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CHEBYSHEV-LIKE POLYNOMIALS, CONIC DISTRIBUTION OF
ROOTS, AND CONTINUED FRACTIONS

BY

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DISSERTATION

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ABSTRACT

In the first part of this thesis, we show that a wide range of the properties of the roots of translated Chebyshev polynomials of the first kind (call these complex numbers Chebyshev points), are illuminated by the study of geometric properties of the ellipse and conversely. This includes the following; Chebyshev points lie on certain ellipses centered at the origin, and conversely, every such ellipse contains infinitely many sets of Chebyshev points. In special cases, the products of all curvatures at Chebyshev points can be expressed in terms of Fibonacci and Lucas numbers. Sets of certain Chebyshev points interlace on an ellipse. We also show that Chebyshev points generated by Chebyshev polynomials of the second kind lie very close to certain high order algebraic curves that are “nearly” ellipses. Next, we examine roots of linear combinations of Chebyshev polynomials. Here we use continued fractions to give estimates for the roots that do not lie in the interval $(-1, 1)$. We then show the connection between polynomials with roots on concentric circles to polynomials with roots on ellipses. In particular, we construct a sequence of polynomials satisfying a fourth-order recurrence relation with a parameter c , by replacing coefficients of finite geometric series by Chebyshev polynomials in a simple way. We show that for c real, the roots of these polynomials lie on two concentric but inversely related circles. The associated $n \times n$ Hankel determinants are also determined. We conclude the first part of the thesis by studying some polynomials that are related to Chebyshev polynomials, and having their roots on hyperbolas.

In the second part of the thesis, we study continued fractions of quadratic irrationals. We construct a class of rationals and quadratic irrationals having continued fractions whose period has length $n \geq 2$, and with “small” partial quotients for which certain integer multiples have continued fractions of period 1, 2 or 4 with “large” partial quotients. We then show that numbers in the period of the new continued fraction are simple functions of the

numbers in the periods of the original continued fraction. We give generalizations of these continued fractions and show that polynomials arising from these generalizations are related to Chebyshev and Fibonacci polynomials. Finally, we examine non-periodic and non-simple continued fractions formed by two interlacing geometric series and show that in certain cases, they converge to quadratic irrationalities. This phenomenon is connected with certain sequences of polynomials whose properties we examine further.

To Sally, Samson, Ken and Tabitha with love.

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INTRODUCTION

The first part of this thesis examines the geometry of Chebyshev-like polynomials that satisfy a fourth-order linear recurrence relation, as well as linear combinations of Chebyshev polynomials. In particular, we show the connection between root distributions of these polynomials and the geometry of conics.

There has been recent progress in the study of zeros of polynomials that are rationally generated. Tran [1] describes the root distribution of polynomials generated by the rational function

$$\frac{1}{1 + B(z)t + A(z)t^n} = \sum_{n=0}^{\infty} H_n(z)t^n.$$

These satisfy a three-term recurrence relation. For more on root distribution of polynomials generated by similar rational functions, see for example [1, 2, 3]. For an arbitrary rational generating function, (i.e., polynomials that solve recurrences of arbitrary order) the problems are quite open. Here, for a real number c , we study the root distribution of the translated Chebyshev polynomials

$$T_n(x) - c \quad \text{and} \quad U_n(x) - c$$

and the polynomials

$$P_n(c, z) := \sum_{k=0}^{n-1} U_k(c/2) (z^{2n-k} + z^k) + U_n(c/2)z^n$$

arising from modifying finite geometric series by replacing coefficients with Chebyshev polynomials. We can choose c such that these polynomials satisfy four-term recurrence relations and their rational generating functions have a reducible denominator.

We will show that a wide range of the properties of the roots of translated

Chebyshev polynomials of the first kind $T_n(x) - c$, are illuminated by the study of geometric properties of the ellipse and conversely. One could in general ask, for what polynomials $p(x)$ do the roots of $p(x) - c$ lie on a “simply describable” family of curves as c varies. There are two obvious cases: if $p(x) = x^n$ then the curves are circles, and if $p(x) = (x^2 - a)^n$ then the curves are ovals of Cassini. It is less obvious that if $p(x) = T_n(x)$, the curves are ellipses. Chapter 1 answers this question in some detail. For the classical $c = 0$ case, the ellipse degenerates to the line $[-1, 1]$.

In general, for a given $p(x)$, it seems quite hard to determine for which c the roots of $p(x) - c$ lie on the boundary of a convex curve. However, asymptotic and approximate results are within reach. For $|c|$ large, it is clear that the roots are “near” to the vertices of a regular polygon with $\deg(p(x))$ vertices. For $p(x) = T_n(x)$, the convexity problem is of course solved by what was indicated above. For $p(x) = U_n(x)$, the n th Chebyshev U -polynomial, the $p(x) + c$ roots in general do not lie on an ellipse, but as we will show in Section 1.6, their arrangement is “nearly” elliptic. In fact, the roots are close to algebraic curves of the form

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\kappa^2} = \left(\frac{c(x, y)}{\lambda} \right)^{\frac{1}{n+1}},$$

where $c(x, y) = (\kappa^4 x^2 + \lambda^4 y^2)/\lambda\kappa$ is an expression that arises in the study of curvatures at Chebyshev points (roots of translated Chebyshev polynomials). In both the T and U cases, we encounter in connection with their root distribution some notable and apparently new factorizations: see equations (1.1.12) and (1.1.14). We add that some non-convex curves have also been encountered in the study of the distribution of certain modified Chebyshev polynomials. See for example [4].

In the opposite direction, given a “simply described” family of curves, is there a nicely structured family of polynomials whose roots lie on these curves? In Chapter 1, we show that for any ellipse of the form

$$E(a, b) : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we can find a translated Chebyshev polynomial whose roots are arbitrarily dense on it.

There is an overlap of some of the results in this thesis and properties of

Chebyshev polynomials studied by Kim and Lee [5]. Independently, they proved that for a constant a , the roots of

$$QT_n(a, z) := \frac{T_n(z) - T_n(a)}{z - a}$$

lie on the ellipse $x^2/a^2 + y^2/(a^2 - 1) = 1$, see Theorem 7 in [5]. This is a special case of Theorem 15 with $r = T_n(a)$ and $\alpha = 1$. It is also a special case of Theorems 2 and 3 with $\alpha = 1$. Our proof of Theorems 2 and 3 employs the “new” factorizations (1.1.12) and (1.1.14). An added advantage of our approach is in proving a similar but approximate statement for Chebyshev U-polynomials and other related polynomials whose roots lie close to the ellipses.

Two important geometric invariants are curvature and cross ratio. The cross ratio is fundamental to the study of conics, as Chasles’ Theorem [6, p. 133] indicates. Thus it is expected that the study of how Chebyshev polynomials and conic Sections interact will involve both of these concepts. In fact this interaction leads to the Fibonacci and Lucas number sequences. The connection with curvature and cross ratio is examined in Section 1.3. Other properties of the geometry of the ellipse $E(\sqrt{5}, 1)$ in relation to Chebyshev points of $T_n(x/2) - T_n(i/2)$ and $T_n(x/2) - T_n(\sqrt{5}/2)$ are studied in Section 1.2.

Let $p_n(x)$ be a polynomial of degree n with real roots, and k is a fixed real number. We call $p_n(x) - kp_{n-1}(x)$ a linear combination of the $p_n(x)$ polynomials is of the “first kind”. What can be said of the roots of $p_n(x) - kp_{n-1}(x)$ as k varies? If $p_n(x)$ is monic and k is sufficiently large, $p_n(x) - kp_{n-1}(x)$ will have a root close to k by Rouches’ Theorem. If $p_n(x)$ is a Chebyshev polynomial, we will have a real root of $p_n(x) - kp_{n-1}(x)$ outside $(-1, 1)$ and all other roots in this interval whenever $k > 1$. In Chapter 2, we use the continued fraction expansion of the largest root of $p_n(x) - kp_{n-1}(x)$, when $p_n(x)$ is the n th Chebyshev polynomial and k is an integer, to study the nature of this root lying outside $(-1, 1)$. We then give bounds for this root for any real k . We give a generalization of the “first kind” linear combination of Chebyshev polynomials $T_{n+r}(x) - kT_n(x)$ and $U_{n+r}(x) - kU_n(x)$, and show that they have r roots close to a fixed ellipse. We will give a remarkable approximation for these r roots.

Now call $T_n(x) - \frac{1}{2}kU_{n-2}(x)$ a “second kind” linear combination of Cheby-

shev polynomials. Graphs show that the Chebyshev polynomials $T_n(x) - \frac{1}{2}kU_{n-2}(x)$ have $n - 2$ real roots in $(-1, 1)$ and two roots outside this interval. What can be said of these two roots outside $(-1, 1)$? Using asymptotic approximations, we show that these two roots are close to the roots of the quadratic $4(k + 1)x^2 - (k + 2)^2$.

Circles can be conformally mapped to and from ellipses, see for example [7, § 1.4.1]. In a similar way, some polynomials with bi-concyclic roots can be mapped to polynomials with roots on ellipses in a very natural way. In Chapter 3, we construct a sequence of polynomials $P_n(c, z)$ defined by

$$P_n(c, z) := \sum_{k=0}^{n-1} U_k(c/2) (z^{2n-k} + z^k) + U_n(c/2)z^n,$$

where c is a fixed real number. It is easy to see that the $P_n(c, z)$ polynomials are just generalizations of finite geometric series, where the coefficients are replaced by Chebyshev polynomials. Indeed when $c = 2$, $P_n(c, z)$ reduces to the square of a geometric series since $U_n(1) = n + 1$. We show that the roots of $P_n(c, z)$ always lie on two concentric and inversely related circles that do not coincide unless $-2 \leq c \leq 2$, in which case the zeros lie on the unit circle. Since $P_n(c, z)$ satisfies a fourth-order recurrence relation, their $k \times k$ Hankel determinants are of interest. Note that for $k \geq 5$, Hankel the dereminants are identically zero. We give polynomial identities for the $k \times k$ Hankel determinants of $P_n(c, z)$ from which we conjecture their curious root distributions.

The main theme of Chapter 4 is to describe Chebyshev-like polynomials with roots on hyperbolas. First we construct a sequence of non-trivial polynomials from convergents of periodic continued fractions and show that under some specialised conditions, the roots of these polynomials lie on hyperbolas. We then show that any polynomial with real roots can be lifted to polynomials with roots on hyperbolas by what we call “a hyperbolic mapping” defined by

$$z \mapsto ib(z^2 + a) + c$$

where a, b, c are real and $i = \sqrt{-1}$. If a polynomial has real roots and the formula for the roots is known, we can use the hyperbolic mapping to give a formula for the roots on the hyperbola. We then give non trivial examples of Chebyshev-like polynomials with roots on hyperbolas.

In 1971, Zaremba [8] conjectured that given any integer $m > 1$, there is a constant B (say $=5$) such that for some integer $a < m$ with a relatively prime to m , the simple continued fraction $[0, c_1, c_2, \dots, c_r]$ of a/m has $c_i < B$, $i = 1, \dots, r$. This conjecture was confirmed by Bourgain and Kontorovich [9] in 2014 for a set of density one. For more on Zaremba's conjecture, see also [10, 11].

In Chapter 5, we construct rationals $N_n(a, b)/D_n(a, b)$ (a and b are fixed integers) whose continued fraction $[c_0, c_1, c_2, \dots, c_r]$ has bounded length, i.e., $r \leq 4$, and whose partial quotients c_i , $i = 1, \dots, r$, are unbounded (they depend on $N_n(a, b)$). Dividing the numerator $N_n(a, b)$ by $N_{n-1}(a, 0)$, we get a rational number that has a periodic continued fraction $[c_0, c_1, c_2, \dots, c_r]$ whose length depends on n in $N_n(a, b)$, and with partial quotients $c_i \leq a + b$.

If α is a quadratic irrational, then clearly for any positive integer N , $N\alpha$ is still a quadratic irrational and its continued fraction will be periodic. In [12], Cusick presents an algorithm for obtaining the continued fraction of $N\alpha$ and uses it to give estimates for the length of the period of the expansion of $N\alpha$ in relation to that of α . In Chapter 5, we also construct a quadratic irrational α such that when N is a Fibonacci or Lucas number, the continued fraction of $N\alpha$ has period of length 1, 2 or 4. As we will see, the length of the period of the new continued fraction depends on the parity of n , where n is the length of the period of the original continued fraction. We will prove using elementary methods that the numbers in the period of the new continued fraction are simple functions of numbers in the periods of the original continued fraction.

It is well known (by a Theorem of Lagrange, see for example [13] or [14]) that any quadratic irrational has a continued fraction expansion which is periodic after a certain stage, and conversely, every periodic continued fraction represents a quadratic irrational.

First, we introduce the continued fraction

$$F(x, y) = [x, y^{-1}, x^2, y^{-2}, x^3, y^{-3}, x^4, y^{-4}, \dots]$$

where x and y are integers. Despite its simple form, this two variable continued fraction does not seem to appear in the literature. Some preliminary computer experiments suggest;

Conjecture 1. *If x and y are positive integers, and $x \neq y$, then $F(x, y)$ is transcendental.*

However the case where $x = y$ is different. We will prove in Chapter 6 that when x is a positive integer, $F(x, x)$ is the largest root of the quadratic $z^2 - (2x - 1)z - x$.

The original motivation for this work came from the study of simple continued fractions with many large partial quotients. A classical example is the continued fraction expansion for $\exp(\frac{1}{x})$ given by Euler in the 1730's, namely;

$$\sum_{k=0}^{\infty} \frac{x^{-k}}{k!} = [1, x-1, 1, 1, 3x-1, 1, 1, 5x-1, 1, \dots, 1, (2n+1)x-1, 1, \dots]. \quad (0.0.1)$$

For x a large integer, there will be infinitely many large partial quotients. These are partial quotients that are linear in x .

The phenomenon of large partial quotients also occurs for polynomial functions of x . For example, one may simply truncate the series of (0.0.1). For another example, consider the polynomial

$$P(a, x) = x^2 + 3a(x + 1) + 3x + 5$$

where a is a very large positive integer. If we set $a = 10^6$, it has a root x_0 near $x = -1$ and a root x_1 near $x = -3000000$. The continued fraction of $-x_0$ begins

$$[1, 10^6, 3, \frac{10^6 - 1}{3}, 9, \frac{10^6 - 1}{9}, 27, \frac{10^6 - 1}{27}, 81, 12345, 1, 2, 26, 1, 2, 4114, 1, 8, \dots]. \quad (0.0.2)$$

By Lagrange's Theorem these will eventually be periodic, with perhaps a very long period. However the remarkable initial pattern suggests the study of $F(x, y)$, at least for $x = y$.

We generalize the continued fraction $F(x, x)$ and prove that for nonzero positive integers x and s , the continued fraction

$$F_k(x, s) := [x, \frac{s}{x}, x^2, \frac{s}{x^2}, x^3, \frac{s}{x^3}, x^4, \frac{s}{x^4}, \dots, x^k, \frac{s}{x^k}]. \quad (0.0.3)$$

converges to the largest root of the quadratic $sz^2 - ((s + 1)x - 1)z - x$.

We then show some (rather surprising) connections between the continued fraction $F(1, y^{-1})$ and some q -series studied by Auluck [15], and a Ramanujan q -series [16, p. 18].

Finally, a byproduct of this thesis is the augmentation of known examples

of polynomials and pairs of polynomials that have reasonably compact formulas for their resultants and discriminants. For ready reference, we provide the following list; Page 38 Theorems 24, 25, 26, 29. Page 63, Theorems 38, 41, 42, and Page 83, Theorems 50, 53 and 54.

The calculations leading up to the resultant formula of Theorem 42 are somewhat intricate. They require the introduction of a new sequence of polynomials we denote by $b_n(k)$. The $b_n(k)$ polynomials complement the $a_n(k)$ polynomials introduced by Dilcher-Stolarsky [17], and later studied also by Ismail-Gishe [18].

CHAPTER 1

TRANSLATED CHEBYSHEV POLYNOMIALS AND ELLIPSES

Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$ are sequences of orthogonal polynomials that solve the recurrence relation

$$f_{n+1}(x) = 2xf_n(x) - f_{n-1}(x) \quad (1.0.1)$$

under the initial conditions $T_0(x) = U_0(x) = 1$, $T_1(x) = x$ and $U_1(x) = 2x$. They are also defined by the explicit formulas

$$T_n(x) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right), \quad (1.0.2)$$

$$U_n(x) = \frac{1}{2\sqrt{x^2 - 1}} \left((x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1} \right), \quad (1.0.3)$$

and are generated by the rational functions

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n \quad \text{and} \quad \frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x)t^n. \quad (1.0.4)$$

It is well known [19] and [7], that Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$ have real roots in $(-1, 1)$, respectively given by

$$\cos \left(\frac{2k-1}{2n} \pi \right) \quad \text{and} \quad \cos \left(\frac{k}{n+1} \pi \right), \quad (1.0.5)$$

where $k = 1, 2, \dots, n$.

There is a close connection between Chebyshev polynomials and ellipses. The line segment joining -1 to 1 can be thought of as a degenerate ellipse with these points as foci. It is in this interval that one has the usual orthogonality properties for the Chebyshev polynomials. Chebyshev polynomials are also orthogonal inside non-degenerate two dimensional ellipses. See for example [20, p. 258] .

In this chapter we show that the roots of a translated Chebyshev polyno-

mial $T_n(x) + c$, where c is a real number, lie on an explicitly defined ellipse. We first consider two cases. First, if c is a Fibonacci or Lucas ‘type’ number then the ellipse on which the roots lie is independent of the degree of the polynomial. The second case is a generalization of the first case. Here we fix c to be any real number, and the ellipse on which the roots of $T_n(x) + c$ lie depends on the degree of the polynomial. The converse of the statements is also true. Given an ellipse $E(a, b)$ centered at the origin, we can get a sequence of translated Chebyshev polynomials whose roots are arbitrarily dense on the ellipse.

We then consider a specific case of the translated Chebyshev polynomials of the first kind and show that a wide range of the properties of the roots of these polynomials are illuminated by the study of geometric properties of the ellipse and conversely.

1.1 Translating using Fibonacci and Lucas ‘like’ numbers

In this section, we prove that for all n , the roots of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ lie on the ellipse $E(\sqrt{5}, 1) : x^2/5 + y^2 = 1$ and are dense on this curve as $n \rightarrow \infty$. We will also show that if $n \not\equiv 0 \pmod{4}$ then these polynomials have a strict interlacing on $E(\sqrt{5}, 1)$. It is easy to deduce from (1.0.2), (1.0.3) and from Binet’s formulas (see for example [21, p. 88]) that for all n , $T_n(i/2) = \frac{i^n}{2} L_n$, $T_n(\sqrt{5}/2) = \frac{1}{2} L_n$ when n even and n odd, $T_n(\sqrt{5}/2) = \frac{\sqrt{5}}{2} F_n$, where F_n and L_n are Fibonacci and Lucas numbers respectively.

We first examine the more general $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ polynomials, where λ and κ are non-zero real numbers with $\alpha^2 = \lambda^2 - \kappa^2$ also non-zero.

From the generating function of Chebyshev polynomials (1.0.4), we find that the generating functions of $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ to be respectively given by

$$\frac{(z - i\kappa)(1 - t^2)}{(\alpha - 2ikt + \alpha t^2)(\alpha - 2zt + \alpha t^2)} = \sum_{n=0}^{\infty} (T_n(z/\alpha) - T_n(i\kappa/\alpha)) t^n \quad (1.1.1)$$

and

$$\frac{(z - \lambda)(1 - t^2)}{(\alpha - 2\lambda t + \alpha t^2)(\alpha - 2zt + \alpha t^2)} = \sum_{n=0}^{\infty} (T_n(z/\alpha) - T_n(\lambda/\alpha)) t^n. \quad (1.1.2)$$

From the rational generating functions (1.1.1) and (5.5.4) it is clear that $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ satisfy a four term recurrence relation.

Theorem 2. *Let $\lambda, \kappa \in \mathbb{R}$ be non-zero with $\lambda^2 - \kappa^2 = \alpha^2 > 0$. Then for all n , the roots of $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ lie on the ellipse $E(\lambda, \kappa) : x^2/\lambda^2 + y^2/\kappa^2 = 1$.*

Proof. Starting with the given ellipse

$$E(\lambda, \kappa) : \frac{x^2}{\lambda^2} + \frac{y^2}{\kappa^2} = 1, \quad (1.1.3)$$

we have the factorization over $\mathbb{C}$ given by

$$\left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right) \left(\frac{x}{\lambda} - i\frac{y}{\kappa}\right) = 1. \quad (1.1.4)$$

We rewrite the ellipse (1.1.3) as

$$(\lambda^2 - \kappa^2) \left(\frac{x^2}{\lambda^2} + \frac{y^2}{\kappa^2}\right) = \lambda^2 - \kappa^2,$$

or

$$x^2 - y^2 - (\lambda^2 - \kappa^2) = \kappa^2 \lambda^2 \left(\frac{x^2}{\lambda^4} - \frac{y^2}{\kappa^4}\right).$$

Completing the square on the right gives

$$x^2 - y^2 + 2ixy - (\lambda^2 - \kappa^2) = \kappa^2 \lambda^2 \left(\frac{x}{\lambda^2} + i\frac{y}{\kappa^2}\right)^2,$$

or

$$z^2 - (\lambda^2 - \kappa^2) = \kappa^2 \lambda^2 \left(\frac{x}{\lambda^2} + i\frac{y}{\kappa^2}\right)^2, \quad (1.1.5)$$

where $z = x + iy$. Now

$$\begin{aligned} \frac{z + \sqrt{z^2 - (\lambda^2 - \kappa^2)}}{2} &= \frac{1}{2} \left(x + iy + \kappa \lambda \left(\frac{x}{\lambda^2} + i\frac{y}{\kappa^2} \right) \right) \\ &= \frac{\lambda + \kappa}{2} \left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right) \end{aligned} \quad (1.1.6)$$

and

$$\begin{aligned}\frac{z - \sqrt{z^2 - (\lambda^2 - \kappa^2)}}{2} &= \frac{1}{2} \left(x + iy - \kappa \lambda \left(\frac{x}{\lambda^2} + i \frac{y}{\kappa^2} \right) \right) \\ &= \frac{\lambda - \kappa}{2} \left(\frac{x}{\lambda} - i \frac{y}{\kappa} \right).\end{aligned}\tag{1.1.7}$$

To simplify notation, let

$$\phi = \frac{\lambda + \kappa}{2}, \quad \text{and} \quad \hat{\phi} = \frac{\lambda - \kappa}{2},$$

and recall that

$$T_n(z) = \frac{1}{2} \left((z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n \right).$$

Now define

$$\tau_n(z) := \frac{1}{2} \left(\left(\frac{z + \sqrt{z^2 - (\lambda^2 - \kappa^2)}}{2} \right)^n + \left(\frac{z - \sqrt{z^2 - (\lambda^2 - \kappa^2)}}{2} \right)^n \right).\tag{1.1.8}$$

It is easy to see that

$$T_n(z/\alpha) = (2/\alpha)^n \tau_n(z),\tag{1.1.9}$$

and

$$\tau_n(i\kappa) = \frac{i^n}{2} \left(\phi^n + (-\hat{\phi})^n \right).\tag{1.1.10}$$

It follows that

$$\tau_n(z) - \tau_n(i\kappa) = \frac{1}{2} \left[\phi^n \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - i^n \right) + \hat{\phi}^n \left(\left(\frac{x}{\lambda} - i \frac{y}{\kappa} \right)^n - (-i)^n \right) \right].\tag{1.1.11}$$

Now multiply both sides of (1.1.11) by $\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n$, and use

$$\begin{aligned}\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n \left(\left(\frac{x}{\lambda} - i \frac{y}{\kappa} \right)^n - (-i)^n \right) &= 1 - (-i)^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n \\ &= -(-i)^n \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - i^n \right).\end{aligned}$$

The result is the remarkable factorization

$$\begin{aligned}
& \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n (\tau_n(z) - \tau_n(i\kappa)) \\
&= \frac{\phi^n}{2} \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - i^n \right) - \frac{i^n (-\hat{\phi})^n}{2} \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - i^n \right) \\
&= \frac{\phi^n}{2} \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - i^n \right) \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - i^n \left(\frac{-\hat{\phi}}{\phi} \right)^n \right),
\end{aligned}$$

which can be written as

$$\tau_n(z) - \tau_n(i\kappa) = \frac{\phi^n}{2} \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - i^n \right) \left(1 - i^n \left(\frac{-\hat{\phi}}{\phi} \right)^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^{-n} \right). \quad (1.1.12)$$

For $\kappa, \lambda > 0$ and z on the ellipse (1.1.3), the last sum has the form $1 - u$ where $|u| = \left| \hat{\phi}/\phi \right|^n < 1$ since from (1.1.3), $\left| \frac{x}{\lambda} + i \frac{y}{\kappa} \right| = 1$. By the triangle inequality, the last factor of (1.1.12) is non-zero, so $\tau_n(z) - \tau_n(i\kappa) = 0$ is equivalent to $\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n = i^n$. In particular this means that each root satisfies $\left| \frac{x}{\lambda} + i \frac{y}{\kappa} \right| = 1$, and hence lies on the ellipse (1.1.3). Similarly, from equation (1.1.8)

$$\tau_n(\lambda) = \frac{1}{2} \left(\phi^n + \hat{\phi}^n \right).$$

It follows that

$$\tau_n(z) - \tau_n(\lambda) = \frac{1}{2} \left[\phi^n \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - 1 \right) + \hat{\phi}^n \left(\left(\frac{x}{\lambda} - i \frac{y}{\kappa} \right)^n - 1 \right) \right]. \quad (1.1.13)$$

Now multiply both sides of (1.1.13) by $\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n$, and use

$$\begin{aligned}
\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n \left(\left(\frac{x}{\lambda} - i \frac{y}{\kappa} \right)^n - 1 \right) &= 1 - \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n \\
&= - \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - 1 \right).
\end{aligned}$$

Again, we get a remarkable factorization

$$\begin{aligned}
& \left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n (\tau_n(z) - \tau_n(\lambda)) \\
&= \frac{\phi^n}{2} \left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n \left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n - 1\right) - \frac{\hat{\phi}^n}{2} \left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n - 1\right) \\
&= \frac{\phi^n}{2} \left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n - 1\right) \left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n - \left(\frac{\hat{\phi}}{\phi}\right)^n\right),
\end{aligned}$$

which can be written as

$$\tau_n(z) - \tau_n(\lambda) = \frac{\phi^n}{2} \left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n - 1\right) \left(1 - \left(\frac{\hat{\phi}}{\phi}\right)^n \left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^{-n}\right). \quad (1.1.14)$$

For $\lambda > 0$ and z on the ellipse (1.1.3), the last sum has the form $1 - u$ where $|u| = \left|\hat{\phi}/\phi\right|^n < 1$. By the triangle inequality, the last factor of (1.1.14) is non-zero, and hence $\tau_n(z) - \tau_n(\lambda) = 0$ is equivalent to $\left(\frac{x}{\lambda} + i\frac{y}{\kappa}\right)^n = 1$. In particular this means that each root satisfies $\left|\frac{x}{\lambda} + i\frac{y}{\kappa}\right| = 1$, and hence lie on the ellipse (1.1.3). Since $T_n(z/\alpha) = (2/\alpha)^n \tau_n(z)$, the result follows. $\square$

Theorem 2 also implies that given an arbitrary ellipse centered at the origin, we can always obtain at least two sequences of translated Chebyshev polynomials with roots on the ellipse.

Example 1. Set $\lambda = \sqrt{5}$ and $\kappa = 1$, to get $\alpha = 2$. By Theorem 2 the roots of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ lies on $E(\sqrt{5}, 1)$. Figure 4.1 illustrates this for $n = 33$ from which we also see the interlacing property.

The case for $n \equiv 0 \pmod{4}$ is however different since $T_{4n}(i/2) = T_{4n}(\sqrt{5}/2) = \frac{1}{2}L_{4n}$ where L_n is the n^{th} Lucas' number. This implies that the roots of $T_{4n}(z/2) - T_{4n}(i/2)$ coincide with the roots of $T_{4n}(z/2) - T_{4n}(\sqrt{5}/2)$.

The interlacing property of these polynomials on the ellipse easily follows from the explicit formula for the roots. It is possible to get an explicit formula for the roots due to the 'nice' factorization of the polynomials on the ellipse i.e. in equations (1.1.12) and (1.1.14).

Theorem 3. The roots of $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ are respectively given by $z_j = \lambda \sin(2\pi j/n) + i\kappa \cos(2\pi j/n)$ and $\tilde{z}_j = \lambda \cos(2\pi j/n) + i\kappa \sin(2\pi j/n)$ for $j = 0, 1, 2, \dots, n-1$.

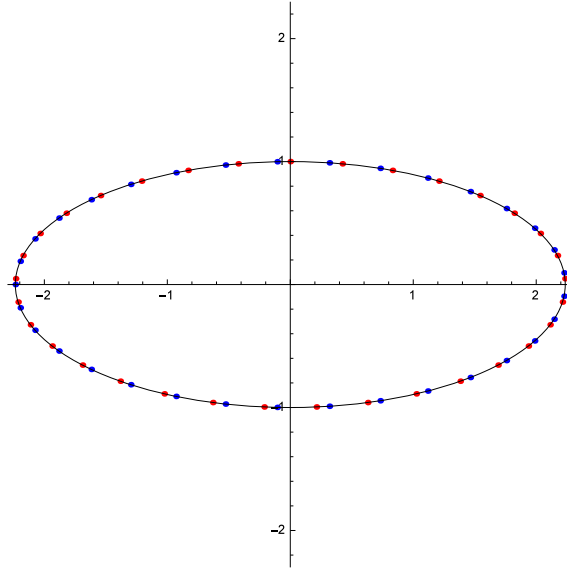


Figure 1.1: Roots of $T_{33}(z/2) - T_{33}(i/2)$ in red and $T_{33}(z/2) - T_{33}(\sqrt{5}/2)$ in blue lie on the ellipse $E(\sqrt{5}, 1)$.

Proof. If $z(\theta) = \lambda \sin \theta + i\kappa \cos \theta$, then $z(\theta) = i(\phi e^{-i\theta} - \hat{\phi} e^{i\theta})$, where $\phi = (\lambda + \kappa)/2$ and $\hat{\phi} = (\lambda - \kappa)/2$. Recall that $\alpha = \sqrt{\lambda^2 - \kappa^2}$, hence $\alpha^2 = 4\phi\hat{\phi}$. If $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ and $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, then

$$\begin{aligned} z^2 - \alpha^2 &= -(\phi^2 e^{-2i\theta} - 2\phi\hat{\phi} + \hat{\phi}^2 e^{2i\theta}) - 4\phi\hat{\phi} \\ &= -(\phi e^{-i\theta} + \hat{\phi} e^{i\theta})^2, \end{aligned}$$

so $\sqrt{z^2 - \alpha^2} = i(\phi e^{-i\theta} + \hat{\phi} e^{i\theta})$. Now we have

$$\begin{aligned} (z + \sqrt{z^2 - \alpha^2})/2 &= i\phi e^{-i\theta}, \\ (z - \sqrt{z^2 - \alpha^2})/2 &= -i\hat{\phi} e^{i\theta}. \end{aligned}$$

Recall that $T_n(z) = 1/2((z + \sqrt{z^2 - \alpha^2})^n + (z - \sqrt{z^2 - \alpha^2})^n)$, hence

$$\begin{aligned}
T_n(z/\alpha) - T_n(i\kappa/\alpha) &= \frac{2^{n-1}}{\alpha^n} \left(i^n \phi^n e^{-ni\theta} + (-i)^n \hat{\phi}^n e^{ni\theta} - i^n \phi^n - (-i)^n \hat{\phi}^n \right) \\
&= \frac{2^{n-1} i^n}{\alpha^n} \left(\phi^n (e^{-ni\theta} - 1) + (-\hat{\phi})^n (e^{ni\theta} - 1) \right) \\
&= \frac{2^{n-1} i^n}{\alpha^n} (e^{ni\theta} - 1) \left((-\hat{\phi})^n - \phi^n e^{-ni\theta} \right) \\
&= \frac{2^{n-1} i^n \phi^n}{\alpha^n} (e^{ni\theta} - 1) \left((-\hat{\phi}/\phi)^n - e^{-ni\theta} \right),
\end{aligned}$$

Since $|\hat{\phi}/\phi| < 1$, by the triangle inequality, the last term on the right hand side is non zero. We must have that $T_n(z/\alpha) - T_n(i\kappa/\alpha) = 0$ implies that $e^{in\theta} - 1 = 0$ or $n\theta = 2\pi k$ for an integer k . Precisely, $\theta = \frac{2\pi k}{n}$ for $k = 0, 1, \dots, n-1$, giving the n distinct roots of $T_n(z/\alpha) - T_n(i\kappa/\alpha)$.

Similarly, if $y(\theta) = \lambda \cos \theta + i\kappa \sin \theta$, then $y(\theta) = \phi e^{i\theta} + \hat{\phi} e^{-i\theta}$, and

$$\begin{aligned}
y^2 - \alpha^2 &= (\phi^2 e^{2i\theta} + 2\phi\hat{\phi} + \hat{\phi}^2 e^{-2i\theta}) - 4\phi\hat{\phi} \\
&= (\phi e^{i\theta} - \hat{\phi} e^{-i\theta})^2,
\end{aligned}$$

so $\sqrt{y^2 - \alpha^2} = \phi e^{i\theta} - \hat{\phi} e^{-i\theta}$. Now we have

$$\begin{aligned}
(y + \sqrt{y^2 - \alpha^2})/2 &= \phi e^{i\theta}, \\
(y - \sqrt{y^2 - \alpha^2})/2 &= \hat{\phi} e^{-i\theta},
\end{aligned}$$

hence

$$\begin{aligned}
T_n(y/\alpha) - T_n(\lambda/\alpha) &= \frac{2^{n-1}}{\alpha^n} \left(\phi^n e^{ni\theta} + \hat{\phi}^n e^{-ni\theta} - \phi^n - \hat{\phi}^n \right) \\
&= \frac{2^{n-1}}{\alpha^n} \left(\phi^n (e^{ni\theta} - 1) + \hat{\phi}^n (e^{-ni\theta} - 1) \right) \\
&= \frac{2^{n-1}}{\alpha^n} (e^{ni\theta} - 1) \left(\phi^n - \hat{\phi}^n e^{-ni\theta} \right) \\
&= \frac{2^{n-1} \phi^n}{\alpha^n} (e^{ni\theta} - 1) \left(1 - (\hat{\phi}/\phi)^n e^{-ni\theta} \right).
\end{aligned}$$

Since $|\hat{\phi}/\phi| < 1$, by the triangle inequality, the last term on the right hand side is non zero. As before, we must have that if $T_n(y/\alpha) - T_n(\lambda/\alpha) = 0$ then $e^{in\theta} - 1 = 0$ or $n\theta = 2\pi j$ for an integer j . Precisely, $\theta = \frac{2\pi j}{n}$ for $j = 0, 1, \dots, n-1$ giving the n distinct root of $T_n(y/\alpha) - T_n(\lambda/\alpha)$. $\square$

From similar calculations and with $y(\theta) = \lambda \cos \theta + i\kappa \sin \theta$, it is easy to see that $T_n(y/\alpha) - T_n(i\kappa/\alpha)$ factors as

$$T_n(y/\alpha) - T_n(i\kappa/\alpha) = \frac{2^{n-1}\phi^n}{\alpha^n} (e^{ni\theta} - i^n) \left(1 - (-i)^n (\hat{\phi}/\phi)^n e^{-ni\theta}\right).$$

The last term on the right is clearly non-zero by the triangle inequality, and hence we have four cases for the roots of $T_n(y/\alpha) - T_n(i\kappa/\alpha)$:

1. If $n \equiv 0 \pmod{4}$, then $\theta = 2\pi j/n$, and the roots coincide with the roots of $T_n(z/\alpha) - T_n(\lambda/\alpha)$,
2. If $n \equiv 1 \pmod{4}$, then $\theta = \pi(2j + 1/2)/n$,
3. If $n \equiv 2 \pmod{4}$, then $\theta = \pi(2j + 1)/n$ and
4. If $n \equiv -1 \pmod{4}$ then $\theta = \pi(2j + 3/2)/n$.

Here the values $j = 0, 1, 2, \dots, n-1$ give the n distinct roots z_j of $T_n(y/\alpha) - T_n(i\kappa/\alpha)$.

Now if $n \not\equiv 0 \pmod{4}$, $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ each have n distinct roots $z_j = \lambda \cos(\pi(2j + t)/n) + i\kappa \sin(\pi(2j + t)/n)$ and $\tilde{z}_j = \lambda \cos(2\pi j/n) + i\kappa \sin(2\pi j/n)$ respectively for $j = 0, 1, \dots, n-1$, where $t = 0.5, 1, 1.5$ when $n \equiv 1, 2, 3 \pmod{4}$. All the roots lie on $E(\lambda, \kappa)$, are simple and do not coincide. Moreover, for these values of t , these two polynomials have roots that strictly interlace on the ellipse.

The density of the zeros of $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ on $E(\lambda, \kappa)$ as $n \rightarrow \infty$ clearly follows from the explicit formulas for the roots.

In the next section, we are going to study properties of the roots of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ that lie on $E(\sqrt{5}, 1)$. The formula for Chebyshev points described in Theorem 3 will be extremely useful.

Corollary 4. *The roots of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ are respectively given by*

$$z_j = \sqrt{5} \sin(2\pi j/n) + i \cos(2\pi j/n), \quad (1.1.15)$$

and

$$\tilde{z}_j = \sqrt{5} \cos(2\pi j/n) + i \sin(2\pi j/n), \quad (1.1.16)$$

for $j = 0, 1, 2, \dots, n-1$.

Proof. Set $\lambda = \sqrt{5}$ and $\kappa = 1$ in Theorem 3. If $n \equiv 0 \pmod{4}$, $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ coincide. There is a strict interlacing of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ on $E(\sqrt{5}, 1)$ whenever $n \not\equiv 0 \pmod{4}$. $\square$

1.2 Properties of the Chebyshev points of

$$T_n(z/2) - T_n(i/2) \text{ and } T_n(z/2) - T_n(\sqrt{5}/2)$$

We deduce these properties from the explicit formula for the roots of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$. Naturally, these properties can be extended to Chebyshev points of the more general $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ polynomials.

Theorem 5. *For $n = 2m + 1$ an odd integer, let $x, \tilde{y} \neq 0$, where $z = x + iy$ is a root of $T_n(z/2) - T_n(i/2)$ and $\tilde{z} = \tilde{x} + i\tilde{y}$ is a root of $T_n(z/2) - T_n(\sqrt{5}/2)$. Then x and $-y$ are roots of*

$$U_m\left(\frac{2x^2}{5} - 1\right) - U_{m-1}\left(\frac{2x^2}{5} - 1\right) \quad \text{and} \quad U_m(y) - U_{m-1}(y),$$

respectively while $\tilde{x}$ and $\tilde{y}$ are respectively roots of

$$U_m\left(\frac{2x^2}{5} - 1\right) + U_{m-1}\left(\frac{2x^2}{5} - 1\right) \quad \text{and} \quad U_m(y) + U_{m-1}(y).$$

Proof. Let $x = \sqrt{5} \sin \theta$; then $2x^2/5 - 1 = -\cos 2\theta$. It suffices to show that for $\theta = (2\pi k)/n$, $U_m(-\cos \theta) - U_{m-1}(-\cos \theta) = 0$. First recall that

$$U_m(x) = \frac{1}{2\sqrt{x^2 - 1}} \left((x + \sqrt{x^2 - 1})^{m+1} - (x - \sqrt{x^2 - 1})^{m+1} \right),$$

hence $U_m(x) - U_{m-1}(x) = 0$ implies

$$(x + \sqrt{x^2 - 1})^m (x + \sqrt{x^2 - 1} - 1) - (x - \sqrt{x^2 - 1})^m (x - \sqrt{x^2 - 1} - 1) = 0$$

which on multiplying by $(x + \sqrt{x^2 - 1})^m$ gives

$$(x + \sqrt{x^2 - 1})^{2m} (x + \sqrt{x^2 - 1} - 1) - (x - \sqrt{x^2 - 1} - 1) = 0. \quad (1.2.1)$$

Now let $x = -\cos \theta$ in (1.2.1). Making this change of variable, we find that

$$(-\cos \theta + i \sin \theta)^{2m}(-\cos \theta + i \sin \theta - 1) - (-\cos \theta - i \sin \theta - 1) = 0,$$

or

$$e^{-2mi\theta}(-e^{-i\theta} - 1) + (e^{i\theta} + 1) = 0$$

which simplifies to

$$(e^{-(2m+1)i\theta} - 1)(e^{i\theta} + 1) = 0.$$

We have a trivial zero of $U_m(x) - U_{m-1}(x) = 0$ when $\cos \theta = -1$, or $x = 1$ and the other zeros are at $\cos(2m+1)\theta = 1$ or when $\theta = \frac{2\pi k}{2m+1}$. The proof for the other two cases is identical. $\square$

Ellipses are often defined as the locus of all points in the plane such that the sum of distances r_1 and r_2 from each point to two fixed points F_1 and F_2 is a constant. We now describe the distances from Chebyshev points of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ to the foci of $E(\sqrt{5}, 1)$.

Theorem 6. *Let $n = 2m + 1$ be an odd integer, then the distances from the Chebyshev points of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ to the foci of $E(\sqrt{5}, 1)$ are respectively roots of $U_{2m} \left(\frac{1}{2} \sqrt{x(2\sqrt{5} - x) - 1} \right)$ and $U_{2m} \left(\frac{1}{2}(x - \sqrt{5}) \right)$.*

Proof. First recall that $U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$. If $\frac{1}{2} \sqrt{x(2\sqrt{5} - x) - 1} = \cos \theta$, then $U_{2m} \left(\frac{1}{2} \sqrt{x(2\sqrt{5} - x) - 1} \right) = 0$ implies that $\sin(2m+1)\theta = 0$, or $\theta = (2\pi k)/(2m+1)$. Let $\frac{1}{2} \sqrt{x(2\sqrt{5} - x) - 1} = \cos \theta$; then $x(2\sqrt{5} - x) - 1 = 4\cos^2 \theta$. We have a quadratic $x^2 - 2\sqrt{5}x - (4\cos^2 \theta + 1)$ with roots

$$x = \sqrt{5} \pm 2 \sin \theta.$$

We show that when $\theta = (2\pi k)/(2m+1)$, the values of x are the distances from Chebyshev points of $T_{2m+1}(z/2) - T_{2m+1}(i/2)$ to the foci of $E(\sqrt{5}, 1)$.

By Corollary 4, for n odd, the roots of $T_n(z/2) - T_n(i/2)$ are given by $z_j = \sqrt{5} \sin \theta + i \cos \theta$ where $\theta = 2\pi j/n$. The distance d from z_j to the foci

$(\pm 2, 0)$ is given by

$$\begin{aligned} d^2 &= \left(\sqrt{5} \sin \theta \pm 2 \right)^2 + \cos^2 \theta \\ &= 5 \sin^2 \theta \pm 4\sqrt{5} \sin \theta + 4 + \cos^2 \theta \\ &= 4 \sin^2 \theta + 5 \pm 4\sqrt{5} \sin \theta, \end{aligned}$$

or $d = \sqrt{5} \pm 2 \sin \theta$. The result follows.

Similarly if $\frac{1}{2}(x - \sqrt{5}) = \cos \theta$, then $x = \sqrt{5} \pm 2 \cos \theta$. It can be shown that if z_j is a root of $T_n(z/2) - T_n(\sqrt{5}/2)$, then the distance d from z_j to the foci $(\pm 2, 0)$ is given by $d = \sqrt{5} \pm 2 \cos \theta$ where $\theta = 2\pi j/n$. $\square$

The ellipse $E(a, b)$ has the pedal equation

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1$$

with respect to the focus F . Here, for $P = (x, y)$, r is the distance from F to P , and p is the distance from F to the line tangent to the ellipse at P . Thus P is the foot of a perpendicular drawn from F . Consider the case $a = \sqrt{5}$ and $b = 1$. Here, the pedal equation is given by

$$\frac{1}{p^2} = \frac{2\sqrt{5}}{r} - 1.$$

Theorem 6 describes r in terms of Chebyshev polynomials. Using the formula for the roots of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ given in Corollary 4, we have that $r = \sqrt{5} \pm 2 \sin \theta$ and $r = \sqrt{5} \pm 2 \cos \theta$ for Chebyshev points of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ respectively. Substituting the values of r gives

$$p^2 = \frac{\sqrt{5} - \sin \theta}{\sqrt{5} + \sin \theta} \quad \text{and} \quad p^2 = \frac{\sqrt{5} - \cos \theta}{\sqrt{5} + \cos \theta} \quad (1.2.2)$$

for Chebyshev points P of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ respectively. This leads to the following Theorem,

Theorem 7. *Let $\phi(x) := 2\sqrt{-1 + 3x - x^2}/(1+x)$, $\varphi(x) := \sqrt{5}(1-x)/(1+x)$ and p be as defined above. For an odd integer $n = 2m + 1$, the roots of $U_{2m}\left(\frac{\phi(x)}{2}\right)$ are values of p^2 for Chebyshev points P of $T_n(z/2) - T_n(i/2)$,*

while the roots of $U_{2m}\left(\frac{\phi(x)}{2}\right)$ are values for p^2 for Chebyshev points P of $T_n(z/2) - T_n(\sqrt{5}/2)$.

Proof. The proof follows the same pattern as that of Theorem 6. If $\phi(x)/2 = \cos \theta$, then $U_{2m}\left(\frac{\phi(x)}{2}\right) = 0$ implies that $\sin(2m+1)\theta = 0$, or $\theta = (2\pi k)/(2m+1)$. Let $\phi(x)/2 = \cos \theta$, then $-4(x^2 - 3x + 1)/(x+1)^2 = 4\cos^2 \theta$, or $\sqrt{5}(x-1)/(x+1) = \sin \theta$. Hence

$$x = \frac{\sqrt{5} - \sin \theta}{\sqrt{5} + \sin \theta}.$$

The result now follows from the argument above. The proof of the second statement is very similar. $\square$

We are now going to see that if z and $\tilde{z}$ are Chebyshev points of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ respectively, then $|z|^2$ and $|\tilde{z}|^2$ are roots of a two term linear combination of Chebyshev polynomials.

Define $R_m(x)$ and $S_m(x)$ as

$$R_m(x) = U_m\left(\frac{x-3}{2}\right) + U_{m-1}\left(\frac{x-3}{2}\right), \quad (1.2.3)$$

$$S_m(x) = U_m\left(\frac{x-3}{2}\right) - U_{m-1}\left(\frac{x-3}{2}\right). \quad (1.2.4)$$

It can be shown that

$$R_m(0) = U_m(-3/2) + U_{m-1}(-3/2) = (-1)^m F_{2m+1}$$

and

$$S_m(0) = U_m(-3/2) - U_{m-1}(-3/2) = (-1)^m L_{2m+1},$$

where F_m and L_m are the m^{th} Fibonacci and the m^{th} Lucas numbers respectively. It is also easy to show that $R_m(1) = (-1)^m$, $S_m(1) = (-1)^m(2m+1)$, $R_m(5) = 2m+1$ and $S_m(5) = 1$.

A less obvious fact is that the roots of these polynomials are all real and lie in the interval $(1, 5)$. This will become clear once we establish a relationship to the $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ polynomials.

Let $\theta = 2\pi k/n$. By Corollary 4, if z is a root of $T_n(z/2) - T_n(i/2)$ and $\tilde{z}$ a root of $T_n(z/2) - T_n(\sqrt{5}/2)$, then

$$\frac{|z|^2 - 3}{2} = -\cos 2\theta \quad \text{and} \quad \frac{|\tilde{z}|^2 - 3}{2} = \cos 2\theta.$$

Following the same construction as in the proof of Theorem 5 we have that when $n = 2m + 1$, $|z|^2$ is a root of $S_m(x)$ and $|\tilde{z}|^2$ a root of $R_m(x)$. This proves the following,

Theorem 8. *If z and $\tilde{z}$ are Chebyshev points of $T_n(x/2) - T_n(i/2)$ and $T_n(x/2) - T_n(\sqrt{5}/2)$ respectively that are not real (or purely imaginary), then for each odd integer $n = 2m + 1$, $|z|^2$ and $|\tilde{z}|^2$ are respectively roots of $S_m(x)$ and $R_m(x)$.*

From Corollary 4, the slope on $E(\sqrt{5}, 1)$ at a Chebyshev point of $T_n(x/2) - T_n(i/2)$ or $T_n(x/2) - T_n(\sqrt{5}/2)$ is given by $\frac{1}{\sqrt{5}} \cot(2\pi k/n)$ or $\frac{1}{\sqrt{5}} \tan(2\pi k/n)$ respectively. We are going to show that the minimal polynomials for these slopes are factors of $G_m(x)$ and $H_m(x)$, where $G_m(x)$ and $H_m(x)$ are generated by

$$\frac{1 + (5x^2 + 1)t}{1 - 2(5x^2 - 1)t + (5x^2 + 1)^2 t^2} = \sum_{m=0}^{\infty} G_m(x) t^m, \quad (1.2.5)$$

$$\frac{1 - (5x^2 + 1)t}{1 - 2(5x^2 - 1)t + (5x^2 + 1)^2 t^2} = \sum_{m=0}^{\infty} H_m(x) t^m. \quad (1.2.6)$$

Theorem 9. *Let $n = 2m + 1$ be odd, and $G_m(x)$ and $H_m(x)$ be as defined in (1.2.5) and (1.2.6). Then the minimal polynomial for the slope at a Chebyshev point of $T_n(x/2) - T_n(i/2)$ (or of $T_n(x/2) - T_n(\sqrt{5}/2)$) on $E(\sqrt{5}, 1)$ is a non-trivial factor of $G_m(x)$ (or a non-trivial factor of $H_m(x)$).*

Proof. First comparing the generating functions (1.2.5) and (1.2.6) for $G_m(x)$ and $H_m(x)$ to the generating function (1.0.4) of Chebyshev polynomials of the second kind, we deduce that

$$\begin{aligned} G_m(x) &= (5x^2 + 1)^m \left(U_m \left(\frac{5x^2 - 1}{5x^2 + 1} \right) + U_{m-1} \left(\frac{5x^2 - 1}{5x^2 + 1} \right) \right), \\ H_m(x) &= (5x^2 + 1)^m \left(U_m \left(\frac{5x^2 - 1}{5x^2 + 1} \right) - U_{m-1} \left(\frac{5x^2 - 1}{5x^2 + 1} \right) \right). \end{aligned}$$

For $x = \frac{1}{\sqrt{5}} \cot \theta$,

$$\begin{aligned}
\frac{5x^2 - 1}{5x^2 + 1} &= \frac{\cot^2 \theta - 1}{\cot^2 \theta + 1} \\
&= \sin^2 \theta (\cot^2 \theta - 1) \\
&= \cos^2 \theta - \sin^2 \theta \\
&= \cos 2\theta,
\end{aligned}$$

so that

$$\begin{aligned}
U_m(x) + U_{m-1}(x) &= U_m(\cos 2\theta) + U_{m-1}(\cos 2\theta) \\
&= U_{2m+1}(\cos \theta) + U_{2m-1}(\cos \theta) \\
&= 2 \cos(\theta) U_{2m}(\cos \theta) \\
&= 2 \cos(\theta) \frac{\sin(2m+1)\theta}{\sin(\theta)},
\end{aligned}$$

which is zero whenever $\theta = \frac{2\pi k}{2m+1}$. The roots of $G_m(x)$ are slopes on the ellipse $E(\sqrt{5}, 1)$ at points that are roots of $T_n(x/2) - T_n(i/2)$. Similarly let $y = \frac{1}{\sqrt{5}} \tan \theta$,

$$\begin{aligned}
\frac{5y^2 - 1}{5y^2 + 1} &= \frac{\tan^2 \theta - 1}{\tan^2 \theta + 1} \\
&= \cos^2 \theta (\tan^2 \theta - 1) \\
&= \sin^2 \theta - \cos^2 \theta \\
&= -\cos 2\theta,
\end{aligned}$$

Recall that $U_m(-x) = (-1)^m U_m(x)$. We now have

$$\begin{aligned}
U_m(y) - U_{m-1}(y) &= U_m(-\cos 2\theta) - U_{m-1}(-\cos 2\theta) \\
&= (-1)^m (U_m(\cos 2\theta) + U_{m-1}(\cos 2\theta)) \\
&= (-1)^m 2 \cos(\theta) \frac{\sin(2m+1)\theta}{\sin(\theta)},
\end{aligned}$$

which, as before, is zero whenever $\theta = \frac{2\pi k}{2m+1}$. The roots of $H_m(x)$ are slopes on the ellipse $E(\sqrt{5}, 1)$ at points that are roots of $T_n(x/2) - T_n(\sqrt{5}/2)$. $\square$

For $k \neq j$, define $\mathfrak{L}_{k,j}$ by

$$\mathfrak{L}_{k,j} = \left| \text{Det} \begin{pmatrix} x_k & \tilde{x}_j \\ y_k & \tilde{y}_j \end{pmatrix} \right|; \quad (1.2.7)$$

$\mathfrak{L}_{k,j}$ is the area of the parallelogram enclosed by the vectors (x_k, y_k) and $(\tilde{x}_j, \tilde{y}_j)$ where $z_k = x_k + iy_k$ and $\tilde{z}_k = \tilde{x}_k + i\tilde{y}_k$ are distinct roots of $T_n(x/2) - T_n(i/2)$ or $T_n(x/2) - T_n(\sqrt{5}/2)$ that lie on the ellipse $E(\sqrt{5}, 1)$.

Theorem 10. *Let $\mathfrak{L}_{k,j}$ be as defined above. The minimum polynomial for $\mathfrak{L}_{k,j}$ divides the polynomial $Q_n(x)$ defined by the generating function*

$$\frac{1 - 5t}{1 - 2(2x^2 - 5)t + 25t^2} = \sum_{n=0}^{\infty} Q_n(x)t^n. \quad (1.2.8)$$

Proof. By Corollary 4, ζ_k is either given by $\sqrt{5}\sin(2\pi k/n) + i\cos(2\pi k/n)$ or $\sqrt{5}\cos(2\pi k/n) + i\sin(2\pi k/n)$. In either case, $\mathfrak{L}_{k,j} = \sqrt{5}|\sin(2\pi(k-j)/n)|$ which is non-zero if $k \neq j$. $\mathfrak{L}_{k,j} = \text{Re}z_k$ where z_k is a root of $T_n(x/2) - T_n(i/2)$. From Theorem 5, we have that whenever $k \neq j$, $\mathfrak{L}_{k,j}$ is a root of $U_m(2x^2/5 - 1) - U_{m-1}(2x^2/5 - 1)$. Also, from the generating function (1.2.8) we deduce

$$Q_n(x) = 5^n (U_m(2x^2/5 - 1) - U_{m-1}(2x^2/5 - 1)). \quad (1.2.9)$$

□

Corollary 11. *For a fixed $j \in \{1, 2, \dots, m\}$, $\mathfrak{L}_{k,k-j}$ is a constant as k varies through the $2m + 1$ roots of $T_n(x/2) - T_n(i/2)$ or $T_n(x/2) - T_n(\sqrt{5}/2)$.*

Proof. For a fixed j , $\mathfrak{L}_{k,k-j} = \sqrt{5}|\sin(2\pi j/n)|$, a constant. □

This Corollary implies that the area of the parallelogram enclosed by any two roots of $T_n(x/2) - T_n(i/2)$ or $T_n(x/2) - T_n(\sqrt{5}/2)$ that are a ‘distance’ j apart is a constant.

Remark 1. *If n is an odd prime, $R_n(x)$, $S_n(x)$, $V_m(x)$, $W_m(x)$ and $r_n(x)$ are irreducible.*

1.3 Curvature and cross ratio at Chebyshev points of $T_n(x/2) - T_n(i/2)$ and $T_n(x/2) - T_n(\sqrt{5}/2)$

The curvature at any point on $E(a, b)$ is given by

$$\kappa = \frac{-(ab)^4}{(a^4y^2 + b^4x^2)^{3/2}}.$$

For our case where $a = \sqrt{5}$ and $b = 1$ we have that

$$\frac{1}{\kappa^2} = \frac{1}{5^4}(25y^2 + x^2)^3 = \frac{1}{5^4}(25 - 4x^2)^3. \quad (1.3.1)$$

We are going to prove that the product of reciprocals of squares of curvatures at Chebyshev points of $T_n(x/2) - T_n(i/2)$ or $T_n(x/2) - T_n(\sqrt{5}/2)$ on $E(\sqrt{5}, 1)$ are just scalar multiples of cubes of Fibonacci numbers of Lucas numbers.

Theorem 12. *Let κ_j be the curvature at a Chebyshev point (x_j, y_j) of $T_{2n+1}(x/2) - T_{2n+1}(i/2)$ on $E(\sqrt{5}, 1)$. Then*

$$\prod_{j=1}^n \frac{1}{\kappa_j^2} = \left(\frac{-1}{5}\right)^n F_{2n+1}^3.$$

Similarly let κ_j be the curvature at a Chebyshev point (x_j, y_j) of $T_{2n+1}(x/2) - T_{2n+1}(\sqrt{5}/2)$ on $E(\sqrt{5}, 1)$. Then

$$\prod_{j=1}^n \frac{1}{\kappa_j^2} = \left(\frac{-1}{5}\right)^n L_{2n+1}^3.$$

On the ellipse $E(\sqrt{5}, 1)$, $-1 \leq y \leq 1$ and $-\sqrt{5} \leq x \leq \sqrt{5}$; hence $\frac{1}{25} \leq \kappa \leq 5$.

To study, in the context of Chebyshev polynomials, the curvature at Chebyshev points of $T_n(x/2) - T_n(i/2)$ or $T_n(x/2) - T_n(\sqrt{5}/2)$, the first step would be to find the minimal polynomials for $25y^2 + x^2$. Here, $x + iy$ is a root of $T_n(x/2) - T_n(i/2)$ or $T_n(x/2) - T_n(\sqrt{5}/2)$.

Consider the monic polynomials defined by

$$V_n(x) = 5^n U_n\left(\frac{x-15}{10}\right) + 5^n U_{n-1}\left(\frac{x-15}{10}\right), \quad (1.3.2)$$

$$W_n(x) = 5^n U_n\left(\frac{x-15}{10}\right) - 5^n U_{n-1}\left(\frac{x-15}{10}\right). \quad (1.3.3)$$

We will later see that the roots of $V_n(x)$ and $W_n(x)$ are all real and lie in the interval $(5, 25)$. For n odd, let $z = x + iy$ be a root of $T_n(x/2) - T_n(i/2)$ that is not purely imaginary. Then, by the explicit formula (1.1.15) for the roots,

$$((25y^2 + x^2) - 15)/10 = \cos(2\theta).$$

Similarly if $\tilde{z} = \tilde{x} + i\tilde{y}$ is a root of $T_n(x/2) - T_n(\sqrt{5}/2)$ that is not real, then by the explicit formula (1.1.16) for the roots,

$$((25\tilde{y}^2 + \tilde{x}^2) - 15)/10 = -\cos(2\theta).$$

Here $\theta = 2\pi k/n$.

Theorem 13. *If $z_k = x_k + iy_k$ and $\tilde{z}_k = \tilde{x}_k + i\tilde{y}_k$ are respectively roots of $T_n(x/2) - T_n(i/2)$ and $T_n(x/2) - T_n(\sqrt{5}/2)$ that are not real or purely imaginary, then for $n = 2m + 1$ odd, $25y_k^2 + x_k^2$ and $25\tilde{y}_k^2 + \tilde{x}_k^2$ are respectively roots of $V_m(x)$ and $W_m(x)$.*

Proof. The proof is similar to that of Theorem 5. □

Since $-\sqrt{5} \leq x \leq \sqrt{5}$ and $-1 \leq y \leq 1$ it is immediate that for all n , the roots of $V_n(x)$ and $W_n(x)$ are all real and lie in the interval $(5, 25)$.

Now, what can be said of the minimal polynomials for $1/\kappa^2$? Tables 1.1 and 1.2 give the minimal polynomials for $1/\kappa^2$ in the case of Chebyshev points of $T_n(x/2) - T_n(i/2)$ and $T_n(x/2) - T_n(\sqrt{5}/2)$ respectively.

From tables 1.1 and 1.2, the constant term is a power of Fibonacci or Lucas numbers. To see this, recall that

$$U_n(x) = \frac{1}{2\sqrt{x^2 - 1}} \left((x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1} \right)$$

and Binet's formula for Fibonacci and Lucas numbers

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}, \quad L_n = \frac{(1 + \sqrt{5})^n + (1 - \sqrt{5})^n}{2^n}.$$

It can easily be shown from (1.4.15) and (1.4.16) and using the Binet's formulas that

$$V_n(0) = (-5)^n F_{2n+1} \quad \text{and} \quad W_n(0) = (-5)^n L_{2n+1},$$

n	Minimal polynomial of $1/\kappa^2$ in the case $T_n(x/2) - T_n(i/2)$.
1	F_1^3
3	$-F_3^3 + 5x$
5	$F_5^3 - 250x + 25x^2$
7	$-F_7^3 + 190x - 2375x^2 + 125x^3$
9	$F_9^3 - 171565x + 118275x^2 - 17500x^3 + 625x^4$
11	$-F_{11}^3 + 3793260x - 4279875x^2 + 1256125x^3 - 115625x^4 + 3125x^5$
13	$F_{13}^3 - 80610900x + 129201525x^2 - 62900000x^3 + 10870000x^4 -$ $718750x^5 + 15625x^6$
15	$-F_{15}^3 + 1670240125x - 3490686825x^2 + 2509115000x^3 - 700906250x^4 +$ $83625000x^5 - 4296875x^6 + 78125x^7$
17	$F_{17}^3 - 33980195390x + 87699469175x^2 - 85832500750x^3 + 35228626875x^4 -$ $6615843750x^5 + 596140625x^6 - 25000000x^7 + 390625x^8$

Table 1.1: Minimal polynomial of $1/\kappa^2$ for the Chebyshev points of $T_n(x/2) - T_n(i/2)$ that lie on $E(\sqrt{5}, 1)$.

n	Minimal polynomial of $1/\kappa^2$ in the case $T_n(x/2) - T_n(\sqrt{5}/2)$.
1	L_1^3
3	$-L_3^3 + 5x$
5	$L_5^3 - 560x + 25x^2$
7	$-L_7^3 + 26720x - 3925x^2 + 125x^3$
9	$L_9^3 - 864695x + 285675x^2 - 25250x^3 + 625x^4$
11	$-L_{11}^3 + 23025720x - 13756575x^2 + 2441875x^3 - 154375x^4 + 3125x^5$
13	$L_{13}^3 - 551197560x + 514207575x^2 - 156031750x^3 + 18542500x^4 -$ $912500x^5 + 15625x^6$
15	$-L_{15}^3 + 12382906465x - 16305723975x^2 + 7594587500x^3 - 1473775000x^4 +$ $130706250x^5 - 5265625x^6 + 78125x^7$
17	$L_{17}^3 - 266986683040x + 462595724675x^2 - 305539103000x^3 + 88570288125x^4 -$ $12408581250x^5 + 875140625x^6 - 29843750x^7 + 390625x^8$

Table 1.2: Minimal polynomial of $1/\kappa^2$ for the Chebyshev points of $T_n(x/2) - T_n(\sqrt{5}/2)$ that lie on $E(\sqrt{5}, 1)$.

where $n = 2m + 1$. Since $V_n(x)$ and $W_n(x)$ are monic, the product of their roots are just the constant terms $V_n(0)$ and $W_n(0)$. Now from (1.3.1), the product of reciprocals of squares of curvatures at points on the ellipse $E(\sqrt{5}, 1)$ which are roots of $T_n(x/2) - T_n(i/2)$ is given by $(\frac{-1}{5})^n F_{2n+1}^3$ while the product when the roots are of $T_n(x/2) - T_n(\sqrt{5}/2)$ is given by $(\frac{-1}{5})^n L_{2n+1}^3$. This proves Theorem 12.

Let A , B , C and D be the intersection between a line l and the ray through a point P on $E(\sqrt{5}, 1)$ and any four distinct and fixed Chebyshev points on $E(\sqrt{5}, 1)$. Chasles' Theorem asserts that the cross ratio

$$(A, B; C, D) := \frac{C - A}{B - C} \bigg/ \frac{D - A}{B - D}$$

is independent of P and the line l . Here, we choose l so that it's not parallel to any of the four rays. We are going to show that in the case of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$, there exists four Chebyshev points such that the minimal polynomial of the cross ratio $(A, B; C, D)$ is a Chebyshev like polynomial.

Recall (see Theorem 3) that the roots of $T_n(z/2) - T_n(i/2)$ and $T_n(z/2) - T_n(\sqrt{5}/2)$ are respectively given by $z_j = \lambda \sin(2\pi j/n) + i\kappa \cos(2\pi j/n)$ and $\tilde{z}_j = \lambda \cos(2\pi j/n) + i\kappa \sin(2\pi j/n)$ for $j = 0, 1, 2, \dots, n-1$.

We first consider the case where the four Chebyshev points are consecutive on $E(\sqrt{5}, 1)$, and show that in these cases, we have the same cross ratio. Draw a ray from $P = \sqrt{5} + i0$ passing through the k th root $\tilde{z}_k$ of $T_n(z/2) - T_n(\sqrt{5}/2)$. The intersection of this ray with the line $x = -\sqrt{5}$ is

$$A_k := -\sqrt{5} + 2i \frac{\sin(2\pi k/n)}{1 - \cos(2\pi k/n)}.$$

Now take the four collinear points to be A_k , A_{k+1} , A_{k+2} and A_{k+3} . The cross ratio of the four points is

$$\begin{aligned} (A_k, A_{k+1}; A_{k+2}, A_{k+3}) &= \frac{4 \cos(\pi/n)^2}{1 + 2 \cos(2\pi/n)} \\ &= \frac{4 \cos(\pi/n)^2}{4 \cos(\pi/n)^2 - 1}. \end{aligned}$$

Similarly, draw a ray from $P = 0 + i$ passing through the k th root z_k of

$T_n(z/2) - T_n(i/2)$. The intersection of this ray with the line $y = -i$ is

$$B_k := \frac{2\sqrt{5}\sin(2\pi k/n)}{1 - \cos(2\pi k/n)} - i.$$

Now take the four collinear points to be B_k, B_{k+1}, B_{k+2} and B_{k+3} . The cross ratio of the four points is

$$(B_k, B_{k+1}; B_{k+2}, B_{k+3}) = \frac{4\cos(\pi/n)^2}{4\cos(\pi/n)^2 - 1}.$$

Theorem 14. *Let $n = 2m + 1$ be odd and $(A_k, A_{k+1}; A_{k+2}, A_{k+3})$ be as defined above. Then the minimal polynomial of $(A_k, A_{k+1}; A_{k+2}, A_{k+3})$ is a non trivial factor of*

$$(x - 1)^m \left(U_m \left(\frac{2 - x}{2(x - 1)} \right) + U_{m-1} \left(\frac{2 - x}{2(x - 1)} \right) \right).$$

Proof. Let $x = \frac{4\cos(\pi/n)^2}{4\cos(\pi/n)^2 - 1}$. Then $\frac{2-x}{2(x-1)} = 2\cos(\pi/n)^2 - 1$. It is easy to prove the identity

$$U_m(y) + U_{m-1}(y) = U_{2m} \left(\sqrt{\frac{x+1}{2}} \right)$$

from the generating function (1.0.4) of Chebyshev polynomials. This implies that

$$\begin{aligned} U_m(2\cos(\pi/n)^2 - 1) + U_{m-1}(2\cos(\pi/n)^2 - 1) &= U_{2m}(\cos(\pi/(2m+1))) \\ &= \frac{\sin(\pi)}{\sin(\pi/(2m+1))} \\ &= 0, \end{aligned}$$

completing the proof. □

Four collinear points can have at most six different cross ratios. In table 1.3, we give these six cross ratios (when $n = 2m + 1$ is odd) and the polynomials $\mathcal{R}_m(x)$ of which the minimal polynomial for the cross ratios is a non-trivial factor. The proof for the minimal polynomials is exactly the same as the proof of Theorem 14 above.

Now consider the case where p is fixed at the center of the ellipse $E(\sqrt{5}, 1)$.

Cross ratio	$\mathcal{R}_m(x)$.
$(A_k, A_{k+1}; A_{k+2}, A_{k+3}) = \frac{4 \cos(\pi/n)^2}{4 \cos(\pi/n)^2 - 1}$	$(x-1)^m U_{2m} \left(\frac{1}{2} \sqrt{\frac{x}{x-1}} \right)$
$(A_k, A_{k+2}; A_{k+1}, A_{k+3}) = \frac{1}{1 - 4 \cos(\pi/n)^2}$	$x^m U_{2m} \left(\sqrt{\frac{x-1}{4x}} \right)$
$(A_k, A_{k+3}; A_{k+1}, A_{k+2}) = \frac{1}{4 \cos(\pi/n)^2}$	$x^m U_{2m} \left(\frac{1}{2\sqrt{x}} \right)$
$(A_k, A_{k+3}; A_{k+2}, A_{k+1}) = 4 \cos(\pi/n)^2$	$U_{2m} \left(\frac{\sqrt{x}}{2} \right)$
$(A_k, A_{k+1}; A_{k+3}, A_{k+2}) = 1 - \frac{1}{4 \cos(\pi/n)^2}$	$(1-x)^m U_{2m} \left(\frac{1}{2\sqrt{1-x}} \right)$
$(A_k, A_{k+3}; A_{k+2}, A_{k+1}) = 1 - 4 \cos(\pi/n)^2$	$U_{2m} \left(\frac{1}{2} \sqrt{1-x} \right)$

Table 1.3: Cross ratios when p is on $E(\sqrt{5}, 1)$, and respective polynomials $\mathcal{R}_m(x)$ of which the minimal polynomial of the cross ratio is a non trivial factor.

Draw a ray from $P = 0 + i0$ passing through the k th root $\tilde{z}_k$ of $T_n(z/2) - T_n(\sqrt{5}/2)$. The intersection of this ray with the line $x = -\sqrt{5}$ is

$$C_k := -\sqrt{5} + i \tan(2\pi k/n).$$

The cross ratio of the four collinear points C_k, C_{k+1}, C_{k+2} and C_{k+3} is

$$(C_k, C_{k+1}; C_{k+2}, C_{k+3}) = \frac{4 \cos(2\pi/n)^2}{4 \cos(2\pi/n)^2 - 1}.$$

The ray from $P = 0 + i0$ passing through the k th root z_k of $T_n(z/2) - T_n(i/2)$ intersects the line $y = -i$ at

$$D_k := 2\sqrt{5} \tan(2\pi k/n) - i.$$

The cross ratio of the four collinear points D_k, D_{k+1}, D_{k+2} and D_{k+3} is

$$(D_k, D_{k+1}; D_{k+2}, D_{k+3}) = \frac{4 \cos(2\pi/n)^2}{4 \cos(2\pi/n)^2 - 1}.$$

It turns out that $(C_k, C_{k+1}; C_{k+2}, C_{k+3})$ and $(A_k, A_{k+1}; A_{k+2}, A_{k+3})$ have the same minimal polynomial. Table 1.4 gives the six cross ratios when p is at the center of $E(\sqrt{5}, 1)$ and when $n = 2m + 1$ is odd.

Cross ratio	$\mathcal{R}_m(x)$.
$(C_k, C_{k+1}; C_{k+2}, C_{k+3}) = \frac{4 \cos(2\pi/n)^2}{4 \cos(2\pi/n)^2 - 1}$	$(x-1)^m U_{2m} \left(\frac{1}{2} \sqrt{\frac{x}{x-1}} \right)$
$(C_k, C_{k+2}; C_{k+1}, C_{k+3}) = \frac{1}{1 - 4 \cos(2\pi/n)^2}$	$x^m U_{2m} \left(\sqrt{\frac{x-1}{4x}} \right)$
$(C_k, C_{k+3}; C_{k+1}, C_{k+2}) = \frac{1}{4 \cos(2\pi/n)^2}$	$x^m U_{2m} \left(\frac{1}{2\sqrt{x}} \right)$
$(C_k, C_{k+3}; C_{k+2}, C_{k+1}) = 4 \cos(2\pi/n)^2$	$U_{2m} \left(\frac{\sqrt{x}}{2} \right)$
$(C_k, C_{k+1}; C_{k+3}, C_{k+2}) = 1 - \frac{1}{4 \cos(2\pi/n)^2}$	$(1-x)^m U_{2m} \left(\frac{1}{2\sqrt{1-x}} \right)$
$(C_k, C_{k+3}; C_{k+2}, C_{k+1}) = 1 - 4 \cos(2\pi/n)^2$	$U_{2m} \left(\frac{1}{2} \sqrt{1-x} \right)$

Table 1.4: Cross ratios when p is at the center of $E(\sqrt{5}, 1)$, and respective polynomials $\mathcal{R}_m(x)$ of which the minimal polynomial of the cross ratio is a non trivial factor.

1.4 Translating using a fixed number

In this section, we give a generalization of Theorems 2 and 3. Let r be a real number. We will show that if $|r| > 1$, $T_n(x) - r$ will have all its roots on an ellipse determined by r and the degree n of the polynomial. We will also show that in some cases, the roots can be given by an explicit formula from which the interlacing property of the polynomials as well as other properties described in the previous section follows. For $r \in \mathbb{R}$, define the polynomials $A_n(r, z)$ by

$$A_n(r, z) := \alpha^n (T_n(z/\alpha) - r). \quad (1.4.1)$$

We know that the roots of $T_n(z)$ are all real and lie in the interval $(-1, 1)$ and are explicitly given by (1.0.5). Naturally, for very small r , we expect the roots of $A_n(r, z)$ to be very close to the roots of $T_n(z)$. Surprisingly for $|r| \leq 1$ the roots are all real, as will be seen from the explicit formula. Define

$$\lambda := \alpha \cos \left(\frac{\arccos(r)}{n} \right) \quad \text{and} \quad i\kappa = \alpha \sin \left(\frac{\arccos(r)}{n} \right), \quad (1.4.2)$$

where $i = \sqrt{-1}$. For $n \geq 1$ define $T_{1/n}(x)$ as

$$T_{1/n}(x) := \cos \left(\frac{\arccos(x)}{n} \right). \quad (1.4.3)$$

We then have $\lambda = \alpha T_{1/n}(r)$, $\kappa = \sqrt{\lambda^2 - \alpha^2}$ from which it is immediate that λ is a real root of $A_n(r, z)$. Now by (1.4.2), for a fixed r , $\lambda^2 - \kappa^2 = \alpha^2$ and

as $n \rightarrow \infty$, $i\kappa \rightarrow 0$ so that $\lambda \sim \alpha$.

Theorem 15. *Let λ and κ be as defined in (1.4.2), then for all $r \in \mathbb{R}$ with $|r| > 1$, the roots of $T_n(z/\alpha) - r$ lie on the ellipse*

$$E(\lambda, \kappa) : \frac{x^2}{\lambda^2} + \frac{y^2}{\kappa^2} = 1.$$

Proof. The proof is very similar to that of Theorem 2 and employs the same factorization. Recall from (1.1.8) the definition of $\tau_n(z)$ and the relationship between $\tau_n(z)$ and $T_n(z/\alpha)$ from (1.1.9). Making this change, $T_n(z/\alpha) - r = 0$ implies that $\tau_n - r(\alpha/2)^n = 0$, or precisely

$$\left(\frac{z + \sqrt{z^2 - (\lambda^2 - \kappa^2)}}{2} \right)^n + \left(\frac{z - \sqrt{z^2 - (\lambda^2 - \kappa^2)}}{2} \right)^n - 2r(\alpha/2)^n = 0.$$

Since λ is a root of $T_n(z) - r$, we have by equations (1.1.6) and (1.1.7) that

$$\phi^n + \hat{\phi}^n - 2r(\alpha/2)^n = 0, \quad (1.4.4)$$

where, as before, $\phi = (\lambda + \kappa)/2$ and $\hat{\phi} = (\lambda - \kappa)/2$. It is clear that $\hat{\phi} < \phi$, and

$$\tau_n(z) - r(\alpha/2)^n = \frac{1}{2} \left(\phi^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n + \hat{\phi}^n \left(\frac{x}{\lambda} - i \frac{y}{\kappa} \right)^n - 2r(\alpha/2)^n \right).$$

Now, $\tau_n(z) - r(\alpha/2)^n = 0$ implies that

$$\phi^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^{2n} - 2r(\alpha/2)^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n + \hat{\phi}^n = 0.$$

Let

$$w := \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n.$$

Making the substitution in the above equation, we get the quadratic

$$\phi^n w^2 - 2r(\alpha/2)^n w + \hat{\phi}^n = 0.$$

Now solving the quadratic using (1.4.4), we get $w = 1$ or $w = (\hat{\phi}/\phi)^n$. This

implies that $\tau_n(z) - r(\alpha/2)^n$ factors as

$$\tau_n(z) - r(\alpha/2)^n = \frac{\phi^n}{2} \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n - 1 \right) \left(1 - \left(\frac{\hat{\phi}}{\phi} \right)^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^{-n} \right). \quad (1.4.5)$$

The last term on the right of (1.4.5) is of the form $1 - u$ where $|u| < 1$, and is therefore non-zero by the triangle inequality. Hence $\tau_n(z) - r(\alpha/2)^n = 0$ implies that $\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n = 1$, or that every root satisfies $\left| \frac{x}{\lambda} + i \frac{y}{\kappa} \right| = 1$.

The proof for the roots of $\tau_n(z) + r(\alpha/2)^n$ is similar but there are two cases to consider. For n even, $i\kappa$ is a root of $T_n(x) + r$ while for n odd, $-\lambda$ is a root. These two conditions give relation (1.4.4) for all n . It can be shown that

$$\tau_n(z) + r(\alpha/2)^n = \frac{1}{2} \left(\phi^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n + \hat{\phi}^n \left(\frac{x}{\lambda} - i \frac{y}{\kappa} \right)^n + 2r(\alpha/2)^n \right),$$

from which, $\tau_{2n}(z) + r(\alpha/2)^n = 0$ implies that

$$\phi^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^{2n} + 2r(\alpha/2)^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n + \hat{\phi}^n = 0.$$

We thus obtain the quadratic

$$\phi^n w^2 + 2rw + \hat{\phi}^n = 0,$$

which we solve using (1.4.4) to deduce that $w = -1$ or $w = -(\hat{\phi}/\phi)^n$. This implies that

$$\tau_n(z) + r(\alpha/2)^n = \frac{\phi^n}{2} \left(\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n + 1 \right) \left(1 + \left(\frac{\hat{\phi}}{\phi} \right)^n \left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^{-n} \right). \quad (1.4.6)$$

As before, the last term on the right-hand side of (1.4.6) is non-zero by the triangle inequality. Hence $\tau_n(z) + r(\alpha/2)^n = 0$ implies that $\left(\frac{x}{\lambda} + i \frac{y}{\kappa} \right)^n = -1$, or that every root satisfies $\left| \frac{x}{\lambda} + i \frac{y}{\kappa} \right| = 1$. For all $r \in \mathbb{R}$ with $|r| > 1$, the roots of $T_n(x/\alpha) - r$ lie on the ellipse $E(\lambda, \kappa)$. $\square$

The main reason for scaling by α is to have a better view of the ellipse on which the roots of $T_n(x/\alpha) - r$ lie. A large enough α scales the roots to a larger ellipse centered at the origin, and it is much easier to see the behavior of the roots on the ellipse. Hence if we set $\alpha = 1$, we have translated Chebyshev

polynomials. The following Corollary is immediate.

Corollary 16. *If $T_{1/n}(r)$ is as defined in (1.4.3), then for all $r \in \mathbb{R}$ with $|r| > 1$, the roots of $T_n(z) - r$ lie on the ellipse*

$$E\left(T_{1/n}(r), \sqrt{T_{1/n}^2(r) - 1}\right) : \frac{x^2}{T_{1/n}^2(r)} + \frac{y^2}{T_{1/n}^2(r) - 1} = 1.$$

We are now going to use the same factorization for the proof of Theorem 15 to prove a formula for the roots of $T_n(z/\alpha) - r$. The added advantage of the exact formula is its validity for all $r \in \mathbb{R}$ and all $n \geq 1$. From the formula of the roots, we can deduce the interlacing of $T_n(z/\alpha) - r$ and $T_n(z/\alpha) + r$ on the ellipse $E(\lambda, \kappa)$ as well as the density of their roots.

Theorem 17. *If λ and κ are as defined in (1.4.2), then for all $r \in \mathbb{R}$, the roots of $T_n(z/\alpha) - r$ and $T_n(z/\alpha) + r$ are respectively given by*

$$z_j = \lambda \cos(2\pi j/n) + i\kappa \sin(2\pi j/n), \quad (1.4.7)$$

and

$$\tilde{z}_j = \lambda \cos((2j+1)\pi/n) + i\kappa \sin((2j+1)\pi/n), \quad (1.4.8)$$

for $j = 0, 1, 2, \dots, n-1$, and they interlace on the ellipse $E(\lambda, \kappa)$.

Proof. The proof is almost identical to that of Theorem 3. If $z(\theta) = \lambda \cos \theta + i\kappa \sin \theta$, then $z(\theta) = (\phi e^{i\theta} + \hat{\phi} e^{-i\theta})$, where, as before, $\phi = (\lambda + \kappa)/2$ and $\hat{\phi} = (\lambda - \kappa)/2$. Also from (1.4.2), $\alpha^2 = \lambda^2 - \kappa^2$, hence $\alpha^2 = 4\phi\hat{\phi}$. Write $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ and $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. Then

$$\begin{aligned} z^2 - \alpha^2 &= (\phi^2 e^{2i\theta} + 2\phi\hat{\phi} + \hat{\phi}^2 e^{-2i\theta}) - 4\phi\hat{\phi}, \\ &= (\phi e^{i\theta} - \hat{\phi} e^{-i\theta})^2, \end{aligned}$$

so $\sqrt{z^2 - \alpha^2} = \phi e^{i\theta} - \hat{\phi} e^{-i\theta}$. Now we have

$$\begin{aligned} (z + \sqrt{z^2 - \alpha^2})/2 &= \phi e^{i\theta}, \\ (z - \sqrt{z^2 - \alpha^2})/2 &= \hat{\phi} e^{-i\theta}. \end{aligned}$$

Recall that $\tau_n(z) = 1/2((z + \sqrt{z^2 - \alpha^2})^n + (z - \sqrt{z^2 - \alpha^2})^n)$. Hence

$$\begin{aligned}
T_n(z/\alpha) - r &= \frac{2^{n-1}}{\alpha^n} (\tau_n(z) - 2r(\alpha/2)^n) \\
&= \frac{2^{n-1}}{\alpha^n} (\phi^n e^{ni\theta} + \hat{\phi}^n e^{-ni\theta} - \phi^n - \hat{\phi}^n) \\
&= \frac{2^{n-1}}{\alpha^n} (\phi^n (e^{ni\theta} - 1) - \hat{\phi}^n e^{-ni\theta} (e^{ni\theta} - 1)) \\
&= \frac{2^{n-1}}{\alpha^n} (e^{ni\theta} - 1) (\phi^n - \hat{\phi}^n e^{-ni\theta}) \\
&= \frac{2^{n-1} \phi^n}{\alpha^n} (e^{ni\theta} - 1) \left(1 - (\hat{\phi}/\phi)^n e^{-ni\theta}\right),
\end{aligned}$$

where we have used (1.4.4) in the second line. Since $|\hat{\phi}/\phi| < 1$, by the triangle inequality, the last term on the right hand side is non zero. Hence $T_n(z/\alpha) - T_n(i\kappa/\alpha) = 0$ implies that $e^{in\theta} - 1 = 0$ or $n\theta = 2j\pi$ for $j = 0, 1, 2, \dots, n-1$. This gives the n distinct roots of $T_n(z/\alpha) - r$ as stated in the Theorem.

Similarly, if $y(\theta) = \lambda \cos \theta + i\kappa \sin \theta$, then $y(\theta) = \phi e^{i\theta} + \hat{\phi} e^{-i\theta}$ and

$$\begin{aligned}
y^2 - \alpha^2 &= (\phi^2 e^{2i\theta} + 2\phi\hat{\phi} + \hat{\phi}^2 e^{-2i\theta}) - 4\phi\hat{\phi} \\
&= (\phi e^{i\theta} - \hat{\phi} e^{-i\theta})^2,
\end{aligned}$$

so $\sqrt{y^2 - \alpha^2} = \phi e^{i\theta} - \hat{\phi} e^{-i\theta}$. Now we have

$$\begin{aligned}
(y + \sqrt{y^2 - \alpha^2})/2 &= \phi e^{i\theta}, \\
(y - \sqrt{y^2 - \alpha^2})/2 &= \hat{\phi} e^{-i\theta},
\end{aligned}$$

and

$$\begin{aligned}
T_n(y/\alpha) + r &= \frac{2^{n-1}}{\alpha^n} (\tau_n(y) + 2r(\alpha/2)^n) \\
&= \frac{2^{n-1}}{\alpha^n} (\phi^n e^{ni\theta} + \hat{\phi}^n e^{-ni\theta} + \phi^n + \hat{\phi}^n) \\
&= \frac{2^{n-1}}{\alpha^n} (\phi^n (e^{ni\theta} + 1) + \hat{\phi}^n (e^{-ni\theta} + 1)) \\
&= \frac{2^{n-1}}{\alpha^n} (e^{ni\theta} + 1) (\phi^n + \hat{\phi}^n e^{-ni\theta}) \\
&= \frac{2^{n-1} \phi^n}{\alpha^n} (e^{ni\theta} + 1) \left(1 + (\hat{\phi}/\phi)^n e^{-ni\theta}\right).
\end{aligned}$$

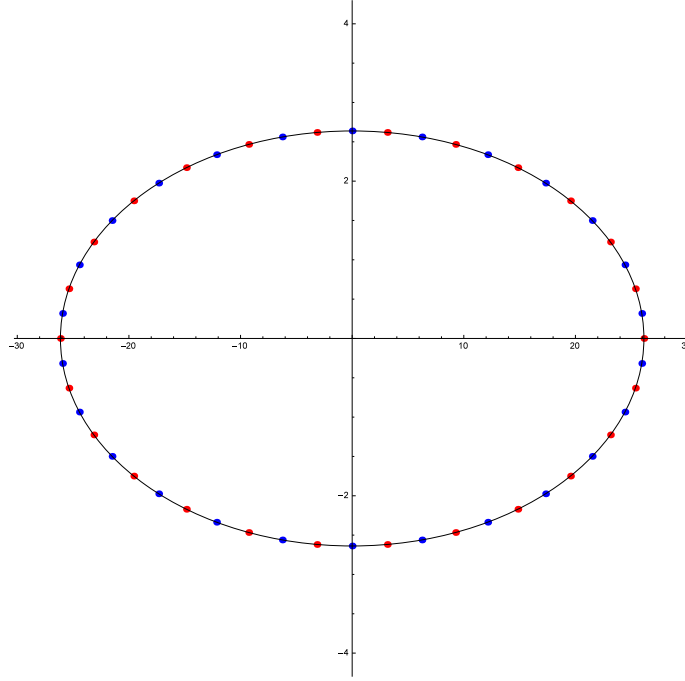


Figure 1.2: Roots of $A_{13}(7, z)$ in red and $A_{13}(-7, z)$ in blue with $\alpha = 13$.

Since $|\hat{\phi}/\phi| < 1$, by the triangle inequality, the last term on the right hand side is non zero. So $T_n(y/\alpha) + r = 0$ implies that $e^{in\theta} + 1 = 0$ or $n\theta = (2j+1)\pi$ for $j = 0, 1, 2, \dots, n-1$. The interlacing property is immediate from the formulas (1.4.7) and (1.4.8) of the roots. $\square$

Corollary 18. *For all $n \geq 1$ and $r \in \mathbb{R}$, the roots of*

$$(U_{n+1}(x) - U_n(x))^2(x+1) - (r+1)$$

lie on the ellipse $E\left(T_{1/n}(r), \sqrt{T_{1/n}^2(r) - 1}\right)$.

Proof. We will use the well known identity for products of Chebyshev polynomials

$$2T_n(x)T_m(x) = T_{m+n}(x) + T_{|n-m|}(x), \quad (1.4.9)$$

the Pell equation

$$T_n^2(x) - (x^2 - 1)U_{n-1}^2(x) = 1 \quad (1.4.10)$$

and the identity

$$2U_m(x)U_n(x) = \frac{T_{m+n}(x) - T_{|m-n|}(x)}{x^2 - 1}. \quad (1.4.11)$$

Expand $(U_{n+1}(x) - U_n(x))^2(x+1) - (r+1)$ and simplify using the identities (1.4.11), (1.4.10) and (1.4.9) to find that

$$(U_{n+1}(x) - U_n(x))^2(x+1) - (r+1) = T_{2n+3}(x) - r. \quad (1.4.12)$$

The proof follows by applying Corollary 16. $\square$

By equation (1.4.12),

$$\frac{1}{2}(U_{n+1}(ax) - U_n(ax))^2(ax+1) - r = \frac{1}{2}(T_{2n+3}(ax) - (2r-1)), \quad (1.4.13)$$

and its roots lie on the scaled ellipse

$$\frac{x^2}{1/a^2 \left(T_{1/n}^2(2r-1) \right)} + \frac{y^2}{1/a^2 \left(T_{1/n}^2(2r-1) - 1 \right)} = 1.$$

The effect of making the change $x \mapsto ax$ in $T_n(x) - r$ for any non-zero real a is to scale the roots of $T_n(x) - r$ by a factor of $1/a$. Clearly the minimal polynomial for $1/a T_{1/n}(2r-1)$ will be a non-trivial factor of $(U_{n+1}(ax) - U_n(ax))^2(ax+1) - 2r$.

We have seen that for any real number r , the roots of the polynomial $T_n(z) + r$ lie on an ellipse whose eccentricity depends on the degree of the polynomial and on r . Hence as r runs through $\mathbb{R}$, the eccentricities of these ellipses are dense in $[-1, 1]$.

For the case where $r \in \mathbb{C}$, numerical calculations suggests that the roots of $T_n(z) - r$ lie on the same ellipse as the roots of $T_n(z) - |r|$.

Conjecture 19. *If $r \in \mathbb{C}$ is non-zero, then for all $n \geq 1$, the roots of $T_n(z) - r$ lie on the ellipse $E\left(T_{1/n}(|r|), \sqrt{T_{1/n}^2(|r|) - 1}\right)$, are all simple and are given by the formula*

$$z_k = T_{1/n}(r) \cos(2\pi k/n) + i\sqrt{r^2 - 1} U_{1/n-1}(r) \sin(2\pi k/n). \quad (1.4.14)$$

It is easy to see that for any non-zero $a \in \mathbb{R}$, a change of variable $T_n(ax) - r$

has the effect of scaling the roots of $T_n(ax) - r$. The roots of $T_n(ax) - r$ will now lie on the scaled ellipse.

By construction, the minimal polynomial for $\frac{1}{a}T_{1/n}(r)$ is obviously a non-trivial factor of $T_n(ax) - r$. What can be said of the minimal polynomial for $\frac{1}{a^2}T_{1/n}^2(r)$?

Proposition 20. *For $n \geq 1$ odd, the minimal polynomial for $\frac{1}{a^2}T_{1/n}^2(r)$ divides $T_n(2a^2x - 1) - 2r^2 + 1$. If n is even then the minimal polynomial is a non-trivial factor of $T_n(2a^2x - 1) - 2r^2 + 1$.*

Proof. We will use the identities

$$T_n(2x^2 - 1) = 2T_n^2(x) - 1, \quad (1.4.15)$$

$$T_n(T_m(x)) = T_{nm}(x). \quad (1.4.16)$$

Consider the polynomial $T_n(x) - 2r^2 + 1$. It has a zero at $x_1 = T_{1/n}(2r^2 - 1)$, and by the identity (1.4.15), $x_1 = 2T_{1/n}^2(r) - 1$. Since $x_1 = 2T_{1/n}^2(r) - 1$ is a zero of $T_n(x) - 2r^2 + 1$, it follows that $\frac{1}{a^2}T_{1/n}^2(r)$ is a zero of $T_n(2a^2x - 1) - 2r^2 + 1$. If n is an odd prime, this polynomial is irreducible. On the other hand if n is even, write $n = 2m$ and use the identity $T_{2m}(x) + 1 = 2T_m^2(x)$ to deduce that

$$T_{2m}(2a^2x - 1) - 2r^2 + 1 = 2(T_m^2(2a^2x - 1) - r^2),$$

which factors as

$$T_{2m}(2a^2x - 1) - 2r^2 + 1 = 2(T_m(2a^2x - 1) - r)(T_m(2a^2x - 1) + r).$$

Hence the minimum polynomial for $\frac{1}{a^2}T_{1/2m}^2(r)$ divides $T_m(2a^2x - 1) - r$. $\square$

Notice that $T_2(r) = 2r^2 - 1$. To identify the ellipse on which the roots of $T_n(2a^2x - 1) - T_2(r)$ lie, observe that the roots of $T_n(2a^2x - 1) - T_2(r)$ are just the roots of $T_n(2a^2x) - T_2(r)$ shifted along the real axis by $\frac{1}{2a^2}$ units. Also by Corollary 16, the roots of $T_n(2a^2x) - T_2(r)$ lie on the scaled ellipse

$$\frac{x^2}{1/(2a^2)^2 T_{1/n}^2(T_2(r))} + \frac{y^2}{1/(2a^2)^2 (T_{1/n}^2(T_2(r)) - 1)} = 1.$$

In Proposition 68, replace r by $T_2(r)$ and replace a by $2a^2$ to deduce that

that the minimal polynomial for $1/(2a^2)^2 T_{1/n}^2(T_2(r))$ divides the polynomial

$$T_n(8a^4x - 1) - T_2(T_2(r)),$$

which by identity (1.4.16) can be written as

$$T_n(8a^4x - 1) - T_4(r).$$

By the argument above, the roots of $T_n(8a^4x - 1) - T_4(r)$ lie on the shifted ellipse

$$\frac{\left(x - \frac{1}{8a^4}\right)^2}{1/(8a^4)^2 T_{1/n}^2(T_4(r))} + \frac{y^2}{1/(8a^4)^2 \left(T_{1/n}^2(T_4(r)) - 1\right)} = 1.$$

Theorem 21. *For each non-zero n , let $a \neq 0$, $r \in \mathbb{R}$ and $k \geq 0$. Then the minimal polynomial for $2^{-2^{k+1}+2}a^{-2^{k+1}}T_{1/n}^2(T_{2^k}(r))$ divides*

$$T_n\left(2^{2^{k+1}-1}a^{2^{k+1}}x - 1\right) - T_{2^{k+1}}(r), \quad (1.4.17)$$

whose roots lie on the shifted ellipse

$$\frac{\left(2^{2^{k+1}-1}a^{2^{k+1}}x - 1\right)^2}{T_{1/n}^2(T_{2^{k+1}}(r))} + \frac{\left(2^{2^{k+1}-1}a^{2^{k+1}}y\right)^2}{T_{1/n}^2(T_{2^{k+1}}(r)) - 1} = 1. \quad (1.4.18)$$

Proof. We have seen in Proposition 68 that the statement is true for $k = 0$. It is also true for $k = 1$ by the argument above. Now substitute r by $T_{2^{k+1}}(r)$ and substitute a by $2^{2^{k+2}-2}a^{2^{k+2}}$ in proposition 68 to find that the minimal polynomial for $2^{-2^{k+2}+2}a^{-2^{k+2}}T_{1/n}^2(T_{2^{k+1}}(r))$ divides

$$T_n\left(2(2^{2^{k+2}-2}a^{2^{k+2}})x - 1\right) - T_2(T_{2^{k+1}}(r)),$$

which by identity (1.4.16) can be written as

$$T_n\left(2^{2^{k+2}-1}a^{2^{k+2}}x - 1\right) - T_{2^{k+2}}(r).$$

The roots of this polynomial will then lie on the shifted ellipse given by the

cartesian equation

$$\frac{\left(2^{2^{k+2}-1}a^{2^{k+2}}x - 1\right)^2}{T_{1/n}^2(T_{2^{k+2}}(r))} + \frac{\left(2^{2^{k+2}-1}a^{2^{k+2}}y\right)^2}{T_{1/n}^2(T_{2^{k+2}}(r)) - 1} = 1. \quad (1.4.19)$$

□

1.5 Resultants and discriminants

There has been a great deal of study on the resultants of Chebyshev polynomials. See for example [22, 23, 24, 17, 18]. In this section we give identities for discriminants and resultants of some of the polynomials that we have studied in this section. First we state some standard identities for resultants and discriminants that will be used.

Let $f(x)$ and $g(x)$ be two polynomials of degree m and n respectively, and $\alpha_1, \dots, \alpha_m$ be the roots of $f(x)$. Then the resultant of $f(x)$ and $g(x)$ with respect to x is given by

$$Res_x(f(x), g(x)) = a_m^n \prod_{i=1}^m g(\alpha_i), \quad (1.5.1)$$

where a_m is the leading coefficient for $f(x)$.

$$Res_x(g(x), f(x)) = (-1)^{mn} Res_x(f(x), g(x)). \quad (1.5.2)$$

If $g(x) = f(x)q(x) + h(x)$ where $\deg(h(x)) = l < m$, then

$$Res_x(f(x), g(x)) = a_m^{n-l} Res_x(f(x), h(x)), \quad (1.5.3)$$

from which we get

$$Res_x(f(x), \alpha) = \alpha^n, \quad (1.5.4)$$

where α is a non zero scalar. If $g(x)$ factors as $g_1(x)g_2(x)$,

$$Res_x(f(x), g_1(x)g_2(x)) = Res_x(f(x), g_1(x))Res_x(f(x), g_2(x)). \quad (1.5.5)$$

We will also use the chain rule technique (see [25]) for evaluating resultants. Let $f(x)$, $g(x)$ and $h(x)$ be polynomials, and let h_0 be the leading coefficient

of $h(x)$. Then

$$Res_x(f(h), g(h)) = \left[h_0^{\deg f \times \deg g} Res_x(f, g) \right]^{\deg h} \quad (1.5.6)$$

unless $h(x)$ is (a constant which is) a common root of $f(x)$ and $g(x)$. The discriminant of $f(x)$ with respect to x is given by

$$Disc_x(f(x)) = \frac{(-1)^{\frac{m(m-1)}{2}}}{a_m} Res_x(f(x), f'(x)), \quad (1.5.7)$$

where $f'(x)$ is the derivative of $f(x)$ with respect to x .

We will apply an identity for the resultants and discriminants of Chebyshev polynomials by Stolarsky and Dilcher (see [17]) stated below, in later proofs.

Theorem 22. [17, Theorem 2] For all $n \geq 2$, we have

$$Res_x(U_n(x) + kU_{n-1}(x), U_{n-1}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)} \quad (1.5.8)$$

Theorem 23. [17, Theorem 4] For all $n \geq 1$, we have

$$Disc_x(U_n(x) + kU_{n-1}(x)) = 2^{n(n-1)} a_{n-1}(k), \quad (1.5.9)$$

where

$$a_{n-1}(k) = (-1)^n \frac{(2n+1)^n k^n}{(n+1)^2 - n^2 k^2} \left[U_n \left(-\frac{n+1+nk^2}{(2n+1)k} \right) + kU_{n-1} \left(-\frac{n+1+nk^2}{(2n+1)k} \right) \right] \quad (1.5.10)$$

is an even polynomial in k of degree $2n - 2$ with positive integer coefficients.

Another proof of this Theorem was given by Gishe and Ismail in [18].

Theorem 24. For all $n \geq 0$ the discriminant of $R_n(x)$ and $S_n(x)$ is given by

$$Disc_x(R_n(x)) = Disc_x(S_n(x)) = (2n+1)^{n-1}. \quad (1.5.11)$$

Proof. If $h = (x - 3)/2$, then using (1.5.9) with $k = \pm 1$, we find that

$$a_{n-1}(-1) = a_{n-1}(1) = (2n+1)^{n-1} \quad (1.5.12)$$

and $Disc_h(R_n(h)) = Disc_h(S_n(h)) = 2^{n(n-1)} (2n+1)^{n-1}$, from which by ap-

plying (1.5.7) with $f = S_n(h)$ and $a_0 = 2^n$, we find that

$$\begin{aligned} Res_h(S_n(h), S'_n(h)) &= (-1)^{n(n-1)/2} a_n Disc_h(S_n(h)) \\ &= (-1)^{n(n-1)/2} 2^{n^2} (2n+1)^{n-1}. \end{aligned}$$

Now apply (1.5.6) with $h_0 = 1/2$ and $\frac{d}{dx} S_n(h) = \frac{1}{2} \frac{d}{dh} S_n(h) = \frac{1}{2} S'_n(h)$ to deduce that

$$\begin{aligned} Res_x(S_n(x), S'_n(x)) &= Res_x(S_n(h), \frac{1}{2} S'_n(h)) \\ &= \frac{1}{2^n} Res_x(S_n(h), S'_n(h)) \\ &= (-1)^{n(n-1)/2} (2n+1)^{n-1}. \end{aligned}$$

Finally apply (1.5.7) with $a_m = 1$ to find that

$$Disc_x(S_n(x)) = (2n+1)^{n-1}.$$

Calculations for $Disc_x(R_n(x))$ are similar to those of $Disc_x(S_n(x))$. $\square$

Theorem 25. *The discriminant of $r_n(x)$ is given by*

$$Disc_x(Q_n(x)) = 5^{n(2n-1)} 2^{4n^2} (2n+1)^{2n-1}.$$

Proof. In the construction of the proof, we use $Q_n(x)$ in the form of equation (1.2.9). Let $h = \frac{2}{5}x^2 - 1$, and note that $S_n(h) = U_n(h) - U_{n-1}(h)$ so $Q_n(h) = 5^n S_n(h)$. Hence,

$$\begin{aligned} Res_h(Q_n(h), Q'_n(h)) &= Res_h(5^n S_n(h), 5^n S'_n(h)) \\ &= (-1)^{n(n-1)/2} 2^{n^2} 5^{n(2n-1)} (2n+1)^{n-1}. \end{aligned}$$

Also note that $\frac{d}{dx} Q_n(x) = \frac{4x}{5} \frac{d}{dh} Q_n(h) = \frac{4x}{5} Q'_n(h)$. We now have

$$\begin{aligned} Res_x(Q_n(x), Q'_n(x)) &= Res_x\left(Q_n(h), \frac{4x}{5} Q'_n(h)\right) \\ &= Res_x\left(Q_n(h), \frac{4x}{5}\right) Res_x(Q_n(h), Q'_n(h)) \\ &= (-1)^{n(n-1)/2} 2^{4n} 5^{-n} (2n+1) Res_x(Q_n(h), Q'_n(h)). \end{aligned}$$

To evaluate $Res_x(Q_n(h), \frac{4x}{5})$ we used (1.5.1) with $f = Q_n(x)$ which has

degree $2n$ and $g = 4x/5$. To evaluate $Res_x(Q_n(h), Q'_n(h))$ we apply (1.5.6) with $h_0 = 2/5$, $Deg(h) = 2$ and $Deg(Q_n(h)) = n$ to find that

$$\begin{aligned} Res_x(Q_n(h), Q'_n(h)) &= \left[h_0^{n(n-1)} Res_h(Q_n(h), Q'_n(h)) \right]^2 \\ &= 2^{2n(2n-1)} 5^{2n^2} (2n+1)^{2n-2}. \end{aligned}$$

We now have that

$$Res_x(Q_n(x), Q'_n(x)) = (-1)^{n(n-1)/2} 2^{2n(2n+1)} 5^{n(2n-1)} (2n+1)^{2n-1}.$$

Finally apply (1.5.7) with $a_m = 2^{2n}$ to find that

$$Disc_x(Q_n(x)) = 2^{4n^2} 5^{n(2n-1)} (2n+1)^{2n-1}.$$

□

To evaluate the discriminant of the $A_n(r, z)$ polynomials, we will use the Pell equation for Chebyshev polynomials (1.4.10), the derivative of $T_n(x)$ given by $\frac{d}{dx}T_n(x) = nU_{n-1}(x)$, and (1.4.9).

Theorem 26. *For $n \geq 1$ even, the discriminant of $A_n(r, z)$ with respect to z is given by*

$$D_z(A_n(r, z)) = 2^{(n-1)^2} n^n \alpha^{n(n-1)} (1+r) (1-r^2)^{n/2-1}.$$

Proof. We first write $A_{2n}(r, z) = \alpha^{2n}(2T_n^2(z/\alpha) - 1 - r)$ using (1.4.9). Its derivative

$$\frac{d}{dz}A_{2n}(r, z) = \frac{4n}{\alpha} \alpha^{2n} T_n(z/\alpha) U_{n-1}(z/\alpha),$$

has leading coefficient $2^{2n}n$. Now taking resultants, we find that

$$\begin{aligned} Res_z(A_{2n}(r, z), A'_{2n}(r, z)) &= Res_z(A_{2n}(r, z), 4n\alpha^n T_n(z/\alpha)) Res_z(A_{2n}(r, z), \alpha^{n-1} U_{n-1}(z/\alpha)) \\ &= (-1)^n 2^{2n^2} (2n)^{2n} \alpha^{2n^2} (r+1)^n Res_z(A_{2n}(r, z), n^{n-1} U_{n-1}(z/\alpha)). \end{aligned}$$

To evaluate $Res_z(A_{2n}(r, z), \alpha^{n-1}U_{n-1}(z/\alpha))$, we use (1.4.10) to deduce that

$$\begin{aligned} A_{2n}(r, z) &= \alpha^{2n} \left(2 + 2 \left(\frac{z^2}{\alpha^2} - 1 \right) U_{n-1}^2(z/\alpha) - 1 - r \right) \\ &= 2\alpha^{2n} \left(\frac{z^2}{\alpha^2} - 1 \right) U_{n-1}^2(z/\alpha) + \alpha^{2n}(1 - r). \end{aligned}$$

Since $\alpha^{n-1}U_{n-1}(z/\alpha)$ has 2^{n-1} as the leading coefficient, we find that

$$Res_z(A_{2n}(r, z), \alpha^{n-1}U_{n-1}(z/\alpha)) = 2^{2n(n-1)}\alpha^{2n(n-1)}(1 - r)^{n-1}.$$

Hence

$$Res_z(A_{2n}(r, z), A'_{2n}(r, z)) = (-1)^n 2^{2n(2n-1)} (2n)^{(2n)} \alpha^{2n(2n-1)} (1+r)(1-r^2)^{n-1}.$$

Finally apply (1.5.7) with $a_0 = 2^{2n-1}$ to complete the proof. $\square$

For n odd, the identity for the discriminants of $A_n(r, z)$ is not as straight forward. We first state some identities for Chebyshev polynomials that will be used in the proof.

Lemma 27. *The even Chebyshev polynomials of the second kind have a factorization over $\mathbb{Z}$ given by*

$$U_{2d}(x) = (U_d(x) - U_{d-1}(x))(U_d(x) + U_{d-1}(x)). \quad (1.5.13)$$

Proof. If $x = \cos \theta$, then using trigonometric identities,

$$\begin{aligned} U_d^2(x) - U_{d-1}^2(x) &= \sin^{-2} \theta (\sin^2(d+1)\theta - \sin^2 d\theta) \\ &= \frac{\sin(2d+1)\theta}{\sin \theta} \\ &= U_{2d}(x). \end{aligned}$$

$\square$

Lemma 28. *The odd Chebyshev polynomials of the first kind and Chebyshev polynomials of the second kind are also related by the identities,*

$$T_{2d+1}(x) + 1 = (x+1)(U_d(x) - U_{d-1}(x))^2, \quad (1.5.14)$$

$$T_{2d+1}(x) - 1 = (x-1)(U_d(x) + U_{d-1}(x))^2. \quad (1.5.15)$$

Proof. If $x = \cos \theta$, then using well known trigonometric identities, we find that

$$\begin{aligned} (x+1)(U_d(x) - U_{d-1}(x))^2 &= \frac{(\cos \theta + 1)}{\sin^2 \theta} (\sin(d+1)\theta - \sin d\theta)^2 \\ &= 1 + \cos(2d+1)\theta \\ &= 1 + T_{2d+1}(x). \end{aligned}$$

By (1.4.10) and Lemma 34,

$$T_{2d+1}^2(x) - 1 = (x^2 - 1)(U_d^2(x) - U_{d-1}^2(x))^2.$$

Using the difference of squares on both sides and substituting in the values of (1.5.14), we find (1.5.15). $\square$

Theorem 29. *For $n \geq 1$ odd, the discriminant of $A_n(r, z)$ with respect to z is given by*

$$D_z(A_n(r, z)) = 2^{(n-1)^2} n^n \alpha^{n(n-1)} (1 - r^2)^{(n-1)/2}.$$

Proof. For $n = 2k + 1$, then, $A_{2k+1}(r, z)$ has leading coefficient 2^{2k} . The derivative is given by

$$\frac{d}{dz} A_{2k+1}(r, z) = (2k+1) \alpha^{2k} U_{2k}(z/\alpha),$$

and has leading coefficient $2^{2k}(2k+1)$. By equation (1.5.13),

$$U_{2k}(z/\alpha) = (U_k(z/\alpha) - U_{k-1}(z/\alpha))(U_k(z/\alpha) + U_{k-1}(z/\alpha))$$

and by equations (1.5.14) and (1.5.15) we can write $A_{2k+1}(r, z)$ as

$$A_{2k+1}(r, z) = \alpha^{2k+1}(z/\alpha+1) (U_k(z/\alpha) - U_{k-1}(z/\alpha))^2 - \alpha^{2k+1}(1+r), \quad (1.5.16)$$

or

$$A_{2k+1}(r, z) = \alpha^{2k+1}(z/\alpha+1) (U_k(z/\alpha) + U_{k-1}(z/\alpha))^2 + \alpha^{2k+1}(1-r). \quad (1.5.17)$$

To evaluate the resultant of $A_{2k+1}(r, z)$ and $\frac{d}{dz} A_{2k+1}(r, z)$ with respect to

z , we will apply (1.5.5) with

$$g_1(x) = \alpha^k (U_k(z/\alpha) - U_{k-1}(z/\alpha))$$

and

$$g_2(x) = \alpha^k (U_k(z/\alpha) + U_{k-1}(z/\alpha)).$$

Notice that $g_1(x)$ and $g_2(x)$ have leading coefficient 2^k .

First using $A_{2k+1}(r, z)$ as in (1.5.16), we have

$$\text{Res}_z(A_{2k+1}(r, z), g_1) = (-1)^k 2^{k(2k+1)} \alpha^{k(2k+1)} (1+r)^k,$$

and then using $A_{2k+1}(r, z)$ as in (1.5.17), we have

$$\text{Res}_z(A_{2k+1}(r, z), g_2) = 2^{k(2k+1)} \alpha^{k(2k+1)} (1-r)^k.$$

Since $\frac{d}{dz} A_{2k+1}(r, z) = (2k+1)g_1g_2$, we find that

$$\text{Res}_z \left(A_{2k+1}(r, z), \frac{d}{dz} A_{2k+1}(r, z) \right) = (-1)^k 2^{2k(2k+1)} (2k+1)^{2k+1} \alpha^{2k(2k+1)} (1-r^2)^k.$$

Finally apply (1.5.7) with $a_0 = 2^{2k}$ to get

$$D_z(A_{2k+1}(r, z)) = 2^{4k^2} (2k+1)^{2k+1} \alpha^{2k(2k+1)} (1-r^2)^k.$$

□

Corollary 30. *For $n \geq 1$, the discriminant of $T_n(z/\alpha) - T_n(\lambda/\alpha)$ with respect to z is given by*

$$\begin{aligned} & D_z(T_n(z/\alpha) - T_n(\lambda/\alpha)) \\ &= \begin{cases} 2^{(n-1)^2} n^n \alpha^{-\frac{1}{2}n(2n-1)+1} \kappa^{n-2} (1 + T_n(\lambda/\alpha)) U_{n-1}(\lambda/\alpha)^{n-2}, & \text{for } n \text{ even,} \\ 2^{(n-1)^2} n^n \alpha^{-\frac{1}{2}n(2n-1)+1} \kappa^{n-1} U_n(\lambda/\alpha)^{n-1}, & \text{for } n \text{ odd.} \end{cases} \end{aligned} \tag{1.5.18}$$

Proof. Recall that $\lambda^2 + \kappa^2 = \alpha^2$, and it is easy to show that for all $n \geq 0$,

$$1 - T_n(x)^2 = (1 - x^2) U_{n-1}(x)^2.$$

The proof is now immediate by identifying that

$$T_n(z/\alpha) - T_n(\lambda/\alpha) = \frac{1}{\alpha^n} A_n(T_n(\lambda/\alpha), z).$$

□

1.6 Translations of $U_n(x)$ and remarks

Translated Chebyshev polynomials of the second kind display a similar root distribution as those of the first kind when their degrees are sufficiently large. We will show that the ellipse $E(\lambda, \kappa)$ is the limiting curve for the set of zeros of $U_n(z/\alpha) - U_n(i\kappa/\alpha)$ and $U_n(z/\alpha) - U_n(\lambda/\alpha)$.

From equations (1.1.6) and (1.1.7), we have

$$\sqrt{z - \alpha^2} = \frac{\kappa}{\lambda}x + \frac{i\lambda}{\kappa}y,$$

and together with the standard equation for Chebyshev polynomials of the second kind (1.0.3) we deduce after simplification

$$U_n(z/\alpha) = \frac{2^n \lambda \kappa}{\alpha^n (\kappa^2 x + \lambda^2 y)} \left(\phi^{n+1} \left(\frac{x}{\lambda} + \frac{iy}{\kappa} \right)^{n+1} - \hat{\phi}^{n+1} \left(\frac{x}{\lambda} - \frac{iy}{\kappa} \right)^{n+1} \right),$$

$$U_n(i\kappa/\alpha) = \frac{2^n i^n}{\alpha^n \lambda} \left(\phi^{n+1} + (-1)^n \hat{\phi}^{n+1} \right)$$

and

$$U_n(\lambda/\alpha) = \frac{2^n}{\alpha^n \kappa} \left(\phi^{n+1} - \hat{\phi}^{n+1} \right).$$

It follows that

$$\begin{aligned} U_n(z/\alpha) - U_n(i\kappa/\alpha) &= \frac{2^n \phi^{n+1} \lambda \kappa}{\alpha^n (\kappa^2 x + i\lambda^2 y)} \left(\left\{ \left(\frac{x}{\lambda} + \frac{iy}{\kappa} \right)^{n+1} - i^n \frac{\kappa^2 x + \lambda^2 y}{\lambda^2 \kappa} \right\} \right. \\ &\quad \left. - \left(\frac{\hat{\phi}}{\phi} \right)^{n+1} \left\{ \left(\frac{x}{\lambda} - \frac{iy}{\kappa} \right)^{n+1} - (-i)^n \frac{\kappa^2 x + \lambda^2 y}{\lambda^2 \kappa} \right\} \right), \end{aligned}$$

and

$$U_n(z/\alpha) - U_n(\lambda/\alpha) = \frac{2^n \phi^{n+1} \lambda \kappa}{\alpha^n (\kappa^2 x + i \lambda^2 y)} \left(\left\{ \left(\frac{x}{\lambda} + \frac{iy}{\kappa} \right)^{n+1} - \frac{\kappa^2 x + \lambda^2 y}{\lambda \kappa^2} \right\} - \left(\frac{\hat{\phi}}{\phi} \right)^{n+1} \left\{ \left(\frac{x}{\lambda} - \frac{iy}{\kappa} \right)^{n+1} - \frac{\kappa^2 x + \lambda^2 y}{\lambda \kappa^2} \right\} \right).$$

Since $\hat{\phi}/\phi < 1$, in the limit as $n \rightarrow \infty$, the roots of $U_n(z/\alpha) - U_n(i\kappa/\alpha)$ and $U_n(z/\alpha) - U_n(\lambda/\alpha)$ are respectively distributed close to the curves

$$\frac{\lambda \kappa}{\kappa^2 x + i \lambda^2 y} \left(\frac{x}{\lambda} + \frac{iy}{\kappa} \right)^{n+1} - \frac{i^n}{\lambda}$$

and

$$\frac{\lambda \kappa}{\kappa^2 x + i \lambda^2 y} \left(\frac{x}{\lambda} + \frac{iy}{\kappa} \right)^{n+1} - \frac{1}{\kappa}.$$

These curves can also be written as

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\kappa^2} = \left(\frac{c(x, y)}{\lambda} \right)^{\frac{1}{n+1}} \quad \text{and} \quad \frac{x^2}{\lambda^2} + \frac{y^2}{\kappa^2} = \left(\frac{c(x, y)}{\kappa} \right)^{\frac{1}{n+1}}$$

respectively, where $c(x, y) = (\kappa^4 x^2 + \lambda^4 y^2)/\lambda \kappa$. It is easy to see that the limiting curve is indeed the ellipse $E(\lambda, \kappa)$. For z on $E(\lambda, \kappa)$ we have $c(x, y) = c(x) = (\lambda^4 - \alpha^2 x^2)\kappa/\lambda$ and from the conditions $\lambda > \kappa$ we find that if $|x| \leq \lambda$,

$$(c(x)/\lambda)^{1/(n+1)} \leq 1 \leq (c(x)/\kappa)^{1/(n+1)},$$

so that the roots of $U_n(z/\alpha) - U_n(i\kappa/\alpha)$ lie inside but very close to the ellipse $E(\lambda, \kappa)$, while the roots of $U_n(z/\alpha) - U_n(\lambda/\alpha)$ lie outside but very close to the ellipse $E(\lambda, \kappa)$.

It turns out that in certain cases, “products” of the $T_n(z/\alpha) - T_n(i\kappa/\alpha)$ and $T_n(z/\alpha) - T_n(\lambda/\alpha)$ polynomials have their roots close to $E(\lambda, \kappa)$ where as before, λ and κ are non-zero real numbers and $\alpha^2 = \lambda^2 - \kappa^2$ is non-zero. For a non-negative integer m , consider the polynomials

$$\mathcal{P}_{n,m}(z) := T_n(z/\alpha)T_m(z/\alpha) - T_n(i\kappa/\alpha)T_m(i\kappa/\alpha),$$

and

$$\mathcal{Q}_{n,m}(z) := T_n(z/\alpha)T_m(z/\alpha) - T_n(\lambda/\alpha)T_m(\lambda/\alpha).$$

These can respectively be written as linear combinations

$$\mathcal{P}_{N,r}(z) := T_N(z/\alpha) - T_N(i\kappa/\alpha) + T_r(z/\alpha) - T_r(i\kappa/\alpha),$$

and

$$\mathcal{Q}_{N,r}(z) := T_N(z/\alpha) - T_N(\lambda/\alpha) + T_r(z/\alpha) - T_r(\lambda/\alpha)$$

where $N = n + m$ and $r = |n - m|$.

Using equation (1.1.9), we have a factorization similar to that in equations (1.1.12) and (1.1.14) given by

$$\begin{aligned} \mathcal{P}_{N,r}(z) &= \frac{2^{N-1}\phi^N}{\alpha^N} \left(\left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^N - i^N \right) \left(1 - i^N \left(\frac{-\hat{\phi}}{\phi} \right)^N \left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^{-N} \right) \right. \\ &\quad \left. + \frac{\alpha^{N-r}}{2^{N-r}\phi^{N-r}} \left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^r - i^r \right) \left(1 - i^r \left(\frac{-\hat{\phi}}{\phi} \right)^r \left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^{-r} \right) \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_{N,r}(z) &= \frac{2^{N-1}\phi^N}{\alpha^N} \left(\left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^N - 1 \right) \left(1 - \left(\frac{\hat{\phi}}{\phi} \right)^N \left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^{-N} \right) \right. \\ &\quad \left. + \frac{\alpha^{N-r}}{2^{N-r}\phi^{N-r}} \left(\left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^r - 1 \right) \left(1 - \left(\frac{\hat{\phi}}{\phi} \right)^r \left(\frac{x}{\lambda} + i\frac{y}{\kappa} \right)^{-r} \right) \right). \end{aligned}$$

Since $\hat{\phi}/\phi < 1$ and $\frac{\alpha}{2\phi} < 1$, this implies that for a fixed r , and N sufficiently large, the roots of $\mathcal{P}_{N,r}(z)$ and $\mathcal{Q}_{N,r}(z)$ lie close to the ellipse $E(\lambda, \kappa)$. In particular if $\lambda = \sqrt{5}$, $\kappa = 1$ and n is sufficiently large, the roots of $T_n(z/2)T_{n+r}(z/2) - T_n(i/2)T_{n+r}(i/2)$ and $T_n(z/2)T_{n+r}(z/2) - T_n(\sqrt{5}/2)T_{n+r}(\sqrt{5}/2)$ lie close to the ellipse $x^2/5 + y^2 = 1$.

CHAPTER 2

LINEAR COMBINATION OF CHEBYSHEV POLYNOMIALS

In this chapter we study roots of a linear combination of two Chebyshev polynomials of the first kind, i.e., $T_n(x) - kT_{n-1}(x)$ as well as of the second kind i.e., $U_n(x) - kU_{n-1}(x)$. We then consider the case where the linear combination involves Chebyshev polynomials of each kind, i.e., a polynomial of the form $T_n(x) - kU_{n-1}(x)$ where k is a real number. We employ continued fractions in this investigation. When k is an integer, the continued fraction of the largest root (in absolute value) begins in a very interesting way which we can easily describe. We then use convergents of the continued fraction to give estimates for the bounds of the largest root of these linear combinations.

Generalizations $T_{n+r}(x) - kT_{n-1}(x)$ and $U_{n+r}(x) - kU_{n-1}(x)$ of these linear combinations of Chebyshev polynomials have interesting root distributions. We show that in both cases, some of the roots of these generalized linear combinations lie arbitrarily close to ellipses studied in chapter 1.

In the second part of this chapter, we study linear combinations of the form $T_n(x) - \frac{1}{2}kU_{n-2}(x)$. Again, we give approximations for the largest (and smallest) root that lies outside $(-1, 1)$. This polynomial in particular has other interesting identities that will be presented.

We conclude the chapter by giving identities for resultants and discriminants of some of the polynomials studied. Of particular interest is the discriminant of $T_n(x) - \frac{1}{2}kU_{n-2}(x)$. We introduce the polynomials $b_n(k)$ (see equation (2.3.12)) in the calculation of the discriminant of $T_n(x) - \frac{1}{2}kU_{n-2}(x)$ when n is even. These complement the polynomials $a_n(k)$ introduced by Dilcher-Stolarsky [17].

2.1 “First kind linear combination” of Chebyshev polynomials

For $n \geq 1$ and $k \in \mathbb{R}$ non zero, let

$$R_n(k, x) := U_n(x) - kU_{n-1}(x), \quad (2.1.1)$$

and

$$S_n(k, x) := T_n(x) - kT_{n-1}(x). \quad (2.1.2)$$

The roots of $R_n(k, x)$ are the points of intersection of $U_n(x)$ and $kU_{n-1}(x)$. Scaling a polynomial does not change the number or location of its roots. The number of critical points is also unchanged, but the location of critical points depends on the scalar. We consider two cases for the number of intersections of $U_n(x)$ and $U_{n-1}(x)$ in $(-1, 1)$ and with $k > 1$. For n odd, $U_n(x)$ has a root at $x = 0$ and has $\frac{n-1}{2}$ critical points in $(0, 1)$ and in $(-1, 0)$. Similarly, $kU_{n-1}(x)$ has a critical point at $x = 0$ and $\frac{n-3}{2}$ critical points in $(0, 1)$ and in $(-1, 0)$. So the two polynomials have exactly $\frac{n-1}{2}$ intersections in $(0, 1)$ and by symmetry $n - 1$ intersections in $(-1, 1)$.

On the other hand for n even, $U_n(x)$ has a critical point at $x = 0$ and $\frac{n}{2} - 1$ critical points in $(0, 1)$ and by symmetry in $(-1, 0)$. Similarly, $kU_{n-1}(x)$ has a root at $x = 0$ and $\frac{n}{2} - 1$ critical points in $(0, 1)$, and by symmetry in $(-1, 0)$. The two polynomials have exactly $\frac{n}{2} - 1$ intersections in $(0, 1)$, and $\frac{n}{2} + 1$ intersections in $(-1, 0)$. As x approaches -1 , $U_{2n}(x)$ approaches $2n + 1$ and $U_{2n-1}(x)$ approaches $-2n$ giving the extra point of intersection, and hence a root of $R_n(k, x)$ in $(-1, 0)$. We now deduce that $R_n(k, x)$ has $n - 1$ roots in $(-1, 1)$. By a similar argument, $S_n(k, x)$ has $n - 1$ roots in $(-1, 1)$.

What can be said of the root of $R_n(k, x)$ and $S_n(k, x)$ that lie $(-1, 1)$? Call these roots r_n and s_n respectively. A quick observation is that when $k > 1$, r_n and s_n are positive and when $k < 1$, r_n and s_n are negative. Also when $-1 \leq k \leq 1$, all the roots of $R_n(k, x)$ lie in $(-1, 1)$. We can also observe that $S_n(1, x)$ has a root at $x = 1$ and $S_n(-1, x)$ has a root at $x = -1$, since $T_n(1) = 1$ and $T_n(-1) = (-1)^n$ for all n . When $-1 < k < 1$, all roots of $S_n(k, x)$ lie in $(-1, 1)$.

Let $k > 1$ be an odd integer. The continued fraction of r_n starts off as

$$\left[\frac{k-1}{2}, 1, 1, \frac{k-1}{2}, 2q_n(k), \left\lfloor \frac{k-1}{2(n-1)} \right\rfloor, \dots \right] \quad (2.1.3)$$

where

$$\begin{aligned} q_n(k) &= \sum_{j=1}^n (n-j)k^{(2j-1)} \\ &= \frac{k(k^{2n}-1)}{(k^2-1)^2} - \frac{nk}{k^2-1}. \end{aligned} \quad (2.1.4)$$

On the other hand, let $k > 1$ be an even integer. The continued fraction of r_n starts off as

$$\left[\frac{1}{2}k, 2k, \frac{1}{2}q_n(k), \left\lfloor \frac{k}{n-1} \right\rfloor, \dots \right]. \quad (2.1.5)$$

If $[a_0, a_1, a_2, \dots, a_k, \dots]$ is a continued fraction of a positive real number α with convergents

$$\frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}},$$

where $p_0 = a_0$, $p_{-1} = 1$, $q_0 = 1$ and $q_{-1} = 0$, then by Theorem 7.8 of [26, p 193],

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \dots < \alpha < \dots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

In both cases (i.e. (2.1.3) and (2.1.5)), we have

$$\frac{k + (k^2 + 1)q_n(k)}{2 + 2kq_n(k)} < r_n < \frac{k^2 + 1}{2k}.$$

Since $q_n(k)$ is a polynomial with non-negative coefficients, whenever $k > 1$, $q_n(k)$ grows without bound. This implies that

$$\lim_{n \rightarrow \infty} r_n = \frac{k^2 + 1}{2k}.$$

Theorem 31. *Let $|k| > 1$ and r_n be the largest root of $R_n(k, x) := U_n(x) - kU_{n-1}(x)$ in absolute value that is outside $(-1, 1)$. Then for all $n > 1$,*

$$\frac{k + (k^2 + 1)q_n(k)}{2 + 2kq_n(k)} < r_n < \frac{k^2 + 1}{2k}.$$

Proof. To simplify the notation, let

$$B = q_n(k) = \frac{k(k^{2n} - 1)}{(k^2 - 1)^2} - \frac{nk}{k^2 - 1} \quad (2.1.6)$$

and

$$\nu = \frac{k + (k^2 + 1)B}{2(kB + 1)}. \quad (2.1.7)$$

First, we show that for all $k > 1$, $R_n(k, \nu)$ is a very small (in absolute value) negative number.

First,

$$\nu^2 - 1 = \frac{(k^2 - 1)^2 B^2}{4(Bk + 1)^2} \left(1 + \frac{2k(k^2 - 3)}{(k^2 - 1)^2 B} + \frac{k^2 - 4}{(k^2 - 1)^2 B^2} \right), \quad (2.1.8)$$

and by the Taylor series expansion of the square-root we find that

$$\begin{aligned} \sqrt{\nu^2 - 1} &= \frac{(k^2 - 1)B}{2(Bk + 1)} \left(1 + \frac{k(k^2 - 3)}{(k^2 - 1)^2 B} - \frac{2}{(k^2 - 1)^4 B^2} \right. \\ &\quad \left. - \frac{k(k^2 - 3)(k^2 - 4)}{4(k^2 - 1)^4 B^3} + \mathcal{O}(B^{-4}) \right) \\ &= \frac{(k^2 - 1)B}{2(Bk + 1)} + \frac{k(k^2 - 3)}{2(k^2 - 1)(Bk + 1)} - \frac{1}{(k^2 - 1)^3(Bk + 1)B} \\ &\quad - \frac{k(k^2 - 3)(k^2 - 4)}{8(k^2 - 1)^3(Bk + 1)B^2} + \mathcal{O}(B^{-4}), \end{aligned}$$

so that

$$\begin{aligned} \nu + \sqrt{\nu^2 - 1} &= k - \frac{k}{(k^2 - 1)(Bk + 1)} - \frac{1}{(k^2 - 1)^3(Bk + 1)B} \\ &\quad - \frac{k(k^2 - 3)(k^2 - 4)}{8(k^2 - 1)^3(Bk + 1)B^2} + \mathcal{O}(B^{-4}), \end{aligned}$$

and

$$\begin{aligned} \nu - \sqrt{\nu^2 - 1} &= \frac{1}{k} + \frac{1}{k(k^2 - 1)(Bk + 1)} + \frac{1}{(k^2 - 1)^3(Bk + 1)B} \\ &\quad + \frac{k(k^2 - 3)(k^2 - 4)}{8(k^2 - 1)^3(Bk + 1)B^2} + \mathcal{O}(B^{-4}). \end{aligned}$$

We now have

$$\begin{aligned} (\nu + \sqrt{\nu^2 - 1})^n &= k^n - \frac{nk^n}{(k^2 - 1)(Bk + 1)} - \frac{nk^{n-1}}{(k^2 - 1)^3(Bk + 1)B} \\ &\quad + \frac{n(n-1)k^n}{2(k^2 - 1)^2(Bk + 1)^2} + \mathcal{O}(B^{-3}), \end{aligned}$$

and

$$\begin{aligned} (\nu - \sqrt{\nu^2 - 1})^n &= k^{-n} + \frac{nk^{-n}}{(k^2 - 1)(Bk + 1)} + \frac{nk^{-n+1}}{(k^2 - 1)^3(Bk + 1)B} \\ &\quad + \frac{n(n-1)k^{-n}}{(k^2 - 1)^2(Bk + 1)^2} + \mathcal{O}(B^{-3}). \end{aligned}$$

Again we have

$$\begin{aligned} (\nu + \sqrt{\nu^2 - 1})^{n+1} - k(\nu + \sqrt{\nu^2 - 1})^n &= \frac{-k^{n+1}}{(k^2 - 1)(Bk + 1)} - \frac{k^n}{(k^2 - 1)^3(Bk + 1)B} \\ &\quad + \frac{nk^{n+1}}{(k^2 - 1)^2(Bk + 1)^2} + \mathcal{O}(B^{-3}) \quad (2.1.9) \end{aligned}$$

and

$$\begin{aligned} (\nu - \sqrt{\nu^2 - 1})^{n+1} - k(\nu - \sqrt{\nu^2 - 1})^n &= \frac{-(k^2 - 1)}{k^{n+1}} + \frac{(1 - n(k^2 - 1))k^{-(n+1)}}{(k^2 - 1)(Bk + 1)} \\ &\quad + \frac{nk^{-n+1}(n + 1 - (n - 1)k^2)}{(k^2 - 1)^2(Bk + 1)^2} + \mathcal{O}(B^{-3}). \quad (2.1.10) \end{aligned}$$

Finally Subtracting (2.1.10) from (2.1.9) and using (1.0.3) we find that

$$\begin{aligned} 2\sqrt{\nu^2 - 1}R_n(k, \nu) &= \frac{k^2 - 1}{k^{n+1}} - \frac{k^{2(n+1)} + 1 - n(k^2 - 1)}{k^{n+1}(k^2 - 1)(Bk + 1)} \\ &\quad + \frac{n(2k^{2n} + (n - 1)k^2 - n - 1)}{k^{n-1}(k^2 - 1)^2(Bk + 1)^2} \\ &\quad - \frac{k^{2n} - nk^2 + n + 1}{k^n(k^2 - 1)^3(Bk + 1)B} + \mathcal{O}(B^{-3}). \end{aligned}$$

For all $n, k > 1$,

$$\frac{k^2 - 1}{k^{n+1}} - \frac{k^{2(n+1)} + 1 - n(k^2 - 1)}{k^{n+1}(k^2 - 1)(Bk + 1)} < 0,$$

and since $\sqrt{\nu^2 - 1} > 0$, we must have that $R_n(k, \nu) < 0$ and is arbitrarily close to zero.

The proof for the upper bound is much easier. Let $\tilde{\nu}$ be the ratio of the arithmetic mean to the geometric mean of k^2 and 1,

$$\tilde{\nu} = \frac{k^2 + 1}{2k}. \quad (2.1.11)$$

Then $\sqrt{\tilde{\nu}^2 - 1} = \frac{k^2 - 1}{2k}$ and

$$\tilde{\nu} + \sqrt{\tilde{\nu}^2 - 1} = k \quad \text{and} \quad \tilde{\nu} - \sqrt{\tilde{\nu}^2 - 1} = \frac{1}{k}.$$

By (1.0.3),

$$U_n(\tilde{\nu}) = \frac{k}{k^2 - 1} (k^{n+1} - k^{-(n+1)}),$$

and substituting in (2.1.1) gives

$$R_n(k, \tilde{\nu}) = \frac{1}{k^n} > 0.$$

For the case when $k < 1$ notice that if r_n is a root of $R_n(k, x)$, then $-r_n$ is a root of $R_n(-k, x)$. The proof is now identical to the case when k is positive. $\square$

Let $k > 1$ be an odd integer, the continued fraction of s_n starts off as

$$\left[\frac{k-1}{2}, 1, 1, \frac{k-3}{2}, 1, 2q_{n-1}(k) - 1, \dots \right], \quad (2.1.12)$$

while for an even integer $k > 1$, the continued fraction starts off as

$$\left[\frac{1}{2}k, 2k-1, 1, \frac{1}{2}q_{n-1}(k) - 1, \dots \right], \quad (2.1.13)$$

where $q_n(k)$ is as defined in (2.1.4).

In both cases (i.e. for all positive integers k) and for all $n > 2$, we have

$$\frac{k^2 + 1}{2k} < s_n < \frac{(k^2 + 1)q_{n-1}(k) - k}{2kq_{n-1}(k) - 2}.$$

This implies that

$$\lim_{n \rightarrow \infty} s_n = \frac{k^2 + 1}{2k}.$$

Theorem 32. *Let $|k| > 1$ and s_n be the largest root of $S_n(k, x) := T_n(x) - kT_{n-1}(x)$ in absolute value that is outside $(-1, 1)$. Then for all $n > 2$,*

$$\frac{k^2 + 1}{2k} < s_n < \frac{(k^2 + 1)q_{n-1}(k) - k}{2kq_{n-1}(k) - 2}.$$

Proof. To simplify notation, let

$$B = q_{n-1}(k) = \frac{k(k^{2(n-1)} - 1)}{(k^2 - 1)^2} - \frac{(n-1)k}{k^2 - 1}, \quad (2.1.14)$$

and

$$\mu = \frac{(k^2 + 1)B - k}{2(Bk - 1)}. \quad (2.1.15)$$

We first show that when $k > 1$, $S_n(k, \mu)$ is a very small negative number. By (2.1.15),

$$\mu^2 - 1 = \frac{(k^2 - 1)^2 B^2}{4(Bk - 1)^2} \left(1 - \frac{2k(k^2 - 3)}{(k^2 - 1)^2 B} + \frac{k^2 - 4}{(k^2 - 1)^2 B^2} \right), \quad (2.1.16)$$

and by the Taylor series expansion of the square-root we get

$$\begin{aligned} \sqrt{\mu^2 - 1} &= \frac{(k^2 - 1)B}{2(Bk - 1)} \left(1 - \frac{k(k^2 - 3)}{(k^2 - 1)^2 B} - \frac{2}{(k^2 - 1)^4 B^2} + \mathcal{O}(B^{-3}) \right) \\ &= \frac{(k^2 - 1)B}{2(Bk - 1)} - \frac{k(k^2 - 3)}{2(k^2 - 1)(Bk - 1)} - \frac{1}{(k^2 - 1)^3 (Bk - 1)B} + \mathcal{O}(B^{-3}), \end{aligned}$$

so that

$$\mu + \sqrt{\mu^2 - 1} = k + \frac{k}{(k^2 - 1)(Bk - 1)} - \frac{1}{(k^2 - 1)^3 (Bk - 1)B} + \mathcal{O}(B^{-3}),$$

and

$$\mu - \sqrt{\mu^2 - 1} = k^{-1} - \frac{k^{-1}}{(k^2 - 1)(Bk - 1)} + \frac{1}{(k^2 - 1)^3 (Bk - 1)B} + \mathcal{O}(B^{-3}).$$

We now have

$$\begin{aligned} (\mu + \sqrt{\mu^2 - 1})^n &= k^n + \frac{nk^n}{(k^2 - 1)(Bk - 1)} - \frac{nk^{n-1}}{(k^2 - 1)^3(Bk - 1)B} \\ &\quad + \frac{n(n-1)k^n}{2(k^2 - 1)^2(Bk - 1)^2} + \mathcal{O}(B^{-3}), \end{aligned}$$

and

$$\begin{aligned} (\mu - \sqrt{\mu^2 - 1})^n &= k^{-n} - \frac{nk^{-n}}{(k^2 - 1)(Bk - 1)} + \frac{nk^{-n+1}}{(k^2 - 1)^3(Bk - 1)B} \\ &\quad + \frac{n(n-1)k^{-n}}{2(k^2 - 1)^2(Bk - 1)^2} + \mathcal{O}(B^{-3}). \end{aligned}$$

Again we have

$$\begin{aligned} (\mu + \sqrt{\mu^2 - 1})^n - k(\mu + \sqrt{\mu^2 - 1})^{n-1} &= \frac{k^n}{(k^2 - 1)(Bk - 1)} - \frac{k^{n-1}}{(k^2 - 1)^3(Bk - 1)B} \\ &\quad + \frac{(n-1)k^n}{(k^2 - 1)^2(Bk - 1)^2} + \mathcal{O}(B^{-3}), \quad (2.1.17) \end{aligned}$$

and

$$\begin{aligned} (\mu - \sqrt{\mu^2 - 1})^n - k(\mu - \sqrt{\mu^2 - 1})^{n-1} &= \frac{-(k^2 - 1)}{k^n} + \frac{(n-1)k^2 - n}{k^n(k^2 - 1)(Bk - 1)} \\ &\quad - \frac{(n-1)k^2 - n}{k^{n-1}(k^2 - 1)^3(Bk - 1)B} \quad (2.1.18) \\ &\quad + \frac{(n-1)(n-(n-2)k^2)}{k^n(k^2 - 1)^2(Bk - 1)^2} + \mathcal{O}(B^{-3}). \end{aligned}$$

Finally adding (2.1.17) to (2.1.18) and using (1.0.2) we find that

$$\begin{aligned} 2S_n(k, \mu) &= \frac{1 - k^2}{k^n} + \frac{k^{2n} + (n-1)k^2 - n}{k^n(k^2 - 1)(Bk - 1)} \\ &\quad + \frac{(n-1)(k^{2n} - (n-2)k^2 + n)}{k^n(k^2 - 1)^2(Bk - 1)^2} \\ &\quad - \frac{k^{2(n-1)} + (n-1)k^2 - n}{k^{n-1}(k^2 - 1)^3(Bk - 1)B} + \mathcal{O}(B^{-3}). \end{aligned}$$

It can be verified numerically that for all $n \geq 3$ and $k > 1$,

$$0 < \frac{1 - k^2}{k^n} + \frac{k^{2n} + (n-1)k^2 - n}{k^n(k^2 - 1)(Bk - 1)},$$

so that for $n \geq 3$, $S_n(k, \mu) > 0$ and is arbitrarily close to zero.

Note that this upper bound for the largest zero of $S_n(k, \mu)$ does not work for $n = 1, 2$ because $q_0(a, k) = q_1(k) = 0$. It is however easy to determine the largest zero as k and $(k + \sqrt{k^2 + 8})/4$ when $n = 1, 2$ respectively.

The proof for the lower bound is much easier. Let $\tilde{\nu}$ be as defined in (2.1.11). Then by (1.0.2)

$$T_n(\tilde{\nu}) = \frac{1}{2} \left(k^n + \frac{1}{k^n} \right).$$

Finally substituting into (2.1.3) gives

$$S_n(k, \tilde{\nu}) = \frac{1 - k^2}{2k^n}.$$

For the case when $k < 1$ notice that if s_n is a root of $S_n(k, x)$, then $-s_n$ is a root of $S_n(-k, x)$. $\square$

Remark 2. For all $n \geq 1$, r_n and s_n are irrationals. Theorems 31 and 32 show that when n is sufficiently large, $\frac{k^2+1}{2k}$ is the best rational approximation of their absolute value.

2.1.1 A generalization of the $R_n(k, x)$ and $S_n(k, x)$ polynomials

In this section, we give a generalization of the $R_n(k, x)$ and $S_n(k, x)$ polynomials and describe their roots. Let $k \geq 0$ and $r \geq 1$, and define

$$R_{n,m}(k, x) := U_{n+m}(x) - kU_n(x) \quad (2.1.19)$$

and

$$S_{n,m}(k, x) := T_{n+m}(x) - kT_n(x). \quad (2.1.20)$$

It can be shown (same as for $R_n(k, x)$ and $S_n(k, x)$) that $R_{n,m}(k, x)$ and $S_{n,m}(k, x)$ have n real roots in $(-1, 1)$. We are now interested in the remaining m roots.

Theorem 33. For n sufficiently large, $S_n(k, x)$ and $R_{n,m}(k, x)$ have m roots close to the m roots of $t_m(k, x)$ defined by

$$t_m(k, x) := k^2 + 1 - 2kT_m(x). \quad (2.1.21)$$

Proof. From the explicit formula (1.0.2) of $T_n(x)$, $S_{n,r}(k, x) = 0$ implies that

$$(x + \sqrt{x^2 - 1})^{n+m} + (x - \sqrt{x^2 - 1})^{n+m} - k(x + \sqrt{x^2 - 1})^n - k(x - \sqrt{x^2 - 1})^n = 0,$$

which implies

$$(x + \sqrt{x^2 - 1})^n((x + \sqrt{x^2 - 1})^m - k) + (x - \sqrt{x^2 - 1})^n((x - \sqrt{x^2 - 1})^m - k) = 0,$$

which also implies

$$\frac{(x - \sqrt{x^2 - 1})^n}{(x + \sqrt{x^2 - 1})^n} = \frac{-((x + \sqrt{x^2 - 1})^m - k)}{(x - \sqrt{x^2 - 1})^m - k}.$$

Rationalizing the numerators on both sides of the above equation and simplifying we find that

$$\frac{1}{(x + \sqrt{x^2 - 1})^{2n}} = \frac{-(k^2 + 1 - 2kT_m(x))}{((x - \sqrt{x^2 - 1})^m - k)^2}.$$

For $x \geq 1$, $\sqrt{x^2 - 1} \geq x - \frac{1}{x}$ so that as n gets larger, the left-hand side vanishes, and the m zeros of $S_{n,r}(k, x)$ approach the zeros of $t_m(k, x)$.

Similarly, using the explicit formula (1.0.3) of $U_n(x)$ in $R_{n,r}(k, x)$, we deduce that $R_{n,r}(k, x) = 0$ implies

$$(x + \sqrt{x^2 - 1})^{n+m+1} - (x - \sqrt{x^2 - 1})^{n+m+1} - k(x + \sqrt{x^2 - 1})^{n+1} + k(x - \sqrt{x^2 - 1})^{n+1} = 0,$$

which implies that

$$(x + \sqrt{x^2 - 1})^{n+1}((x + \sqrt{x^2 - 1})^m - k) - (x - \sqrt{x^2 - 1})^{n+1}((x - \sqrt{x^2 - 1})^m - k) = 0,$$

from which we get

$$\frac{(x - \sqrt{x^2 - 1})^{n+1}}{(x + \sqrt{x^2 - 1})^{n+1}} = \frac{(x + \sqrt{x^2 - 1})^m - k}{(x - \sqrt{x^2 - 1})^m - k}.$$

Rationalizing the numerators on both sides of the above equation and sim-

plifying we find that

$$\frac{1}{(x + \sqrt{x^2 - 1})^{2(n+1)}} = \frac{k^2 + 1 - 2kT_m(x)}{((x - \sqrt{x^2 - 1})^m - k)^2}.$$

So that, as before, the m zeros of $R_{n,r}(k, x)$ approach the zeros of $t_m(k, x)$ as n advances. $\square$

By corollary 16, the roots of $T_m(x) - \alpha_k$, for $\alpha_k > 1$ lie on the ellipse

$$E\left(\lambda_n(\alpha_k), \sqrt{\lambda_n^2(\alpha_k) - 1}\right) : \frac{x^2}{\lambda_n^2(\alpha_k)} + \frac{y^2}{\lambda_n^2(\alpha_k) - 1} = 1,$$

where

$$\lambda_n(\alpha_k) = T_{1/n}(\alpha_k) := \cos\left(\frac{1}{n} \arccos(\alpha_k)\right).$$

If we set

$$\alpha_k = \frac{k^2 + 1}{2k},$$

then

$$\lambda_n(\alpha_k) = \frac{1}{2} (k^{1/n} + k^{-1/n}) \quad \text{and} \quad \sqrt{\lambda_n^2(\alpha_k) - 1} = \frac{1}{2} (k^{1/n} - k^{-1/n}) \quad .$$

This shows that the m roots of $R_{n,m}(k, x)$ and $S_{n,m}(k, x)$ lie close to an ellipse. In fact, a formula for the roots of $t_m(k, x)$ can be deduced from Theorem 17.

Figures 2.1 and 2.2 show the roots of $R_{n,7}(3, x)$ in blue, $S_{n,7}(3, x)$ in red and $t_7(3, x)$ in green with $n = 3$ and with $n = 25$. Notice that for n as small as 25 the roots seem to coincide on the ellipse $E\left(\lambda_n(\alpha_k), \sqrt{\lambda_n^2(\alpha_k) - 1}\right)$.

2.2 “Second kind linear combination” of Chebyshev polynomials

Let $k \in \mathbb{R}$ be non-zero and consider the polynomial

$$P_n(k, x) = T_n(x) - \frac{1}{2}kU_{n-2}(x). \tag{2.2.1}$$

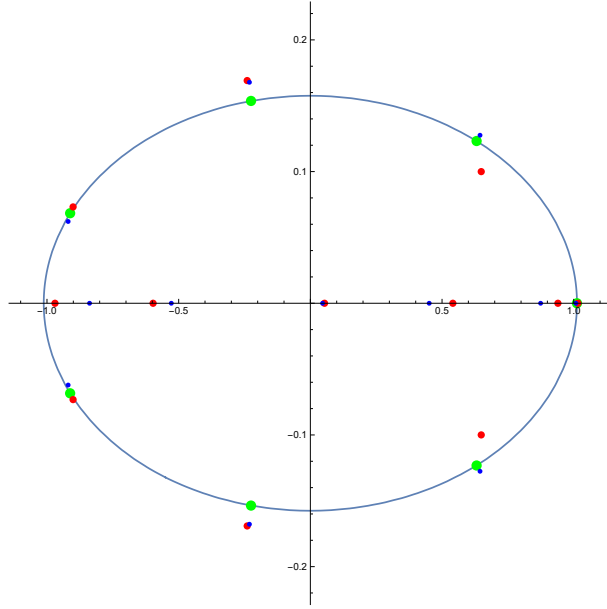


Figure 2.1: Roots of $R_{3,7}(3, x)$ in red, $S_{3,7}(3, x)$ in blue and $t_7(3, x)$ in green.

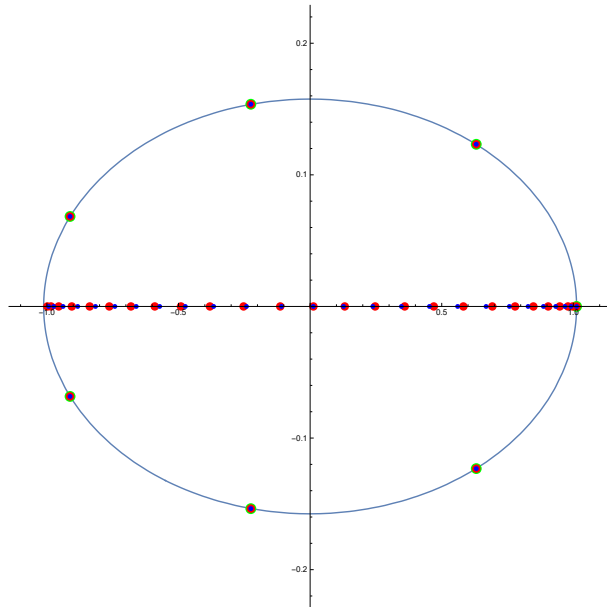


Figure 2.2: Roots of $R_{25,7}(3, x)$ in red, $S_{25,7}(3, x)$ in blue and $t_7(3, x)$ in green.

Using the identities

$$\begin{aligned} T_n(x) &= U_n(x) - xU_{n-1}(x) \\ &= \frac{1}{2}(U_n(x) - U_{n-2}(x)), \end{aligned} \quad (2.2.2)$$

we can rewrite $P_n(k, x)$ as

$$P_n(k, x) = \frac{1}{2}(U_n(x) - (k+1)U_{n-2}(x)). \quad (2.2.3)$$

In this section, we study some interesting properties of the polynomials $P_n(k, x)$. First, we give an identity involving $P_n(k, x)$ when n is an odd integer.

2.2.1 An identity involving $P_n(k, x)$

We first show that $k^n P_{2n+1}(k-1, x) - x$ factors into a cubic and an irreducible polynomial. Let $k \in \mathbb{R}$ and define

$$G_n(k, x) := \frac{k^n (2T_{2n+1}(x) - (k-1)U_{2n-1}(x)) - 2x}{2x(4kx^2 - (k+1)^2)}, \quad (2.2.4)$$

so that

$$G_n(k, x) = \frac{k^n P_{2n+1}(k-1, x) - x}{x(4kx^2 - (k+1)^2)}.$$

We will show that $G_n(k, x)$ is a polynomial in k and x , but first we give a recurrence relation for $G_n(k, x)$.

Lemma 34. *For all $k \in \mathbb{R}$, $G_n(k, x)$ has a recurrence relation given by*

$$G_n(k, x) = k^{n-1}U_{2n-2}(x) + (1-k)G_{n-1}(k, x) + kG_{n-2}(k, x). \quad (2.2.5)$$

Proof. We use the definition of $G_n(k, x)$ to show that the right hand side of

(2.2.5) is just

$$\begin{aligned}
& k^{n-1}U_{2n-2}(x) \\
& + \frac{(1-k)(k^{n-1}P_{2n-1}(k-1, x) - x) + k(k^{n-2}P_{2n-3}(k-1, x) - x)}{x(4kx^2 - (k+1)^2)} \\
& = k^{n-1}U_{2n-2}(x) + \frac{(1-k)k^{n-1}P_{2n-1}(k-1, x) + k^{n-1}P_{2n-3}(k-1, x) - x}{x(4kx^2 - (k+1)^2)}.
\end{aligned}$$

It suffices to show that

$$\begin{aligned}
k^n P_{2n+1}(k-1, x) &= (1-k)k^{n-1}P_{2n-1}(k-1, x) + k^{n-1}P_{2n-3}(k-1, x) \\
&+ k^{n-1}x(4kx^2 - (k+1)^2)U_{2n-2}(x). \tag{2.2.6}
\end{aligned}$$

We use $P_n(k, x)$ as given by (2.2.3) together with the recurrence relation (1.0.1) in the form

$$U_{n+1}(x) = (4x^2 - 2)U_{n-1}(x) - U_{n-3}(x)$$

on the right hand side of (6.1.5) to obtain

$$\begin{aligned}
& (1-k)k^{n-1}P_{2n-1}(k-1, x) + k^{n-1}P_{2n-3}(k-1, x) + k^{n-1}x(4kx^2 - (k+1)^2)U_{2n-2}(x) \\
& = \frac{1}{2}(k^{n-1} - k^n)(U_{2n-1}(x) - kU_{2n-3}(x)) + \frac{1}{2}k^{n-1}(U_{2n-3}(x) - kU_{2n-5}(x)) \\
& \quad + \frac{1}{2}k^{n-1}(4kx^2 - (k+1)^2)U_{2n-2}(x)(U_{2n-1}(x) - kU_{2n-3}(x)) \\
& = \frac{1}{2}(k^{n-1} - k^n)(U_{2n-1}(x) - kU_{2n-3}(x)) + \frac{1}{2}k^{n-1}U_{2n-3}(x) \\
& \quad - \frac{1}{2}k^n((4x^2 - 2)U_{2n-3}(x) - U_{2n-1}(x)) \\
& \quad + \frac{1}{2}k^{n-1}(4kx^2 - (k+1)^2)U_{2n-2}(x)(U_{2n-1}(x) - kU_{2n-3}(x)) \\
& = \frac{1}{2}(k^n(4x^2 - 2) - k^{n+1})U_{2n-1}(x) - \frac{1}{2}k^nU_{2n-3}(x) \\
& = \frac{1}{2}k^n(U_{2n+1}(x) + U_{2n-3}(x)) - \frac{1}{2}k^{n+1}U_{2n-1}(x) - \frac{1}{2}k^nU_{2n-3}(x) \\
& = \frac{1}{2}k^n(U_{2n+1}(x) - kU_{2n-1}(x)),
\end{aligned}$$

as desired. □

We now use the recurrence relation (2.2.5) to prove another identity for $G_n(k, x)$ from which it becomes clear that it is a polynomial.

Theorem 35. *Let $k \in \mathbb{R}$ and $G_n(k, x)$ be as defined in (2.2.4). Then for all $n \geq 1$,*

$$G_n(k, x) = \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{i+j} k^j U_{2i}(x). \quad (2.2.7)$$

Proof. We show that

$$g_n(k, x) := \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{i+j} k^j U_{2i}(x)$$

solves the recurrence relation (2.2.5) under the same initial conditions as those of $G_n(k, x)$ and hence they must be equal. For $n = 0$ it is easy to see that $g_0(k, x) = 0 = G_0(k, x)$ and $g_1(k, x) = U_0(x) = G_1(k, x)$,

$$\begin{aligned} g_{n+1}(k, x) &= \sum_{i=0}^n \sum_{j=i}^n (-1)^{i+j} k^j U_{2i}(x) \\ &= k^n U_{2n}(x) + \sum_{i=0}^{n-2} \left((-1)^i \left\{ \sum_{j=i}^{n-2} (-k)^j + (-k)^{n-1} (1-k) \right\} \right) U_{2i}(x) \\ &= k^n U_{2n}(x) + g_{n-1}(k, x) + k^{n-1} (k-1) \sum_{i=0}^{n-1} (-1)^{n+i} U_{2i}(x). \end{aligned} \quad (2.2.8)$$

We can also write

$$\begin{aligned} g_n(k, x) &= k^{n-1} U_{2n-2}(x) + \sum_{i=0}^{n-2} \left((-1)^i \left\{ \sum_{j=i}^{n-2} (-k)^j + (-k)^{n-1} \right\} \right) U_{2i}(x) \\ &= k^{n-1} U_{2n-2}(x) + g_{n-1}(k, x) - k^{n-1} \sum_{i=0}^{n-2} (-1)^{n+i} U_{2i}(x) \\ &= g_{n-1}(k, x) - k^{n-1} \sum_{i=0}^{n-1} (-1)^{n+i} U_{2i}(x), \end{aligned}$$

so that

$$k^{n-1} \sum_{i=0}^{n-1} (-1)^{n+i} U_{2i}(x) = g_{n-1}(k, x) - g_n(k, x). \quad (2.2.9)$$

Making the substitution in (2.2.8), we obtain

$$\begin{aligned} g_{n+1}(k, x) &= k^n U_{2n}(x) + g_{n-1}(k, x) + (k-1)(g_{n-1}(k, x) - g_n(x)) \\ &= k^n U_{2n}(x) + (1-k)g_n(k, x) + kg_{n-1}(k, x), \end{aligned}$$

as desired. $\square$

The following corollary is immediate.

Corollary 36. *For all $k \in \mathbb{R}$ and all $n \geq 1$,*

$$\frac{k^n (2T_{2n+1}(x) - kU_{2n-1}(x)) - 2x}{2x(4(k+1)x^2 - (k+2)^2)} = \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{i+j} (k+1)^j U_{2i}(x). \quad (2.2.10)$$

2.2.2 Roots of $P_n(k, x)$

Recall that $U_n(-x) = (-1)^n U_n(x)$ so that if r_n is a root of $P_n(k, x)$, so is $-r_n$. We will show that when $k > 1$ and $n \geq 3$, $P_n(k, x)$ has $n-2$ roots in $(-1, 1)$ and two roots outside this interval. We need $n \geq 3$ because $P_1(k, x) = x$ and $P_2(k, x) = 2x^2 - \frac{1}{2}(k+2)$ and these are trivial cases. We then give estimates for the bounds of the roots that lie outside $(-1, 1)$.

The quadratic $4(k+1)x^2 - (k+2)^2$ that we encountered in the previous section has connections to the roots of $P_n(k, x)$. In fact, we will see that the two roots of $P_n(k, x)$ that are outside $(-1, 1)$ are very close to the roots of this quadratic.

The roots of $P_n(k, x)$ in $(-1, 1)$ are the intersection of $U_n(x)$ and $(k+1)U_{n-2}(x)$ in $(-1, 1)$. For n odd, both $U_n(x)$ and $U_{n-2}(x)$ have a root at $x = 0$ which is also a root of $P_n(k, x)$. $U_n(x)$ has $\frac{n-1}{2}$ critical points in $(0, 1)$ and in $(-1, 0)$. Similarly, $U_{n-2}(x)$ has $\frac{n-3}{2}$ critical points in $(0, 1)$ and in $(-1, 0)$. Clearly, $U_n(x)$ and $U_{n-2}(x)$ have $\frac{n-3}{2}$ intersections in $(0, 1)$ and in $(-1, 0)$, which are the $n-3$ non trivial roots of $P_n(k, x)$.

On the other hand, when n is even both $U_n(x)$ and $U_{n-2}(x)$ have a critical point at $x = 0$ with one being a maximum and the other a minimum. $U_n(x)$ and $U_{n-2}(x)$ respectively have $\frac{n}{2} - 1$ and $\frac{n}{2} - 2$ critical points in $(0, 1)$ as well as in $(-1, 0)$ by symmetry. This implies that $U_n(x)$ and $U_{n-2}(x)$ have $\frac{n}{2} - 1$ intersection points in $(0, 1)$, and by symmetry in $(-1, 0)$, which together

make the $n - 2$ roots of $P_n(k, x)$. We now study the remaining two roots of $P_n(k, x)$.

Theorem 37. *Let $n \geq 3$, and let r_n be the largest root (in absolute value) of $P_n(k, x)$. Then for all $k > 0$,*

$$\frac{k+2}{2\sqrt{k+1}} - \frac{\sqrt{k}}{4(k+1)^{n-1}} < r_n < \frac{k+2}{2\sqrt{k+1}}.$$

Proof. To simplify the notation, if

$$\nu = \frac{k+2}{2\sqrt{k+1}} \quad \text{and} \quad \mu = \frac{k+2}{2\sqrt{k+1}} - \frac{\sqrt{k}}{4(k+1)^{n-1}}, \quad (2.2.11)$$

then

$$\nu + \sqrt{\nu^2 - 1} = \sqrt{k+1}, \quad \text{and} \quad \nu - \sqrt{\nu^2 - 1} = \frac{1}{\sqrt{k+1}}.$$

We also have $\frac{k}{2\sqrt{\nu^2-1}} = \sqrt{k+1}$, so that by (1.0.2) and (1.0.3),

$$\begin{aligned} 2p_n(k, \nu) &= \left((k+1)^{n/2} + \frac{1}{(k+1)^{n/2}} \right) - \sqrt{k+1} \left((k+1)^{\frac{n-1}{2}} - \frac{1}{(k+1)^{\frac{n-1}{2}}} \right) \\ &= \frac{k+2}{(k+1)^{n/2}} > 0 \end{aligned}$$

for all $k > 0$ and for all $n \geq 1$. Now

$$\mu^2 - 1 = \frac{k^2}{4(k+1)} \left(1 - \frac{k+2}{k^{\frac{3}{2}}(k+1)^{n-\frac{3}{2}}} + \frac{1}{4k(k+1)^{2n-3}} \right).$$

By series expansion,

$$\begin{aligned} &\sqrt{\mu^2 - 1} \\ &= \frac{k}{2\sqrt{k+1}} \left(1 - \frac{k+2}{2k^{\frac{3}{2}}(k+1)^{n-\frac{3}{2}}} - \frac{1}{2k^3(k+1)^{2n-4}} + \mathcal{O}_k \left(\frac{k+2}{k^{\frac{5}{2}}(k+1)^{3n-\frac{9}{2}}} \right) \right) \\ &= \frac{k}{2\sqrt{k+1}} - \frac{k+2}{4k^{\frac{1}{2}}(k+1)^{n-1}} - \frac{1}{4k^2(k+1)^{2n-\frac{7}{2}}} + \mathcal{O}_k \left(\frac{1}{k^{\frac{3}{2}}(k+1)^{3n-5}} \right), \end{aligned} \quad (2.2.12)$$

and

$$\begin{aligned}
& \frac{k}{2\sqrt{\mu^2-1}} \\
&= \sqrt{k+1} \left(1 + \frac{k+2}{4k^{\frac{3}{2}}(k+1)^{n-\frac{3}{2}}} + \frac{12(k+1)-k^2}{32k^3(k+1)^{2n-3}} + \mathcal{O}_k \left(\frac{k+2}{k^{\frac{5}{2}}(k+1)^{3n-\frac{9}{2}}} \right) \right) \\
&= \sqrt{k+1} + \frac{k+2}{4k^{\frac{3}{2}}(k+1)^{n-2}} + \frac{12(k+1)-k^2}{32k^3(k+1)^{2n-\frac{7}{2}}} + \mathcal{O}_k \left(\frac{1}{k^{\frac{5}{2}}(k+1)^{3n-6}} \right).
\end{aligned} \tag{2.2.13}$$

We also obtain the expansions

$$\begin{aligned}
& \left(\mu + \sqrt{\mu^2-1} \right)^n \\
&= \left(\sqrt{k+1} - \frac{1}{2k^{\frac{1}{2}}(k+1)^{n-2}} - \frac{1}{4k^2(k+1)^{2n-\frac{7}{2}}} + \mathcal{O}_k \left(\frac{1}{k^{\frac{3}{2}}(k+1)^{3n-5}} \right) \right)^n \\
&= (k+1)^{\frac{n}{2}} - \frac{n}{2k^{\frac{1}{2}}(k+1)^{\frac{n}{2}-\frac{3}{2}}} + \frac{n(n-1)k-2n}{8k^2(k+1)^{\frac{3n}{2}-3}} + \mathcal{O}_k \left(\frac{n}{k^{\frac{3}{2}}(k+1)^{\frac{5n}{2}-\frac{9}{2}}} \right),
\end{aligned} \tag{2.2.14}$$

and

$$\begin{aligned}
& \left(\mu - \sqrt{\mu^2-1} \right)^n \\
&= \left(\frac{1}{\sqrt{k+1}} + \frac{1}{2k^{\frac{1}{2}}(k+1)^{n-1}} + \frac{1}{4k^2(k+1)^{2n-\frac{7}{2}}} + \mathcal{O}_k \left(\frac{1}{k^{\frac{3}{2}}(k+1)^{3n-5}} \right) \right)^n \\
&= \frac{1}{(k+1)^{\frac{n}{2}}} + \frac{n}{2k^{\frac{1}{2}}(k+1)^{\frac{3n}{2}-\frac{3}{2}}} + \mathcal{O}_k \left(\frac{n}{k^2(k+1)^{\frac{5n}{2}-4}} \right).
\end{aligned} \tag{2.2.15}$$

From (2.2.13) we find that

$$\begin{aligned}
\frac{k}{2\sqrt{\mu^2-1}} \left(\mu + \sqrt{\mu^2-1} \right)^{n-1} &= (k+1)^{\frac{n}{2}} + \frac{2(k+1)-nk}{2k^{\frac{3}{2}}(k+1)^{\frac{n}{2}-\frac{3}{2}}} \\
&+ \frac{4n^2-20n+15}{32k(k+1)^{\frac{3n}{2}-3}} + \mathcal{O}_k \left(\frac{n}{k^2(k+1)^{\frac{5n}{2}-3}} \right),
\end{aligned} \tag{2.2.16}$$

and

$$\frac{k}{2\sqrt{\mu^2-1}} \left(\mu - \sqrt{\mu^2-1} \right)^{n-1} = \frac{1}{(k+1)^{\frac{n}{2}-1}} + \frac{nk+2}{2k^{\frac{3}{2}}(k+1)^{\frac{3n}{2}-\frac{5}{2}}} + \mathcal{O}_k \left(\frac{n}{k^2(k+1)^{\frac{5n}{2}-5}} \right). \tag{2.2.17}$$

Now subtract (2.2.16) from (2.2.14) to obtain

$$\frac{-(k+1)^{\frac{5}{2}}}{k^{\frac{3}{2}}(k+1)^{\frac{n}{2}}} - \frac{4n^2 - 20n + 15}{32k(k+1)^{\frac{3n}{2}-3}} + \mathcal{O}_k \left(\frac{n}{k^2(k+1)^{\frac{3n}{2}-3}} \right), \quad (2.2.18)$$

and add (2.2.17) to (2.2.15) to obtain

$$\frac{k+2}{(k+1)^{\frac{n}{2}}} + \frac{(nk+2)(k+1) + nk}{2k^{\frac{3}{2}}(k+1)^{\frac{3n}{2}-\frac{3}{2}}} + \mathcal{O}_k \left(\frac{n}{k^2(k+1)^{\frac{5n}{2}-5}} \right). \quad (2.2.19)$$

Finally add (2.2.18) to (2.2.19) to find that

$$\begin{aligned} & 2P_n(k, \mu) \\ &= \frac{k^{\frac{3}{2}}(k+2) - (k+1)^{\frac{5}{2}}}{k^{\frac{3}{2}}(k+1)^{\frac{n}{2}}} + \frac{16(nk+2)(k+1)^{\frac{5}{6}} - k^{\frac{1}{2}}(4n^2 - 20n + 15)}{32k^{\frac{3}{2}}(k+1)^{\frac{3n}{2}-3}} + \mathcal{O}_k \left(\frac{n}{k^2(k+1)^{\frac{3n}{2}-3}} \right) \end{aligned}$$

It is easy to check that for all $k \geq 0$

$$k^{\frac{3}{2}}(k+2) - (k+1)^{\frac{5}{2}} < 0$$

and is $\approx -\frac{1}{2}k^{3/2} - k^{1/2}$ for k large. □

2.3 Resultants and discriminants

In this section we give identities for resultants and discriminants of some of the polynomials studied in this chapter. Identities for resultants are particularly important in understanding the relationship between some of the polynomials.

Theorem 38. *If $k \in \mathbb{R}$, then for all $n \geq 1$, the resultants of $S_n(k, x)$ and $R_n(k, x)$ with respect to x are given by*

$$\text{Res}_x(S_n(k, x), R_n(k, x)) = 2^{n(n-1)}$$

for n even, and

$$\text{Res}_x(S_n(k, x), R_n(k, x)) = 2^{n(n-1)}k$$

for n odd.

Proof. Recall that

$$T_n(x) = U_n(x) - xU_{n-1}(x),$$

so that

$$S_n(k, x) = R_n(k, x) - xR_{n-1}(k, x).$$

By (1.5.3),

$$\text{Res}_x(S_n(k, x), R_n(k, x)) = \text{Res}_x(-xR_{n-1}(k, x), R_n(k, x)).$$

It is easy to check that $R_n(k, 0) = (-1)^{n/2}$ for n even and $R_n(k, 0) = (-1)^{\frac{n+1}{2}}k$ for n odd. Now by (1.5.1) and (1.5.5),

$$\text{Res}_x(S_n(k, x), R_n(k, x)) = (-1)^{n/2-1} \text{Res}_x(R_n(k, x), R_{n-1}(k, x)),$$

for n even, and for n odd we have

$$\text{Res}_x(S_n(k, x), R_n(k, x)) = (-1)^{\frac{n+1}{2}}k \text{Res}_x(R_n(k, x), R_{n-1}(k, x)).$$

It remains to show that for all $n \geq 1$

$$\text{Res}_x(R_n(k, x), R_{n-1}(k, x)) = (-1)^j 2^{n(n-1)},$$

where $j = n/2$ for n even, and for n odd, $j = (n-1)/2$. It is easy to check that $\text{Res}_x(R_3(k, x), R_2(k, x)) = -2^6$, $\text{Res}_x(R_2(k, x), R_1(k, x)) = -2^2$, and $\text{Res}_x(R_1(k, x), R_0(k, x)) = 2^0$. Using the recurrence relation (1.0.1) we obtain

$$R_n(k, x) = 2xR_{n-1}(k, x) - R_{n-2}(k, x),$$

so that by (1.5.3) and (1.5.1)

$$\begin{aligned} \text{Res}_x(R_n(k, x), R_{n-1}(k, x)) &= \text{Res}_x(2xR_{n-1}(k, x) - R_{n-2}(k, x), R_{n-1}(k, x)) \\ &= (-1)^{n-1} 2^{2(n-1)} \text{Res}_x(R_{n-1}(k, x), R_{n-2}(k, x)). \end{aligned}$$

Similarly

$$\begin{aligned}
& Res_x(R_{n-1}(k, x), R_{n-2}(k, x)) \\
&= Res_x(2xR_{n-2}(k, x) - R_{n-3}(k, x), R_{n-2}(k, x)) \\
&= (-1)^{n-2} 2^{2(n-2)} Res_x(R_{n-2}(k, x), R_{n-3}(k, x)).
\end{aligned}$$

The proof follows by iterating this process $n - 3$ times and using the initial values $Res_x(R_3(k, x), R_2(k, x)) = -2^6$, $Res_x(R_2(k, x), R_1(k, x)) = -2^2$, and $Res_x(R_1(k, x), R_0(k, x)) = 2^0$. $\square$

Conjecture 39. *The resultant of $S_{n,r}(k, x)$ and $t_r(k, x)$ with respect to x is given by*

$$Res_x(S_{n,r}(k, x), t_r(k, x)) = (-1)^r 2^{r(n+r-1)} (k^2 - 1)^r.$$

Conjecture 40. *The resultant of $R_{n,r}(k, x)$ and $t_r(k, x)$ with respect to x is given by*

$$Res_x(R_{n,r}(k, x), t_r(k, x)) = 2^{r(n+r)} (k^2 - 1)^{r-1}$$

for r odd, and

$$Res_x(R_{n,r}(k, x), t_r(k, x)) = 2^{r(n+r)} (k - 1)^{r-2} (k + 1)^r$$

for r even.

Theorem 41. *For all $k \in \mathbb{R}$ and $n \geq 1$, the discriminant of $P_{2n+1}(k - 1, x)$ is given by*

$$Disc_x(P_{2n+1}(k - 1, x)) = 2^{4n^2} ((2n + 1) + n(k - 1))^3 a_{n-1}(k)^2,$$

where $a_n(k)$ is given by (1.5.10).

Proof. It can easily be shown (see [7, p. 10]) that Chebyshev polynomials of the second kind satisfy the identity

$$U_{2n+1}(x) = 2xU_n(2x^2 - 1).$$

We use this identity to write

$$P_{2n+1}(k - 1, x) = x (U_n(2x^2 - 1) - kU_{n-1}(2x^2 - 1)).$$

To simplify the notation, let

$$h_n(k, x) := U_n(2x^2 - 1) - kU_{n-1}(2x^2 - 1).$$

The derivative of Chebyshev polynomials is given by

$$\begin{aligned} \frac{d}{dx}T_n(x) &= nU_{n-1}(x) \\ \frac{d}{dx}U_n(x) &= \frac{-nxU_n(x) + (n+1)U_{n-1}(x)}{1-x^2}. \end{aligned} \quad (2.3.1)$$

To calculate the derivative of $h_n(k, x)$, we use (2.3.1) and simplify using the recurrence relation (1.0.1) to get

$$x(1-x^2)h'_n(k, x) = (k-2x^2+1)nh_n(k, x) + l_n(k, x)U_{n-1}(2x^2-1) \quad (2.3.2)$$

where

$$l_n(k, x) = nk^2 + n + 1 - k(2n+1)(2x^2-1).$$

Using the derivative of $h_n(k, x)$ given by (2.3.2), we find that the derivative of $P_{2n+1}(k-1, x)$ to be given (after rearrangement) by

$$\begin{aligned} (1-x^2)P'_{2n+1}(k-1, x) &= (1-x^2)h_n(k, x) + (k-2x^2+1)nh_n(k, x) \\ &\quad + l_n(k, x)U_{n-1}(2x^2-1) \\ &= ((n+1+nk) - (2n+1)x^2)h_n(k, x) \\ &\quad + l_n(k, x)U_{n-1}(2x^2-1). \end{aligned}$$

Apply (1.5.3) with $a_m = 2^{2n}$ and the role of $n-l$ played by $(2n+2)-2n=2$ to deduce that

$$\begin{aligned} Res_x(xh_n(k, x), (1-x^2)(xh_n(k, x))') &= \\ 2^{4n}Res_x(xh_n(x), (n+1+nk)h_n(k, x) + S_n(x)U_{n-1}(2x^2-1)). \end{aligned} \quad (2.3.3)$$

Now, evaluate the left-hand side of (2.3.3) using the special values

$$U_n(1) = n+1, \quad U_n(-1) = (-1)^n(n+1) \quad (2.3.4)$$

(see e.g., [27, p. 777]), and using identities (1.5.1) and (1.5.5) to find that

$$\begin{aligned}
Res_x(xh_n(k, x), 1 - x^2) &= (-1)^{2n+1}h_n(k, 1)(-1)h_n(k, -1) \\
&= (U_n(1) - kU_{n-1}(1))^2 \\
&= (n + 1 - nk)^2,
\end{aligned}$$

so that

$$Res_x(xh_n(k, x), (xh_n(k, x))') = \frac{Res_x(xh_n(k, x), (1 - x^2)(xh_n(k, x))')}{(n + 1 - nk)^2}. \quad (2.3.5)$$

To evaluate the right-hand side of (2.3.3), we first note that

$$\begin{aligned}
h_n(k, 0) &= U_n(-1) - kU_{n-1}(-1) \\
&= (-1)^n(n + 1 + nk),
\end{aligned}$$

and

$$l_n(k, 0)U_{n-1}(-1) = (-1)^{n-1}n((2n + 1)k + n + 1 + nk^2),$$

so that

$$\begin{aligned}
Res_x(x, (n + 1 + nk)h_n(k, x) + l_n(k, x)U_{n-1}(2x^2 - 1)) \\
= (n + 1 + nk)h_n(k, 0) + l_n(k, 0)U_{n-1}(-1) \\
= (-1)^n(n + 1 + nk).
\end{aligned}$$

Again by using (1.5.3) with $a_m = 2^{2n}$ and $n - l = 0$, we have

$$\begin{aligned}
Res_x(h_n(k, x), (n + 1 + nk)h_n(k, x) + l_n(k, x)U_{n-1}(2x^2 - 1)) \\
= Res_x(h_n(k, x), l_n(k, x)U_{n-1}(2x^2 - 1)).
\end{aligned}$$

Using the chain rule for resultants (1.5.6) with $\deg f = \deg g = n$, and $t(x) = 2x^2 - 1$, we find that

$$Res_x(h_n(k, t(x)), l_n(k, t(x))U_{n-1}(t)) = 2^{2n^2} [Res_x(h_n(k, x), l_n(k, x)U_{n-1}(x))]^2. \quad (2.3.6)$$

Now

$$\begin{aligned}
& Res_x(h_n(k, x), l_n(k, x)) \\
&= Res_x(U_n(x) - kU_{n-1}(x), -(2n+1)kx + n+1 + nk^2) \\
&= (-1)^n(2n+1)^n k^n \left[U_n\left(\frac{n+1+nk^2}{(2n+1)k}\right) - kU_{n-1}\left(\frac{n+1+nk^2}{(2n+1)k}\right) \right] \\
&= ((n+1)^2 - n^2k^2)a_{n-1}(-k) \\
&= ((n+1)^2 - n^2k^2)a_{n-1}(k), \quad (2.3.7)
\end{aligned}$$

which follows from (1.5.10) and the fact that $a_n(k)$ is an even polynomial in k . On the other hand, from (1.5.10)

$$Res_x(U_n(x) - kU_{n-1}(x), U_{n-1}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)}. \quad (2.3.8)$$

Substituting (2.3.7) and (2.3.8) in (2.3.6) and together with (2.3.5), we find the right-hand side of (2.3.6) to be equal to

$$(-1)^n 2^{4n^2-2n} (n+1+nk)((n+1)^2 - n^2k^2)a_{n-1}(k)^2.$$

By (2.3.5),

$$Res_x(P_{2n+1}(k-1, x), P'_{2n+1}(k-1, x)) = (-1)^n (n+1+nk)^3 2^{4n^2+2n} a_{n-1}(k)^2. \quad (2.3.9)$$

Finally apply (1.5.7) with $a_m = 2^{2n}$ and $(-1)^{\frac{n(n-1)}{2}} = (-1)^{2n^2+n}$ to complete the proof. $\square$

The discriminant of $P_{2n}(k-1, x)$ with respect to x involves some polynomials $b_n(k)$ that seem to behave in a similar manner to the $a_n(k)$ polynomials. Recall that the $a_n(k)$ polynomials first appeared in the work of Dilcher and Stolarsky [17] and later in the work of Gishe and Ismail [18]. Below is a list

of the first few $a_n(k)$ polynomials

$$\begin{aligned}
a_1(k) &= 1 \\
a_2(k) &= k^2 + 4 \\
a_3(k) &= 4k^4 + 13k^2 + 32 \\
a_4(k) &= 32k^6 + 93k^4 + 204k^2 + 400 \\
a_5(k) &= 400k^8 + 1084k^6 + 2213k^4 + 4032k^2 + 6912 \\
a_6(k) &= 6912k^{10} + 17856k^8 + 34709k^6 + 60152k^4 + 98000k^2 + 153664 \\
a_7(k) &= 153664k^{12} + 383376k^{10} + \dots + 2850816k^2 + 4194304
\end{aligned} \tag{2.3.10}$$

This is the first time we are encountering the $b_n(k)$ polynomials. The first few terms of this polynomial sequence are

$$\begin{aligned}
b_1(k) &= 1 \\
b_2(k) &= k^2 + 2k + 5 \\
b_3(k) &= 5k^4 + 8k^3 + 18k^2 + 28k + 49 \\
b_4(k) &= 49k^6 + 70k^5 + 151k^4 + 212k^3 + 351k^2 + 486k + 729 \\
b_5(k) &= 729k^8 + 972k^7 + 2052k^6 + 2704k^5 + 4350k^4 + 5676k^3 + \\
&\quad 8228k^2 + 10648k + 14641 \\
b_6(k) &= 14641k^{10} + 18634k^9 + 38841k^8 + 49016k^7 + 77522k^6 + 97116k^5 + \\
&\quad 137930k^4 + 171704k^3 + 230685k^2 + 285610k + 371293 \\
b_7(k) &= 371293k^{12} + 456976k^{11} + \dots + 9112500k + 11390625
\end{aligned} \tag{2.3.11}$$

Observe that just like the $a_n(k)$ polynomials, the leading coefficient of $b_n(k)$ coincides with the constant coefficient of $b_{n-1}(k)$.

Theorem 42. *For all $k \in \mathbb{R}$ and $n \geq 1$, the discriminant of $P_{2n}(k-1, x)$ is given by*

$$Disc_x(P_{2n}(k-1, x)) = 2^{(2n-1)^2+1}(k+1)b_n(k)^2,$$

where $b_n(k)$ are defined by

$$\begin{aligned}
b_n(k) &:= \frac{-2^n n^n k^n}{(2n-1)k - (2n+1)} \left[U_n \left(\frac{(2n-1)k^2 + (2n+1)}{4nk} \right) - \frac{(2n-1)k+1}{2n} U_{n-1} \left(\frac{(2n-1)k^2 + (2n+1)}{4nk} \right) \right].
\end{aligned} \tag{2.3.12}$$

Proof. We first use the identity

$$U_{2n}(x) = U_n(2x^2 - 1) + U_{n-1}(2x^2 - 1)$$

together with the recurrence relation (1.0.1) in (2.2.3) to obtain

$$P_{2n}(k-1, x) = \frac{1}{2} \left((k+1)U_n(2x^2 - 1) + (k+1 - 4kx^2)U_{n-1}(2x^2 - 1) \right). \quad (2.3.13)$$

Using the derivative for Chebyshev polynomials (2.3.1) we have

$$\begin{aligned} & \frac{d}{dx} U_n(2x^2 - 1) \\ &= \frac{4x}{1 - (2x^2 - 1)^2} \left(-n(2x^2 - 1)U_n(2x^2 - 1) + (n+1)U_{n-1}(2x^2 - 1) \right) \\ &= \frac{1}{x(1 - x^2)} \left(-n(2x^2 - 1)U_n(2x^2 - 1) + (n+1)U_{n-1}(2x^2 - 1) \right). \end{aligned} \quad (2.3.14)$$

Now apply (2.3.14) in (2.3.13) to find that

$$\begin{aligned} & x(1 - x^2)P'_{2n}(k-1, x) \\ &= \frac{1}{2}(k+1) \left(-n(2x^2 - 1)U_n(2x^2 - 1) + (n+1)U_{n-1}(2x^2 - 1) \right) \\ &+ \frac{1}{2}(k+1 - 4kx^2) \left(-(n-1)(2x^2 - 1)U_{n-1}(2x^2 - 1) + nU_{n-2}(2x^2 - 1) \right) \\ &\quad - 4k(1 - x^2)x^2U_{n-1}(2x^2 - 1). \end{aligned} \quad (2.3.15)$$

We now simplify (2.3.15) using the recurrence relation (1.0.1) in the form

$$U_{n-2}(2x^2 - 1) = 2(2x^2 - 1)U_{n-1}(2x^2 - 1) - U_n(2x^2 - 1)$$

to arrive at

$$\begin{aligned} x(1 - x^2)P'_{2n}(k-1, x) &= n(k-1)x^2U_n(2x^2 - 1) \\ &+ \left((3nk - k + n + 1)x^2 - 4n k x^4 \right) U_{n-1}(2x^2 - 1). \end{aligned} \quad (2.3.16)$$

We now verify that

$$x(1 - x^2)P'_{2n}(k-1, x) = \frac{2n(k-1)}{k+1}x^2P_{2n}(k-1, x) + \frac{x^2}{k+1}q_n(k, x)U_{n-1}(2x^2 - 1), \quad (2.3.17)$$

where

$$q_n(k, x) := (k+1)((2n-1)k+2n+1) - 8nkx^2 \quad (2.3.18)$$

is equivalent to (2.3.16) by applying the formula (2.3.13) for $P_{2n}(k-1, x)$ and expanding.

Using (1.5.2) with $f(x) = x^2/(k+1)$ and $g(x) = P_{2n}(k-1, x)$ we find that

$$\begin{aligned} \text{Res}_x \left(P_{2n}(k-1, x), \frac{x^2}{k+1} \right) \\ = \frac{1}{(k+1)^{2n}} \left(\frac{1}{2} ((k+1)U_n(-1) + (k+1)U_{n-1}(-1)) \right)^2 \\ = \frac{1}{4(k+1)^{2n-2}}, \end{aligned} \quad (2.3.19)$$

where we have used (2.3.13). Next apply (1.5.3) with $g(x) = P_{2n}(k-1, x)$ and $f(x) = U_{n-1}(2x^2 - 1)$ to arrive at

$$\begin{aligned} \text{Res}_x (P_{2n}(k-1, x), U_{n-1}(2x^2 - 1)) \\ = \left(\frac{1}{2}(k+1) \right)^{2n-2} \text{Res}_x (U_n(2x^2 - 1), U_{n-1}(2x^2 - 1)). \end{aligned} \quad (2.3.20)$$

Now using (2.3.8) and the chain-rule for resultants (1.5.6) in (2.3.20), we deduce that

$$\text{Res}_x (P_{2n}(k-1, x), U_{n-1}(2x^2 - 1)) = 2^{2(n-1)(2n-1)}(k+1)^{2n-2}. \quad (2.3.21)$$

Let

$$r(n, k)^2 = \frac{(k+1)((2n-1)k+2n+1)}{8nk}, \quad (2.3.22)$$

Then $r(n, k)$ is a root of $q_n(k, x)$. If $x = \pm r(n, k)$, then $2x^2 - 1$ is the

$$z(m, k) := \frac{(2m-1)k^2 + (2m+1)}{4mk} \quad (2.3.23)$$

expression that appears in the definition of the polynomials $b_n(k)$. Apply (2.3.22) to (2.3.13) to obtain

$$P_{2n}(k-1, r(n, k)) = \frac{-1}{2^{n+1}n^n k^n} (k+1)((2n-1)k - (2n+1))b_n(k), \quad (2.3.24)$$

where $b_n(k)$ is as defined in (2.3.12). Using (1.5.1) with $f(x) = q_n(k, x)$ and $g(x) = (P_{2n}(k-1, x))$ together with (2.3.24), we obtain upon simplification

$$\begin{aligned} Res_x(P_{2n}(k-1, x), q_n(k, x)) &= (-8nk)^{2n} P_{2n}(k-1, r(n, k))^2 \\ &= 2^{2(2n-1)}(k+1)^2((2n-1)k - (2n+1))^2 b_n(k)^2. \end{aligned} \quad (2.3.25)$$

Combining (2.3.19), (2.3.21) and (2.3.25), and applying (1.5.5) we find that

$$\begin{aligned} Res_x(P_{2n}(k-1, x), x(1-x^2)P'_{2n}(k-1, x)) &= 2^{2n(2n-1)-2}(k+1)^2((2n-1)k - (2n+1))^2 b_n(k)^2. \end{aligned} \quad (2.3.26)$$

Now

$$\begin{aligned} Res_x(P_{2n}(k-1, x), x(1-x^2)) &= P_{2n}(k-1, 0)P_{2n}(k-1, 1)P_{2n}(k-1, -1) \\ &= (-1)^n \frac{1}{2^3} (k+1)((2n-1)k - (2n+1))^2, \end{aligned} \quad (2.3.27)$$

where we have used the special values (2.3.4) in the evaluation of the resultant. Using (1.5.5) again, together with (2.3.26) and (2.3.27), we find that

$$\begin{aligned} Res_x(P_{2n}(k-1, x), P'_{2n}(k-1, x)) &= \frac{Res_x(P_{2n}(k-1, x), x(1-x^2)P'_{2n}(k-1, x))}{Res_x(P_{2n}(k-1, x), x(1-x^2))} \\ &= (-1)^n 2^{2n(2n-1)+1} (k+1) b_n(k)^2. \end{aligned} \quad (2.3.28)$$

Finally use (2.3.28) in the formula for discriminants (1.5.5) with $f(x) = P_{2n}(k-1, x)$ and $a_0 = 2^{2n-1}$ to obtain

$$\begin{aligned} Disc_x(P_{2n}(k-1, x)) &= \frac{1}{2^{2n-1}} (-1)^{n(2n-1)} (-1)^n 2^{2n(2n-1)+1} (k+1) b_n(k)^2 \\ &= 2^{(2n-1)^2+1} (k+1) b_n(k)^2 \end{aligned}$$

as desired. □

Remark 3. *It can easily be shown that $U_n\left(\frac{k^2+1}{2k}\right)$ as a polynomial in k has*

all roots on the unit circle. Here,

$$\frac{k^2 + 1}{2k} = \lim_{m \rightarrow \infty} z(m, k),$$

where $z(m, k)$ is as defined in (2.3.23).

Recall that $U_n(\cos(j\pi/(n+1))) = 0$ for all $n \geq 1$ and $j = 1, 2, \dots, n$. So if we set $z(m, k) = \cos \theta$ where $\theta = j\pi/(n+1)$, we will have

$$U_n(z(m, k)) = 0.$$

We now show that for $m > \frac{1}{2 \sin(\pi/(n+1))}$, all the roots of $U_n(z(m, k))$ lie on the circle of radius $\sqrt{\frac{2m+1}{2m-1}}$. Setting $z(m, k) = \cos \theta$, we have

$$k = \frac{1}{2m-1} \left(2m \cos \theta \pm (1 - 4m^2(\sin \theta)^2)^{1/2} \right)$$

and k is non-real whenever $m > 1/(2 \sin \theta)$. It is straightforward to check that k lies on the circle of radius $\sqrt{\frac{2m+1}{2m-1}}$.

Corollary 43. *Let $b_n(k)$ be as defined in (2.3.12). Then the leading coefficient of this polynomial is $(2n-1)^{n-2}$. Further,*

$$b_n(0) = (2n-3)^{n-3}, \quad (2.3.29)$$

$$b_n(1) = 2^{n-1}n^n, \quad (2.3.30)$$

$$b_n(-1) = 2^{n-1}n^{n-1}. \quad (2.3.31)$$

Proof. The first statement and (2.3.29) follow from the fact that $U_n(x)$ has leading coefficient 2^n . Using the special values (2.3.4), we find that (2.3.30) and (2.3.31) follow quite easily. $\square$

CHAPTER 3

POLYNOMIALS WITH CHEBYSHEV COEFFICIENTS AND BI-CONCYCLIC ZEROS, AND THEIR HANKEL DETERMINANTS

Let $c > 0$ and consider the polynomials $K_n(c, x)$ defined as

$$K_n(c, x) := \sum_{k=0}^{n-1} T_k(c/2)(z^{2n-k} + z^k) + T_n(c/2)z^n. \quad (3.0.1)$$

We show in Section 3.1 that the generating function for $K_n(c, z)$ is

$$\frac{(1 - z^2 t^2)(2 - ctz)}{2(t-1)(tz^2 - 1)(z^2 t^2 - czt + 1)} = \sum_{n=0}^{\infty} K_n(c, z)t^n. \quad (3.0.2)$$

Since $T_n(1) = 1$ for all n , then $K_n(c, z)$ reduces to

$$\sum_{i=0}^{2n} z^i$$

for $c = 2$, a self-reciprocal polynomial which of course has all roots on the unit circle. So as we will see in Section 3.1, for $c = 2 - \epsilon$ where ϵ is real and very small, all roots must remain on the unit circle. It is easily seen that as c varies through real values, no zero can move off $|z| = 1$ until it becomes a multiple root by coinciding with another zero. Thus we want to know the discriminant $Disc_z K_n(c, z)$ and its roots. This discriminant has no known “nice form”. For $n = 13$, its peculiar root distribution is shown in figure 3.1. Indeed as c decreases from 2 through real values, multiple zeros occur and a complete description of the root description even for $-2 \leq c \leq 2$ seems elusive.

It would seem even more difficult to handle a corresponding modification of

$$\left(\sum_{i=0}^{2n} z^i \right)^2.$$

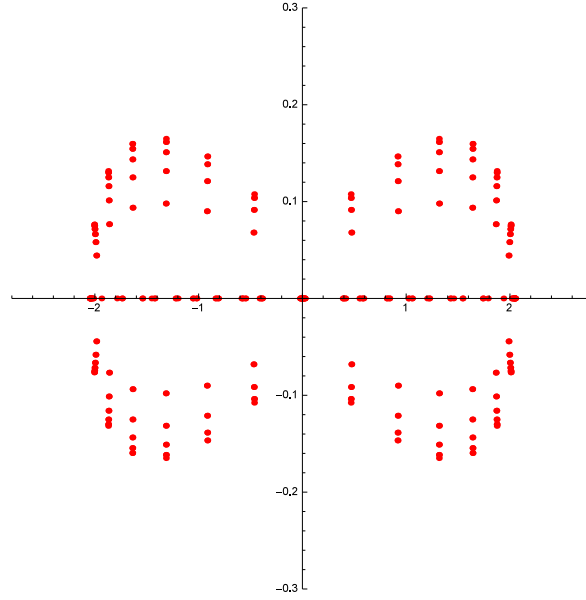


Figure 3.1: Roots of $Disc_z K_{13}(c, z)$.

For example

$$\left(\sum_{i=0}^4 z^i \right)^2 = \sum_{i=0}^3 (i+1)(z^{8-i} + z^i) + 5z^4,$$

so the most natural corresponding Chebyshev modification would be $P_n(c, z)$ defined by

$$P_n(c, z) := \sum_{k=0}^{n-1} U_k(c/2) (z^{2n-k} + z^k) + U_n(c/2) z^n. \quad (3.0.3)$$

In this chapter, we study properties of $P_n(c, z)$ in great detail. Of particular interest is the root distributions and Hankel determinants of $P_n(c, z)$.

3.1 Generating function and zeros of $P_n(c, z)$

We start by showing that the polynomials generated by the rational function

$$\frac{1 - z^2 t^2}{(t-1)(tz^2 - 1)(z^2 t^2 - czt + 1)} = \sum_{n=0}^{\infty} P_n(c, z) t^n \quad (3.1.1)$$

n	$P_n(c, z)$
0	1
1	$z^2 + cz + 1$
2	$z^4 + cz^3 + (c^2 - 1)z^2 + cz + 1$
3	$z^6 + cz^5 + (c^2 - 1)z^4 + (c^3 - 2c)z^3 + (c^2 - 1)z^2 + cz + 1$
4	$z^8 + cz^7 + (c^2 - 1)z^6 + (c^3 - 2c)z^5 + (c^4 - 3c^2 + 1)z^4 + (c^3 - 2c)z^3 + (c^2 - 1)z^2 + cz + 1$
5	$z^{10} + cz^9 + (c^2 - 1)z^8 + (c^3 - 2c)z^7 + (c^4 - 3c^2 + 1)z^6 + (c^5 - 4c^3 + 3c)z^5 + (c^4 - 3c^2 + 1)z^4 + (c^3 - 2c)z^3 + (c^2 - 1)z^2 + cz + 1$
6	$z^{12} + cz^{11} + (c^2 - 1)z^{10} + (c^3 - 2c)z^9 + (c^4 - 3c^2 + 1)z^8 + (c^5 - 4c^3 + 3c)z^7 + (c^6 - 5c^4 + 6c^2 - 1)z^6 + (c^5 - 4c^3 + 3c)z^5 + (c^4 - 3c^2 + 1)z^4 + (c^3 - 2c)z^3 + (c^2 - 1)z^2 + cz + 1$

Table 3.1: Table of the first seven polynomials $P_n(c, z)$.

are indeed the polynomials defined by equation (3.0.3). From the generating function (3.1.1), we deduce that these polynomials have a five term recurrence relation given by

$$\begin{aligned}
P_{n+1}(c, z) = & (z^2 + cz + 1)P_n(c, z) - z(cz^2 + 2z + c)P_{n-1}(c, z) + \\
& z^2(z^2 + cz + 1)P_{n-2}(c, z) - z^4P_{n-3}(c, z). \quad (3.1.2)
\end{aligned}$$

The first seven polynomials in this sequence are given in the table below;

We can see that they are palindromes with Chebyshev polynomials in $c/2$ as their coefficients. The following Theorem makes this precise.

Theorem 44. *For all $n \geq 0$ and $c \in \mathbb{R}$,*

$$P_n(c, z) = \sum_{k=0}^{n-1} U_k(c/2) (z^{2n-k} + z^k) + U_n(c/2)z^n. \quad (3.1.3)$$

Proof. It is easy to check the right hand side of (3.1.3) for $n = 0, 1$ and 2 . It suffices to show that (3.1.3) satisfies the recurrence relation (3.1.2). We will proceed by induction on n using the recurrence relation for Chebyshev polynomials (1.0.1) in the form

$$U_{n+1} = cU_n - U_{n-1}. \quad (3.1.4)$$

From (3.1.2), we have

$$\begin{aligned} P_{n+1} &= (z^2 + 1)(P_n + z^2 P_{n-2}) - 2z^2 P_{n-1} - z^4 P_{n-3} \\ &\quad + c \{z P_n + z^3 P_{n-2} - (z^3 + z) P_{n-1}\}. \end{aligned} \quad (3.1.5)$$

First we evaluate the sum of ‘terms without c ’ in (3.1.5) above. By (3.1.3),

$$\begin{aligned} &(z^2 + 1)(P_n + z^2 P_{n-2}) - 2z^2 P_{n-1} - z^4 P_{n-3} \\ &= (z^2 + 1) \left\{ \sum_{k=0}^{n-1} U_k (z^{2n-k} + z^k) + \sum_{k=0}^{n-3} U_k (z^{2n-2-k} + z^{2+k}) + U_n z^n + U_{n-2} z^n \right\} \\ &\quad - 2 \sum_{k=0}^{n-2} U_k (z^{2n-k} + z^{2+k}) - \sum_{k=0}^{n-4} U_k (z^{2n-2-k} + z^{4+k}) - 2U_{n-1} z^{n+1} - U_{n-3} z^{n+1}. \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned} &\sum_{k=0}^{n-1} U_k (z^{2n+2-k} + z^k) + \sum_{k=0}^{n-1} U_k (z^{2n-k} + z^{2+k}) + \sum_{k=0}^{n-3} U_k (z^{2n-k} + z^{2+k}) \\ &\quad - 2 \sum_{k=0}^{n-2} U_k (z^{2n-k} + z^{2+k}) + \sum_{k=0}^{n-3} U_k (z^{2n-2-k} + z^{4+k}) \\ &\quad - \sum_{k=0}^{n-4} U_k (z^{2n-2-k} + z^{4+k}) + c U_{n-1} (z^{n+2} + z^n) \\ &\quad - U_{n-1} z^{n+1} - c U_{n-2} z^{n+1}. \end{aligned}$$

Note we have used (3.1.4). And after cancellations we obtain

$$\begin{aligned} &\sum_{k=0}^{n-1} U_k (z^{2n+2-k} + z^k) + 2U_{n-1} z^{n+1} + 2U_{n-3} z^{n+1} - U_{n-2} (z^{n+2} + z^n) \\ &\quad + c U_{n-1} (z^{n+2} + z^n) - U_{n-1} z^{n+1} - c U_{n-2} z^{n+1}. \end{aligned}$$

Using (3.1.4) once again, we deduce that

$$\begin{aligned} &\sum_{k=0}^{n-1} U_k (z^{2n+2-k} + z^k) - U_{n-2} (z^{n+2} + z^n) - U_{n-1} z^{n+1} \\ &\quad + c \{U_{n-1} (z^{n+2} + z^n) + U_{n-2} z^{n+1}\}. \end{aligned} \quad (3.1.6)$$

Now combining ‘terms with c ’ in (3.1.6) together with those in (3.1.5) and

use (3.1.3) to find that

$$\begin{aligned}
& zP_n + z^3P_{n-2} - (z^3 + z)P_{n-1} + U_{n-1}(z^{n+2} + z^n) + U_{n-2}z^{n+1} \\
&= \sum_{k=0}^{n-1} U_k (z^{2n+1-k} + z^{1+k}) + \sum_{k=0}^{n-3} U_k (z^{2n-1-k} + z^{3+k}) \\
&\quad - (z^3 + z) \sum_{k=0}^{n-2} U_k (z^{2n-2-k} + z^k) + U_n z^{n+1} + 2U_{n-2}z^{n+1}.
\end{aligned}$$

Upon rearranging and simplifying, we obtain

$$U_{n-1}(z^{n+2} + z^n) + U_n z^{n+1}.$$

Using the recurrence relation (3.1.4), we find that the total contribution of ‘terms with c ’ is given by

$$\begin{aligned}
& c \{U_{n-1}(z^{n+2} + z^n) + U_n z^{n+1}\} \\
&= (U_n + U_{n-2})(z^{n+2} + z^n) + (U_{n+1} + U_{n-1})z^{n+1}. \quad (3.1.7)
\end{aligned}$$

Finally add ‘terms without c ’ in (3.1.6) together with (3.1.7) to get

$$\begin{aligned}
P_{n+1} &= \sum_{k=0}^{n-1} U_k (z^{2n+2-k} + z^k) - U_{n-2}(z^{n+2} + z^n) - U_{n-1}z^{n+1} \\
&\quad + (U_n + U_{n-2})(z^{n+2} + z^n) + (U_{n+1} + U_{n-1})z^{n+1} \\
&= \sum_{k=0}^n U_k (z^{2n+2-k} + z^k) + U_{n+1}z^{n+1},
\end{aligned}$$

completing the proof. $\square$

From the generating function (3.1.1) of $P_n(c, z)$, we easily get the generating function of $K_n(c, z)$. First recall (see (2.2.2)) that the Chebyshev polynomials of the first kind and those of the second kind are related by the identity

$$T_n(x) = U_n(x) - xU_{n-1}(x). \quad (3.1.8)$$

Using this identity in (3.0.1) and (3.0.3), we deduce the identity

$$K_n(c, z) = P_n(c, z) - \frac{1}{2}czP_{n-1}(c, z). \quad (3.1.9)$$

Corollary 45. For all $n \geq 0$, $K_n(c, z)$ are generated by the rational function

$$\frac{(1 - z^2 t^2)(2 - ctz)}{2(t-1)(tz^2 - 1)(z^2 t^2 - czt + 1)} = \sum_{n=0}^{\infty} K_n(c, z) t^n.$$

Proof. Straightforward using the generating function (3.1.1) and the identity (3.1.9). $\square$

To calculate the zeros of $P_n(c, z)$, we first transform the polynomials to a much simpler expression. We do this by multiplying each polynomial $P_n(c, z)$ by the quadratic $z^2 - cz + 1$.

Lemma 46. For each $n \geq 0$,

$$(z^2 - cz + 1)P_n(c, z) = z^{2n+2} - 2T_{n+1}(c/2)z^{n+1} + 1 := \hat{P}_n(c, z). \quad (3.1.10)$$

Proof. We are going to use (1.0.1) and (3.1.8). From the definition of $\hat{P}_n(c, z)$,

$$\begin{aligned} \hat{P}_n(c, z) &= (1 - cz + z^2) \sum_{k=0}^{n-1} U_k(c/2) (z^{2n-k} + z^k) + U_n(c/2) z^n \\ &= \sum_{k=0}^{n-1} U_k(c/2) (z^{2n+2-k} + z^{k+2} + z^{2n-k} + z^k) + U_n(c/2) (z^{n+2} + z^n) \\ &\quad - \sum_{k=0}^{n-1} (U_{k+1}(c/2) + U_{k-1}(c/2)) (z^{2n-k+1} + z^{k+1}) \\ &\quad - (U_{n+1}(c/2) + U_{n-1}(c/2)) z^{n+1} \\ &= \sum_{k=0}^{n-1} U_k(c/2) (z^{2n+2-k} + z^{k+2} + z^{2n-k} + z^k) + U_n(c/2) (z^{n+2} + z^n) \\ &\quad - \sum_{k=1}^n (U_k(c/2) + U_{k-2}(c/2)) (z^{2n-k+2} + z^k) \\ &\quad - (U_{n+1}(c/2) + U_{n-1}(c/2)) z^{n+1}, \end{aligned}$$

where we have used (1.0.1) with $x = c/2$. After cancellations we get

$$\begin{aligned} \hat{P}_n(c, z) &= z^{2n+2} + 1 + 2U_{n-1}(c/2)z^{n+1} - (U_{n+1}(c/2) + U_{n-1}(c/2)) z^{n+1} \\ &= z^{2n+2} - (U_{n+1}(c/2) - U_{n-1}(c/2)) z^{n+1} + 1 \\ &= z^{2n+2} - 2T_{n+1}(c/2)z^{n+1} + 1, \end{aligned}$$

where the last line above follows from (1.0.1) and (3.1.8). $\square$

From this simple expression for the polynomials $\hat{P}_n(c, z)$, we can easily show how their roots are distributed.

Theorem 47. *Let $c \in \mathbb{R}$ be non-zero, and let $\hat{P}_n(c, z)$ be as defined above. For all $n \geq 0$, $\hat{P}_n(c, z)$ has all roots on T , the union of two concentric and inversely related circles with radii*

$$r_1 = \left| \frac{c + \sqrt{c^2 - 4}}{2} \right| \quad \text{and} \quad r_2 = \left| \frac{c - \sqrt{c^2 - 4}}{2} \right|.$$

Proof. Set $z^{n+1} = y$ in (3.1.10) to obtain the quadratic

$$y^2 - 2T_{n+1}(c/2)y + 1.$$

The roots of this quadratic are

$$\begin{aligned} y &= T_{n+1}(c/2) \pm (T_{n+1}^2(c/2) - 1)^{1/2} \\ &= T_{n+1}(c/2) \pm \frac{1}{2}\sqrt{c^2 - 4}U_n(c/2), \end{aligned}$$

where the last equality follows from Pell's equation for Chebyshev polynomials

$$T_n^2(x) - (x^2 - 1)U_{n-1}^2(x) = 1. \quad (3.1.11)$$

Now use the explicit formulas (1.0.2) and (1.0.3) for Chebyshev polynomials to deduce the two roots of the quadratic $y^2 - 2T_{n+1}(c/2)y + 1$ to be given by

$$\begin{aligned} y_1 &= T_{n+1}(c/2) + \frac{1}{2}\sqrt{c^2 - 4}U_n(c/2) \\ &= \left(\frac{c + \sqrt{c^2 - 4}}{2} \right)^{n+1}, \end{aligned}$$

and

$$\begin{aligned} y_2 &= T_{n+1}(c/2) - \frac{1}{2}\sqrt{c^2 - 4}U_n(c/2) \\ &= \left(\frac{c - \sqrt{c^2 - 4}}{2} \right)^{n+1}. \end{aligned}$$

Recall that $y = z^{n+1}$, so $\hat{P}_n(c, z)$ factors as

$$\hat{P}_n(c, z) = (z^{n+1} - y_1)(z^{n+1} - y_2)$$

with roots $z_j = r_1 e^{2\pi j/(n+1)}$ and $z_k = r_2 e^{2\pi k/(n+1)}$ for $j, k = 0, 1, 2, \dots, n$ where $r_1 = \frac{c + \sqrt{c^2 - 4}}{2}$ and $r_2 = \frac{c - \sqrt{c^2 - 4}}{2}$. Clearly all the zeros of $\hat{P}_n(c, z)$ lie on T and are dense on T as $n \rightarrow \infty$. Moreover from the explicit formula for the zeros, it is easy to see that they are evenly distributed on T with an equal number of zeros on each circle. $\square$

The explicit formula for the roots of $\hat{P}_n(c, z)$ implies that for each $c \in \mathbb{R}$, the roots of $\hat{P}_n(c, z)$ come in both conjugate and ‘angular’ pairs.

Corollary 48. *Let $c \in \mathbb{R}$ be non-zero, then for all $n \geq 0$, $P_n(c, z)$ has all roots on T , the union of two concentric and inversely related circles with radii*

$$r_1 = \left| \frac{c + \sqrt{c^2 - 4}}{2} \right| \quad \text{and} \quad r_2 = \left| \frac{c - \sqrt{c^2 - 4}}{2} \right|.$$

Since the quadratic $z^2 - cz + 1$ has its roots on T , the proof is immediate from Theorem 47.

We are now going to see how $P_n(c, z)$ can be conformally lifted (in a very natural way) to a polynomial with roots on the ellipse

$$E(c, \sqrt{c^2 - 4}) : \frac{x^2}{c^2} + \frac{y^2}{c^2 - 4} = 1.$$

For more on the conformal mappings between circles and ellipses, see for example [7, § 1.4.1].

From the generating function (3.1.1) we have

$$\frac{1 - t^2}{(1 - (z + 1/z)t + t^2)(1 - ct + t^2)} = \sum_{n=0}^{\infty} P_n(c, z) \frac{1}{z^n} t^n, \quad (3.1.12)$$

and by setting

$$w := z + \frac{1}{z}$$

we find that

$$P_n(c, w) = \frac{1}{z^n} P_n(c, z).$$

Comparing the generating function (3.1.12) for $P_n(c, w)$ to the rational gen-

erating function

$$\frac{(w-c)(1-t^2)}{4(1-wt+t^2)(1-ct+t^2)} = \sum_{n=0}^{\infty} (T_n(w/2) - T_n(c/2)) t^n \quad (3.1.13)$$

for translated Chebyshev polynomials, it is easy to see that

$$P_n(c, w) = \frac{4}{w-c} (T_n(w/2) - T_n(c/2)).$$

From Theorem 3 with $\lambda = c$ and $\alpha = 2$, the roots of $P_n(c, w)$ are given by

$$w_j = c \cos\left(\frac{2\pi j}{n}\right) + i\sqrt{c^2 - 4} \sin\left(\frac{2\pi j}{n}\right)$$

for $1 \leq j \leq n-1$ and lie on the ellipse $E(c, \sqrt{c^2 - 4})$ whenever $|c| > 2$. For $|c| \leq 2$ the roots are all real.

3.2 Resultants and discriminants

Our main goal is to give identities for the discriminants of $P_n(c, z)$ with respect to c and with respect to z . A knowledge of the resultants of cyclotomic polynomials will be useful. A classic article on resultants of cyclotomic polynomials is [28], and an elementary proof of an identity for cyclotomic polynomials was given by Dresden [29].

We start by proving a trivial (but rather useful) identity for the resultant of a product of cyclotomic polynomials. Recall that

$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

where

$$\Phi_d(x) = \prod_{(k,d)=1} (x - e(k/d)),$$

and $e(k/d)$ is the shorthand notation for $\exp(2\pi i k/d)$.

Lemma 49. *For $m \geq 1$ and $n = m - 1$,*

$$\text{Res}_x((ax)^m - 1, (bx)^n - 1) = (-1)^n (a^{mn} - b^{mn}). \quad (3.2.1)$$

Proof. The roots of $(bx)^n - 1$ are $\frac{1}{b}e(k/n)$ for $k = 1, 2, \dots, n$. Now apply (1.5.1) with $f = (bx)^m - 1$ and $g = (bx)^n - 1$ to deduce that

$$\begin{aligned}
Res_x(f, g) &= b^{mn} \prod_{k=1}^n \left(\left(\frac{a}{b} e(k/n) \right)^m - 1 \right) \\
&= a^{mn} \prod_{k=1}^n \left(e(km/n) - \frac{b^m}{a^m} \right) \\
&= (-1)^n a^{mn} \prod_{k=1}^n \left(\frac{b^m}{a^m} - e(k/n) \right) \\
&= (-1)^n a^{mn} \left(\frac{b^{mn}}{a^{mn}} - 1 \right).
\end{aligned}$$

Here we have used $m = n + 1$ and $x^n - 1 = \prod_{k=1}^n (x - e(k/n))$. $\square$

For $n \geq 0$ and $a \in \mathbb{C}$, let

$$\begin{aligned}
Q_n(a, x) &:= x^{2n} - 2ax^n + 1 \\
&= (x^n - r)(x^n - r^{-1}),
\end{aligned} \tag{3.2.2}$$

where $r = a + \sqrt{a^2 - 1}$. We can now write

$$Q_n(a, x) = ((x/r^{1/n})^n - 1)((xr^{1/n})^n - 1). \tag{3.2.3}$$

Theorem 50. For $n \geq 1$,

$$Res_x(Q_n(a, x), Q_{n-1}(b, x)) = 4(T_n(b) - T_{n-1}(a))^2. \tag{3.2.4}$$

Proof. If we write

$$Q_{n-1}(b, x) = ((x/s^{1/(n-1)})^{n-1} - 1)((xs^{1/(n-1)})^{n-1} - 1), \tag{3.2.5}$$

then by (1.5.5), the resultant of $Q_n(a, x)$ and $Q_{n-1}(b, x)$ is a product of four resultants. Using (3.2.1) with $a = r^{-1/n}$ and $b = s^{-1/(n-1)}$, we have

$$Res_x(((x/r^{1/n})^n - 1), ((x/s^{1/(n-1)})^{n-1} - 1)) = (-1)^n (s^{-n} - r^{-(n-1)}). \tag{3.2.6}$$

Next apply (3.2.1) with $a = r^{-1/n}$ and $b = s^{1/(n-1)}$ to find that

$$Res_x \left(((x/r^{1/n})^n - 1), ((xs^{1/(n-1)})^{n-1} - 1) \right) = (-1)^n (s^n - r^{-(n-1)}). \quad (3.2.7)$$

Now apply (3.2.1) with $a = r^{1/n}$ and $b = s^{-1/(n-1)}$ to find that

$$Res_x \left(((xr^{1/n})^n - 1), ((x/s^{1/(n-1)})^{n-1} - 1) \right) = (-1)^n (s^{-n} - r^{(n-1)}). \quad (3.2.8)$$

Finally apply (3.2.1) with $a = r^{1/n}$ and $b = s^{1/(n-1)}$ to obtain

$$Res_x \left(((xr^{1/n})^n - 1), ((xs^{1/(n-1)})^{n-1} - 1) \right) = (-1)^n (s^n - r^{(n-1)}). \quad (3.2.9)$$

Multiplying the right-hand sides of (3.2.6) and (3.2.7), we obtain

$$1 - r^{-(n-1)}(s^n + s^{-n}) + r^{-2(n-1)}, \quad (3.2.10)$$

and multiplying the right-hand sides of (3.2.8) and (3.2.9), we obtain

$$1 - r^{(n-1)}(s^n + s^{-n}) + r^{2(n-1)}. \quad (3.2.11)$$

Now note that by definition,

$$2T_n(a) = r^n + r^{-n},$$

and has the identity

$$2T_n(a)^2 = T_{2n}(a) + 1.$$

Multiply (3.2.10) and (3.2.11) to obtain after simplification

$$\begin{aligned} Res_x (Q_n(a, x), Q_{n-1}(b, x)) &= 2 + T_{2n-2}(a) - 8T_{n-1}(a)T_n(b) + 4T_n(b)^2 \\ &= 4(T_n(b) - T_{n-1}(a))^2. \end{aligned}$$

□

We have an immediate corollary

Corollary 51. *For all $n \geq 1$,*

$$Res_x \left(\hat{P}_n(c, x), \hat{P}_{n-1}(c, x) \right) = 0$$

This can be seen directly from the definition of $\hat{P}_n(c, x)$. We can also apply the Theorem with $a = T_n(c/2)$ and $b = T_{n-1}(c/2)$. By the well known identity (see for example [19, p. 5])

$$T_n(T_m(x)) = T_{nm}(x),$$

the corollary follows.

Corollary 52. *For all $n \geq 1$,*

$$\text{Res}_x(Q_n(a, x), Q_{n-1}(a, x)) = 4(a-1)^2(U_{n-1}(a) + U_{n-2}(a))^2$$

Proof. Set $b = a$ in (3.2.4) and use the recurrence relation (1.0.1) for Chebyshev polynomials together with the identity (3.1.8). $\square$

Theorem 53. *The discriminant of $\hat{P}_n(c, z)$ with respect to z is a polynomial in c given by the identity,*

$$\text{Disc}_z(\hat{P}_n(c, z)) = (n+1)^{2(n+1)}(c^2 - 4)^{n+1}U_n(c/2)^{2(n+1)}. \quad (3.2.12)$$

Proof. First using (3.1.10), we have

$$\frac{d}{dz}\hat{P}_n(c, z) = 2(n+1)z^n(z^{n+1} - T_{n+1}(c/2)),$$

so that

$$\begin{aligned} & \text{Res}_z\left(\hat{P}_n(c, z), \frac{d}{dz}\hat{P}_n(c, z)\right) \\ &= \text{Res}_z\left(\hat{P}_n(c, z), 2(n+1)z^n\right) \text{Res}_z\left(\hat{P}_n(c, z), (z^{n+1} - T_{n+1}(c/2))\right). \end{aligned}$$

Clearly

$$\text{Res}_z\left(\hat{P}_n(c, z), 2(n+1)z^n\right) = (2(n+1))^{2(n+1)},$$

while on the other hand,

$$\begin{aligned} \text{Res}_z\left(\hat{P}_n(c, z), (z^{n+1} - T_{n+1}(c/2))\right) &= (T_{n+1}(c/2)^2 - 2T_{n+1}(c/2)^2 + 1)^{2(n+1)} \\ &= (-1)^{n+1}(T_{n+1}(c/2)^2 - 1)^{2(n+1)} \\ &= 2^{-2(n+1)}(c^2 - 4)^{n+1}U_n(c/2)^{2(n+1)}, \end{aligned}$$

where we have used Pell's equation for Chebyshev polynomials (3.1.11) to find $(T_{n+1}(c/2)^2 - 1) = -i/2\sqrt{c^2 - 4}U_n(c/2)$. Putting everything together, we find that

$$\text{Res}_z \left(\hat{P}_n(c, z), \frac{d}{dz} \hat{P}_n(c, z) \right) = (n+1)^{2(n+1)} (c^2 - 4)^{n+1} U_n(c/2)^{2(n+1)}.$$

Now use (1.5.7) with $f(x) = \hat{P}_n(c, z)$ and $a_0 = 1$ to get the desired identity,

$$\text{Disc}_z(\hat{P}_n(c, z)) = (n+1)^{2(n+1)} (c^2 - 4)^{n+1} U_n(c/2)^{2(n+1)}.$$

□

Since the roots of $U_n(x)$ are all real and in the interval $(-1, 1)$, the roots of $\text{Disc}_z(\hat{P}_n(c, z))$ are all real and in the interval $[-2, 2]$.

Theorem 54. *The discriminant of $\hat{P}_n(c, z)$ with respect to c is a polynomial in z given by the identity,*

$$\text{Disc}_c(\hat{P}_{2n}(c, z)) = (2n+1)^{2n+1} z^{2n(2n+1)} (z^{4n+2} - 1)^{2n}, \quad (3.2.13)$$

for n even and

$$\text{Disc}_c(\hat{P}_{2n+1}(c, z)) = (2n+2)^{2n+2} z^{2(2n+1)(n+1)} (z^{2n+2} + 1)^2 (z^{4n+4} - 1)^{2n}, \quad (3.2.14)$$

for n odd.

Proof. Recall the definition of $\hat{P}_n(c, z)$ in (3.1.10). As a polynomial in c ,

$$\hat{P}_n(c, z) = -2T_{n+1}(c/2)z^{n+1} + z^{2n+2} + 1,$$

and has leading coefficient z^{n+1} . Its derivative with respect to c is given by

$$\frac{d}{dc} \hat{P}_n(c, z) = -(n+1)U_n(c/2)z^{n+1}.$$

This implies that

$$\begin{aligned} \text{Res}_c \left(\hat{P}_n(c, z), \frac{d}{dc} \hat{P}_n(c, z) \right) = \\ (-1)^{n+1} (n+1)^{n+1} z^{(n+1)^2} \text{Res}_z \left(\hat{P}_n(c, z), U_n(c/2) \right). \end{aligned}$$

Applying (1.5.7) with $f(x) = \hat{P}_n(c, z)$ and $a_0 = z^{n+1}$, we have

$$Disc_c \left(\hat{P}_n(c, z) \right) = (n+1)^{n+1} z^{n+1} Res_c \left(\hat{P}_n(c, z), U_n(c/2) \right).$$

It suffices to prove that for n even,

$$Res_c \left(\hat{P}_n(c, z), U_n(c/2) \right) = (z^{2n+2} - 1)^n$$

and for n odd

$$Res_c \left(\hat{P}_n(c, z), U_n(c/2) \right) = (z^{n+1} + 1)^2 (z^{2n+2} - 1)^{n-1}.$$

From (3.1.8) we get the identity $T_{n+1}(x) = xU_n(x) - U_{n-1}(x)$ which we use to find that

$$\begin{aligned} Res_c \left(\hat{P}_n(c, z), U_n(c/2) \right) &= Res_z \left((2U_{n-1}(c/2) - cU_n(c/2))z^{n+1} \right. \\ &\quad \left. + z^{2n+2} + 1, U_n(c/2) \right) \\ &= Res_z \left(2U_{n-1}(c/2)z^{n+1} + z^{2n+2} + 1, U_n(c/2) \right) \\ &= \prod_{k=1}^n \left(2U_{n-1} \left(\cos \frac{\pi k}{n+1} \right) z^{n+1} + z^{2n+2} + 1 \right) \\ &= \prod_{k=1}^n \left(z^{2n+2} + 2 \frac{\sin(nk\pi/(n+1))}{\sin(k\pi/(n+1))} z^{n+1} + 1 \right). \end{aligned}$$

We have used the well known identity (see [7, p. 15])

$$U_{n-1}(\cos \theta) = \frac{\sin(n\theta)}{\sin(\theta)}.$$

Recall that

$$\sin(nk\pi/(n+1)) = \begin{cases} \sin(k\pi/(n+1)), & \text{if } k \text{ is odd} \\ -\sin(k\pi/(n+1)), & \text{if } k \text{ is even} \end{cases}.$$

The theorem follows. □

Corollary 55. *For $n \geq 1$, let $p_n(x)$ be defined as*

$$p_n(x) := U_n(x) + U_{n-1}(x). \tag{3.2.15}$$

Then for $2n + 1$ prime,

$$p_n(T_m(x)) \bmod p_n(x) \equiv \begin{cases} 0, & \text{if } m \not\equiv 0 \pmod{2n+1} \\ m, & \text{if } m \equiv 0 \pmod{2n+1}. \end{cases} \quad (3.2.16)$$

3.3 Hankel determinants

For a sequence $\{a_0, a_1, a_3, \dots\}$, its $n \times n$ Hankel matrix is given by the symmetric matrix

$$H_n := \begin{pmatrix} a_0 & \dots & a_{n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \dots & a_{2n-1} \end{pmatrix}. \quad (3.3.1)$$

In this Section, we are going to give identities for the determinant of the $k \times k$ Hankel matrix of the sequence $\{P_n(c, z)\}$. Since $\hat{P}_n(c, z)$ has a much simpler form, it is easier to calculate for the Hankel determinants of $\{\hat{P}_n(c, z)\}$ from which the Hankel determinants of $\{P_n(c, z)\}$ are easily obtained.

Denote by $m_n^{(k)}$ the $k \times k$ Hankel matrix

$$m_n^{(k)} := \begin{pmatrix} \hat{P}_{n-1}(c, z) & \dots & \hat{P}_{n+k-2}(c, z) \\ \vdots & \ddots & \vdots \\ \hat{P}_{n+k-2}(c, z) & \dots & \hat{P}_{n+2k-3}(c, z) \end{pmatrix} \quad (3.3.2)$$

of $\{\hat{P}_n(c, z)\}$. We start with the most challenging calculation, the 2×2 Hankel determinant of $\hat{P}_n(c, z)$.

3.3.1 The 2×2 Hankel determinants of $\{P_n(c, z)\}$

We give the identity for the 2×2 Hankel determinants of $\{P_n(c, z)\}$, a multivariate polynomial in c and z , and conjecture some interesting root distribution properties of this polynomial.

Theorem 56. *The 2×2 Hankel determinant for $P_n(c, z)$ is given by $-z^n J_n(c, z)$ where*

$$J_n(c, z) = \sum_{k=0}^{n-1} 2(k+1)T_{n-k}(c/2)(z^{2n-k} + z^k) + (2n+1)z^n. \quad (3.3.3)$$

Proof. To simplify calculations for the 2×2 Hankel determinants of $\{P_n(c, z)\}$, we will first calculate for $m_n^{(2)}(c, z)$, the determinant of $m_n^{(2)}$ and show that

$$m_n^{(2)}(c, z) = -z^n(z^2 - cz + 1)^2 J_n(c, z). \quad (3.3.4)$$

The Theorem will be immediate by applying (3.1.10) and some basic properties of determinants.

From the definition of $\hat{P}_n(c, z)$ in (3.1.10), the determinant of $m_n^{(2)}$ is given (after simplification) by,

$$\begin{aligned} & -z^n (2T_n(c/2)(z^{2n+4} + 1) - 4T_{n+1}(c/2)(z^{2n+3} + z) \\ & + 2T_{n+2}(c/2)(z^{2n+2} + z^2) - 2T_2 z^{n+2} - (z^{n+4} - 4z^{n+2} + z^n)). \end{aligned} \quad (3.3.5)$$

We now have to show that

$$2T_n(z^{2n+4} + 1) - 4T_{n+1}(z^{2n+3} + z) + 2T_{n+2}(z^{2n+2} + z^2) - 2T_2 z^{n+2} - (z^{n+4} - 4z^{n+2} + z^n) \quad (3.3.6)$$

is equal to $(z^2 - cz + 1)^2 J_n(c, z)$. Write

$$(z^2 - cz + 1)^2 = z^4 - 4T_1 z^3 + (2T_2 + 4)z^2 - 4T_1 z + 1,$$

and use it to expand the product

$$(z^2 - cz + 1)^2 (2(k+1)T_{n-k}(z^{2n-k} + z^k) + (2n+1)z^n)$$

to deduce that

$$\begin{aligned} = & 2(k+1)T_{n-k}(z^{2n+4-k} + z^{4+k}) - 8(k+1)T_{n-k}T_1(z^{2n+3-k} + z^{3+k}) \\ & + 4(k+1)T_{n-k}T_2(z^{2n+2-k} + z^{2+k}) + 8(k+1)T_{n-k}(z^{2n+2-k} + z^{2+k}) \\ & - 8(k+1)T_{n-k}T_1(z^{2n+1-k} + z^{1+k}) + 2(k+1)T_{n-k}(z^{2n-k} + z^k) \\ & + (2n+1)z^{n+4} - 4(2n+1)T_1 z^{n+3} + 2(2n+1)T_2 z^{n+2} + 4(2n+1)z^{n+2} \\ & - 4(2n+1)T_1 z^{n+1} + (2n+1)z^n. \end{aligned} \quad (3.3.7)$$

Now use the identity (see for example [19, p. 5])

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{|n-m|}(x)$$

in (3.3.7) to find that

$$\begin{aligned}
&= 2(k+1)T_{n-k}(z^{2n+4-k} + z^{4+k}) - 4(2n+1)T_1z^{n+1} + (2n+1)z^n \\
&\quad - 4(k+1)(T_{n+1-k} + T_{|n-1-k|})(z^{2n+3-k} + z^{3+k}) \\
&\quad + 2(k+1)(T_{n+2-k} + T_{|n-2-k|})(z^{2n+2-k} + z^{2+k}) \\
&\quad + 8(k+1)T_{n-k}(z^{2n+2-k} + z^{2+k}) + 2(k+1)T_{n-k}(z^{2n-k} + z^k) \\
&\quad - 4(k+1)(T_{n+1-k} + T_{|n-1-k|})(z^{2n+1-k} + z^{1+k}) + (2n+1)z^{n+4} \\
&\quad - 4(2n+1)T_1z^{n+3} + 2(2n+1)T_2z^{n+2} + 4(2n+1)z^{n+2}. \tag{3.3.8}
\end{aligned}$$

We now compare coefficients of T_j in (3.3.6) and those of T_j in (3.3.8) for $0 \leq j \leq n-1$. We get coefficients for T_n when $k=0, 1, 2$ and these are

$$\begin{aligned}
T_n : \quad & 2(z^{2n+4} + z^4) - 8(z^{2n+2} + z^4) + 6(z^{2n} + z^4) + 8(z^{2n+2} + z^2) \\
& - 8(z^{2n+2} + z^2) + 2(z^{2n} + 1) \\
& = 2(z^{2n+4} + 1).
\end{aligned}$$

For $k=0, 1$ we get coefficients of T_{n+1} . These are

$$\begin{aligned}
T_{n+1} : \quad & -4(z^{2n+3} + z^3) + 4(z^{2n+1} + z^3) - 4(z^{2n+1} + z) \\
& = -4(z^{2n+3} + z).
\end{aligned}$$

Coefficients for T_{n+2} appear only when $k=0$, and these are

$$T_{n+2} : \quad 2(z^{2n+2} + z^2).$$

We now examine the coefficients of T_{n-m} for $1 \leq m \leq n-3$ and show that they are all zero. These are

$$\begin{aligned}
T_{n-m} : \quad & 2(m+1)(z^{2n+4-m} + z^{4+m}) - 4(m+2)(z^{2n+2-m} + z^{4+m}) \\
& - 4m(z^{2n+4-m} + z^{2+m}) + 2(m+3)(z^{2n-m} + z^{4+m}) \\
& + 2(m-1)(z^{2n+4-m} + z^m) + 8(m+1)(z^{2n+2-m} + z^{2+m}) - \\
& - 4(m+2)(z^{2n-m} + z^{2+m}) - 4m(z^{2n+2-m} + z^m) \\
& + 2(m+1)(z^{2n-m} + z^m) \\
& = 0.
\end{aligned}$$

The coefficients of T_2 come from the values of $k = n - 4, n - 3, n - 2, n - 1$. These are

$$\begin{aligned}
T_2 : \quad & 2(n-1)(z^{n+6} + z^{n+2}) - 4n(z^{n+4} + z^{n+2}) - 4(n-2)(z^{n+6} + z^n) \\
& + 2(n-3)(z^{n+6} + z^{n-2}) + 8(n-1)(z^{n+4} + z^n) - 4n(z^{n+2} + z^n) \\
& - 4(n-2)(z^{n+4} + z^{n-2}) + 2(n-1)(z^{n+2} + z^{n-2}) + 2(2n+1)z^{n+2} \\
& = -2z^{n+2}.
\end{aligned}$$

Coefficients of T_1 come from the values of $k = n - 3, n - 2, n - 1$. These are

$$\begin{aligned}
T_1 : \quad & 2n(z^{n+5} + z^{n+3}) - 4(n-1)(z^{n+5} + z^{n+1}) + 2n(z^{n+3} + z^{n+1}) \\
& + 2(n-2)(z^{n+5} + z^{n-1}) + 8n(z^{n+3} + z^{n+1}) - 4(n-1)(z^{n+3} + z^{n-1}) \\
& + 2n(z^{n+1} + z^{n-1}) - 4(2n+1)z^{n+3} - 4(2n+1)z^{n+1} \\
& = 0.
\end{aligned}$$

Finally the coefficients independent of T_n when $n > 0$ are given by

$$\begin{aligned}
T_0 : \quad & -4n(z^{n+4} + z^{n+2}) + 2(n-1)(z^{n+4} + z^n) - 4n(z^{n+2} + z^n) \\
& + (2n+1)z^{n+4} + 4(2n+1)z^{n+2} + (2n+1)z^n \\
& - z^{n+4} + 4z^{n+2} - z^n.
\end{aligned}$$

All the coefficients of T_j for $0 \leq j \leq n+2$ match, hence the two polynomials must be equal. $\square$

Numerical calculations suggest that for all $n \geq 1$ and all real c with $|c| \geq 2$, the roots of $J_n(c, z)$ all lie on the unit circle.

Conjecture 57. *For $n = 2k + 1$ odd, all the coefficients of c in $J_n(c, z)$ are polynomials in z whose non trivial roots lie on the unit circle.*

Let n be the degree of $J_n(c, z)$ as a polynomial in the variable c . We easily see that

$$\lim_{n \rightarrow \infty} \frac{1}{c^n} J_n(c, z) = z^{2n} + 1$$

has all roots on the unit circle. Of course the truth of the conjecture requires this. We now ask about the coefficients of the other powers of c . For n odd, they seem to have all non-zero roots on the unit circle, in spite of not satisfying in general any standard sufficient condition for this to be true.

The coefficient of the zeroth power of c in $J_{2k+1}(c, z)$ is given by $zJ_k^{(0)}(z)$ where

$$\begin{aligned} J_k^{(0)}(z) &:= 4 \sum_{j=0}^{k-1} (-1)^j (j+1) (z^{4k-2j} + z^{2j}) + (-1)^k (4n+3) z^{2k} \\ &= 4 \left(\frac{z^{2k+2} + (-1)^k}{z^2 + 1} \right)^2 - (-1)^k z^{2k} \\ &= \frac{1}{(z^2 + 1)^2} (4z^{4k+4} - (-1)^k (z^4 - 6z^2 + 1) z^{2k} + 4). \quad (3.3.9) \end{aligned}$$

The coefficient of c in $J_{2k+1}(c, z)$ is given by $(z^{2k+2} + (-1)^k) J_k^{(1)}(z)$ where

$$J_k^{(1)}(z) := \sum_{j=0}^k (-1)^j (2k+1-2j)(2j+1) z^{2j}.$$

The coefficient of c^2 in $J_{2k+1}(c, z)$ is given by $(-1)^{k+1} 2z J_k^{(2)}(z)$ where

$$J_k^{(2)}(z) := \sum_{j=0}^k (-1)^j (k-j)^2 (j+1) (z^{2k-2j} + z^{2j}).$$

The coefficient of c^3 in $J_{2k+1}(c, z)$ is given by $(-1)^{k+1} J_k^{(3)}(z)$ where

$$J_k^{(3)}(z) := \frac{1}{6} \sum_{j=0}^k (-1)^j (k-j)(k-j+1)(2(k-j)+1)(2j+1) (z^{4k+2-2j} + z^{2j}).$$

The coefficient of c^4 in $J_{2k+1}(c, z)$ is given by $(-1)^k 2z(z^2 + 1) J_k^{(4)}(z)$ where

$$J_k^{(4)}(z) := \frac{1}{12} \sum_{j=0}^k (-1)^j (k-j)^2 ((k-j)^2 - 1) (j+1) (z^{4k-2j} + z^{2j}).$$

3.3.2 Other Hankel determinants of $P_n(c, z)$

The following lemma will be useful in the proof of the next two Theorems.

Lemma 58. *Let $\hat{P}_n(c, z)$ be as defined in (3.1.10). Then for all $n \geq 0$ and all $j \geq 0$*

$$\hat{P}_{n+j}(c, z) - cz \hat{P}_{n+j-1}(c, z) + z^2 \hat{P}_{n+j-2}(c, z) = (z^{2n+2j} + 1)(z^2 - cz + 1). \quad (3.3.10)$$

Proof. From the definition of $\hat{P}_n(c, z)$ in (3.1.10),

$$\begin{aligned}
& \hat{P}_{n+j}(c, z) - cz\hat{P}_n(c, z) + z^2\hat{P}_{n-j}(c, z) \\
&= (z^{2(n+j+1)} - 2T_{n+j+1}z^{n+j+1} + 1) - cz(z^{2(n+j)} - 2T_{n+j}z^{n+j} + 1) \\
&\quad + z^2(z^{2(n+j-1)} - 2T_{n+j-1}z^{n+j-1} + 1) \\
&= z^{2(n+j)}(z^2 - cz + 1) - 2z^{n+j+1}(T_{n+j+1} - cT_{n+j} + T_{n+j-1}) + (1 - cz + z^2) \\
&= (z^{2n+2j} + 1)(z^2 - cz + 1).
\end{aligned}$$

□

Theorem 59. *The 3×3 Hankel determinant of $P_n(c, z)$ is given by $z^{2n+2}H_n(c, z)$ where $H_n(c, z)$ is a palindrome in z of degree $2n + 2$ given by the identity*

$$\begin{aligned}
H_n(c, z) &= \sum_{k=0}^{n-1} 2(T_{k+2}(c/2) - T_k(c/2))(z^{2n+2-k} + z^k) - 2T_n(c/2)(z^{n+2} + z^n) \\
&\quad - 4T_{n+1}(c/2)z^{n+1}.
\end{aligned} \tag{3.3.11}$$

Proof. First, we show that the determinant of $m_n^{(3)}$ is given by

$$m_n^{(3)}(c, z) := z^{2n+2}(z^2 - cz + 1)^3 H_n(c, z) \tag{3.3.12}$$

from which the Hankel determinant of $P_n(c, z)$ follows. We first write $m_n^{(3)}(c, z)$ as

$$m_n^{(3)}(c, z) = \begin{vmatrix} \hat{P}_{n-1}(c, z) & \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) \\ \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) \\ \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) & \hat{P}_{n+3}(c, z) \end{vmatrix}$$

In the matrix, perform the row operation $r_3 - czr_2 + z^2r_1$ using (3.3.10) and factor out $z^2 - cz + 1$ from the new row r_3 to find that

$$\frac{m_n^{(3)}(c, z)}{z^2 - cz + 1} = \begin{vmatrix} \hat{P}_{n-1}(c, z) & \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) \\ \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) \\ z^{2n+2} + 1 & z^{2n+4} + 1 & z^{2n+6} + 1 \end{vmatrix}.$$

Next, perform the column operations $C_3 - czC_2 + z^2C_1$ using (3.3.10) and

factor out $z^2 - cz + 1$ from the new column C_3 to find that

$$\frac{m_n^{(3)}(c, z)}{(z^2 - cz + 1)^2} = \begin{vmatrix} \hat{P}_{n-1}(c, z) & \hat{P}_n(c, z) & z^{2n+2} + 1 \\ \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & z^{2n+4} + 1 \\ z^{2n+2} + 1 & z^{2n+4} + 1 & z^{2n+4} + 1 \end{vmatrix}.$$

Next, perform the column operations $C_2 - C_3$ to obtain a new column C_2 followed by the row operation $r_2 - r_3$ to obtain a new row r_2 to find that

$$\frac{m_n^{(3)}(c, z)}{(z^2 - cz + 1)^2} = \begin{vmatrix} z^{2n} + 1 - 2T_n z^n & -2T_{n+1} z^{n+1} & z^{2n+2} + 1 \\ -2T_{n+1} z^{n+1} & -2T_{n+2} z^{n+2} & 0 \\ z^{2n+2} + 1 & 0 & z^{2n+4} + 1 \end{vmatrix}.$$

We can simplify the first row(column) further by $r_1 - z^{-2}r_3$ to obtain a new row r_1

$$\frac{m_n^{(3)}(c, z)}{(z^2 - cz + 1)^2} = \begin{vmatrix} 1 - z^{-2} - 2T_n z^n & -2T_{n+1} z^{n+1} & 1 - z^{-2} \\ -2T_{n+1} z^{n+1} & -2T_{n+2} z^{n+2} & 0 \\ z^{2n+2} + 1 & 0 & z^{2n+4} + 1 \end{vmatrix},$$

and finally $C_1 - C_3$ to obtain the new column C_1 to obtain a much simpler matrix

$$\frac{m_n^{(3)}(c, z)}{(z^2 - cz + 1)^2} = \begin{vmatrix} -2T_n z^n & -2T_{n+1} z^{n+1} & z^{-2}(z^2 - 1) \\ -2T_{n+1} z^{n+1} & -2T_{n+2} z^{n+2} & 0 \\ -z^{2n+2}(z^2 - 1) & 0 & z^{2n+4} + 1 \end{vmatrix}.$$

Now

$$\begin{aligned} & \begin{vmatrix} -2T_n z^n & -2T_{n+1} z^{n+1} & z^{-2}(z^2 - 1) \\ -2T_{n+1} z^{n+1} & -2T_{n+2} z^{n+2} & 0 \\ -z^{2n+2}(z^2 - 1) & 0 & z^{2n+4} + 1 \end{vmatrix} \\ &= -2z^{3n+2}(z^2 - 1) \begin{vmatrix} zT_{n+1} & z^{-2}(z^2 - 1) \\ z^2T_{n+2} & 0 \end{vmatrix} + 4(z^{2n+4} + 1)z^{2n+2} \begin{vmatrix} T_n & T_{n+1} \\ T_{n+1} & T_{n+2} \end{vmatrix} \\ &= z^{2n+2} \{ 4(z^{2n+4} + 1)(T_n T_{n+2} - T_{n+1}^2) - 2z^n(z^2 - 1)^2 T_{n+2} \} \\ &= z^{2n+2} \{ (z^{2n+4} + 1)(c^2 - 4) - 2z^n(z^2 - 1)^2 T_{n+2} \}, \end{aligned}$$

where the last step follows from

$$\begin{aligned}
T_n T_{n+2} - T_{n+1}^2 &= T_n (cT_{n+1} - T_n) - T_{n+1}^2 \\
&= -T_{n+1}^2 + cT_n T_{n+1} - T_n^2 \\
&= -T_{n+1} (cT_n - T_{n-1}) + cT_n T_{n+1} - T_n^2 \\
&= T_{n+1} T_{n-1} - T_n^2 \\
&\vdots \\
&= T_2 T_0 - T_1^2 \\
&= \frac{c^2}{4} - 1.
\end{aligned} \tag{3.3.13}$$

it suffices to show that

$$(z^{2n+4} + 1)(c^2 - 4) - 2z^n(z^2 - 1)^2 T_{n+2} = (z^2 - cz + 1)H_n(c, z). \tag{3.3.14}$$

First

$$\begin{aligned}
&-2z^n(z^2 - 1)^2 T_{n+2} + (z^2 - cz + 1)(2T_n(c/2)(z^{n+2} + z^n) + 4T_{n+1}(c/2)z^{n+1}) \\
&= -2z^n \{ (cz^4 - 2z^3 - 2z + c)T_{n+1} - (2z^4 - cz^3 - cz + 2)T_n \}.
\end{aligned}$$

For $k \geq 0$, let $F_k = F_{n,k}(c, z)$ be defined as

$$\begin{aligned}
F_k &:= -2z^{n-k} \{ (cz^{4+2k} - 2z^{3+2k} - 2z + c)T_{n-k+1} \\
&\quad - (2z^{4+2k} - cz^{3+2k} - cz + 2)T_{n-k} \}
\end{aligned} \tag{3.3.15}$$

We show that

$$F_k = 2(z^2 - cz + 1)(z^{n+k+3} + z^{n-k-1})(T_{n-k+1} - T_{n-k-1}) + F_{k+1}.$$

Indeed

$$\begin{aligned}
F_k - F_{k+1} &= -2z^{n-k} \{ (cz^{4+2k} - 2z^{3+2k} - 2z + c)T_{n-k+1} \\
&\quad - (2z^{4+2k} - cz^{3+2k} - cz + 2)T_{n-k} \} \\
&\quad + 2z^{n-k-1} \{ (cz^{6+2k} - 2z^{5+2k} - 2z + c)T_{n-k} \\
&\quad - (2z^{6+2k} - cz^{5+2k} - cz + 2)T_{n-k-1} \} \\
&= -2z^{n-k-1} \{ (cz^{5+2k} - 2z^{4+2k} - 2z^2 + cz)T_{n-k+1} \\
&\quad + (2z^{6+2k} - cz^{5+2k} - cz + 2)T_{n-k-1} \\
&\quad + (-z^{6+2k} + z^{4+2k} + z^2 - 1)cT_{n-k} \} \\
&= -2(z^2 - cz + 1)z^{n-k-1} \{ -(z^{4+2k} + 1)T_{n-k+1} + \\
&\quad (z^{4+2k} + 1)T_{n-k-1} \} \\
&= 2(z^2 - cz + 1)z^{n-k-1}(z^{4+2k} + 1)(T_{n-k+1} - T_{n-k-1}),
\end{aligned}$$

since $cT_n = T_{n+1} + T_{n-1}$. We can now write F_k as

$$F_k = 2(z^2 - cz + 1) \sum_{k=0}^{n-1} (T_{n-k+1} - T_{n-k-1})(z^{n+k+3} + z^{n-k-1}) + F_n,$$

where

$$\begin{aligned}
F_n &= -2(cz^{4+2n} - 2z^{3+2n} - 2z + c)T_1 + 2(2z^{4+2n} - cz^{3+2n} - cz + 2)T_0 \\
&= -(c^2 - 4)(z^{2n+4} + 1),
\end{aligned}$$

which conveniently cancels with the remaining term on the left-hand side of (3.3.14). $\square$

Numerical computations suggest a rather interesting behavior for the roots of $H_n(c, z)$. For n sufficiently large and $|c| \geq 2$ real, $H_n(c, z)$ has a pair of complex conjugate roots close to -1 , and a pair of real reciprocal roots close to 1 . The remaining roots lie close to T , the union of two concentric and inversely related circles with radii

$$r_1 = \left| \frac{c + \sqrt{c^2 - 4}}{2} \right| \quad \text{and} \quad r_2 = \left| \frac{c - \sqrt{c^2 - 4}}{2} \right|.$$

Theorem 60. *The 4×4 Hankel determinant of $P_n(c, z)$ is given by*

$$(z^2 - 1)^2(c^2 - 4)z^{4n+6}.$$

Proof. We show by elementary row and column operations that the determinant of $m_4^{(k)}$ is given by the identity

$$m_n^{(4)}(c, z) := (z^2 - cz + 1)^4(z^2 - 1)^2(c^2 - 4)z^{4n+6}. \quad (3.3.16)$$

First write $m_4^{(k)}(c, z)$ as

$$m_n^{(4)}(c, z) = \begin{vmatrix} \hat{P}_{n-1}(c, z) & \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) \\ \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) & \hat{P}_{n+3}(c, z) \\ \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) & \hat{P}_{n+3}(c, z) & \hat{P}_{n+4}(c, z) \\ \hat{P}_{n+2}(c, z) & \hat{P}_{n+3}(c, z) & \hat{P}_{n+4}(c, z) & \hat{P}_{n+5}(c, z) \end{vmatrix}$$

In the matrix, perform the row operations $r_3 - czr_2 + z^2r_1 \rightarrow r_3$ and $r_4 - czr_3 + z^2r_2 \rightarrow r_4$ using (3.3.10) and factor out $z^2 - cz + 1$ from each of the new rows to get

$$\frac{m_n^{(4)}(c, z)}{(z^2 - cz + 1)^2} = \begin{vmatrix} \hat{P}_{n-1}(c, z) & \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) \\ \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & \hat{P}_{n+2}(c, z) & \hat{P}_{n+3}(c, z) \\ z^{2n+2} + 1 & z^{2n+4} + 1 & z^{2n+6} + 1 & z^{2n+8} + 1 \\ z^{2n+4} + 1 & z^{2n+6} + 1 & z^{2n+8} + 1 & z^{2n+10} + 1 \end{vmatrix}$$

Next, perform the column operations $C_3 - czC_2 + z^2C_1 \rightarrow C_3$ and $C_4 - czC_3 + z^2C_2 \rightarrow C_4$ using (3.3.10) and factor out $z^2 - cz + 1$ from each the new columns to get

$$\frac{m_n^{(4)}(c, z)}{(z^2 - cz + 1)^4} = \begin{vmatrix} \hat{P}_{n-1}(c, z) & \hat{P}_n(c, z) & z^{2n+2} + 1 & z^{2n+4} + 1 \\ \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & z^{2n+4} + 1 & z^{2n+6} + 1 \\ z^{2n+2} + 1 & z^{2n+4} + 1 & z^{2n+4} + 1 & z^{2n+6} + 1 \\ z^{2n+4} + 1 & z^{2n+6} + 1 & z^{2n+6} + 1 & z^{2n+8} + 1 \end{vmatrix}.$$

Now follow the last step by $r_4 - z^2r_3 \rightarrow r_4$ and factor $1 - z^2$ from the new

row. And $C_4 - z^2 C_3 \rightarrow C_4$, and factor $1 - z^2$ from the new column to get

$$\frac{m_n^{(4)}(c, z)}{(z^2 - cz + 1)^4(1 - z^2)^2} = \begin{vmatrix} \hat{P}_{n-1}(c, z) & \hat{P}_n(c, z) & z^{2n+2} + 1 & 1 \\ \hat{P}_n(c, z) & \hat{P}_{n+1}(c, z) & z^{2n+4} + 1 & 1 \\ z^{2n+2} + 1 & z^{2n+4} + 1 & z^{2n+4} + 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

Further simplify the matrix by the column operations $C_j - C_4 \rightarrow C_j$ followed by factoring z^{n+j} from C_j respectively for $j = 1, 2, 3$, and z^2 from r_3 to get

$$\frac{m_n^{(4)}(c, z)}{(z^2 - cz + 1)^4(1 - z^2)^2 z^{4n+5}} = \begin{vmatrix} z^n - 2T_n & z^{n+1} - 2T_{n+1} & 1 & 1 \\ z^{n+2} - 2zT_{n+1} & z^{n+3} - 2zT_{n+2} & z^2 & 1 \\ z^n & z^{n+1} & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

This is now a determinant of a 3×3 matrix. Now $r_2 - z^2 r_1 \rightarrow r_2$ and $r_3 - r_1 \rightarrow r_1$ gives

$$\begin{aligned} \frac{m_n^{(4)}(c, z)}{(z^2 - cz + 1)^4(1 - z^2)^2 z^{4n+5}} &= \begin{vmatrix} z^n - 2T_n & z^{n+1} - 2T_{n+1} & 1 \\ 2z^2 T_n - 2zT_{n+1} & 2z^2 T_{n+1} - 2zT_{n+2} & 0 \\ 2T_n & 2T_{n+1} & 0 \end{vmatrix} \\ &= 4z \begin{vmatrix} zT_n - T_{n+1} & zT_{n+1} - T_{n+2} \\ T_n & T_{n+1} \end{vmatrix} \\ &= 4z(T_n T_{n+2} - T_{n+1}^2). \end{aligned}$$

The theorem follows by applying (3.3.13). □

3.4 Remarks

Let

$$S_{n,m}(x) = \left(\sum_{k=0}^{n+m} x^k \right) \left(\sum_{j=0}^n x^j \right).$$

One might ask about the root distribution of the polynomials formed by first expanding $S_{n,m}(x)$ and then replacing each integer coefficient $k+1$, for $k \geq 0$, by $U_k(c/2)$. The case $n = 6$, $m = 4$ and $c = 5$ already shows that the zeros need not lie on the union of two circles. In fact we get 12 zeros on the union

of two inversely related circles and 4 zeros on the unit circle. Is it always true that we will have $2n$ zeros on the union of two inversely related circles, and m zeros on the unit circle? We leave open the problem of finding the root distribution in this more general case.

Another interesting problem comes from modifying the numerator of the generating function for $P_n(c, z)$ to obtain

$$\frac{1 + z^2 t^2}{(t - 1)(tz^2 - 1)(z^2 t^2 - czt + 1)} = \sum_{n=0}^{\infty} B_n(c, z) t^n.$$

The zeros of $B_n(c, z)$ seem to lie close to the zeros of $P_n(c, z)$. Unlike the polynomials $P_n(c, z)$ from which we could get a simpler polynomial, it is not clear how the same can be done to $B_n(c, z)$. Also, the polynomials $B_n(c, z)$ do not have simple identities for their discriminants with respect to both z and c . What relationships can we draw from $P_n(c, z)$ and $B_n(c, z)$? We also leave open the problem of finding the root distribution as well as identities for the discriminants of $B_n(c, z)$.

CHAPTER 4

Chebyshev-Like Polynomials with Roots on Hyperbolas

We have so far just connected Chebyshev polynomials with ellipses. What about other conic sections? It turns out that a family of parabolas formed from the roots of a generalized cyclotomic polynomial has parameters whose various minimal polynomials are of interest. Some of them lead immediately (but conjecturally) to Chebyshev T-polynomials. We conjecture specific generating functions for the others. Various specializations of these generating functions lead to rational functions with quartic denominators whose power series coefficient ratios connect with periodic continued fractions (even though periodic continued fractions are traditionally only associated with quadratic polynomials!)

We then proceed in another direction, motivated by a connection between one of the quartics, and another quartic which then becomes our primary focus. The connection here might appear tenuous, but this quartic leads us to periodic continued fractions that we find of considerable interest. These depend on a parameter x , namely

$$[2x, \overline{2x(x^2 - 2)}].$$

When 2 is replaced by a parameter s , we obtain polynomials with many (and in many cases all) roots on hyperbolas, and which are expressible in terms of Chebyshev U-polynomials. We thus have connected Chebyshev polynomials to all types of conic sections.

In section 4.2 of the chapter, we construct from polynomials with real roots, polynomials with roots on hyperbolas. Here, we show how any polynomial with real roots can be lifted to a polynomial with all roots on a hyperbola.

4.1 Chebyshev-like polynomials from continued fractions with roots on hyperbolas

4.1.1 Parabolas and the unit circle

This section contains mostly conjectural statements. However, the generating functions encountered here raise interesting questions about rationally generated integer sequences and periodic continued fractions. Consider the polynomial

$$G_n(m, x) := m + x + x^2 + x^3 + \cdots + x^{2n-1} + mx^{2n} \quad (4.1.1)$$

where m is real. By a theorem of Weiyu Chen [30], $G_n(m, x)$ has all its roots on the unit circle whenever $m \geq 1$.

Conjecture 61. *For $m \geq 1/2$, all roots of $G_n(m, x) = 0$ are on $|x| = 1$.*

Obviously when $m = 0$, $G_n(m, x)$ has a trivial root and the remaining roots lie on $|x| = 1$. For $m = 1/2$,

$$G_n(1/2, x) = \frac{(1+x)^2(1-x^{2n})}{2(1-x^2)},$$

which clearly has all roots on $|x| = 1$. As m increases from 0 to $1/2$, the two real (and inversely related) roots of $G_n(m, x)$ move towards $|x| = 1$, and at $m = 1/2$, the two real roots coincide at $x = -1$. As m further increases, the double root splits into complex conjugate roots on $|x| = 1$. In the limit as $m \rightarrow \infty$, the $2n$ roots of $G_n(m, x)$ approach the roots of $1 + x^{2n}$.

Let

$$y^2 = h(2x - h) \quad (4.1.2)$$

be the parabola that goes through (x, y) and $(x, -y)$, the left-most pair of conjugate roots of $G_n(m, x)$. For non-real roots $x \pm iy$ with $0 \leq y \leq x$ there are two distinct real values for h , call them h_1 and h_2 . Let $H_n(m, x)$ be the minimal polynomial for $h_1 h_2$.

Conjecture 62. *For n odd,*

$$H_n(m, x) = m^2 T_n(x)^2 - (2m - 1) \frac{1}{\sqrt{x}} T_{2n-1}(x) - 2m(2m - 1), \quad (4.1.3)$$

while for n even,

$$H_n(m, x) = m^2 T_n(x)^2 - (2m - 1) \frac{1}{\sqrt{x}} T_{2n-1}(x) - 2m. \quad (4.1.4)$$

It is easy to prove the conjecture when $m = 1$, $m = 0$ and $m = 1/2$. From the right hand side of (4.1.3) and (4.1.4),

$$H_n(1, x) = \frac{1}{\sqrt{x}} T_{2n+1}(\sqrt{x}), \quad H_n(0, x) = \frac{1}{\sqrt{x}} T_{2n-1}(\sqrt{x}),$$

and

$$H_n(1/2, x) = \begin{cases} T_n(\sqrt{x})^2 & \text{for } n \text{ odd} \\ \frac{1}{2}(T_{2n}(\sqrt{x}) - 1) & \text{for } n \text{ even.} \end{cases}$$

The two non-real and left-most roots of $G_n(1, x)$ are given by

$$-\cos(\pi/(2n+1)) \pm i \sin(\pi/(2n+1)).$$

Solving the quadratic (4.1.2) for h using these roots we find that

$$\begin{aligned} h_1 &= -\cos(\pi/(2n+1)) - \sqrt{\cos(2\pi/(2n+1))}, \\ h_2 &= -\cos(\pi/(2n+1)) + \sqrt{\cos(2\pi/(2n+1))}, \end{aligned}$$

so that

$$h_1 h_2 = \sin(\pi/(2n+1))^2.$$

To see that

$$\frac{1}{\sqrt{x}} T_{2n+1}(\sqrt{x})$$

is the minimal polynomial for $h_1 h_2$, recall that

$$T_n(x) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right).$$

Now we have

$$T_n(\sin(x)) = \frac{i^n}{2} (1 + (-1)^n) \cos(nx) - \frac{i^{n+1}}{2} (1 - (-1)^n) \sin(nx).$$

It is clear that for n odd, $T_n(\sin(\pi/n)) = 0$. The proof is now straightforward.

The case where $m = 0$ is very similar.

Now

$$h_1 + h_2 = -2 \cos(\pi/(2n+1))^2,$$

and its minimal polynomial divides $U_n(x/2) + U_{n-1}(x/2)$. To see this, first note that $h_1 + h_2$ is a root of $T_{2n+1}(x/2) - 1$. It remains to show that $U_n(x/2) + U_{n-1}(x/2)$ is a non-trivial factor of $T_{2n+1}(x/2) - 1$.

Lemma 63. *For all $n \geq 1$, $T_{2n+1}(x/2) - 1$ factors as*

$$T_{2n+1}(x) - 1 = (x - 1)(U_n(x) + U_{n-1}(x))^2. \quad (4.1.5)$$

Proof. First we apply Turan's identity for Chebyshev polynomials to the right-hand side of (4.1.5) to deduce that

$$\begin{aligned} (x - 1)(U_n(x/2) + U_{n-1}(x/2))^2 &= (x - 1)(2(x + 1)U_n(x)U_{n-1}(x) + 1) \\ &= 2(x^2 - 1)U_n(x)U_{n-1}(x) + x - 1. \end{aligned}$$

Let $x = \cos \theta$ and use the well known trigonometric identity

$$\cos(a + b) - \cos(a - b) = -2 \sin(a) \sin(b)$$

to find that

$$\begin{aligned} -2 \sin^2 \theta U_n(\cos \theta) U_{n-1}(\cos \theta) + \cos \theta - 1 &= -2 \sin(n + 1)\theta \sin n\theta + \cos \theta - 1 \\ &= \cos(2n + 1)\theta - \cos \theta + \cos \theta - 1 \\ &= T_{2n+1}(\cos \theta). \end{aligned}$$

□

For the case where $m = 1/2$, the two non-real and left-most roots of $G_n(1/2, x)$ are given by

$$-\cos(\pi/n) \pm i \sin(\pi/n).$$

Solving the quadratic (4.1.2) for h using these roots we find

$$\begin{aligned} h_1 &= -\cos(\pi/n) - \sqrt{\cos(2\pi/n)}, \\ h_2 &= -\cos(\pi/n) + \sqrt{\cos(2\pi/n)}, \end{aligned}$$

so that $h_1 h_2 = \sin(\pi/n)^2$. It is now easy to show that for n odd, the minimal polynomial will be a non-trivial factor of $\frac{1}{\sqrt{x}} T_n(\sqrt{x})$. Recall that for n even, $T_n(\sin(x)) = i^n \cos(nx)$. This implies that $T_{2k}(\sin(\pi/2k)) = (-1)^{k+1}$ and hence the minimal polynomial will be a non-trivial factor of $T_n(\sqrt{x})^2 - 1$.

Now $h_1 + h_2 = -2 \cos(\pi/n)$ and its minimal polynomial is a non-trivial factor of $(-1)^n T_n(x/2) + 1$. When $n = 2k$ is even,

$$T_n(x) + 1 = 2T_k(x)^2,$$

so the minimal polynomial of $h_1 + h_2$ will be a non trivial factor of $T_k(x/2)$. On the other hand, when $n = 2k + 1$ is odd, we have from Lemma 63 that the minimal polynomial of $h_1 + h_2$ will divide $U_k(x/2) + U_{k-1}(x/2)$.

What can be said of the minimal polynomials for h_1 and h_2 ? The first observation is that h_1 and h_2 have the same minimal polynomial. If we let

$$(x - h_1)(x - h_2) = x^2 + 2x \cos(\pi/n) + \sin^2(\pi/n),$$

we find that the minimal polynomial for h_1 (or h_2) is a polynomial in $\mathbb{Q}[x]$ for which

$$4x^2 + 4(\zeta_n + \zeta_n^{-1})x - (\zeta_n^2 - 2 + \zeta_n^{-2}),$$

where $\zeta_n = e^{\frac{\pi i}{n}}$ is a factor in $\mathbb{C}[x]$.

Conjecture 64. *Let $m_n(x)$ be the required minimal polynomial. Then for $n = 2j + 1$ odd, $m_n(x)$ is a non trivial factor of $M_j(x)$ where*

$$J(x, t) := \frac{1 + (4x - 1)t + (4x - 1)t^2 + t^3}{1 - 4(1 + x^2)t + 6(1 + 4x^2)t^2 - 4(1 + x^2)t^3 + t^4} = \sum_{j=0}^{\infty} M_j(x)t^j. \quad (4.1.6)$$

The first 6 polynomials in this sequence are given in the table below;

Conjecture 65. *For all $j \geq 1$, the two leading coefficients of $M_j(x)$ are 2^{2j} . $M_j(x)$ is irreducible if and only if j is prime. If $p|j$ and p is prime, then $M_p(x)|M_j(x)$.*

Conjecture 66. *If p is prime, the discriminant $\Delta_x(M_p(x))$ is divisible by p (and in fact by p^{p-3}). If $q|\Delta_x(M_p(x))$, where p and q are prime, then $q \equiv \pm 1 \pmod{p}$.*

j	$M_j(x)$
0	1
1	$3 + 4x + 4x^2$
2	$5 + 20x + 4x^2 + 16x^3 + 16x^4$
3	$7 + 56x - 56x^2 + 48x^3 - 16x^4 + 64x^5 + 64x^6$
4	$9 + 120x - 312x^2 - 144x^3 - 464x^4 + 64x^5 - 192x^6 + 256x^7 + 256x^8$
5	$11 + 220x - 1012x^2 - 1584x^3 - 1584x^4 - 1792x^5 - 2560x^6 - 256x^7 - 1280x^8 + 1024x^9 + 1024x^{10}$

Table 4.1: Table of the first six polynomials $M_j(x)$.

j	$Y_j(x)$
0	1
1	$4 + 4x^2$
2	$10 + 8x^2 + 16x^4$
3	$20 - 44x^2 + 64x^6$
4	$35 - 352x^2 - 448x^4 - 128x^6 + 256x^8$
5	$56 - 1416x^2 - 2048x^4 - 2624x^6 - 1024x^8 + 1024x^{10}$
6	$84 - 4272x^2 - 2912x^4 - 6912x^6 - 12800x^8 - 6144x^{10} + 4096x^{12}$

Table 4.2: Table of the first seven polynomials $Y_j(x)$.

Now set

$$K(x, t) := \frac{1}{1 - 4(1 + x^2)t + 6(1 + 4x^2)t^2 - 4(1 + x^2)t^3 + t^4} = \sum_{j=0}^{\infty} Y_j(x)t^j \quad (4.1.7)$$

The first few polynomials in the sequence are given by

Then $M_j(x)$ and $Y_j(x)$ have many properties in common.

Conjecture 67. *The roots of the polynomials generated by $J(x, t)$ and $K(x, t)$ are either real in the interval $(-2, 2)$, or lie on the ellipse*

$$E(1/\sqrt{2}, 1) : \quad 2x^2 + y^2 = 1.$$

The real roots are either “significantly” inside or outside the ellipse. So none are very close to $\pm \frac{1}{\sqrt{2}}$, the points at which the ellipse intersects the real axis. The roots on the ellipse cluster somewhat at the top and bottom of the ellipse.

Recall that the transfinite diameter of $E(\sqrt{5}, 1)$ equals the golden ration. The ellipse $E(1/\sqrt{2}, 1)$ is also very special in the sense that its orthogonal trajectories are parabolas. To see this, Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be an arbitrary ellipse. Then $y' = -\frac{b^2}{a^2} \frac{x}{y}$, so the orthogonal slope is $\frac{a^2}{b^2} \frac{y}{x}$. Now $y' = \frac{a^2}{b^2} \frac{y}{x}$ has the solution $y = kx^{a^2/b^2}$, and this is a parabola when $a = 1/\sqrt{2}$ and $b = 1$.

4.1.2 Rationally generated integer sequences and periodic continued fractions

Given any integer sequence $\{C_n\}_{n \geq 0}$, the ratios of consecutive elements of the sequence C_{n+1}/C_n are rational numbers with finite continued fraction expansion. Suppose the integer sequence is rationally generated. When is the ratio C_{n+1}/C_n a truncation of a periodic continued fraction? Put another way, what rational functions generate integer sequences such that the ratio C_{n+1}/C_n of consecutive elements is a truncation of periodic continued fractions?

As an example, the rational function $\frac{t}{1-t-t^2}$ generates Fibonacci numbers and clearly the ratio of consecutive Fibonacci numbers is the truncation of $\frac{1}{2}(1 + \sqrt{5})$, the golden ratio.

Remark 4. *It is easy to show that integers generated by the rational functions $\frac{1+bt}{1-at+t^2}$ and $\frac{1+bt}{1-at-t^2}$, where a and b are integers, have this property, i.e., the ratio of consecutive integers is a truncation of a periodic continued fraction. We will revisit these integer sequences in chapter 5 where we will look at integer multiples of continued fractions.*

In this section, we look at examples of rationally generated integers of the form

$$\frac{N(t)}{D(a, t)} = \sum_{n=0}^{\infty} C_n(a) t^n \quad (4.1.8)$$

where $D(a, t)$ is a quartic in t , and a is an integer. We show that for specific values of a , $C_{n+1}(a)/C_n(a)$ is a truncation of a periodic continued fraction.

Revisiting the rational function $J(x, t)$ defined in equation (4.1.6), we note that the values $x = 1, i, 1/\sqrt{2}$ are special in this rational function. Consider for instance

$$J(1/\sqrt{2}, t) = \frac{(1+t)(1-2(1-\sqrt{2})t+t^2)}{1-6t+18t^2-6t^3+t^4}.$$

Its denominator factors over $\mathbb{Q}(\sqrt{7})$ as

$$\left((8 + 3\sqrt{7}) - (3 + \sqrt{7})t + t^2\right) \left((8 - 3\sqrt{7}) - (3 - \sqrt{7})t + t^2\right).$$

Below are some phenomena connected with $\mathbb{Q}(\sqrt{7})$.

Proposition 68. *Expand $J(1/\sqrt{2}, t)$ and replace every $a + b\sqrt{2}$ by the coefficient a . The resulting sequence is generated by*

$$\frac{(1-t)^2(1+t)}{1-6t+18t^2-6t^3+t^4} = \sum_{n=0}^{\infty} C_n t^n.$$

The C_{n+1}/C_n in this case are essentially truncations of the continued fractions of $8 + 3\sqrt{7}$, $2 + \sqrt{7}$, $-(2 + \sqrt{7})/3$ and $(4 + \sqrt{7})/3$.

Proof. We are going to show that for all $n \geq 0$,

$$\begin{aligned} \frac{C_{4n+1}}{C_{4n}} &= [4, 1, \{1, 1, 4, 1, 1, 1, 4, 1\}^{(n)}], \\ \frac{C_{4n+2}}{C_{4n+1}} &= [2, 4, 1, \{1, 1, 4, 1, 1, 1, 4, 1\}^{(n)}], \\ -\frac{C_{4n+3}}{C_{4n+2}} &= [1, 1, 4, 1, \{1, 1, 4, 1, 1, 1, 4, 1\}^{(n)}], \\ \frac{C_{4n+4}}{C_{4n+3}} &= [15, 1, \{14, 1, 14, 1\}^{(n)}, 15, 1], \end{aligned}$$

and these are truncations of $2 + \sqrt{7}$, $\frac{1}{3}(4 + \sqrt{7})$, $\frac{1}{3}(2 + \sqrt{7})$ and $8 + 3\sqrt{7}$ respectively. The notation $\{c_1, c_2, \dots, c_k\}^{(j)}$ means the word $c_1, c_2, \dots, c_k$ has been repeated j times.

The sequence C_{n+1}/C_n starts off as

$$5, \frac{11}{5}, -\frac{17}{11}, \frac{271}{17}, \frac{1259}{271}, \frac{2789}{1259}, -\frac{4319}{2789}, \frac{68833}{4319}, \frac{319781}{68833}, \dots$$

We first divide the sequence C_n into four classes as shown below:

$$\begin{aligned} \frac{1-17t}{1+254t+t^2} &= \sum_{n=0}^{\infty} C_{4n} t^n, & \frac{5+11t}{1+254t+t^2} &= \sum_{n=0}^{\infty} C_{4n+1} t^n, \\ \frac{-17+t}{1+254t+t^2} &= \sum_{n=0}^{\infty} C_{4n+3} t^n, & \frac{11+5t}{1+254t+t^2} &= \sum_{n=0}^{\infty} C_{4n+2} t^n. \end{aligned}$$

Now define the sequence l_n by

$$\frac{1}{1 + 254t + t^2} = \sum_{n=0}^{\infty} l_n t^n. \quad (4.1.9)$$

From (4.1.9), we get the relations

$$l_{n+1} = -254l_n - l_{n-1}, \quad (4.1.10)$$

$$C_{4n} = l_n - 17l_{n-1}, \quad (4.1.11)$$

$$C_{4n+1} = 5l_n + 11l_{n-1}, \quad (4.1.12)$$

$$C_{4n+3} = -17l_n + l_{n-1}. \quad (4.1.13)$$

Now

$$\begin{aligned} [4, 1, \{1, 1, 4, 1, 1, 1, 4, 1\}^{(n+1)}] &= [4, 1, 1, 1, 4, 1, 1, 1, \frac{C_{4n+1}}{C_{4n}}] \\ &= \frac{223C_{4n+1} + 144C_{4n}}{48C_{4n+1} + 31C_{4n}}. \end{aligned}$$

Using (4.1.10), (4.1.11) and (4.1.12), we have

$$\begin{aligned} 223C_{4n+1} + 144C_{4n} &= 223(5l_n + 11l_{n-1}) + 144(l_n - 17l_{n-1}) \\ &= 5 \times 254l_n - 11l_n + 5l_{n-1} \\ &= 5(-l_{n+1} - l_{n-1}) - 11l_n + 5l_{n-1} \\ &= -(5l_{n+1} + 11l_{n+1}) \\ &= -C_{4n+5}, \end{aligned}$$

and

$$\begin{aligned} 48C_{4n+1} + 31C_{4n} &= 271l_n + l_{n-1} \\ &= (-l_{n+1} - l_{n-1}) + 17l_n + l_{n-1} \\ &= -(l_{n+1} - 17l_n) \\ &= -C_{4n+4}. \end{aligned}$$

Hence

$$[4, 1, \{1, 1, 4, 1, 1, 1, 4, 1\}^{(n+1)}] = \frac{C_{4n+5}}{C_{4n+4}}.$$

The proofs for $\frac{C_{4n+2}}{C_{4n+1}}$ and $-\frac{C_{4n+3}}{C_{4n+2}}$ are very similar and will be omitted. Suppose that

$$\frac{C_{4n+4}}{C_{4n+3}} = [15, 1, \{14, 1, 14, 1\}^{(n)}, 15, 1].$$

Then

$$\frac{C_{4n+4} - 15C_{4n+3}}{-C_{4n+4} + 16C_{4n+3}} = [\{14, 1, 14, 1\}^{(n)}, 15, 1],$$

so that

$$\begin{aligned} [15, 1, \{14, 1, 14, 1\}^{(n+1)}, 15, 1] &= [15, 1, 14, 1, 14, 1, \frac{C_{4n+4} - 15C_{4n+3}}{-C_{4n+4} + 16C_{4n+3}}] \\ &= \frac{255C_{4n+4} - 16C_{4n+3}}{16C_{4n+4} - C_{4n+3}}. \end{aligned}$$

Using (4.1.10), (4.1.11) and (4.1.13), we have

$$\begin{aligned} 255C_{4n+4} - 16C_{4n+3} &= 255l_{n+1} - 4063l_n - 16l_{n-1} \\ &= (-l_{n+2} - l_n) + l_{n+1} + 16(l_{n+1} + l_{n-1}) + l_n - 16l_{n-1} \\ &= -l_{n+2} + 17l_{n+1} \\ &= -C_{4n+8}. \end{aligned}$$

Similarly,

$$\begin{aligned} 16C_{4n+4} - C_{4n+3} &= 16l_{n+1} - 255l_n - l_{n-1} \\ &= 16l_{n+1} + (l_{n+1} + l_{n-1}) - l_n - l_{n-1} \\ &= 17l_{n+1} - l_n \\ &= -C_{4n+7}. \end{aligned}$$

This implies that

$$[15, 1, \{14, 1, 14, 1\}^{(n+1)}, 15, 1] = \frac{C_{4n+8}}{C_{4n+7}}.$$

The proof follows by induction on n . □

Proposition 69. *Expand $J(1/\sqrt{2}, t)$ and replace every $a + b\sqrt{2}$ by the coef-*

ficient b . The resulting sequence is generated by

$$\frac{2t(1+t)}{1-6t+18t^2-6t^3+t^4} = \sum_{n=0}^{\infty} B_n t^n.$$

The B_{n+1}/B_n in this case are essentially truncations of the continued fractions of $5+2\sqrt{7}$, $(11+4\sqrt{7})/3$, $-(5+2\sqrt{7})/3$ and also, 1 is a ratio infinitely often.

Proof. We are going to show that for all $n \geq 1$, $\frac{B_{4n+4}}{B_{4n+3}} = 1$ and

$$\begin{aligned} \frac{B_{4n+3}}{B_{4n+2}} &= [3, 2, 3, \{10, 3, 2, 3\}^{(n)}], \\ \frac{B_{4n+2}}{B_{4n+1}} &= [7, \{5, 6, 1, 6\}^{(n)}], \\ -\frac{B_{4n+1}}{B_{4n}} &= [\{10, 3, 2, 3\}^{(n)}], \end{aligned}$$

and these are truncations of $\frac{1}{3}(5+2\sqrt{7})$, $\frac{1}{3}(11+4\sqrt{7})$ and $5+2\sqrt{7}$ respectively. The sequence B_{n+1}/B_n starts off as

$$7, \frac{24}{7}, 1, -\frac{247}{24}, \frac{1777}{247}, \frac{6096}{1777}, 1, -\frac{62737}{6096}, \frac{451351}{62737}, \dots$$

We divide the sequence B_n into four classes as shown below

$$\begin{aligned} \frac{48t}{1+254t+t^2} &= \sum_{n=0}^{\infty} B_{4n} t^n, & \frac{2+14t}{1+254t+t^2} &= \sum_{n=0}^{\infty} B_{4n+1} t^n, \\ \frac{48}{1+254t+t^2} &= \sum_{n=0}^{\infty} B_{4n+3} t^n, & \frac{14+2t}{1+254t+t^2} &= \sum_{n=0}^{\infty} B_{4n+2} t^n. \end{aligned}$$

and from (4.1.9), we get the relations

$$B_{4n+4} = B_{4n+3} = 48l_n, \tag{4.1.14}$$

$$B_{4n+1} = 2l_n + 14l_{n-1}, \tag{4.1.15}$$

$$B_{4n+2} = 14l_n + 2l_{n-1}. \tag{4.1.16}$$

Clearly, $\frac{B_{4n+4}}{B_{4n+3}} = 1$ follows from (4.1.14). Now suppose

$$\frac{B_{4n+3}}{B_{4n+2}} = [3, 2, 3, \{10, 3, 2, 3\}^{(n)}],$$

we find that

$$\begin{aligned} [3, 2, 3, \{10, 3, 2, 3\}^{(n+1)}] &= [3, 2, 3, 10, \frac{B_{4n+3}}{B_{4n+2}}] \\ &= \frac{247B_{4n+3} + 24B_{4n+2}}{72B_{4n+3} + 7B_{4n+2}}. \end{aligned}$$

Using (4.1.10), (4.1.14) and (4.1.16), we have

$$\begin{aligned} 247B_{4n+3} + 24B_{4n+2} &= 48(254l_n + l_{n-1}) \\ &= -48l_{n+1} \\ &= -B_{4n+7}, \end{aligned}$$

and

$$\begin{aligned} 72B_{4n+3} + 7B_{4n+2} &= 3554l_n + 14l_{n-1} \\ &= 14(-l_{n+1} - l_{n-1}) - 2l_n + 14l_{n-1} \\ &= -(14l_{n+1} + 2l_n) \\ &= -B_{4n+6}, \end{aligned}$$

which implies that

$$[3, 2, 3, \{10, 3, 2, 3\}^{(n+1)}] = \frac{B_{4n+7}}{B_{4n+6}}.$$

Next suppose that

$$-\frac{B_{4n+1}}{B_{4n}} = [\{10, 3, 2, 3\}^{(n)}].$$

Then

$$\begin{aligned} [\{10, 3, 2, 3\}^{(n+1)}] &= [10, 3, 2, 3, -\frac{B_{4n+1}}{B_{4n}}] \\ &= \frac{-247B_{4n+1} + 72B_{4n}}{-24B_{4n+1} + 7B_{4n}}. \end{aligned}$$

Using (4.1.10), (4.1.14) and (4.1.15), we have

$$\begin{aligned}
-247B_{4n+1} + 72B_{4n} &= -494l_n - 2l_{n-1} \\
&= 2(l_{n+1} + l_{n-1}) + 14l_n - 2l_{n-1} \\
&= 2l_{n+1} + 14l_n \\
&= B_{4n+5},
\end{aligned}$$

and

$$\begin{aligned}
-24B_{4n+1} + 7B_{4n} &= -48l_n \\
&= -B_{4n+4}.
\end{aligned}$$

Hence

$$[\{10, 3, 2, 3\}^{(n+1)}] = -\frac{B_{4n+5}}{B_{4n+4}}.$$

Finally, suppose that

$$\frac{B_{4n+2}}{B_{4n+1}} = [7, \{5, 6, 1, 6\}^{(n)}].$$

Then

$$\frac{B_{4n+1}}{B_{4n+2} - 7B_{4n+1}} = [\{5, 6, 1, 6\}^{(n)}].$$

We now find that

$$\begin{aligned}
[7, \{5, 6, 1, 6\}^{(n+1)}] &= [7, 5, 6, 1, 6, \frac{B_{4n+1}}{B_{4n+2} - 7B_{4n+1}}] \\
&= \frac{259B_{4n+2} - 36B_{4n+1}}{36B_{4n+2} - 5B_{4n+1}}.
\end{aligned}$$

Using (4.1.10), (4.1.15) and (4.1.16), we have

$$\begin{aligned}
259B_{4n+2} - 36B_{4n+1} &= 3554l_n + 14l_n \\
&= 14(-l_{n+1} - l_{n-1}) - 2l_n + 14l_{n-1} \\
&= -14l_{n+1} - 2l_n \\
&= -B_{4n+6},
\end{aligned}$$

and

$$\begin{aligned}
36B_{4n+2} - 5B_{4n+1} &= 494l_n + 2l_{n-1} \\
&= 2(-l_{n+1} - l_{n-1}) - 14l_n + 2l_{n-1} \\
&= -2l_{n+1} - 14l_n \\
&= -B_{4n+5},
\end{aligned}$$

which implies that

$$[7, \{5, 6, 1, 6\}^{(n+1)}] = \frac{B_{4n+6}}{B_{4n+5}}.$$

The proof follows by induction on n . □

Now consider the rational function

$$\frac{(1-t^2)^2}{1-6t+at^2-6t^3+t^4} = \sum_{n=0}^{\infty} C_n(a)t^n. \quad (4.1.17)$$

For what values of a do we get integer sequences $\{C_n(a)\}$ such that the ratios of consecutive elements of the sequence is a truncation of a periodic continued fraction?

First, $C_{2n}(a)$ and $C_{2n+1}(a)$ are respectively generated by the rational functions

$$\frac{(1-t)^2(1+at+t^2)}{1+2(a-18)t+(a^2-70)t^2+2(a-18)t^3+t^4}$$

and

$$\frac{6(1-t)^2(1+t)}{1+2(a-18)t+(a^2-70)t^2+2(a-18)t^3+t^4}.$$

Taking the resultant of the numerator and the denominator of the above rational functions with respect to t we obtain

$$2^4 3^4 (a-2)^2 (a-10)^2 (a+14)^2,$$

which implies that the numbers 2, 10, -14, 18 and $\sqrt{70}$ are of interest. For $a = 10$, we have

$$\begin{aligned}
\frac{1+10t+t^2}{1-14t+t^2} &= \sum_{n=0}^{\infty} C_{2n}(10)t^n, \\
\frac{6(1+t)}{1-14t+t^2} &= \sum_{n=0}^{\infty} C_{2n+1}(10)t^n.
\end{aligned}$$

Proposition 70. *For all $n \geq 0$, the consecutive ratios $C_{n+1}(10)/C_n(10)$ are truncations of periodic continued fractions. Precisely,*

$$\begin{aligned}\frac{C_{4n+2}(10)}{C_{4n+1}(10)} &= [3, 1, \{2, 1, 2, 1, 2, 1, 2, 1\}^{(n)}], \\ \frac{C_{4n+3}(10)}{C_{4n+2}(10)} &= [3, 1, 2, 1, \{2, 1, 2, 1, 2, 1, 2, 1\}^{(n)}], \\ \frac{C_{4n+4}(10)}{C_{4n+3}(10)} &= [3, 1, 2, 1, 2, 1, \{2, 1, 2, 1, 2, 1, 2, 1\}^{(n)}], \\ \frac{C_{4n+5}(10)}{C_{4n+4}(10)} &= [3, 1, 2, 1, 2, 1, 2, 1, \{2, 1, 2, 1, 2, 1, 2, 1\}^{(n)}].\end{aligned}$$

These continued fractions are truncations of $2 + \sqrt{3}$.

For $a = -14$ we have

$$\begin{aligned}\frac{1 - 14t + t^2}{1 - 62t + t^2} &= \sum_{n=0}^{\infty} C_{2n}(-14)t^n, \\ \frac{6(1+t)}{1 - 62t + t^2} &= \sum_{n=0}^{\infty} C_{2n+1}(-14)t^n.\end{aligned}$$

Proposition 71. *For all $n \geq 0$, the consecutive ratios $C_{n+1}(-14)/C_n(-14)$ are truncations of periodic continued fractions. Precisely,*

$$\begin{aligned}\frac{C_{4n+2}(-14)}{C_{4n+1}(-14)} &= [7, 1, \{6, 1, 6, 1, 6, 1, 6, 1\}^{(n)}], \\ \frac{C_{4n+3}(-14)}{C_{4n+2}(-14)} &= [7, 1, 6, 1, \{6, 1, 6, 1, 6, 1, 6, 1\}^{(n)}], \\ \frac{C_{4n+4}(-14)}{C_{4n+3}(-14)} &= [7, 1, 6, 1, 6, 1, \{6, 1, 6, 1, 6, 1, 6, 1\}^{(n)}], \\ \frac{C_{4n+5}(-14)}{C_{4n+4}(-14)} &= [7, 1, 6, 1, 6, 1, 6, 1, \{6, 1, 6, 1, 6, 1, 6, 1\}^{(n)}].\end{aligned}$$

These continued fractions are truncations of $4 + \sqrt{15}$.

For $a = 18$ we have

$$\begin{aligned}\frac{(1-t)^2(1+18t+t^2)}{1+254t^2+t^4} &= \sum_{n=0}^{\infty} C_{2n}(18)t^n, \\ \frac{6(1-t)^2(1+t)}{1+254t^2+t^4} &= \sum_{n=0}^{\infty} C_{2n+1}(18)t^n.\end{aligned}$$

Proposition 72. *For all $n \geq 0$, the consecutive ratios $C_{n+1}(18)/C_n(18)$ are truncations of periodic continued fractions. Precisely,*

$$\begin{aligned}\frac{C_{4n+2}(18)}{C_{4n+1}(18)} &= [2, 1, 1, 1, \{4, 1, 1, 1, 4, 1, 1\}^{(n)}], \\ -\frac{C_{4n+3}(18)}{C_{4n+2}(18)} &= [0, 2, 1, 1, 1, \{4, 1, 1, 1, 4, 1, 1\}^{(n)}].\end{aligned}$$

These continued fractions are truncations of $\sqrt{7}$ and $1/\sqrt{7}$ respectively.

The continued fractions of $C_{4n+1}(18)/C_{4n}(18)$ and $C_{4n+4}(18)/C_{4n+3}(18)$ do not have a simple description but are truncations of $[\overline{6, 7}]$ and $[47, \overline{1, 4, 3, 4, 1, 46}]$ respectively.

The proofs of these propositions are similar to the proofs of propositions 68 and 69 and will thus be omitted. Another interesting observation is that the ratios $C_{n+1}(a)/C_{n-1}(a)$ and $C_{n+2}(a)/C_{n-1}(a)$, where $a = 10, -14, 18$, also seem to be truncations of periodic continued fractions. As an example, for $n \geq 1$,

$$\begin{aligned}\frac{C_{4n+1}(10)}{C_{4n-1}(10)} &= [13, 1, \{12, 1\}^{(2n-2)}, 13, 1], \\ \frac{C_{4n+2}(10)}{C_{4n}(10)} &= [13, 1, \{12, 1\}^{(2n-1)}], \\ \frac{C_{4n+3}(10)}{C_{4n+1}(10)} &= [13, 1, \{12, 1\}^{(2n-1)}, 13, 1], \\ \frac{C_{4n+4}(10)}{C_{4n+2}(10)} &= [13, 1, \{12, 1\}^{(2n)}].\end{aligned}$$

Similarly

$$\begin{aligned}\frac{C_{4n+1}(-14)}{C_{4n-1}(-14)} &= [61, 1, \{60, 1\}^{(2n-2)}, 61, 1], \\ \frac{C_{4n+2}(-14)}{C_{4n}(-14)} &= [61, 1, \{60, 1\}^{(2n-1)}], \\ \frac{C_{4n+3}(-14)}{C_{4n+1}(-14)} &= [61, 1, \{60, 1\}^{(2n-1)}, 61, 1], \\ \frac{C_{4n+4}(-14)}{C_{4n+2}(-14)} &= [61, 1, \{60, 1\}^{(2n)}].\end{aligned}$$

Continued fractions of this type will be studied in chapter 5.

The continued fraction of $c_{n+1}(a)/c_n(a)$ when $a = 2$ does not have a pattern

that can easily be described, but there is periodicity in the continued fraction. For $a = \sqrt{70}$, $c_{n+1}(a)/c_n(a)$ is a quadratic irrational with an infinite periodic continued fraction.

How about any other value of a ? Numerical calculations show that if we use any other value of a that is not 2, 10, -14, 18 or $\sqrt{70}$, the continued fraction of $c_{n+1}(a)/c_n(a)$ does not seem to have any periodicity and hence cannot be a truncation of a periodic continued fraction. We attempted the case for $a = 0$ and $a = 1$ and for the first 2700 partial quotients in the continued fractions, periodicity was not detected. This leads to the conclusion that either the period is too long (or much farther out in the continued fraction), or they are truncations of non periodic continued fractions.

Another rational function of interest is

$$\frac{1}{1 - 6t + at^2 - 6t^3 + t^4} = \sum_{n=0}^{\infty} B_n(a)t^n. \quad (4.1.18)$$

It can easily be shown that $B_{2n}(a)$ and $B_{2n+1}(a)$ are respectively generated by the rational functions

$$\frac{1 + at + t^2}{1 + 2(a - 18)t + (a^2 - 70)t^2 + 2(a - 18)t^3 + t^4}$$

and

$$\frac{6(1 + t)}{1 + 2(a - 18)t + (a^2 - 70)t^2 + 2(a - 18)t^3 + t^4}.$$

Taking the resultant of the numerator and the denominator of the above rational functions with respect to t we find

$$2^4 3^4 (a - 2)^2,$$

which implies that the numbers 2, 18 and $\sqrt{70}$ are of interest. The case where $a = 18$ will be studied in detail in the next section. When $a = 2$,

$$\begin{aligned} \frac{1}{1 - 34t + t^2} &= \sum_{n=0}^{\infty} B_{2n}(2)t^n, \\ \frac{6}{(1 + t)(1 - 34t + t^2)} &= \sum_{n=0}^{\infty} B_{2n+1}(2)t^n. \end{aligned}$$

It can be shown (just like in the previous cases) that

$$\begin{aligned}\frac{B_{4n+1}(2)}{B_{4n}(2)} &= [5, 1, \{4, 1, 4, 1\}^{(n)}], \\ \frac{B_{4n+2}(2)}{B_{4n+1}(2)} &= [5, 1, \{4, 1, 4, 1\}^{(n)}, 2], \\ \frac{B_{4n+3}(2)}{B_{4n+2}(2)} &= [5, 1, 4, 1, \{4, 1, 4, 1\}^{(n)}, 2], \\ \frac{B_{4n+4}(2)}{B_{4n+3}(2)} &= [5, 1, 4, 1, \{4, 1, 4, 1\}^{(n)}],\end{aligned}$$

for all $n \geq 0$.

4.1.3 Polynomials from convergents of a generalized periodic continued fraction

Let

$$Q_1(x, t) = 1 - 4(1 + x^2)t + 6(1 + 4x^2)t^2 - 4(1 + x^2)t^3 + t^4,$$

and

$$Q_2(x, t) = 1 - 2xt + 2x^2t^2 - 2xt^3 + t^4.$$

Recall that $Q_1(t, x)$ is the denominator to the rational functions $J(x, t)$ and $K(x, t)$ in (4.1.6) and (4.1.7) respectively, and connects with the ellipse $E(1/\sqrt{2}, 1)$.

Observe that

$$\frac{(2p)^2}{2} - 2p^2 = 0$$

and that

$$\frac{4(1 + x^2)^2}{2} - 6(1 + 4x^2) = 2(-1 + 2x^2)^2.$$

Thus

$$Q_1(1/\sqrt{2}, t) = Q_2(3, t).$$

The polynomial $Q_2(x, t)$ has a very interesting denominator expansion. Write

$$\frac{1}{Q_2(x, t)} = \sum_{n=0}^{\infty} c_n(x) t^n. \quad (4.1.19)$$

Here, the $c_n(x)$ are remarkably composite. Below is a table of the first 10 polynomials in this sequence.

n	$c_n(x)$
0	1
1	$2x$
2	$2x^2$
3	$2x$
4	$-(2x^2 - 2x - 1)(2x^2 + 2x - 1)$
5	$-4x(2x^4 - 4x^2 + 1)$
6	$-4x^2(2x^4 - 4x^2 + 1)$
7	$-4x(2x^4 - 4x^2 + 1)$
8	$(4x^4 - 4x^3 - 6x^2 + 6x - 1)(4x^4 + 4x^3 - 6x^2 - 6x - 1)$
9	$2x(2x^2 - 3)(2x^2 - 1)(2x^2 - 2x - 1)(2x^2 + 2x - 1)$
10	$2x^2(2x^2 - 3)(2x^2 - 1)(2x^2 - 2x - 1)(2x^2 + 2x - 1)$

Table 4.3: Table of the first seven polynomials $c_n(x)$.

Remark 5. $Q_2(x, t)$ factors over $\mathbb{Q}(\sqrt{2 - x^2})$ as

$$Q_2(x, t) = \left(t^2 - t \left(x + \sqrt{2 - x^2} \right) + 1 \right) \left(t^2 - t \left(x - \sqrt{2 - x^2} \right) + 1 \right).$$

Proposition 73. The ratios $c_{n+1}(x)/c_n(x)$ are truncations of the periodic continued fraction

$$E(2, x) := \overline{[2x, 2x(x^2 - 2)]},$$

which converges to

$$\alpha := x + \frac{x^2 - 1}{\sqrt{x^2 - 2}}.$$

Here,

$$\alpha \bar{\alpha} = \frac{1}{2 - x^2}.$$

Proof. We will show that for all $n \geq 0$,

$$\frac{c_{4n+1}(x)}{c_{4n}(x)} = [2x, \{2x(x^2 - 2), 2x\}^{(n)}]$$

and

$$-\frac{c_{4n}(x)}{c_{4n-1}(x)} = [\{2x(x^2 - 2), 2x\}^{(n)}],$$

where the notation $\{c_1, c_2, \dots, c_k\}^{(j)}$ means the word $c_1, c_2, \dots, c_k$ has been repeated j times. First we break down the $c_n(x)$ polynomials into four classes. Recall that $c_n(x)$ are generated by the rational function in (4.1.19). It is easy

to show that

$$\begin{aligned}\frac{2x}{1 + \xi(2, x)t + t^2} &= \sum_{n=0}^{\infty} c_{4n+1}(x)t^n, \\ \frac{1+t}{1 + \xi(2, x)t + t^2} &= \sum_{n=0}^{\infty} c_{4n}(x)t^n,\end{aligned}\tag{4.1.20}$$

where

$$\xi(s, x) := 2 - s^3x^2 + s^2x^4\tag{4.1.21}$$

is a quartic that plays a key role in proving the root distribution of $c_n(s, x)$ and other related polynomials that we will encounter later. Moreover,

$$xc_{4n+1}(x) = xc_{4n+3}(x) = c_{4n+2}(x)$$

which implies that for all n ,

$$\frac{c_{4n+3}(x)}{c_{4n+2}(x)} = x \quad \text{and} \quad \frac{c_{4n+2}(x)}{c_{4n+1}(x)} = \frac{1}{x}.$$

Let

$$\frac{1}{1 + \xi(2, x)t + t^2} = \sum_{n=0}^{\infty} b_n(x)t^n.\tag{4.1.22}$$

Then from (5.5.2) we obtain the recurrence relation

$$b_{n+1}(x) = -\xi(2, x)b_n(x) - b_{n-1}(x)\tag{4.1.23}$$

together with the relations

$$xc_{4n+1}(x) = xc_{4n+3}(x) = c_{4n+2}(x) = 2x^2b_n(x),\tag{4.1.24}$$

$$c_{4n}(x) = b_n(x) + b_{n-1}(x).\tag{4.1.25}$$

From the initial list, $c_1(x)/c_0(x) = 2x$ and $-c_4(x)/c_3(x) = 2x(x^2-2)+1/(2x)$.

Suppose $\frac{c_{4n+1}(x)}{c_{4n}(x)} = [2x, \{2x(x^2 - 2), 2x\}^{(n)}]$, then

$$\begin{aligned}
[2x, \{2x(x^2 - 2), 2x\}^{(n+1)}] &= [2x, 2x(x^2 - 2), 2x, \{2x(x^2 - 2), 2x\}^{(n)}] \\
&= [2x, 2x(x^2 - 2), \frac{2xb_n(x)}{b_n(x) + b_{n-1}(x)}] \\
&= \frac{2x(\xi(2, x)b_n(x) + b_{n-1}(x))}{\xi(2, x)b_n(x) - b_n(x) + b_{n-1}(x)} \\
&= \frac{-2xb_{n+1}(x)}{-b_{n+1}(x) - b_n(x)} \\
&= \frac{c_{4n+5}(x)}{c_{4n+4}(x)},
\end{aligned}$$

where we have used (4.1.23), (4.1.24) and (4.1.25).

Now suppose $-\frac{c_{4n}(x)}{c_{4n-1}(x)} = [\{2x(x^2 - 2), 2x\}^{(n)}]$. Using the same relations as above, we deduce that

$$\begin{aligned}
[\{2x(x^2 - 2), 2x\}^{(n+1)}] &= [2x(x^2 - 2), 2x, \{2x(x^2 - 2), 2x\}^{(n)}] \\
&= [2x(x^2 - 2), 2x, -\frac{b_n(x) + b_{n-1}(x)}{2xb_{n-1}(x)}] \\
&= \frac{\xi(2, x)b_n(x) - b_n(x) + b_{n-1}(x)}{2xb_n(x)} \\
&= \frac{-b_{n+1}(x) - b_n(x)}{2xb_n(x)} \\
&= -\frac{c_{4n+4}(x)}{c_{4n+3}(x)}.
\end{aligned}$$

□

Corollary 74. *The continued fraction of $Y_{4n+1}(1/\sqrt{2})/Y_{4n}(1/\sqrt{2})$ and of $-Y_{4n}(1/\sqrt{2})/Y_{4n-1}(1/\sqrt{2})$ are respectively truncations of the continued fractions of $3 + \frac{8}{7}\sqrt{7}$ and $21 + 8\sqrt{7}$.*

Proof. Recall that $Q_1(1/\sqrt{2}, t) = Q_2(3, t)$ and hence $c_n(3) = Y_n(1/\sqrt{2})$ for all n . The proof follows by setting $x = 3$ in Proposition 73. □

Proposition 75. *Let $c_n(x)$ be as defined in (4.1.19). Then the ratios $c_{n+1}(x)/c_{n-1}(x)$ are identically 1 or are truncations of periodic continued fractions.*

Proof. Following from the proof of Proposition 73, it is clear that for all n ,

$c_{4n+3}(x)/c_{4n+1}(x) = 1$. We are going to show that for all $n \geq 1$,

$$\begin{aligned} -\frac{c_{4n+1}(x)}{c_{4n-1}(x)} &= [\xi(2, x) - 1, 1, \{\xi(2, x) - 2, 1\}^{(n-1)}], \\ -\frac{c_{4n}(x)}{c_{4n-2}(x)} &= [\{2(x^2 - 2), 2x^2\}^{(n)}], \end{aligned}$$

and for $n \geq 0$,

$$\frac{c_{4n+2}(x)}{c_{4n}(x)} = [2x^2, \{2(x^2 - 2), 2x^2\}^{(n)}].$$

From the initial sequence, $\frac{c_2(x)}{c_0(x)} = 2x$, $\frac{c_3(x)}{c_1(x)} = 1$, $-\frac{c_4(x)}{c_2(x)} = 2(x^2 - 2) + \frac{1}{2x^2}$ and $-\frac{c_5(x)}{c_3(x)} = \xi(2, x)$.

Suppose $-\frac{c_{4n+1}(x)}{c_{4n-1}(x)} = [\xi(2, x) - 1, 1, \{\xi(2, x) - 2, 1\}^{(n-1)}]$. Then

$$\begin{aligned} &[\xi(2, x) - 1, 1, \{\xi(2, x) - 2, 1\}^{(n)}] \\ &= [\xi(2, x) - 1, 1, \xi(2, x) - 2, 1, \{\xi(2, x) - 2, 1\}^{(n-1)}] \\ &= [\xi(2, x) - 1, 1, \xi(2, x) - 2, \frac{-b_n(x)}{b_n(x) + (\xi(2, x) - 1)b_{n-1}(x)}] \\ &= \frac{\xi(2, x)b_n(x) + b_{n-1}(x)}{b_n(x)} \\ &= \frac{-b_{n+1}(x)}{b_n(x)} \\ &= -\frac{c_{4n+5}(x)}{c_{4n+3}(x)} \end{aligned}$$

as desired. To simplify the calculations, we used (4.1.23), (4.1.24) and (4.1.25). Using the same relations, suppose $-\frac{c_{4n}(x)}{c_{4n-2}(x)} = [\{2(x^2 - 2), 2x^2\}^{(n)}]$.

Then

$$\begin{aligned}
[\{2(x^2 - 2), 2x^2\}^{(n+1)}] &= [2(x^2 - 2), 2x^2, \{2(x^2 - 2), 2x^2\}^{(n)}] \\
&= [2(x^2 - 2), 2x^2, -\frac{b_n(x) + b_{n-1}(x)}{2x^2 b_{n-1}(x)}] \\
&= \frac{(\xi(2, x) - 1)b_n(x) + b_{n-1}(x)}{2x^2 b_n(x)} \\
&= \frac{-b_{n+1}(x) - b_n(x)}{2x^2 b_n(x)} \\
&= -\frac{c_{4n+4}(x)}{c_{4n+2}(x)}.
\end{aligned}$$

Finally suppose $\frac{c_{4n+2}(x)}{c_{4n}(x)} = [2x^2, \{2(x^2 - 2), 2x^2\}^{(n)}]$. Then

$$\begin{aligned}
[2x^2, \{2(x^2 - 2), 2x^2\}^{(n+1)}] &= [2x^2, 2(x^2 - 2), 2x^2, \{2(x^2 - 2), 2x^2\}^{(n)}] \\
&= [2x^2, 2(x^2 - 2), \frac{2x^2 b_n(x)}{b_n(x) + b_{n-1}(x)}] \\
&= \frac{2x^2(\xi(2, x)b_n(x) + b_{n-1}(x))}{(\xi(2, x) - 1)b_n(x) + b_{n-1}(x)} \\
&= \frac{-2x^2 b_{n+1}(x)}{-b_{n+1}(x) - b_n(x)} \\
&= \frac{c_{4n+6}(x)}{c_{4n+4}(x)}.
\end{aligned}$$

The proof for all the cases easily follows by induction on n . □

If $E(2, x)$ is truncated, we seem to have numerator and denominator polynomials

$$E_m(2, x) = \frac{w_m(x)}{v_m(x)},$$

with real roots lying in the interval $[-\sqrt{2}, \sqrt{2}]$, and concentrated towards the endpoints of the interval and at $x = 0$.

For example,

$$\begin{aligned}
E_4(2, x) &= [\{2x, 2x(x^2 - 2)\}^{(4)}] \\
&= \frac{(1 - 8x^2 + 4x^4)(1 - 72x^2 + 420x^4 - 896x^6 + 864x^8 - 384x^{10} + 64x^{12})}{8x(-2 + x^2)(1 - 4x^2 + 2x^4)(1 - 16x^2 + 40x^4 - 32x^6 + 8x^8)}
\end{aligned}$$

The numerator of $E_4(x)$ factors further over $\mathbb{Q}(x)$. As shown in this case,

$w_4(x)$ and $v_4(x)$ are very composite. What about

$$E(3, x) := [\overline{3x, 3x(x^2 - 3)}]?$$

Here it seems that all roots are real and in $[-\sqrt{3}, \sqrt{3}]$. The roots seem to be symmetric about $x = 0$ and are concentrated at the ends of the interval and at $x = 0$. This begs the question; What is the behavior in general, of the numerator and denominator polynomials of the convergents of

$$E(s, x) := [\overline{sx, sx(x^2 - s)}]?$$

Theorem 76. *For $s \geq 2$, all the roots of the numerators and denominators of the convergents of $[\overline{sx, sx(x^2 - s)}]$ are real and lie in the interval $[-\sqrt{s}, \sqrt{s}]$. The roots are symmetric about $x = 0$ and are concentrated at the ends of the interval and at $x = 0$. Moreover for large s , the roots approach $\pm\sqrt{s}$ and 0. For $s < 2$, we have real roots are in $[-\sqrt{s}, \sqrt{s}]$ and its non-real roots lie on the hyperbola*

$$H\left(\sqrt{s/2}, \sqrt{s/2}\right) : 2x^2 - 2y^2 = s. \quad (4.1.26)$$

As s further decreases to 0, all the roots become arbitrarily close to the asymptotes of the hyperbola.

First we describe the polynomials that appear in the numerators and denominators of the convergents of $E(s, x)$. It can easily be shown that $E(s, x)$ converges to

$$\alpha(s, x) := \frac{1}{2}sx + \frac{\sqrt{s^2x^2(x^2 - s) + 4}}{2\sqrt{x^2 - s}},$$

the largest root of the quadratic

$$(x^2 - s)y^2 - sx(x^2 - s)y - 1 = 0.$$

The convergents $p_n(s, x)/q_n(s, x)$ of $[\{sx, sx(x^2 - s)\}]$ for $0 \leq n \leq 3$ are given by

$$sx, \frac{1 - s^3x^2 + s^2x^4}{sx(x^2 - s)}, \frac{sx(2 - s^3x^2 + s^2x^4)}{1 - s^3x^2 + s^2x^4}, \frac{1 - 3s^3x^2 + 3s^2x^4 + s^6x^4 - 2s^5x^6 + s^4x^8}{sx(x^2 - s)(2 - s^3x^2 + s^2x^4)}.$$

Lemma 77. For all $n \geq 0$,

$$\begin{aligned} q_{2n}(s, x) &= p_{2n-1}(s, x), \\ q_{2n+1}(s, x) &= (x^2 - s)p_{2n}(s, x). \end{aligned}$$

Proof. We can see from the convergents above that $q_2(s, x) = p_1(s, x)$ and $q_3(s, x) = (x^2 - s)p_2(s, x)$. We are going to use the well known relations for convergents of continued fractions given by

$$\begin{aligned} p_n(s, x) &= a_n(s, x)p_{n-1}(s, x) + p_{n-2}(s, x), \\ q_n(s, x) &= a_n(s, x)q_{n-1}(s, x) + q_{n-2}(s, x), \end{aligned} \tag{4.1.27}$$

where the $a_n(s, x)$'s are partial quotients of the continued fraction. Note that for our continued fraction $E(s, x)$, we have $a_{2n} = sx$ and $a_{2n+1} = sx(x^2 - s)$ for all $n \geq 0$.

Assume that the statements of the lemma are true for all $j = 0, 1, 2, \dots, 2n$. Then

$$\begin{aligned} q_{2n+1}(s, x) &= a_{2n+1}(s, x)q_{2n}(s, x) + q_{2n-1}(s, x) \\ &= sx(x^2 - s)p_{2n-1}(s, x) + (x^2 - s)p_{2n-2}(s, x) \\ &= (x^2 - s)(sx p_{2n-1}(s, x) + p_{2n-2}(s, x)) \\ &= (x^2 - s)p_{2n}(s, x). \end{aligned}$$

Similarly,

$$\begin{aligned} q_{2n+2}(s, x) &= a_{2n+2}(s, x)q_{2n+1}(s, x) + q_{2n}(s, x) \\ &= sx(x^2 - s)p_{2n}(s, x) + p_{2n-1}(s, x) \\ &= a_{2n+1}(s, x)p_{2n}(s, x) + p_{2n-1}(s, x) \\ &= p_{2n+1}(s, x). \end{aligned}$$

Hence by induction, the statements of the Lemma are true for all n . $\square$

It is now enough to study only the numerators of the convergents of $E(s, x)$. To simplify the notation, write

$$p_{2n}(s, x) := sxw_n(s, x) \quad \text{and} \quad p_{2n+1}(s, x) := v_n(s, x).$$

n	$w_n(s, x)$	$v_n(s, x)$
0	1	1
1	$\xi(s, x)$	$1 + \xi(s, x)$
2	$-1 + \xi(s, x)^2$	$-1 + \xi(s, x) + \xi(s, x)^2$
3	$-2\xi(s, x) + \xi(s, x)^3$	$-1 - 2\xi(s, x) + \xi(s, x)^2 + \xi(s, x)^3$
4	$1 - 3\xi(s, x)^2 + \xi(s, x)^4$	$1 - 2\xi(s, x) - 3\xi(s, x)^2 + \xi(s, x)^3 + \xi(s, x)^4$

Table 4.4: Table of the first five polynomials $w_n(s, x)$ and $v_n(s, x)$ where $\xi(s, x) = 2 - s^3x^2 + s^2x^4$.

The first five polynomials in each sequence are given in terms of the $\xi(s, x)$ of (4.1.21) by the tables below.

From (4.1.27),

$$p_{n+2}(s, x) = \xi(s, x)p_n(s, x) - p_{n-2}(s, x),$$

from which we obtain the recurrence relations

$$\begin{aligned} w_n(s, x) &= \xi(s, x)w_{n-1}(s, x) - w_{n-2}(s, x), \\ v_n(s, x) &= \xi(s, x)v_{n-1}(s, x) - v_{n-2}(s, x). \end{aligned}$$

From the initial conditions and the recurrence relations above, it is easy to show that $w_n(s, x)$ and $v_n(s, x)$ are generated by the rational functions

$$\frac{1}{1 - \xi(s, x)t + t^2} = \sum_{n=0}^{\infty} w_n(s, x)t^n, \quad (4.1.28)$$

$$\frac{1 - t}{1 - \xi(s, x)t + t^2} = \sum_{n=0}^{\infty} v_n(s, x)t^n. \quad (4.1.29)$$

It is immediate from the generating functions that

$$v_n(s, x) = w_n(s, x) - w_{n-1}(s, x), \quad (4.1.30)$$

$$w_n(s, x) = U_n\left(\frac{1}{2}\xi(s, x)\right). \quad (4.1.31)$$

For each root $\cos\left(\frac{k}{n+1}\pi\right)$ of $U_n(x)$, $k = 1, 2, \dots, n$, there are four correspond-

ing roots of $w_n(s, x)$ which are the roots of the quartic

$$2 - s^3 x^2 + s^2 x^4 - 2r_{n,k} = 0, \quad (4.1.32)$$

and are given by

$$x = \pm \frac{1}{\sqrt{2s}} \sqrt{s^2 \pm \sqrt{s^4 - 8 \left(1 - \cos\left(\frac{k}{n+1}\pi\right)\right)}}. \quad (4.1.33)$$

Since the roots of $U_n(x)$ are symmetric about $x = 0$, the roots of $w_n(s, x)$ are symmetric about $x = 0$ as well. Hence from (4.1.33) and $\left|\cos\left(\frac{k}{n+1}\pi\right)\right| < 1$, it is clear that whenever $s \geq 2$, all the roots of $w_n(s, x)$ are real and lie in the interval $[-\sqrt{s}, \sqrt{s}]$. Write (4.1.32) as

$$x^2(s - x^2) = \frac{2}{s^2} \left(1 - \cos\left(\frac{k}{n+1}\pi\right)\right).$$

This shows that for large s , the roots of $w_n(s, x)$ are concentrated at $x = 0$ and near the end points of the interval $[-\sqrt{s}, \sqrt{s}]$. It also shows that as s increases, the roots of $w_n(s, x)$ approach $x = 0$ and $x = \pm\sqrt{s}$.

What about the case $s < 2$? Let

$$\gamma = \left\lfloor \frac{n+1}{\pi} \cos^{-1} \left(\frac{s^4}{8} - 1 \right) \right\rfloor. \quad (4.1.34)$$

The non-real roots of $w_n(s, x)$ are given by (4.1.33) for $k = 1, 2, \dots, \gamma$ while the real roots are given by (4.1.33) for $k = \gamma + 1, \dots, n$. It remains to show that the non-real roots lie on the hyperbola $H\left(\sqrt{s/2}, \sqrt{s/2}\right)$ defined in (4.1.26). If $8 \left(1 - \cos\left(\frac{k}{n+1}\pi\right)\right) > s^4$, then we have the square-root

$$\begin{aligned} x &= \frac{\pm 1}{\sqrt{2s}} \sqrt{s^2 \pm i \sqrt{8 \left(1 - \cos\left(\frac{k}{n+1}\pi\right)\right)} - s^4} \\ &= \frac{\pm 1}{2\sqrt{s}} \left(\sqrt{2\sqrt{2 \left(1 - \cos\left(\frac{k}{n+1}\pi\right)\right)} + s^2} \pm i \sqrt{2\sqrt{2 \left(1 - \cos\left(\frac{k}{n+1}\pi\right)\right)} - s^2} \right). \end{aligned} \quad (4.1.35)$$

From (4.1.35),

$$2\mathcal{R}(x)^2 - 2\mathcal{I}(x)^2 = s,$$

showing that all the non-real roots of $w_n(s, x)$ lie on $H\left(\sqrt{s/2}, \sqrt{s/2}\right)$. See figures 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6 below for an illustration.

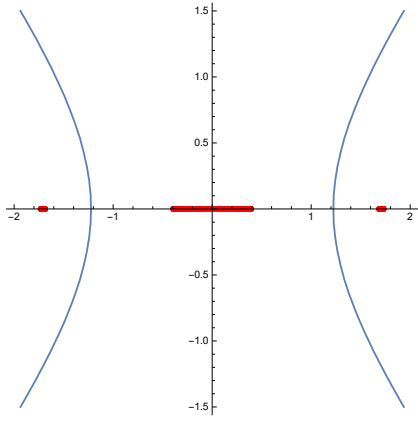


Figure 4.1: Roots of $w_{39}(3, x)$.

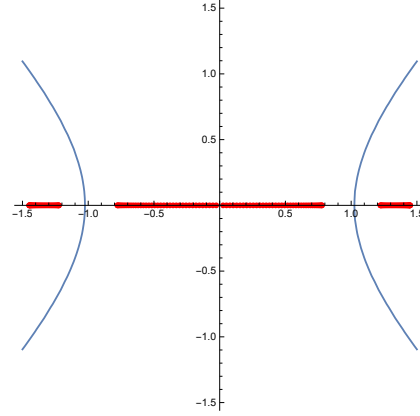


Figure 4.2: Roots of $w_{39}(2.1, x)$.

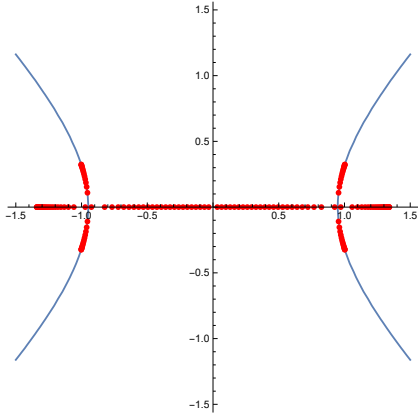


Figure 4.3: Roots of $w_{39}(1.8, x)$.

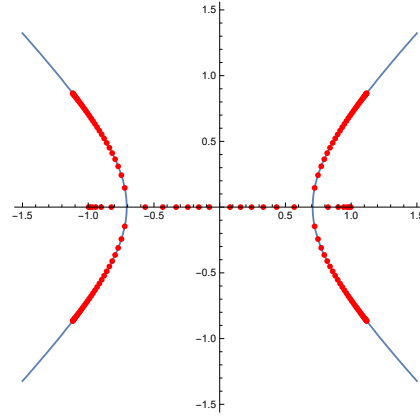


Figure 4.4: Roots of $w_{39}(1.0, x)$.

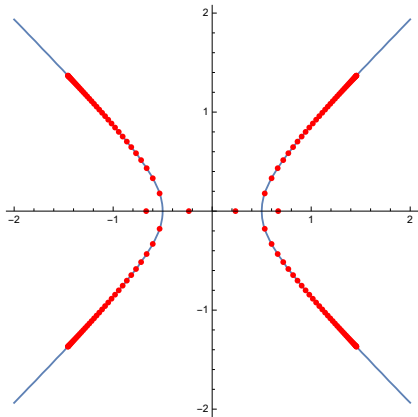


Figure 4.5: Roots of $w_{39}(0.5, x)$.

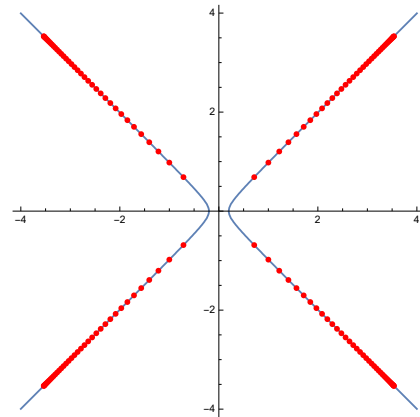


Figure 4.6: Roots of $w_{39}(0.08, x)$.

The roots of $v_n(s, x)$ are distributed in exactly the same way as the roots of $w_n(s, x)$. First we describe the roots of $U_n(x) - U_{n-1}(x)$ which are the intersections of $U_n(x)$ and $U_{n-1}(x)$ in the interval $(-1, 1)$. We consider two cases (depending on the parity of n) for the number of intersections. For n odd, $U_n(x)$ has a root at $x = 0$ and has $\frac{n}{2}$ critical points in $(0, 1)$ and in $(-1, 0)$. From graphs, $U_{n-1}(x)$ has a critical point at $x = 0$ and $\frac{n}{2} - 1$ critical points in $(0, 1)$ and in $(-1, 0)$, so the two polynomials have exactly $\frac{n-1}{2}$ intersection points in $(-1, 0)$. There are $\frac{n-1}{2} + 1$ point of intersection in $(0, 1)$, with the additional intersection between $\cos(\pi/(n+1))$ and 1 where $\cos(\pi/(n+1))$ is the largest root of $U_n(x)$.

For n even, $U_n(x)$ has a critical point at $x = 0$, $\frac{n}{2} - 1$ critical points in $(0, 1)$ and in $(-1, 0)$. Now, $U_{n-1}(x)$ has a root at $x = 0$ and $\frac{n}{2} - 1$ critical points in $(0, 1)$, and in $(-1, 0)$. The two polynomials have exactly $\frac{n}{2}$ points of intersection in $(0, 1)$, and in $(-1, 0)$. The largest root of $U_n(x) - U_{n-1}(x)$ is the intersection of $U_n(x)$ and $U_{n-1}(x)$ which occurs between $\cos(\pi/(n+1))$ and 1. In both cases, $U_n(x)$ and $U_n(x) - U_{n-1}(x)$ are strictly interlacing on the interval $(-1, 1)$.

For each root β of $U_n(x) - U_{n-1}(x)$, there are four roots of $v_n(s, x)$, which are the roots of the quartic

$$2 - s^3x^2 + s^2x^4 - 2\beta = 0 \quad (4.1.36)$$

and are given by

$$x = \pm \frac{1}{\sqrt{2}s} \sqrt{s^2 \pm \sqrt{s^4 - 8(1 - \beta)}}. \quad (4.1.37)$$

This explains the symmetry of the roots of $v_n(s, x)$. The explanation of the distribution of the real roots and the non-real roots of $v_n(s, x)$ on the hyperbola is exactly the same as that of $w_n(s, x)$.

Corollary 78. *Let $\xi(s, x)$ be as defined in (4.1.21) and $T_n(x)$ be the n th Chebyshev polynomial of the first kind. Then for $s \geq 2$, all the roots of $T_n(\frac{1}{2}\xi(s, x))$ and $T_n(\frac{1}{2}\xi(s, x)) \pm T_{n-1}(\frac{1}{2}\xi(s, x))$ are real and lie in the interval $[-\sqrt{s}, \sqrt{s}]$. The roots are symmetric about $x = 0$ and are concentrated at the ends of the interval and at $x = 0$. Moreover for large s , The roots approach $\pm\sqrt{s}$ and 0.*

For $s < 2$, we have real roots in $[-\sqrt{s}, \sqrt{s}]$ and non-real roots on the

hyperbola $H\left(\sqrt{s/2}, \sqrt{s/2}\right)$. As s further decreases to 0, all the roots become arbitrarily close to the asymptotes of the hyperbola $H\left(\sqrt{s/2}, \sqrt{s/2}\right)$.

The proof is very similar to that of Theorem 76.

Corollary 79. *All roots of $c_n(x)$ are real and lie in the interval $(-\sqrt{2}, \sqrt{2})$.*

Proof. Recall that $c_n(x)$ is defined by (4.1.19). Comparing the generating functions (4.1.20) and (4.1.29) we find that $c_{4n+1}(x) = c_{4n+3}(x) = (-1)^n 2xw_n(2, x)$ and $c_{4n}(x) = (-1)^n v_n(2, x)$. Hence the proof for the root distribution of $c_n(x)$ is exactly the same as that for $v_n(s, x)$ and $w_n(s, x)$ when $s = 2$. $\square$

4.2 Chebyshev-like polynomials with roots on hyperbolas

All polynomials with real roots can be lifted to polynomials with roots on hyperbolas. Let $a, b, c \in \mathbb{R}$ with $a \neq 0$ and consider the map

$$z \mapsto ia(z^2 + b) + c. \quad (4.2.1)$$

If $g(x)$ is a polynomial with real roots, then $g(ib(z^2 + a) + c)$ will have all its roots on the hyperbola

$$\mathcal{H}_b : \quad y^2 - x^2 = b. \quad (4.2.2)$$

To see this, write

$$g(x) = c_n \prod_{k=1}^n (x - x_i),$$

where $x_i \in \mathbb{R}$. Then

$$g(ia(z^2 + a) + c) = c_n \prod_{k=1}^n (ia(z^2 + b) + c - x_i) = 0$$

implies that

$$z^2 + b + \frac{i}{a}(x_i - c) = 0$$

and hence

$$z = \pm \frac{1}{\sqrt{2(r-b)}} \left(r - b - \frac{i}{a}(x_i - c) \right)$$

where $r = \left(b^2 + \frac{(x_i - c)^2}{a^2}\right)^{1/2}$. It is easy to see that these points lie on the hyperbola $\mathcal{H}_b$. We make this precise in the following Lemma.

Lemma 80. *Let $a, b, c \in \mathbb{R}$ with $a \neq 0$, and $g(x)$ be any polynomial whose roots are all real. Then all the roots of $g(ia(x^2 + b) + c)$ lie on the hyperbola $\mathcal{H}_b : y^2 - x^2 = b$.*

Since $T_n(x)$ and $U_n(x)$ have real roots for all $n \geq 1$, by Lemma 80, $T_n(ia(z^2 + b) + c)$ and $U_n(ia(z^2 + b) + c)$ have all their roots on the hyperbola $\mathcal{H}_b$. We will give a formula for the points on the hyperbola $\mathcal{H}_b$ which are roots of $T_n(ia(z^2 + b) + c)$ and $U_n(ia(z^2 + b) + c)$.

First, consider the Chebyshevs $T_n(a(x^2 - 1) + 1)$ and $U_n(a(x^2 - 1) + 1)$. Recall that the zeros of $T_n(x)$ and $U_n(x)$ are respectively given by

$$\cos\left(\frac{2k-1}{2n}\pi\right) \quad \text{and} \quad \cos\left(\frac{k\pi}{n+1}\right),$$

where $k = 1, 2, \dots, n$. This implies that for $a \in \mathbb{R}$ non-zero, the zeros of $T_n(a(x^2 - 1) + 1)$ and $U_n(a(x^2 - 1) + 1)$ are respectively given by the square roots of

$$\frac{1}{a} (a - 1 + \cos\left(\frac{2k-1}{2n}\pi\right)) \quad \text{and} \quad \frac{1}{a} (a - 1 + \cos\left(\frac{k\pi}{n+1}\right)). \quad (4.2.3)$$

Clearly, $a = 0$ is the degenerate case where the two polynomials are constant polynomials. For a ‘very small’, the zeros of the two polynomials are all purely imaginary and large in modulus. To make this precise, whenever $0 < a < 1 - \cos\left(\frac{\pi}{2n}\right)$ and $0 < a < 1 - \cos\left(\frac{\pi}{n+1}\right)$ respectively in $T_n(a(x^2 - 1) + 1)$ and $U_n(a(x^2 - 1) + 1)$ their zeros are purely imaginary and large in modulus.

As a increases, the moduli of the zeros decrease and a zero of minimal modulus drops to 0 to coincide with its conjugate. The double 0 then splits, as a further increases, into two real zeros moving in opposite directions along the real axis. This process continues until when at $a = 1$, where with the exception of the trivial zero, we have an equal number of real and purely imaginary zeros.

Before $a = 2$ is reached, and when $a \geq 1 - \cos\left(\frac{\pi}{2n}\right)$ and $a \geq 1 - \cos\left(\frac{\pi}{n+1}\right)$ respectively in $T_n(a(x^2 - 1) + 1)$ and $U_n(a(x^2 - 1) + 1)$ their zeros are all real. Past $a = 2$ the zeros continue moving towards 1 and (the other half) to -1 . Hence given $\epsilon > 0$, for $N = N(\epsilon)$ large enough, $a > N$ implies all zeros lie in $(-1, -1 + \epsilon)$ and $(1 - \epsilon, 1)$.

When a is purely imaginary, the two polynomials have a hyperbolic distribution of their zeros.

Proposition 81. *For all $a \neq 0$ and $n \geq 1$, the zeros of $T_n(ai(x^2 - 1) + 1)$ and $U_n(ai(x^2 - 1) + 1)$ all lie on the hyperbola $x^2 - y^2 = 1$ and are respectively given by $x_k(n, a) = \cosh \theta_k(n, a) \pm i \sinh \theta_k(n, a)$ and $\tilde{x}_k(n, a) = \cosh \varphi_k(n, a) \pm i \sinh \varphi_k(n, a)$, where*

$$\begin{aligned}\theta_k(n, a) &:= \frac{1}{2} \log \frac{1}{|a|} \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) + \sqrt{a^2 + \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} \right), \\ \varphi_k(n, a) &:= \frac{1}{2} \log \frac{1}{|a|} \left(1 - \cos \left(\frac{k\pi}{n+1} \right) + \sqrt{a^2 + \left(1 - \cos \left(\frac{k\pi}{n+1} \right) \right)^2} \right),\end{aligned}$$

and $k = 1, 2, \dots, n$.

Proof. Recall that $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. From the definition of $\theta_k(n, a)$ we have

$$\begin{aligned}\sinh \theta_k(n, a) &= \frac{1}{2\sqrt{|a|}} \left\{ \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) + \sqrt{a^2 + \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} \right)^{1/2} \right. \\ &\quad \left. - a^2 \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) + \sqrt{a^2 + \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} \right)^{-1/2} \right\} \\ &= \frac{1}{2\sqrt{|a|}} \left\{ \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) + \sqrt{a^2 + \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} \right)^{1/2} \right. \\ &\quad \left. - \left(\cos \left(\frac{2k-1}{2n} \pi \right) - 1 + \sqrt{a^2 + \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} \right)^{-1/2} \right\},\end{aligned}$$

and by taking the square on both sides,

$$\sinh^2 \theta_k(n, a) = \frac{1}{2} \left(\sqrt{1 + \frac{1}{a^2} \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} - 1 \right).$$

Finally taking square-roots on both sides of the above equation we obtain

$$\sinh \theta_k(n, a) := \frac{1}{\sqrt{2}} \left(\sqrt{1 + \frac{1}{a^2} \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} - 1 \right)^{1/2}.$$

Similarly

$$\cosh \theta_k(n, a) := \frac{1}{\sqrt{2}} \left(\sqrt{1 + \frac{1}{a^2} \left(1 - \cos \left(\frac{2k-1}{2n} \pi \right) \right)^2} + 1 \right)^{1/2}.$$

Now it is easy to check that $ai(x_k(n, \theta)^2 - 1) + 1 = \cos \left(\frac{2k-1}{2n} \pi \right)$ and hence

the roots of $T_n(ai(x^2 - 1) + 1)$ are given by

$$x_k(n, a) = \cosh \theta_k(n, a) + i \sinh \theta_k(n, a).$$

In exactly the same way, it can also be shown that if $\tilde{x}_k(n, a)$ is a zero of $U_n(ai(x^2 - 1) + 1)$ then

$$\tilde{x}_k(n, a) = \cosh \varphi_k(n, a) + i \sinh \varphi_k(n, a).$$

where

$$\begin{aligned} \cosh \varphi_k(n, a) &= \frac{1}{\sqrt{2}} \left(\sqrt{1 + \frac{1}{a^2} \left(1 - \cos\left(\frac{k\pi}{n+1}\right)\right)^2} + 1 \right)^{1/2}, \\ \sinh \varphi_k(n, a) &= \frac{1}{\sqrt{2}} \left(\sqrt{1 + \frac{1}{a^2} \left(1 - \cos\left(\frac{k\pi}{n+1}\right)\right)^2} - 1 \right)^{1/2}. \end{aligned}$$

From these calculations, we can see that if a is sufficiently large, both $\sinh \phi_k(n, a)$ and $\sinh \varphi_k(n, a)$ are arbitrarily close to zero, while $\cosh \phi_k(n, a)$ and $\cosh \varphi_k(n, a)$ are arbitrarily close to ± 1 . $\square$

Remark 6. For a ‘sufficiently’ large, there exists an $\epsilon(a)$ such that $T_n(ai(z^2 - 1) + 1)$ and $U_n(ai(z^2 - 1) + 1)$ have all their zeros in the union of disjoint balls $B_{\epsilon(a)}(-1)$ and $B_{\epsilon(a)}(1)$. We can take $\epsilon(a)$ to be $\frac{1}{2a}$.

In general if $a, b, c \in \mathbb{R}$ with $a \neq 0$ then $T_n(ai(x^2 - b) + c)$ and $U_n(ai(x^2 - b) + c)$ have their roots on the hyperbola $x^2 - y^2 = b$ and are respectively given by

$$x_k(n, a, b, c) := \cosh \theta_k(n, a, b, c) + i \sinh \theta_k(n, a, b, c)$$

where

$$\begin{aligned} \cosh \theta_k(n, a, b, c) &:= \frac{1}{\sqrt{2}} \left(\sqrt{b + \frac{1}{a^2} \left(c - \cos\left(\frac{2k-1}{2n}\pi\right)\right)^2} + b \right)^{1/2}, \\ \sinh \theta_k(n, a, b, c) &:= \frac{1}{\sqrt{2}} \left(\sqrt{b + \frac{1}{a^2} \left(c - \cos\left(\frac{2k-1}{2n}\pi\right)\right)^2} - b \right)^{1/2}, \end{aligned}$$

and

$$\tilde{x}_k(n, a, b, c) := \cosh \varphi_k(n, a, b, c) + i \sinh \varphi_k(n, a, b, c),$$

where

$$\begin{aligned}\cosh \varphi_k(n, a, b, c) &= \frac{1}{\sqrt{2}} \left(\sqrt{b + \frac{1}{a^2} \left(c - \cos \left(\frac{k\pi}{n+1} \right) \right)^2} + b \right)^{1/2}, \\ \sinh \varphi_k(n, a, b, c) &= \frac{1}{\sqrt{2}} \left(\sqrt{b + \frac{1}{a^2} \left(c - \cos \left(\frac{k\pi}{n+1} \right) \right)^2} - b \right)^{1/2},\end{aligned}$$

for $k = 1, 2, \dots, n$.

We have already seen that for all $k \in \mathbb{R}$, $R_n(k, x) = U_n(x) - kU_{n-1}(x)$ and $S_n(k, x) = T_n(x) - kT_{n-1}(x)$ have real roots. By Lemma 81, if a is purely imaginary, $R_n(k, a(x^2 - 1) + 1)$ and $S_n(k, a(x^2 - 1) + 1)$ have all their roots on the hyperbola $x^2 - y^2 = 1$. We are now going to give a formula for the roots of $R_n(1, a(x^2 - 1) + 1)$.

Proposition 82. *For a fixed n and $a \in \mathbb{R}$, the roots of $R_n(1, a(x^2 - 1) + 1)$ are given by*

$$y_k = \left(1 - \frac{2}{a} \sin^2 \left(\frac{2k-1}{2(2n+1)} \pi \right) \right)^{1/2}.$$

When a is purely imaginary, the roots are given by

$$\begin{aligned}z_k(a) &= \frac{1}{\sqrt{2}} \left(\sqrt{1 + \frac{4}{a^2} \sin^4 \left(\frac{2k-1}{2(2n+1)} \pi \right)} + 1 \right)^{1/2} + \frac{i}{\sqrt{2}} \left(\sqrt{1 + \frac{4}{a^2} \sin^4 \left(\frac{2k-1}{2(2n+1)} \pi \right)} - 1 \right)^{1/2}.\end{aligned}$$

for $k = 1, 2, \dots, n$.

From the generating functions (1.0.4), $R_n(k, a(x^2 - 1) + 1)$ is generated by

$$\frac{1 - kt}{1 - 2(a(x^2 - 1) + 1)t + t^2} = \sum_{n=0}^{\infty} R_n(k, a(x^2 - 1) + 1)t^n,$$

and comparing it to the rational generating function of odd Chebyshev polynomials of the first kind

$$\frac{(1-t)x}{1 - 2(2x^2 - 1)t + t^2} = \sum_{n=0}^{\infty} T_{2n+1}(x)t^n,$$

we obtain

$$xR_n(1, 2x^2 - 1) = T_{2n+1}(x). \quad (4.2.4)$$

We are going to use this relationship to give a formula for the roots of

$$\hat{R}_n(a, x) := R_n(1, a(x^2 - 1) + 1).$$

For a fixed n , and $a \neq 0$ real, $\hat{R}_n(a, x) = 0$ implies that $a(x^2 - 1) + 1 = 2x_k^2 - 1$ where $x_k = \pm \cos\left(\frac{2k-1}{2(2n+1)}\pi\right)$ is a non-trivial root of $T_{2n+1}(x)$.

This gives the roots of $\hat{R}_n(a, x)$ as

$$y_k = \left(1 - \frac{2}{a} \sin^2\left(\frac{2k-1}{2(2n+1)}\pi\right)\right)^{1/2},$$

for $k = 1, 2, \dots, n$. We can see that the roots of $\hat{R}_n(a, x)$ are all purely imaginary and very large in modulus whenever $a < 2(1 - x_k^2)$. The behavior of the roots when a is real can easily be verified from the formula for the roots.

Now when a is purely imaginary, the construction of the roots is very similar to the case when a is real. For clarity in calculations, write $\hat{R}_n(a, x)$ as $R_n(ia(x^2 - 1) + 1)$. Then $R_n(ia(x^2 - 1) + 1) = 0$ implies that $ia(x^2 - 1) + 1 = 2x_k^2 - 1$ where x_k is a non-trivial zero of $T_{2n+1}(x)$. Let $x = \cosh \theta + i \sinh \theta$, then $x^2 = 1 + i \sinh(2\theta)$ so that if x is a root of $R_n(ia(x^2 - 1) + 1)$, then $\sinh(2\theta) = \frac{2}{a}(1 - x_k^2)$. Using the identities

$$\sinh x = \frac{1}{\sqrt{2}}(\sqrt{1 + \sinh^2 2x} - 1)^{1/2} \quad \text{and} \quad \cosh x = \frac{1}{\sqrt{2}}(\sqrt{1 + \sinh^2 2x} + 1)^{1/2}$$

we find that

$$\begin{aligned} \cosh \theta_k &= \frac{1}{\sqrt{2}} \left(\sqrt{1 + \frac{4}{a^2} \sin^4\left(\frac{2k-1}{2(2n+1)}\pi\right)} + 1 \right)^{1/2}, \\ \sinh \theta_k &= \frac{1}{\sqrt{2}} \left(\sqrt{1 + \frac{4}{a^2} \sin^4\left(\frac{2k-1}{2(2n+1)}\pi\right)} - 1 \right)^{1/2}, \end{aligned}$$

and the Proposition follows.

Define the operator J_x as

$$J_x := \frac{1}{x} \frac{d}{dx}. \tag{4.2.5}$$

If we apply J_x to an even polynomial $g(x)$, we will still obtain an even polynomial. In fact, if $g(x)$ is an even polynomial with degree $2n$, then applying J_x to $g(x)$ $n - 1$ times will give a polynomial of degree 2, i.e., $J_x^k g(x)$ has degree $2n - 2k$ for $1 \leq k < n$. The following lemma is immediate.

Lemma 83. *Let $a, b, c \in \mathbb{R}$ with $a \neq 0$, and $g(x)$ be any polynomial of degree n whose roots are all real, then all the roots of $J_x^k g(ia(x^2 + b) + c)$ for $1 \leq k < n$ lie on the hyperbola $\mathcal{H}_b: y^2 - x^2 = b$.*

Proof. By the Gauss-Lucas theorem, the convex hull of the roots of $g(x)$ contains the roots of $g'(x)$. It follows that all the derivatives $g^{(k)}(x)$ will have real roots. If we define $f(x) := g^{(k)}(x)$ then $J_x^k g(ia(x^2 + b) + c) = (ia)^k f(ia(x^2 + b) + c)$, and by Lemma 80, $J_x^k g(ia(x^2 + b) + c)$ has all its roots on $y^2 - x^2 = b$. $\square$

Corollary 84. *If a is purely imaginary, then $J_x^k T_n(a(x^2 + b) + c)$, $J_x^k U_n(a(x^2 + b) + c)$, $J_x^k R_n(k, a(x^2 + b) + c)$ and $J_x^k S_n(k, a(x^2 + b) + c)$ have their roots on $y^2 - x^2 = b$ for $1 \leq k < n$.*

CHAPTER 5

MULTIPLES OF LONG PERIOD SMALL ELEMENT CONTINUED FRACTIONS TO SHORT PERIOD LARGE ELEMENTS CONTINUED FRACTIONS

Fibonacci numbers and Lucas numbers are integers that solve the the recurrence relation

$$f_{n+1} = f_n + f_{n-1}$$

under the initial conditions $F_0 = 0$, $F_1 = F_2 = 1$, $L_0 = 2$ and $L_1 = 1$. Fibonacci numbers can also be defined by evaluating the Fibonacci polynomials $F_n(x)$ at 1, i.e., $F_n(1)$. The sequence of Fibonacci polynomials is generated by the rational function

$$\frac{t}{1 - xt - t^2} = \sum_{k=1}^{\infty} F_k(x)t^k. \quad (5.0.1)$$

In the first section of this chapter, we show that certain families of rationally generated integers have the property that the ratio of consecutive integers in the sequence is finite periodic with the length of the continued fraction dependent on the integers, and the partial quotients of the continued fraction are bounded. We then show that certain integer multiples of these continued fractions give a continued fraction of fixed length, but whose partial quotients are functions of the integers.

In the second section of the chapter, we construct Fibonacci and Lucas ‘like’ integers $\tilde{F}_n(m)$ and $\tilde{L}_n(m)$, and use them to construct a quadratic irrational $\alpha_n(N)$, the largest root of the quadratic

$$x^2 - 2Nh_nx - h_n\tilde{h}_n(2N)$$

where N is a positive integer and h_n is either a Fibonacci or Lucas-like integer. Of interest is the continued fraction of $\alpha_n(N)$. In particular we will show that the continued fraction of α_n is related to the continued fractions

$$F_n \cdot [2N, 1^{(n-1)}]$$

n	$r_n(a, b)$	$s_n(a, b)$
0	1	1
1	$a + b$	$a + b$
2	$a^2 + ab - 1$	$a^2 + ab + 1$
3	$a^3 + a^2b - 2a - b$	$a^3 + a^2b + 2a + b$
4	$a^4 + a^3b - 3a^2 - 2ab + 1$	$a^4 + a^3b + 3a^2 + 2ab + 1$
5	$a^5 + a^4b - 4a^3 - 3a^2b + 3a + b$	$a^5 + a^4b + 4a^3 + 3a^2b + 3a + b$
6	$a^6 + a^5b - 5a^4 - 4a^3b + 6a^2 + 3ab - 1$	$a^6 + a^5b + 5a^4 + 4a^3b + 6a^2 + 3ab + 1$

Table 5.1: Table of the first seven polynomials $r_n(a, b)$ and $s_n(a, b)$.

and

$$L_n \cdot \overline{[2N, 1^{(n-2)}, 2, 1, 2k, 1, 2, 1^{(n-2)}]}.$$

5.1 On Integer multiples of finite periodic continued fractions

Let a and b be fixed integers, and define the integer sequences $r_n(a, b)$ and $s_n(a, b)$ by the generating functions

$$\frac{1 + bt}{1 - at + t^2} = \sum_{n=0}^{\infty} r_n(a, b)t^n \quad (5.1.1)$$

and

$$\frac{1 + bt}{1 - at - t^2} = \sum_{n=0}^{\infty} s_n(a, b)t^n. \quad (5.1.2)$$

These integers are clearly related to Fibonacci and Lucas numbers. The first seven integers in each sequence are given in the table below

Lemma 85. For $n \geq 1$,

$$r_{n-1}(a, 0)r_{n+1}(a, 0) - r_n(a, 0)^2 = -1, \quad (5.1.3)$$

$$s_{n-1}(a, 0)s_{n+1}(a, 0) - s_n(a, 0)^2 = (-1)^{n-1} \quad (5.1.4)$$

Proof. we can easily prove (5.1.3) and (5.1.4) by induction on n . $\square$

From the generating functions (5.1.1) and (5.1.2), we obtain the recurrence

relations

$$r_{n+1}(a, b) = ar_n(a, b) - r_{n-1}(a, b), \quad (5.1.5)$$

$$s_{n+1}(a, b) = as_n(a, b) + s_{n-1}(a, b) \quad (5.1.6)$$

for the integer sequences $r_n(a, b)$ and $s_n(a, b)$.

Proposition 86. *For $n \geq 1$,*

$$\frac{r_{n+1}(a, b)}{r_n(a, b)} = [a - 1, 1, \{a - 2, 1\}^{(n-1)}, a + b - 1]$$

and

$$\frac{s_{n+1}(a, b)}{s_n(a, b)} = [\{a\}^{(n)}, a + b]$$

where $\{c_1, \dots, c_k\}^{(n)}$ means that the word $c_1, \dots, c_k$ has been repeated n times.

Proof. Clearly for $n = 1$,

$$[a - 1, 1, a + b - 1] = \frac{a^2 + ab - 1}{a + b} = \frac{r_2(a, b)}{r_1(a, b)}$$

and

$$[a, a + b] = \frac{a^2 + ab + 1}{a + b} = \frac{s_2(a, b)}{s_1(a, b)}.$$

If $\frac{r_{n+1}(a, b)}{r_n(a, b)} = [a - 1, 1, \{a - 2, 1\}^{(n-1)}, a + b - 1]$, then

$$\begin{aligned} [a - 1, 1, \{a - 2, 1\}^{(n)}, a + b - 1] &= [a - 1, 1, a - 2, 1, \{a - 2, 1\}^{(n-1)}, a + b - 1] \\ &= [a - 1, 1, a - 2, \frac{r_n(a, b)}{r_{n+1}(a, b) - (a-1)r_n(a, b)}] \\ &= \frac{ar_{n+1}(a, b) - r_n(a, b)}{r_{n+1}(a, b)} \\ &= \frac{r_{n+2}(a, b)}{r_{n+1}(a, b)}. \end{aligned}$$

Similarly, if $\frac{s_{n+1}(a,b)}{s_n(a,b)} = [\{a\}^{(n)}, a+b]$, then

$$\begin{aligned}
[\{a\}^{(n+1)}, a+b] &= [a, \{a\}^{(n)}, a+b] \\
&= [a, \frac{s_{n+1}(a,b)}{s_n(a,b)}] \\
&= \frac{as_{n+1}(a,b) + s_n(a,b)}{s_{n+1}(a,b)} \\
&= \frac{s_{n+2}(a,b)}{s_{n+1}(a,b)}
\end{aligned}$$

as desired. The proof follows by induction on n . $\square$

Clearly, the length of the continued fractions of $r_{n+1}(a,b)/r_n(a,b)$ and $s_{n+1}(a,b)/s_n(a,b)$ depends on n . We are going to see that by multiplying these ratios by $r_{n-1}(a,0)$ and $s_{n-1}(a,0)$ respectively, the continued fractions of the new rational numbers will be of fixed length, but the partial quotients will now depend on $r_n(a,b)$ and $s_n(a,b)$ respectively.

Proposition 87. *For $n \geq 1$,*

$$r_{n-1}(a,0) \frac{r_{n+1}(a,b)}{r_n(a,b)} = [r_n(a,0) - 1, 1, r_n(a,b) - 1]$$

and

$$s_{n-1}(a,0) \frac{s_{n+1}(a,b)}{s_n(a,b)} = \begin{cases} [s_n(a,0), s_n(a,b)] & \text{for } n \text{ odd,} \\ [s_n(a,0) - 1, 1, s_n(a,b) - 1] & \text{for } n \text{ even.} \end{cases}$$

Proof. For $n \geq 1$,

$$[r_n(a,0) - 1, 1, r_n(a,b) - 1] = \frac{r_n(a,0)r_n(a,b) - 1}{r_n(a,b)},$$

and

$$\begin{aligned}
r_n(a,0)r_n(a,b) - 1 &= r_n(a,0)(r_n(a,0) + br_{n-1}(a,0)) - 1 \\
&= r_n(a,0)^2 + br_n(a,0)r_{n-1}(a,0) - 1 \\
&= r_{n-1}(a,0)r_{n+1}(a,0) + br_{n-1}(a,0)r_n(a,0) \\
&= r_{n-1}(a,0)r_{n+1}(a,b),
\end{aligned}$$

where we have used (5.1.3). Similarly, using (5.1.4) we find that for n odd,

$$[s_n(a, 0), s_n(a, b)] = \frac{s_n(a, 0)s_n(a, b) + 1}{s_n(a, b)},$$

and

$$\begin{aligned} s_n(a, 0)s_n(a, b) + 1 &= s_n(a, 0)(s_n(a, 0) + bs_{n-1}(a, 0)) + 1 \\ &= s_n(a, 0)^2 + bs_n(a, 0)s_{n-1}(a, 0) + 1 \\ &= s_{n-1}(a, 0)s_{n+1}(a, 0) + bs_{n-1}(a, 0)s_n(a, 0) \\ &= s_{n-1}(a, 0)s_{n+1}(a, b). \end{aligned}$$

The proof for n even is exactly the same. $\square$

Theorem 88. *Let $r_n(a, b)$ and $s_n(a, b)$ be as defined by the generating functions (5.1.1) and (5.1.2). Then for all $n \geq 1$*

$$r_{n-1}(a, 0) \cdot [a - 1, 1, \{a - 2, 1\}^{(n-1)}, a + b - 1] = [r_n(a, 0) - 1, 1, r_n(a, b) - 1], \quad (5.1.7)$$

and

$$s_{n-1}(a, 0) \cdot [\{a\}^{(n)}, a + b] = \begin{cases} [s_n(a, 0), s_n(a, b)] & \text{for } n \text{ odd,} \\ [s_n(a, 0) - 1, 1, s_n(a, b) - 1] & \text{for } n \text{ even.} \end{cases} \quad (5.1.8)$$

Proof. The proofs of (5.1.7) and (5.1.8) from Propositions 86 and 87. $\square$

The most interesting observation from the above theorem is that the left hand side of equations (5.1.7) and (5.1.8) have finite continued fractions of variable length n , and multiplying these continued fractions by certain integers give continued fractions of fixed length, but with large partial quotients.

5.2 Fibonacci-like numbers and quadratic irrationals

Let m be a positive integer and define the sequence $\tilde{F}_n(m)$ by the recurrence relation

$$\tilde{F}_n(m) = \tilde{F}_{n-1}(m) + \tilde{F}_{n-2}(m) \quad (5.2.1)$$

under the initial conditions $\tilde{F}_0(m) = 0$, $\tilde{F}_1(m) = 1$ and $\tilde{F}_2(m) = m$. It is easy to see that they are generated by the rational function

$$\frac{t + (m-1)t^2}{1-t-t^2} = \sum_{n=0}^{\infty} \tilde{F}_n(m)t^n. \quad (5.2.2)$$

Clearly, for $m = 1$ we get the usual Fibonacci sequence. From the generating function (5.2.2), and the generating function of Fibonacci numbers, we deduce the relation

$$\begin{aligned} \tilde{F}_{n+1}(m) &= F_{n+1} + (m-1)F_n \\ &= mF_n + F_{n-1}. \end{aligned} \quad (5.2.3)$$

For a fixed positive integer N , consider the quadratic

$$x^2 - 2NF_nx - F_n\tilde{F}_n(2N). \quad (5.2.4)$$

The roots of the quadratic are given by

$$\begin{aligned} \alpha_n(N) &= NF_n + \sqrt{N^2F_n^2 + F_n\tilde{F}_n(2N)}, \\ \bar{\alpha}_n(N) &= NF_n - \sqrt{N^2F_n^2 + F_n\tilde{F}_n(2N)}. \end{aligned} \quad (5.2.5)$$

We study the continued fraction of these quadratic irrationals.

A quadratic irrational α is said to be reduced if $\alpha > 1$ and $-1/\bar{\alpha} > 1$. If α is reduced, then its continued fraction is purely periodic (see [31] Theorem 2.48). First we show that for n odd, $1/(\alpha_n(N) - \tilde{F}_{n+1}(2N))$ is reduced.

If $B_n(N) = N^2F_n^2 + F_n\tilde{F}_n(2N)$, then

$$\begin{aligned} 1/(\alpha_n(N) - \tilde{F}_{n+1}(2N)) &= -1/(NF_n + F_{n-1} - \sqrt{B_n(N)}) \\ &= \frac{-(NF_n + F_{n-1} + \sqrt{B_n(N)})}{F_{n-1}^2 - F_nF_{n-2}} \\ &= NF_n + F_{n-1} + \sqrt{B_n(N)} > 1. \end{aligned}$$

On the other hand,

$$\tilde{F}_{n+1}(2N) - \bar{\alpha}_n(N) = NF_n + F_{n-1} + \sqrt{B_n(N)} > 1.$$

We have used the identities (5.2.3) and the well known identity for Fibonacci numbers

$$F_{n-1}^2 - F_n F_{n-2} = (-1)^n. \quad (5.2.6)$$

Let $x_n(N) = NF_n + F_{n-1} + \sqrt{B_n(N)}$. By using (5.2.3) and the Euclidean algorithm we get

$$\begin{aligned} x_n(N) &= 2\tilde{F}_{n+1}(N) - (NF_n + F_{n-1} - \sqrt{B_n(N)}) \\ &= 2\tilde{F}_{n+1}(N) + \frac{1}{x_n(N)}. \end{aligned}$$

We now have

$$\begin{aligned} \alpha_n(N) &= x_n - F_{n-1} \\ &= \tilde{F}_{n+1}(2N) + \frac{1}{x_n(N)} \\ &= [\tilde{F}_{n+1}(2N), \overline{2\tilde{F}_{n+1}(N)}]. \end{aligned}$$

For n even, the calculations for the continued fraction expansion of α_n are a bit involved due to the identity (5.2.6). Thus,

$$\begin{aligned} 1/(\alpha_n(N) - \tilde{F}_{n+1}(2N) + 1) &= \frac{1}{1 - (NF_n + F_{n-1} - \sqrt{B_n(N)})} \\ &= \frac{NF_n + F_{n-1} + \sqrt{B_n(N)}}{NF_n + F_{n-1} + \sqrt{B_n(N)} - (F_{n-1}^2 - F_n F_{n-2})} \\ &= \frac{NF_n + F_{n-1} + \sqrt{B_n(N)}}{NF_n + F_{n-1} + \sqrt{B_n(N)} - 1} > 1. \end{aligned}$$

We also have that

$$\tilde{F}_{n+1}(2N) - 1 - \overline{\alpha_n} = NF_n + F_{n-1} + \sqrt{B_n(N)} - 1 > 1.$$

Now if $x_n(N) = 1/(1 - (NF_n + F_{n-1} - \sqrt{B_n(N)}))$, then

$$\begin{aligned} x_n(N) &= 1 + \frac{1}{NF_n + F_{n-1} + \sqrt{B_n(N)} - 1} \\ &= 1 + \frac{1}{2\tilde{F}_{n+1}(N) - 2 + (1 - (NF_n + F_{n-1} - \sqrt{B_n(N)}))} \\ &= 1 + \frac{1}{2\tilde{F}_{n+1}(N) - 2 + 1/x_n(N)}. \end{aligned}$$

This implies that for n even,

$$\begin{aligned}\alpha_n(N) &= \tilde{F}_{n+1}(2N) - 1 + \frac{1}{x_n(N)} \\ &= [\tilde{F}_{n+1}(2N) - 1, \overline{1, 2\tilde{F}_{n+1}(N) - 2}].\end{aligned}$$

In each case, the length of the period in the continued fraction expansion of $\alpha_n(N)$ is independent of n . We have proved the following theorem

Theorem 89. *Let $\alpha_n(N)$ be the largest root of the quadratic (5.2.4) and $N > 1$ be a positive integer. Then the continued fraction of $\alpha_n(N)$ for n odd is given by*

$$\alpha_n(N) = [\tilde{F}_{n+1}(2N), \overline{2\tilde{F}_{n+1}(N)}], \quad (5.2.7)$$

while for n even it is given by

$$\alpha_n(N) = [\tilde{F}_{n+1}(2N) - 1, \overline{1, 2\tilde{F}_{n+1}(N) - 2}]. \quad (5.2.8)$$

We now show the relationship between the continued fraction of $\alpha_n(N)$ and $F_n \cdot [2N, 1^{(n-1)}]$ where $1^{(n)} = 1, 1, \dots, 1$ n times. Let α have the simple continued fraction

$$\alpha = [c_0, c_1, c_2, \dots, c_{k-1}, c_k] = \frac{p_k}{q_k}, \quad (5.2.9)$$

where $p_0 = a_0 = \lfloor \alpha \rfloor$, $p_{-1} = 1$, $q_0 = 1$ and $q_{-1} = 0$. We will use the correspondence (see [31, Lemma 2.8])

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}. \quad (5.2.10)$$

Taking the determinant on both sides of (5.2.10) gives the identity

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}. \quad (5.2.11)$$

Also, taking the transpose of the matrices on both sides of (5.2.10) we find

that

$$\begin{pmatrix} c_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{pmatrix}, \quad (5.2.12)$$

from which we deduce

$$\begin{aligned} [c_k, c_{k-1}, \dots, c_1, c_0] &= \frac{p_k}{p_{k-1}}, \\ [c_k, c_{k-1}, \dots, c_1] &= \frac{q_k}{q_{k-1}}. \end{aligned} \quad (5.2.13)$$

Lemma 90. *For all $n \geq 2$, we have the continued fraction,*

$$\frac{\tilde{F}_{n+1}(m)}{\tilde{F}_n(m)} = [1^{(n-1)}, m], \quad (5.2.14)$$

and for $n \geq 3$, we have the continued fraction

$$\frac{\tilde{F}_{n+2}(m)}{\tilde{F}_n(m)} = [2, 1^{(n-2)}, m], \quad (5.2.15)$$

where $1^{(n)} = 1, 1, \dots, 1$ n times.

Proof. We can easily show by induction on k that for all $k \geq 1$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k-1} \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{F}_{k+1}(m) & F_k \\ \tilde{F}_k(m) & F_{k-1} \end{pmatrix},$$

where we have used (5.2.3). It follows by (5.2.9), and the correspondence (5.2.10) that $[1^{(k-1)}, m] = \frac{\tilde{F}_{k+1}(m)}{\tilde{F}_k(m)}$.

Similarly,

$$\begin{aligned} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k-2} \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} F_{k+1} & F_k \\ F_{k-1} & F_{k-2} \end{pmatrix} \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{F}_{k+2}(m) & F_{k+1} \\ \tilde{F}_k(m) & F_{k-1} \end{pmatrix}, \end{aligned}$$

from which we deduce that $[2, 1^{(k-2)}, m] = \frac{\tilde{F}_{k+2}(m)}{\tilde{F}_k(m)}$. We can observe that all the partial quotients of $[1^{(n)}, m]$ are bounded, and for $j \leq n$, its sequence of convergents $\{p_j/q_j\}$ is the same as the sequence of convergents for the continued fraction of F_{n+1}/F_n , which converges to the golden ratio. By comparison, for any positive integer m , $\tilde{F}_{n+1}(m)/\tilde{F}_n(m)$ also converges to the golden ratio. $\square$

We now show another interesting property of the continued fraction of $\alpha_n(N)$.

Theorem 91. *For $n \geq 1$, let*

$$\lambda_n(N) := F_n \cdot \overline{[2N, 1^{(n-1)}]},$$

where the $1^{(n)}$ means that 1 has been repeated n times, and F_n is the n th Fibonacci number. Then $\lambda_n(N)$ is an algebraic integer, and for n odd

$$\lambda_n(N) = [\tilde{F}_{n+1}(2N), \overline{2\tilde{F}_{n+1}(N)}],$$

while for n even

$$\lambda_n(N) = [\tilde{F}_{n+1}(2N) - 1, \overline{1, 2\tilde{F}_{n+1}(N) - 2}].$$

One way of approaching this question is by the use of Chatelet algorithm for integer multiples of a continued fraction described by Cusik in [12] in order to derive the continued fraction of $\lambda_n(N)$. This approach is tedious and very involved. We will employ a rather straightforward approach by using the correspondence (5.2.10).

Let $x = \overline{[2N, 1^{(n-1)}]}$, and recall

$$x = [2N, 1^{(n-1)}, x] = \frac{xp_n + p_{n-1}}{xq_n + q_{n-1}},$$

where $[2N, 1^{(n-1)}] = \frac{p_n}{q_n}$. By the correspondence (5.2.10), we find that

$$\begin{aligned} \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} &= \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} F_{n+1}(2N) & F_n(2N) \\ F_n & F_{n-1} \end{pmatrix}, \end{aligned}$$

from which we deduce that

$$[2N, 1^{(n-1)}] = \frac{\tilde{F}_{n+1}(2N)}{F_n},$$

and

$$x = \frac{x\tilde{F}_{n+1}(2N) + \tilde{F}_n(2N)}{x\tilde{F}_n + F_{n-1}}.$$

Clearly x is the largest root of the quadratic

$$F_n x^2 - 2N F_n x - \tilde{F}_n(2N)$$

given by

$$x = N + \sqrt{N^2 + \tilde{F}_n(2N)/F_n}.$$

It is also clear that x is an algebraic integer, and so is $\lambda_n = F_n x$. Also, $F_n x = \alpha_n$ as defined in (5.2.5) and whose continued fraction has been described in theorem 89.

5.3 Lucas-like numbers and quadratic irrationals

Let m be a positive integer and define the Lucas-like sequence $\tilde{L}_n(m)$ by

$$\tilde{L}_n(m) = mL_{n-1} + L_{n-2}, \quad (5.3.1)$$

where $\tilde{L}_n(1)$ is the usual Lucas sequence. The numbers $\tilde{L}_n(m)$ solve the recurrence relation

$$\tilde{L}_n(m) = \tilde{L}_{n-1}(m) + \tilde{L}_{n-2}(m) \quad (5.3.2)$$

with initial conditions $\tilde{L}_0(m) = 3 - m$, $\tilde{L}_1(m) = -1 + 2m$ and $\tilde{L}_2(m) = 2 + m$.

For a fixed positive integer N , consider the quadratic

$$x^2 - 2NL_nx - L_n\tilde{L}_n(2N). \quad (5.3.3)$$

The roots of the quadratic are given by

$$\begin{aligned} \beta_n(N) &:= NL_n + \sqrt{N^2L_n^2 + L_n\tilde{L}_n(2N)}, \\ \bar{\beta}_n(N) &= NL_n - \sqrt{N^2L_n^2 + L_n\tilde{L}_n(2N)}. \end{aligned} \quad (5.3.4)$$

Just like in the previous section, we are going to examine the continued fractions of these quadratic irrationals. We will use the well known identity

$$L_{n-1}^2 - L_nL_{n-2} = (-1)^{n-1}5, \quad (5.3.5)$$

as well as identities (see for example [21, p. 93]) that relate the Fibonacci sequence to the Lucas sequence

$$L_n = F_{n+1} + F_{n-1} \quad (5.3.6)$$

and

$$L_n + 2L_{n-1} = 5F_n. \quad (5.3.7)$$

First we show that for n even, $1/(\beta_n(N) - \tilde{L}_{n+1}(2N))$ is reduced.

Let $C_n(N) := N^2L_n^2 + L_n\tilde{L}_n(2N)$. Then

$$\begin{aligned} 1/(\beta_n(N) - \tilde{L}_{n+1}(2N)) &= -1/\left(NL_n + L_{n-1} - \sqrt{C_n(N)}\right) \\ &= \frac{-\left(NL_n + L_{n-1} + \sqrt{C_n(N)}\right)}{L_{n-1}^2 - L_nL_{n-2}} \\ &= \frac{1}{5}\left(NL_n + L_{n-1} + \sqrt{C_n(N)}\right) > 1, \end{aligned}$$

where we have used (5.3.5). On the other hand,

$$\tilde{L}_{n+1}(2N) - \bar{\beta}_n(N) = NL_n + L_{n-1} + \sqrt{C_n(N)} > 1.$$

For an integer $k \geq 0$, let $N = 5k + 3$ and $y_n(N) := 1/(\beta_n(N) - \tilde{L}_{n+1}(2N))$. Using (5.3.1) and the Euclidean algorithm we find that

$$\begin{aligned}
y_n(N) &= 2kL_n + 2F_{n+1} + \frac{1}{5} \left(\sqrt{C_n(N)} - NL_n - L_{n-1} \right) \\
&= 2kL_n + 2F_{n+1} + \frac{-(L_{n-1}^2 - L_n L_{n-2})}{5 \left(NL_n + L_{n-1} + \sqrt{C_n(N)} \right)} \\
&= 2kL_n + 2F_{n+1} + \frac{1}{NL_n + L_{n-1} + \sqrt{C_n(N)}}.
\end{aligned}$$

We also have that

$$\begin{aligned}
NL_n + L_{n-1} + \sqrt{C_n(N)} &= 2\tilde{L}_{n+1}(2N) + \left(\sqrt{C_n(N)} - NL_n - L_{n-1} \right) \\
&= 2\tilde{L}_{n+1}(2N) + \frac{-(L_{n-1}^2 - L_n L_{n-2})}{NL_n + L_{n-1} + \sqrt{C_n(N)}} \\
&= 2\tilde{L}_{n+1}(N) + \frac{1}{y_n(N)}.
\end{aligned}$$

We now have that for n even,

$$\begin{aligned}
\beta_n(N) &= 5y_n(N) - L_{n-1} \\
&= \tilde{L}_{n+1}(2N) + \frac{1}{2kL_n + 2F_{n+1} + \frac{1}{2\tilde{L}_{n+1}(N) + \frac{1}{y_n(N)}}} \\
&= [\tilde{L}_{n+1}(2N), \overline{2kL_n + 2F_{n+1}, 2\tilde{L}_{n+1}(N)}].
\end{aligned}$$

For n odd, the calculations for the continued fraction expansion of β_n are a bit delicate due to the identity (5.3.5). First,

$$\begin{aligned}
1/(\beta_n(N) - \tilde{L}_{n+1}(2N) + 1) &= \frac{1}{1 - \left(NL_n + L_{n-1} - \sqrt{C_n(N)} \right)} \\
&= \frac{NL_n + L_{n-1} + \sqrt{C_n(N)}}{NL_n + L_{n-1} + \sqrt{C_n(N)} - (L_{n-1}^2 - L_n L_{n-2})} \\
&= \frac{NL_n + L_{n-1} + \sqrt{C_n(N)}}{NL_n + L_{n-1} + \sqrt{C_n(N)} - 5} > 1,
\end{aligned}$$

for $n > 1$. We also have that

$$\tilde{L}_{n+1}(2N) - 1 - \overline{\beta_n} = NL_n + L_{n-1} + \sqrt{C_n(N)} - 1 > 1.$$

It can easily be verified that for all $n \geq 1$, $1 < y_n(N) < 2$. Let $N = 5k + 3$ where $k \geq 0$ is an integer, and

$$y_n(N) = 1 / \left(1 - \left(NL_n + L_{n-1} - \sqrt{C_n(N)} \right) \right).$$

Then,

$$\begin{aligned} y_n(N) &= 1 + \frac{1}{\frac{1}{5} \left(NL_n + L_{n-1} + \sqrt{C_n(N)} - 5 \right)} \\ &= 1 + \frac{1}{2kL_n + 2F_{n+1} - 2 + \frac{1}{5} \left(5 - NL_n - L_{n-1} + \sqrt{C_n(N)} \right)}. \end{aligned}$$

It is easy to show (by induction on n) that

$$0 < \frac{1}{5} \left(5 - NL_n - L_{n-1} + \sqrt{C_n(N)} \right) < 1.$$

Now

$$\begin{aligned} \frac{1}{\frac{1}{5} \left(5 - NL_n - L_{n-1} + \sqrt{C_n(N)} \right)} &= 1 + \frac{1}{NL_n + L_{n-1} + \sqrt{C_n(N)} - 1} \\ &= 1 + \frac{1}{2\tilde{L}_{n+1}(N) - 2 + \frac{1}{y_n(N)}}. \end{aligned}$$

This implies that for n odd,

$$\begin{aligned} \beta_n(N) &= \tilde{L}_{n+1}(2N) - 1 + \frac{1}{y_n(N)} \\ &= [\tilde{L}_{n+1}(2N) - 1, 1, 2kL_n + 2F_{n+1} - 2, 1, 2\tilde{L}_{n+1}(N) - 2], \end{aligned}$$

where $N = 5k + 3$. Using (5.3.1), we can also make the simplification

$$\begin{aligned} 2kL_n + 2F_{n+1} &= 2kL_n + \frac{2}{5}(3L_n + L_{n-1}) \\ &= \frac{2}{5}((5k + 3)L_n + L_{n-1}) \\ &= \frac{2}{5}\tilde{L}_{n+1}(N) \end{aligned}$$

in the two statements above. We have proved the following theorem.

Theorem 92. *Let $N = 5k + 3$ for $k \geq 0$ and $\beta_n(k)$ be the largest root of the quadratic (5.3.3). Then the continued fraction of $\beta_n(k)$ for n odd is given by*

$$\beta_n(N) = [\tilde{L}_{n+1}(2N) - 1, 1, \overline{\frac{2}{5}L_{n+1}(N) - 2, 1, 2\tilde{L}_{n+1}(N) - 2}], \quad (5.3.8)$$

while for n even it is given by

$$\beta_n(N) = [\tilde{L}_{n+1}(2N), \overline{\frac{2}{5}L_{n+1}(N), 2\tilde{L}_{n+1}(N)}]. \quad (5.3.9)$$

There are some purely periodic continued fractions with periods of arbitrary length, which when multiplied by L_n , give periodic continued fractions with period of length 2 or 4 depending on the parity of n . We examine these continued fractions in the theorem below.

Theorem 93. *Let $n \geq 2$, $k \geq 0$, and*

$$\mu_n(k) := L_n \cdot [\overline{2(5k + 3), 1^{(n-2)}, 2, 1, 2k, 1, 2, 1^{(n-2)}}],$$

where the $1^{(n)}$ means that 1 has been repeated n times, and L_n is the n th Lucas number. Then $\mu_n(k)$ is an algebraic integer, and for n odd

$$\mu_n(k) = [\tilde{L}_{n+1}(10k + 6) - 1, 1, \overline{\frac{2}{5}L_{n+1}(5k + 3) - 2, 1, 2\tilde{L}_{n+1}(5k + 3) - 2}], \quad (5.3.10)$$

while for n even it is given by

$$\mu_n(k) = [\tilde{L}_{n+1}(10k + 6), \overline{\frac{2}{5}L_{n+1}(5k + 3), 2\tilde{L}_{n+1}(5k + 3)}]. \quad (5.3.11)$$

Proof. Fix $N = 5k + 3$ and let

$$y = \overline{[2N, 1^{(n-2)}, 2, 1, 2k, 1, 2, 1^{(n-2)}]}.$$

Then

$$y = [2N, 1^{(n-2)}, 2, 1, 2k, 1, 2, 1^{(n-2)}, y] = \frac{yp_n + p_{n-1}}{yq_n + q_{n-1}},$$

where $[2N, 1^{(n-2)}, 2, 1, 2k, 1, 2, 1^{(n-2)}, y] = \frac{p_n}{q_n}$. By the correspondence (5.2.10),

$$\begin{aligned} & \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{(n-2)} \\ &= \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} & F_{n-2} \\ F_{n-2} & F_{n-3} \end{pmatrix} \begin{pmatrix} 6(3k+2) & 6k+5 \\ 6k+5 & 2k+2 \end{pmatrix} \begin{pmatrix} F_{n-1} & F_{n-2} \\ F_{n-2} & F_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}, \end{aligned}$$

with p_n , p_{n-1} , q_n and q_{n-1} given by

$$\begin{aligned} p_n &= \frac{4}{5}N^2L_n^2 + \frac{6}{5}NL_nL_{n-1} + \frac{1}{5}(L_n^2 - L_nL_{n-1} + L_{n-1}^2), \\ p_{n-1} &= \frac{4}{5}N^2L_nL_{n-1} + \frac{2}{5}N(L_n^2 - L_nL_{n-1}) + \frac{2}{5}L_nL_{n-1} + (4k+2)L_{n-1}^2, \\ &= \frac{2}{5}(2N-1)L_{n-1}\tilde{L}_{n+1}(N) + \frac{2}{5}L_n\tilde{L}_{n+1}(N), \\ &= \frac{2}{5}\tilde{L}_{n+1}(N)(2NL_{n-1} - L_{n-1} + L_n), \\ &= \frac{2}{5}\tilde{L}_{n+1}(N)\tilde{L}_n(2N), \\ q_n &= \frac{2}{5}NL_n^2 + \frac{2}{5}L_nL_{n-1} \\ &= \frac{2}{5}L_n\tilde{L}_{n+1}(N), \\ q_{n-1} &= (2k+1)L_nL_{n-1} + \frac{1}{5}(L_n^2 + L_{n-1}^2). \end{aligned}$$

In evaluating the matrix multiplication, we used the identities (5.3.1), (5.3.6) and (5.3.7). Clearly, y is the largest root of the quadratic

$$y^2q_n + (q_{n-1} - p_n)y - p_{n-1}.$$

Here,

$$q_{n-1} - p_n = -\frac{4}{5}NL_n\tilde{L}_{n+1}(N),$$

and so the quadratic can also be written as

$$\frac{2}{5}\tilde{L}_{n+1}(N) \left(y^2 L_n - 2NL_n y - \tilde{L}_n(2N) \right).$$

The largest root of the quadratic is given by

$$y = \frac{1}{L_n} \left(NL_n + \sqrt{N^2 L_n^2 + L_n \tilde{L}_n(2N)} \right),$$

from which it is clear that y and $\mu_n(k) = L_n y$ are algebraic integers. The continued fraction of $\mu_n(k) = L_n y$ follows from theorem 92. $\square$

5.4 Generalizations

We now examine the continued fraction of $G_k(N, x)$ defined by

$$G_k(N, x) := F_k(x) \cdot \overline{[2N, x^{(k-1)}]}, \quad (5.4.1)$$

where $x \geq 1$, $x^{(k)}$ means the word $x, x, \dots, x$ is repeated k times and N is a non-zero positive integer.

Let $\tilde{G}_k(N, x) = \overline{[2N, x^{(k-1)}]}$, so that

$$\tilde{G}_k(N, x) = [2N, x^{(k-1)}, \tilde{G}_k(N, x)] = \frac{\tilde{G}_k(N, x)p_k(x) + p_{k-1}(x)}{\tilde{G}_k(N, x)q_k(x) + q_{k-1}(x)},$$

where $[2N, x^{(k-1)}] = \frac{p_k(x)}{q_k(x)}$. It can easily be shown by induction on k that

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1}(x) & F_k(x) \\ F_k(x) & F_{k-1}(x) \end{pmatrix},$$

from which we deduce that

$$\begin{aligned} \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^{k-1} &= \begin{pmatrix} 2N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k(x) & F_{k-1}(x) \\ F_{k-1}(x) & F_{k-2}(x) \end{pmatrix} \\ &= \begin{pmatrix} 2NF_k(x) + F_{k-1}(x) & 2NF_{k-1}(x) + F_{k-2}(x) \\ F_k(x) & F_{k-1}(x) \end{pmatrix}. \end{aligned}$$

This implies that

$$[2N, x^{(k-1)}] = \frac{2NF_k(x) + F_{k-1}(x)}{F_k(x)},$$

and

$$\tilde{G}_k(N, x) = \frac{\tilde{G}_k(N, x) (2NF_k(x) + F_{k-1}(x)) + 2NF_{k-1}(x) + F_{k-2}(x)}{\tilde{G}_k(N, x)F_k(x) + F_{k-1}(x)}.$$

Clearly, $\tilde{G}_k(N, x)$ is the largest root of the quadratic

$$F_k(x)z^2 - 2NF_k(x)z - (2NF_{k-1}(x) + F_{k-2}(x))$$

and is given by

$$\tilde{G}_k(N, x) = N + \sqrt{N^2 + (2NF_{k-1}(x) + F_{k-2}(x))/F_k(x)}.$$

We can now write

$$G_k(N, x) = NF_k(x) + \sqrt{N^2 F_k^2(x) + F_k(x)(2NF_{k-1}(x) + F_{k-2}(x))}. \quad (5.4.2)$$

As we will see below, the continued fraction of $G_k(N, x)$ depends on the parity of k because of the identity

$$F_{k-1}^2(x) - F_k(x)F_{k-2}(x) = (-1)^k. \quad (5.4.3)$$

To simplify the notation, let

$$\beta_k(N, x) := N^2 F_k^2(x) + F_k(x)(2NF_{k-1}(x) + F_{k-2}(x)),$$

so that $G_k(N, x) = NF_k(x) + \sqrt{\beta_k(N, x)}$. We first examine the continued fraction of $G_k(N, x)$ for k odd. For all $k \geq 1$ and $x \geq 1$, we have by (5.4.3)

that $\beta_k(N, x) = (NF_k(x) + F_{k-1}(x))^2 + 1$ so that

$$G_k(N, x) - (2NF_k(x) + F_{k-1}(x)) = \sqrt{\beta_k(N, x)} - (NF_k(x) + F_{k-1}(x)) > 0$$

and

$$\begin{aligned} & 1/(G_k(N, x) - (2NF_k(x) + F_{k-1}(x))) \\ &= \frac{NF_k(x) + F_{k-1}(x) + \sqrt{\beta_k(N, x)}}{-(F_{k-1}^2(x) - F_k(x)F_{k-2}(x))} \\ &= NF_k(x) + F_{k-1}(x) + \sqrt{\beta_k(N, x)} > 0. \end{aligned}$$

If $G_k^{(1)}(N, x) := NF_k(x) + F_{k-1}(x) + \sqrt{\beta_k(N, x)}$, then

$$G_k^{(1)}(N, x) - (2NF_k(x) + 2F_{k-1}(x)) = \sqrt{\beta_k(N, x)} - (NF_k(x) + F_{k-1}(x)),$$

and from the above calculations,

$$1/(G_k^{(1)}(N, x) - (2NF_k(x) + 2F_{k-1}(x))) = G_k^{(1)}(N, x).$$

Hence the continued fraction of $G_k^{(1)}(N, x)$ is purely periodic with period of length 1 given by $\{2NF_k(x) + 2F_{k-1}(x)\}$. For all $k > 1$ odd, the continued fraction of $G_k(N, x)$ is given by,

$$G_k(N, x) = [2NF_k(x) + F_{k-1}(x), \overline{2NF_k(x) + 2F_{k-1}(x)}]. \quad (5.4.4)$$

Now for the case of $k > 1$ even. From (5.4.3) we have $\beta_k(N, x) = (NF_k(x) + F_{k-1}(x))^2 - 1$, so that

$$G_k(N, x) - (2NF_k(x) + F_{k-1}(x) - 1) = \sqrt{\beta_k(N, x)} - (NF_k(x) + F_{k-1}(x)) + 1 > 0.$$

If $G_k^{(2)}(N, x) := \sqrt{\beta_k(N, x)} - (NF_k(x) + F_{k-1}(x)) + 1$, then

$$\begin{aligned}
1/G_k^{(2)}(N, x) &= 1 + \frac{NF_k(x) + F_{k-1}(x) - \sqrt{\beta_k(N, x)}}{G_k^{(2)}(N, x)} \\
&= 1 + \frac{F_{k-1}^2(x) - F_k(x)F_{k-2}(x)}{NF_k(x) + F_{k-1}(x) + \sqrt{\beta_k(N, x)} - (F_{k-1}^2(x) - F_k(x)F_{k-2}(x))} \\
&= 1 + \frac{1}{NF_k(x) + F_{k-1}(x) + \sqrt{\beta_k(N, x)} - 1} \\
&= 1 + \frac{1}{2NF_k(x) + 2F_{k-1}(x) - 2 + G_k^{(2)}(N, x)}.
\end{aligned}$$

We can now see that $1/G_k^{(2)}(N, x)$ is purely periodic with a period of length 2 given by $\{1, 2NF_k(x) + 2F_{k-1}(x) - 2\}$. For all $k > 1$ even, the continued fraction of $G_k(N, x)$ is given by,

$$G_k(N, x) = [2NF_k(x) + F_{k-1}(x) - 1, \overline{1, 2NF_k(x) + 2F_{k-1}(x) - 2}]. \quad (5.4.5)$$

We have proved the following theorem.

Theorem 94. *For all $k \geq 1$ and $x \geq 1$, the product of the k th Fibonacci polynomial $F_k(x)$ with the periodic continued fraction $\overline{[2N, x^{(k-1)}]}$ gives the periodic continued fraction*

$$[2NF_k(x) + F_{k-1}(x), \overline{2NF_k(x) + 2F_{k-1}(x)}]$$

for k odd and

$$[2NF_k(x) + F_{k-1}(x) - 1, \overline{1, 2NF_k(x) + 2F_{k-1}(x) - 2}]$$

for k even.

If we set $x = 1$ we get the result of theorem 97.

5.5 Polynomials from convergents of $\overline{[N, x^{(k)}]}$

In this section, we show how polynomials arising from the convergents of $\overline{[N, x^{(k)}]}$ are related to Chebyshev and Fibonacci polynomials. We describe

the polynomials for $k = 1, 2$ and 3 and show how the roots of these polynomials are distributed.

First, for a fixed $k = 1$, let $p_n(N, x)/q_n(N, x)$ be the convergents of $[\overline{N, x}]$. When $n \equiv 0 \pmod{2}$ and $n \equiv 1 \pmod{2}$, $q_{2n}(N, x)$ and $q_{2n+1}(N, x)$ are respectively generated by the rational functions

$$\frac{1-t}{1-(Nx+2)t+t^2} \quad \text{and} \quad \frac{x}{1-(Nx+2)t+t^2}. \quad (5.5.1)$$

Comparing these generating functions to the generating function of Chebyshev polynomials of the second kind below

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \quad (5.5.2)$$

we have

$$q_{2n}(N, x) = U_n\left(\frac{1}{2}Nx+1\right) - U_{n-1}\left(\frac{1}{2}Nx+1\right) \quad \text{and} \quad q_{2n+1}(N, x) = xU_n\left(\frac{1}{2}Nx+1\right).$$

To determine the roots of $q_k(x)$, first note that $U_n(x)$ has all its roots in the interval $(-1, 1)$ given by $x_k = \cos\left(\frac{k}{n+1}\pi\right)$ (see for example [7, Section 2.2]). It follows that $q_{2n+1}(N, x)$ has real roots in the interval $(-4/N, 0]$ given by $x_k = \frac{2}{N}\left(\cos\left(\frac{k}{n+1}\pi\right) - 1\right)$. It can also be shown that $q_{2n}(N, x)$ has $n-1$ real roots in the interval $(-4/N, 0]$ and one real root outside this interval.

Now for the case when $k = 2$, let $p_n(N, x)/q_n(N, x)$ be the convergents of $[\overline{N, x, x}]$. When $n \equiv 0 \pmod{3}$ and $n \equiv 1 \pmod{3}$, $q_{3n}(N, x)$ and $q_{3n+1}(N, x)$ are respectively generated by the rational functions

$$\frac{x+t}{1-(Nx^2+2x+N)t-t^2} \quad \text{and} \quad \frac{x}{1-((Nx^2+2x+N))t-t^2}, \quad (5.5.3)$$

and their zeros seem to lie close to the hyperbola $y^2 - x^2 = 1 - 1/N^2$.

Let $p_n(N, x)/q_n(N, x)$ be the convergents of $[\overline{N, x, x}]$. When $n \equiv 2 \pmod{3}$, $q_n(N, x)$ are generated by the rational function

$$\frac{1+x^2}{1-(Nx^2+2x+N)t-t^2}$$

and have a factor of $x^2 + 1$. By eliminating this factor and making a change of variable $x \mapsto x - 1/N$ we obtain (after clearing denominators) polynomials

$Q_n(N, x)$ that are generated by

$$\frac{1}{1 - (N^2x^2 + N^2 - 1)t - N^2t^2} = \sum_{n=0}^{\infty} Q_n(N, x)t^n. \quad (5.5.4)$$

Comparing this generating function to that of the Fibonacci polynomials (5.0.1), we have

$$Q_k(N, x) = N^k F_k(Nx^2 + N + 1/N).$$

Theorem 95. *For any non-zero $k \in \mathbb{R}$, let $Q_n(N, x)$ be as defined in equation (5.5.4). Then for all $n \geq 1$, all the zeros $Q_n(N, x)$ lie on the hyperbola*

$$H1: \quad y^2 - x^2 = \frac{N^2 - 1}{N^2}.$$

Proof. Bicknell and Hoggatt proved in [32] that if $F_n(x) = 0$ then $x = 2i \cos(j\pi/n)$ for $j = 1, 2, \dots, n-1$. Let

$$z = \frac{\sqrt{N^2 - 1}}{N} (\sinh \phi + i \cosh \phi)$$

where $0 < N$. Then

$$\begin{aligned} Nz^2 + N - 1/N &= (N - 1/N)(2i \sinh \phi \cosh \phi - 1) + N - 1/N \\ &= (N - 1/N)i \sinh 2\phi. \end{aligned}$$

Now $Q_n(N, z) = 0$ implies that

$$\sinh 2\phi = \frac{2N}{N^2 - 1} \cos \theta_j$$

where $\theta_j = j\pi/(n+1)$ for $j = 1, 2, \dots, n-1$. Using the identity

$$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1}), \quad -\infty < x < \infty, \quad (5.5.5)$$

in which we consider the principal branch of the log, we obtain

$$\phi_j = \frac{1}{2} \log \left| 2N \cos \theta_j + \sqrt{4N^2 \cos^2 \theta_j + (N^2 - 1)^2} \right| - \frac{1}{2} \log |N^2 - 1|. \quad (5.5.6)$$

The $2n$ zeros of $Q_n(N, z)$ are now given by

$$z_j = \pm \sqrt{(1 - 1/N^2)}(\sinh \phi_j + i \cosh \phi_j) \quad (5.5.7)$$

for $j = 1, 2, \dots, n-1$ where ϕ_j is given by (5.5.6). It is straightforward to check that all these points lie on $H1$. $\square$

Numerical calculations suggest that the polynomials generated by the rational functions (5.5.4) have their roots close to $H1$.

These polynomials arising from the convergents of $[\overline{N, x^{(k)}}]$ seem to get more complicated as k gets larger. For example, let $k = 3$, and $p_n(N, x)/q_n(N, x)$ be the convergents of $[\overline{N, x, x, x}]$. Then $q_{4n+3}(N, x)$ are generated by

$$\frac{x(x^2 + 2)}{1 - (Nx^3 + 2x^2 + 2Nx + 2)t + t^2}.$$

Now for $k = 4$, $q_{5n+4}(N, x)$ are generated by

$$\frac{x^4 + 3x^2 + 1}{1 - (Nx^4 + 2x^3 + 3Nx^2 + 4x + N)t - t^2}.$$

In general, for a fixed k , let $q_n(N, x)$ be the denominator of the convergents of $[\overline{N, x^{(k)}}]$ when $n \equiv -1 \pmod{(k+1)}$. To simplify the notation, call them $Q_m(N, x)$. Then $Q_m(N, x)$ are generated by

$$\frac{F_{k+1}(x)}{1 - g_k(N, x)t - (-1)^k t^2} = \sum_{m=0}^{\infty} Q_m(N, x)t^m, \quad (5.5.8)$$

where $g_k(N, x)$ are defined by the generating function

$$\frac{N + 2t}{1 - xt - t^2} = \sum_{k=0}^{\infty} g_k(N, x)t^k. \quad (5.5.9)$$

By (5.0.1),

$$g_k(N, x) = NF_{k+1}(x) + 2F_k(x).$$

We can apply a theorem of Khang Tran (see [1, Theorem 1]) to determine the curve on which the roots of $Q_m(N, x)$ lie. As k increases however, so does the difficulty in describing this curve. As an example, for $k = 4$ the roots of

$Q_m(3, x)/F_5(x)$ lie on the curve given by

$$3x^4 - 18x^2y^2 + 3y^4 + 2x^3 - 6xy^2 + 9x^2 - 9y^2 + 4x + 3 = 0.$$

The relationship to Fibonacci and Chebyshev polynomials of the second kind follows from the generating function (5.5.8). For a fixed k that is even,

$$Q_n(N, x) = F_{k+1}(x)F_n(g_k(N, x)),$$

while for a fixed k that is odd,

$$Q_n(N, x) = F_{k+1}(x)U_n(g_k(N, x)).$$

5.6 Conclusion

In conclusion, we pose a question. Are there any other integer sequences f_n such that $\tilde{f}_{n+1}(m) := mf_n + f_{n-1}$, for which the continued fraction of

$$\frac{1}{f_n} \left(Nf_n + \sqrt{N^2f_n^2 + f_n\tilde{f}_n(2N)} \right)$$

is purely periodic with length of the period depending on n , and the continued fraction of

$$Nf_n + \sqrt{N^2f_n^2 + f_n\tilde{f}_n(2N)}$$

is periodic with a fixed period length independent of n ?

CHAPTER 6

NON-PERIODIC CONTINUED FRACTIONS FOR QUADRATIC IRRATIONALITIES

Consider the continued fraction

$$[1, 10^6, 3, \frac{10^6 - 1}{3}, 9, \frac{10^6 - 1}{9}, 27, \frac{10^6 - 1}{27}, 81, 12345, 1, 2, 26, 1, 2, 4114, 1, 8, \dots] \quad (6.0.1)$$

of the largest root (in absolute value) of the quadratic

$$P(a, x) = x^2 - 3a(x + 1) - 3x + 5$$

with $a = 10^6$.

In the language of van der Poorten and Shallit [33], we call a partial quotient (other than ‘the first’ which may be zero) ‘inadmissible’ if it is zero, negative or a fraction. Hence all partial quotients of a regular continued fraction expansion of a real number are admissible. From the initial pattern in (6.0.1) we are led to an elegant continued fraction,

$$[1, 10^6, 3, \frac{10^6 - 1}{3}, 3^2, \frac{10^6 - 1}{3^2}, 3^3, \frac{10^6 - 1}{3^3}, 3^4, \frac{10^6 - 1}{3^4}, \dots, 3^k, \frac{10^6 - 1}{3^k}, \dots] \quad (6.0.2)$$

which is non-simple and has partial quotients that are inadmissible from some point on. From computations in Mathematica it seems to converge to the same real number as (6.0.1). Consider more generally, for nonzero positive integers x and s , the continued fraction

$$F_k(x, s) := [x, \frac{s}{x}, x^2, \frac{s}{x^2}, x^3, \frac{s}{x^3}, x^4, \frac{s}{x^4}, \dots, x^k, \frac{s}{x^k}]. \quad (6.0.3)$$

In this chapter, we will prove the following theorem.

Theorem 96. *The continued fraction $F_k(x, s)$ converges to the largest root of the polynomial $P(x, z, s) = sz^2 - ((s + 1)x - 1)z - x$.*

Of special interest is the rate at which $P(x, F_k(x, s), s)$ approaches zero as

$k \rightarrow \infty$; see equations (6.2.11) and (6.2.12) that occur in the proof of Theorem 108. Additional results on (6.0.3) are presented in Section 6.3. We shall however begin with the case $s = 1$,

$$F(x) := [x, \frac{1}{x}, x^2, \frac{1}{x^2}, x^3, \frac{1}{x^3}, x^4, \frac{1}{x^4}, \dots, x^k, \frac{1}{x^k}, \dots]. \quad (6.0.4)$$

Results of this chapter are contained in the paper [34].

6.1 Convergent and generating functions

This section investigates the convergents of (6.0.4), which give two sequences $(A_j(x))$ and $(B_j(x))$ of polynomials with positive integer coefficients. The recurrence relations and generating functions of these sequences are studied. It is rather difficult to prove results on (6.0.4) in its given form, but it becomes easier when we transform it into an ‘equivalent’ continued fraction of the form;

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots + \frac{a_j}{b_j + \dots}}} \quad (6.1.1)$$

We use the familiar notation

$$K\left(\frac{a_j}{b_j}\right) = K_{j=1}^{\infty}\left(\frac{a_j}{b_j}\right) := \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots + \frac{a_j}{b_j + \dots}}}.$$

Two continued fractions are said to be **equivalent** i.e. $K(\frac{a_j}{b_j}) \sim K(\frac{c_j}{d_j})$, if they have the same sequence of classical approximants (or convergents) ([35], pp. 77). It is straightforward to verify that $\sim$ is indeed an equivalence relation.

Theorem 97. $K\left(\frac{a_j}{b_j}\right) \sim K\left(\frac{c_j}{d_j}\right)$ if and only if there exists a sequence $\{r_j\}$ of complex numbers with $r_0 = 1$ and $r_j \neq 0$ for all $j \in \mathbb{N}$, such that,

$$c_j = r_{j-1}r_j a_j, \quad d_j = r_j b_j \quad \text{for all } j \in \mathbb{N}$$

Theorem 98. If $F(x)$ is defined by (6.0.4), then $F(x)$ is equivalent to $\tilde{F}(x)$ defined by

$$\tilde{F}(x) := x + \frac{x}{1+x} \frac{1}{1+\frac{x}{1+\dots+\frac{1}{x+1}+\dots}}. \quad (6.1.2)$$

j	A_j	B_j
0	x	1
1	$2x$	1
2	$x(2x + 1)$	$x + 1$
3	$x(4x + 1)$	$2x + 1$
4	$x(4x^2 + 3x + 1)$	$2x^2 + 2x + 1$
5	$x(8x^2 + 4x + 1)$	$4x^2 + 3x + 1$
6	$x(8x^3 + 8x^2 + 4x + 1)$	$4x^3 + 5x^2 + 3x + 1$
7	$x(16x^3 + 12x^2 + 5x + 1)$	$8x^3 + 8x^2 + 4x + 1$
8	$x(16x^4 + 20x^3 + 13x^2 + 5x + 1)$	$8x^4 + 12x^3 + 9x^2 + 4x + 1$
9	$x(32x^4 + 32x^3 + 18x^2 + 6x + 1)$	$16x^4 + 20x^3 + 13x^2 + 5x + 1$
10	$x(32x^5 + 48x^4 + 38x^3 + 19x^2 + 6x + 1)$	$16x^5 + 28x^4 + 25x^3 + 14x^2 + 5x + 1$

Table 6.1: First 11 polynomials for A_j and B_j .

Proof. In the notation of (6.1.1), $F(x)$ is expressed as

$$F(x) := x + \frac{1}{1/x + x^2} + \frac{1}{1/x^2 + \dots} + \frac{1}{x^k + 1/x^k} + \dots$$

Let $K\left(\frac{a_j}{b_j}\right) = F(x) - x$ and apply Theorem 97 with the sequence $\{r_j\}$ defined as

$$r_j := \begin{cases} x^{\frac{j+1}{2}} & \text{for } j \text{ odd,} \\ x^{-\frac{j}{2}} & \text{for } j \text{ even.} \end{cases} \quad (6.1.3)$$

□

From the relation $\frac{A_{j+1}}{B_{j+1}} = \frac{b_{j+1}A_j + a_{j+1}A_{j-1}}{b_{j+1}B_j + a_{j+1}B_{j-1}}$ of the convergents $\frac{A_j}{B_j}$ of (6.1.1), we have the recurrence relations

$$\begin{aligned} A_{j+1}(x) &= b_{j+1}A_j(x) + a_{j+1}A_{j-1}(x), \\ B_{j+1}(x) &= b_{j+1}B_j(x) + a_{j+1}B_{j-1}(x). \end{aligned} \quad (6.1.4)$$

This will be useful in proving a similar result for (6.0.4). The first 11 terms for $A_j(x)$ and $B_j(x)$ are given in Table 6.1.

Theorem 99. *The polynomials $A_j(x)$ and $B_j(x)$ from the convergents $\frac{A_j(x)}{B_j(x)}$ of (6.1.2) are given by the recurrence relations;*

$$\begin{aligned} A_j(x) &= (2x + 1)A_{j-2}(x) - xA_{j-4}(x) \\ B_j(x) &= (2x + 1)B_{j-2}(x) - xB_{j-4}(x) \end{aligned} \quad (6.1.5)$$

Proof. To simplify the notation, set we will use $A_j := A_j(x)$ and $B_j := B_j(x)$. From (6.1.2) first observe that for j odd, $a_j = x$ and $b_j = 1$, and for j even, $a_j = 1$ and $b_j = x$. Let $\frac{A_j}{B_j}$ be the convergents of $\tilde{F}(x)$. The A_j and B_j have initial conditions $A_0 = x$, $A_1 = 2x$, $A_2 = x(2x + 1)$, $B_0 = B_1 = 1$ and $B_2 = x + 1$. Also for $j = 4$, (6.1.5) holds. Suppose that for all $i \leq j$ (6.1.5) is true. Then for j even,

$$\begin{aligned} \frac{A_{j+2}}{B_{j+2}} := \tilde{z}_{j+2}(x) &= x + \frac{x}{1+x} \frac{1}{1+x} \frac{x}{1+x} \frac{1}{1+x} \frac{x^2}{1+x} \\ &= \frac{(x+1)A_j + x^2A_{j-1}}{(x+1)B_j + x^2B_{j-1}}. \end{aligned}$$

Since j is even, by (6.1.4), $A_j = xA_{j-1} + A_{j-2}$ and $B_j = xB_{j-1} + B_{j-2}$, so that,

$$\begin{aligned} A_{j+2} &= (x+1)A_j + x(A_j - A_{j-2}) \\ &= (2x+1)A_j - xA_{j-2}, \end{aligned}$$

and

$$\begin{aligned} B_{j+2} &= (x+1)B_j + x(B_j - B_{j-2}) \\ &= (2x+1)B_j - xB_{j-2}. \end{aligned}$$

Similarly for j odd,

$$\begin{aligned} \frac{A_{j+2}}{B_{j+2}} := \tilde{z}_{j+2}(x) &= x + \frac{x}{1+x} \frac{1}{1+x} \frac{x}{1+x} \frac{x}{1+x} \frac{1}{1+x} \\ &= \frac{2xA_j + A_{j-1}}{2xB_j + B_{j-1}}, \end{aligned}$$

and by (6.1.4), $A_j = A_{j-1} + xA_{j-2}$ and $B_j = B_{j-1} + xB_{j-2}$. Thus

$$\begin{aligned} A_{j+2} &= 2xA_j + (A_j - xA_{j-2}) \\ &= (2x+1)A_j - xA_{j-2}, \end{aligned}$$

and

$$\begin{aligned} B_{j+2} &= 2xB_j + (B_j - xB_{j-2}) \\ &= (2x+1)B_j - xB_{j-2}. \end{aligned}$$

We have the same recurrence for j odd and j even. Hence by induction the recurrence holds for all j . This is the recurrence for $F(x)$ by the equivalence $F(x) \sim \tilde{F}(x)$. $\square$

It is clear from the initial conditions and from the recurrence relations (6.1.5) that the A_j 's and B_j 's are polynomials in x with positive integer coefficients. We now give their generating functions.

Proposition 100. *The polynomials A_{2j} and A_{2j+1} are respectively generated by*

$$\frac{x}{1 - (2x + 1)t + xt^2} = \sum_{j=0}^{\infty} A_{2j}(x)t^j, \quad (6.1.6)$$

$$\frac{x(2 - t)}{1 - (2x + 1)t + xt^2} = \sum_{j=0}^{\infty} A_{2j+1}(x)t^j, \quad (6.1.7)$$

while the polynomials B_{2j} and B_{2j+1} are generated by

$$\frac{1}{1 - (2x + 1)t + xt^2} = \sum_{j=0}^{\infty} B_{2j+1}(x)t^j, \quad (6.1.8)$$

$$\frac{(1 - xt)}{1 - (2x + 1)t + xt^2} = \sum_{j=0}^{\infty} B_{2j}(x)t^j. \quad (6.1.9)$$

Proof. From equation (6.1.5)

$$\begin{aligned} \sum_{j=2}^{\infty} A_{2j}t^j &= \sum_{j=2}^{\infty} ((2x + 1)A_{2j-2} - xA_{2j-4})t^j \\ &= (2x + 1)x \sum_{j=2}^{\infty} A_{2j-2}t^{j-1} - xt^2 \sum_{j=2}^{\infty} A_{2j-4}t^{j-2} \\ &= (2x + 1)x \sum_{j=1}^{\infty} A_{2j-2}t^{j-1} - xt^2 \sum_{j=2}^{\infty} A_{2j-4}t^{j-2} - (2x + 1)tA_0. \end{aligned}$$

Hence

$$\begin{aligned} (1 - (2x + 1)t + xt^2) \sum_{j=0}^{\infty} A_{2j}t^j &= -(2x + 1)tA_0 + A_0 + tA_2. \\ &= x. \end{aligned}$$

The proofs of (6.1.7), (6.1.8) and (6.1.9) follow the same pattern as above. $\square$

From the generating functions, we can observe relationships between the polynomials A_j and B_j . Here are some notable ones.

Corollary 101. *For all $j \geq 1$,*

$$A_{2j+1} = 2A_{2j} - A_{2j-2}, \quad (6.1.10)$$

$$B_{2j+1} = \frac{1}{x}A_{2j}, \quad (6.1.11)$$

$$B_{2j} = \frac{1}{x}A_{2j} - A_{2j-2}, \quad (6.1.12)$$

$$= B_{2j+1} - xB_{2j-1}. \quad (6.1.13)$$

There are other relationships of interest between the polynomials B_j 's and $\frac{1}{x}A_j$'s. The following lemma is an example.

Lemma 102. *Let the polynomials A_j and B_j be as defined in Theorem 99. Then*

$$B_j = \frac{1}{2x}A_j + \frac{1}{2}B_{j-2}. \quad (6.1.14)$$

Proof. Proceed by induction on j with base case $j = 2$ using the initial conditions for A_j and B_j as well as their recurrence relations. For $j = 2$, $x + 1 = \frac{1}{2x}A_2 + \frac{1}{2}B_0 = B_2$. Suppose for all $i \leq j$, (6.1.14) is true. Then

$$\begin{aligned} B_{j+1} &= (2x + 1)B_{j-1} - xB_{j-3} \\ &= (2x + 1) \left(\frac{1}{2x}A_{j-1} + \frac{1}{2}B_{j-3} \right) - x \left(\frac{1}{2x}A_{j-3} + \frac{1}{2}B_{j-5} \right) \\ &= \frac{1}{2x}((2x + 1)A_{j-1} - xA_{j-3}) - \frac{1}{2}((2x + 1)B_{j-3} - xB_{j-5}) \\ &= \frac{1}{2x}A_{j+1} + \frac{1}{2}B_{j-1}. \end{aligned}$$

□

We use the above lemma to show that the polynomials B_j can be expressed in terms of the polynomials A_j .

Theorem 103. *Let the polynomials A_j and B_j be as defined in Theorem 99.*

Then

$$B_{2k} = \frac{A_0}{2^k x} + \frac{1}{x} \sum_{j=1}^k \frac{A_{2j}}{2^{k-j+1}}, \quad (6.1.15)$$

$$B_{2k+1} = \frac{1}{x} \sum_{j=0}^k \frac{A_{2j+1}}{2^{k-j+1}}. \quad (6.1.16)$$

Proof. From the initial conditions, $\frac{A_0}{x} = B_0$ and $\frac{A_0}{2x} + \frac{A_2}{2x} = B_2$. Suppose for $i \leq j$, (6.1.15) is true. Then by (6.1.14)

$$\begin{aligned} B_{2k+2} &= \frac{1}{2x} A_{2k+2} + \frac{1}{2} B_{2k} \\ &= \frac{1}{2x} A_{2k+2} + \frac{1}{2} \left(\frac{A_0}{2^k x} + \frac{1}{x} \sum_{j=1}^k \frac{A_{2j}}{2^{k-j+1}} \right) \\ &= \frac{A_0}{2^{k+1} x} + \frac{1}{x} \left(\frac{A_{2k+2}}{2} + \sum_{j=1}^k \frac{A_{2j}}{2^{k-j+2}} \right) \\ &= \frac{A_0}{2^{k+1} x} + \frac{1}{x} \sum_{j=1}^{k+1} \frac{A_{2j}}{2^{k-j+2}}. \end{aligned}$$

Similarly, $\frac{A_1}{2x} = B_1$ and suppose for $i \leq j$, (6.1.16) is true. Then by (6.1.14)

$$\begin{aligned} B_{2k+3} &= \frac{1}{2x} A_{2k+3} + \frac{1}{2} B_{2k+1} \\ &= \frac{1}{2x} A_{2k+3} + \frac{1}{x} \sum_{j=0}^k \frac{A_{2j+1}}{2^{k-j+2}} \\ &= \frac{1}{x} \sum_{j=0}^{k+1} \frac{A_{2j+1}}{2^{k-j+2}}, \end{aligned}$$

so by induction on j , (6.1.15) and (6.1.16) are true for all j . $\square$

6.2 Convergence of $F_k(x)$

In this section we prove that $F_k(x)$ converges to a quadratic irrational. We present two different proofs for the convergence. Since $\tilde{F}(x)$ is periodic, if it converges, the proof of its convergence is straightforward. By the equivalence proved in Theorem 98, $F(x)$ will converge to the same limit.

Proposition 104. *If $\tilde{F}(x)$ converges, then the limit is a root of the quadratic $P(x, z) = z^2 - (2x - 1)z - x$.*

Proof. Suppose $\tilde{F}(x)$ converge to z . Then,

$$\begin{aligned} z &= x + \frac{x}{1+x} + \frac{1}{1+x} + \frac{x}{1+x} + \frac{1}{1+x} + \dots \\ &= x + \frac{x}{1+z}, \end{aligned}$$

from which we deduce that $z^2 + (1 - 2x)z - x = 0$. $\square$

Let α be an irrational number with a simple continued fraction

$$\alpha = [a_0, a_1, a_2, \dots].$$

We know by Lagrange (see [13, p. 48]) that the continued fraction is periodic and does not terminate. Let p_n/q_n be the n th convergent of the continued fraction for α . Call p_n/q_n an even convergent if n is even, and an odd convergent if n is odd. By Theorem 7.8 of [26, p. 193],

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \dots < \alpha < \dots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Even convergents are strictly increasing while odd convergent are strictly decreasing.

We now prove the convergence of $F_k(x)$ by solving the recurrences (6.1.5) by the method of characteristic roots (see for example [36, p. 300]). In particular, we treat even and odd convergents of $F_k(x)$ separately.

Theorem 105. *Let $F_k(x)$ be defined for $k \geq 1$ by*

$$F_k(x) := [x, \frac{1}{x}, x^2, \frac{1}{x^2}, x^3, \frac{1}{x^3}, x^4, \frac{1}{x^4}, \dots, x^k, \frac{1}{x^k}]. \quad (6.2.1)$$

Then $F_k(x)$ converges to a quadratic irrational that is the positive root of the polynomial $P(x, z) = z^2 - (2x - 1)z - x$.

Proof. Since odd convergents are increasing and even convergents are decreasing, they will be treated separately. Let $a_k = A_{2k}$. Then from (6.1.5) we deduce the recurrence $a_k = (2x + 1)a_{k-1} - xa_{k-2}$ with characteristic polynomial $\lambda^2 - (2x + 1)\lambda + x = 0$. The roots of this polynomial are

$\lambda_1 = \frac{1}{2}(2x + 1 + \sqrt{4x^2 + 1})$ and $\lambda_2 = \frac{1}{2}(2x + 1 - \sqrt{4x^2 + 1})$, both positive with $\lambda_1 > \lambda_2$ and $a_k = \alpha\lambda_1^k + \beta\lambda_2^k$.

From the initial conditions $A_0 = x$ and $A_2 = 2x^2 + x$, we obtain the values of the constants $\alpha = \frac{\lambda_2 x - 2x^2 - x}{-(\lambda_1 - \lambda_2)}$ and $\beta = \frac{\lambda_1 x - 2x^2 - x}{\lambda_1 - \lambda_2}$ so that

$$a_k = \frac{\lambda_2 x - 2x^2 - x}{-(\lambda_1 - \lambda_2)} \lambda_1^k + \frac{\lambda_1 x - 2x^2 - x}{\lambda_1 - \lambda_2} \lambda_2^k.$$

Similarly let $b_k = B_{2k}$. Then from (6.1.5) and the fact that the recurrences are the same, $b_k = \alpha'\lambda_1^k + \beta'\lambda_2^k$. From the initial conditions $B_0 = 1$ and $B_2 = x + 1$, we find that $\alpha' = \frac{\lambda_2 - x - 1}{-(\lambda_1 - \lambda_2)}$ and $\beta' = \frac{\lambda_1 - x - 1}{\lambda_1 - \lambda_2}$ so that

$$b_k = \frac{\lambda_2 - x - 1}{-(\lambda_1 - \lambda_2)} \lambda_1^k + \frac{\lambda_1 - x - 1}{\lambda_1 - \lambda_2} \lambda_2^k,$$

and

$$\begin{aligned} \frac{a_k}{b_k} &= \frac{(\lambda_2 x - 2x^2 - x)\lambda_1^k - (\lambda_1 x - 2x^2 - x)\lambda_2^k}{(\lambda_2 - x - 1)\lambda_1^k - (\lambda_1 - x - 1)\lambda_2^k} \\ &= \frac{(\lambda_2 x - 2x^2 - x) - (\lambda_1 x - 2x^2 - x)(\lambda_2/\lambda_1)^k}{(\lambda_2 - x - 1) - (\lambda_1 - x - 1)(\lambda_2/\lambda_1)^k}, \end{aligned}$$

and recalling that $\lambda_2 < \lambda_1$, we deduce that

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{\lambda_2 x - 2x^2 - x}{\lambda_2 - x - 1}. \quad (6.2.2)$$

Substitute for λ_2 in equation (6.2.2) and rationalize the denominator to find that

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{2x - 1 + \sqrt{4x^2 + 1}}{2}.$$

On the other hand, let $a_k = A_{2k-1}$. Since we have the same recurrence relation as before, $a_k = \gamma\lambda_1^k + \delta\lambda_2^k$. From the initial conditions $A_1 = 2x$ and $A_3 = 4x^2 + x$, we find that $\gamma = \frac{2x\lambda_2 - 4x^2 - x}{(\lambda_1\lambda_2 - \lambda_1^2)}$ and $\delta = \frac{2x\lambda_1 - 4x^2 - x}{\lambda_1\lambda_2 - \lambda_2^2}$. Thus

$$a_k = \frac{2x\lambda_2 - 4x^2 - x}{\lambda_1\lambda_2 - \lambda_1^2} \lambda_1^k + \frac{2x\lambda_1 - 4x^2 - x}{\lambda_1\lambda_2 - \lambda_2^2} \lambda_2^k.$$

Similarly let $b_k = B_{2k-1}$. From (6.1.5) and the fact that the recurrences are the same, $b_k = \gamma'\lambda_1^k + \delta'\lambda_2^k$. From the initial conditions $B_1 = 1$ and

$B_3 = 2x + 1$, we find that $\gamma' = \frac{\lambda_2 - 2x - 1}{\lambda_1 \lambda_2 - \lambda_1^2}$ and $\delta' = \frac{\lambda_1 - 2x - 1}{\lambda_1 \lambda_2 - \lambda_2^2}$. Thus

$$b_k = \frac{\lambda_2 - 2x - 1}{\lambda_1 \lambda_2 - \lambda_1^2} \lambda_1^k + \frac{\lambda_1 - 2x - 1}{\lambda_1 \lambda_2 - \lambda_2^2} \lambda_2^k.$$

Hence

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{2x\lambda_2 - 4x^2 - x}{\lambda_2 - 2x - 1}. \quad (6.2.3)$$

Substitute λ_2 into (6.2.3) and rationalize the denominator to deduce that

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{2x - 1 + \sqrt{4x^2 + 1}}{2}.$$

This completes the proof. $\square$

In preparation for an inductive proof of Theorem 105, we give some results on linear combinations of products of the polynomials A_j and B_j that yield monomials.

Lemma 106. *Let $\frac{A_j}{B_j}$ be the convergents of $F(n)$. For $j \geq 1$,*

$$\begin{aligned} A_{2j-2}A_{2j+2} - A_{2j}^2 &= -x^{j+2}, \\ B_{2j-2}B_{2j+2} - B_{2j}^2 &= x^{j+1}, \end{aligned} \quad (6.2.4)$$

and for $j \geq 2$,

$$\begin{aligned} A_{2j-3}A_{2j+1} - A_{2j-1}^2 &= x^j, \\ B_{2j-3}B_{2j+1} - B_{2j-1}^2 &= -x^{j-1}. \end{aligned} \quad (6.2.5)$$

Proof. From the initial conditions $A_0 = x$, $A_2 = 2x^2 + x$ and $A_4 = 4x^3 + 3x^2 + x$, $A_0A_4 - A_2^2 = -x^3$. Suppose (6.2.4) holds for all integers up to j . Then

$$\begin{aligned} A_{2j}A_{2j+4} - A_{2j+2}^2 &= A_{2j}\{(2x+1)A_{2j+2} - xA_{2j}\} - A_{2j+2}^2 \\ &= A_{2j+2}\{(2x+1)A_{2j}\} - xA_{2j}^2 - A_{2j+2}^2 \\ &= A_{2j+2}\{A_{2j+2} + xA_{2j-2}\} - xA_{2j}^2 - A_{2j+2}^2 \\ &= xA_{2j-2}A_{2j+2} - xA_{2j}^2 \\ &= -x^{j+3}, \end{aligned}$$

where we have used the recurrence relations (6.1.5) for A_j . By induction on j , (6.2.4) is true for all j . The proofs of the other relations are identical to the above proof with the only difference being the initial conditions. $\square$

Lemma 107. *Let the polynomials A_j and B_j be as defined in Theorem 99. Then for $j \geq 0$,*

$$A_{2j}^2 - (2x - 1)A_{2j}B_{2j} - xB_{2j}^2 = -x^{j+2}, \quad (6.2.6)$$

and for $j \geq 1$,

$$A_{2j-1}^2 - (2x - 1)A_{2j-1}B_{2j-1} - xB_{2j-1}^2 = x^j. \quad (6.2.7)$$

Proof. From the initial conditions $A_0 = x$ and $B_0 = 1$, $A_0^2 - (2x - 1)A_0B_0 - xB_0^2 = -x^2$. Suppose (6.2.6) is true for all $i \leq j$. From the recurrence relations (6.1.5), we find that

$$\begin{aligned} & A_{2j+2}^2 - (2x - 1)A_{2j+2}B_{2j+2} - xB_{2j+2}^2 \\ &= [(2x + 1)A_{2j} - xA_{2j-2}]^2 - x((2x + 1)B_{2j} - xB_{2j-2})^2 \\ &\quad - (2x - 1)((2x + 1)A_{2j} - xA_{2j-2})((2x + 1)B_{2j} - xB_{2j-2}) \\ &= (2x + 1)^2(A_{2j}^2 - (2x - 1)A_{2j}B_{2j} - xB_{2j}^2) - 2x^2(2x + 1)B_{2j-2}B_{2j} \\ &\quad + x^2(A_{2j-2}^2 - (2x - 1)A_{2j-2}B_{2j-2} - xB_{2j-2}^2) \\ &\quad - (2x + 1)(2xA_{2j-2}A_{2j} - (2x - 1)(xA_{2j}B_{2j-2} + xA_{2j-2}B_{2j})) \\ &= -(2x + 1)^2x^{j+2} - x^{j+3} - (2x + 1)[2xA_{2j-2}A_{2j} \\ &\quad - (2x - 1)(xA_{2j}B_{2j-2} + xA_{2j-2}B_{2j}) - 2x^2B_{2j-2}B_{2j}]. \end{aligned} \quad (6.2.8)$$

It suffices to show that the expressions in the square brackets equals $-(2n + 1)n^{j+2}$. Using (6.1.12) and (6.2.4) we find that

$$\begin{aligned}
& 2xA_{2j-2}A_{2j} - (2x-1)(xA_{2j}B_{2j-2} + xA_{2j-2}B_{2j}) - 2x^2B_{2j-2}B_{2j} \\
&= 2xA_{2j-2}A_{2j} - (2x-1)(A_{2j}(A_{2j-2} - xA_{2j-4}) + A_{2j-2}(A_{2j} - xA_{2j-2})) \\
&\quad - 2(A_{2j-2} - xA_{2j-4})(A_{2j} - xA_{2j-2}) \\
&= -2xA_{2j-2}A_{2j} + x(2x+1)(A_{2j-2}^2 + A_{2j-4}A_{2j}) - 2x^2A_{2j-4}A_{2j-2} \\
&= -2xA_{2j-2}(A_{2j} + xA_{2j-4}) + x(2x+1)(2A_{2j-2} - x^{j+1}) \\
&= -2xA_{2j-2}((2x+1)A_{2j-2}) + x(2x+1)(2A_{2j-2} - x^{j+1}) \\
&= -(2x+1)x^{j+2}.
\end{aligned}$$

Substituting in (6.2.8) we obtain

$$A_{2j+2}^2 - (2x-1)A_{2j+2}B_{2j+2} - xB_{2j+2}^2 = -x^{j+3}.$$

Similarly, $A_1 = 2x$ and $B_1 = 1$ so that $A_1^2 - (2x-1)A_1B_1 - xB_1^2 = x$. Suppose (6.2.7) is true for all $i \leq j$. From the recurrence relations (6.1.5), we find that

$$\begin{aligned}
& A_{2j+1}^2 - (2x-1)A_{2j+1}B_{2j+1} - xB_{2j+1}^2 \\
&= ((2x+1)A_{2j-1} - xA_{2j-3})^2 - x((2x+1)B_{2j-1} - xB_{2j-3})^2 \\
&\quad - (2x-1)((2x+1)A_{2j-1} - xA_{2j-3})((2x+1)B_{2j-1} - xB_{2j-3}) \\
&= (2x+1)^2(A_{2j-1}^2 - (2x-1)A_{2j-1}B_{2j-1} - xB_{2j-1}^2) - 2x^2B_{2j-3}B_{2j-1} \\
&\quad + x^2(A_{2j-3}^2 - (2x-1)A_{2j-3}B_{2j-3} - xB_{2j-3}^2) - 2x(2x+1)A_{2j-3}A_{2j-1} \\
&\quad + (2x+1)(2x-1)(xA_{2j-1}B_{2j-3} + xA_{2j-3}B_{2j-1}) \\
&= (2x+1)^2x^j + x^{j+1} + (2x+1) [2x^2B_{2j-3}B_{2j-1} \\
&\quad - 2xA_{2j-3}A_{2j-1} + (2x-1)(xA_{2j-1}B_{2j-3} + xA_{2j-3}B_{2j-1})].
\end{aligned}$$

It suffices to show that the expressions in the square brackets in the equation above equals $(2x+1)x^j$. Using (6.1.10), (6.1.11) and (6.2.5), we find that

$$\begin{aligned}
& 2xA_{2j-3}A_{2j-1} - (2x-1)(xA_{2j-1}B_{2j-3} + xA_{2j-3}B_{2j-1}) - 2x^2B_{2j-3}B_{2j-1} \\
&= 2xA_{2j-3}A_{2j-1} - 2(A_{2j-3} - xA_{2j-5})(A_{2j-1} - xA_{2j-3}) \\
&\quad - (2x-1)(A_{2j-1}(A_{2j-3} - xA_{2j-5}) + A_{2j-3}(A_{2j-1} - xA_{2j-3})) \\
&= -2xA_{2j-3}A_{2j-1} + x(2x+1)(A_{2j-3}^2 + A_{2j-5}A_{2j-1}) - 2x^2A_{2j-5}A_{2j-3} \\
&= -2xA_{2j-2}(A_{2j} + xA_{2j-4}) + x(2x+1)(2A_{2j-2} - x^{j+1}) \\
&= -2xA_{2j-3}((2x+1)A_{2j-3}) + x(2x+1)(2A_{2j-3} + x^{j-1}) \\
&= (2x+1)x^j.
\end{aligned}$$

Substituting in (6.2.9) we obtain

$$A_{2j+1}^2 - (2x-1)A_{2j+1}B_{2j+1} - xB_{2j+1}^2 = x^{j+1}.$$

By induction on j , (6.2.6) and (6.2.7) are true for all j . $\square$

We now use lemma 107 to prove the convergence of $F_k(x)$ in a manner that quantifies how close the convergents are to the limit.

Theorem 108. *For $k \geq 1$, let*

$$F_k(x) := [x, \frac{1}{x}, x^2, \frac{1}{x^2}, x^3, \frac{1}{x^3}, x^4, \frac{1}{x^4}, \dots, x^k, \frac{1}{x^k}], \quad (6.2.9)$$

and

$$F_k^*(x) := [x, \frac{1}{x}, x^2, \frac{1}{x^2}, x^3, \frac{1}{x^3}, x^4, \frac{1}{x^4}, \dots, x^k] \quad (6.2.10)$$

be truncations of $F(x)$ to give odd and even convergents respectively. Then $F_k(x)$ and $F_k^*(x)$ converge to the positive root of $P(x, z) = z^2 - (2x-1)z - x$.

Proof. Since $F_k(x) = \frac{A_{2k-1}}{B_{2k-1}}$, equation (6.2.7) yields

$$\begin{aligned}
P(x, F_k(x)) &= \frac{A_{2k-1}^2 - (2x-1)A_{2k-1}B_{2k-1} - xB_{2k-1}^2}{B_{2k-1}^2} \\
&= \frac{x^k}{B_{2k-1}^2(x)}.
\end{aligned} \quad (6.2.11)$$

Here $B_{2k-1}(x)$ has positive integer coefficients and $\deg(B_{2k-1}(x)) = k-1$, so for $x > 1$,

$$\lim_{k \rightarrow \infty} P(x, F_k(x)) = \lim_{k \rightarrow \infty} \frac{x^k}{B_{2k-1}^2(x)} \leq \lim_{k \rightarrow \infty} \frac{1}{x^{k^2-3k+1}} = 0.$$

Since $F_k^*(x) = \frac{A_{2k-2}}{B_{2k-2}}$, equation (6.2.6) yields

$$\begin{aligned} P(x, F_k^*(x)) &= \frac{A_{2k-2}^2 - (2x-1)A_{2k-2}B_{2k-2} - xB_{2k-2}^2}{B_{2k-2}^2} \\ &= \frac{-x^{k+1}}{B_{2k-2}^2(x)}. \end{aligned} \quad (6.2.12)$$

As before, $B_{2k}(x)$ has positive integer coefficients and $\deg(B_{2k-2}(x)) = k-1$, so for $x > 1$,

$$\lim_{k \rightarrow \infty} P(x, F_k^*(x)) = \lim_{k \rightarrow \infty} \frac{-x^{k+1}}{B_{2k-2}^2(x)} \leq \lim_{k \rightarrow \infty} \frac{-1}{x^{k^2-3k}} = 0.$$

To show that they converge to the positive root of $P(x, z)$, write $P(x, z) = (z-z_1)(z-z_2)$ where $z_1 = \frac{1}{2}(2x-1+\sqrt{4x^2+1})$ and $z_2 = \frac{1}{2}(2x-1-\sqrt{4x^2+1})$. Note that $F_k(x)$ and $F_k^*(x)$ are all positive and so they converge to z_1 . $\square$

The evaluations of the function $P(x, z)$ at the convergents of the continued fraction is analogous to the known result that $u^2 - xu - 1$ evaluated at $[x, \dots, x]$ with n partial quotients of x is

$$\frac{(-1)^n}{F_n^2(x)}$$

where $F_n(x)$ is the Fibonacci polynomial of degree n . This result implies that $[x, x, x, \dots]$ converges to $\frac{1}{2}(x + \sqrt{4 + x^2})$, a root of $u^2 - xu - 1$, for $x \geq 1$.

6.3 A generalization of $F_k(x)$

We now give a generalization of (6.0.4) by introducing a parameter ‘ s ’ to obtain a continued fraction of the form (6.0.2). In this section, we just present results and omit the proofs since they have the same construction as the proofs already presented. For a positive integer s , define $F(x, s)$ as

$$F(x, s) := [x, \frac{s}{x}, x^2, \frac{s}{x^2}, x^3, \frac{s}{x^3}, x^4, \frac{s}{x^4}, \dots, x^k, \frac{s}{x^k}, \dots], \quad (6.3.1)$$

then $F(x, s)$ has an equivalent presentation

$$\tilde{F}(x, s) := x + \frac{x}{s+x} \frac{1}{s+x} \frac{x}{s+x} \dots \frac{1}{s+x} \frac{x}{s+x} \dots. \quad (6.3.2)$$

With the use of Theorem (97), it is straightforward to verify the equivalence with the sequence $\{r_j\}$ defined by (6.1.3). Clearly, $\tilde{F}(x, s)$ is periodic and converges to a root of the polynomial $P(x, z, s) = sz^2 - ((s+1)x - 1)z - x$. By the equivalence to $\tilde{F}(x, s)$, $F(x, s)$ converges to the same limit. To see this, consider the convergents $\frac{A_j(x, s)}{B_j(x, s)}$ of $F(x, s)$. The polynomials $A_j(x, s)$ and $B_j(x, s)$ have initial conditions $A_0 = x$, $A_1 = x(s+1)$, $A_2 = x((s+1)x+1)$, $B_0 = 1$, $B_1 = s$ and $B_2 = xs+1$ with recurrence relations

$$\begin{aligned} A_j(x, s) &= ((s+1)x+1)A_{j-2}(x, s) - xA_{j-4}(x, s), \\ B_j(x, s) &= ((s+1)x+1)B_{j-2}(x, s) - xB_{j-4}(x, s). \end{aligned} \quad (6.3.3)$$

From the recurrence relations (6.3.3), we deduce the generating functions

$$\begin{aligned} \frac{x}{1 - ((s+1)x+1)t + xt^2} &= \sum_{j=0}^{\infty} A_{2j}(x, s)t^j, \\ \frac{x((s+1)-t)}{1 - ((s+1)x+1)t + xt^2} &= \sum_{j=0}^{\infty} A_{2j+1}(x, s)t^j, \\ \frac{s}{1 - ((s+1)x+1)t + xt^2} &= \sum_{j=0}^{\infty} B_{2j+1}(x, s)t^j, \\ \frac{(1-xt)}{1 - ((s+1)x+1)t + xt^2} &= \sum_{j=0}^{\infty} B_{2j}(x, s)t^j. \end{aligned}$$

From the generating functions above, we establish relationships between the polynomials $A_j(x, s)$ and $B_j(x, s)$ given by

$$\begin{aligned} A_{2j+1}(x, s) &= (s+1)A_{2j}(x, s) - A_{2j-2}(x, s), \\ xB_{2j+1}(x, s) &= sA_{2j}(x, s), \\ sB_{2j}(x, s) &= B_{2j+1}(x, s) - xB_{2j-1}(x, s), \\ &= \frac{1}{x}A_{2j}(x, s) - A_{2j-2}(x, s). \end{aligned}$$

It can be shown that for $j \geq 0$,

$$A_{2j}(x, s)^2 - (2x - 1)A_{2j}(x, s)B_{2j}(x, s) - xB_{2j}(x, s)^2 = -x^{j+2}, \quad (6.3.4)$$

and for $j \geq 1$,

$$A_{2j-1}(x, s)^2 - (2x - 1)A_{2j-1}(x, s)B_{2j-1}(x, s) - xB_{2j-1}(x, s)^2 = x^j s. \quad (6.3.5)$$

Thus one finds from equation (6.3.5) that for odd convergents $F_k(x, s)$

$$P(x, F_k(x, s)) = \frac{x^k s}{B_{2k-1}^2(x, s)}. \quad (6.3.6)$$

Here, $B_{2k-1}(x, s)$ has positive integer coefficients, has s as a factor and $\deg(B_{2k-1}(x)) = k - 1$. Hence for $x > 1$,

$$\lim_{k \rightarrow \infty} P(x, F_k(x, s)) = \lim_{k \rightarrow \infty} \frac{x^k s}{B_{2k-1}^2(x)} \leq \lim_{k \rightarrow \infty} \frac{1}{x^{k^2-3k+1} s} = 0.$$

Similarly one finds from equation (6.3.4) that for even convergents $F_k^*(x, s)$

$$P(x, F_k^*(x, s)) = \frac{-x^{k+1}}{B_{2k-2}^2(x, s)}. \quad (6.3.7)$$

As before, $B_{2k}(x, s)$ has positive integer coefficients and $\deg(B_{2k-2}(x)) = k - 1$. Hence for $x > 1$,

$$\lim_{k \rightarrow \infty} P(x, F_k^*(x, s)) = \lim_{k \rightarrow \infty} \frac{-x^{k+1}}{B_{2k-2}^2(x, s)} \leq \lim_{k \rightarrow \infty} \frac{-1}{x^{k^2-3k}} = 0.$$

Since for all k , $F_k(x, s)$ and $F_k^*(x, s)$ are positive, $F(x, s)$ is positive and converges to the positive root of the quadratic

$$P(x, z, s) = sz^2 - ((s + 1)x - 1)z - x.$$

We can then deduce the following theorem.

Theorem 109. *The continued fraction*

$$F(x, s) := [x, \frac{s}{x}, x^2, \frac{s}{x^2}, x^3, \frac{s}{x^3}, x^4, \frac{s}{x^4}, \dots, x^k, \frac{s}{x^k}, \dots]$$

converges to the positive root of the quadratic $P(x, z, s) = sz^2 - ((s + 1)x - 1)z - x$.

We conclude by drawing a relationship between the continued fraction $F(1, x^{-1})$ and some q -series studied by Auluck, [15] and a Ramanujan q -series. Define a continued fraction with m interlacing geometric sequences by

$$F(m; x_1, \dots, x_m) := [x_1, \dots, x_m, x_1^2, \dots, x_m^2, x_1^3, \dots, x_m^3, \dots] \quad (6.3.8)$$

Aside from the present work, nothing seems to be known for $m \geq 2$. We also don't see anything of immediate interest for $m \geq 3$, or even the case $m = 1$ (despite a superficial resemblance to the famous Rogers-Ramanujan continued fraction). However the case $m = 2$, $x_1 = 1$ and $x_2 = x$, namely;

$$F(2; 1, x) := [1, x, 1, x^2, 1, x^3, 1, x^4, \dots] \quad (6.3.9)$$

does seem of immediate interest. Write the k^{th} truncation of $F(2; 1, x)$ in reduced form as $\frac{N_k(x)}{D_k(x)}$. Here $(D_k(x))$ seems to converge, as a formal power series, to the q -series

$$\sum_{k=1}^{\infty} \frac{x^{\frac{k^2+k}{2}}}{((1-x)(1-x^2) \dots (1-x^{k-1}))^2 (1-x^k)}$$

studied by Auluck in [15]. Coefficients of this power series are A001524¹. On the other hand, reversing the coefficients of $D_{2k+1}(x)$, the sequence $(D_k(x))$ seems to converge as a formal power series to another q -series studied by Auluck whose coefficients are A005576².

The sequence of polynomials $(N_{2k+1}(x))$ seems to converge, as a formal power series, to a Ramanujan q -series

$$\sum_{k=1}^{\infty} \frac{x^{\frac{k^2+k}{2}}}{((1-x)(1-x^2) \dots (1-x^k))^2}$$

which has coefficients A143184³. This q -series appears in part II of Ramanujan's Lost Notebooks [16, p. 18].

¹As given in the Online Encyclopedia of Integer Sequences <https://oeis.org/A001524>

²As given in the Online Encyclopedia of Integer Sequences <https://oeis.org/A005576>

³As given in the Online Encyclopedia of Integer Sequences <https://oeis.org/A143184>

Remark 7. *The root distribution of the $A_n(x, s)$ and $B_n(x, s)$ polynomials easily follows from a theorem of Khang Tran (see [1, Theorem 1] or [2]).*

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