitions

3.1 Introduction

This chapter contains joint work with Axel Brandt, Michael Ferrara, Mohit Kumbhat, Derrick Stolee, and Matthew Yancey.

A k-coloring $c\colon V(G) \to \{1, \ldots, k\}$ of a graph G is *proper* if c assigns distinct colors to adjacent vertices. The *chromatic number* of G is the minimum k such that G has a proper k-coloring.

Acyclic coloring was first introduced by Grünbaum [22]. A proper vertex coloring is *acyclic* if the union of any two color classes induces a forest. The least k such that G has an acyclic k-coloring is the *acyclic chromatic number* of G, denoted $\chi_a(G)$. An acyclic k-coloring of G is a *star k-coloring* if the components of the forest induced by the union of two color classes are stars; the least k such that G has a star k-coloring is the star chromatic number of G, denoted $\chi_s(G)$. It follows immediately that $\chi(G) \leq \chi_a(G) \leq \chi_s(G)$ for any graph G, although is not difficult to see that $\chi \neq \chi_a$ in general by considering, for instance, any bipartite graph containing a cycle. We refer the reader to the thorough survey of Borodin [13] for additional results on acyclic and star colorings beyond what we present here.

In this chapter, we are interested in the problem of star-coloring planar graphs. The well-known Four Color Theorem of Appel and Haken [7, 8] states that $\chi(G) \leq 4$ if G is planar, while Grünbaum [22] constructed a planar graph with no acyclic 4-coloring (and so, in particular, no star 4-coloring). Subsequently, Borodin [12] showed $\chi_a(G) \leq 5$ for all planar G. Albertson, Chappell, Kierstead, Kündgen, and Ramamurthi [2] showed that every planar graph G satisfies $\chi_s(G) \leq 20$ and also constructed a planar graph with star chromatic number at least 10. Kündgen and Timmons [29] proved that every planar graph of girth 6 (respectively 7 and 8) can be star-colored with 8 (respectively 7 and 6) colors. Kierstead, Kündgen, and Timmons [25] showed that every bipartitie planar graph can be star 14-colored, and they constructed a bipartite planar graph with star chromatic number 8. It is worthwhile to note that, while not our focus here, the results in [29] and [25] hold for the natural extension of star-colorings to a list coloring framework.

Given the Four Color Theorem, it is natural to search for conditions ensuring that a planar graph can

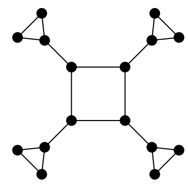


Figure 3.1: Graph demonstrating that Theorem 3.1.3 is sharp.

be star 4-colored. Albertson *et al.* [2] also showed that for every girth g, there exists a graph G_g with girth at least g and $\chi_s(G_g) = 4$, and further that there is some girth g such that every planar graph of girth at least g is star 4-colorable. Timmons [41] showed that $g = 14$ is sufficient and also gave a planar graph with girth 7 and star chromatic number 5. Bu, Cranston, Montassier, Raspaud, and Wang [15] improved upon Timmons' result by showing that every planar graph with girth $g \geq 13$ has a star 4-coloring.

The *maximum average degree* of a graph G, denoted $\mathrm{Mad}(G)$, is $\max\limits_{H\subseteq G} \frac{2|E(H)|}{|V(H)|}$. The main result of this paper is the following.

Theorem 3.1.1. *If G is a graph with $\mathrm{Mad}(G) < \frac{5}{2}$, then $\chi_s(G) \leq 4$.*

A straightforward application of Euler's formula shows that if G is a planar graph with girth at least g, then $\mathrm{Mad}(G) < \frac{2g}{g-2}$. Thus, as a corollary to Theorem 3.1.1 we have the following improvement on [15].

Corollary 3.1.2. *If G is a planar graph with girth at least 10, then $\chi_s(G) \leq 4$.*

To prove Theorem 3.1.1 we will use I,F-partitions, which were first introduced in [2]. A *2-independent set* in G is a set of vertices that have pairwise distance greater than 2. An I,F-*partition* of a graph G is a partition of $V(G)$ as $(\mathcal{I}, \mathcal{F})$ where $\mathcal{I}$ is a 2-independent set in G and $G[\mathcal{F}]$ is a forest. Albertson *et al.* [2] observed that if G has an I,F-partition $(\mathcal{I}, \mathcal{F})$, then $\chi_s(G) \leq 4$; because $\chi_s(T) \leq 3$ for any tree T there is a 3-coloring of $G[\mathcal{F}]$ which can be extended to all of G by assigning the vertices in $\mathcal{I}$ a new color. Note that the converse does not hold; for example, $\chi_s(K_{3,3}) = 4$, but $K_{3,3}$ has no I,F-partition. Timmons [41] and Bu *et al.* [15] showed that maximum average degree less than $\frac{7}{3}$ and $\frac{26}{11}$, respectively, imply the existence of an I,F-partition, which in turn imply the above mentioned girth bounds sufficient for a planar graph to be star 4-colorable. We strengthen their results (weakening the hypothesis on $\mathrm{Mad}(G)$) by proving the following theorem, which implies Theorem 3.1.1.

Theorem 3.1.3. *If G is a graph with $\mathrm{Mad}(G) < \frac{5}{2}$, then G has an* I,F-*partition.*

Theorem 3.1.3 is sharp in the sense that there are graphs with maximum average degree $\frac{5}{2}$ that do not have an I,F-partition. Indeed, given a cycle C, for each vertex v in the cycle add a 3-cycle $a_v b_v c_v$ and the edge va_v. Figure 3.1 shows the construction when the intial cycle is C_4. To see that such a graph, which has maximum average degree $\frac{5}{2}$, does not have an I,F-partition, simply note that no vertex v on the cycle C can be in the 2-independent set, as then $a_v b_v c_v$ would necessarily have to be in the forest $\mathcal{F}$, an impossibility. However, this then implies that every vertex on C must be in $\mathcal{F}$, which is also impossible.

To prove this result, we use the method of potentials as utilized by Kostochka and Yancey [27, 28], Borodin, Kostochka, and Yancey [14], and Chen, Kim, Kostochka, West, and Zhu [16]. We also generalize the problem of finding an I,F-partition by allowing some vertices to be initially assigned to $\mathcal{I}$ and $\mathcal{F}$ and modifying the condition on maximum average degree to account for the preassigned vertices.

Going forward, when $X_1, \ldots, X_t$ form a partition of a set X, then we will write $X = (X_1, \ldots, X_t)$. If G is a graph with $V(G) = (I, F, U)$, we say that G is a *partially assigned graph*. This terminology arises from the fact that we will require vertices in I and F to belong to the sets $\mathcal{I}$ and $\mathcal{F}$ respectively of an I,F-partition of G. If H is a subgraph of a partially assigned graph G, then let $I(H) = I \cap V(H)$, $F(H) = F \cap V(H)$, and $U(H) = U \cap V(H)$. Let the *potential* of H in G, denoted $\rho_G(H)$, be

$$\rho_G(H) = |I(H)| + 4|F(H)| + 5|U(H)| - 4|E(H)|.$$

When the context is clear, we omit the subscript.

Theorem 3.1.4. *Let G be a partially assigned graph with vertex set partitioned as (I, F, U). If $\rho_G(H) > 0$ for all nonempty subgraphs $H \subseteq G$, then G has an* I,F*-partition $(\mathcal{I}, \mathcal{F})$ such that $I \subseteq \mathcal{I}$ and $F \subseteq \mathcal{F}$.*

The coefficients of $|U(H)|$ and $|E(H)|$ in $\rho(H)$ are chosen to make $\rho(H) > 0$ equivalent to $\overline{d}(H) < \frac{5}{2}$ when $V(H) = U(H)$. The coefficients for $|I(H)|$ and $|F(H)|$ are chosen to so that Theorem 3.1.3 and the Theorem 3.1.4 equivalent and are further explained in Section 3.2. For now, we comment that if G is a partially assigned graph where two vertices in I are adjacent or have a common neighbor, then G has a subgraph with nonpositive potential. Similarly, a cycle of vertices in F form a subgraph of nonpositive potential. Neither structure therefore appears as a subgraph of any graph satisfying the hypotheses of Theorem 3.1.4. Also, adding edges between vertices in a subgraph of G only decreases the potential. Thus we need only consider induced subgraphs when minimizing the potential across all subgraphs of G. Reflecting that, for $S \subseteq V(G)$, we define the *potential* of S, $\rho(S)$ as $\rho(G[S])$.

In Section 3.2, we demonstrate that Theorem 3.1.3 and Theorem 3.1.4 are equivalent. Hence Theorems 3.1.3 and 3.1.1 and Corollary 3.1.2 all follow from the proof of Theorem 3.1.4, which appears in Section 3.4. Section 3.3 contains the lemmas we will use to prove Theorem 3.1.4. Even though Theorem 3.1.3 is equivalent to Theorem 3.1.4, the inductive proof of Theorem 3.1.4 is simplified by the use of partially assigned graphs. The use of partially assigned graphs allows us to have a nicer ordering of graphs in the inductive proof. They also allow us to improve over the results of Bu *et al.* [15] by allowing us to specify that some vertices end up in a particular part of the I,F-partition.

We conclude this section with some further notation.

A k-vertex is a vertex of degree k and a k^+-vertex is a vertex of degree at least k. For a vertex v, $N(v)$ is the neighborhood of v. We use I, F, and U as sets of the vertex partition of a partially assigned graph.

An I,F-partition $(\mathcal{I}, \mathcal{F})$ *extends* an assignment (I, F, U) if $I \subseteq \mathcal{I}$ and $F \subseteq \mathcal{F}$. For a partially assigned graph H, we say H has an I,F-partition only if H has an I,F-partition that extends (I, F, U). For an I,F-partition of H, let $H_{\mathcal{F}}$ be the subgraph of H induced by vertices assigned to $\mathcal{F}$ and let $H_{\mathcal{I}}$ be the subgraph of H induced by vertices assigned to $\mathcal{I}$.

3.2 Proof that Theorem 3.1.3 and Theorem 3.1.4 are equivalent

In this section, we demonstrate that Theorems 3.1.3 and 3.1.4 are equivalent, and in the process we demonstrate some of the rationale that led to the coefficients in the potential function ρ.

Proposition 3.2.1. *Theorem 3.1.4 implies Theorem 3.1.3.*

Proof. Let G be an ordinary graph (as opposed to a partially assigned graph) with $\mathrm{Mad}(G) < \frac{5}{2}$. We can view G as a partially assigned graph by setting $U(G) = V(G)$. Since $\overline{d}(H) < \frac{5}{2}$ is equivalent to $5|V(H)| - 4|E(H)| > 0$, we have that $\rho(H) > 0$ for every $H \subseteq G$. By Theorem 3.1.4, G has an I,F-partition. $\square$

Say a partially assigned graph G is *feasible* if $\rho(H) > 0$ for every $H \subseteq G$.

The proof of Proposition 3.2.2 uses the F-gadget and I-gadget shown in Figure 3.2, where the vertices are in U. Note that the potential of an F-gadget is 4 and that the potential of an I-gadget is 1. In $\rho(H)$, these agree with the coefficients on $|F(H)|$ and $|I(H)|$.

Proposition 3.2.2. *Theorem 3.1.3 implies Theorem 3.1.4.*

Proof. Assume G is a feasible partially assigned graph with the vertex partition (I, F, U).

Let an *F-gadget* consist of a vertex u joined to a 3-cycle abc via edge ua (see Figure 3.2a). Let an *I-gadget* consist of vertex u joined to vertices d and e, with additional 3-cycles abc and fgh, and path $adef$ (see Figure 3.2b). In an F-gadget or I-gadget, all vertices are placed in U. We say that a vertex v is *replaced* by a gadget if we remove v from G and add the gadget to G by making u adjacent to the vertices in $N_G(v)$.

(a) An F-gadget forces u to be assigned $\mathcal{F}$ in an I,F-partition.

(b) An I-gadget forces u to be assigned $\mathcal{I}$ in an I,F-partition.

Figure 3.2: The F- and I-gadgets.

First we argue that the replacement of a vertex in v in $F(G)$ by an F-gadget preserves feasibility. Let G_F be the result of such a replacement. Let $H' \subseteq G_F$ such that $\rho_{G_F}(H')$ is minimal. If $H' \subseteq G$, then $\rho_{G_F}(H') = \rho_G(H') > 0$. Thus, we may assume that $H' \not\subseteq G$.

Thus H' contains some vertices of the F-gadget. The only subgraph of minimum potential within the triangle abc is the cycle itself, which has potential 3. Thus if any of $\{a, b, c\}$ are in $V(H')$, then the entire cycle in in H'. If this is the entirety of H', then $\rho_{G_F}(H') > 0$. If $u \in V(H')$, then the addition of the edge ua and the cycle abc does not change the potential of H', so we may assume that H' contains the entire F-gadget. Thus we may assume that H' has positive potential, or contains the entire F-gadget. In the later case, let $H'' \subseteq G$, be the graph with with $G[H'' - v] = G[H' - \{u, a, b, c\}]$ and $N_{H''}(v) = N_{H'}(u) - \{a\}$. Now,

$$\rho_{G_F}(H') = \rho_G(H'') - \rho_G(\{v\}) + \rho_{G_F}(\{u, a, b, c\}) = \rho_G(H'') > 0$$

and G_F is feasible.

Next we argue that the replacement of a vertex $vinI(G)$ by an I-gadget preserves feasibility. Let G_I be the result of such a replacement. Let $H' \subseteq G_I$ such that $\rho_{G_I}(H')$ is minimal. If $H' \subseteq G$, then $\rho_{G_I}(H') = \rho_G(H') > 0$. Thus, we may assume that $H' \not\subseteq G$.

Thus H' contains some vertices of the I-gadget. The minimality of $\rho_{G_I}(H')$ and structure of the I-gadget forces one of two possibilities. In one case, H' is contained in the I-gadget and $\rho_{G_I}(H') > 0$. In the other, H' contains the entire I-gadget, but is not contained in the I-gadget. In the later case, let $H'' \subseteq G$, be the graph with with $G[H'' - v] = G[H' - \{u, a, \ldots, h\}]$ and $N_{H''}(v) = N_{H'}(u) - \{d, e\}$. Now,

$$\rho_{G_I}(H') = \rho_G(H'') - \rho_G(\{v\}) + \rho_{G_I}(\{u, a, \ldots, h\}) = \rho_G(H'') > 0$$

and G_F is feasible.

Since replacing vertices in F by F-gadgets and vertices in I by I-gadgets preserves feasibility, the graph G' resulting in making all possible such replacements is feasible. Since G' is a partially assigned graph with with $U(G') = V(G')$, we have $\rho_{G'}(H) = 5|V(H)| - 4|E(H)|$ for all $H \subseteq G'$. Thus $\mathrm{Mad}(G') < \frac{5}{2}$ and, by the assumption of Theorem 3.1.3, G' has an I,F-partition $(\mathcal{I}', \mathcal{F}')$.

Given an F-gadget $\{u, a, b, c\}$ in G', one of the vertices a, b, c must be in $\mathcal{I}$, which in turn forces u to be in $\mathcal{F}$. Similarly, for any I-gadget $\{u, a, \ldots, h\}$ in G' one of a, b or c and one of f, g or h must be in I, implying that d and e must be in F. Consequently, $u \in \mathcal{I}$, as desired. By letting $\mathcal{I} = (\mathcal{I}' \cap V(G)) \cup I$ and $\mathcal{F} = (\mathcal{F}' \cap V(G)) \cup F$ we thus obtain an I,F-partition of G extending (I, F, U). $\square$

3.3 Some useful claims

In this section, we use our minimality assumptions to restrict the structure of a minimal counterexample to Theorem 3.1.4. The proof of Theorem 3.1.4 will then be completed in Section 3.4 using the discharging method.

An *ℓ-thread* is a path P in a partially assigned graph H of ℓ vertices in U that have degree 2 in H such that the neighbors of the endpoints of P in $H - V(P)$, which we say *border* the thread P, are 3^+-vertices or are in $I \cup F$. Define an *open* thread to be a thread with two bordering vertices and a *closed* thread to be a thread with one bordering vertex. In counting the incidences of threads with a vertex, open threads contribute once to the count and closed threads contribute twice. Note that it is traditional to define an ℓ-thread in a graph G as a trail of length $\ell + 1$ in G whose ℓ internal vertices have degree 2 in the full graph G. In both definitions, the number of 2-vertices in an ℓ-thread is ℓ, but here we require these 2-vertices be in U. In addition, our requirement on bordering vertices means that an ℓ-thread does not contain any shorter thread.

Recall that, for $H' \subseteq H$ in a partially assigned graph H, we maintain the preassignment on H' so that, for example, $I(H') = I(H) \cap V(H')$. Also, for a partially assigned graph H, we only consider I,F-partitions $(\mathcal{I}, \mathcal{F})$ that satisfy $I(H) \subseteq \mathcal{I}$ and $F(H) \subseteq \mathcal{F}$.

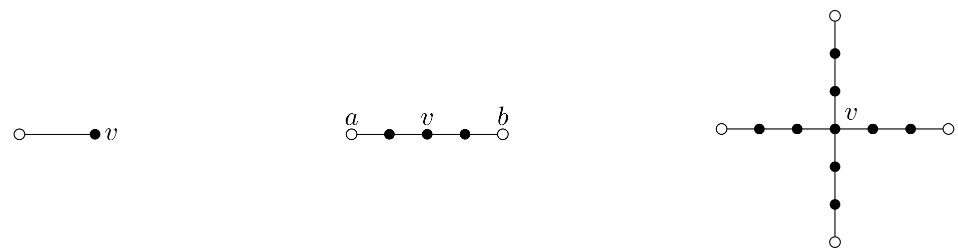
If $|V(H')| + |E(H')| < |V(H)| + |E(H)|$, then we say that H' is *smaller* than H.

Definition. A *minimal counterexample* is a feasible partially assigned graph G that has no I,F-partition extending $(I(G), F(G), U(G))$ such that every smaller feasible partially assigned graph G' has an I,F-partition extending $(I(G'), F(G'), U(G'))$. For the remainder of this section, assume that a minimal counterexample exists, and let G be that minimal counterexample.

We include the proof of the following claim about G for completeness.

Claim 3.3.1 (Timmons [41], Bu *et. al.*[15])**.** *None of the following appear in G:*

(C1) A 1-vertex in U.



(a) The subgraph described in (C1). (b) The subgraph described in (C2). (c) A subgraph described in (C3).

Figure 3.3: Subgraphs described in Claim 3.3.1 where all solid vertices are in U.

(C2) A 3^+-thread.

(C3) A 4-vertex in U incident to four 2-threads.

Proof. Suppose (C1) appears in G as shown in Figure 3.3a where $v \in U$. Then $G - v$ is smaller than G. By the minimality of G, $G-v$ therefore has an I,F-partition $(\mathcal{I}, \mathcal{F})$ that extends $(I(G-v), F(G-v), U(G-v))$. Extend $(\mathcal{I}, \mathcal{F})$ to G by assigning v to $\mathcal{F}$. Doing so does not decrease the distance in G between vertices of $G_{\mathcal{I}}$ and does not create a cycle in $G_{\mathcal{F}}$ since $d(v) = 1$. Thus, this extension is an I,F-partition of G, contradicting the choice of G.

Next, suppose (C2) appears in G as shown in Figure 3.3b. It is possible that a or b are internal to a larger thread containing v, or that $a = b$. Obtain G' from G by deleting v and its neighbors, and note that G' is smaller than G, which implies G' has an I,F-partition $(\mathcal{I}, \mathcal{F})$ extending $(I(G'), F(G'), U(G'))$.

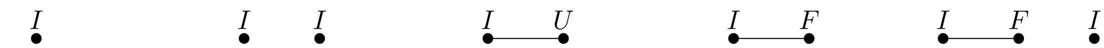
If at least one of a or b is in $G_{\mathcal{I}}$, then assigning the deleted vertices to $\mathcal{F}$ does not create an $\mathcal{F}$-cycle. Otherwise, v is at distance at least 3 from a $\mathcal{I}$. Thus assigning v to $\mathcal{I}$ and the neighbors of v to $\mathcal{F}$ preserves the distance requirement for vertices in $\mathcal{I}$ and does not introduce any $\mathcal{F}$-cycles. In either case, $(\mathcal{I}, \mathcal{F})$ extends to an I,F-partition of G, again a contradiction.

Finally, assume that (C3) appears in G with 4-vertex v. See Figure 3.3c. Note that we neither assume that the threads incident to v are open, nor that the boundary vertices of these threads are distinct. The graph G' obtained by deleting v and its incident threads has G' smaller than G, and once again has an I,F-partition $(\mathcal{I}, \mathcal{F})$ that extends $(I(G'), F(G'), U(G'))$. Notice that v is at distance at least 3 from any vertex in $\mathcal{I}$, so assigning v to $\mathcal{I}$ and the other deleted vertices to $\mathcal{F}$ extends $(\mathcal{I}, \mathcal{F})$ to an I,F-partition of G. □

Before proceeding to our key claims, we have the following claims about cut sets in G and the structure of small sets of small potential. For $S \subseteq V(G)$, an *S-lobe* of G is an induced subgraph of G whose vertex set consists of S and the vertices of some component of $G - S$.

Claim 3.3.2. *If $R \subseteq I$, then $G - R$ is connected.*

Proof. Otherwise, every R-lobe G_i is a proper subgraph of G and, by the minimality of G, there exists an I,F-partition $(\mathcal{I}_i, \mathcal{F}_i)$ of G_i with $I(G_i) \subseteq \mathcal{I}_i$ and $F(G_i) \subseteq \mathcal{F}_i$. Consider $\mathcal{I} = \cup \mathcal{I}_i$ and $\mathcal{F} = \cup \mathcal{F}_i$. Since $R \subseteq I$, $\mathcal{I}$ has no two vertices within distance two and $\mathcal{F}$ contains no cycles. Hence the partition $(\mathcal{I}, \mathcal{F})$ is an I,F-partition of G extending (I, F, U), a contradiction. □



(a) $\rho(K_1) = 1$ (b) $\rho(2K_1) = 2$ (c) $\rho(K_2) = 2$ (d) $\rho(K_2) = 1$ (e) $\rho(K_2 + K_1) = 2$

Figure 3.4: The five induced proper subgraphs H of G with $\rho(H) < 3$ and $|V(H)| + |E(H)| \leq 4$.

Claim 3.3.3. *If H is an induced proper subgraph of G with $\rho(H) < 3$ and $|V(H)| + |E(H)| \leq 4$, then H is one of the five partially assigned graphs shown in Figure 3.4.*

Proof. Let H be an induced proper subgraph of G with $\rho(H) < 3$ and $|V(H)| + |E(H)| \leq 4$. If $|E(H)| = 2$ then $|V(H)| + |E(H)| \geq 5$, so $|E(H)| \in \{0, 1\}$. If $|E(H)| = 0$, then as $\rho(H) < 3$, every vertex must be assigned to I and there are either one or two such vertices as in Figures 3.4a and 3.4b. Otherwise, if $|E(H)| = 1$, then the combined potential of the (at least two) vertices can be at most 6 and must be at least 5 since $\rho(H) > 0$. Hence exactly one vertex of H is in either U or F. Now, if one vertex is in U, then these conditions force H to have exactly one other vertex in I as depicted in Figure 3.4c. If instead one vertex is in F, then H has either one or two additional vertices in I. As vertices in I are necessarily nonadjacent, this leaves Figures 3.4d and 3.4e as the remaining feasible configurations. □

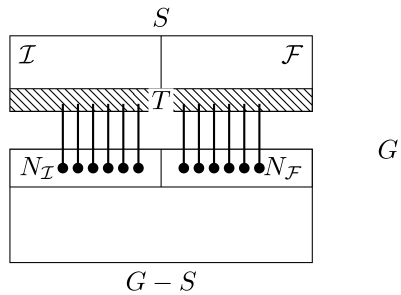
Claim 3.3.4, which we prove next, states that G has no large proper subsets of small potential. On the other hand, Claim 3.3.3 enumerates all the small sets of small potential. Together these are used in the proof of Claim 3.3.6 which allow us to reassign vertices from U to F.

Claim 3.3.4. *Let S be a proper subset of G and let $H = G[S]$. If $|S| + |E(H)| \geq 5$, then $\rho(S) \geq 3$.*

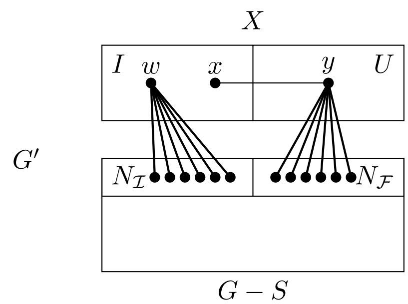
Proof. Suppose otherwise, so there is a proper subset S with $|S| + |E(H)| \geq 5$ and $\rho_G(S) < 3$. Select such S to minimize $\rho_G(S)$, and recall that $\rho_G(S) > 0$. Further, let

$$T = \{v \in S \colon N(v) \cap \overline{S} \neq \emptyset\}.$$

By Claim 3.3.2, if $T \subseteq I$ then $S = T$. As vertices in I are pairwise nonadjacent, we then would have $|S| + |E(H)| = |S| \geq 5$ and thus $\rho_G(H) \geq 5$, a contradiction. Therefore, at least one vertex in T is in $F \cup U$.



(a) Vertex sets and some adjacencies of G.



(b) Vertex sets and some adjacencies of G'.

Figure 3.5: The construction of G' from G as described in Claim 3.3.4.

If $\rho_G(S) = 2$ and $T \cap F = \emptyset$, then modify the partially assigned graph H to a partially assigned graph H_0 by changing a vertex of $T \cap U$ from U to F. Note that for all $S' \subseteq S$, we have $\rho_{H_0}(S') \geq \rho_H(S') - 5 + 4 = \rho_H(S') - 1$. By the minimality of $\rho_G(S)$, either $\rho_H(S') = \rho_G(S') \geq 2$ or S' induces one of the subgraphs given in Claim 3.3.3, of which only (c) has a vertex in U. Thus $\rho_{H_0}(S') \geq 1$ for all $S' \subseteq S$. In particular, moving a vertex to F does not complete a cycle in F since the potential of such cycles is 0. Otherwise, let $H_0 = H$, and our assumptions guarantee $\rho_{H_0}(S') \geq 1$ for all $S' \subseteq S$. This alteration, if necessary, will aid in the construction of an auxiliary graph, which we describe below.

By the minimality of G, the fact that potentials are minimized by induced subgraphs, and $S \subsetneq V(G)$, there is an I,F-partition $(\mathcal{I}_{H_0}, \mathcal{F}_{H_0})$ of H_0 such that $I(H_0) \subseteq \mathcal{I}_{H_0}$ and $F(H_0) \subseteq \mathcal{F}_{H_0}$. Let $N_{\mathcal{I}}$ be the set of vertices in $\overline{S}$ adjacent to a vertex $t \in T$ with $t \in \mathcal{I}_{H_0}$ and $N_{\mathcal{F}}$ be the set of vertices in $\overline{S}$ adjacent to a vertex $t \in T$ with $t \in \mathcal{F}_{H_0}$. We claim that $N_{\mathcal{I}} \cup N_{\mathcal{F}} \neq \emptyset$. Indeed, $N_{\mathcal{I}}$ and $N_{\mathcal{F}}$ are defined relative to the given I,F-partition of H_0, so we must have that $N_{\mathcal{I}} \cup N_{\mathcal{F}} = N(S) - S$, which is nonempty as G is connected and S is a nonempty proper subset of $V(G)$.

Construct an auxiliary graph G' (see Figure 3.5 for a visual representation) by adding vertices to $G - S$ as follows. If $N_{\mathcal{I}} \neq \emptyset$, add a new vertex w that is adjacent to every vertex in $N_{\mathcal{I}}$. If $N_{\mathcal{F}} \neq \emptyset$, then add adjacent vertices x and y and connect y to each vertex in $N_{\mathcal{F}}$. Add w and/or x to I, and y to U and let X denote those of w, x and y that are added to G'.

Observe the following statements about G':

(Ob 0) $|N_G(v) \cap T| \leq 1$ for all $v \in \overline{S}$, as otherwise $\rho_G(S \cup \{v\}) \leq \rho_G(S) + \rho_G(v) - 4 \cdot 2 \leq 3 + 5 - 8 = 0$, contradicting the hypothesis on G; hence $|N_G(v) \cap S| = |N_{G'}(v) \cap X|$.

(Ob 1) X is nonempty, as $N_{\mathcal{I}} \cup N_{\mathcal{F}} \neq \emptyset$.

(Ob 2) If $\rho_G(S) = 2$, then $T \cap F(H_0) \neq \emptyset$ by construction, and hence $\{x, y\} \subseteq X$.

Since $|X| + |E(G'[X])| \leq 4$ and $|S| + |E(H)| \geq 5$, $G'$is smaller than G. By the assignment of vertices in X under the construction of G', if G' has an I,F-partition $(\mathcal{I}_{G'}, \mathcal{F}_{G'})$, then specifically $y \in \mathcal{F}_{G'}$ if $\{x, y\} \subseteq X$. Observe that $(\mathcal{I}_{G'-X} \cup \mathcal{I}_{H_0}, \mathcal{F}_{G'-X} \cup \mathcal{F}_{H_0})$ is an I,F-partition of G because an $\mathcal{F}$-cycle cannot be formed and the construction of X implies $\mathcal{I}$ is necessarily a 2-independent set. Thus, by minimality of G, there is instead some $W \subseteq V(G')$ with $\rho_{G'}(W) \leq 0$. Select $W \subseteq V(G')$ to minimize $\rho_{G'}(W)$. Notice that if $W \cap X = \emptyset$, then $W \subset G$ and $\rho_G(W) = \rho_{G'}(W) \leq 0$, a contradiction. We may therefore assume $W \cap X \neq \emptyset$. Observe that $\rho_{G'}(W \cap X) \geq 1$.

The minimality of $\rho_{G'}(W)$ and the assignment of vertices in X imply that if $W \cap N_{\mathcal{I}} \neq \emptyset$, then $w \in W$, and that if $W \cap N_{\mathcal{F}} \neq \emptyset$, then $\{x, y\} \subseteq W$. Since every edge between $W \setminus X$ and X in G' corresponds to an edge between $W \setminus S$ and S in G by (Ob 0), we have

$$0 < \rho_G((W - X) \cup S) = \rho_{G'}(W) - \rho_{G'}(W \cap X) + \rho_{H_0}(S).$$

Since $\rho_{G'}(W) \leq 0$, it follows that

$$\rho_{G'}(W \cap X) < \rho_{H_0}(S).$$

We have two cases to consider. First, suppose that $\overline{S} \not\subseteq W$. Define $S' = (W - X) \cup S$ and recall $\rho_G(S') \leq \rho_{G'}(W) - \rho_{G'}(W \cap X) + \rho_{H_0}(S)$, which implies $\rho_G(S') < \rho_{H_0}(S) \leq \rho_G(S)$ since $\rho_{G'}(W) \leq 0$ and $\rho_{G'}(W \cap X) > 0$. Since $\overline{S} \not\subseteq W$, we have $S' \subsetneq V(G)$, which contradicts the minimality of $\rho(S)$.

Now suppose $\overline{S} \subseteq W$. By the minimality of $\rho_{G'}(W)$, $W \cap X = X$ and hence $W = V(G')$. Since $\rho_{G'}(X) \in \{1, 2\}$, $\rho_G(S) \in \{1, 2\}$, (3.3) requires $\rho_G(S) = 2$, and $\rho_{G'}(X) = 1$. However, by (Ob 2) we have that $\rho_G(S) = 2$ implies $\rho_{G'}(X) \geq 2$, a contradiction. $\square$

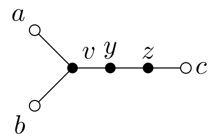
We can immediately use Claim 3.3.4 to prove a claim about vertices in F.

Claim 3.3.5. *Every vertex of F is a 3^+-vertex.*

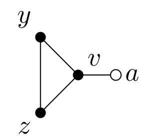
Proof. Let v be a 1-vertex in F. Then $G - v$ is smaller than G and is feasible. Consequently, $G - v$ has an I,F-partition $(\mathcal{I}, \mathcal{F})$ which can be extended to an I,F-partition of G by assigning v to $\mathcal{F}$.

If, instead, v is a 2-vertex in F, let u be a neighbor of v. If there is an edge joining the neighbors of v, then the subgraph induced by v and its neighbors has three vertices, three edges and potential at most 2. This contradicts Claim 3.3.4, so the neighbors of v are not adjacent.

Let G' be the partially assigned graph formed from G by contracting the edge uv into a vertex labeled uv, and assign uv to the same set in (I, F, U) that u was assigned to. If $S \subseteq V(G')$ is a nonempty subset with $\rho_{G'}(S) \leq 0$, then necessarily $uv \in S$. Let $S' = (S \setminus \{uv\}) \cup \{u, v\}$, and observe that $\rho_G(S') = \rho_{G'}(S')$,



(a)



(b)

Figure 3.6: A 3-vertex v incident to a 2-thread.

a contradiction. Thus G' is feasible, and since G' is smaller than G, the minimality of G implies G' has an I,F-partition $(\mathcal{I}, \mathcal{F})$. Reversing the contraction does not decrease the distance in G between vertices in $G_{\mathcal{I}}$, and $G_{\mathcal{F}}$ remains a forest after adding v to $\mathcal{F}$ since there are no cycles in G that are not in G'. □

Before proceeding to Claim 3.3.6, note that by Claim 3.3.3 a copy of K_2 with a vertex in I and the other in U, as seen in Figure 3.4c, is the only possible induced proper subgraph H of G with a vertex in U that satisfies $\rho(H) < 3$. Claim 3.3.6 will allow us to move up to two vertices in U to F and still get an I,F-partition for any proper subgraph of G.

Claim 3.3.6. *Let S be a nonempty proper subset of $V(G)$ and let G' be obtained from $G[S]$ by reassigning up to two vertices u and v from U to F, then G' has an* I,F*-partitionthat extends $(I(G'), F(G'), U(G'))$.*

Proof. Since S is a proper subset of $V(G)$, we G' is smaller than G. Thus G' has an I,F-partition unless reassigning u and v resulted in G' no longer being feasible. If $W \subset V(G')$ has $\rho_{G'}(W) \leq 0$, then $W \cap \{u, v\} \neq \emptyset$, since otherwise $\rho_{G'}(W) = \rho_G(W) > 0$.

As a vertex in F has lower potential by 1 than a vertex in U, $\rho_{G'}(W) \geq \rho_G(W) - 2$. Thus, $\rho_G(W) \leq 2$ and by Claim 3.3.4, $|W| + |E(G'[W])| \leq 4$. However, Claim 3.3.3 gives the set of such partially assigned graphs, and we find that W cannot contain both u and v. Thus $\rho_{G'}(W) = \rho_G(W) - 1$. However, as W contains a vertex of U and is one of the graphs from Claim 3.3.3, $\rho_G(W) = 2$. Hence $\rho_{G'}(W) \geq 1$. □

The following two claims restrict the local structure around 3-vertices in $F \cup U$.

Claim 3.3.7. *If v is a 3-vertex in $F \cup U$ with no neighbors in I, then v is not incident to a 2-thread in G.*

Proof. Let v be a 3-vertex in $F \cup U$ that is incident to a 2-thread but has no neighbors in I. We consider two cases, depending on whether a 2-thread incident to v is open or closed.

First, suppose that an open 2-thread with vertices y and z is incident to v as in Figure 3.6a; let a and b be the neighbors of v not in this 2-thread. Let $S = V(G) \setminus \{y, z\}$, and let G' be the partially assigned graph obtained from $G[S]$ by placing a and b in $F(G')$. By Claim 3.3.6, G' has an I,F-partition $(\mathcal{I}, \mathcal{F})$. If v or c is

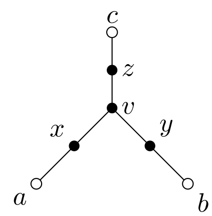


Figure 3.7: A 3-vertex v incident to three 1-threads.

in $\mathcal{I}$, adding y and z to $\mathcal{F}$ extends $(\mathcal{I}, \mathcal{F})$ to G without creating any cycles in $G_{\mathcal{F}}$. Otherwise $v, c \in \mathcal{F}$ and adding y to $\mathcal{I}$ and z to $\mathcal{F}$ extends $(\mathcal{I}, \mathcal{F})$ to G.

Second, suppose that a closed 2-thread with vertices y and z is incident to v as in Figure 3.6b; let a be the neighbor of v not in this 2-thread. Let $S = V(G) \setminus \{v, y, z\}$, and let G' be the partially assigned graph obtained from $G[S]$ by placing $a \in F(G')$. By Claim 3.3.6, G' has an I,F-partition $(\mathcal{I}, \mathcal{F})$. Adding v and z to $\mathcal{F}$ and y to $\mathcal{I}$ extends $(\mathcal{I}, \mathcal{F})$ from G' to G even in the case that $v \in F$. □

Claim 3.3.8. *A 3-vertex in U incident to three 1-threads with bordering vertices in $F \cup U$ does not appear in G.*

Proof. Let v be a 3-vertex in U as shown in Figure 3.7 where $x, y, z \in U$ are the internal vertices of the 1-threads and $a, b, c \in F \cup U$ are the other endpoints of the 1-threads.

Suppose first that at most two of a, b, and c are assigned to U; say $c \in F$. Let $S = V(G) \setminus \{v, x, y, z\}$, and let G' be the partially assigned graph obtained from $G[S]$ and placing a and b in $F(G')$. Claim 3.3.6 implies that there exists an I,F-partition $(\mathcal{I}, \mathcal{F})$ of G' that extends to an I,F-partition of G by adding v to $\mathcal{I}$ and x, y, and z to $\mathcal{F}$.

Thus we may assume that a, b, and c are all assigned to U in G. Let $G' = G - \{v, x, y, z\}$ and reassign a, b and c to F in G'. Since G' is smaller than G, G' has an I,F-partition unless there is some $W \subseteq V(G')$ with $\rho_{G'}(W) \leq 0$ and $|W \cap \{a, b, c\}| = 3$. From the reassignment of a, b, and c, $\rho_G(W) \leq \rho_{G'}(W) + 3 \cdot (5 - 4) \leq 3$, which then implies that $\rho_G(W \cup \{v, x, y, z\}) = \rho_G(W) + 4 \cdot 5 + 6 \cdot (-4) \leq -1$, contradicting the choice of G. □

Our final claim restricts the structure around 4-vertices in U.

Claim 3.3.9. *G has no 4-vertex in U incident to three 2-threads and a 1-thread whose bordering vertex is in $F \cup U$.*

Figure 3.8: A 4-vertex v incident to three 2-threads and one 1-thread.

Proof. Let v be a 4-vertex in U incident with three 2-threads and one 1-thread, and let $a \in F \cup U$ be the other vertex bordering the 1-thread. Let T be the set of internal vertices in the threads incident to v. At most one of the 2-threads may be closed, as depicted in Figure 3.8.

Let $S = V(G) \setminus (T \cup \{v\})$, and let G' be the partially assigned graph obtained from $G[S]$ by assigning a to $F(G')$. Claim 3.3.6 implies that there exists an I,F-partition $(\mathcal{I}, \mathcal{F})$ of G', necessarily with $a \in \mathcal{F}$. Adding v to $\mathcal{I}$ and the vertices of T to $\mathcal{F}$ extends $(\mathcal{I}, \mathcal{F})$ to G so that vertices in $\mathcal{I}$ have pairwise distance at least two in G and $G_{\mathcal{F}}$ is a forest. Notice that since any cycle in $G_{\mathcal{F}}$ that would be created must use v, $G_{\mathcal{F}}$ is a forest. $\square$

3.4 Proof of Theorem 3.1.4

We use discharging to prove the following lemma. As we previously demonstrated that the minimal counterexample to Theorem 3.1.4 satisfies Claims 3.3.1, 3.3.5 and 3.3.7–3.3.9, this proves that no counterexample exists.

Lemma 3.4.1. *If G is a graph satisfying Claims 3.3.1, 3.3.5 and 3.3.7–3.3.9, then $\rho_G(G) \leq 0$.*

Proof. Suppose that G satisfies Claims 3.3.1, 3.3.5 and 3.3.7–3.3.9. Assign an initial charge μ to vertices of G as follows:

$$\mu(v) = 2d(v) - \begin{cases} 1, & v \in I \\ 4, & v \in F \\ 5, & v \in U. \end{cases}$$

Observe that $\sum_{v \in V(G)} \mu(v) = 4|E(G)| - |I(G)| - 4|F(G)| - 5|U(G)| = -\rho(G)$.

We distribute charge using three rules, (R1), (R2), and (R3), in order. Let $\mu^*(v)$ denote the final charge on a vertex v.

(R1) If $v \in V(G)$ satisfies $\mu(v) \geq d(v)$, then v sends charge 1 to each neighbor $u \in N(v)$.

(R2) If v is the internal vertex of a 1-thread and v did not receive charge under (R1), then v pulls charge $\frac{1}{2}$ from each of its neighbors.

(R3) If v is an internal vertex of a 2-thread and v did not receive charge under (R1), then v pulls charge 1 from its neighbor on the border of the thread.

We will demonstrate $\mu^*(v) \geq 0$ for all $v \in V(G)$, which implies

$$-\rho(G) = \sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} \mu^*(v) \geq 0.$$

By Claims 3.3.1 and 3.3.5, the only vertices v with $\mu_0(v) < 0$ are vertices in U of degree 2, which have $\mu_0(v) = -1$. If v is the internal vertex of a thread and receives charge under (R1), then $\mu_1(v) = 0$ and v receives no charge under (R2) and (R3). If $\mu_1(v) < 0$, and v receives any charge under (R2), then $\mu_2(v) = 0$ and v receives no charge under (R3). Thus, we have that a vertex receives charge under at most one rule and that each vertex v gives to each neighbor u under at most one rule.

If v is a vertex in I, a vertex in F with $d(v) \geq 4$, or a vertex in U with $d(v) \geq 5$, then $\mu(v) \geq d(v)$. Consequently, $\mu_1(v) = \mu_2(v) = \mu_3(v) \geq 0$ since v sends charge $d(v)$ during (R1) and does not send charge using (R2) or (R3).

Next, consider a vertex $v \in F$ with $d(v) \leq 3$. By Claim 3.3.5, $d(v) \geq 3$. If v is incident to a 2-thread, then by Claim 3.3.7, v has a neighbor in I. This neighbor sends charge 1 to v by (R1) and v sends charge at most 1 to each other neighbor by (R2) or (R3), so $\mu^*(v) \geq \mu(v) + 1 - 2 \geq 0$. Otherwise, from (R2), v sends charge at most $\frac{3}{2}$ to incident 1-threads and $\mu^*(v) \geq \mu(v) - 3 \cdot \frac{1}{2} = \frac{1}{2} > 0$, as desired.

Recall that by Claim 3.3.1, no vertex in U has degree less than 2. Since G contains no 3^+-thread by Claim 3.3.1, if v is a 2-vertex in U, then $\mu_0(v) = -1$ and $\mu_3(v) \geq 0$ by either (R1), (R2), or (R3).

Suppose next that $v \in U$ with $d(v) = 3$. If v is incident to a 2-thread, then by Claim 3.3.7, v has a neighbor in I. This neighbor sends charge 1 to v by (R1), and v sends charge at most 1 to each other neighbor by (R2) or (R3), so $\mu^*(v) \geq \mu(v) + 1 - 2 \geq 0$, as desired. If v is not incident to any 2-threads and is incident to fewer than three 1-threads, then $\mu^*(v) \geq \mu(v) - 2 \cdot \frac{1}{2} \geq 0$, as desired. Otherwise, v is incident to exactly three 1-threads, and at least one of the 1-threads is bordered by a vertex a in I by Claim 3.3.8. Since a sends charge 1 to the internal vertex of the 1-thread by (R1), v sends charge at most $\frac{1}{2}$ to the other neighbors by (R2), and hence $\mu^*(v) \geq \mu(v) - 2 \cdot \frac{1}{2} \geq 0$, as desired.

Finally, consider $v \in U$ with $d(v) = 4$. By Claim 3.3.1, v is not incident to four 2-threads. By Claim 3.3.9, if v is incident to three 2-threads and a 1-thread, then the other vertex a bordering the 1-thread is in I. Since a sends charge 1 to the 1-thread by (R1), v sends charge at most 1 to at most three neighbors by

(R2) and (R3), and hence $\mu^*(v) \geq \mu(v) - 3 \cdot 1 \geq 0$, as desired. If v is incident to three 2-threads and no other thread, then $\mu^*(v) \geq \mu(v) - 3 \cdot 1 \geq 0$. Finally, if v is incident to at most two 2-threads and up to two 1-threads, then $\mu^*(v) \geq \mu(v) - 2 \cdot 1 - 2 \cdot \frac{1}{2} \geq 0$.

Therefore, every vertex in G has non-negative final charge, which completes the proof. $\square$

Chapter 4

3-Dynamic coloring of planar and toroidal graphs

4.1 Introduction

This chapter contains joint work with Thomas Mahoney, Benjamin Reiniger, and Jennifer Wise.

For a graph G and positive integer r, an *r-dynamic coloring* of G is a proper vertex coloring such that for each $v \in V(G)$, at least $\min\{r, d(v)\}$ distinct colors appear in $N_G(v)$. The *r-dynamic chromatic number*, denoted $\chi_r(G)$, is the minimum k such that G admits an r-dynamic k-coloring. Montgomery [35] introduced 2-dynamic coloring and the generalization to r-dynamic coloring.

List coloring was introduced independently by Vizing [42] and by Erdős, Rubin, and Taylor [19]. A *list assignment* L for G assigns to each vertex v a list $L(v)$ of permissible colors. Given a list assignment L for a graph G, if a proper coloring f can be chosen so that $f(v) \in L(v)$ for all $v \in V(G)$, then G is *L-colorable*. The *choosability* of G is the least k such that G is L-colorable for any list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$. We consider the r-dynamic version of this parameter. For further work on dynamic coloring, see [1, 23, 24]. A graph G is *r-dynamically L-colorable* when an r-dynamic coloring can be chosen from the list assignment L. The *r-dynamic choosability* of G, denoted $\mathrm{ch}_r(G)$, is the least k such that G is *r-dynamically L-colorable* for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$.

The *square* of a graph G, denoted G^2, is the graph resulting from adding an edge joining any two vertices separated by distance 2. For any graph G,

$$\begin{aligned}
\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \cdots = \chi(G^2), \\
\mathrm{ch}(G) = \mathrm{ch}_1(G) \leq \mathrm{ch}_2(G) \leq \cdots \leq \mathrm{ch}_{\Delta(G)}(G) = \cdots = \mathrm{ch}(G^2),
\end{aligned}$$

and that $\chi_r(G) \leq \mathrm{ch}_r(G)$ for all r. Thus we can think of r-dynamic coloring as bridging the gap between coloring a graph and coloring its square.

Wegner [45] conjectured bounds for the chromatic number of squares of planar graphs in terms of their maximum degree. For a graph G with $\Delta(G) \leq 3$, proper colorings of G^2 and 3-dynamic colorings of G are equivalent. Thomassen [40] proved Wegner's conjecture for maximum degree 3, showing that $\chi_3(G) \leq 7$ for any planar subcubic graph G. Cranston and Kim [17] studied the list coloring version and proved that when G is a planar subcubic graph, $\mathrm{ch}_3(G) \leq 7$ if the girth is at least 7 and $\mathrm{ch}_3(G) \leq 6$ if the girth is at least 9.

Thomassen [39] proved that planar graphs are 5-choosable, and Voigt [43] proved that this is sharp. Kim, Lee, and Park [26] proved that planar graphs are actually 2-dynamically 5-choosable. Their proof involves showing that every planar graph has a planar supergraph with an edge in the neighborhood of every vertex. They then invoke Thomassen's result that planar graphs are 5-choosable to obtain their result.

Our main results are on the 3-dynamic chromatic number and choice number for planar and toroidal graphs. A graph is *toroidal* if it can be drawn on the torus without crossing edges; in particular, planar graphs are also toroidal.

Theorem 4.1.1. *If G is a toroidal graph, then $\chi_3(G) \leq \mathrm{ch}_3(G) \leq 10$.*

Theorem 4.1.1 is sharp: the Petersen graph P has maximum degree 3 and diameter 2, so $\chi_3(P) = \chi(P^2) = \chi(K_{10}) = 10$.

Corollary 4.1.2. *If G is a planar graph, then $\chi_3(G) \leq \mathrm{ch}_3(G) \leq 10$.*

We do not believe that Corollary 4.1.2 is sharp. An example of a planar graph G with $\chi_3(G) = 7$ is the graph obtained from K_4 by subdividing the three edges incident to one vertex. Note that G has maximum degree 3 and diameter 2, so $\chi_3(G) = \chi(G^2) = \chi(K_7)$.

Our proofs use the Discharging Method.

In Section 4.2, we show that several configurations cannot occur in a minimal counterexample to Theorem 4.1.1. In Section 4.3, we complete the proof of Theorem 4.1.1 by using the Discharging Method to show that the configurations listed in Section 4.2 form a set that is unavoidable in a toroidal graph.

4.2 Structure of a minimal counterexample

In this section, we use our minimality assumptions to restrict the structure of a minimal counterexample to Theorem 4.1.1. The proof of Theorem 4.1.1 is completed in Section 4.3.

Definition. A *minimal counterexample* is a graph G that is not 3-dynamic 10-choosable such that every graph G' with $|V(G')| < |V(G)|$ is 3-dynamic 10-choosable.

For the remainder of this section, fix a minimal counterexample G. In addition, fix a list assignment L for G such that $|L(v)| \geq 10$ for all $v \in V(G)$, yet G is not 3-dynamic L-choosable. Finally, fix an embedding for G.

Note that the claims in this section do not require G to be a toroidal graph. However, we do consider a fixed embedding of G so that we can discuss the faces of G and have an orientation (generally clockwise) for the neighbors of a vertex.

In each claim, we obtain a graph G' with $V(G') \subsetneq V(G)$. By minimality, G' is 3-dynamically 10-choosable. Thus G' has a 3-dynamic L-coloring, where we use the list L on G restricted to the vertices of G'. In our proofs, all colorings of G' and G discussed are with respect to L. We retain the coloring on $V(G')$ when extending the coloring to G.

Note that if two vertices are on the same face of an embedding, then we can add an edge joining them while maintaining the embedding. Whenever we add edges to G', the endpoints of the edge are on the same face after we preform the vertex deletions given in producing G' from G. If our construction of G' "adds" edges, then we do not add edges that are already present in G. For example, in Claim 4.2.2, if $y_1z_2, y_2z_2 \in E(G)$, then $G' = G - \{v_1, v_2\}$.

In the figures for the claims, the thick gray edges represent the edges possibly added by $E'(G)$, and the dashed lines enclose the vertices deleted from G.

Say that a vertex is *properly colored* in a coloring of it receives a color distinct from the colors on its neighbors. Say that a vertex $w \in V(G)$ is *full* in a coloring if w has at least $\min\{3, d_G(w)\}$ different colors appear on $N_G(w)$.

Claim 4.2.1. *Every vertex in G has degree at least 3.*

Proof. Let $v \in V(G)$ be a 1-vertex with neighbor u (Figure 4.1(i)). Let $G' = G - v$ and obtain a 3-dynamic L-coloring of G'.

We can extend the coloring by avoiding the color on u, and, if u is not full, up to two colors appearing on neighbors of u. With ten colors available for v, we can extend the coloring.

Let $v \in V(G)$ be a 2-vertex with neighbors y and z (Figure 4.1(ii)). Let $G' = (G - v) \cup \{yz\}$. Because any two neighbors of a 2-vertex lie on the same face in an embedding, we may add the edge yz to G'. Obtain a 3-dynamic L-coloring of G'. Since yz is in $E(G')$, the colors on y and z will be distinct in the coloring chosen from L on G'. To extend the coloring, we must give v a color distinct from the colors of y and z, and avoiding up to two colors each in the neighborhoods of y and z. Since there are at most six colors to avoid on v, we may extend the coloring. □

Claim 4.2.2. *No 3-vertex in G is adjacent to a 3-vertex.*

Proof. Let v_1 and v_2 be adjacent 3-vertices. Let y_i and z_i be the other neighbors of v_i (Figure 4.2). Let $G' = (G - \{v_1, v_2\}) \cup \{y_1z_1, y_2z_2\}$ and obtain a 3-dynamic L-coloring of G'.

(i) (ii)

Figure 4.1: Configurations for Claim 4.2.1.

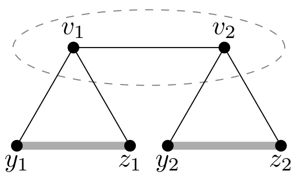


Figure 4.2: Configuration for Claim 4.2.2.

To extend the coloring, we may first color v_1 by avoiding the colors on y_1, z_1, y_2, z_2 and up to two colors each in the neighborhoods of y_1 and z_1 to make them full. Then we may color v_2 by avoiding the colors on y_1, z_1, y_2, z_2, and up to two colors each in the neighborhoods of y_2 and z_2. Here y_1 must avoid at most eight colors and y_2 must avoid at most nine colors. By avoiding the colors of $y_1 z_1, y_2$, and z_2 when coloring both v_1 and v_2, we ensure that both v_1 and v_2 are full. $\square$

We say that a 3-face is *expensive* when it has an incident 3-vertex; a 4-face is *expensive* when it has two incident 3-vertices. Claim 4.2.2 implies that the 3-vertices on an expensive 4-face must be nonadjacent.

Claim 4.2.3. *Let v be a d-vertex in G that is contained in e_3 expensive 3-faces and e_4 expensive 4-faces. Let k be the number of 3-neighbors of v. If $k \geq 2$, then and $d + k - e_3 - e_4 \geq 10$.*

Proof. Since $k \geq 2$, Claims 4.2.1 and 4.2.2 imply $d \geq 4$. Let $x_1, \ldots, x_k$ be the 3-neighbors of v in clockwise order, and let v, y_i, z_i be the neighbors of x_i for $i \in [k]$ (Figure 4.3). Let $S = \{v, x_1, \ldots, x_k\}$ and $E' = \{y_i z_i : i \in [k]\}$, and let $G' = (G - S) \cup E'$ and obtain a 3-dynamic L-coloring of G'.

To extend the coloring, we first color v by avoiding the colors on $N_G(v) - \{x_1, ..., x_k\}$ and $\{y_i, z_i : i \in [k]\}$. The first guarantees v is properly colored and the second guarantees that each x_i will be full. When x_i is on an expensive 3-face, then one of y_i or z_i is in $N_G(v) - \{x_1, ..., x_k\}$. When x_i and x_j are on an expensive 4-face, we have $\{y_i, z_i\} \cap \{y_j, z_j\} \neq \emptyset$. Thus

$$|(N_G(v) - \{x_1, ..., x_k\}) \cup \{y_i, z_i : i \in [k]\}\,| \leq d - k + 2k - e_3 - e_4.$$

Since $d - 2k - e_3 - e_4 < 10$, there is a color available for v.

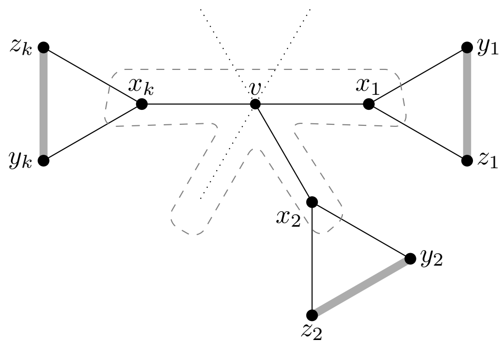


Figure 4.3: Configuration for Claim 4.2.3.

If $k = 2$, and $d(v) \geq 3$, let x' be a vertex in $N(v) - \{x_1, x_2\}$. If x' is defined, then to make v full we avoid the color of x' when coloring x_1 and x_2. This is in addition to the colors that are listed as avoided below. If $k \geq 3$, then x_1, x_2 and x_3 will make v full.

We color x_1 by avoiding the colors on v, y_1, z_1 and up to two colors each in the neighborhoods of y_1 and z_1 to make y_1 and z_1 full. We then color x_2 by a the colors on v, x_1, y_2, z_2 and up to two colors each in the neighborhoods of y_2 and z_2 to make y_2 and z_2 full. If $k \geq 3$, we then color x_3 by avoiding the colors on v, x_1, x_2, y_3, z_3 and up to two colors each in the neighborhoods of y_3 and z_3 to make y_3 and z_3 full. Finally, if $k \geq 4$, then we color x_i for $i \geq 4$ by avoiding the colors on v, y_i, z_i, and up to two colors each in the neighborhoods of y_i and z_i to make y_i and z_i full. In this process, we avoid up to nine colors at each vertex. In each case, avoiding the colors v, y_i, and z_i guarantees that x_i is properly colored, and avoiding colors in the neighborhoods of y_i and z_i, guarantees that these vertices are full. Finally, avoiding the colors on x_1 and x_2 as appropriate guarantees that v is full. $\square$

Claim 4.2.4. *No 4-vertex in G is adjacent to a 3-vertex.*

Proof. By Claim 4.2.3, it suffices to consider a 4-vertex v_1 with exactly one 3-neighbor v_2. Let y_1 and z_1 be neighbors of v_1, and let y_2 and z_2 be the neighbors of v_2 other than v_1 (Figure 4.4). Because y_1 and z_1 are on the same face, we may add the edge y_1z_1 to G'. Let $G' = (G - \{v_1, v_2\}) \cup \{y_1z_1, y_2z_2\}$ and obtain a 3-dynamic L-coloring of G'.

To extend the coloring, first color v_1 by avoiding the colors on y_1, z_1, y_2, z_2, the color on the other neighbor of v_1, and the colors on up to two neighbors of y_1 and z_1 to make them full. Note that since v_1 had exactly one 3^--neighbor, the other neighbor of v_1 has degree at least 4 and is thus either one of y_2 or z_2, or is full. With at most nine colors to avoid, we can color v_1.

Finally, we color v_2 by avoiding the colors on v_1, y_1, z_1, y_2, z_2, and the colors on up to two neighbors of

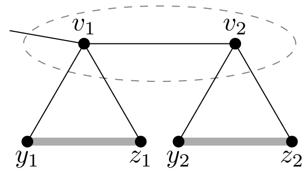


Figure 4.4: Configuration for Claim 4.2.4.

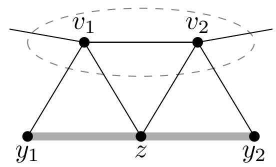


Figure 4.5: Configuration for Claim 4.2.5.

y_2 and z_2. □

Claim 4.2.5. *Every 3-cycle in G has at least two 5^+-vertices.*

Proof. If a 3-cycle has an incident 3-vertex, then Claim 4.2.4 implies the result. Thus we may assume that the 3-cycle has two incident 4-vertices v_1 and v_2, Let z be the common neighbor of v_1 and v_2 on the 3-cycle. Let y_i be a neighbor of v_i such that y_i, v_i, z are consecutive vertices on a face and $y_i \neq v_{3-i}$, (Figure 4.5). Let $G' = (G - \{v_1, v_2\}) \cup \{y_1z, y_2z\}$ and obtain a 3-dynamic L-coloring of G'.

For $i \in \{1, 2\}$, when coloring v_i we avoid the colors on $N(v_i), z, y_{3-i}$, up to one color in the neighborhood of z to make it full, and up to two colors in the neighborhood of y_i to make it full. Each vertex avoids up to nine colors. We have that y_i is full because of v_i and z is full because of v_1 and v_2. In addition, v_i is full since y_i, v_{3-i} and z have distinct colors. □

Claim 4.2.6. *Let uv be the common edge of two adjacent 3-faces uvy and uvz. Then $d(v) \geq 5$, and if equality holds, then one of the vertices in $N(v) - \{u, y, z\}$ is a 3-vertex.*

Proof. Assume that $d(v) \leq 5$ and $N(v) - \{u, y, z\}$ contains only 4^+-vertices. (Figure 4.6). Note that if v is a 4^--vertex, then Claim 4.2.4 implies that any other neighbor of v is a 4^+-vertex; in this case, we will show that such vertices are not incident to two adjacent 3-faces. Let $G' = (G - v) \cup \{yz\}$ and obtain a 3-dynamic L-coloring of G'. Note that the vertices in $N(v) - \{u, y, z\}$ have at least three colors in their neighborhood.

To extend the coloring, we color v by avoiding the colors on vertices in $N(v)$ and also up to two colors in the neighborhoods of y and z to make them full. Since v needs to avoid at most nine colors, we can extend the coloring. □

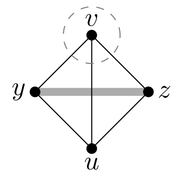


Figure 4.6: Configuration for Claim 4.2.6.

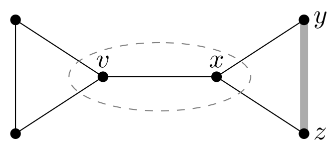


Figure 4.7: Configuration for Claim 4.2.7.

The conclusion of Claim 4.2.6 also applies to both vertices on two adjacent 3-faces.

Claim 4.2.7. *Let v be a 7^--vertex with a 4^--neighbor x. If v has no 3-neighbors (aside from possibly x), then v is on at most one 3-face. Furthermore, if v is on a 3-face, then x is also on that 3-face.*

Proof. We show that there cannot be a 3-face containing v but not x. After showing this, it follows that any 3-face containing v must use the edge xv. Claim 4.2.6 implies there is at most one such face. Let y and z be other neighbors of x that are on a shared face with x (Figure 4.7). Let $G' = (G - \{v, x\}) \cup \{yz\}$ and obtain a 3-dynamic L-coloring of G'.

To extend the coloring, we first color x by avoiding the colors appearing on $N_G(x)$ and up to two colors each in the neighborhoods of v, y, or z to make sure they are full. We then color v by avoiding the colors on $N_G(v) \cup \{y, z\}$. Since each of x and v avoids at most nine colors when we color it, we can extend the coloring. □

Claim 4.2.8. *If G has vertices u, v, x, y, z forming 3-faces vzx, vxy, and vyu, then $d(v) \geq 7$.*

Proof. Suppose $d(v) \leq 6$. See Figure 4.8. Claim 4.2.6 implies $d_G(x) \geq 5$ and $d_G(y) \geq 5$. v has at most four 3-neighbors. Claim 4.2.3 implies that vyu is not an expensive 3-face; in particular $d_G(u) > 3$. Let $G' = (G - v) \cup \{yz\}$ and obtain a 3-dynamic L-coloring of G'. Note y and z are incident to a common face in the embedding of $G - v$ so the edge may be added.

Since $d(x)$ and $d(u)$ are greater than 3, both u and x are full. In addition, we know that y, z, and x received distinct colors, so v will be full in any coloring. When extending the coloring, we avoid the colors on $N_G(v)$ when coloring v. If we blindly avoid two colors each in the neighborhoods of y or z to make them

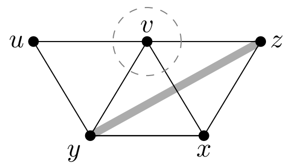


Figure 4.8: Configuration for Claim 4.2.8.

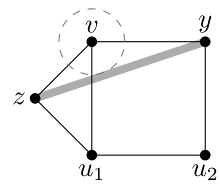


Figure 4.9: Configuration for Claim 4.2.9.

full, then we may have specified $6 + 2 + 2$ colors to avoid. Instead, since y and z share a common neighbor x, we avoid the color on x and up to one other color in the neighborhoods of y and z to make them full. $\square$

Claim 4.2.9. *No expensive 4-face in G shares an edge with a 3-face.*

Proof. Claim 4.2.2 implies that the 3-vertices of the expensive 4-face are not adjacent, and Claim 4.2.4 implies that the other vertices on the 4-face are 5^+-vertices. Let vu_1z be the 3-face sharing an edge with the 4-face vyu_2u_1 where $d_G(v) = d_G(u_2) = 3$ (Figure 4.9). Let $G' = (G - v) \cup \{yz\}$ and obtain a 3-dynamic L-coloring of G'. Observe that, in order to make u_2 full, u_1 and y receive distinct colors. Thus v will be full in any coloring since u_1, y, and z receive distinct colors.

We extend the coloring by coloring v to avoid the colors on vertices of $N_G(v)$ and up to two colors each in the neighborhoods of y and z to make them full. Since v must avoid at most seven colors, we may extend the coloring. $\square$

Claim 4.2.10. *Every 4-face in G has a 5^+-vertex.*

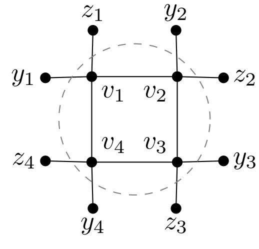


Figure 4.10: Configuration for Claim 4.2.10.

Proof. Claim 4.2.4 implies that it suffices to consider a face $v_1v_2v_3v_4$ in which all vertices have degree 4. For $i \in [4]$, let y_i and z_i be neighbors of v_i not on the 4-face, taken in clockwise order (Figure 4.10). Let $S = \{v_1, \dots, v_4\}$, and let $G' = G - S$ and obtain a 3-dynamic L-coloring of G'.

Claim 4.2.4 implies that $d_G(w) \geq 4$ for each $w \in N_G(S)$ and Claim 4.2.5 implies that $z_i \neq y_{i+1}$ where indices are taken mod 4. Thus if $d_{G'}(y_i) = 2$ or $d_{G'}(z_i) = 2$ for some $i \in [4]$, then that vertex is adjacent to v_{i+2} with indices taken mod 4.

We color v_1 by avoiding the colors of y_1 and z_1, of the neighbors of y_1 if $d_{G'}(y_1) = 2$, of the neighbors z_1 if $d_{G'}(z_1) = 2$, the color on y_2 and z_2 if they are the same, and the color on y_4 and z_4 if they are the same.

We then color v_2 by avoiding the colors of v_1, y_2, and z_2, of the neighbors of y_2 if $d_{G'}(y_2) = 2$, of the neighbors of z_2 if $d_{G'}(z_2) = 2$, the color on y_1 and z_1 if they are the same, and the color on y_3 and z_3 if they are the same.

We color v_3 by avoiding the colors on $v_1, v_2, y_2, z_2, y_3, z_3, y_4$, and z_4. Finally, we color v_4 by avoiding the colors on $v_1, v_2, v_3, y_1, z_1, y_3, z_3, y_4$, and z_4. For $i \in [4]$, we avoid at most nine colors when coloring v_i.

To verify that this produces a 3-dynamic coloring of G, we note that distinct colors are given to v_1, v_2, v_3, and v_4, and that each v_i avoids the colors on y_i and z_i. Thus the coloring is proper.

If any $w \in N_G(S)$ satisfies $d_{G'}(w) = 2$, then its neighbors provide two colors to $N_G(w)$, and either v_1 or v_2 will provide a third color. For any $w \in N_G(S)$ with $d_{G'}(w) \geq 3$, the coloring of G' results in at least three colors appearing in $N_G(w) - S$. Since the colors on v_1, v_2, v_3, and v_4 are all distinct, it suffices to show that for $i \in [4]$, either y_i or z_i has a color distinct from both v_{i-1} and v_{i+1} with indices taken mod 4. This is true based on the vertices whose colors are avoided when coloring v_3 and v_4 and the fourth set of avoided colors for v_1 and v_2. $\square$

4.3 Discharging for toroidal graphs

In this section, we give the discharging argument for Theorem 4.1.1.

Theorem 4.3.1 completes the proof of Theorem 4.1.1. We note that if G is a planar graph, then (R9) and Claim 4.2.10 are not needed.

Theorem 4.3.1. *A graph toroidal graph cannot satisfy all claims from Section 4.2.*

Proof. Suppose the claim is false, and let G be a graph embedded in the torus satisfying the claims from Section 4.2.

For $x \in V(G) \cup F(G)$, let $\mu(x)$ be the initial charge on x, and let $\mu^*(x)$ be the final charge on x. We use *face charging*: for a vertex v we set $\mu(v) = 2d(v) - 6$, and for a face f we set $\mu(f) = \ell(f) - 6$. By Euler's formula, $|V(G)| - |E(G)| + |F(G)|$ is 0 for an embedding on the plane and 2 for an embedding on the torus. Thus the total initial charge is

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (\ell(f) - 6) = 4|E(G)| - 6|V(G)| + 2|E(G)| - 6|F(G)| \leq 0.$$

Under the assumption that G satisfies the claims of Section 4.2, we will argue that after following the rules below the final charge on each vertex and face of G is nonnegative. For a planar graph, this is sufficient to obtain a contradiction. For a graph embedded on the torus, we make a final analysis of the faces and vertices with positive final charge to obtain a contradiction. This contradiction shows that there is no toroidal graph satisfying the claims of Section 4.2 and thus that every toroidal graph is 3-dynamically 10-choosable.

By Claim 4.2.1, G has no 2^--vertices, so all vertices start with nonnegative charge. Thus our discharging rules consist of vertices giving charge to the faces. Recall that a 3-face is expensive if it has a 3-vertex and a 4-face is expensive if it has two 3-vertices. Say that a 3-face is *intermediate* when it has an incident 4-vertex.

By Claim 4.2.5 a 3-face has at most one 4^--vertex. By Claim 4.2.2 a 4-face has at most two 3-vertices. Also, if a 4-face has two 3-vertices, then Claim 4.2.4 implies that these vertices are nonadjacent, and since Claim 4.2.3 applied to a 5^--vertex v implies that v is not on any expensive face, we have that the other vertices of an expensive 4-face are 6^+-vertices. If a 4-face has exactly one 3-vertex, then Claim 4.2.4 implies that its neighbors are 5^+-vertices. Finally, Claim 4.2.2 implies that a 5-face has at most two 3-vertices and two 3-vertices on a 5-face are nonadjacent. Figure 4.11 illustrates all possible 5^-faces and the discharging rules used on those faces.

We move charge according to the following rules which are illustrated in Figure 4.11 for all faces receiving charge.

(R1) A 3-face with a 3-vertex takes $\frac{3}{2}$ from each of its 5^+-vertices.

(R2) A 3-face with a 4-vertex takes $\frac{1}{2}$ its 4-vertex and $\frac{5}{4}$ charge from each of its 5^+-vertices.

(R3) A 3-face with no 4^--vertices takes 1 from each of its vertices.

(R4) A 4-face with exactly two 3-vertices takes 1 from each of its 6^+-vertices.

(R5) A 4-face with exactly one 3-vertex takes $\frac{1}{2}$ from its vertex opposite the 3-vertex and $\frac{3}{4}$ from each of its other two vertices.

(R6) A 4-face with no 3-vertices takes $\frac{1}{2}$ from each of its vertices.

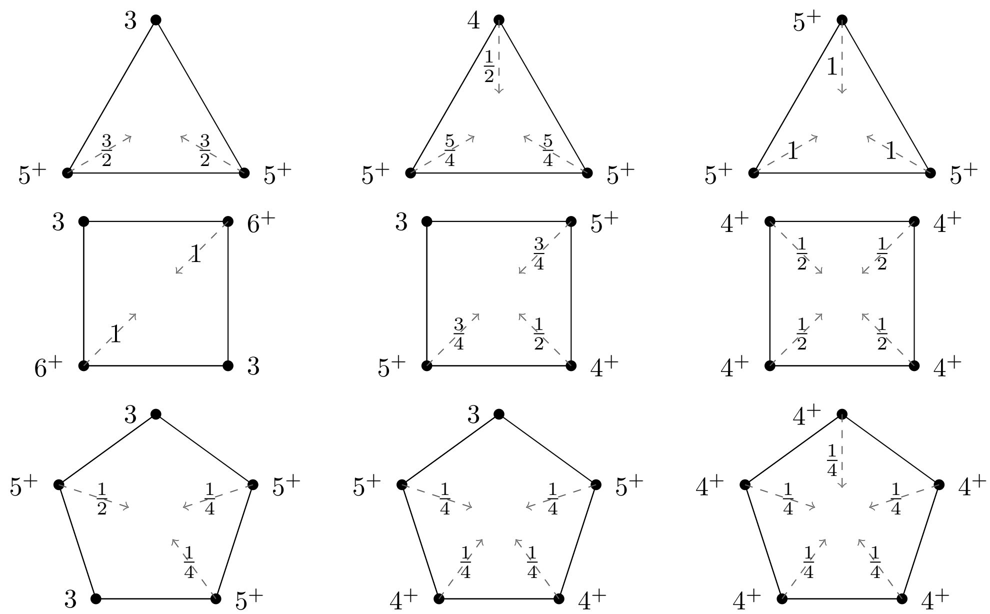


Figure 4.11: Discharging rules for 5^--faces.

(R7) A 5-face with two 3-vertices takes $\frac{1}{2}$ from their common neighbor on the face and $\frac{1}{4}$ from its two other vertices.

(R8) A 5-face with at most one 3-vertex takes $\frac{1}{4}$ from each of its 4^+-vertices.

(R9) A 6^+-face takes $\frac{1}{4}$ from each incident 4^+-vertex.

First we argue that every face ends with nonnegative charge. Let f be a face.

Suppose $\ell(f) = 3$, so $\mu(f) = -3$. Since f has at most one 4^--vertex, exactly one of (R1)–(R3) applies to f. Under that rule, f receives 3, so $\mu^*(f) = 0$.

Suppose $\ell(f) = 4$, so $\mu(f) = -2$. The cases that f has two, one, or zero 3-vertices are covered by (R4)–(R6). Under the relevant rule, f receives 2, so $\mu^*(f) = 0$.

Suppose $\ell(f) = 5$, so $\mu(f) = -1$. The cases that f has two 3-vertices or at most one 3-vertex are covered by (R7) and (R8). Under either, f receives at least 1, so $\mu^*(f) \geq 0$.

Suppose $\ell(f) \geq 6$, so $\mu(f) \geq 0$. Since faces never lose charge, and by Claim 4.2.2 f gains charge from at least one vertex, so $\mu^*(f) > 0$.

It remains to show that every vertex ends with nonnegative charge. We further show that 5^+-vertices end with positive charge. Let w be a vertex. By Claim 4.2.1, $d(w) \geq 3$.

Case 1: $d(w) = 3$.

We have $\mu(w) = 0$, and w neither gives nor receives charge. Thus $\mu^*(w) = 0$.

Figure 4.12: Cases 3b and 4b: $d(w) \in \{5, 6\}$; w has exactly one 3-neighbor.

Case 2: $d(w) = 4$.

We have $\mu(w) = 2$ and that w gives charge only under (R2), (R5), (R6), (R8) and (R9). Since w gives at most $\frac{1}{2}$ to each of four incident faces, $\mu^*(w) \geq 2 - 4 \cdot \frac{1}{2} = 0$.

Case 3: $d(w) = 5$.

We have $\mu(w) = 4$, and Claim 4.2.3 implies that if w has at least two 3-neighbors, then all neighbors of w are 3-vertices. Thus w can have zero, one, or five 3-neighbors. We break into cases based on the number of 3-neighbors of w.

Case 3a: w has no 3-neighbors.

By (R2) and (R3), w gives at most $\frac{5}{4}$ to intermediate 3-faces and at most 1 to other 3-faces; by (R5), (R6), (R8), and (R9) w gives at most $\frac{1}{2}$ to each incident 4^+-face. By Claim 4.2.6, w is on at most two 3-faces. If w is not on any intermediate 3-face, then $\mu^*(w) \geq 4 - 2 \cdot 1 - 3 \cdot \frac{1}{2} > 0$. If w is on a intermediate 3-face, then by Claim 4.2.7 it is on no other 3-face and so $\mu^*(w) \geq 4 - 1 \cdot \frac{5}{4} - 4 \cdot \frac{1}{2} > 0$.

Case 3b: w has exactly one 3-neighbor x.

Claim 4.2.7 implies that w is incident to at most one 3-face as in Figure 4.12. If w is not on a 3-face, then by (R5)–(R9), it gives at most $\frac{3}{4}$ to each incident face. Thus $\mu^*(w) \geq 4 - 5 \cdot \frac{3}{4} > 0$.

If w is on a 3-face f with x, then (R1) implies that w gives $\frac{3}{2}$ to f. By (R5)–(R9), w gives at most $\frac{3}{4}$ to the other face shared with x and at most $\frac{1}{2}$ to each other incident face. Thus $\mu^*(w) \geq 4 - 1 \cdot \frac{3}{2} - 1 \cdot \frac{3}{4} - 3 \cdot \frac{1}{2} > 0$. If w is on a 3-face f that does not contain x, then (R5)–(R9) implies that w gives at most $\frac{3}{4}$ to each face shared with x, at most $\frac{5}{4}$ to f, and at most $\frac{1}{2}$ to the remaining incident faces. Thus $\mu^*(w) \geq 4 - 2 \cdot \frac{3}{4} - 1 \cdot \frac{5}{4} - 2 \cdot \frac{1}{2} > 0$.

Case 3c: w has five 3-neighbors.

Claim 4.2.2 implies w is not on a 3-face and Claim 4.2.3 implies w is not on a 4-face, because such a face would be expensive. So by (R7)–(R9) w gives at most $\frac{1}{2}$ to each face, and thus $\mu^*(w) \geq 4 - 5 \cdot \frac{1}{2} > 0$.

Case 4: $d(w) = 6$.

We have $\mu(w) = 6$. Claim 4.2.3 implies that w does not have exactly two or three 3-neighbors. We break into cases based on the number of 3-neighbors of w.

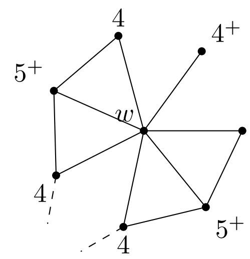


Figure 4.13: Case 5a: $d(w) = 7$; w has no 3-neighbors; subcase where w is on four intermediate 3-faces.

Case 4a: w has no 3-neighbors.

Claim 4.2.8 implies that w is on at most four 3-faces. By (R3), w gives 1 to each incident 3-face and by (R5), (R6), (R8), and (R9) w gives at most $\frac{1}{2}$ to each other incident face. Thus $\mu^*(w) \geq 6 - 4 \cdot 1 - 2 \cdot \frac{1}{2} > 0$.

Case 4b: w has exactly one 3-neighbor x.

Claim 4.2.7 implies that w is incident to at most one 3-face, as in Figure 4.12. If w is on a 3-face f, then f is incident to x, and by (R1) w gives $\frac{3}{2}$ to f. By (R5) and (R7)–(R9), w gives at most $\frac{3}{4}$ to the other incident face containing x and at most $\frac{1}{2}$ to each other incident face. So $\mu^*(w) \geq 6 - 1 \cdot \frac{3}{2} - 1 \cdot \frac{3}{4} - 4 \cdot \frac{1}{2} > 0$. If w has no incident 3-faces, then by (R5)–(R9) w gives at most $\frac{3}{4}$ to the faces with x and at most $\frac{1}{2}$ to each other incident face. Thus $\mu^*(w) \geq 6 - 2 \cdot \frac{3}{4} - 4 \cdot \frac{1}{2} > 0$.

Case 4c: w has k 3-neighbors with $k \geq 4$.

Claim 4.2.3 implies that w is on at most $k - 5$ expensive faces.

If $k \in \{4, 5\}$, then w is on at most one 3-face ($k = 4$), or one expensive face ($k = 5$). If w is on these faces, w gives them at most $\frac{3}{2}$ and w gives at most $\frac{3}{4}$ to any other incident face. Thus $\mu^*(v) \geq 6 - 1 \cdot \frac{3}{2} - 5 \cdot \frac{3}{4} > 0$.

If $k = 6$, then by Claim 4.2.2 any expensive faces incident to v are 4-faces. So w gives at most 1 to each, and at most $\frac{3}{4}$ to any other incident face. Thus $\mu^*(v) \geq 6 - 2 \cdot 1 - 4 \cdot \frac{3}{4} > 0$.

Case 5: $d(w) = 7$.

We have $\mu(w) = 8$, and Claim 4.2.3 implies that w does not have exactly two 3-neighbors. We break into cases based on the number of 3-neighbors of w.

Case 5a: w has no 3-neighbors.

By (R2), w gives at most $\frac{5}{4}$ to a intermediate 3-face, and w is incident to at most 4 such faces. By (R3),and (R5)–(R9), w gives at most 1 to each other incident face. Also, if w is on four intermediate 3-faces (Figure 4.13), then Claim 4.2.5 implies that at least one of the other faces containing w is not a 3-face and hence by (R5)–(R9) takes at most $\frac{1}{2}$ from w. Thus either $\mu^*(w) \geq 8 - 3 \cdot \frac{5}{4} - 4 \cdot 1 > 0$ or $\mu^*(w) \geq 8 - 4 \cdot \frac{5}{4} - 2 \cdot 1 - 1 \cdot \frac{1}{2} > 0$ based on whether or not w is on four intermediate 3-faces.

Case 5b: w has exactly one 3-neighbor x.

Figure 4.14: Case 5b: $d(w) = 7$; w has exactly one 3-neighbor x.

Figure 4.15: Case 5b continued: $d(w) = 7$; w has exactly one 3-neighbor.

Claim 4.2.7 implies that at most one 3-face is incident to w and x and that this is the only possible 3-face containing w. If such a face exists (Figure 4.14(i)), then w gives $\frac{3}{2}$ to it by (R1). By (R5)–(R9), w gives at most $\frac{3}{4}$ to any other face incident to x and w. Claim 4.2.6 implies that w is incident to at most four intermediate 3-faces. By (R2), w gives $\frac{5}{4}$ to each intermediate 3-face. Finally, by (R3) and (R6)–(R9), w gives at most 1 to each other incident face. If w is on at most three intermediate 3-faces, then $\mu^*(w) \geq 8 - 1 \cdot \frac{3}{2} - 1 \cdot \frac{3}{4} - 2 \cdot \frac{5}{4} - 2 \cdot 1 = 0$. Otherwise, w is on four intermediate 3-faces (Figure 4.14(ii)), and then neither of the faces containing x are 3-faces. Thus $\mu^*(w) \geq 8 - 2 \cdot \frac{3}{4} - 4 \cdot \frac{5}{4} - 1 \cdot 1 > 0$.

To show that $\mu^*(w) > 0$, we must only consider the case that x and w share an expensive 3-face and a 4-face, and that w is on exactly three intermediate 3-faces; possible instances of this configuration are shown in Figure 4.15. However, Claim 4.2.5 implies that if w is contained in three intermediate 3-faces, then w is contained in a 4^+-face, which takes at most $\frac{1}{2}$ from w. Thus $\mu^*(w) \geq \frac{1}{2}$.

Case 5c: w has k 3-neighbors with $k \geq 3$.

Claim 4.2.3 implies that w is on at most $k - 3$ expensive faces.

If $k = 3$, then w is on at most two intermediate 3-faces (Figure 4.16). By (R2), w gives $\frac{5}{4}$ to any intermediate 3-face and by (R3), and by (R5)–(R9) w gives at most 1 to each other incident face. Thus $\mu^*(w) \geq 8 - 2 \cdot \frac{5}{4} - 5 \cdot 1 > 0$.

If $k = 4$, then w is on at most one expensive face and at most two intermediate 3-faces. So $\mu^*(w) \geq 8 - 1 \cdot \frac{3}{2} - 2 \cdot \frac{5}{4} - 4 \cdot 1 = 0$. To show $\mu^*(w) > 0$, we need only consider the case that w is on one expensive

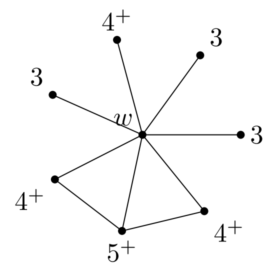


Figure 4.16: Case 5c: $d(w) = 7$; w has exactly three 3-neighbors.

3-face, no expensive 4-faces, and two intermediate 3-faces. However, if w is contained in two intermediate 3-faces, then the four 3-neighbors of w are consecutive in the cyclic order around w. Thus w does not have an expensive 3-face, and $\mu^*(w) > 0$.

If $k = 5$, then w is on at most two expensive faces. By considering the cyclic ordering of the 3-neighbors and the number of expensive faces, we have that w is on at most three 3-faces. If w is on three 3-faces, then at least one is neither expensive nor intermediate. If w is on two expensive 3-faces, then $\mu^*(w) \geq 8 - 2 \cdot \frac{3}{2} - 1 - 4 \cdot \frac{3}{4} > 0$. If w is on at most one expensive 3-face, then $\mu^*(w) \geq 8 - 1 \cdot \frac{3}{2} - 1 \cdot \frac{5}{4} - 5 \cdot 1 > 0$.

If $k = 6$, then w is on at most three expensive faces and at most one 3-face. Thus $\mu^*(w) \geq 8 - 3 \cdot \frac{3}{2} - 1 \cdot 1 - 3 \cdot \frac{3}{4} > 0$.

Finally, if $k = 7$, then Claim 4.2.2 implies that w is on no 3-faces, so $\mu^*(w) \geq 8 - 7 \cdot 1 = 1$, by (R4) and (R7).

Case 6: $d(w) = 8$.
We have $\mu(w) = 10$.

Consider the faces in cyclic order around w. A *run* of 3-faces around w is a set of 3-faces $f_1, \ldots, f_k$ such that for $i \in [k-1]$, faces f_i and f_{i+1} share an edge of the form wu for some vertex u. We consider the maximal runs of 3-faces. If there are at least three consecutive 3-faces around w, then Claim 4.2.6 implies that the 3-faces that have two adjacent 3-faces around w have only 5^+-vertices. By (R3), w gives at most 1 to these 3-faces, and by (R1)–(R3) w gives at most $\frac{3}{2}$ to other 3-faces. Lastly, by (R4)–(R9) w gives at most 1 to each incident 4^+-face. With this in mind, we break into cases by the configurations of 3-faces around w.

Case 6a: w is on at most four 3-faces.
By (R1)–(R3), w gives at most $\frac{3}{2}$ to each of them, and by (R4)–(R9) w gives at most 1 to each other incident face. Thus $\mu^*(w) \geq 10 - 4 \cdot \frac{3}{2} - 4 \cdot 1 = 0$. To show $\mu^*(w) > 0$, we need only consider the case that w is on exactly four 3-faces, all of which are expensive; possible instances of this configuration are shown in Figure 4.17. Claims 4.2.6 and 4.2.9 imply that at least one of the 4^+-faces containing w and a 3-neighbor of

Figure 4.17: Case 6a: $d(w) = 8$; subcase where w is on exactly four 3-faces, all of which are expensive.

Figure 4.18: Case 6b: $d(w) = 8$; subcase where w is on at least five 3-faces.

w takes at most $\frac{3}{4}$ from w. Thus $\mu^*(w) \geq 10 - 4 \cdot \frac{3}{2} - 1 \cdot \frac{3}{4} - 3 \cdot 1 > 0$.

Case 6b: w has at most two maximal runs of 3-faces.

Thus $\mu^*(w) \geq 10 - 4 \cdot \frac{3}{2} - 4 \cdot 1 = 0$. Note that if w is on at least six 3-faces, then we are in this case.

To show $\mu^*(w) > 0$, we need only consider the case that w is on at least five 3-faces, there are two maximal runs of 3-faces, and the first and last 3-faces on the runs are expensive; possible instances of this configuration are shown in Figure 4.18. Claim 4.2.9 implies that the runs end with 4^+-faces that take at most $\frac{3}{4}$ from w under (R5). Thus $\mu^*(w) \geq 10 - 4 \cdot \frac{3}{2} - 1 \cdot \frac{3}{4} - 3 \cdot 1 > 0$.

Case 6c: w is on exactly five 3-faces that form exactly three maximal runs.

By Claim 4.2.9, none of the 4^+-faces are expensive, so w gives at most $\frac{3}{4}$ to each. Thus $\mu^*(w) \geq 10 - 5 \cdot \frac{3}{2} - 3 \cdot \frac{3}{4} > 0$.

Case 7: w is a 9^+-vertex.

By Claim 4.2.6, the number of 3-faces containing w and a 4^--vertex is at most $\lfloor 2d(w)/3 \rfloor$. By (R1), and (R2), w gives at most $\frac{3}{2}$ to each such 3-face, and (R3)–(R9) imply that w gives at most 1 to each other incident face. If $d(w) = 9$, then Claim 4.2.9 implies that w cannot be on six expensive 3-faces and three expensive 4-faces, so $\mu^*(w) > 12 - 6 \cdot \frac{3}{2} - 3 \cdot 1 = 0$. If $d(w) \geq 10$, then

$$\mu^*(w) \geq (2d(w) - 6) - d(w) \cdot 1 - \frac{2d(w)}{3} \cdot \frac{1}{2} = \frac{2d(w)}{3} - 6 > 0.$$

Finally, we know that

$$\sum_{x \in V(G) \cup F(G)} \mu^*(x) = \sum_{x \in V(G) \cup F(G)} \mu(x) \leq 0,$$

and we have shown that every vertex and face ends with nonnegative charge. Furthermore, we have shown that only vertices of degree 3 or 4 can end with zero charge, so we conclude that every vertex of G has degree 3 or 4. Claim 4.2.4 now implies that G is 4-regular. Since every 6^+-face ends with positive charge, and Claims 4.2.5 and 4.2.10 imply that G has no 3-faces or 4-faces, we conclude that every face of G is a 5-face. However, in this case (R8) implies that every face of G ends with positive charge, a contradiction. $\square$

While we do not use this in the proof, there is no 4-regular graph that embeds on the torus with only 5-faces. From Euler's Formula, we have $|V(G)| - |E(G)| + |F(G)| = 0$. However, double counting yields $2|E(G)| = 4|V(G)|$ and $2|E(G)| = 5|F(G)|$. These three equations together have the single solution with $|V(G)| = |E(G)| = |F(G)| = 0$.

Chapter 5

Neighbor sum distinguishing total colorings

5.1 Introduction

This chapter contains joint work with Yunfang Tang. The results were independently obtained by Jakub Przybyło and appear in a joint paper with him.

For an edge-coloring c, define the *sum value* $s_c(v)$ of a vertex v by $\sum_{u\in N(v)} c(uv)$. An edge-coloring is a *proper edge-weighting* if s_c forms a proper coloring. The least k such that G has a proper k-edge-coloring that is a proper edge-weighting is the *neighbor sum distinguishing edge-chromatic number* of a graph, denoted $\chi'_\Sigma(G)$. This parameter is well defined only for graphs with no isolated edges. Clearly, $\chi'_\Sigma(G) \geq \chi'(G) \geq \Delta(G)$. Flandrin, Marczyk, Przybyło, Saclé, and Woźniak [20] conjectured that:

Conjecture 5.1.1 ([20]). *If G is a connected graph with at least three vertices other than C_5, then $\chi'_\Sigma(G) \leq \Delta(G) + 2$.*

Przybyło [37] proved an asymptotically optimal upper bound for graphs with large maximum degree. Specifically, he showed:

Theorem 5.1.2 ([37]). *If G is a connected graph with $\Delta(G)$ sufficiently large, then $\chi'_\Sigma(G) \leq \Delta(G) + 50\Delta(G)^{5/6}\ln^{1/6}\Delta(G)$.*

A *proper total k-coloring* of G is a function $c : V(G) \cup E(G) \to [k]$ such that c restricted to $V(G)$ is a proper coloring, c restricted to $E(G)$ is a proper edge-coloring, and the color on each vertex is different from the color on its incident edges. For a total coloring c, define the *sum value* $s_c(v)$ of a vertex v by $c(v) + \sum_{uv\in E(G)} c(uv)$. A total coloring is a *proper total weighting* if s_c is a proper coloring. The least k such that G has a proper total k-coloring that is a proper total weighting is the *neighbor sum distinguishing total chromatic number* of G, denoted $\chi''_\Sigma(G)$. Clearly, $\chi''_\Sigma(G) \geq \chi''(G) \geq \Delta(G) + 1$. Pilśniak and Woźniak [36] conjectured that

Conjecture 5.1.3 ([36]). *If G is a connected graph with maximum degree $\Delta(G)$, then $\chi''_\Sigma(G) \leq \Delta(G) + 3$.*

Pilśniak and Woźniak [36] proved that Conjecture 5.1.3 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. Using the Combinatorial Nullstellensatz, Wang, Ma, and Han [44] proved that the conjecture holds for triangle-free planar graphs with maximum degree at least 7. Dong and Wang [18] showed that Conjecture 5.1.3 holds for graphs with $\mathrm{Mad}(G) < 3$ and $\Delta(G) \geq 4$, and Li, Liu, and Wang [31] proved that the conjecture holds for K_4-minor-free graphs. Li, Ding, Liu, and Wang [30] also confirmed Conjecture 5.1.3 for planar graphs with maximum degree at least 13. Finally, Xu, Wu, and Xu [46] proved $\chi''_\Sigma(G) \leq \Delta(G) + 2$ for graphs G with $\Delta(G) \geq 14$ that can be embedded in a surface of nonnegative Euler characteristic.

By modifying Przybyło's proof that Conjecture 5.1.1 is asymptotically correct for graphs with large maximum degree, we confirm that Conjecture 5.1.3 is also asymptotically correct.

Theorem 5.1.4. *If G is a connected graph with $\Delta(G)$ sufficiently large, then*

$$\chi''_\Sigma(G) \leq \Delta(G) + 50\Delta(G)^{5/6} \ln^{1/6} \Delta(G).$$

Przybyło's proof uses Vizing's Theorem at one step of the production of the desired coloring. To produce a total coloring, we start with a vertex coloring and then produce a compatible edge coloring. In doing so, we use a result on list-edge coloring in place of Vizing's Theorem to guarantee the compatibility of the colorings. To use the list-edge coloring result, we increase a lower order term in the primary lemma. This increase does not alter the proof of the lemma.

5.2 Ideas

We color the vertices of the graph and produce an edge-coloring such that the combined total coloring is a proper total weighting. For a coloring g and an edge-coloring h, let (g, h) be the total coloring produced by combining g and h.

The main work is in producing the desired edge-coloring. Our Lemma 5.2.3 serves a similar purpose to Lemma 6 of Przybyło [37]. Lemma 5.2.3 guarantees (not necessarily proper) colorings c_1 and c_2 of the vertices and edges respectively. Statement (Q1) guarantees that the vertex coloring has the property that each color appears roughly the expected number of times in the neighborhood of each high degree vertex. Statement (Q2) guarantees that the edge coloring has the property that each coloring appears roughly the expected number of times on the edges incident to high degree vertices. These colorings are used to produce an initial (also improper) edge-coloring c' by $c'(uv) = c_1(u)+c_1(v)+c_2(uv)$. Statement (Q3) of Lemma 5.2.3 guarantees that the colors used by c' do not appear too often among edges incident to a fixed vertex. Finally, statement (Q4) of Lemma 5.2.3 will be used to guarantee that the final sum values for the vertices form a proper coloring. The proof uses the Lovász Local Lemma and the Chernoff Bound in the forms below.

Theorem 5.2.1 (Lovász Local Lemma [5]). *Let $A_1, \ldots, A_n$ be events in a probability space. Suppose that each event A_i is mutually independent of a set of all but at most d others of these events, and that $\Pr(A_i) \le p$ for all $1 \le i \le n$. If $ep(d+1) \le 1$, then $\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.*

Theorem 5.2.2 (Chernoff Bound [33]). *If $0 \le t \le np$, then*

$$\Pr\left(|\operatorname{BIN}(n,p) - np| > t\right) < 2e^{-t^2/3np},$$

where $\operatorname{BIN}(n,p)$ is a binomial random variable with n independent trials having success probability p.

Lemma 5.2.3 defines a function S on the vertices. While the lemma is phrased more generally, we will apply the lemma using the function in (5.12). This function represents the dominant term in the sum value of v in our final coloring.

Lemma 5.2.3. *Let H be a graph of maximum degree at most D and let $C_V = \left\lceil D^{1/6} \ln^{-1/6} D \right\rceil$ and $C_E = \left\lceil D^{1/3} \ln^{-1/3} D \right\rceil$. For a coloring $c_1\colon V(H) \to [C_V]$, let*

$$S(v) = \left(\left\lceil D^{2/3} \ln^{1/3} D \right\rceil + \left\lceil D^{1/2} \right\rceil\right) d(v) c_1(v) + R(d(v), D)$$

for all $v \in V(H)$, where R is a non-negative integer function of two variables such that $S(v) \le D^2$ for all $v \in V(H)$.

If D is sufficiently large, then there exist colorings $c_1 : V(H) \to [C_V]$ and $c_2 : E(H) \to [C_E]$, such that for every vertex $v \in V(H)$:

(Q1) if $d(v) \ge D^{5/6} \ln^{1/6} D$, then the number of vertices adjacent to v having any given color is within $3D^{5/12} \ln^{7/12} D$ of $d(v)/C_V$;

(Q2) if $d(v) \ge D^{5/6} \ln^{1/6} D$, then the number of edges incident with v having any given color is within $3D^{1/3} \ln^{2/3} D$ of $d(v)/C_E$;

(Q3) for $3 \le c \le 2C_V + C_E$, the number of edges uv incident with v such that $c_1(u) + c_1(v) + c_2(uv) = c$ is at most $2D^{1/3} \ln^{1/3} D$ if $d(v) < D^{2/3}$, or at most $D^{2/3} \ln^{1/3} D + 3D^{1/3} \ln^{2/3} D$ otherwise;

(Q4) if $d(v) \ge 3D^{5/6} \ln^{1/6} D$, then for every integer $\alpha \ge 1$, the number of neighbors u of v such that

$\frac{d}{2} \le d(u) \le 2d$ and $S(u) \in I_{d(v),\alpha,D}$ is at most at most $d\left(D^{1/6}\ln^{-1/6} D\right)^{-1} + 3D^{5/12}\ln^{7/12} D$, where

$$I_{d,\alpha,D} = \left((\alpha-1)\frac{d}{2}D^{2/3}\ln^{1/3} D, \alpha\frac{d}{2}D^{2/3}\ln^{1/3} D\right].$$

We will prove Lemma 5.2.3 in section Section 5.3. Here we give a sketch of the argument. The only difference between Lemma 6 of [37] and Lemma 5.2.3 is that Przybyło writes:

$$S(v) = \left(\left\lceil D^{2/3}\ln^{1/3} D\right\rceil + 4\left\lceil D^{1/3}\ln^{2/3} D\right\rceil\right) dc_1(v) + R(d, D)$$

while we have

$$S(v) = \left(\left\lceil D^{2/3}\ln^{1/3} D\right\rceil + \left\lceil D^{1/2}\right\rceil\right) dc_1(v) + R(d, D).$$

Our increase to the lower order term in $S(v)$ accommodates an increase in the number of possible colors used on the edges and is made to permit a total coloring.

Start with colorings c_1 and c_2 where the color assigned to each vertex and edge is chosen independently and uniformly at random from $[C_V]$ and $[C_E]$ respectively. For each vertex v, define four events corresponding to v violating each of (Q1), (Q2), (Q3), and (Q4). For each bad event A among these, because of the way that C_V and C_E are defined in terms of D, the Chernoff bound shows that the probability of A is less than $D^{-5/2}$. Since all events for a vertex v are mutually independent of those corresponding to vertices having distance at least 3 from v, each event is mutually independent of all but at most $3 + 4D^2$ events. Finally, since

$$eD^{-5/2}(4 + 4D^2) < 1,$$

the Lovász Local Lemma gives that there is some selection of c_1 and c_2 such that none of the events occur.

To form a total coloring, we start with a coloring of the vertices and extend it to a total coloring. To guarantee that the total coloring is proper, we use a result of Molloy and Reed [34]. A *list assignment* L for $E(G)$ assigns to each vertex e a list $L(e)$ of permissible colors. Given a list assignment L for the edges of G, if a proper edge-coloring c can be chosen so that $c(e) \in L(v)$ for all $e \in E(G)$, then we say that G is *L-edge-colorable.* The *list edge-chromatic number* $\chi'_\ell(G)$ of G is the least k such that G is L-edge-colorable for any list assignment L satisfying $|L(e)| \ge k$ for all $e \in E(G)$.

Theorem 5.2.4 (Molloy and Reed [34])**.** *There is a constant k such that for every graph G,*

$$\chi'_\ell(G) \le \Delta(G) + k\Delta(G)^{1/2}(\log \Delta(G))^4.$$

The list coloring conjecture states that $\chi'_\ell(G) = \chi'(G)$ for every graph G. Since $\chi'(G) \leq \Delta(G) + 1$, Theorem 5.2.4 gives an asymptotic version of this conjecture.

5.3 Proof of Lemma 5.2.3

Let $d_0 = D^{5/6} \ln^{1/6} D$; this is the threshold for the degree of vertices to which (Q1) and (Q2) apply; (Q1) and (Q2) impose no condition on vertices of smaller degree.

For every vertex $v \in V(H)$, we choose a color $c_1(v) \in [C_v]$ independently and uniformly at random. For each edge $e \in E(H)$, we choose a color $c_2(e) \in [C_E]$ independently and uniformly at random. In the following, whenever needed we assume that D is sufficiently large.

For each vertex $v \in V(H)$ of degree d with $d \geq d_0$, let A_v^1 denote the event that for at least one value $a_1 \in [C_V]$, the number of neighbors u of v with $c_1(u) = a_1$ is outside the range in (Q1). Similarly, for each vertex $v \in V(H)$ of degree d with $d \geq d_0$, let A_v^2 denote the event for at least one value $a_2 \in [C_E]$, the number of neighbors u of v with $c_2(uv) = a_2$ is outside the range in (Q2). For $a_1 \in [C_V]$ and a_2 in $[C_E]$, let X_{v,a_1} and Y_{v,a_2} be the random variables of the number of neighbors u of v with $c_1(u) = a_1$ and $c_2(uv) = a_2$ respectively. We have $X_{v,a_1} \sim \mathrm{BIN}(d, 1/C_V)$ and $Y_{v,a_2} \sim \mathrm{BIN}(d, 1/C_E)$. By the Chernoff Bound, for $d \geq d_0$,

$$\begin{aligned}\Pr\left(|X_{v,a_1} - d/C_V| > 3D^{5/12} \ln^{7/12}\right) &< 2\exp(-3D^{5/6} \ln^{7/6} D\, C_V/d) \\ &\leq 2\exp(-3D \ln D/d) \leq 2\exp(-3\ln D) = 2D^{-3}\end{aligned}$$

and

$$\begin{aligned}\Pr\left(|Y_{v,a_2} - d/C_E| > 3D^{1/3} \ln^{2/3} D\right) &< 2\exp(-3D^{2/3} \ln^{4/3} D\, C_E/d) \\ &\leq 2\exp(-3D \ln D/d) \leq 2\exp(-3\ln D) = 2D^{-3}.\end{aligned}$$

Thus,

$$\Pr\left(A_v^1\right) \leq C_V \cdot 2D^{-3} < D^{-17/6} \tag{5.1}$$

and

$$\Pr\left(A_v^2\right) \leq C_E \cdot 2D^{-3} < D^{-8/3}. \tag{5.2}$$

For a vertex $v \in V(H)$ of degree d, let A_v^3 be the event that there is a value a_3 such that the number of edges uv for which $c_1(u) + c_1(v) + c_2(uv) = a_3$ is more than $D^{2/3}\ln^{1/3} D + 3D^{1/3}\ln^{2/3}$ if $d \geq D^{2/3}$ or more than $2D^{1/3}\ln^{1/3} D$ if $d < D^{2/3}$. For each value $a_3 \in \{3, \dots, 2C_V + C_E\}$, let Z_{v,a_3} be the random variable counting the edges uv with $c_1(u)+c_1(v)+c_2(uv) = a_3$. For any fixed coloring c_1 and $u \in N(v)$, the probability $c_1(u)+c_1(v)+c_2(uv) = a_3$ (that is $c_2 = a_3 - c_1(u) - c_1(v)$), is at most $1/C_E$. Since the selections of $c_2(uv)$ are independent and we get equivalent selections for any fixed c_1, for any integer $d' \geq D^{2/3}$ with $d' \geq d$, the Chernoff Bound yields

$$\begin{aligned}
\Pr\left(Z_{v,a_3} > \frac{d'}{C_E} + 3\sqrt{\frac{d'}{C_E}}\ln^{1/2} D\right) &\leq \Pr\left(\mathrm{BIN}(d', C_E^{-1}) > \frac{d'}{C_E} + 3\sqrt{\frac{d'}{C_E}}\ln^{1/2} D\right)\\
&\leq \Pr\left(\left|\mathrm{BIN}(d', C_E^{-1}) - \frac{d'}{C_E}\right| > 3\sqrt{\frac{d'}{C_E}}\ln^{1/2} D\right)\\
&< 2\exp -9\ln D/3 = 2D^{-3}.
\end{aligned}$$

Thus if $D^{2/3} \leq d \leq D$, then

$$\Pr\left(Z_{v,a_3} > 2D^{2/3}\ln^{1/3} D + 3D^{1/3}\ln^{2/3} D\right) \leq \Pr\left(Z_{v,a_3} > \frac{D}{C_E} + 3\sqrt{\frac{D}{C_E}}\ln^{1/2} D\right) < 2D^{-3},$$

while for $d < D^{2/3}$

$$\Pr\left(Z_{v,a_3} > 2D^{1/3}\ln^{1/3} D\right) \leq \Pr\left(Z_{v,a_3} > \frac{\lfloor D^{2/3}\rfloor}{C_E} + 3\sqrt{\frac{\lfloor D^{2/3}\rfloor}{C_E}}\ln^{1/2} D\right) < 2D^{-3}.$$

Thus, regardless of the degree of v,

$$\Pr\left(A_v^3\right) < (2C_V + C_E)2D^{-3} \leq D^{-8/3} \tag{5.3}$$

for D sufficiently large.

For a vertex v of degree d with $d \geq 3d_0$, let A_v^4 be the event corresponding to the violation of (Q4). That is, for some integer $\alpha \in \left[\lceil \frac{2D^{4/3}}{d \ln^{1/3} D} \rceil\right]$, the number of neighbors u of v with $\frac{d}{2} \le d(u) \le 2d$ and $S(u) \in I_{d,\alpha,D}$ is greater than

$$\frac{d}{D^{1/6} \ln^{-1/6} D} + 3D^{5/12} \ln^{7/12} D.$$

Note that for $\alpha \ge \lceil \frac{2D^{4/3}}{d \ln^{1/3} D} \rceil + 1$, we have $I_{d,\alpha,D} \subset (D^2, \infty)$. Because we assumed that $S(u) \le D^2$, allowing $\alpha \in \left[\lceil \frac{2D^{4/3}}{d \ln^{1/3} D} \rceil\right]$ is sufficient.

For a given vertex v of degree d with $d \ge d_0$ and value $\alpha \in \left[\lceil \frac{2D^{4/3}}{d \ln^{1/3} D} \rceil\right]$, let $W_{v,\alpha}$ be the random variable counting the vertices u such that $u \in N(v)$, $\frac{d}{2} \le d(u) \le 2d$, and $S(u) \in I_{d,\alpha,D}$.

Recall that, for each vertex u, $S(u)$ is a random variable based on $c_1(u)$. For every $u \in N(v)$ with $\frac{d}{2} \le d(u) \le 2d$, by the definitions of $I_{d,\alpha,D}$ and $S(u)$ we have

$$\Pr\left(S(u) \in I_{d,\alpha,D}\right) = \Pr\left(\left(\left\lceil D^{2/3} \ln^{1/3} D \right\rceil + \left\lceil D^{1/2} \right\rceil\right) d(u) c_1(v) \in I_{d,\alpha',D}\right)$$

for some real number α'. Here the change from α to α' "shrinks" the interval to allow us to drop the additive $R(d(u), D)$ term in $S(u)$.Since

$$\frac{\frac{d}{2} D^{2/3} \ln^{1/3} D}{\left(\left\lceil D^{2/3} \ln^{1/3} D \right\rceil + \lceil D^{1/2} \rceil\right) d(u)} \le 1,$$

the probability that $S(u) \in I_{d,\alpha,D}$ can be bounded as follows for a real number α'':

$$\Pr\left(S(u) \in I_{d,\alpha,D}\right) \le \Pr\left(c_1(u) \in (\alpha'' - 1, \alpha'']\right) = \Pr\left(c_1(u) = \lfloor \alpha'' \rfloor\right) \le \frac{1}{C_V} \le D^{-1/6} \ln^{1/6} D.$$

Since the value of $S(u)$ depends only on the choice of $c_1(u)$, in our random process, the random variables $S(u)$ for $u \in V(H)$ are independent. Therefore, by the Chernoff Bound,

$$\begin{aligned}
&\Pr\left(W_{v,\alpha} > \frac{d}{D^{1/6} \ln^{-1/6} D} + 3D^{5/12} \ln^{7/12} D\right) \\
&\le \Pr\left(\mathrm{BIN}(d, D^{-1/6} \ln^{1/6} D) > \frac{d}{D^{1/6} \ln^{-1/6} D} + 3D^{5/12} \ln^{7/12} D\right) \\
&\le \Pr\left(\left|\mathrm{BIN}(d, D^{-1/6} \ln^{1/6} D) - \frac{d}{D^{1/6} \ln^{-1/6} D}\right| > 3D^{5/12} \ln^{7/12} D\right) \\
&< 2\exp(-3D^{5/6} \ln^{7/6} D \frac{D^{1/6} \ln^{-1/6} D}{d}) = 2\exp(-3D \ln D / d) \le 2\exp(-3 \ln D) = 2D^{-3}.
\end{aligned}$$

Hence for $d \geq 3d_0$,

$$\begin{aligned}\Pr\left(A_v^4\right) &\leq \sum_{\alpha=1}^{\left\lceil \frac{2D^{4/3}}{d\ln^{1/3} D}\right\rceil} \Pr\left(W_{v,\alpha} > \frac{d}{D^{1/6}\ln^{-1/6} D} + 3D^{5/12}\ln^{7/12} D\right) \\ &< \left\lceil \frac{2D^{4/3}}{d\ln^{1/3} D}\right\rceil \cdot 2D^{-3} \leq \left\lceil \frac{2D^{4/3}}{3d_0\ln^{1/3} D}\right\rceil \cdot 2D^{-3} \leq D^{-5/2}. \qquad (5.4)\end{aligned}$$

Note that since each of the events A_v^1, A_v^2, A_v^3 and A_v^4 depends only on the random colors of v, its neighbors, and its incident edges, each event corresponding to v is mutually independent of all other events corresponding to vertices v' at distance at least 3 from v. Hence the events are mutually independent of all others except at most $3 + 4D^2$ other events. Moreover, by (5.1), (5.2), (5.3), and (5.4), the probability of each event is at most $D^{-5/2}$. Since

$$eD^{-5/2}(4 + 4D^2) < 1,$$

by the Lovász Local Lemma we obtain

$$\Pr\left(\bigcap_{v\in V(H), i\in[4]} \overline{A_v^i}\right) > 0.$$

We can thus select colorings c_1 and c_2 satisfying (Q1) – (Q4).

5.4 Proof of Theorem 5.1.4

We first give an outline of the proof.

Suppose that $g : V(G) \to [\Delta(G) + 1]$ is a proper coloring of a graph G. We work to produce a proper edge-coloring h such that (g, h) is a proper total weighting. Let M be a maximal matching in G. Producing h takes three steps: the first two steps focus on producing an edge-coloring of $G - M$, and the final step assigns colors to M.

More specifically, in Step 1, we use Theorem 5.2.4 to define an edge-coloring h_1 for $E(G) - M$ such that (g, h_1) is a proper total coloring of $G - M$. To do this, we using the (improper) colorings from Lemma 5.2.3, a "stretch factor" to distribute the colors, and Theorem 5.2.4 to provide a proper coloring. In Step 2, we modify h_1 to obtain an edge-coloring h_2 on $E(G) - M$ so that (g, h_2) is a proper total coloring and $s_{(g,h_2)}(u) \neq s_{(g,h_2)}(v)$ whenever $uv \in M$. In Step 3, we extend h_2 to M to yield a coloring h of $E(G)$ such that (g, h) is a proper total coloring that is also a proper total weighting of G. Extending the coloring to M changes the values on the vertices incident to an edge in M. Since every edge uv is either in M (and therefore u and v have distinct sum values by Step 2) or has that one of the endpoints is incident to an edge in M, extending the coloring to M will be sufficient to make (g, h) a proper total weighting.

Proof. Let G be a graph with maximum degree D. Let M be a maximal matching in G, and define G' by $V(G') = V(G)$ and $E(G') = E(G) - M$. Let $g : V(G) \to [D+1]$ be a proper coloring of G (and thus of G'). In the following, we use $d'(v)$ for the degree of v in G'.

Let $C_V = \left\lceil D^{1/6} \ln^{-1/6} D \right\rceil$ and $C_E = \left\lceil D^{1/3} \ln^{-1/3} D \right\rceil$. These are the numbers of colors used in the coloring c_1 and the edge-coloring c_2 guaranteed by Lemma 5.2.3. Let $d_0 = D^{5/6} \ln^{1/6} D$ so that d_0 is the the degree threshold in Lemma 5.2.3 (Q4). Let $C_M = \lceil 47 d_0 \rceil$; we will color M from $[C_M]$. The dominant term in the "stretch factor" used to produce a proper edge-coloring is $\left\lceil D^{2/3} \ln^{1/3} D \right\rceil$, which we abbreviate as C.

Step 1: The coloring h_1 for $E(G) - M$ is defined in several phases that guarantee (g, h_1) is a proper total coloring of $G - M$. Our argument follows that of Sections 5.1 and 5.2 in [37], with modifications to produce a total coloring rather than an edge-coloring.

Let $c_1 : V(G') \to [C_V]$ and $c_2 : E(G') \to [C_E]$ be the colorings of $V(G')$ and $E(G')$ guaranteed by Lemma 5.2.3, where the function $R(d, D)$ used to define S in the application of the lemma agrees with the function S defined in (5.12).

Give $uv \in E(G')$ a tentative color $c'(uv)$ defined by

$$c'(uv) = [c_1(u) + c_1(v) + c_2(uv)] \left(C + \left\lceil D^{1/2} \right\rceil \right) + C_M.$$

This coloring is not a proper edge-coloring. However, by Lemma 5.2.3 (Q3), the colors are distributed so that we will be able to modify them to produce a proper edge-coloring h_1. The additive factor of C_M means that the colors 1 through C_M are not used; these colors will not be used until Step 3, when they are used on M.

For each $\beta \in \{3, \dots, 2C_V + C_E\}$, the $C + \lceil D^{1/2} \rceil$ colors starting with $\left(C + \lceil D^{1/2} \rceil\right) + C_M$ shall be called the *palette* corresponding to β. In c', only the smallest member of each color class may appear. The color of each edge in $E(G')$ will remain in the same palette in h_1 and h_2. We will define $h_1(e) = c'(e) + a_1(e)$, where $a_1(e)$ specifies which element from the palette associated with e is given to e. To this end, let $P = \{0, \dots, C + \lceil D^{1/2} \rceil - 1\}$. We divide P into lower and upper portions P^- and P^+ with $P^- = \{0, \dots, C + \lceil D^{1/2}/2 \rceil\}$ and $P^+ = \{C + \lceil D^{1/2}/2 \rceil + 1, \dots, C + \lceil D^{1/2} \rceil - 1\}$. Note that in h_1 only the lower portion of the elements from each palette is used. The remaining colors in the color class are used in Step 2.

In our specification of the final coloring h, we will have $h(e) - c'(e) \leq C + \lceil D^{1/2} \rceil$. Thus, if D is sufficiently large, then

$$h(e) \leq (2C_V + C_E)(C + \left\lceil D^{1/2} \right\rceil) + C_M < D + 50D^{5/6} \ln^{1/6} D = D + o(D). \tag{5.5}$$

To choose $a_1(e)$, we give a list assignment and use Theorem 5.2.4. Let G_β be the spanning subgraph of G' with $E(G_\beta) = \{e \in E(G') : c'(e) = \beta\}$. By Lemma 5.2.3 (Q3), $\Delta(G_\beta) \leq D^{2/3} \ln^{1/3} D + 3D^{1/3} \ln^{2/3} D$. For an edge uv, let $T(uv)$ be the colors on u and v in the palette associated with uv, that is, $T(uv) = \{g(u) - c'(uv), g(v) - c'(uv)\}$. To guarantee that edges receive colors distinct from the colors of their endpoints, let $L(uv) = P^- - T(uv)$. For D sufficiently large, we have

$$|L(uv)| \geq C + \left\lceil D^{1/2}/2 \right\rceil - 2 \geq \Delta(G_\beta) + k\Delta(G_\beta)^{1/2} \log^6(\Delta(G_\beta).$$

Let $a_{1,\beta}$ be the L-edge-coloring for G_β guaranteed by Theorem 5.2.4, and let a_1 agree with $a_{1,\beta}$ for every β.

The definition of a_1 guarantees that under h_1 no color is used on two incident edges. Thus h_1 is a proper edge-coloring. Furthermore, we have $h_1(uv) \notin \{g(u), g(v)\}$, so (g, h_1) is a proper total coloring of G'.

Step 2: This step has two phases, with no substantial difference between our argument and that of Section 5.3 in [37]. In this step, we produce a coloring h_2 on $E(G) - M$ so that (g, h_2) is a proper total coloring and $s_{(g,h_2)}(u) \neq s_{(g,h_2)}(v)$ whenever $uv \in M$. Since Step 3 will extend h_2 to $E(G)$ but not otherwise modify h_2, this step is necessary to guarantee that endpoints of edges in M receive distinct sum values in our final coloring.

First we choose a subgraph of G' containing edges incident with all edges of relatively large degree. Independently for every vertex of degree larger than $D^{2/3}$, we choose one of its incident edges randomly and uniformly. Let H be the graph with $V(H) = V(G')$ and the edges chosen. Note that $d_H(v) \geq 1$ for such vertices. Let F_v be the event $d_H(v) - 1 > 2D^{1/3}$. Given a vertex $v \in V(G')$ of degree d, the probability that an edge $uv \in E(G')$ was chosen by a neighbor u of v is at most $D^{-2/3}$. Taking into account the one more edge that may have been chosen by v, the Chernoff Bound yields

$$\begin{aligned}
\Pr(F_v) &\leq \Pr\left(\text{BIN}(d, D^{-2/3}) > 2D^{1/3}\right) \\
&\leq \Pr\left(\text{BIN}(D, D^{-2/3}) > 2D^{1/3}\right) \\
&\leq \Pr\left(\left|\text{BIN}(D, D^{-2/3}) - D^{1/3}\right| > D^{1/3}\right) \\
&< 2\exp(-D^{1/3}/3) \leq D^{-3}
\end{aligned}$$

Note that F_v is mutually independent of all events F_u for u having distance at least three from v, that is, of all but at most D^2 other events. By the Lovász Local Lemma, we can thus choose these edges so that, for every $v \in V(G)$,

$$d_H(v) - 1 \leq 2D^{1/3} \leq \frac{1}{2}D^{1/2} - 4.$$

Fix a graph H such that each vertex has a specified incident edge and $d_H(v) - 1 \leq \frac{1}{2}D^{1/2} - 4$. We examine the edges of H one by one (in any order). When we reach the last edge of H incident with any edge (or two edges) of M, we modify the color on it. Let uv be this edge. We pick $a_2'(uv) \in P^+ - T(uv)$ so that replacing $a_1(uv)$ with $a_2'(uv)$ for such an edge uv maintains a proper total coloring. The bound on the maximum degree of H makes this possible. Let $a_2'(e) = a_1(e)$ for all other edges, and let $h_2'(e) = c'(e)+a_2'(e)$. The coloring (g, h_2') assigns distinct sum values to endpoints of edges in M as long as one of the vertices has large degree.

Let M' be the set of edges $uu' \in M$ such that $s_{(g,h_2')}(u) = s_{(g,h_2')}(u')$. For each edge $uu' \in M'$, we pick an edge $uw \in E(G')$ incident with uu' in G. By the choice of h_2', we have $d'(u) < D^{2/3}$, so by Lemma 5.2.3 (Q3) there are at most $2D^{1/3}\ln^{1/3} D$ members of the palette associated to $c'(uw)$ incident with u and at most $C + 3D^{1/3}\ln^{2/3} D$ members of the color class $c'(uu')$ incident with w. We may thus easily pick $a_2(uw) \in P - T(uw)$ so that the values of u and u' are different and that if $ww' \in M$, then the values of w and w' are also different.

For all other edges, set $a_2(e) = a_2'(e)$ and let $h_2(e) = c'(e) + a_2(e)$.

Step 3: This step follows the argument in Section 5.4 of [37]. In this step, we extend h_2 to M to yield a coloring h of $E(G)$ such that (g, h) is a proper total coloring that is also a proper total weighting of G.

Before we define h, we need to know (roughly) the current sum value of the vertices. For v be a vertex of degree d with $d \geq d_0$:

$$\begin{aligned}
s(v) &= g(v) + \sum_{u\in N(v)} h_2(uv) \\
&= g(v) + \sum_{u\in N(v)} \Big([c_1(u) + c_1(v) + c_2(uv)](C + \lceil D^{1/2}\rceil) + C_M + a_2(uv)\Big) \\
&= g(v) + dC_M + \sum_{u\in N(v)} (a_2(uv)) + \left(dc_1(v) + \sum_{u\in N(v)} c_1(u) + \sum_{u\in N(v)} c_2(uv)\right)(C + \lceil D^{1/2}\rceil). \qquad (5.6)
\end{aligned}$$

We bound the sums in (5.6). By our selection of $a_2(uv)$, we have

$$0 \leq \sum_{u\in N(v)} a_2(u) \leq D(C + \lceil D^{1/2}\rceil). \qquad (5.7)$$

By Lemma 5.2.3 (Q1), we can write

$$\begin{aligned}\sum_{u\in N(v)} c_1(u) &= \sum_{i=1}^{C_E}\left(\frac{d}{C_E}+f_{1,i}(v)\right) i \\ &= \left(\frac{d}{C_E}+f_1(v)\right)\binom{C_E+1}{2}, \end{aligned} \tag{5.8}$$

where $f_{1,i}(v)$ and $f_1(v)$ result from the error terms in Lemma 5.2.3 (Q1). From this,

$$|f_1(v)| \le 3D^{5/12}\ln^{7/12} D. \tag{5.9}$$

Finally, by Lemma 5.2.3 (Q1), we can write

$$\begin{aligned}\sum_{u\in N(v)} c_2(uv) &= \sum_{i=1}^{C_V}\left(\frac{d}{C_E}+f_{2,i}(v)\right) i \\ &= \left(\frac{d}{C_V}+f_2(v)\right)\binom{C_V+1}{2}, \end{aligned} \tag{5.10}$$

where $f_2, i(v)$ and $f_2(v)$ result from the error terms in Lemma 5.2.3 (Q2). From this,

$$|f_2(v)| \le 3D^{1/3}\ln^{2/3} D. \tag{5.11}$$

By (5.6), (5.8), and (5.10), we may break $s(v)$ into a dominant term $S(v)$ and an error term $F(v)$. Specifically, we write $s(v) = S(v) + F(v)$ where

$$S(v) = g(v) + dC_M + \left(C + \left\lceil D^{1/2}\right\rceil\right) \times \left[dc_1(v) + \frac{d}{C_V}\binom{C_V+1}{2} + \frac{d}{C_E}\binom{C_E+1}{2}\right] \tag{5.12}$$

and

$$F(v) = \sum_{u\in N(v)} a_2(uv) + \left(C + \left\lceil D^{1/2}\right\rceil\right) \times \left[f_1(v)\binom{C_V+1}{2} + f_2(v)\binom{C_E+1}{2}\right].$$

We defined R so that this $S(v)$ is the one needed to apply Lemma 5.2.3. Note that $S(v) \le D^2$ when D is sufficiently large.

By (5.7), (5.9), and (5.11), we have

$$\begin{aligned}|F(v)| &\le \left(C + \lceil D^{1/2}\rceil\right)\left(D + 3\binom{C_E+1}{2}D^{5/12}\ln^{7/12} D + 3\binom{C_V+1}{2}D^{1/3}\ln^{2/3} D\right) \\ &= \left(\frac{5}{2}+o(1)\right) D^{5/3}\ln^{1/3} D. \end{aligned} \tag{5.13}$$

Thus (5.12) implies that

$$\frac{3}{4}\frac{d'(v)D}{2} < s_{(g,h_2)}(v) < \frac{5}{2}\frac{d'(v)D}{2} \tag{5.14}$$

when D is sufficiently large.

Consequentially, every vertex v of degree d with $d \geq 3d_0$ will have a sum value distinct from its neighbors u of degree at least d_0 satisfying $d'(u) \leq \frac{d}{2}$ or $d'(u) \geq 2d$, even if we later increase the sum values of some of the vertices by (an irrelevant additive factor of) at most C_M. Moreover, by (5.5), the sum value of every neighbor u of v with $d'(u) < d_0$ shall never exceed $d'(u)\cdot(D+50D^{5/6}\ln^{1/6} D)+C_M < \frac{3}{8}dD$. Thus, by (5.14), the only neighbors u of v whose sum values might eventually land in conflict with the sum value of v are those with $\frac{d}{2} \leq d'(u) \leq 2d$. However, only some part of these might, at this point, have their sum values close enough to the sum values of v to threaten a conflict with v. We obtain an upper bound for the number of such neighbors of v.

Consider a vertex v with degree d such that $d \geq 3d_0$ with a neighbor u with $\frac{d}{2} \leq d'(u) \leq 2d$ (Hence $d > d_0$). By (5.13),

$$(F(v)+C_M)+(F(u)+C_M) \leq (5+o(1))D^{5/3}\ln^{1/3} D = (10+o(1))\frac{D}{d}\left(\frac{d}{2}D^{2/3}\ln^{1/3} D\right)$$

where $\frac{d}{2}D^{2/3}\ln^{1/3} D$ is the length of one of the intervals $I_{d,\alpha,D}$ in Lemma 5.2.3 (Q4). Thus $s(v)$ may only potentially land in conflict with its neighbors u having $\frac{d}{2} \leq d'(u) \leq 2d$ such that $S(u)$ falls into one of at most $2[(10+o(1))\frac{D}{d}+1]+1 \leq (23+o(1))\frac{D}{d}$ intervals $I_{d,\alpha,D}$ for consecutive integers α. Letting U_v be the number of such neighbors, by Lemma 5.2.3 (Q4) we have

$$\begin{aligned}|U_v| &\leq \left((23+o(1))\frac{D}{d}\right)\left(\frac{d}{D^{1/6}\ln^{1/6} D}+3D^5 12\ln^7 12\right)\\ &= (23+o(1))D^{5/6}\ln^{1/6} D\\ &= (23+o(1))d_0\end{aligned}$$

To define a set U_v for each vertex, for those vertices v with $d'(v) < 3d_0$, let $U_v = N_{G'}(v)$. Trivially, $|U_v| \leq 3d_0$ for these vertices.

Finally, we choose colors from $[C_M]$ for edges of M so that all neighbors in G have distinct sum values. To do so, we analyze one by one every edge of M. For $xy \in M$, we choose the color so that each of x and y have distinct sum value from its at most $(23+o(1))d_0$ neighbors in each U_x and U_y. Then, for $uv \in E(G)$, if $uv \notin M$, the sum values of u and v are distinct, since at least one of u and v is incident to an edge in M. If $uv \notin M$, then the sum values of u and v are distinct by our choice of coloring in Step 2.

We may thus extend h_2 to h using colors in $[C_M]$ so that (g, h) is a proper total coloring that is a proper total weighting.

$\square$

References

[1] S. Akbari, M. Ghanbari, and S. Jahanbekam, *On the list dynamic coloring of graphs*, Discrete Appl. Math. **157** (2009), no. 14, 3005–3007. MR 2553387 (2010i:05289)

[2] M. Albertson, G. Chappell, H. A. Kierstead, André Kündgen, and Radhika Ramamurthi, *Coloring with no 2-colored P_4's*, Electron. J. Combin. **11** (2004), no. 1, Research Paper 26, 13pp.

[3] N. Alon, *High girth graphs with large fractional separation dimension*, draft, 2016.

[4] N. Alon and F. R. K. Chung, *Explicit construction of linear sized tolerant networks*, Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986), vol. 72, 1988, pp. 15–19. MR 975519

[5] N. Alon and J.H. Spencer, *The probabilistic method*, Wiley Series in Discrete Mathematics and Optimization, Wiley, 2011.

[6] Noga Alon, Manu Basavaraju, L. Sunil Chandran, Rogers Mathew, and Deepak Rajendraprasad, *Separation dimension of bounded degree graphs*, SIAM J. Discrete Math. **29** (2015), no. 1, 59–64. MR 3295685

[7] Kenneth Appel, Wolfgang Haken, et al., *Every planar map is four colorable. Part I: Discharging*, Illinois Journal of Mathematics **21** (1977), no. 3, 429–490.

[8] Kenneth Appel, Wolfgang Haken, John Koch, et al., *Every planar map is four colorable. Part II: Reducibility*, Illinois Journal of Mathematics **21** (1977), no. 3, 491–567.

[9] Manu Basavaraju, L. Sunil Chandran, Martin Charles Golumbic, Rogers Mathew, and Deepak Rajendraprasad, *Boxicity and separation dimension*, Graph-theoretic concepts in computer science, Lecture Notes in Comput. Sci., vol. 8747, Springer, Cham, 2014, pp. 81–92. MR 3295860

[10] ———, *Separation dimension of graphs and hypergraphs*, Algorithmica **75** (2016), no. 1, 187–204. MR 3492062

[11] Manu Basavaraju, L. Sunil Chandran, Rogers Mathew, and Deepak Rajendraprasad, *Pairwise suitable family of permutations and boxicity*, arXiv:1212.6756 [math.CO].

[12] O. V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979), no. 3, 211–236.

[13] ———, *Colorings of plane graphs: a survey*, Discrete Mathematics **313** (2013), no. 4, 517–539.

[14] Oleg V Borodin, A Kostochka, and Matthew Yancey, *On 1-improper 2-coloring of sparse graphs*, Discrete Mathematics **313** (2013), no. 22, 2638–2649.

[15] Y. Bu, D. Cranston, M. Montassier, A. Raspaud, and W. Wang, *Star coloring of sparse graphs*, J. Graph Theory **62** (2009), no. 3, 201–219.

[16] M. Chen, S. Kim, A. Kostochka, D. B. West, and X. Zhu, *Decomposition of sparse graphs into forests: The nine dragon tree conjecture for $k \leq 2$*, arXiv:1502.04755 [math.CO].

[17] Daniel W. Cranston and Seog-Jin Kim, *List-coloring the square of a subcubic graph*, J. Graph Theory **57** (2008), no. 1, 65–87. MR 2370185 (2008k:05072)

[18] AiJun Dong and GuangHui Wang, *Neighbor sum distinguishing total colorings of graphs with bounded maximum average degree*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 4, 703–709. MR 3176924

[19] Paul Erdős, Arthur L. Rubin, and Herbert Taylor, *Choosability in graphs*, Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), Congress. Numer., XXVI, Utilitas Math., Winnipeg, Man., 1980, pp. 125–157. MR 593902 (82f:05038)

[20] Evelyne Flandrin, Antoni Marczyk, Jakub Przybyło, Jean-François Saclé, and Mariusz Woźniak, *Neighbor sum distingishing index*, Graphs Combin. **29** (2013), no. 5, 1329–1336. MR 3091391

[21] Martin Charles Golumbic, Rogers Mathew, and Deepak Rajendraprasad, *Induced separation dimension*, preprint.

[22] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973), 390–408.

[23] Sogol Jahanbekam, Jaehoon Kim, Suil O, and Douglas B West, *On r-dynamic coloring of graphs*, submitted.

[24] Ross Kang, Tobias Müller, and Douglas B. West, *On r-dynamic coloring of grids*, submitted.

[25] H. A. Kierstead, A. Kündgen, and C. Timmons, *Star coloring bipartite planar graphs*, J. Graph Theory **60** (2009), no. 1, 1–10.

[26] Seog-Jin Kim, Sang June Lee, and Won-Jin Park, *Dynamic coloring and list dynamic coloring of planar graphs*, Discrete Appl. Math. **161** (2013), no. 13-14, 2207–2212. MR 3057028

[27] Alexandr Kostochka and Matthew Yancey, *A brooks-type result for sparse critical graphs*, Combinatorica **34** (2014), no. 3, 323–329.

[28] ———, *Ore's conjecture on color-critical graphs is almost true*, Journal of Combinatorial Theory, Series B **109** (2014), 73–101.

[29] A. Kündgen and C. Timmons, *Star coloring planar graphs from small lists*, J. Graph Theory **63** (2010), no. 4, 324–337.

[30] Hualong Li, Laihao Ding, Bingqiang Liu, and Guanghui Wang, *Neighbor sum distinguishing total colorings of planar graphs*, Journal of Combinatorial Optimization (2013), 1–14 (English).

[31] Hualong Li, Bingqiang Liu, and Guanghui Wang, *Neighbor sum distinguishing total colorings of K_4-minor free graphs*, Front. Math. China **8** (2013), no. 6, 1351–1366. MR 3122315

[32] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277. MR 963118

[33] M. Molloy and B. Reed, *Graph colouring and the probabilistic method*, Algorithms and Combinatorics, Springer, 2002.

[34] Michael Molloy and Bruce Reed, *Near-optimal list colorings*, Proceedings of the Ninth International Conference "Random Structures and Algorithms" (Poznan, 1999), vol. 17, 2000, pp. 376–402. MR 1801140 (2002b:05062)

[35] Bruce Montgomery, *Dynamic coloring of graphs*, ProQuest LLC, Ann Arbor, MI, 2001, Thesis (Ph.D.)–West Virginia University. MR 2702379

[36] Monika Pilśniak and Mariusz Woźniak, *On the total-neighbor-distinguishing index by sums*, Graphs and Combinatorics (2013), 1–12 (English).

[37] Jakub Przybyło, *Asymptotically optimal neighbour sum distinguishing colourings of graphs*, Random Structures & Algorithms (2014), n/a–n/a.

[38] Edward R. Scheinerman and Daniel H. Ullman, *Fractional graph theory*, Dover Publications, Inc., Mineola, NY, 2011, A rational approach to the theory of graphs, With a foreword by Claude Berge, Reprint of the 1997 original. MR 2963519

[39] Carsten Thomassen, *Every planar graph is* 5*-choosable*, J. Combin. Theory Ser. B **62** (1994), no. 1, 180–181. MR 1290638 (95f:05045)

[40] ______, *The square of a planar cubic graph is 7-colorable*, Manuscript, 2006.

[41] C. Timmons, *Star coloring high girth planar graphs*, Electron. J. Combin. **15** (2008), no. 1, Research Paper 124, 17.

[42] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors*, Diskret. Analiz (1976), no. 29 Metody Diskret. Anal. v Teorii Kodov i Shem, 3–10, 101. MR 0498216 (58 #16371)

[43] Margit Voigt, *List colourings of planar graphs*, Discrete Math. **120** (1993), no. 1-3, 215–219. MR 1235909 (94d:05061)

[44] JiHui Wang, QiaoLing Ma, and Xue Han, *Neighbor sum distinguishing total colorings of triangle free planar graphs*, Acta Math. Sin. (Engl. Ser.) **31** (2015), no. 2, 216–224. MR 3297846

[45] Gerd Wegner, *Graphs with given diameter and a coloring problem*, Tech. report, University of Dortmund, 1977.

[46] Renyu Xu, Jianliang Wu, and Jin Xu, *Neighbor sum distinguishing total coloring of graphs embedded in surfaces of nonnegative euler characteristic*, Journal of Combinatorial Optimization (2015), 1–13 (English).