GLUING CONSTRUCTIONS FOR HIGGS BUNDLES OVER A COMPLEX CONNECTED SUM

BY

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DISSERTATION

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ABSTRACT

For a compact Riemann surface of genus $g \geq 2$, the components of the moduli space of Sp(4,$\mathbb{R}$)-Higgs bundles, or equivalently the Sp(4,$\mathbb{R}$)-character variety, are partially labeled by an integer d known as the Toledo invariant. The subspace for which this integer attains a maximum has been shown to have $3 \cdot 2^{2g} + 2g - 4$ many components. A gluing construction between parabolic Higgs bundles over a connected sum of Riemann surfaces provides model Higgs bundles in a subfamily of particular significance. This construction is formulated in terms of solutions to the Hitchin equations, using the linearization of a relevant elliptic operator.

Στην οικογένειά μου και τους δασκάλους μου

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Τῶν γονέων τοὺς διδασκάλους προτιμᾶν,
οἱ μὲν γὰρ τοῦ ζῆν, οἱ δὲ τοῦ εὖ ζῆν γεγόνασιν αἴτιοι.

Ισοκρατης, Αριστοτελης, Αλεξανδρος

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SYNOPSIS

Let Σ be a closed connected and oriented surface of genus $g \geq 2$ and G be a connected semisimple Lie group. The moduli space of reductive representations of $\pi_1(\Sigma)$ into G modulo conjugation

$$\mathcal{R}(G) = \mathrm{Hom}^+(\pi_1(\Sigma), G)/G$$

has been an object of extensive study and interest. Fixing a complex structure J on the surface Σ transforms this into a Riemann surface $X = (\Sigma, J)$ and opens the way for holomorphic techniques using the theory of Higgs bundles. The non-abelian Hodge theory correspondence provides a real-analytic isomorphism between the character variety $\mathcal{R}(G)$ and the moduli space $\mathcal{M}(G)$ of polystable G-Higgs bundles. In this dissertation we are primarily interested in the case when $G = \mathrm{Sp}(4,\mathbb{R})$. The precise definition of an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle over a compact Riemann surface X reads as follows:

Definition 1. Let $K = T^*X$ be the canonical line bundle over X. An *Sp(4,$\mathbb{R}$)-Higgs bundle* over X is defined as a triple (V, β, γ), where V is a rank 2 holomorphic vector bundle over X and β, γ are symmetric homomorphisms

$$\beta : V^* \to V \otimes K \text{ and } \gamma : V \to V^* \otimes K$$

The embedding $\mathrm{Sp}(4,\mathbb{R}) \hookrightarrow \mathrm{SL}(4,\mathbb{C})$ allows one to reinterpret the defining $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data as special $\mathrm{SL}(4,\mathbb{C})$-data in the original sense of N. Hitchin [26]. In particular, an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle is alternatively defined as a pair (E, Φ), where

1. $E = V \oplus V^*$ is a rank 4 holomorphic vector bundle over X and
2. $\Phi : E \to E \otimes K$ is a holomorphic K-valued endomorphism of E with $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

A basic topological invariant for the tuples (V, β, γ) is given by the degree of the underlying rank 2 bundle

$$d = \deg(V)$$

This invariant, called the *Toledo invariant*, ranges between $2-2g$ and $2g-2$ and the corresponding representations in the character variety are of particular interest for the extremal cases, that is when $|d|=2g-2$. The subspace of maximal Sp(4,$\mathbb{R}$)-Higgs bundles $\mathcal{M}^{\max}=\mathcal{M}_{2g-2}\simeq\mathcal{M}_{2-2g}$ has been shown to have $3\cdot 2^{2g}+2g-4$ connected components [21].

Among the connected components of $\mathcal{M}^{\max}\simeq\mathcal{R}^{\max}$, there are $2g-3$ *exceptional* components of this moduli space. These components are all smooth but topologically non-trivial, and representations in these do not factor through any proper reductive subgroup of $\mathrm{Sp}(4,\mathbb{R})$, thus have Zariski-dense image in $\mathrm{Sp}(4,\mathbb{R})$. On the other hand, for the remaining $3\cdot 2^{2g}-1$ components, model Higgs bundles can be obtained by embedding stable SL(2,$\mathbb{R}$)-Higgs data into Sp(4,$\mathbb{R}$), using appropriate embeddings $\phi:\mathrm{SL}(2,\mathbb{R})\hookrightarrow\mathrm{Sp}(4,\mathbb{R})$ (see [9]). This method, however, will obviously not apply for finding model Higgs bundles in the $2g-3$ exceptional ones. The *construction* of Sp(4,$\mathbb{R}$)-Higgs bundles that lie in these exceptional components is the principal objective of this dissertation.

From the point of view of the character variety $\mathcal{R}^{\max}$, model representations in a subfamily of the $2g-3$ special components have been effectively constructed by O. Guichard and A. Wienhard in [22] by means of a certain topological *gluing construction*, which we briefly describe next: Let $\Sigma=\Sigma_l\cup_\gamma\Sigma_r$ be a decomposition of the surface Σ along a simple, closed, oriented, separating geodesic γ into two subsurfaces Σ_l and Σ_r. Pick $\rho_{irr}:\pi_1(\Sigma)\to\mathrm{SL}(2,\mathbb{R})\xrightarrow{\phi_{irr}}\mathrm{Sp}(4,\mathbb{R})$ an irreducible Fuchsian representation and $\rho_\Delta:\pi_1(\Sigma)\to\mathrm{SL}(2,\mathbb{R})\xrightarrow{\Delta}\mathrm{SL}(2,\mathbb{R})^2\to\mathrm{Sp}(4,\mathbb{R})$ a diagonal Fuchsian representation. One could amalgamate the restriction of the irreducible Fuchsian representation ρ_{irr} to Σ_l with the restriction of the diagonal Fuchsian representation ρ_Δ to Σ_r, however the holonomies of those along γ a priori do not agree. A deformation of ρ_Δ on $\pi_1(\Sigma)$ can be considered, such that the holonomies would agree along γ, thus allowing the amalgamation operation. This introduces new representations by gluing:

Definition 2. A *hybrid representation* is defined as the amalgamated representation

$$\rho:=\rho_l\big|_{\pi_1(\Sigma_l)}*\rho_r\big|_{\pi_1(\Sigma_r)}:\pi_1(\Sigma)\simeq\pi_1(\Sigma_l)*_{\langle\gamma\rangle}\pi_1(\Sigma_r)\to\mathrm{Sp}(4,\mathbb{R})$$

O. Guichard and A. Wienhard also introduce appropriate topological invariants for Anosov representations, a special case of which are the maximal symplectic surface group representations. An explicit computation of the invariants for the hybrid representations provides that these serve as models to the *odd-indexed* exceptional components of $\mathcal{M}^{\max}$, while the actual component in which a particular hybrid representation lies, depends entirely on the genus of the surface Σ_l appearing in the decomposition of Σ along a closed, separating geodesic.

Motivated by the topological gluing construction described above, we aim at developing a gluing construction for (poly)stable G-Higgs bundles over a complex connected sum of Riemann surfaces. The establishment of such a technique may have a wider applicability in constructing points in the interior of moduli of G-Higgs bundles.

In this dissertation, we formulate the gluing construction for the case when $G = \mathrm{Sp}(4,\mathbb{R})$. We also point out how one can choose Sp(4,$\mathbb{R}$)-Higgs bundle data over a pair of Riemann surfaces so that the resulting *hybrid Higgs bundle* obtained by gluing lies in one of the $2g-3$ exceptional components of $\mathcal{M}^{\max}$. Even further, we describe how the choices of the initial gluing data can provide model Higgs bundles in *all* exceptional components. The latter completes the description of a specific relation between the Higgs bundle topological invariants and the topological invariants for Anosov representations for maximal symplectic surface group representations.

The first step in this direction is to understand the objects corresponding to Sp(4,$\mathbb{R}$)-representations over a surface with boundary with fixed arbitrary holonomy around the boundary. These objects are Higgs bundles defined over a Riemann surface with a divisor, together with a weighted flag on the fibers over the points in the divisor, namely *parabolic Higgs bundles*. Indeed, a non-abelian Hodge correspondence was established by C. Simpson in the non-compact case [40] and later on, a Hitchin-Kobayashi correspondence was provided by O. Biquard, O. García-Prada and I. Mundet i Riera for parabolic G-Higgs bundles [5]. We define these appropriate holomorphic objects as follows:

Definition 3. Let X be a compact Riemann surface of genus g and consider the divisor $D := \{x_1, \ldots, x_s\}$ of s-many distinct points on X, assuming that $2g-2+s > 0$. A *parabolic Sp(4,$\mathbb{R}$)-Higgs bundle* over X is defined as a triple (V, β, γ), where

- V is a rank 2 bundle on X, equipped with a parabolic structure at each point $x \in D$ given by the flag
$$V_x \supset L_x \supset 0$$
and weights
$$0 \leq \alpha_1(x) < \alpha_2(x) < 1$$
- $\beta : V^\vee \to V \otimes K \otimes \iota$ and $\gamma : V \to V^\vee \otimes K \otimes \iota$ are strongly parabolic morphisms, where $V^\vee$ denotes the parabolic dual of V, $K = T^*X$ and $\iota = \mathcal{O}_X(D)$ is a fixed line bundle over the divisor D.

A notion of *parabolic Toledo invariant* of a parabolic $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundle is defined as the rational number

$$\tau = par\deg(V) = \deg(V) + \sum_{x\in D}(\alpha_1(x) + \alpha_2(x))$$

and a Milnor-Wood type inequality for this invariant can still be established:

Proposition 1. [Proposition 2.4.2] Let (E, Φ) be a semistable parabolic $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundle. Then

$$|\tau| \leq 2g - 2 + s$$

where s is the number of points in the divisor D.

As in the non-parabolic case, the parabolic $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundles with parabolic Toledo invariant $\tau = 2g - 2 + s$ will be called *maximal* and we denote the components containing such triples (V, β, γ) by $\mathcal{M}_{par}^{\max}$.

Let X_1, X_2 be two distinct compact Riemann surfaces with a divisor of s-many distinct points on each, and consider a pair of parabolic Sp(4,$\mathbb{R}$)-Higgs bundles over X_1, X_2 respectively. The complex connected sum $X_\# = X_1 \# X_2$ of the Riemann surfaces is constructed using a biholomorphism between annuli around pairs of points, one on each of X_1 and X_2. It is important that a gluing construction of parabolic Higgs bundles over the connected sum $X_\#$ is formulated so that the gluing of *stable* parabolic pairs is providing a *polystable* Higgs bundle over $X_\#$. Moreover, in order to construct new models in the components of $\mathcal{M}(X_\#, \mathrm{Sp}(4,\mathbb{R}))$, the parabolic gluing data over X_1 and X_2 are chosen to be coming from different embeddings of SL(2,$\mathbb{R}$)-parabolic data into Sp(4,$\mathbb{R}$), and so a priori *do not agree* over disks around the points in the divisors. We choose to switch to the language of solutions to Hitchin's equations and make use of the analytic techniques of C. Taubes for gluing instantons over 4-manifolds in order to control the stability condition. This involves viewing our stable parabolic Sp(4,$\mathbb{R}$)-Higgs bundles over the punctured Riemann surfaces X_1 and X_2 as solutions to the Sp(4,$\mathbb{R}$)-Hitchin equations.

The problem now involves perturbing this initial data into model solutions which are *identified* locally over the annuli around the points in the divisors, thus allowing the construction of a pair over $X_\#$ that combines the initial data over X_1 and X_2. The existence of these perturbations in terms of appropriate gauge transformations is initially provided for SL(2,$\mathbb{R}$)-data, and we next use the embeddings of SL(2,$\mathbb{R}$) into Sp(4,$\mathbb{R}$) to extend this deformation argument for our initial pairs. This produces an approximate solution to the Sp(4,$\mathbb{R}$)-Hitchin equations $(A_R^{app}, \Phi_R^{app})$ over $X_\#$, with respect to a parameter $R > 0$ which describes the size of the neck region in the construction of $X_\#$. The pair $(A_R^{app}, \Phi_R^{app})$ coincides with the initial data over each hand side Riemann surface and with the model over the neck region.

By construction, this pair is complex gauge equivalent to an exact solution of the Hitchin equations, so the second equation is preserved, while the first equation is satisfied up to an error which we have good control of:

Lemma. [Lemma 3.4.4] The approximate solution $(A_R^{app}, \Phi_R^{app})$ to the parameter $0 < R < 1$ satisfies

$$\left\| *F_{A_R^{app}} + *\left[\Phi_R^{app}, -\tau\left(\Phi_R^{app}\right)\right] \right\|_{C^0} \leq CR^{\delta''}$$

for some constants $\delta'' > 0$ and $C = C(\delta'')$ not depending on R.

The next important step is to correct this approximate solution to an exact solution of the Sp(4,ℝ)-Hitchin equations over the complex connected sum of Riemann surfaces. In other words, we seek for a complex gauge transformation g such that $g^*(A_R^{app}, \Phi_R^{app})$ is an exact solution of the Sp(4,ℝ)-Hitchin equations. The argument providing the existence of such a gauge is translated into a Banach fixed point theorem argument and involves the study of the linearization of a relevant elliptic operator. For Higgs bundles this was first studied by R. Mazzeo, J. Swoboda, H. Weiss and F. Witt in [29], who described solutions to the $\mathrm{SL}(2,\mathbb{C})$-Hitchin equations near the ends of the moduli space. For the complex connected sum $X_\#$ we consider the nonlinear *G-Hitchin operator* at a pair (A, Φ),

$$\mathcal{H}(A, \Phi) = \left(F(A) - [\Phi, \tau(\Phi)], \bar{\partial}_A \Phi\right)$$

to work with. A crucial step in this argument is to show that the linearization of this operator at our approximate solution $(A_R^{app}, \Phi_R^{app})$ is *invertible*; this is obtained by showing that an appropriate self-adjoint Dirac-type operator has no small eigenvalues. This method was also used by J. Swoboda in [42] to produce a family of smooth solutions of the SL(2,ℂ)-Hitchin equations, which may be viewed as desingularizing a solution with logarithmic singularities over a noded Riemann surface. Modifying the analytic techniques from [42], we extend the main theorem from that article to solutions of the Sp(4,ℝ)-Hitchin equations, and moreover obtain our main result:

Theorem. [Theorem 3.8.4] Let X_1 be a closed Riemann surface of genus g_1 and $D_1 = \{p_1, \ldots, p_s\}$ be a collection of s-many distinct points on X_1. Consider respectively a closed Riemann surface X_2 of genus g_2 and a collection of also s-many distinct points $D_2 = \{q_1, \ldots, q_s\}$ on X_2. Let $(E_1, \Phi_1) \to X_1$ and $(E_2, \Phi_2) \to X_2$ be parabolic stable Sp(4,ℝ)-Higgs bundles with corresponding solutions to the Hitchin equations (A_1, Φ_1) and (A_2, Φ_2). Assume that these solutions agree with model solutions $\left(A_{1,p_i}^{\mathrm{mod}}, \Phi_{1,p_i}^{\mathrm{mod}}\right)$ and $\left(A_{2,q_j}^{\mathrm{mod}}, \Phi_{2,q_j}^{\mathrm{mod}}\right)$ near the points $p_i \in D_1$ and $q_j \in D_2$, and that the model solutions satisfy $\left(A_{1,p_i}^{\mathrm{mod}}, \Phi_{1,p_i}^{\mathrm{mod}}\right) = -\left(A_{2,q_j}^{\mathrm{mod}}, \Phi_{2,q_j}^{\mathrm{mod}}\right)$, for s-many possible pairs of points (p_i, q_j). Then there is a polystable

Sp(4,$\mathbb{R}$)-Higgs bundle $(E_{\#}, \Phi_{\#}) \to X_{\#}$, constructed over the connected sum of Riemann surfaces $X_{\#} = X_1 \# X_2$ of genus $g_1 + g_2 + s - 1$, which agrees with the initial data over $X_{\#} \backslash X_1$ and $X_{\#} \backslash X_2$.

Definition 4. We call an Sp(4,$\mathbb{R}$)-Higgs bundle constructed by the preceding construction a *hybrid Sp(4,$\mathbb{R}$)-Higgs bundle*.

Subsequently, the goal is to identify the connected component of the moduli space a hybrid Higgs bundle lies, given a choice of stable parabolic ingredients to glue. For this purpose, we need to look at how do the Higgs bundle topological invariants behave under the complex connected sum operation. We first show the following:

Proposition 2. [Proposition 4.1.1] Let $X_{\#} = X_1 \# X_2$ be the complex connected sum of two closed Riemann surfaces X_1 and X_2 with divisors D_1 and D_2 of s-many distinct points on each surface, and let V_1, V_2 be parabolic principal $H^{\mathbb{C}}$-bundles over X_1 and X_2 respectively. For a parabolic subgroup $P \subset H^{\mathbb{C}}$, a holomorphic reduction σ of the structure group of E from $H^{\mathbb{C}}$ to P and an antidominant character χ of P, the following identity holds:

$$\deg\left(V_1 \# V_2\right)(\sigma, \chi) = par\deg_{\alpha_1}\left(V_1\right)(\sigma, \chi) + par\deg_{\alpha_2}\left(V_2\right)(\sigma, \chi)$$

This proposition implies that the connected sum of maximal parabolic Sp(4,$\mathbb{R}$)-Higgs bundles is again a maximal (non-parabolic) Sp(4,$\mathbb{R}$)-Higgs bundle. Note that an analogous *additivity property* for the Toledo invariant was established by M. Burger, A. Iozzi and A. Wienhard in [12] from the point of view of fundamental group representations.

In order to obtain model hybrid Higgs bundles inside the exceptional $2g-3$ components of $\mathcal{M}^{\max}$, we construct appropriate model maximal parabolic Sp(4,$\mathbb{R}$)-Higgs bundles extending maximal parabolic SL(2,$\mathbb{R}$)-data through the embeddings ϕ_{irr} and Δ used in the topological construction of a hybrid representation; let these particular parabolic models be denoted by (V_1, β_1, γ_1) and (V_2, β_2, γ_2) over the Riemann surfaces X_1 and X_2 respectively. We can then keep track of the Higgs bundle topological invariants under this grafting procedure and deduce the following two propositions:

Proposition 3. [Proposition 4.2.4] Let L_0 be a square root of the canonical line bundle $K_{\#}$ over the complex connected sum surface $X_{\#}$. The hybrid Higgs bundle $(V_{\#}, \Phi_{\#})$ constructed by gluing the maximal parabolic Higgs bundles (V_1, β_1, γ_1) and (V_2, β_2, γ_2) is maximal with a corresponding Cayley partner $W_{\#} := V_{\#}^{*} \otimes L_0$ for which it is $w_1\left(W_{\#}\right) = 0$ and $W_{\#} = L_{\#} \oplus L_{\#}^{-1}$, for some line bundle $L_{\#}$ over $X_{\#}$.

Proposition 4. [Proposition 4.2.6] Let $\iota_1 = \mathcal{O}_{X_1}(D_1)$ be the line bundle over a divisor in X_1. For the line bundle $L_\#$ appearing in the decomposition $W_\# = L_\# \oplus L_\#^{-1}$ of the Cayley partner, it is

$$\deg(L_\#) = par\deg K_{X_1} \otimes \iota_1$$

The last two propositions assert that the hybrid Higgs bundles constructed are modeling *all* exceptional components of $\mathcal{M}^{\max}$. These components are fully distinguished by the calculation of the degree of the line bundle $L_\#$. Moreover, for the case $G = \mathrm{Sp}(4,\mathbb{R})$, taking all the possible decompositions of a surface Σ along a simple, closed, separating geodesic is sufficient in order to obtain representations in the desired components of $\mathcal{M}^{\max}$. This result also allows, for the first time, to compare the invariants of maximal Higgs bundles to the topological invariants for Anosov representations constructed by O. Guichard and A. Wienhard.

CHAPTER 1

SP(4,$\mathbb{R}$)-HIGGS BUNDLES

1.1 G-Higgs bundles

Let X be a compact Riemann surface and let G be a real reductive group. The latter involves considering *Cartan data* (G, H, θ, B), where $H \subset G$ is a maximal compact subgroup, $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution and B is a non-degenerate bilinear form on $\mathfrak{g}$, which is $\mathrm{Ad}\,(G)$-invariant and θ-invariant. Moreover, the data (G, H, θ, B) have to satisfy the following:

1. The Lie algebra $\mathfrak{g}$ of the group G is reductive
2. θ gives a decomposition (called the *Cartan decomposition*)
$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$
into its ± 1-eigenspaces, where $\mathfrak{h}$ is the Lie algebra of H
3. $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal under B and B is positive definite on $\mathfrak{m}$ and negative definite on $\mathfrak{h}$
4. multiplication as a map from $H \times \exp \mathfrak{m}$ into G is an onto diffeomorphism.

Let $H^{\mathbb{C}}$ be the complexification of H and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ be the complexification of the Cartan decomposition. The adjoint action of G on $\mathfrak{g}$ restricts to give a representation (the *isotropy representation*) of H on $\mathfrak{m}$. This is independent of the choice of Cartan decomposition, since any two Cartan decompositions of G are related by a conjugation, using also that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, and the same is true for the complexified isotropy representation

$$\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$$

This introduces the following definition:

Definition 1.1.1. Let K be the canonical line bundle over X. A *G-Higgs bundle* is a pair (E, φ) where

- E is a principal holomorphic $H^{\mathbb{C}}$-bundle over X and
- φ is a holomorphic section of the vector bundle $E\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes K = \left(E \times_{\iota} \mathfrak{m}^{\mathbb{C}}\right) \otimes K$

The section φ is called the *Higgs field*.

Two G-Higgs bundles (E, φ) and (E', φ') are said to be *isomoprhic* if there is a vector bundle isomorphism $E \cong E'$ which takes the induced φ to φ' under the induced isomorphism $E\left(\mathfrak{m}^{\mathbb{C}}\right) \cong E'\left(\mathfrak{m}^{\mathbb{C}}\right)$.

When G is a real *compact* reductive Lie group, the Cartan decomposition of the Lie algebra is

$$\mathfrak{g} = \mathfrak{h}$$

thus the Higgs field φ equals zero. Hence, a G-Higgs bundle in this case is in fact a principal $G^{\mathbb{C}}$-bundle.

When G is a *complex* reductive Lie group, with G^r the underlying real Lie group, the complexification $H^{\mathbb{C}}$ of a maximal compact subgroup coincides with G and since

$$\mathfrak{g}^r = \mathfrak{h} \oplus i\mathfrak{h},$$

the isotropy representation coincides with the adjoint representation of G on its Lie algebra. Hence, Definition 1.1.1 for the underlying real Lie group G^r coincides with the notion of a G-Higgs bundle for a complex reductive Lie group G.

When $G = \mathrm{GL}(n, \mathbb{C})$ in particular, $E\left(\mathfrak{gl}\left(n, \mathbb{C}\right)\right) = \mathrm{End}\left(V\right)$, where V is the rank n vector bundle associated to the principal $\mathrm{GL}(n, \mathbb{C})$-bundle E via the standard representation of $\mathrm{GL}(n, \mathbb{C})$ in $\mathbb{C}^n$. Hence, a G-Higgs bundle in this case is a Higgs bundle in the original sense of N. Hitchin [26].

1.1.1 Stability

To define a moduli space of G-Higgs bundles we need to consider a notion of semistability, stability and polystability. These notions are defined in terms of an antidominant character for a parabolic subgroup $P_A \subseteq H^{\mathbb{C}}$ and a holomorphic reduction σ of the structure group of the bundle E from $H^{\mathbb{C}}$ to P_A. We next summarize the introduction of these notions; for more details see [18], or [1] in the case when $H^{\mathbb{C}}$ is semisimple in particular.

Let H be a compact and connected Lie group and let $H^{\mathbb{C}}$ be its complexification, which is assumed to be a *semisimple* complex Lie group. A subgroup $P \subset H^{\mathbb{C}}$ is said to be *parabolic* if the homogeneous space $H^{\mathbb{C}}/P$ is a projective variety. Consider a Cartan subalgebra $\mathfrak{c}$ of the Lie algebra $\mathfrak{h}$. Finally, let Δ denote a choice of simple roots of $\mathfrak{h}^{\mathbb{C}}$, with respect to the Cartan algebra $\mathfrak{c}$. We can then write the root space decomposition of $\mathfrak{h}^{\mathbb{C}}$ as:

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{c} \oplus \left(\bigoplus_{\delta \in \Delta} \mathfrak{h}_\delta \right)$$

where $\mathfrak{h}_\delta = \mathfrak{h}^{\mathbb{C}}$ is the root space corresponding to δ. Let Δ^+ be the set of positive roots and $\Pi = \{\alpha_1, \ldots \alpha_n\}$ be the set of simple roots. For any subset $A \subset \Pi$ define

$$\Delta_A = \left\{ \delta \in \Delta \,\middle|\, \delta = \sum_{i=1}^{n} m_i \alpha_i \text{ with } m_i \geq 0 \text{ for all } \alpha_i \in A \right\}$$

and let

$$\mathfrak{p}_A = \mathfrak{c} \oplus \left(\bigoplus_{\delta \in \Delta_A} \mathfrak{h}_\delta \right)$$

as a Lie subalgebra of $\mathfrak{h}^{\mathbb{C}}$. If $P_A \subset H^{\mathbb{C}}$ denotes the connected subgroup with Lie algebra $\mathfrak{p}_A$, then P_A is a parabolic subgroup of $H^{\mathbb{C}}$.

An *antidominant character* for the parabolic subgroup P_A is an element of the form

$$\chi = \sum_{\alpha_i \in A} m_i \lambda_i$$

with all $m_i \leq 0$ and for $\{\lambda_1, \ldots, \lambda_n\} \in \mathfrak{c}^*$ defined by the condition $\frac{2\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$, where $\alpha_i \in \Pi$ are simple roots. The character χ is called *strictly antidominant* if $m_i < 0$ for all $\alpha_i \in A$.

Now let (E, φ) be a G-Higgs bundle such that $H^{\mathbb{C}}$ is a semisimple complex Lie group, and consider a parabolic subgroup $P_A \subseteq H^{\mathbb{C}}$ and $L_A \subseteq P_A$ its Levi subgroup. Moreover, for a holomorphic section σ of $E\left(H^{\mathbb{C}}/P_A\right)$, let E_σ be the corresponding reduction of structure group of E from $H^{\mathbb{C}}$ to P_A, i.e. a principal P_A-bundle E_σ such that $E \cong E_\sigma \times_{P_A} H^{\mathbb{C}}$.

If χ is an antidominant character for P_A, let

$$\begin{aligned}
\left(\mathfrak{m}^{\mathbb{C}}\right)_\chi^- &= \left\{ v \in \mathfrak{m}^{\mathbb{C}} \,\middle|\, \iota\left(e^{ts_\chi}\right) v \text{ remains bounded as t} \to \infty \right\} \\
\left(\mathfrak{m}^{\mathbb{C}}\right)_\chi^0 &= \left\{ v \in \mathfrak{m}^{\mathbb{C}} \,\middle|\, \iota\left(e^{ts_\chi}\right) v = v \text{ for any t} \right\} \subset \left(\mathfrak{m}^{\mathbb{C}}\right)_\chi^-
\end{aligned}$$

which are subspaces of $\mathfrak{m}^{\mathbb{C}}$ invariant under the action of P_A and L_A respectively. We have that $E\left(\mathfrak{m}^{\mathbb{C}}\right) \cong E_\sigma \times_{P_A} \mathfrak{m}^{\mathbb{C}}$ and $E\left(\mathfrak{m}^{\mathbb{C}}\right) \cong E_{\sigma_L} \times_{L_A} \mathfrak{m}^{\mathbb{C}}$ and we can thus identify the vector bundles $E_\sigma \times_{P_A} \left(\mathfrak{m}^{\mathbb{C}}\right)^-_\chi$ and $E_{\sigma_L} \times_{L_A} \left(\mathfrak{m}^{\mathbb{C}}\right)^0_\chi$ with two holomorphic subbundles

$$E\left(\mathfrak{m}^{\mathbb{C}}\right)^0_\chi \subseteq E\left(\mathfrak{m}^{\mathbb{C}}\right)^-_\chi \subseteq E\left(\mathfrak{m}^{\mathbb{C}}\right)$$

If $\chi = \sum\limits_{\alpha_i \in A} m_i \lambda_i$, where $\{\lambda_i\} \in \zeta^* \oplus \mathfrak{c}^*$ is the set of fundamental weights associated to simple roots $\Pi = \{\alpha_i\}$, there exists some positive integer n such that for any $\alpha_i \in A$, the morphism of Lie algebras $n\lambda_i : \zeta \oplus \mathfrak{c} \to \mathbb{C}$ gives a morphism of Lie groups $\kappa_{n\alpha_i} : P_A \to \mathbb{C}^*$. The *degree* of the bundle E with respect to a reduction σ and to an antidominant character χ is defined as the real number

$$\deg(E)(\sigma, \chi) = \frac{1}{n} \sum \deg\left(E_\sigma \times_{\kappa_{n\alpha_i}} \mathbb{C}^*\right)$$

We are finally in position to define the stability conditions:

Definition 1.1.2. A G-Higgs bundle (E, φ) is called

- *semistable* if for any parabolic subgroup $P \subset H^{\mathbb{C}}$, any antidominant character χ for P and any holomorphic section $\sigma \in \Gamma\left(E\left(H^{\mathbb{C}}/P\right)\right)$ such that $\varphi \in H^0\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)^-_{\sigma,\chi} \otimes K\right)$, we have
$$\deg(E)(\sigma, \chi) \geq 0$$
- *stable* if it is semistable and furthermore: for any P,χ and σ as above, such that $\varphi \in H^0\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)^-_{\sigma,\chi} \otimes K\right)$ and such that $P \neq H^{\mathbb{C}}$, we have
$$\deg(E)(\sigma, \chi) > 0$$
- *polystable* if it is semistable and furthermore: for any P,χ and σ as above, such that $\varphi \in H^0\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)^-_{\sigma,\chi} \otimes K\right)$, $P \neq H^{\mathbb{C}}$ and χ is strictly antidominant, and such that
$$\deg(E)(\sigma, \chi) = 0,$$
there is a holomorphic reduction of the structure group $\sigma_L \in \Gamma\left(E_\sigma\left(P/L\right)\right)$, where E_σ denotes the principal P-bundle obtained from reduction of structure group σ and $L \subset P$ is the Levi subgroup. Furthermore, under these hypotheses, it is required that $\varphi \in H^0\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)^0_{\sigma_L,\chi} \otimes K\right)$.

These notions can be generalized for the case when the group $H^{\mathbb{C}}$ is reductive but not semisimple. In that case, the notions depend also on an extra parameter $\alpha \in Z\left(\mathfrak{h}^{\mathbb{C}}\right)$ which is equal to zero when $H^{\mathbb{C}}$ is indeed semisimple (cf. [18] for more details). A more workable version of these notions is obtained by giving a description of the objects involved in the definition in terms of filtrations of certain vector bundles:

Let $H^{\mathbb{C}}$ be a classical group, and let $\rho : H^{\mathbb{C}} \to \mathrm{GL}(n,\mathbb{C})$ be the standard representation which associates to E the vector bundle $V = E \times_{\rho} \mathbb{C}^n$. A pair (σ, χ) consisted of a holomorphic reduction of structure group σ and an antidominant character χ for a parabolic subgroup $P_A \subseteq H^{\mathbb{C}}$ can be shown to correspond to a *filtration of vector bundles*

$$\mathcal{V} = (0 \subset V_1 \subset \ldots \subset V_{k-1} \subset V_k = V)$$

and an increasing sequence of real numbers (usually called *weights*)

$$\lambda_1 < \ldots < \lambda_k$$

We define the *degree* of the bundle E with respect to a weighted filtration of vector bundles by

$$\deg(E) = \lambda_k \deg V + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg V_i$$

Definition 1.1.3. A G-Higgs bundle (E, φ) is called *semistable* if for any weighted filtration $\mathcal{V}$, we have $\deg(E) \geq 0$; it is called *stable* if for any $\mathcal{V}$, we have $\deg(E) > 0$ and finally it is called polystable if $\deg(E) = 0$.

When the group G is connected, principal $H^{\mathbb{C}}$-bundles E are topologically classified by a characteristic class $c(E) \in H^2\left(X, \pi_1\left(H^{\mathbb{C}}\right)\right) = \pi_1\left(H^{\mathbb{C}}\right) = \pi_1(H) = \pi_1(G)$.

Definition 1.1.4. For a fixed class $d \in \pi_1(G)$, the *moduli space of polystable G-Higgs bundles* is defined as the set of isomorphism classes of polystable G-Higgs bundles (E, φ) such that $c(E) = d$. We will denote this by $\mathcal{M}(G)$ and when the group G is compact, the moduli space $\mathcal{M}_d(G)$ coincides with $\mathcal{M}_d\left(G^{\mathbb{C}}\right)$.

The following theorem can be shown using the general GIT constructions of A. Schmitt for decorated principal bundles in the case of a real form of a complex reductive algebraic Lie group; see [36], [37] for details.

Theorem 1.1.5. *The moduli space $\mathcal{M}_d(G)$ is a complex analytic variety, which is algebraic when G is algebraic.*

Deformation theory of G-Higgs bundles can be now used to provide a computation of the expected dimension of this moduli space; for further information we refer to [17] and the references therein.

Definition 1.1.6. Let (E,φ) be a G-Higgs bundle. The *deformation complex* of (E,φ) is the following complex of sheaves

$$C^{\bullet}(E,\varphi) : E\left(\mathfrak{h}^{\mathbb{C}}\right) \xrightarrow{\mathrm{ad}(\varphi)} E\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes K$$

The space of infinitesimal deformations of a G-higgs bundle (E,φ) is shown to be naturally isomorphic to the hypercohomology group $\mathbb{H}^1\left(C^{\bullet}(E,\varphi)\right)$. For G semisimple and for a G-Higgs bundle (E,φ) stable and simple, the dimension of the component of the moduli space containing the pair (E,φ) equals the dimension of the infinitesimal deformation space; this is referred to as the *expected dimension* of the moduli space. The Riemann-Roch theorem can be now used to calculate this dimension:

Proposition 1.1.7. *Let G be a connected semisimple real Lie group. Then the expected dimension of the moduli space of G-Higgs bundles is $(g-1)\dim G^{\mathbb{C}}$.*

1.1.2 G-Higgs bundles and Hitchin equations

Let (E,φ) be a G-Higgs bundle over a compact Riemann surface X. By a slight abuse of notation we shall denote the underlying smooth objects of E and φ by the same symbols. The Higgs field can be thus viewed as a $(1,0)$-form $\varphi \in \Omega^{1,0}\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$. Given a reduction h of structure group to H in the smooth $H^{\mathbb{C}}$-bundle E, we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E. Let $\tau_h : \Omega^{1,0}\left(E\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \to \Omega^{0,1}\left(E\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$ be defined by the compact conjugation of $\mathfrak{g}^{\mathbb{C}}$ which is given fiberwise by the reduction h, combined with complex conjugation on complex 1-forms. The next theorem was proved in [18] for an arbitrary reductive real Lie group G.

Theorem 1.1.8. *There exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H satisfying the Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = 0$$

if and only if (E,φ) is polystable.

From the point of view of moduli spaces it is convenient to fix a C^{∞} principal H-bundle $\mathbf{E}_H$ with fixed topological class $d \in \pi_1(H)$ and study the moduli space of solutions to Hitchin's equations for a pair (A,φ) consisting of an H-connection A and $\varphi \in \Omega^{1,0}\left(X, \mathbf{E}_H\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$:

$$F_A - [\varphi, \tau(\varphi)] = 0 \tag{1.1}$$

$$\bar{\partial}_A \varphi = 0 \tag{1.2}$$

where d_A is the covariant derivative associated to A and $\bar{\partial}_A$ is the $(0,1)$-part of d_A, defining the holomorphic structure on $\mathbf{E}_H$. Also, τ is defined by the fixed reduction of structure group $\mathbf{E}_H \hookrightarrow \mathbf{E}_H\left(H^{\mathbb{C}}\right)$. The gauge group $\mathcal{G}_H$ of $\mathbf{E}_H$ acts on the space of solutions by conjugation and the moduli space of solutions is defined by

$$\mathcal{M}_d^{\text{gauge}}(G) := \{(A,\varphi) \text{ satisfying (1.1) and (1.2)}\}/\mathcal{G}_H$$

Now, Theorem 1.1.8 implies the following

Theorem 1.1.9. *There is a homeomorphism*

$$\mathcal{M}_d(G) \cong \mathcal{M}_d^{gauge}(G)$$

Using the one-to-one correspondence between H-connections on $\mathbf{E}_H$ and $\bar{\partial}$-operators on $\mathbf{E}_{H^{\mathbb{C}}}$, the homeomorphism in the above theorem can be interpreted by saying that in the $\mathcal{G}_H^{\mathbb{C}}$-orbit of a polystable G-Higgs bundle $\left(\bar{\partial}_{E_0},\varphi_0\right)$ we can find another Higgs bundle $\left(\bar{\partial}_E,\varphi\right)$ whose corresponding pair (d_A,φ) satisfies the equation $F_A - [\varphi,\tau(\varphi)] = 0$, and this is unique up to H-gauge transformations.

1.1.3 Morse theory on the moduli space of G-Higgs bundles

Morse theoretic techniques for the study of moduli of holomorphic vector bundles were first applied by M. Atiyah and R. Bott in [3]. In the context of moduli of Higgs bundles such techniques were applied by N. Hitchin in [25] and [26]. In order to count the connected components of the moduli space of G-Higgs bundles, a criterion for finding the local minima of a Morse function on $\mathcal{M}(G)$ is of particular importance.

The appropriate Morse function is defined on the moduli space of G-Higgs bundles, when viewed in the context of solutions to the Hitchin equations. From this point of view, define

$$\begin{aligned} f : \mathcal{M}_d(G) &\to \mathbb{R} \\ (d_A,\varphi) &\mapsto \|\varphi\|^2 \end{aligned}$$

where $\|\varphi\|^2 = \int_X |\varphi|^2 d\text{vol}$ is the L^2-norm of φ. This norm is well defined because $|\varphi|^2$ is invariant under H-gauge transformations. An important property of the map f is that away from the singular locus of $\mathcal{M}_d(G)$ it is a moment map for the Hamiltonian S^1-action given by

$$(d_A,\varphi) \mapsto \left(d_A, e^{i\vartheta}\varphi\right)$$

When $\mathcal{M}_d(G)$ is smooth, the map f is a *perfect Morse-Bott function* and the critical points of f are exactly the fixed points of the circle action; the G-Higgs bundles corresponding to fixed points are called *Hodge-bundles*, and for those there is a semisimple element $\psi \in H^0(E(\mathfrak{h}))$ and decompositions $E\left(\mathfrak{h}^{\mathbb{C}}\right) = \bigoplus_k E\left(\mathfrak{h}^{\mathbb{C}}\right)_k$, $E\left(\mathfrak{m}^{\mathbb{C}}\right) = \bigoplus_k E\left(\mathfrak{m}^{\mathbb{C}}\right)_k$ in eigen-bundles for ψ. However, even when $\mathcal{M}_d(G)$ has singularities, the map f can be still used to study the connected components of $\mathcal{M}_d(G)$, due to the next important proposition proved by N. Hitchin in [26] and its following corollary:

Proposition 1.1.10. *The function $f : \mathcal{M}_d(G) \to \mathbb{R}$ is a proper map.*

Corollary 1.1.11. *Let $\mathcal{M} \subseteq \mathcal{M}_d(G)$ be a closed subspace and let $\mathcal{N} \subseteq \mathcal{M}$ be the subspace of local minima of f on $\mathcal{M}$. If $\mathcal{N}$ is connected, then $\mathcal{M}$ is.*

Therefore, in order to study the connected components of $\mathcal{M}_d(G)$, one has to focus on the subspace of local minima of the map f, and the following criterion proven in [10] is used to efficiently identify these local minima:

Theorem 1.1.12. *Let (E, φ) be a stable G-Higgs bundle which represents a non-singular point of $\mathcal{M}_d(G)$. Then (E, φ) represents a local minimum of f if and only if*

$$ad(\varphi) : E\left(\mathfrak{h}^{\mathbb{C}}\right)_k \to E\left(\mathfrak{m}^{\mathbb{C}}\right)_{k+1} \otimes K$$

is an isomorphism for all $k > 0$.

1.1.4 Surface group representations and the non-abelian Hodge theorem

Let Σ be a closed oriented (topological) surface of genus g. The fundamental group of Σ is described by

$$\pi_1(\Sigma) = \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod [a_i, b_i] = 1 \right\rangle$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator. The set of all representations of $\pi_1(\Sigma)$ into a connected reductive real Lie group G, $\mathrm{Hom}(\pi_1(\Sigma), G)$, can be naturally identified with the subset of G^{2g} consisting of $2g$-tuples $(A_1, B_1, \ldots, A_g, B_g)$ satisfying the algebraic equation $\prod [A_i, B_i] = 1$. As such, the set $\mathrm{Hom}(\pi_1(\Sigma), G)$ is a real analytic variety which is algebraic when G is algebraic. The group G acts on $\mathrm{Hom}(\pi_1(\Sigma), G)$ by conjugation

$$(g \cdot \rho) = g\rho(\gamma) g^{-1}$$

where $g \in G$, $\rho \in \mathrm{Hom}(\pi_1(\Sigma), G)$ and $\gamma \in \pi_1(\Sigma)$, and the restriction of this action to the subspace $\mathrm{Hom}^{\mathrm{red}}(\pi_1(\Sigma), G)$ of reductive representations provides that the orbit space is

Hausdorff. Here, by a reductive representation we mean one that composed with the adjoint representation in the Lie algebra of G decomposes as a sum of irreducible representations. When G is algebraic, this is equivalent to the Zariski closure of the image of $\pi_1(\Sigma)$ in G being a reductive group. Define the *moduli space of reductive representations of* $\pi_1(\Sigma)$ *into* G to be the orbit space

$$\mathcal{R}(G) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(\Sigma), G)/G$$

The following theorem from [20] provides this space is a real analytic variety and so $\mathcal{R}(G)$ is usually called the *character variety*:

Theorem 1.1.13. *The moduli space $\mathcal{R}(G)$ has the structure of a real analytic variety, which is algebraic if G is algebraic and is a complex variety if G is complex.*

We can assign a topological invariant to a representation $\rho \in \mathcal{R}(G)$, by considering its corresponding flat G-bundle on Σ, defined as $E_\rho = \tilde{\Sigma} \times_\rho G$. Here $\tilde{\Sigma} \to \Sigma$ is the universal cover and $\pi_1(\Sigma)$ acts on G via ρ. A topological invariant is then given by the characteristic class $c(\rho) := c(E_\rho) \in \pi_1(G) \simeq \pi_1(H)$, for $H \subseteq G$ a maximal compact subgroup of G. For a fixed $d \in \pi_1(G)$ the moduli space of reductive representations with fixed topological invariant d is now defined as the subvariety

$$\mathcal{R}_d(G) := \{[\rho] \in \mathcal{R}(G) \,|\, c(\rho) = d\}$$

Equipping the surface Σ with a complex structure J, there corresponds to a reductive fundamental group representation a polystable G-Higgs bundle over the Riemann surface $X = (\Sigma, J)$. This is seen using that any solution h to Hitchin's equations defines a flat reductive G-connection

$$D = D_h + \varphi - \tau(\varphi), \tag{1.3}$$

where D_h is the unique H-connection on E compatible with its holomorphic structure. Conversely, given a flat reductive connection D in a G-bundle E_G, there exists a harmonic metric, i.e. a reduction of structure group to $H \subset G$ corresponding to a harmonic section of $E_G/H \to X$. This reduction produces a solution to Hitchin's equations such that Equation (1.3) holds. In summary, we have the following seminal result, the **non-abelian Hodge correspondence**; its proof is based on combined work by N. Hitchin [26], C. Simpson [39], [41], S. Donaldson [15] and K. Corlette [14]:

Theorem 1.1.14. *Let G be a connected semisimple real Lie group with maximal compact subgroup $H \subseteq G$ and let $d \in \pi_1(G) \simeq \pi_1(H)$. Then there exists a homeomorphism*

$$\mathcal{R}_d(G) \cong \mathcal{M}_d(G)$$

1.1.5 Reduction of structure group for Higgs bundles

For a real reductive Lie group (G, H, θ, B) we are interested in reformulating in terms of Higgs bundles, what it means for a fundamental group representation into G to factor through a subgroup of G. A *reductive subgroup* of G is a reductive group (G', H', θ', B') where the Cartan data are compatible under the inclusion map $G' \hookrightarrow G$.

Definition 1.1.15. Let G be a real reductive Lie group and let $G' \subset G$ be a reductive subgroup. Let (E, φ) be a G-Higgs bundle. A reduction of (E, φ) to a G'-Higgs bundle is a pair (E', φ') given by the following data:

- a holomorphic reduction of the structure group of E to a principal $H'^{\mathbb{C}}$-bundle $E' \hookrightarrow E$ or, equivalently, a holomorphic section of $E \times_{H^{\mathbb{C}}} \left(H^{\mathbb{C}}/H'^{\mathbb{C}} \right)$ and
- a holomorphic section φ' of $E' \left(\mathfrak{m}'^{\mathbb{C}} \right) \otimes K$ which maps to φ under the embedding $E' \left(\mathfrak{m}'^{\mathbb{C}} \right) \otimes K \to E \left(\mathfrak{m}^{\mathbb{C}} \right) \otimes K$.

The following proposition links the polystability condition for a G-Higgs bundle to the polystability of its structure group reduction.

Proposition 1.1.16. *Let G be a real reductive group and let $G' \subset G$ be a reductive subgroup. Let (E, φ) be a G-Higgs bundle and (E', φ') the corresponding G'-Higgs bundle under reduction of structure group. If (E, φ) is polystable as a G-Higgs bundle, then (E', φ') is polystable as a G'-Higgs bundle.*

The non-abelian Hodge correspondence now implies that the polystable G'-Higgs bundles correspond to fundamental group representations into $G' \subset G$. Therefore, a reductive fundamental group representation into G factors through a reductive representation into G', if and only if the corresponding polystable G-Higgs bundle admits a reduction of structure group to G'; cf. [18] for more details.

1.2 Sp(4,$\mathbb{R}$)-Higgs bundles

1.2.1 Sp($2n$,$\mathbb{R}$)-Higgs bundles

Let us consider now the special case when the structure group is $G = \mathrm{Sp}(2n,\mathbb{R})$ in particular. Then $H = \mathrm{U}(n)$ is a maximal compact subgroup with complexification $H^{\mathbb{C}} = \mathrm{GL}(n,\mathbb{C})$. If

$\mathbb{V}=\mathbb{C}^n$ is the fundamental representation of GL(n,$\mathbb{C}$) then the isotropy representation space is

$$\mathfrak{m}^{\mathbb{C}}=S^2\mathbb{V}\oplus S^2\mathbb{V}^*$$

The definition of a G-Higgs bundle in this case was specialized in [17] to the following:

Definition 1.2.1. Let X be a compact Riemann surface and K be the canonical line bundle over X. An Sp($2n$,$\mathbb{R}$)-Higgs bundle is defined by a triple (V,β,γ), where V is a rank n holomorphic vector bundle and $\beta\in H^0\left(X,S^2V\otimes K\right)$, $\gamma\in H^0\left(X,S^2V^*\otimes K\right)$ are holomorphic sections. To be compatible with the general G-Higgs bundle definition, we may consider $\varphi=\beta+\gamma$.

The stability notion for a G-Higgs bundle in terms of filtrations (Definition 1.1.3) also specializes in the case when $G=$ Sp($2n$,$\mathbb{R}$) to the following:

Definition 1.2.2. An Sp($2n$,$\mathbb{R}$)-Higgs bundle (V,β,γ) over X will be called

- *semistable*, if for any filtration of subbundles

  $$0\subset V_1\subset V_2\subset V$$

  such that $\beta\in H^0\left(K\otimes\left(S^2V_2+V_1\otimes_S V\right)\right)$ and $\gamma\in H^0\left(K\otimes\left(S^2V_1^{\perp}+V_2^{\perp}\otimes_S V^*\right)\right)$ it is $\deg\left(V\right)-\deg\left(V_1\right)-\deg\left(V_2\right)\geq 0$.
  Here, $V_1\otimes_S V$ denotes the subbundle of S^2V which is the image of $V_1\otimes V\subset V\otimes V$ under the symmetrization map $V\otimes V\to S^2V$; similarly for $V_2^{\perp}\otimes_S V^*$.

- *stable*, if for any filtration as above, except the filtration $0=V_1\subset V_2=V$, it is $\deg\left(V\right)-\deg\left(V_1\right)-\deg\left(V_2\right)>0$.

- *polystable*, if for any filtration as above, except the filtration $0=V_1\subset V_2=V$, and with $\deg\left(V\right)-\deg\left(V_1\right)-\deg\left(V_2\right)=0$, there exists an isomorphism of holomorphic vector bundles

  $$\sigma:V\to V_1\oplus V_2/V_1\ \oplus V/V_2$$

  satisfying the following properties:

  1. $V_1=\sigma^{-1}\left(V_1\right),V_2=\sigma^{-1}\left(V_1\oplus V_2/V_1\ \right)$
  2. $\beta\in H^0\left(K\otimes\left(S^2\left(\sigma^{-1}\left(V_2/V_1\ \right)\right)\oplus\sigma^{-1}\left(V_1\right)\otimes_S\sigma^{-1}\left(V/V_2\ \right)\right)\right)$
  3. $\gamma\in H^0\left(K\otimes\left(S^2\left(\sigma^*(V_2/V_1\ )^*\right)\oplus\sigma^*\left(V_1^*\right)\otimes_S\sigma^*(V/V_2\ )^*\right)\right)$

1.2.2 Sp(4,$\mathbb{R}$)-Higgs bundles

The group $G = \mathrm{Sp}(4,\mathbb{R})$ is the semisimple real subgroup of SL(4,$\mathbb{R}$) that preserves a symplectic form on $\mathbb{R}^4$:

$$\mathrm{Sp}(4,\mathbb{R}) = \left\{ A \in \mathrm{SL}(4,\mathbb{R}) \,\middle|\, A^T J_{13} A = J_{13} \right\},$$

where $J_{13} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ defines a symplectic form on $\mathbb{R}^4$, for I_2 the 2×2 identity matrix. The complexification of its Lie algebra

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}\,(4,\mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \,\middle|\, A, B, C \in \mathcal{M}_2\,(\mathbb{C})\,; B^T = B, C^T = C \right\}$$

has split real form $\mathfrak{sp}\,(4,\mathbb{R})$ and compact real form $\mathfrak{sp}\,(2)$.

The Cartan involution $\theta : \mathfrak{sp}\,(4,\mathbb{C}) \to \mathfrak{sp}\,(4,\mathbb{C})$ with $\theta\,(X) = -X^T$ determines a Cartan decomposition for a choice of maximal compact subgroup $H \simeq \mathrm{U}\,(2) \subset \mathrm{Sp}(4,\mathbb{R})$ as follows

$$\mathfrak{sp}\,(4,\mathbb{R}) = \mathfrak{u}\,(2) \oplus \mathfrak{m}$$

with complexification

$$\mathfrak{sp}\,(4,\mathbb{C}) = \mathfrak{gl}\,(2,\mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}}$$

Applying the change of basis on $\mathbb{C}^4$ effected by the mapping $T = \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}$, we can identify the summands in the Cartan decomposition of $\mathfrak{sp}\,(4,\mathbb{C}) \subset \mathfrak{sl}\,(4,\mathbb{C})$ as:

$$\mathfrak{gl}\,(2,\mathbb{C}) = \left\{ \begin{pmatrix} Z & 0 \\ 0 & -Z^T \end{pmatrix} \,\middle|\, Z \in \mathcal{M}_2\,(\mathbb{C}) \right\}$$

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \,\middle|\, \beta, \gamma \in \mathcal{M}_2\,(\mathbb{C})\,; \beta^T = \beta, \gamma^T = \gamma \right\} = \mathrm{Sym}^2\left(\mathbb{C}^2\right) \oplus \mathrm{Sym}^2\left(\left(\mathbb{C}^2\right)^*\right)$$

Let V denote the rank 2 vector bundle associated to a holomorphic principal GL(2,$\mathbb{C}$)-bundle E via the standard representation. Then from the Cartan decomposition for the Lie algebra $\mathfrak{sp}\,(4,\mathbb{C})$ we can identify

$$E\left(\mathfrak{m}^{\mathbb{C}}\right) = \mathrm{Sym}^2\,(V) \oplus \mathrm{Sym}^2\,(V^*)$$

and so the general definition for a G-Higgs bundle specializes to the following:

Definition 1.2.3. An Sp(4,$\mathbb{R}$)-Higgs bundle over a compact Riemann surface X is defined by a triple (V, β, γ), where V is a rank 2 holomorphic vector bundle over X and β, γ are symmetric homomorphisms

$$\beta : V^* \to V \otimes K \text{ and } \gamma : V \to V^* \otimes K$$

where K is the canonical line bundle over X.

The embedding Sp(4,$\mathbb{R}$) $\hookrightarrow$ SL(4,$\mathbb{C}$) allows one to reinterpret the defining Sp(4,$\mathbb{R}$)-data of a Higgs bundle as special SL(4,$\mathbb{C}$)-data in the original sense of N. Hitchin. We can thus consider an Sp(4,$\mathbb{R}$)-Higgs bundle to be defined as a pair (E, Φ), where

1. $E = V \oplus V^*$ is a rank 4 holomorphic vector bundle over X and
2. $\Phi : E \to E \otimes K$ is a Higgs field with $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

1.2.3 Sp(4,$\mathbb{R}$)-Hitchin equations

Remember that a Cartan decomposition $\mathfrak{sp}(4,\mathbb{R}) = \mathfrak{u}(2) \oplus \mathfrak{m}$ for a choice of maximal compact $H \simeq \mathrm{U}(2) \subset$ Sp(4,$\mathbb{R}$) is determined by the Cartan involution

$$\theta : \mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{sp}(4,\mathbb{C}) \text{ with } \theta(X) = -X^T$$

Moreover, the involution $\sigma : \mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{sp}(4,\mathbb{C})$, $\sigma(X) = \bar{X}$ defines the split real form:

$$\begin{aligned} \{X \in \mathfrak{sp}(4,\mathbb{C}) \,|\sigma(X) = X\} &= \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} |A, B, C \in \mathcal{M}_2(\mathbb{R})\,;\, B^T = B, C^T = C \right\} \\ &= \mathfrak{sp}(4,\mathbb{R}) \end{aligned}$$

Now, the involution $\tau : \mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{sp}(4,\mathbb{C})$, $\tau(X) = -X^*$ defines the compact real form. Indeed, we have

$$\mathfrak{u}(4) = \{X \in \mathfrak{gl}(4,\mathbb{C}) \,|X + X^* = 0\} \text{ and Sp(2)=Sp(4,}\mathbb{C}) \cap \mathrm{U}(4).$$

Notice that

$$\begin{aligned} \{X \in \mathfrak{sp}(4,\mathbb{C}) \,|\tau(X) = X\} &= \{X \in \mathfrak{sp}(4,\mathbb{C}) \,|-X^* = X\} \\ &= \mathfrak{sp}(4,\mathbb{C}) \cap \mathfrak{u}(4) = \mathfrak{sp}(2) \end{aligned}$$

Since τ and the Cartan involution commute, we have $\tau\left(\mathfrak{m}^{\mathbb{C}}\right) \subseteq \mathfrak{m}^{\mathbb{C}}$ and then τ preserves the Cartan decomposition $\mathfrak{sp}(4,\mathbb{C}) = \mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}}$. Thus, there is an induced real form on $E\left(\mathfrak{m}^{\mathbb{C}}\right)$ which we shall call τ as well for simplicity. Now, it makes sense to apply τ on a section $\varphi \in \Omega^{1,0}\left(E\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$.

Moreover, for $\varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ notice that

$$-\left[\varphi, \tau\left(\varphi\right)\right] = \left[\varphi, \varphi^*\right] = \begin{pmatrix} \beta\bar{\beta} - \bar{\gamma}\gamma & 0 \\ 0 & \gamma\bar{\gamma} - \bar{\beta}\beta \end{pmatrix}$$

The G-Hitchin equations for $G = \mathrm{Sp}(4,\mathbb{R})$ with maximal compact subgroup $H \simeq \mathrm{U}\left(2\right) \subset \mathrm{Sp}(4,\mathbb{R})$ read

$$\begin{aligned} F_A - \left[\varphi, \tau\left(\varphi\right)\right] &= 0 \\ \bar{\partial}_A \varphi &= 0 \end{aligned}$$

where:

- A is a $\mathrm{U}\left(2\right)$-connection on a fixed smooth principal $\mathrm{U}\left(2\right)$-bundle $\mathbb{E}_H \to X$
- $\varphi \in \Omega^1\left(X, E_{H^{\mathbb{C}}}\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$
- $\tau : \Omega^1\left(X, E_{H^{\mathbb{C}}}\left(\mathfrak{m}^{\mathbb{C}}\right)\right) \to \Omega^1\left(X, E_{H^{\mathbb{C}}}\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$ is the compact real structure considered above.
- $\bar{\partial}_A$ is the $(0,1)$-part of the covariant derivative associated to A.

whereas $\mathcal{G}_H = \mathrm{Aut}\left(\mathbb{E}_H\right) = \Omega^0\left(X, \mathbb{E}_H \times_{Ad} H\right)$ is the gauge group of $\left(\mathbb{E}_H, h\right)$ for $H = \mathrm{U}\left(2\right)$.

1.2.4 Stability of an Sp(4,ℝ)-Higgs bundle

In order to state explicitly the notions of stability, semistability and polystability for an Sp(4,ℝ)-Higgs bundle (V, β, γ), consider the short exact sequence

$$0 \to L^{\perp} \to V^* \to L^* \to 0$$

for any line subbundle $L \subset V$ and for $L^{\perp}$ the subbundle of V^* in the kernel of the projection of L^*. The following two propositions are proven in [18]:

Proposition 1.2.4. *An Sp(4,ℝ)-Higgs bundle (V, β, γ) is semistable if and only if all the following conditions hold:*

1. *If $\beta = 0$, then* $\deg\left(V\right) \geq 0$.

2. *If* $\gamma = 0$, *then* $\deg(V) \leq 0$.

3. *Let* $L \subset V$ *be a line subbundle.*

   a *If* $\beta \in H^0(L \otimes_S V \otimes K)$ *and* $\gamma \in H^0\left(L^{\perp} \otimes_S V^* \otimes K\right)$, *then* $\deg(L) \leq \deg(V)/2$ *.*

   b *If* $\gamma \in H^0\left(\left(L^{\perp}\right)^2 \otimes K\right)$, *then* $\deg(L) \leq 0$.

   c *If* $\beta \in H^0\left(L^2 \otimes K\right)$, *then* $\deg(L) \leq \deg(V)$.

If, in addition, strict inequalities hold in (3), then (V, β, γ) *is stable.*

Proposition 1.2.5. *An Sp(4,ℝ)-Higgs bundle* (V, β, γ) *is polystable, if it is either stable, or there is a decomposition* $V = L_1 \oplus L_2$ *of the bundle* V *as a direct sum of line bundles, such that one of the following conditions is satisfied:*

1. *The Higgs fields satisfy* $\beta = \beta_1 + \beta_2$ *and* $\gamma = \gamma_1 + \gamma_2$, *where*

$$\beta_i \in H^0\left(L_i^2 \otimes K\right) \text{ and } \gamma_i \in H^0\left(L_i^{-2} \otimes K\right), \ i = 1, 2$$

   and the SL(2,ℝ)-Higgs bundles (L_i, β_i, γ_i) *are polystable for* $i = 1, 2$.

2. *The Higgs fields satisfy*

$$\beta \in H^0\left((L_1 L_2 \oplus L_2 L_1) \otimes K\right) \text{ and } \gamma \in H^0\left(\left(L_1^{-1} L_2^{-1} \oplus L_2^{-1} L_1^{-1}\right) \otimes K\right).$$

   Furthermore, $\deg(L_1) = \deg(L_2)$ *and the rank 2 Higgs bundle* $\left(L_1 \oplus L_2^{-1}, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$ *is polystable.*

Having seen that the defining Sp(4,ℝ)-data of a Higgs bundle can be reinterpreted as special SL(4,ℂ)-data in the original sense of N. Hitchin, it is useful to relate the above described stability conditions of an Sp(4,ℝ)-Higgs bundle to the ones for an SL(4,ℂ)-Higgs bundle. Recall that a GL(4,ℂ)-Higgs bundle (E, ϕ) is *stable* if any proper non-zero ϕ-invariant subbundle $F \subseteq E$ satisfies $\mu(F) < \mu(E)$, for $\mu(F) = \deg(F)/\mathrm{rk}(F)$, the slope of the bundle. The following proposition is proven in [17]:

Proposition 1.2.6. *An Sp(4,ℝ)-Higgs bundle* (V, β, γ) *is polystable if and only if the GL(4,ℂ)-Higgs bundle* $\left(V \oplus V^*, \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$ *is polystable. Moreover, even though the polystability conditions coincide, the stability condition for an Sp(4,ℝ)-Higgs bundle is in general weaker than the stability condition for the corresponding GL(4,ℂ)-Higgs bundle.*

1.3 Connected components of $\mathcal{M}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$

1.3.1 The Toledo invariant and Cayley partner

In this section we consider the basic topological invariant of an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle and describe a sharp bound for it. Let $X = (\Sigma, J)$ a compact Riemann surface with underlying topological surface Σ. The locally constant obstruction map

$$o_2 : \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbb{R})) \to H^2(\Sigma, \pi_1(\mathrm{Sp}(4,\mathbb{R})))$$

is an integer valued function, since $H^2(\Sigma, \pi_1(\mathrm{Sp}(4,\mathbb{R}))) \simeq \pi_1(\mathrm{Sp}(4,\mathbb{R})) \simeq \mathbb{Z}$. Now, $o_2(\rho) = c_1(V)$, where V is the rank 2 vector bundle appearing in the Higgs bundle data (V, β, γ) corresponding to ρ via the non-abelian Hodge correspondence. Thus, we have an integer valued function $d = \deg(V) = \langle c_1(V), [\Sigma]\rangle$, whose fibers are unions of connected components.

Definition 1.3.1. The *Toledo invariant* of an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V, β, γ) is defined as the integer

$$d = \deg(V)$$

We use the notation $\mathcal{M}_d = \mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ to denote the moduli space parameterizing isomorphism classes of polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles with $\deg(V) = d$.

Remark 1.3.2.

- For representations of $\pi_1(\Sigma)$ into $\mathrm{SL}(2,\mathbb{R}) \simeq \mathrm{Sp}(2,\mathbb{R})$ the Toledo invariant coincides with the Euler class of the corresponding flat $\mathrm{SL}(2,\mathbb{R})$-bundle. In this case the classical inequality of J. Milnor [31] provides an appropriate bound for this invariant:

  $$|d| = |e(\rho)| \leq -\chi(\Sigma) = 2g - 2$$

  Later on, J. Wood [46] gave a similar bound considering $\mathrm{SU}(1,1)$-bundles, and so this is usually now called the *Milnor-Wood inequality* in describing a sharp bound for the topological invariant, also for representations into more general Lie groups G.

- T. Hartnick and A. Ott describe in [23] how the generalized Milnor-Wood inequality of M. Burger and A. Iozzi [12] translates under the non-abelian Hodge correspondence to an inequality for topological invariants of Higgs bundles.

The sharp bound below for the Toledo invariant when $G = \mathrm{Sp}(4,\mathbb{R})$ was first given by V. Turaev [44]. We include here however a proof by P. Gothen [21] in the Higgs bundle context, as this proof will be particularly instructive for the sequel.

Proposition 1.3.3. *(Milnor-Wood inequality) Let (V, β, γ) be a semistable $Sp(4, \mathbb{R})$-Higgs bundle. Then $|d| \leq 2g-2$.*

Proof. For this proof, it is more convenient to consider the interpretation of the defining data for an $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundle as data for a special $\mathrm{SL}(4, \mathbb{C})$-Higgs bundle. Moreover, the map $(V, \beta, \gamma) \mapsto (V^*, \gamma^t, \beta^t)$ provides an isomorphism $\mathcal{M}_d \simeq \mathcal{M}_{-d}$, thus we can restrict our attention to the case $d \geq 0$.

Let (E, Φ) with $E = V \oplus V^*$, $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ be a semistable $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundle and $d = \deg(V) \geq 0$. Then $\gamma \neq 0$, as otherwise V would be Φ-invariant and so would violate the stability condition, since

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{\deg(V \oplus V^*)}{\mathrm{rk}(E)} = 0 \text{ and } \mu(V) = \frac{\deg(V)}{\mathrm{rk}(V)} = \frac{d}{2} \geq 0$$

Consider the bundles $N = \ker(\gamma)$ and $I = \mathrm{Im}(\gamma) \otimes K^{-1} \leq V^*$.

We thus get an exact sequence of bundles

$$0 \to N \to V \to I \otimes K \to 0$$

and so

$$\begin{aligned} \deg(V) = & \deg(N) + \deg(I \otimes K) \\ = & \deg(N) + \deg(I) + \mathrm{rk}(I)(2g-2) \end{aligned}$$

using that $\deg K = 2g - 2$.

Now, the bundles $N, V \oplus I \subset E$ are both Φ-invariant subbundles of E, thus from the semistability of (E, Φ) we get $\mu(N) \leq \mu(E)$ and $\mu(V \oplus I) \leq \mu(E)$. Therefore

$$\deg(N) \leq 0 \text{ and } d + \deg(I) \leq 0$$

We have also seen that

$$d = \deg(N) + \deg(I) + \mathrm{rk}(I)(2g-2)$$

so from these relations we get

$$2d \leq \mathrm{rk}(I)(2g-2)$$

and since $\mathrm{rk}(I) = \mathrm{rk}(\gamma) \leq 2$, we imply the desired inequality. $\square$

Definition 1.3.4. We shall call $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundles with Toledo invariant $d = 2g - 2$

maximal and denote the components of $\mathcal{M}(\mathrm{Sp}(4,\mathbb{R}))$ with maximal positive Toledo invariant by $\mathcal{M}^{\max} \simeq \mathcal{M}_{2g-2}$.

The Higgs bundle proof of Proposition 1.3.3 opens the way to considering new topological invariants for our Higgs bundles in order to successfully compute the number of components of $\mathcal{M}^{\max}$. Namely, we see from this proof that for a maximal semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ), the map $\gamma : V \to V^* \otimes K$ is an *isomorphism*. Moreover, since γ is symmetric, it equips V with a K-valued non-degenerate quadratic form.

Remark 1.3.5. Having considered $-(2g-2) \leq d \leq 0$ in the proof of the proposition, then $\beta : V^* \to V \otimes K$ would be an isomorphism.

Fix a square root of the canonical bundle K, i.e. pick a line bundle L_0 such that $L_0^2 = K$ and define

$$W := V^* \otimes L_0$$

Then the map

$$q_W := \gamma \otimes I_{L_0^{-1}} : W^* \to W$$

defines a symmetric, non-degenerate form on W; in other words (W, q_W) defines an $O(2,\mathbb{C})$-*holomorphic bundle*. Moreover, the map β in (V,β,γ) defines a K^2-twisted endomorphism

$$\theta := (\gamma \otimes I_{K\otimes L_0}) \circ (\beta \circ I_{L_0}) : W \to W \otimes K^2$$

which is q_W-symmetric, i.e takes values in the isotropy representation for $GL(2,\mathbb{R})$. We say that (W,θ) defines a *K^2-twisted Higgs pair* with structure group $GL(2,\mathbb{R})$, i.e. θ takes values in $E\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes K^2$.

Definition 1.3.6. We call (W, q_W, θ) the *Cayley partner* of the $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V,β,γ).

The original $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data can clearly be recovered from the defining data of its Cayley partner, so the previous construction describes a well-defined correspondence $(V,\beta,\gamma) \mapsto (W, q_W, \theta)$. A careful comparison of the semistability condition for the maximal $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles (V,β,γ) and the one for their Cayley partners provides the following:

Theorem 1.3.7. *Let $\mathcal{M}^{\max}$ be the moduli space of polystable $Sp(4,\mathbb{R})$-Higgs bundles with degree $d = 2g-2$ and let $\mathcal{M}'$ be the moduli space of polystable K^2-twisted $GL(2,\mathbb{R})$-Higgs pairs. The map $(V,\beta,\gamma) \mapsto (W, q_W, \theta)$ defines an isomorphism of complex algebraic varieties*

$$\mathcal{M}^{\max} \simeq \mathcal{M}'$$

Proof. see [17], Theorem 4.3. □

Remark 1.3.8. The theorem holds for polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles with $n \geq 2$ in general, and the correspondence discussed is referred to as the *Cayley correspondence*.

The Cayley correspondence brings in new topological invariants for our triples (V,β,γ), namely the first and second Stiefel-Whitney classes of the orthogonal bundle (W,q_W) underlying the Cayley partner:

$$w_1(W,q_W) \in H^1(X,\mathbb{Z}/2\,) \simeq (\mathbb{Z}/2\,)^{2g}$$

$$w_2(W,q_W) \in H^2(X,\mathbb{Z}/2\,) \simeq \mathbb{Z}/2$$

Therefore, we may define

$$w_i(V,\beta,\gamma) := w_i(W,q_W)\,,\ \ i\text{=}1\text{,}2$$

and these invariants are well defined, because the Stiefel-Whitney classes are independent of the choice of the square root $L_0 = K^{1/2}$ used in the definition of (W,q_W).

1.3.2 The components of $\mathcal{M}^{\max}(X,\mathrm{Sp}(4,\mathbb{R}))$

In the previous section we have seen how $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles can be related to rank 2 orthogonal bundles, and the latter were classified by D. Mumford in [32]. For our purposes we will be needing the following result from that article:

Proposition 1.3.9. *Let (W,q_W) be a rank 2 orthogonal bundle. If $w_1(W,q_W) = 0$, then $W = L \oplus L^{-1}$, where L is a line bundle over X, and $q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

Having this result in hand, we now obtain a first important description of the maximal semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data (cf. §3.6 in [9]):

Proposition 1.3.10. *Let (V,β,γ) be a maximal semistable $Sp(4,\mathbb{R})$-Higgs bundle with $w_1(V,\beta,\gamma) = 0$ and let (W,q_W) be its Cayley partner, so $W = L \oplus L^{-1}$ and $q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then there is a line bundle N such that*

1. *$V = N \oplus N^{-1}K$ and with respect to this decomposition, the Higgs fields are $\beta = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix} \in H^0(S^2V \otimes K)$ and $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in H^0(S^2V^* \otimes K)$*

2. *The degree of N is given by* $\deg(N) = \deg(L) + g - 1$

3. *The degree of L satisfies* $0 \leq \deg(L) \leq 2g-2$ *and for* $\deg(L) > 0$, *it is* $\beta_2 \neq 0$.

4. *When* $\deg(L) > 0$, *N is unique.*
   When $\deg(L) = 0$, *N is unique up to multiplication by a square root of the trivial bundle.*
   When $\deg(L) = 2g-2$, *N satisfies* $N^2 = K^3$.

Proof. (1) Consider $N := L \otimes L_0$. Then $V = W \otimes L_0 = (L \oplus L^{-1}) \otimes L_0 = N \oplus N^{-1}K$. Moreover, $\gamma = q \otimes I_{L_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : (V^* \otimes L_0) \otimes L_0 \to (L_0^* \otimes V) \otimes L_0$ and since $\theta = (\gamma \otimes I_{K \otimes L_0}) \circ \beta \otimes I_{L_0}$ is q_W-symmetric, it turns out that $\beta = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix} : V^* \to V \otimes K$.
(2) Since $N = L \otimes L_0$, then $\deg(N) = \deg(L_0) + \deg(L) = \deg(L) + g - 1$.
(3) Interchanging L with its dual if necessary we may assume that $\deg(L) \geq 0$. Now, whenever $\deg(L) > 0$, the Higgs field θ must induce a non-zero holomorphic map $L \to L^{-1}K^2$ otherwise $L \subset W$ would violate the stability condition, since $\theta : L \oplus L^{-1} \to (L \oplus L^{-1}) \otimes K^2 = LK^2 \oplus L^{-1}K^2$ and θ should not preserve L. Hence global sections exist for the line bundle $L^{-2}K^2$, therefore $\deg(L^{-2}K^2) \geq 0$, i.e. $\deg(L) \leq 2g-2$. The fact that for $\deg(L) > 0$, β_2 is non-zero, follows also from the semistability condition.
(4) When $\deg(L) = 2g-2$, the Higgs field θ induces a non-zero section of the degree 0 line bundle $L^{-2}K^2$, thus $L^2 = K^2$ and so $N^2 = (LL_0)^2 = K^3$. □

Provoked by this proposition, we distinguish the Higgs bundles in $\mathcal{M}^{\max}$ in the following subfamilies:
(i) (V, β, γ) for which $w_1 \neq 0$.
(ii) (V, β, γ) for which $w_1 = 0$, and therefore $V = N \oplus N^{-1}K$ with $N := L \otimes L_0$ for $L_0 = K^{1/2}$ and $0 \leq \deg(L) \leq 2g-2$.
(iii) As a special case of (ii), (V, β, γ) with $\deg(L) = 2g-2$, in which case $N^2 = K^3$; thus such Higgs bundles are parameterized by spin structures $L_0 = K^{1/2}$ on the surface Σ underlying the Riemann surface X.
This motivates considering the following subspaces of the moduli space $\mathcal{M}^{\max}$ and we shall see next that these are actually connected components in $\mathcal{M}^{\max}$.

Definition 1.3.11. Let (V, β, γ) be a maximal $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundle with topological invariants $w_1(W, q_W) \in H^1(X, \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{2g}$ and $w_2(W, q_W) \in H^2(X, \mathbb{Z}/2) \simeq \mathbb{Z}/2$. Define the following subspaces of $\mathcal{M}^{\max}$:

1. $\mathcal{M}_{w_1, w_2} = \{(V, \beta, \gamma) \,|\, w_1 = w_1(V, \beta, \gamma) \neq 0,\ w_2 = w_2(V, \beta, \gamma)\}/\simeq$

2. $\mathcal{M}_c^0 = \{(V,\beta,\gamma)\,|\,w_1(V,\beta,\gamma)=0, 0\le c<2g-2, \text{ for } c{:=}\deg(L)\}/\simeq$

3. $\mathcal{M}^T_{K^{1/2}} = \left\{(V,\beta,\gamma)\,\middle|\,V=N\oplus N^{-1}K \text{ with } N=K^{3/2}\right\}/\simeq$

where $\simeq$ indicates isomorphism classes of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles.

Theorem 1.3.12 (P. Gothen [21])**.** *The subspaces $\mathcal{M}_{w_1,w_2}$, $\mathcal{M}_c^0$, $\mathcal{M}^T_{K^{1/2}}$ are connected. Hence, $\mathcal{M}^{\max}$ decomposes in its connected components as*

$$\mathcal{M}^{\max} = \left(\bigcup_{w_1,w_2}\mathcal{M}_{w_1,w_2}\right)\bigcup\left(\bigcup_{0\le c<2g-2}\mathcal{M}_c^0\right)\bigcup\left(\bigcup_{K^{1/2}}\mathcal{M}^T_{K^{1/2}}\right)$$

and so the total number of connected components of this moduli space is $2\cdot(2^{2g}-1)+2g-2+2^{2g}=3\cdot 2^{2g}+2g-4$.

Remark 1.3.13. From N. Hitchin's fundamental article [25], we knew already that there exists a distinguished component of $\mathcal{M}(\mathrm{Sp}(4,\mathbb{R}))$, the *Hitchin component*, isomorphic to a vector space and containing naturally the Teichmüller space. This actually shows that there are exactly 2^{2g} such components, which are precisely the components $\mathcal{M}^T_{K^{1/2}}$ parameterized by the spin structures on the surface Σ.

Proof. We treat each case separately:
(i) $\mathcal{M}^T_{K^{1/2}}$ is connected. The Cayley partner (W,q_W) of a Higgs bundle $(V,\beta,\gamma)\in\mathcal{M}^T_{K^{1/2}}$ is completely determined by the line bundle L in the decomposition $W=L\oplus L^{-1}$. But here $L=K^{1/2}$ and every (W,q_W) is stable. Hence,

$$\mathcal{M}^T_{K^{1/2}} \simeq H^0\left(\Sigma,\mathrm{End}(W)\otimes K^2\right)$$

and the Higgs field is q_W-symmetric, i.e. $\Phi=\begin{pmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{12} & \Phi_{22}\end{pmatrix}$. Therefore $\mathcal{M}^T_{K^{1/2}}$ is isomorphic to the vector space $H^0(\Sigma,K^2)\oplus H^0(\Sigma,K^2)\oplus H^0(\Sigma,K^4)$.
(ii) $\mathcal{M}_c^0$ is connected. The proof is based on the study of the local minima of the proper Hitchin map on $\mathcal{M}_c^0$.
For $c>0$, the Higgs field Φ must be non-zero, otherwise the subbundle $L\subset W$ for the Cayley partner (W,q_W) would violate the stability condition. Moreover, for the critical points in $\mathcal{M}_c^0$, $\Phi=\begin{pmatrix}0 & 0\\ \tilde{\Phi} & 0\end{pmatrix}$ with $\tilde{\Phi}\in H^0(\Sigma,L^{-2}K^2)$. Now, the subspace of local minima $\mathcal{N}_c^0\subset\mathcal{M}_c^0$ fits into the pullback diagram

$$\begin{array}{ccc}
\mathcal{N}_c^0 & \longrightarrow & \mathrm{Jac}^c(\Sigma) \\
\Big\downarrow \pi & & \Big\downarrow L\to L^{-2}K^2 \\
S^{4g-4-2c}\Sigma & \xrightarrow{D\to[D]} & \mathrm{Jac}^{4g-4-2c}(\Sigma)
\end{array}$$

where $\pi(W, q_W, \Phi) = (\Phi)$.

Thus, $\mathcal{N}_c^0$ is connected, so from the properness of the Hitchin map $f : \mathcal{M}_c^0 \to \mathbb{R}$, it follows that $\mathcal{M}_c^0$ is connected, for $c > 0$.

For $c = 0$, every local minimum of f on $\mathcal{M}_c^0$ has $\Phi = 0$, so the subspace of local minima is isomorphic to the moduli space of polystable (W, q_W), where $W = L \oplus L^{-1}$ with $\deg(L) = 0$. It follows that there is a surjective continuous map $\mathrm{Jac}^0(\Sigma) \to \mathcal{N}_0^0$, with $L \mapsto (W, q_W)$, and so $\mathcal{N}_0^0$ is connected.

(iii) $\mathcal{M}_{w_1,w_2}$ is connected. We shall include here just a sketch; for the complete proof see Theorem 5.8 in [21].

Similarly to the previous part, we are trying to show that the subspace of local minima of the Hitchin map $\mathcal{N}_{w_1,w_2} \subset \mathcal{M}_{w_1,w_2}$ is connected. These subspaces consist of critical points (V, β, γ) with $\beta = 0$ and $\gamma \neq 0$. There is a connected double cover $\tilde{\Sigma} \to \Sigma$ given by $w_1 \in H^1(\Sigma, \mathbb{Z}/2)$. Then it turns out that $\mathcal{N}_{w_1,0} \cup \mathcal{N}_{w_1,1} = \ker(1 + \tau^*)$, where $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ is the involution interchanging the sheets of the covering.

Now, $\ker(1 + \tau^*) = P^+ \cup P^-$ where the two components P^+ and P^- are the abelian varieties associated to the double cover of Σ given by w_1, each of them a translate of the Prym variety of the covering. Then $\mathcal{N}_{w_1,0} \cup \mathcal{N}_{w_1,1} = P^+ \cup P^-$, hence $\mathcal{N}_{w_1,w_2}$ is connected. □

The description of a maximal $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle from the data of its Cayley partner, as well as Proposition 1.3.10 and Theorem 1.3.12, provide a description of the $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data in each connected component of $\mathcal{M}^{\max}$. This information is summarized in the following table:

Table 1.1: $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle data in the connected components of $\mathcal{M}^{\max}$

Component	V	β	γ
$\mathcal{M}_{K^{1/2}}^T$	$K^{3/2} \oplus K^{-1/2}$	$\begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & 1 \end{pmatrix}, \begin{cases} \beta_3 \in H^0(K^2) \\ \beta_1 = \mathrm{const.}(\beta_3)^2 \end{cases}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathcal{M}_c^0$	$V = N \oplus N^{-1}K$, with $g - 1 < \deg(N) < 3g - 3$	$\begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix}$, with $\beta_2 \neq 0$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathcal{M}_0^0$	$V = N \oplus N^{-1}K$, with $\deg(N) = g - 1$	$\begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathcal{M}_{w_1,w_2}$	$V = W \otimes L_0$, with $L_0^2 = K$	$\beta \in H^0(S^2 V \otimes K)$	$\gamma = q_W \otimes I_{L_0}$

So far we have been interested in identifying particular polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles in the connected components of $\mathcal{M}^{\max}$. The non-abelian Hodge theorem provides a homeomorphism

$$\mathcal{R}^{\max} \simeq \mathcal{M}^{\max}$$

to a moduli space of representations $\mathcal{R}^{\max}$, which we briefly introduce next:

Let G be a Hermitian Lie group of non-compact type, that is, the symmetric space associated to G is an irreducible Hermitian symmetric space of non-compact type. Using the identification $H^2(\pi_1(\Sigma), \mathbb{R}) \simeq H^2(\Sigma, \mathbb{R})$, the *Toledo invariant* of a representation $\rho : \pi_1(\Sigma) \to G$ is defined as the integer

$$T_\rho := \langle \rho^*(\kappa_G), [\Sigma] \rangle,$$

where $\rho^*(\kappa_G)$ is the pullback of the Kähler class $\kappa_G \in H_c^2(G, \mathbb{R})$ of G and $[\Sigma] \in H_2(\Sigma, \mathbb{R})$ is the orientation class. The Toledo invariant is bounded in absolute value:

$$|T_\rho| \leq -C(G)\chi(\Sigma),$$

where $C(G)$ is an explicit constant depending only on G; we refer the reader to [12] for more details.

Definition 1.3.14. A representation $\rho : \pi_1(\Sigma) \to G$ is called *maximal* whenever $T_\rho = -C(G)\chi(\Sigma)$.

The moduli space of maximal representations into $\mathrm{Sp}(4,\mathbb{R})$ is now denoted here by $\mathcal{R}^{\max}$, and analogously to the space $\mathcal{M}^{\max}$ we consider its following subspaces

$$\mathcal{R}_{w_1,w_2} \simeq \mathcal{M}_{w_1,w_2}, \quad \mathcal{R}_c^0 \simeq \mathcal{M}_c^0, \quad \mathcal{R}_{K^{1/2}}^T \simeq \mathcal{M}_{K^{1/2}}^T,$$

which are furthermore connected components in $\mathcal{R}^{\max}$.

Now, the possible subgroups of $\mathrm{Sp}(4,\mathbb{R})$ through which a maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ can factor, can be explicitly described:

Proposition 1.3.15. *Let $\rho : \pi_1(\Sigma) \to Sp(4,\mathbb{R})$ be maximal and assume that ρ factors through a proper reductive subgroup $\tilde{G} \subset Sp(4,\mathbb{R})$. Then, up to conjugation, the group $\tilde{G}$ is contained in one of the subgroups G_i, G_Δ and G_p, where*

1. *G_i, the normalizer of the irreducible four-dimensional representation of $SL(2,\mathbb{R})$ into $Sp(4,\mathbb{R})$.*

2. *G_p, the normalizer of the product representation $\rho_p : SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to Sp(4,\mathbb{R})$*

3. *G_Δ, the normalizer of the composition of ρ_p with the diagonal embedding of $SL(2,\mathbb{R})$ into $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.*

Proof. See §4 in [9] and the references therein. $\square$

Defining the group $\mathrm{Sp}(4,\mathbb{R})$ with respect to the symplectic form $J_{12} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, explicit calculations show:

1. $G_i = \mathrm{SL}(2,\mathbb{R})$

2. $G_p = \left\{ \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{R}) \mid \textit{either } Y = Z = 0 \textit{ or } X = T = 0 \right\}$

3. $G_\Delta = \left\{ \begin{pmatrix} xA & yA \\ zA & tA \end{pmatrix} \middle| X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{O}(2) \text{ and } A \in \mathrm{SL}(2,\mathbb{R}) \right\} = \mathrm{O}(2) \otimes \mathrm{SL}(2,\mathbb{R})$

We would like to identify in which connected components of $\mathcal{R}^{\max}$ we can find representations that can factor through one of the subgroups G_i, G_Δ or G_p described above. According to the non-abelian Hodge correspondence, a reductive representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ that factors through a proper reductive subgroup $G_* \subset \mathrm{Sp}(4,\mathbb{R})$ corresponds to a polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle (V, β, γ) for which the structure group reduces to G_* (cf. §1.1.5).

Therefore, for each of the possible reductive subgroups $G_* \subset \mathrm{Sp}(4,\mathbb{R})$, we first need to describe the defining data for the G_*-Higgs bundles, then describe the semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles for which the structure group reduces to G_* and lastly, using the information from Table 1.1 we can see in which connected component these Higgs bundles lie.

Eventually, we get the following picture for the $3 \cdot 2^{2g} + 2g - 4$ many connected components of $\mathcal{M}^{\max}$, regarding particular fundamental group representations in these components:

- <u>2^{2g} *Hitchin components* $\mathcal{M}^T_{K^{1/2}}$</u>

$$\left\{ \begin{array}{c} \mathrm{Sp}(4,\mathbb{R}) - \text{Higgs bundles} \\ \text{str.gp. reduces to } \mathrm{SL}(2,\mathbb{R}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R}) \\ \text{factors through } \mathrm{SL}(2,\mathbb{R}) \end{array} \right\}$$

- $\underline{2 \cdot 2^{2g} - 1 \textit{ components } \mathcal{M}_{w_1,w_2}, \mathcal{M}_0^0}$

$$\left\{\begin{matrix}\mathrm{Sp}\left(4,\mathbb{R}\right)-\text{Higgs bundles}\\ \text{str. gp. reduces to } G_p\\ \text{and}\\ \mathrm{Sp}\left(4,\mathbb{R}\right)-\text{Higgs bundles}\\ \text{str. gp. reduces to } G_\Delta\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\rho:\pi_1\left(\Sigma\right)\to\mathrm{Sp}\left(4,\mathbb{R}\right)\\ \text{factors through } G_p\\ \text{and}\\ \rho:\pi_1\left(\Sigma\right)\to\mathrm{Sp}\left(4,\mathbb{R}\right)\\ \text{factors through } G_\Delta\end{matrix}\right\}$$

- $\underline{2g - 3 \textit{ components } \mathcal{M}_c^0}$

$$\left\{\begin{matrix}\mathrm{Sp}\left(4,\mathbb{R}\right)-\text{Higgs bundles}\\ \text{str.gp. does not reduce}\\ \text{to any } G_*\subset\mathrm{Sp}\left(4,\mathbb{R}\right)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\rho:\pi_1\left(\Sigma\right)\to\mathrm{Sp}\left(4,\mathbb{R}\right)\\ \text{does not factor}\\ \text{through any } G_*\subset\mathrm{Sp}\left(4,\mathbb{R}\right)\end{matrix}\right\}$$

From the investigation summarized in this section we conclude to the following result (cf. [9]):

Theorem 1.3.16. *Among the $3 \cdot 2^{2g} + 2g - 4$ connected components of $\mathcal{M}^{\max} \simeq \mathcal{R}^{\max}$, there are $2g - 3$ components where the corresponding Higgs bundles do not admit a reduction of structure group to any proper reductive subgroup of Sp$\left(4,\mathbb{R}\right)$. Equivalently, the corresponding representations do not factor through any proper reductive subgroup of Sp$\left(4,\mathbb{R}\right)$, thus they have Zariski-dense image in Sp$\left(4,\mathbb{R}\right)$.*

Remark 1.3.17. Quite differently than the $\mathrm{Sp}\left(4,\mathbb{R}\right)$-case, the moduli space of maximal polystable $\mathrm{Sp}\left(2n,\mathbb{R}\right)$-Higgs bundles has $3 \cdot 2^{2g}$ many connected components for every $n \geq 3$, and any $\mathrm{Sp}\left(2n,\mathbb{R}\right)$-Higgs bundle in those can be deformed to a G_*-Higgs bundle for some proper reductive Zariski closed subgroup $G_* \subset \mathrm{Sp}(2n,\mathbb{R})$. This distinction arises from the structure group of the Cayley partner. In general, the Cayley partner of a maximal $\mathrm{Sp}\left(2n,\mathbb{R}\right)$-Higgs bundle is described by an $\mathrm{O}\left(n,\mathbb{C}\right)$-bundle and for vanishing first Stiefel-Whitney class, it admits a reduction of structure group to $\mathrm{SO}\left(n,\mathbb{C}\right)$. For $n = 2$, however, this indicates special cases in the classification of those bundles, leading to extra components in the maximal $\mathrm{Sp}\left(4,\mathbb{R}\right)$-Higgs bundle moduli space (cf. §9 in [9] and §8 in [17]).

Let (V,β,γ) be a maximal semistable $\mathrm{Sp}\left(4,\mathbb{R}\right)$-Higgs bundle in the exceptional $2g-3$ components described above. Next we collect some results concerning these Higgs bundles, the first three of which we have already seen.

1. For the Cayley partner ($W = L \oplus L^{-1}, q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) the first Stiefel-Whitney class vanishes: $w_1\left(V,\beta,\gamma\right) = w_1\left(W,q_W\right) = 0$, where L is a line bundle on X.

2. The bundle V decomposes as $V = N \oplus N^{-1}K$, for a line bundle N with $\deg(N) = \deg(L) + g - 1$ and $g - 1 < \deg(N) < 3g - 3$, in other words $0 < \deg(L) < 2g - 2$.

3. The Higgs fields with respect to this decomposition for V are $\beta = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix} \in H^0(S^2V \otimes K)$, with $\beta_2 \neq 0$ and $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in H^0(S^2V^* \otimes K)$.

4. Furthermore, since $0 < \deg(L) < 2g - 2 = \deg(V)$, all points in the exceptional components are represented by stable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles. From this fact, it follows that these Higgs bundles are smooth points in the moduli space. This is proven using the standard slice method construction used to prove that the moduli space $\mathcal{M}_d(G)$ has the structure of a complex analytic variety (see Proposition 3.18 in [18] and the discussion preceding this). Hence, *the exceptional* $2g-3$ *components are smooth.*

   Remark 1.3.18. Using these same arguments, one shows that all $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles in the 2^{2g}-many Hitchin components $\mathcal{M}^T_{K^{1/2}}$ are stable with $\beta_2 \neq 0$, and smooth as well.

5. Isomorphism classes of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles in the exceptional components can be also described. Considering a representative of such an isomorphism class to be determined by a triple $(N, \beta_1, \beta_2, \beta_3)$, the following holds (see Proposition 3.28 in [9]):

   Proposition 1.3.19. *Fix $c = \deg(L)$ with $0 < c < 2g-2$. Tuples $(N, \beta_1, \beta_2, \beta_3)$ and $(N', \beta'_1, \beta'_2, \beta'_3)$ define the same isomorphism class in $\mathcal{M}^0_c$ if and only if $N = N'$ and $(\beta'_1, \beta'_2, \beta'_3) = (t^2\beta_1, t^{-2}\beta_2, \beta_3)$, for some $t \in \mathbb{C}^*$.*

6. Lastly, there is a fibration of a certain subfamily of the exceptional components over the Jacobian Jac^d of degree d line bundles on X (see Proposition 3.30 in [9]):

   Proposition 1.3.20. *For $0 < c < g-1$, the space $\mathcal{M}^0_c$ fibers over Jac^d with $d = c+g-1$, and the fibers are given by*

$$\mathcal{F}^d = \left[\left(\mathbb{C}^r \oplus (\mathbb{C}^*)^{s+1} \right) / \mathbb{C}^* \right] \times \mathbb{C}^{3g-3}$$

   where $r = 2c + 3g - 3$, $s = 3g - 4 - 2c$ and the $\mathbb{C}^*$-action is given by the relation $t(\vec{z}, \vec{w}) = (t^2\vec{z}, t^{-2}\vec{w})$.

1.4 Maximal fundamental group representations into $\mathrm{Sp}(4,\mathbb{R})$ and topological gluing constructions

In [22], O. Guichard and A. Wienhard describe model maximal fundamental representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ in the components of $\mathcal{R}^{\max}$. These models are distinguished into two subcategories, *standard representations* and *hybrid representations.*

As standard representations are considered the ones which come from homomorphisms of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{Sp}(4,\mathbb{R})$, possibly twisted by a representation of $\pi_1(\Sigma)$ into the centralizer of the image of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{Sp}(4,\mathbb{R})$. In this case, $\rho(\pi_1(\Sigma))$ is contained by construction into a proper closed Lie subgroup of $\mathrm{Sp}(4,\mathbb{R})$. On the other hand, considering $\Sigma = \Sigma_l \cup_\gamma \Sigma_r$ a decomposition of Σ along a simple closed oriented separating geodesic γ into two subsurfaces Σ_l and Σ_r, a hybrid representation is defined to be a representation $\rho = \rho_l * \rho_r$ constructed by amalgamation of two specific representations ρ_l, ρ_r on $\pi_1(\Sigma_l)$, $\pi_1(\Sigma_r)$ respectively, with $\rho_l(\gamma) = \rho_r(\gamma)$.

We now describe these model representations in further detail with particular notice towards the construction of these hybrid representations. Let us first fix a discrete embedding $i : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$.

i) *Irreducible Fuchsian representations*

Choose the symplectic identification $(\mathbb{R}_3[X,Y], -\omega_2) \cong (\mathbb{R}^4, \omega)$ given by $X^3 = e_1$, $X^2Y = -e_2$, $Y^3 = -e_3$, $XY^2 = \frac{-e_4}{\sqrt{3}}$, where ω is the symplectic form given by the antisymmetric matrix $J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$. With respect to this identification the irreducible representation $\phi_{irr} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(4,\mathbb{R})$ is given by

$$\phi_{irr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^3 & -\sqrt{3}a^2b & -b^3 & -\sqrt{3}ab^2 \\ -\sqrt{3}a^2c & 2abc + a^2d & \sqrt{3}b^2d & 2abd + b^2c \\ -c^3 & \sqrt{3}c^2d & d^3 & \sqrt{3}cd^2 \\ -\sqrt{3}ac^2 & 2acd + bc^2 & \sqrt{3}bd^2 & 2bcd + ad^2 \end{pmatrix}$$

Note that this choice has been made so that $(\phi_{irr})_* : \pi_1(\mathrm{SL}(2,\mathbb{R})) \to \pi_1(\mathrm{Sp}(4,\mathbb{R}))$ is the multiplication by 2. Precomposition with $i : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ gives rise to an *irreducible Fuchsian representation*

$$\rho_{irr} : \pi_1(\Sigma) \xrightarrow{i} \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\phi_{irr}} \mathrm{Sp}(4,\mathbb{R})$$

ii) *Diagonal Fuchsian representations*

Let $\mathbb{R}^4 = W_1 \oplus W_2$, with $W_i = \mathrm{span}(e_i, e_{2+i})$ be a symplectic splitting of $\mathbb{R}^4$ with respect to the symplectic basis $(e_i)_{i=1,\dots,4}$. This splitting gives rise to an embedding $\psi : \mathrm{SL}(2,\mathbb{R})^2 \to \mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2) \subset \mathrm{Sp}(4,\mathbb{R})$ given by

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ c & 0 & d & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}$$

Precomposition with the diagonal embedding of $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})^2$ gives rise to the diagonal embedding $\phi_\Delta : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(4,\mathbb{R})$.
Note that the choice of ψ has been made so that $(\phi_\Delta)_*$ is the multiplication by 2.
Precomposition with $i : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ gives now rise to a *diagonal Fuchsian representation*

$$\rho_\Delta : \pi_1(\Sigma) \xrightarrow{i} \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\phi_\Delta} \mathrm{Sp}(4,\mathbb{R})$$

iii) *Twisted diagonal representations*

For any maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ the centralizer $\rho(\pi_1(\Sigma))$ is a subgroup of $\mathrm{O}(2)$. Considering now a representation $\Theta : \pi_1(\Sigma) \to \mathrm{O}(2)$, set

$$\begin{aligned} \rho_\Theta = i \otimes \Theta : \pi_1(\Sigma) &\to \mathrm{Sp}(4,\mathbb{R}) \\ \gamma &\mapsto \phi_\Delta(i(\gamma), \Theta(\gamma)) \end{aligned}$$

Such representations will be called *twisted diagonal representation.*

Remark 1.4.1. The representations in the families (i)-(iii) above are the so-called standard representations.

iv) *Hybrid representations*

The definition of hybrid representations involves a gluing construction for fundamental group representations over a connected sum of surfaces and this will provide the motivation for an analogous construction in the language of Higgs bundles. The following lemma from classical Fricke-Klein theory is crucial in the construction:

Lemma 1.4.2. *Let $\gamma \in \pi_1(\Sigma)$ be a closed separating geodesic on Σ and $i_0 : \pi_1(\Sigma) \to$ SL(2,$\mathbb{R}$) a discrete embedding with $i_0(\gamma) = \begin{pmatrix} e^{\lambda_0} & 0 \\ 0 & e^{-\lambda_0} \end{pmatrix}$ and for $\lambda_0 \in \mathbb{R} \setminus \{0\}$. Let $(\lambda_t)_{t\in[0,1]}$ a continuous path in $\mathbb{R} \setminus \{0\}$. Then there exists a continuous path of discrete embeddings $(i_t)_{t\in[0,1]}$ such that for any $t \in [0,1]$, $i_t(\gamma) = \begin{pmatrix} e^{\lambda_t} & 0 \\ 0 & e^{-\lambda_t} \end{pmatrix}$.*

Let $\Sigma = \Sigma_l \cup_\gamma \Sigma_r$ be a decomposition of Σ along a simple closed oriented separating geodesic γ into two subsurfaces Σ_l and Σ_r. Consider $\rho_{irr} : \pi_1(\Sigma) \xrightarrow{i} \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\phi_{irr}} \mathrm{Sp}(4,\mathbb{R})$ an irreducible Fuchsian representation and $\rho_\Delta : \pi_1(\Sigma) \xrightarrow{i} \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\Delta} \mathrm{SL}(2,\mathbb{R})^2 \xrightarrow{\psi} \mathrm{Sp}(4,\mathbb{R})$ a diagonal Fuchsian representation.

We would like to amalgamate the restriction of the irreducible Fuchsian representation to Σ_l with the restriction of the diagonal Fuchsian representation to Σ_r, however the holonomies of those along γ do not agree. Thus, we are going to consider a deformation of ρ_Δ on $\pi_1(\Sigma)$ such that the holonomies agree along γ and then we will amalgamate the restrictions of those to the left and the right hand side subsurfaces accordingly.

Assume $i(\gamma) = \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix}$ with $m > 0$. There exist continuous paths $(\tau_{1,t})_{t\in[0,1]}$ and $(\tau_{2,t})_{t\in[0,1]}$ of discrete embeddings $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ with initial point $\tau_{1,0} = \tau_{2,0} = i$ and, for all $t \in [0,1]$,

$$\tau_{1,t}(\gamma) = \begin{pmatrix} e^{l_{1,t}} & 0 \\ 0 & e^{-l_{1,t}} \end{pmatrix} \quad \text{and } \tau_{2,t}(\gamma) = \begin{pmatrix} e^{l_{2,t}} & 0 \\ 0 & e^{-l_{2,t}} \end{pmatrix}$$

where $l_{1,t} > 0$ and $l_{2,t} > 0$, $l_{1,0} = l_{2,0} = m$, $l_{1,1} = 3m$ and $l_{2,1} = m$. In other words we are considering a continuous path $(\tau_{1,t}, \tau_{2,t})_{t\in[0,1]}$ of pairs of discrete embeddings starting from (i,i) and terminating at a pair $(\tau_{1,1}, \tau_{2,1})$ having specific behaviour on γ.

Now set

$$\rho_l := \rho_{irr} : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$$

and

$$\rho_r := \psi \circ (\tau_{1,1}, \tau_{2,1}) : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$$

Thus ρ_l and ρ_r are defined *over the whole surface* Σ, with ρ_r a continuous deformation of ρ_Δ satisfying $\rho_l(\gamma) = \rho_r(\gamma)$.

Definition 1.4.3. A *hybrid representation* is defined as the amalgamated representation

$$\rho := \rho_l \big|_{\pi_1(\Sigma_l)} * \rho_r \big|_{\pi_1(\Sigma_r)} : \pi_1(\Sigma) \simeq \pi_1(\Sigma_l) *_{\langle\gamma\rangle} \pi_1(\Sigma_r) \to \mathrm{Sp}(4,\mathbb{R})$$

If $\chi(\Sigma_l) = k$, then we call ρ a *k-hybrid representation.*

The following important result was established in [22]:

Theorem 1.4.4. *Every maximal representation $\rho : \pi_1(\Sigma) \to Sp(4,\mathbb{R})$ can be deformed to a standard representation or a hybrid representation.*

The subsurfaces Σ_l and Σ_r that we are considering here are surfaces with boundary. The Toledo invariant can be also defined for representations over such surfaces and it thus makes sense to talk about maximal representations over surfaces with boundary as well; see [12] for a detailed definition. Moreover, the authors in [12] have established an additivity property for the Toledo invariant over a connected sum of surfaces, which provides that the amalgamated product of two maximal representations is again a maximal representation defined over the compact surface Σ. In particular:

Proposition 1.4.5 ([12], Proposition 3.2). *If $\Sigma = \Sigma_1 \cup_C \Sigma_2$ is the connected sum of two subsurfaces Σ_i along a separating loop C, then*

$$T_\kappa(\Sigma, \rho) = T_\kappa(\Sigma_1, \rho_1) + T_\kappa(\Sigma_2, \rho_2)$$

where $\rho_i = \rho\big|_{\pi_1(\Sigma_i)}$, $i = 1, 2$.

1.5 Topological invariants for maximal symplectic representations

In [22] the authors introduce topological invariants for Anosov representations, a special case of which are the maximal representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbb{R})$ we are interested in. We review the definition of these invariants and describe their values for the hybrid representations in particular; we refer to [22] for more details on the material covered in this section.

Let (M, ϕ_t) a compact manifold with an Anosov flow and G a connected semisimple Lie group. Consider (P^s, P^u) a pair of opposite parabolic subgroups of G, $H := P^s \cap P^u$ and $\mathcal{F}^s := G/P^s$, $\mathcal{F}^u := G/P^u$ the flag varieties associated to P^s, P^u respectively. Let $\mathcal{X} := G/H \subset \mathcal{F}^s \times \mathcal{F}^u$ an open G-orbit inherited by two foliations $\mathcal{E}^s$ and $\mathcal{E}^u$ with corresponding distributions E^s and E^u, that is, $(E^s)_{(f^s,f^u)} \cong T_{f^s}\mathcal{F}^s$ and $(E^u)_{(f^s,f^u)} \cong T_{f^u}\mathcal{F}^u$.

Definition 1.5.1. The flat G-bundle $P_G \to M$ is said to be a (G, H)-*Anosov bundle*, if:

1. P_G admits an H-reduction that is flat along flow lines, i.e.

   i. there exists a section $\sigma : M \to P_G \times_G \mathcal{X}$

   ii. the restriction of σ to every orbit of ϕ_t is locally constant with respect to the induced flat structure on $P_G \times_G \mathcal{X}$.

2. The flows ϕ_t on $\sigma^* E^s$ and $\sigma^* E^u$ are contracting and dilating respectively, that is, there exist continuous families of norms $\left(\|\cdot\|_m\right)_{m\in M}$ on $\sigma^* E^s$ and $\sigma^* E^u$, and constants $A, a > 0$, such that for any e in $\left(\sigma^* E^s\right)_m$ or $\left(\sigma^* E^u\right)_m$ and for any $t > 0$ it holds respectively that

$$\|\phi_t e\|_{\phi_t m} \leq A \exp\left(-at\right) \|e\|_m \text{ or } \|\phi_{-t} e\|_{\phi_{-t} m} \leq A \exp\left(-at\right) \|e\|_m.$$

Definition 1.5.2. A representation $\rho : \pi_1(M) \to G$ is said to be (G, H)-*Anosov*, if the corresponding flat G-bundle P_G is (G, H)-Anosov.

Specializing to the case when $M = T^1\Sigma$, the unit tangent bundle of a closed oriented connected surface Σ with $g \geq 2$ and ϕ_t the geodesic flow on $T^1\Sigma$ with respect to a hyperbolic metric on Σ, we call a flat G-bundle P_G over Σ to be *Anosov* if its pullback $\pi^* P_G \to T^1\Sigma$ is Anosov. A fundamental group representation $\rho : \pi_1(\Sigma) \to G$ is now called *Anosov*, if the composite map

$$\pi_1\left(T^1\Sigma\right) \to \pi_1(\Sigma) \xrightarrow{\rho} G$$

is an Anosov representation.

The following theorem provides that the maximal symplectic group representations we are interested in admit an Anosov structure; see [11], [12] for more details:

Theorem 1.5.3 (M. Burger, A. Iozzi, F. Labourie and A. Wienhard). *A maximal representation $\rho : \pi_1(\Sigma) \to Sp(4,\mathbb{R})$ is an Anosov representation. More precisely, for P the corresponding flat principal $Sp(4,\mathbb{R})$-bundle over $T^1\Sigma$ and E the corresponding flat symplectic $\mathbb{R}^{2n}$-bundle over $T^1\Sigma$, ρ is an $(Sp(2n,\mathbb{R}), GL(n,\mathbb{R}))$-Anosov representation. The canonical $GL(n,\mathbb{R})$-reduction of P is equivalent to a continuous splitting of E into two flow-invariant transverse Lagrangian subbundles*

$$E = L^s(\rho) \oplus L^u(\rho)$$

The next result opens the way for introducing obstruction theory for Anosov representations:

Proposition 1.5.4 ([22], Proposition 4.1). *Let $Hom_{H-Anosov}(\pi_1(M), G)$ denote the set of (G, H)-Anosov representations and $\mathcal{B}_H(M)$ the set of gauge isomorphism classes of H-bundles over M. For any pair (G, H), there is a well-defined locally constant map*

$$Hom_{H-Anosov}(\pi_1(M), G) \to \mathcal{B}_H(M)$$

associating to an Anosov representation its Anosov H-reduction.

This proposition allows one to associate to a maximal representation into $\mathrm{Sp}(2n,\mathbb{R})$, the first and second Stiefel-Whitney classes of the corresponding $\mathrm{GL}(n,\mathbb{R})$-bundle over $T^1\Sigma$:

$$\mathrm{sw}_1 : \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{R})) \to H^1(T^1\Sigma; \mathbb{Z}_2)$$

$$\mathrm{sw}_2 : \mathrm{Hom}_{\max}(\pi_1(\Sigma), \mathrm{Sp}(2n,\mathbb{R})) \to H^2(T^1\Sigma; \mathbb{Z}_2)$$

The values of these Stiefel-Whitney classes for the model symplectic representations of the previous section, were explicitly computed by O. Guichard and A. Wienhard in [22]. A relation between these invariants and the Higgs bundle invariants w_1, w_2 discussed in §1.3 can be deduced from case-by-case considerations for model representations, although these invariants live naturally in different cohomology groups:

Proposition 1.5.5 ([22], Proposition 19)**.** *Let $\rho : \pi_1(\Sigma) \to Sp(2n,\mathbb{R})$ be a maximal representation. Then, for any choice of spin structure v, the following equality holds in $H^i(T^1\Sigma; \mathbb{Z}_2)$:*

$$\begin{aligned} sw_1(\rho) &= w_1(\rho, v) + nv \\ sw_2(\rho) &= w_2(\rho, v) + sw_1(\rho) \cup v + (g-1) \bmod 2 \end{aligned}$$

Even though the first and second Stiefel-Whitney class are enough to distinguish the $3 \cdot 2^{2g}$-many connected components of maximal representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbb{R})$ for $n \geq 3$, for the case $n = 2$ an extra topological invariant needs to be considered in order to distinguish the extra components of $\mathcal{R}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$.

Remark 1.5.6. Note that in the Higgs bundle viewpoint, the degree $\deg(L)$ of the underlying line bundle L in the decomposition of the Cayley partner $W = L \oplus L^{-1}$ whenever $w_1(W) = 0$, was used in order to distinguish the extra connected components of $\mathcal{M}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$.

For $n = 2$, when $\mathrm{sw}_1(\rho) = 0$, the Lagrangian bundle $L^s(\rho)$ is orientable, however, a priori this bundle has no canonical orientation. It is shown in [22] that for every pair $(\rho, L^s(\rho))$ with ρ maximal and with $\mathrm{sw}_1(\rho) = 0$, there is a natural associated *oriented* Lagrangian bundle L_+ over $T^1\Sigma$, and the associated flat $\mathrm{GL}^+(2,\mathbb{R})$-bundle E associated to ρ decomposes to two *oriented* Lagrangian subbundles

$$E = L_+^s(\rho) \oplus L_+^u(\rho)$$

An Euler class $e(\rho, L_+)$ whose image lies in $H^2(T^1\Sigma, \mathbb{Z})$ is now well-defined for the canonical $\mathrm{GL}^+(2,\mathbb{R})$-reduction of the $\mathrm{GL}(2,\mathbb{R})$-Anosov reduction associated to ρ. For any representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ define a representation $\varepsilon \otimes \rho : \widehat{\pi_1(\Sigma)} \to \mathrm{Sp}(4,\mathbb{R})$ by setting $\varepsilon \otimes \rho(x, \gamma) = \varepsilon(x)\rho(\gamma)$, where $\varepsilon : \widehat{\pi_1(\Sigma)} := \{\pm 1\} \times \pi_1(\Sigma) \to \{\pm 1\}$ is the projection onto the first factor. Now, for the hybrid representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ described in §1.4 it holds that $sw_1(\rho) = 0$, whereas

$$e(\varepsilon \otimes \rho, L_+) = -\chi(\Sigma_l)[\Sigma] \in H^2\left(T^1\Sigma, \mathbb{Z}\right)$$

Remember that Σ_l is considered here to be a surface with genus $1 \leq g_l \leq g-1$ and one boundary component, thus its Euler characteristic $\chi(\Sigma_l) = 2 - 2g_l - 1 = 1 - 2g_l$ is odd. Now, $1 \leq g_l \leq g-1$ implies

$$-2g + 3 \leq \chi(\Sigma_l) \leq -1.$$

Moreover, any representation in $\mathrm{Hom}_{\max, sw_1=0}(\pi_1(\Sigma), \mathrm{Sp}(4,\mathbb{R}))$ with Euler class not equal to $(g-1)[\Sigma]$ has Zariski dense image. Therefore, we obtain a model k-hybrid representation for each possible value of the Euler characteristic, and these representations thus distinguish the odd-indexed $2g-3$ exceptional components of $\mathcal{R}^{\max}(\mathrm{Sp}(4,\mathbb{R}))$ (see Theorem 5.8 as well as §5.6 in [22]).

1.6 Statement of the problem

Motivated by the topological gluing construction described above, we aim at developing a gluing construction for (poly)stable Sp(4,$\mathbb{R}$)-Higgs bundles over a complex connected sum of Riemann surfaces. Moreover, we seek for a way to choose the Sp(4,$\mathbb{R}$)-Higgs data on the left and right hand side Riemann surfaces, so that the resulting *hybrid Higgs bundle* will lie in one of the $2g-3$ exceptional components of $\mathcal{M}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$. Even further, we would like to obtain models in *all* these components, thus extending the result of O. Guichard and A. Wienhard to the even-indexed ones. The latter would provide a specific relation between the Higgs bundle topological invariants and the topological invariants for Anosov representations, as defined in [22].

In the following chapters we develop the necessary machinery for the above mentioned purpose. The appropriate analog to a surface group representation into a reductive Lie group G for a surface with boundary is a *parabolic G-Higgs bundle* over a Riemann surface with a divisor. We need to describe the defining data for these holomorphic objects and especially what it would mean to have a maximal parabolic stable Sp(4,$\mathbb{R}$)-Higgs bundle; this is the content of Chapter 2. From this point on, the problem of establishing a gluing construction using such objects (stable but not necessarily maximal) over a complex connected sum of Riemann surfaces is a more complicated procedure compared to its topological counterpart. We choose to switch to the language of solutions to the Hitchin equations and develop a gluing construction in the gauge-theoretic language, adapting into our setting the very effective techniques of C. Taubes for gluing instantons over 4-manifolds. This adaptation involves a good understanding of the linearization of the Hitchin operator when we perform the gluing over a complex connected sum of Riemann surfaces. This is the content of Chapter 3, and we show that by gluing parabolic stable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles we may get a polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle defined over the compact connected sum surface $X_{\#}$. In Chapter 4, we show how to construct model hybrid Higgs bundles in all the exceptional components of $\mathcal{M}^{\max}(X_{\#}, \mathrm{Sp}(4,\mathbb{R}))$. For this purpose, two results need to be established: First, we need to have an additivity property for the Toledo invariant, analogous to the one described in Proposition 1.4.5 for maximal representations; this will provide that gluing maximal parabolic G-Higgs bundles gives a maximal (non-parabolic) G-Higgs bundle. The second is a description of the Higgs bundle invariants under the complex connected sum operation. This will predict the choices that need to be made for gluing parabolic $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles, in order to end up with a model inside a desired component of the maximal $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle moduli space.

CHAPTER 2

PARABOLIC SP(4,$\mathbb{R}$)-HIGGS BUNDLES

Parabolic vector bundles over Riemann surfaces with marked points were introduced by C. Seshadri in [38] and similar to the Narasimhan-Seshadri correspondence, there is an analogous correspondence between stable parabolic bundles and unitary representations of the fundamental group of the punctured surface with fixed holonomy class around each puncture [30]. Later on, C. Simpson in [40] proved a non-abelian Hodge correspondence in the *non-compact case*: Parabolic Higgs bundles are in bijection with meromorphic flat connections, whose holonomy around each puncture defines a conjugacy class of an element in the unitary group described by the weights in the parabolic structure of the bundle. These connections correspond to representations of the fundamental group of the punctured surface in the general linear group, which send a small loop around each parabolic point to an element conjugate to a unitary element. More recently, O. Biquard, O. García-Prada and I. Mundet i Riera provided in [5] a Hitchin-Kobayashi correspondence for parabolic G-Higgs bundles.

In this chapter we include the main definitions for parabolic G-Higgs bundles. We are primarily interested in the case $G = \mathrm{Sp}(4,\mathbb{R})$ and in describing the moduli space of maximal parabolic Sp(4,$\mathbb{R}$)-Higgs bundles. For the latter, a Milnor-Wood bound for an appropriate notion of Toledo invariant is necessary.

2.1 Parabolic GL $(n, \mathbb{C})$-Higgs bundles

For further reference on the material covered in this section see [8] or [19].

Definition 2.1.1. Let X be a closed, connected, smooth Riemann surface of genus $g \geq 2$ with s-many marked points $x_1, \ldots, x_s$ and let a divisor $D = \{x_1, \ldots, x_s\}$. We define a *parabolic vector bundle* E over X to be a holomorphic vector bundle $E \to X$ with *parabolic structure* at each $x \in D$ (*weighted flag* on each fiber E_x):

$$E_x = E_{x,1} \supset E_{x,2} \supset \ldots \supset E_{x,r(x)+1} = \{0\}$$
$$0 \leq \alpha_1(x) < \ldots < \alpha_{r(x)}(x) < 1$$

We usually write (E, α) to denote a vector bundle equipped with a parabolic structure determined by a system of weights $\alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x))$ at each $x \in D$. Moreover, set $k_i(x) = \dim(E_{x,i}/E_{x,i+1})$ be the *multiplicity* of the weight $\alpha_i(x)$. We can also write the weights repeated according to their multiplicity as

$$0 \leq \tilde{\alpha}_1(x) \leq \ldots \leq \tilde{\alpha}_n(x) < 1$$

where now $n = \mathrm{rk}E$. A weighted flag shall be called *full*, if $k_i(x) = 1$ for every i and $x \in D$.

Definition 2.1.2. A holomorphic map $f : E \to E'$ of parabolic vector bundles $(E, \alpha), (E', \alpha')$ is called *parabolic* if $\alpha_i(x) > {\alpha'}_j(x)$ implies $f(E_{x,i}) \subset {E'}_{x,j+1}$, for every $x \in D$.
Furthermore, we call such a map *strongly parabolic* if $\alpha_i(x) \geq {\alpha'}_j(x)$ implies $f(E_{x,i}) \subset {E'}_{x,j+1}$ for every $x \in D$.
We denote by $\mathrm{ParHom}(E, E')$ and $\mathrm{SParHom}(E, E')$ the sheaves of parabolic and strongly parabolic morphisms respectively.

Definition 2.1.3. We define the *parabolic degree* and *parabolic slope* of a vector bundle equipped with a parabolic structure as follows

$$\mathrm{par}\deg(E) = \deg E + \sum_{x \in D} \sum_{i=1}^{r(x)} k_i(x)\, \alpha_i(x)$$

$$\mathrm{par}\mu(\mathrm{E}) = \frac{\mathrm{pardeg}(E)}{\mathrm{rk}(E)}$$

We now describe the basic constructions for parabolic vector bundles that we are going to be considering:

1. *Subbundle and quotient*
   If (E, α) is a parabolic vector bundle then a vector subbundle $F \leq E$ inherits a parabolic structure from E (*induced parabolic structure*) by setting $F_{x,i} = F_x \cap E_{x,i}$ and discarding the weights of multiplicitly zero. Quite similarly, the quotient E/F can be equipped with a parabolic structure inherited from the structure on E.

2. *Direct Sum*
   Let $(V, \alpha), (W, \alpha')$ be parabolic vector bundles. We define the *parabolic direct sum* $(E, \tilde{\alpha})$ of parabolic bundles as the direct sum $E = V \oplus W$ of holomorphic bundles with weight type $\tilde{\alpha}$ consisted of the ordered collection of the weights in α and α', and filtration $E_{x,k} = V_{x,i} \oplus W_{x,j}$ where i (resp. j) is the smallest integer such that

$\tilde{\alpha}_k(x) \leq \alpha_i(x)$ (resp. $\tilde{\alpha}_k(x) \leq \alpha'_j(x)$). Under this definition we can now check that

$$\operatorname{par}\deg(V \oplus W) = \operatorname{par}\deg(V) + \operatorname{par}\deg(W)$$

3. *Dual*

   Let (E, α) be a parabolic vector bundle. There is a well defined notion of a dual $E^\vee$ by considering the bundle $\operatorname{Hom}(E, \mathcal{O}(-D))$ equipped with a parabolic structure defined by the filtration

   $$E_x^\vee = E_{x,1}^\vee \supset \ldots \supset E_{x,r(x)}^\vee \supset \{0\}$$

   where $E_{x,i}^\vee = \operatorname{Hom}\left(E_x / E_{x,r(x)+2-i}, \mathcal{O}(-D)_x\right)$ and weights

   $$1 - \alpha_{r(x)}(x) < \ldots < 1 - \alpha_1(x).$$

   Under this definition we can now check that $E^{\vee\vee} = E$, as well as that

   $$\operatorname{par}\deg(E^\vee) = -\operatorname{par}\deg E$$

4. *Tensor product*

   A notion of parabolic tensor product was defined in [48] in the language of parabolic sheaves. Let E and M be two parabolic vector bundles on X with the same parabolic divisor D and let $\tau : X \backslash D \to X$ be the natural inclusion. Define

   $$\mathcal{E} := \tau_* \tau^* (E \otimes M)$$

   which is a quasi-coherent sheaf over X and now for any $t \in \mathbb{R}$ denote by $\mathcal{E}_t$ the subsheaf of $\mathcal{E}$ generated by all $E_k \otimes M_l$ with $k + l \geq t$. The filtration $(\mathcal{E}_t)_{t \in \mathbb{R}}$ defines a parabolic structure on the coherent sheaf $\mathcal{E}_0$, which is locally free. The parabolic tensor product $E \otimes M$ is defined as the parabolic bundle $\mathcal{E}$ constructed previously; cf. [6] or [48] for more details. We now have

   $$par \deg(E \otimes M) = \operatorname{rk}(M)\, par \deg(E) + \operatorname{rk}(E)\, par \deg(M)$$

Definition 2.1.4. A parabolic vector bundle will be called *stable* (resp. *semistable*) if for every non-trivial proper parabolic subbundle $F \leq E$, it is $\operatorname{par}\mu(F) < \operatorname{par}\mu(E)$, (resp. $\leq$).

Definition 2.1.5. Let K be the canonical bundle over X and E a parabolic vector bundle. The bundle morphism $\Phi : E \to E \otimes K(D)$ will be called a *parabolic Higgs field*, if it preserves the parabolic structure at each point $x \in D$:

$$\Phi\,|_x\,(E_{x,i}) \subset E_{x,i} \otimes K\,(D)\,|_x$$

In particular, we call Φ *strongly parabolic*, if

$$\Phi\,|_x\,(E_{x,i}) \subset E_{x,i+1} \otimes K\,(D)\,|_x$$

in other words, Φ is a meromorphic endomorphism valued 1-form with simple poles along the divisor D, whose residue at $x \in D$ is nilpotent with respect to the filtration.

After these considerations we are in position to define parabolic Higgs bundles.

Definition 2.1.6. Let K be the canonical bundle over X and E a parabolic vector bundle over X. Consider on $E \otimes K\,(D)$ the parabolic structure induced by the tensor product construction.

- A *parabolic $K\,(D)$-pair* is a pair (E, Φ), where E is a parabolic vector bundle and $\Phi : E \to E \otimes K\,(D)$ is a parabolic Higgs field.
- A *parabolic Higgs bundle* is a parabolic $K\,(D)$-pair (E, Φ), where Φ is additionally a strongly parabolic Higgs field.

Analogously to the non-parabolic case, we may define stability as follows:

Definition 2.1.7. A parabolic $K\,(D)$-pair will be called *stable* (resp. *semistable*) if for every Φ-invariant parabolic subbundle $F \leq E$ it is $\mathrm{par}\mu\,(F) < \mathrm{par}\mu\,(E)$ (resp. $\leq$). Furthermore, it will be called *polystable* if it is the direct sum of stable parabolic $K\,(D)$-pairs of the same parabolic slope.

For fixed $n = \mathrm{rk}E$, $d = \deg E$ and weight type α, two moduli spaces can be now obtained given the preceding definitions. In [47] and [48], K. Yokogawa has constructed the *moduli space of $K\,(D)$-pairs* $\mathcal{P}_\alpha$ using geometric invariant theory and has shown that it is a normal, quasi-projective variety of dimension

$$\dim \mathcal{P}_\alpha = (2g - 2 + s)\,n^2 + 1$$

which is smooth at the stable points. Moreover, in [28] H. Konno constructed the *moduli space of parabolic Higgs bundles* $\mathcal{N}_\alpha$ as a hyperkähler quotient. It is contained in $\mathcal{P}_\alpha$ as a closed subvariety of dimension

$$\dim \mathcal{N}_\alpha = 2\,(g-1)\,n^2 + 2 + 2 \sum_{x \in D} f_x$$

where $f_x = \frac{1}{2}\left(n^2 - \sum\limits_{i=1}^{r(x)} (k_i(x))^2\right)$ is the dimension of the associated flag variety.

Remark 2.1.8. In the literature, a parabolic Higgs bundle is sometimes defined by requiring the Higgs field to be just preserving the parabolic structure at each point $x \in D$. For us, a parabolic Higgs bundle will always involve a strongly parabolic Higgs field.

Lastly, we say that the weights of a parabolic Higgs bundle are *generic*, when stability and semistability are equivalent. In this case, there are no properly semistable parabolic Higgs bundles and the moduli space $\mathcal{N}_\alpha$ is *smooth*.

2.2 Parabolic G-Higgs bundles

In [5] the authors introduce parabolic G-Higgs bundles over a punctured Riemann surface for a non-compact real reductive Lie group G. This definition involves a choice for each puncture of an element in the Weyl alcove of a maximal compact subgroup $H \subset G$, handling both cases as if this element lies in the interior of the alcove or if it lies in a 'bad' wall of the alcove. Below we summarize the basic steps towards this definition.

Let X be a compact, connected Riemann surface and let $\{x_1, \ldots, x_s\}$ be a finite set of different points on X with $D = x_1 + \ldots x_s$ be the corresponding effective divisor. Let now $H^{\mathbb{C}}$ be a reductive, complex Lie group. Fix a maximal compact subgroup $H \subset H^{\mathbb{C}}$, and a maximal torus $T \subset H$ with Lie algebra $\mathfrak{t}$. Denote $E\left(H^{\mathbb{C}}\right) = E \times_{H^{\mathbb{C}}} H^{\mathbb{C}} \to X$, the $H^{\mathbb{C}}$-fibration associated to E via the adjoint representation of $H^{\mathbb{C}}$ on itself. Then

$$E\left(H^{\mathbb{C}}\right)_x = \left\{\phi : E_x \to H^{\mathbb{C}} \,\middle|\, \phi(eh) = h^{-1}\phi(e)\,h\ ,\ \forall e \in E_x, h \in H^{\mathbb{C}}\right\}$$

i.e. the fiber can be identified with the set of antiequivariant maps ϕ.

Fix an alcove $\mathcal{A} \subset \mathfrak{t}$ of H containing $0 \in \mathfrak{t}$ and for $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$ let $P_{\alpha_i} \subset H^{\mathbb{C}}$ be the parabolic subgroup defined by the α_i.

Definition 2.2.1. We define a *parabolic structure* of weight α_i on E over a point x_i as the choice of a subgroup $Q_i \subset E\left(H^{\mathbb{C}}\right)_{x_i}$ with the property that there exists a trivialization $e \in E_{x_i}$ for which $P_{\alpha_i} = \{\phi(e) \,|\, \phi \in Q_i\}$.

Given this, we now set the following:

Definition 2.2.2. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a collection of elements in $\sqrt{-1}\bar{\mathcal{A}}$. A *parabolic bundle* over (X, D) of weight α is a holomorphic principal bundle with a choice for any i of a parabolic structure of weight α_i over x_i.

Consider that the parabolic bundle E comes equipped with a holomorphic structure $\bar{\partial}$ and consider a metric $h \in \Gamma(X \backslash D; E/H)$ defined away from the divisor D.

Definition 2.2.3. The metric h is called an *α-adapted metric* if for any parabolic point x_i the following holds: Let $e_i \in E_{x_i}$ be an element belonging to the P_{α_i} orbit specified by the parabolic structure. Choose local holomorphic coordinate z and extend the trivialization e_i into a holomorphic trivialization of E near x_i. Then we can write near x_i

$$h = \left(|z|^{-\alpha_i} e^c\right)^2$$

with $\mathrm{Ad}\left(|z|^{-\alpha_i}\right) c = \mathrm{o}(\log|z|)$, $\mathrm{Ad}\left(|z|^{-\alpha_i}\right) dc \in L^2$ and $\mathrm{Ad}\left(|z|^{-\alpha_i}\right) F_h \in L^1$.

For a real reductive Lie group G with a maximal compact subgroup H, let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the Cartan decomposition of its Lie algebra into its ± 1-eigenspaces, where $\mathfrak{h} = \mathrm{Lie}(H)$ and let $E\left(\mathfrak{m}^{\mathbb{C}}\right)$ be the bundle associated to E via the isotropy representation. Choose a trivialization $e \in E$ near the point x_i, such that near x_i the parabolic weight lies in $\alpha_i \in \sqrt{-1}\bar{\mathcal{A}}$. In the trivialization e, we can decompose the bundle $E\left(\mathfrak{m}^{\mathbb{C}}\right)$ under the eigenvalues of $\mathrm{ad}(\alpha_i)$ acting on $\mathfrak{m}^{\mathbb{C}}$ as

$$E\left(\mathfrak{m}^{\mathbb{C}}\right) = \bigoplus_{\mu} \mathfrak{m}_{\mu}^{\mathbb{C}}$$

Decompose accordingly a section φ of $E\left(\mathfrak{m}^{\mathbb{C}}\right)$ as $\varphi = \sum \varphi_\mu$. After these preliminaries we set the following:

Definition 2.2.4. The sheaf $PE\left(\mathfrak{m}^{\mathbb{C}}\right)$ of *parabolic sections* (resp. the sheaf $NE\left(\mathfrak{m}^{\mathbb{C}}\right)$ of *strictly parabolic sections*) of $E\left(\mathfrak{m}^{\mathbb{C}}\right)$ is consisted of meromorphic sections φ of the bundle $E\left(\mathfrak{m}^{\mathbb{C}}\right)$ with singularities at the points x_i, with φ_μ having order

$$v(\varphi_\mu) \geq -\lfloor -\mu \rfloor \quad (\text{resp. } v(\varphi_\mu) > -\lfloor -\mu \rfloor\,)$$

This means that if $a - 1 < \mu \leq a$ (resp. $a - 1 \leq \mu < a$) for some integer a, then $\varphi_\mu = \mathrm{O}(z^a)$.

We finally have the definition of a parabolic G-Higgs bundle as follows:

Definition 2.2.5. We define a *parabolic G-Higgs bundle* over a Riemann surface with a divisor (X, D) to be a pair (E, φ), where:

1. E is a parabolic principal $H^{\mathbb{C}}$-bundle over (X, D), and
2. φ is a holomorphic section of $PE\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes K(D)$.

The pair (E, φ) will be called a *strictly parabolic G-Higgs bundle* if in addition the Higgs field φ is a section of $NE\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes K(D)$.

For a parabolic principal bundle E over (X, D) with weights α, a notion of parabolic degree was defined in [5] as the sum of two terms, one global and independent of the parabolic structure, and one local and depending on the parabolic structure. Before we state this definition, recall that the degree of a (non-parabolic) bundle can be defined using Chern-Weil theory as follows:

Fix a standard parabolic subgroup $P \subset H^{\mathbb{C}}$, an antidominant character $\chi : \mathfrak{p} \to \mathbb{C}$ and a holomorphic reduction σ of the structure group of E from $H^{\mathbb{C}}$ to P, with E_σ denoting the P-principal bundle corresponding to this reduction σ. Then, the degree of E is given by

$$\deg(E)(\sigma, \chi) := \frac{\sqrt{-1}}{2\pi} \int_X \chi_* (F_A)$$

where F_A is the curvature of any P-connection A on E_σ.

Now, let $Q_i \subset E\big(H^{\mathbb{C}}\big)_{x_i}$ the parabolic subgroups in the definition of the parabolic structure. At each point in the divisor D, there are two parabolic subgroups equipped with an antidominant character, one coming from the parabolic structure (Q_i, α_i) and one coming from the reduction $\big(E_\sigma(P)_{x_i}, \chi\big)$. A relative degree for such a pair of parabolic subgroups $(\mathcal{Q}, \mathcal{P})$ is then defined:

$$\deg((\mathcal{Q}, \sigma), (\mathcal{P}, s)) = \cos \angle_{Tits} (\eta(\sigma), \eta(s))$$

where $\angle_{Tits}$ is the Tits distance on $\partial_\infty \mathcal{X}$ for $\mathcal{X} = H \backslash G$ a symmetric space of non-compact type, and $\eta(s) = \lim\limits_{t \to \infty} * e^{ts} \in \partial_\infty \Sigma$ for s in an H-orbit $\mathcal{O}_H \subset \mathfrak{m}$. The *parabolic degree* is now given by the sum of the two terms described previously:

$$par\deg_\alpha(E)(\sigma, \chi) := \deg(E)(\sigma, \chi) - \sum_i \deg\big((Q_i, \alpha_i), \big(E_\sigma(P)_{x_i}, \chi\big)\big)$$

The definition of polystability is next given with respect to an element $c \in \sqrt{-1}\mathfrak{l}$ for $\mathfrak{l} = \mathrm{Lie}(Z(H))$:

Definition 2.2.6. Let (E, Φ) be a parabolic G-Higgs bundle over (X, D). Then (E, Φ) will be called *polystable* if for every $s \in \sqrt{-1}\mathfrak{h}$ and any holomorphic reduction σ of the structure group of E to P_s, such that $\Phi\big|_{X \backslash D} \in H^0(X \backslash D, E_\sigma(\mathfrak{m}_s) \otimes K)$ it is

$$par\deg(E)(\sigma, \chi_s) - \langle c, s \rangle \geq 0$$

The following theorem proven in [5] establishes a Hitchin-Kobayashi correspondence for parabolic G-Higgs bundles.

Theorem 2.2.7. *Let (E,Φ) be a parabolic G-Higgs bundle equipped with an adapted initial metric h_0. Suppose that* $\operatorname{par}\deg(E)=\chi(c)$ *for all characters of $\mathfrak{g}$. Then (E,Φ) admits an Hermite-Einstein metric h, quasi-isometric to h_0, if and only if (E,Φ) is polystable.*

2.2.1 Deformation theory

The deformation theory of parabolic $K(D)$-pairs was studied by K. Yokogawa in [48]. We now adapt results from that article to the case of parabolic G-Higgs bundles for G semisimple, analogously to the non-parabolic case treated in §3.3 of [18]. For a semisimple Lie group G, with $H\subset G$ a maximal compact subgroup, let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ be a Cartan decomposition so that the Lie algebra structure of $\mathfrak{g}$ satisfies:

$$[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h},\quad [\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m},\quad [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$$

Let $\mathfrak{g}^{\mathbb{C}}=\mathfrak{h}^{\mathbb{C}}\oplus\mathfrak{m}^{\mathbb{C}}$ be the complexification of the Cartan decomposition. The group H acts linearly on $\mathfrak{m}$ through the adjoint representation and this action extends to a linear holomorphic action of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}=\mathfrak{m}\otimes\mathbb{C}$:

$$\iota: H^{\mathbb{C}}\to\operatorname{Aut}\left(\mathfrak{m}^{\mathbb{C}}\right)$$

We consider the deformation complex of a parabolic G-Higgs bundle as follows:

Definition 2.2.8. Let (E,φ) be a parabolic G-Higgs bundle. The *deformation complex* of (E,φ) is the following complex of sheaves

$$C^{\bullet}(E,\varphi): NE\left(\mathfrak{h}^{\mathbb{C}}\right)\xrightarrow{d\iota(\varphi)} NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D).$$

The definition makes sense because φ is a meromorphic section of $NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D)$ and $\left[\mathfrak{m}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}}\right]\subseteq\mathfrak{m}^{\mathbb{C}}$.

The results of K. Yokogawa now readily adapt to provide the following:

Proposition 2.2.9. *The space of infinitesimal deformations of a G-Higgs bundle (E,φ) is naturally isomorphic to the hypercohomology group $\mathbb{H}^1(C^{\bullet}(E,\varphi))$.*

For any G-Higgs bundle there is a natural long exact sequence:

$$0\to\mathbb{H}^0(C^{\bullet}(E,\varphi))\to H^0\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)\xrightarrow{d\iota(\varphi)} H^0\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D)\right)$$

$$\to\mathbb{H}^1(C^{\bullet}(E,\varphi))\to H^1\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)\xrightarrow{d\iota(\varphi)} H^1\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D)\right)\to\mathbb{H}^2(C^{\bullet}(E,\varphi))\to 0,$$

where $d\iota : \mathfrak{h}^{\mathbb{C}} \to \operatorname{End}\left(\mathfrak{m}^{\mathbb{C}}\right)$ is the derivative at the identity of the complexified isotropy representation $\iota = Ad|_{H^{\mathbb{C}}} : H^{\mathbb{C}} \to \operatorname{Aut}\left(\mathfrak{m}^{\mathbb{C}}\right)$.

The Serre duality theorem for parabolic sheaves (Proposition 3.7 in [48]) provides that there are natural isomorphisms:

$$\mathbb{H}^i\left(C^{\bullet}\left(E,\varphi\right)\right) \cong \mathbb{H}^{2-i}\left(C^{\bullet}(E,\varphi)^{\vee} \otimes K\left(D\right)\right)^{\vee},$$

where the dual of the deformation complex $C^{\bullet}\left(E,\varphi\right)$ is defined as

$$C^{\bullet}(E,\varphi)^{\vee} : NE\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes \left(K\left(D\right)\right)^{\vee} \xrightarrow{-d\iota(\varphi)} NE\left(\mathfrak{h}^{\mathbb{C}}\right).$$

An important special case of this is when G is a *complex* group:

Proposition 2.2.10. *Assume that G is a complex semisimple group. Then there is a natural isomorphism:*

$$\mathbb{H}^2\left(C^{\bullet}\left(E,\varphi\right)\right) \cong \mathbb{H}^0(C^{\bullet}\left(E,\varphi\right))^{\vee}$$

Proof. When G is complex, $d\iota = \operatorname{ad}\colon \mathfrak{g} \to \mathfrak{g}$ and the Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$, where $\mathfrak{u} = \operatorname{Lie}(U)$ for $U \subset G$ a maximal compact subgroup. Thus, in this case $\varphi \in NE\left(\mathfrak{g}\right) \otimes K\left(D\right)$. Moreover, for a complex group G the deformation complex is dual to itself, except for a sign in the map, which does not affect the cohomology:

$$C^{\bullet}(E,\varphi)^{\vee} \otimes K\left(D\right) : NE\left(\mathfrak{g}\right) \xrightarrow{-\operatorname{ad}(\varphi)} NE\left(\mathfrak{g}\right) \otimes K\left(D\right)$$

The result now follows from Serre duality. □

The proof of the next proposition is immediate, since $NE\left(\mathfrak{h}^{\mathbb{C}}\right) \oplus NE\left(\mathfrak{m}^{\mathbb{C}}\right) = NE\left(\mathfrak{g}^{\mathbb{C}}\right)$, given the Cartan decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$. The corollary is also immediate from Serre duality:

Proposition 2.2.11. *Let G be a real semisimple group and let $G^{\mathbb{C}}$ be its complexification. Let (E,φ) be a G-Higgs bundle. Then there is an isomorphism of complexes:*

$$C^{\bullet}_{G^{\mathbb{C}}}\left(E,\varphi\right) \cong C^{\bullet}_{G}\left(E,\varphi\right) \oplus C^{\bullet}_{G}(E,\varphi)^{\vee} \otimes K\left(D\right),$$

where $C^{\bullet}_{G^{\mathbb{C}}}\left(E,\varphi\right)$ denotes the deformation complex of (E,φ) viewed as a $G^{\mathbb{C}}$-Higgs bundle, while $C^{\bullet}_{G}\left(E,\varphi\right)$ denotes the deformation complex of (E,φ) viewed as a G-Higgs bundle.

Corollary 2.2.12. *With the same hypotheses as in the previous proposition, there is an isomorphism*

$$\mathbb{H}^0\left(C^{\bullet}_{G^{\mathbb{C}}}\left(E,\varphi\right)\right) \cong \mathbb{H}^0\left(C^{\bullet}_{G}\left(E,\varphi\right)\right) \oplus \mathbb{H}^2(C^{\bullet}_{G}\left(E,\varphi\right))^{\vee}.$$

Under the genericity assumption we made, every parabolic polystable G-pair is stable, thus simple and defines a smooth point in the moduli space. This gives that $\mathbb{H}^0\left(C^\bullet_{G^{\mathbb{C}}}(E,\varphi)\right)=0$ and so

$$\mathbb{H}^0\left(C^\bullet_G(E,\varphi)\right)=0=\mathbb{H}^2\left(C^\bullet_G(E,\varphi)\right)$$

The long exact sequence then provides that

$$\dim \mathbb{H}^1\left(C^\bullet(E,\varphi)\right)=-\chi\left(C^\bullet(E,\varphi)\right)$$

Note that given the genericity assumption this will be the *actual dimension* of the moduli space of parabolic polystable G-Higgs bundles. This dimension can be computed using the Riemann-Roch formula and is independent of the choice of (E,φ):

Proposition 2.2.13. *Under the genericity assumption, the moduli space $\mathcal{M}_{par}(G)$ of stable parabolic G-Higgs bundles is a smooth complex variety of dimension*

$$(g-1)\dim G^{\mathbb{C}}+s\cdot rk\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right),$$

where g is the genus of the Riemann surface X and s is the number of marked points on X.

Proof. Let (E,φ) be any stable parabolic G-Higgs bundle. The long exact sequence for the deformation complex $C^\bullet(E,\varphi)$ of (E,φ) provides that

$$\chi\left(C^\bullet(E,\varphi)\right)-\chi\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)+\chi\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D)\right)=0.$$

The Riemann-Roch formula now gives:

$$\chi\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)=\deg\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)+\mathrm{rk}\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)\cdot(1-g)$$

as well as

$$\chi\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D)\right)=\deg\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D)\right)+\mathrm{rk}\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\otimes K(D)\right)\cdot(1-g)$$

$$\begin{aligned}&=\deg\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)+\mathrm{rk}\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)\cdot(2g-2+s)+\mathrm{rk}\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)\cdot(1-g)\\&=\deg\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)+\mathrm{rk}\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)\cdot(g-1+s),\end{aligned}$$

where we used that $\deg K(D)=2g-2+s$.

Thus, the dimension of the moduli space is: $-\chi\left(C^{\bullet}\left(E,\varphi\right)\right)=$

$$\deg\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)+\operatorname{rk}\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)\cdot\left(g-1+s\right)-\deg\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)-\operatorname{rk}\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)\cdot\left(1-g\right)$$

Moreover, any invariant pairing on $\mathfrak{g}^{\mathbb{C}}$ (i.e. the Killing form) induces isomorphisms $NE\left(\mathfrak{h}^{\mathbb{C}}\right)\simeq NE\left(\mathfrak{h}^{\mathbb{C}}\right)^{*}$ and $NE\left(\mathfrak{m}^{\mathbb{C}}\right)\simeq NE\left(\mathfrak{m}^{\mathbb{C}}\right)^{*}$. Hence,

$$\deg\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)=\deg\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)=0$$

and lastly: $\operatorname{rk}\left(NE\left(\mathfrak{h}^{\mathbb{C}}\right)\right)+\operatorname{rk}\left(NE\left(\mathfrak{m}^{\mathbb{C}}\right)\right)=\dim G^{\mathbb{C}}$. The computation now follows. □

Remark 2.2.14. Notice that when the number of punctures s is zero, this dimension count coincides with the count in Proposition 3.19 of [18] in the non-parabolic case.

2.3 Parabolic Sp(4,$\mathbb{R}$)-Higgs bundles

In this section, we restrict the general parabolic G-Higgs bundle definitions of §2.2 to the case when $G=\mathrm{Sp}(4,\mathbb{R})$ that we are primarily interested in. A maximal compact subgroup of $G=\mathrm{Sp}(4,\mathbb{R})$ is $H=\mathrm{U}\left(2\right)$ and $H^{\mathbb{C}}=\mathrm{GL}(2,\mathbb{C})$, thus the parabolic structure on a GL(2,$\mathbb{C}$)-principal bundle is in this case defined by a weighted filtration. We will first fix some notation before giving the precise definitions.

Let X be a compact Riemann surface of genus g and let the divisor $D:=\{x_1,\ldots,x_s\}$ of s-many distinct points on X. Let $X^{\times}:=X-D$ denote the punctured Riemann surface.

Let K denote the canonical line bundle on X of degree $2g-2$. Define $\iota:=\mathcal{O}_X\left(D\right)$ to be the line bundle on X given by the divisor D. The degree of the line bundle $K\otimes\iota$ is $2g-2+s$, where s is the number of punctures considered, and let us further assume that $2g-2+s>0$, in other words, the punctured Riemann surface $X^{\times}$ is equipped with a hyperbolic metric.

Let V be a rank 2 bundle over X. Equip this with a parabolic structure at each $x\in D$

$$V_x\supset L_x\supset 0$$

$$0\leq\alpha_1\left(x\right)<\alpha_2\left(x\right)<1$$

and denote this parabolic bundle by $V_{par}:=(V,\alpha)$. We will be omitting the subscript *par*, when there is no risk of confusion. Define the *parabolic degree* of the parabolic bundle V_{par}

to be given by the rational number

$$par\deg V_{par} = \deg(V) + \sum_{x\in D}(\alpha_1(x) + \alpha_2(x))$$

Let $\xi = \mathcal{O}_X(-D)$. We may define a notion of parabolic dual of the parabolic bundle V_{par} by $(V_{par})^{\vee} := V^* \otimes \xi$ with weights $1-\alpha$, under which it now holds that

$$\left((V_{par})^{\vee}\right)^{\vee} = V_{par}$$

as well as that

$$par\deg(V_{par})^{\vee} = -par\deg(V_{par})$$

Note however that the underlying vector bundle of $(V_{par})^{\vee}$ *does not coincide* with the usual bundle dual V^* when there is at least one nonzero weight.

For a parabolic principal $H^{\mathbb{C}} = \mathrm{GL}(2,\mathbb{C})$-bundle E, let $E\left(\mathfrak{m}^{\mathbb{C}}\right)$ denote the (parabolic) bundle associated to E via the isotropy representation and, as a bundle,

$$E\left(\mathfrak{m}^{\mathbb{C}}\right) = \mathrm{Sym}^2(V) \oplus \mathrm{Sym}^2(V^*)$$

for V the rank 2 bundle associated to E by the standard representation. In order to describe the parabolic symmetric power of a parabolic bundle V, we note the following:

Let $V \to X$ be a rank 2 bundle defined over the compact surface and let it be equipped with a parabolic structure defined by a trivial flag $V_x \supset \{0\}$ and weight $\frac{1}{2}$ for each V_x and $x \in D$. Then the parabolic symmetric power $V^{\otimes_{par}2}$ is equipped with the trivial flag and weight 1. In order to have a parabolic structure with the weight in the correct interval $[0,1)$, we define the parabolic symmetric square $V^{\otimes_{par}2}$, as the bundle $V^2 \otimes \iota$ equipped with a parabolic structure given by the trivial flag and weight 0. Similarly, the parabolic symmetric power for the parabolic dual $(V^{\vee})^{\otimes_{par}2}$ is defined as the bundle $(V^*)^2 \otimes \xi$ equipped with a parabolic structure given by the trivial flag and weight 0.

Now, the parabolic tensor product $E\left(\mathfrak{m}^{\mathbb{C}}\right) \otimes K(D)$ is expressed as

$$\left[\mathrm{Sym}^2(V) \otimes \iota \otimes K \otimes \iota\right] \oplus \left[\mathrm{Sym}^2(V^*) \otimes \xi \otimes K \otimes \iota\right]$$

equipped with a parabolic structure given by the trivial flag and weight 0.

In other words, the Higgs field according to the definition of a parabolic G-Higgs bundle described in §2.2 will be given by a pair (β, γ), where

$$\beta \in H^0\left(\mathrm{Sym}^2(V) \otimes \iota \otimes K \otimes \iota\right) \text{ or } \beta : V^* \otimes \xi \to V \otimes K \otimes \iota$$

and

$$\gamma \in H^0\left(\mathrm{Sym}^2\left(V^*\right) \otimes \xi \otimes K \otimes \iota\right) \text{ or } \gamma : V \to V^* \otimes \xi \otimes K \otimes \iota$$

Thus, the definition of a parabolic Sp(4,ℝ)-Higgs bundle according to §2.2 specializes to the following:

Definition 2.3.1. Let X be a compact Riemann surface of genus g and let the divisor $D := \{x_1, \ldots, x_s\}$ of s-many distinct points on X, assuming that $2g-2+s>0$. A *parabolic Sp(4,ℝ)-Higgs bundle* is defined as a triple (V, β, γ), where

- V is a rank 2 bundle on X, equipped with a parabolic structure given by a flag $V_x \supset L_x \supset 0$ and weights $0 \leq \alpha_1(x) < \alpha_2(x) < 1$ for every $x \in D$, and
- $\beta : V^\vee \to V \otimes K \otimes \iota$ and $\gamma : V \to V^\vee \otimes K \otimes \iota$ are strongly parabolic morphisms.

Remark 2.3.2. The parabolic structures on V and $(V)^\vee$ now induce a parabolic structure on the parabolic sum $E = V \oplus (V)^\vee$; Moreover, $par\deg E = 0$. We will prefer to think of a parabolic $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle as a pair (E, Φ), where $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$ rather than as a triple (V, β, γ), because it is preferred to use the more workable notion for a stable parabolic Higgs bundle of C. Simpson in order to introduce a notion of maximality for these objects.

2.4 Milnor-Wood type inequality

Definition 2.4.1. The *parabolic Toledo invariant* of a parabolic $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle is defined as the rational number

$$\tau = par\deg(V)$$

Moreover, we get a *Milnor-Wood type inequality* for this topological invariant:

Proposition 2.4.2. *Let (E, Φ) be a semistable parabolic $Sp(4,\mathbb{R})$-Higgs bundle. Then*

$$|\tau| \leq 2g-2+s$$

where s is the number of punctures on the surface X.

Proof. Consider parabolic bundles $N = \ker(\gamma)$ and $I = \mathrm{Im}(\gamma) \otimes (K \otimes \iota)^{-1} \leq (V)^\vee$. We thus get an exact sequence of parabolic bundles

$$0 \to N \to V \to I \otimes K \otimes \iota \to 0$$

and so

$$\begin{aligned}\operatorname{par}\deg(V) &= \operatorname{par}\deg(N) + \operatorname{par}\deg(I \otimes K \otimes \iota) \qquad (2.1)\\ &= \operatorname{par}\deg(N) + \operatorname{par}\deg(I) + \operatorname{rk}(I)(2g-2+s) \qquad (2.2)\end{aligned}$$

using the formula that gives the parabolic degree for the tensor product and the fact that $\operatorname{par}\deg(K \otimes \iota) = 2g - 2 + s$.

I is a subsheaf of $(V)^{\vee}$ and $I \hookrightarrow (V)^{\vee}$ is a parabolic map. Let $\tilde{I} \subset (V)^{\vee}$ be its saturation, which is a subbundle of $(V)^{\vee}$ and endow it with the induced parabolic structure. So $N, V \oplus \tilde{I} \subset E$ are Φ-invariant parabolic subbundles of E. The semistability of (E, Φ) now implies $par\mu(N) \leq par\mu(E)$ and $par\mu(V \oplus I) \leq par\mu\left(V \oplus \tilde{I}\right) \leq par\mu(E)$. However,

$$par\mu(E) = \frac{par\deg(E)}{\operatorname{rk}(E)} = 0$$

thus we have

$$par\deg(N) \leq 0$$

and

$$par\deg(V) + par\deg(I) \leq 0$$

From the last two inequalities, as well as Equation (2.2) we get:

$$par\deg(V) \leq -par\deg(V) + \operatorname{rk}(I)(2g-2+s)$$

In other words, $\tau \leq 2g - 2 + s$, since $\operatorname{rk}(I) \leq 2$.

Lastly, the map $(V, \beta, \gamma) \mapsto \left((V)^{\vee} \otimes \iota, \gamma, \beta\right)$ defines an isomorphism $\mathcal{M}_{-\tau} \cong \mathcal{M}_{\tau}$ providing also the minimal bound $-\tau \leq 2g - 2 + s$. □

Definition 2.4.3. The parabolic $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles with parabolic Toledo invariant $\tau = 2g - 2 + s$ will be called *maximal* and we will denote the components containing such triples by

$$\mathcal{M}_{par}^{\max} := \mathcal{M}_{par}^{2g-2+s}$$

2.5 Non-abelian Hodge correspondence on the punctured disk

In this section we review the non-abelian Hodge correspondence for non-compact surfaces established by C. Simpson in [40] and describe the relation between the parabolic weights for a fixed $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle on the punctured unit disk $\mathbb{D}_0 := \mathbb{D}\backslash\{0\}$ with varying weights, and the parallel transport along a loop around the puncture for the associated flat connection on the bundle. In order to describe this relation, we will need the definition of a parabolic Higgs bundle as a filtered regular Higgs bundle. Moreover, for the construction of the correspondence in this case it is necessary that the harmonic metric on the bundle has at most polynomial growth at the punctures in order to extend the holomorphic Higgs bundles across those points; these notions were introduced in [40] and the necessary growth condition of the hermitian metric, called *tameness*, is related to the algebraic stability of the filtered regular Higgs bundle.

An algebraic vector bundle over a surface X is a bundle given by regular algebraic transition functions over Zariski open sets, in other words, a locally free sheaf of $\mathcal{O}_X$-modules. For a compact Riemann surface X of genus $g \geq 2$ with s-many marked points $D = \{x_1, \ldots, x_s\}$, a filtered vector bundle is defined as follows:

Definition 2.5.1. A *filtered vector bundle* $(E, \{E_{\alpha,x_i}\})$ is an algebraic vector bundle $E \to X \backslash D$ together with a collection of vector bundles E_{α,x_i} indexed by $\alpha \in \mathbb{R}$ and extending E across the punctures x_i, such that

- the extensions form a decreasing left continuous filtration $E_{\alpha,x_i} \subset E_{\beta,x_i}$ for $\alpha \geq \beta$,
- for every α, $E_{\alpha-\varepsilon,x_i} = E_{\alpha,x_i}$ for small ε, and
- if z is a local coordinate vanishing to order one at x_i, then $E_{\alpha+1} = E_\alpha \oplus \mathcal{O}(-x_i)$.

Let $\bar{E}$ denote the bundle over the compact surface X obtained from E using the extensions E_{0,x_i} at all punctures. Then the fiber $\bar{E}_{x_i}$ is a vector space with a filtration $\left(\bar{E}_\alpha\right)_{x_i}$, indexed by $0 \leq \alpha < 1$. The weights of the filtration $\{E_\alpha\}$ are precisely the values where the filtration jumps, so there is a proper filtration

$$\bar{E}_{\alpha_n} \supset \bar{E}_{\alpha_{n-1}} \supset \ldots \supset \bar{E}_{\alpha_1} \supset 0$$

For $\mathrm{Gr}_{\alpha_i}\left(\bar{E}_{x_i}\right) := \left(\bar{E}_{\alpha_i}\right)_{x_i} / \left(\bar{E}_{\alpha_i - 1}\right)_{x_i}$, the *algebraic degree* of a filtered bundle is defined as the rational number

$$\deg(E) = \deg\left(\bar{E}\right) + \sum_{x_i \in D} \sum_{0 \leq \alpha < 1} \alpha \dim\left(\mathrm{Gr}_\alpha\left(\bar{E}_{x_i}\right)\right)$$

A *filtered regular Higgs bundle* $\left((E, \{E_{\alpha,x_i}\}), \Phi\right)$ is now a filtered vector bundle $(E, \{E_{\alpha,x_i}\})$ together with a map $\Phi : E \to E \otimes K_X$ satisfying a regularity condition with respect to the filtrations:

$$\Phi : E_{\alpha,x_i} \to E_{\alpha,x_i} \otimes K_X(D)$$

Definition 2.5.2. We say that a filtered regular Higgs bundle $((E, \{E_{\alpha,x_i}\}), \Phi)$ is *algebraically stable* (resp. *algebraically semistable*), if for any filtered subbundle $F \subset E$ with induced filtration preserved by Φ, it holds that

$$\frac{\deg(\mathrm{F})}{\mathrm{rk}(F)} < \frac{\deg(E)}{\mathrm{rk}(E)}, (\text{ resp } \leq)$$

Remark 2.5.3. Note that the definition of a filtered regular Higgs bundle is equivalent to the definition by V. Mehta and C. Seshadri described in §2.1. Indeed, for a filtered vector bundle $(E, \{E_\alpha\})$ and $E_{x,0}$ the fiber of $E_0 \to X$ over $x \in X$, the vector space $E_{x,0}$ has an induced filtration $\{E_{x,\alpha}\}$ indexed by $0 \leq \alpha < 1$. For each α, let $\mathrm{Gr}_\alpha(E_{x,0})$ be the direct limit of the system $E_{x,\alpha}/E_{x,\beta}$ over all $\beta > \alpha$. The weights of the parabolic structure are the values of $\alpha \in [0,1)$ such that $\dim_{\mathbb{C}}\mathrm{Gr}_\alpha(E_{x,0}) > 0$. Now, in a neighborhood U of the point x with coordinate z around x, such that $z(x) = 0$, the Higgs field Φ locally has the form

$$\varphi(z)\frac{dz}{z}$$

where φ is a holomorphic endomorphism of $E_0|_U$. The residue of Φ at the point x is defined to be $\mathrm{Res}_x\Phi := \varphi(0)$. The condition that Φ preserves the parabolic structure at each point $x \in D$, as in Definition 2.1.5, means that the residue of Φ respects the filtration $\{E_{x,\alpha}\}$ defined above.

A filtered regular Higgs bundle together with the notion of algebraic stability is a purely algebraic object. The topological objects corresponding to those were called by C. Simpson filtered local systems and are defined below:

Definition 2.5.4. For a fixed base point $y \in X$ and a puncture $x_i \in D$, a *filtered local system* is a representation $\rho : \pi_1(X) \to \mathrm{GL}(L_y)$ with filtrations L_{β,x_i} of the fiber L_y, indexed by $\beta \in \mathbb{R}$, such that

- the filtrations are decreasing and left continuous in β, and
- L_{β,x_i} is $\rho(\gamma_{x_i})$-invariant for a loop γ_{x_i} around x_i.

The *degree* of a filtered local system is defined as the rational number

$$\deg(L) = \sum_{x_i \in D} \sum_{\beta} \beta \dim(\mathrm{Gr}_\beta(L_{x_i}))$$

and a filtered local system is called *stable* (resp. *semistable*) if for any subsystem $M \subset L$

with an induced filtration it holds that

$$\frac{\deg(M)}{\mathrm{rk}(M)} < \frac{\deg(L)}{\mathrm{rk}(L)} \quad (\text{resp.} \quad \leq).$$

In order to show a correspondence between these algebraic and topological objects, we need to use a hermitian metric on the bundle with a specific growth condition at the punctures imposed:

Definition 2.5.5. Let $E \to X$ be a holomorphic bundle with a smooth hermitian metric h and let F_h denote the curvature of the associated Chern connection. Let $U \subset X$ be a neighborhood of a puncture x_i on X with coordinate r around the puncture. The metric h on E is called *acceptable*, if $|F_h| \leq f + \frac{1}{r^2(\log r)^2}$ for some $f \in L^p$ with $p > 1$.

The main theorem from [40] is now the following:

Theorem 2.5.6. *There is a one-to-one correspondence between polystable filtered regular Higgs bundles of degree zero and polystable filtered local systems of degree zero.*

In [27] S. Kim and G. Wilkin show that for a stable parabolic Higgs bundle, the metric solving the self-duality equations depends analytically on the choice of weights and stable Higgs bundle in a neighborhood of the initial weight and Higgs bundle. A local version of this theorem provides an explicit description of the relation between the parabolic weights of a stable parabolic Higgs bundle and the holonomy of the associated flat connection around each puncture for the case $G = \mathrm{SL}(2, \mathbb{C})$.

From this point on, we restrict attention to one particular point $p \in D$. Let $\mathbb{D}_0 := \mathbb{D} \setminus \{0\}$ denote the punctured unit disk and choose a branch of log

$$U = \left\{ z = re^{i\gamma} \in \mathbb{D}_0 : \gamma \in (-\pi, \pi) \right\}.$$

Let $E \to \mathbb{D}_0$ a rank 2 complex vector bundle trivialized over U and define a Higgs structure on E taking the trivial holomorphic structure, and defining the Higgs field on the trivialization over U by

$$\Phi(z) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \frac{dz}{z}$$

Let $w^{1,0}$ and $w^{0,1}$ be a basis for the holomorphic sections of E in the trivialization over U such that

$$\Phi(z)\, w^{1,0} = \frac{1}{2} w^{0,1} \frac{dz}{z} \text{ and } \Phi(z)\, w^{0,1} = 0 \tag{2.3}$$

With respect to these sections, let the decomposition $E \cong E^{1,0} \oplus E^{0,1}$ and consider the hermitian metric on E

$$k_\theta(r) = \begin{pmatrix} \frac{1}{2\theta}\left(r^{-\theta} - r^{\theta}\right) & 0 \\ 0 & \frac{2\theta}{r^{-\theta}-r^{\theta}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\theta}\sinh\left(\theta \log r\right) & 0 \\ 0 & -\frac{1}{\frac{1}{\theta}\sinh(\theta\log r)} \end{pmatrix}$$

With respect to this metric,

$$\left|w^{1,0}\right|_{k_\theta} = \frac{r^{-\frac{1}{2}\theta}}{\sqrt{2\theta}}\left(1 - r^{2\theta}\right)^{\frac{1}{2}} = O\left(r^{-\frac{1}{2}\theta}\right), \ \left|w^{0,1}\right|_{k_\theta} = \frac{\sqrt{2\theta}r^{\frac{1}{2}\theta}}{\left(1-r^{2\theta}\right)^{\frac{1}{2}}} = O\left(r^{\frac{1}{2}\theta}\right)$$

thus the weights in the interval $[0,1)$ are

$$\frac{1}{2}\theta \text{ and } 1 - \frac{1}{2}\theta$$

The curvature of k_θ is calculated to be

$$F_{k_\theta} = -\frac{\theta^2}{4r^2\sinh^2\left(\theta\log r\right)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} d\bar{z}dz$$

and so $|F_{k_\theta}| \leq \frac{1}{r^2(\log r)^2}$ in a neighborhood of $r = 0$, that is, the metric is acceptable. Moreover, one sees that $F_{k_\theta} + [\Phi, \Phi^*] = 0$, thus the metric is Hermitian-Einstein for all θ and the associated connection $D_\theta = \bar{\partial} + \partial_\theta + \Phi + \Phi^*$ is flat.

For the basis $w^{1,0}$, $w^{0,1}$ of the holomorphic section of the bundle E considered in Equation (2.3), the holomorphic structure $d''_\theta := \bar{\partial} + \Phi^*$ has holomorphic sections given by $w^{1,0}$ and $v^{0,1}_\theta := w^{0,1} + \theta\coth\left(\theta\log r\right)w^{1,0}$. A calculation from [27,p.11] shows that

$$d'_\theta w^{1,0} = \frac{1}{2}v^{0,1}_\theta\frac{dz}{z} \text{ and } d'_\theta v^{0,1} = \frac{1}{2}\theta^2 w^{1,0}\frac{dz}{z}$$

It turns out that the sections

$$s_1 = z^{-\frac{\theta}{2}}\left(\theta w^{1,0} + v^{0,1}_\theta\right) \text{ and } s_2 = z^{\frac{\theta}{2}}\left(\theta w^{1,0} - v^{0,1}_\theta\right)$$

are flat with respect to the connection $D_\theta = d''_\theta + d'_\theta$. Therefore, the parallel transport along a loop around the puncture with respect to this basis is given by

$$(s_1, s_2) \mapsto \left(e^{-i\pi\theta}s_1, e^{i\pi\theta}s_2\right).$$

In other words, the corresponding representation $\rho_\theta : \mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ maps a generator of the integers to the element $\begin{pmatrix} e^{-i\pi\theta} & 0 \\ 0 & e^{i\pi\theta} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$. S. Kim and G. Wilkin also show that $k_\theta(r)$ depends analytically on $\frac{1}{2}\theta$ and the representations ρ_θ converge to the representation $\rho_0 : \mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$, which maps a generator of $\mathbb{Z}$ to the element $\begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$; cf. [27], §3 for more details.

CHAPTER 3

GLUING CONSTRUCTIONS OVER A COMPLEX CONNECTED SUM OF RIEMANN SURFACES

In this chapter we develop our gluing construction for stable parabolic Sp(4,$\mathbb{R}$)-Higgs bundles to produce a polystable non-parabolic Sp(4,$\mathbb{R}$)-Higgs bundle over the complex connected sum of Riemann surfaces. The necessary condition in order to combine the initial parabolic data over the connected sum operation is that this data is identified over annuli around the points in the divisors of the Riemann surfaces. Aiming to provide new model Higgs bundles in the exceptional components of $\mathcal{M}^{\max}$, we consider parabolic data which around the punctures are *a priori not identified*, but we then look for deformations of those into model solutions of the Hitchin equations which will allow us to combine data over the complex connected sum. This deformation argument uses deformations of SL(2,$\mathbb{R}$)-solutions to the Hitchin equations over a punctured surface and subsequently we extend this for Sp(4,$\mathbb{R}$)-pairs using appropriate embeddings $\phi : \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$. Therefore, our gluing construction involves parabolic Sp(4,$\mathbb{R}$)-pairs which arise from SL(2,$\mathbb{R}$)-pairs via extensions by such embeddings, producing an approximate solution of the Sp(4,$\mathbb{R}$)-equations. We then apply a contraction mapping argument to correct this approximate solution to an exact solution of the equations. The analytic machinery we use to achieve this is based on work by R. Mazzeo, J. Swoboda, H. Weiss and F. Witt [29] and J. Swoboda [42], whereas the analysis worked out in this chapter also provides an extension of the main theorem in [42]. By analogy with the terminology introduced by O. Guichard and A. Wienhard in their construction of hybrid representations, we call the polystable Higgs bundles corresponding to such exact solutions *hybrid*. In the next chapter we deal with the problem of identifying the components such glued objects may lie and see that they do indeed correspond to the Guichard-Wienhard hybrid representations.

3.1 The local model

In this section, we describe the local SL(2,$\mathbb{R}$)-model solutions to the Hitchin equations which are going to serve as a guide for the gluing construction of the parabolic stable Sp(4,$\mathbb{R}$)-Higgs bundles. The description of these models is obtained by studying the behavior of the harmonic map between a surface X with a given complex structure and the surface X with the corresponding Riemannian metric of constant curvature -4, under degeneration of the domain Riemann surface X to a noded surface; cf. [42], [45] for further details.

Let $\mathcal{C} = (S^1)_x \times [1,\infty)_y$ denote the half-infinite cylinder, endowed with the complex coordinate $z = x+iy$ and flat Riemannian metric $g_{\mathcal{C}} = |dz|^2 = dx^2 + dy^2$. For parameter $s > 0$ let

$$N_s = \left[s^{-1}\csc^{-1}\left(s^{-1}\right), \frac{\pi}{s} - s^{-1}\csc^{-1}\left(s^{-1}\right)\right]_u \times \left(S^1\right)_v$$

be the finite cylinder with complex coordinate $w = u+iv$, and carrying the hyperbolic metric $g_s = s^2\csc^2(su)\,|dw|^2$. It is shown in [45] that the one parameter family $w_s : \left(\mathcal{C}, |dz|^2\right) \to (N_s, g_s)$ with $w_s = u_s + iv_s$ and where $v_s(x,y) = x$, $u_s(x,y) = \frac{1}{s}\sin^{-1}\left(\frac{1-B_s(y)}{1+B_s(y)}\right)$, for $B_s(y) = \frac{1-s}{1+s}e^{2s(1-y)}$, serves as a model for harmonic maps with domain a noded Riemann surface and target a smooth Riemann surface containing a long hyperbolic neck with central geodesic of length $2\pi s$.

For a stable SL(2,$\mathbb{R}$)-Higgs bundle (E,Φ) on X with $E = L \oplus L^{-1}$ for L a holomorphic square root of the canonical line bundle over X, endowed with an auxiliary hermitian metric h_0, and $\Phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \in H^0(X, \mathfrak{sl}(E))$ for q a holomorphic quadratic differential, there is an induced hermitian metric $H_0 = h_0 \oplus h_0^{-1}$ on E and $A = A_L \oplus A_L^{-1}$ the associated Chern connection with respect to h. The stability condition implies that there exists a complex gauge transformation g unique up to unitary gauge transformations, such that $(A_{1,s}, \Phi_{1,s}) := g^*(A,\Phi)$ is a solution to the Hitchin equations. Choosing a local holomorphic trivialization on E and assuming that with respect to it the auxiliary hermitian metric h_0 is the standard hermitian metric on $\mathbb{C}^2$, the corresponding hermitian metric for this solution on the bundle $E = L \oplus L^{-1}$ is globally well-defined with respect to the holomorphic splitting of E into line bundles. Calculations worked out in [42] imply that in particular $H_{1,s} = \begin{pmatrix} h_{1,s} & 0 \\ 0 & h_{1,s}^{-1} \end{pmatrix}$, for

$$h_{1,s} = \frac{2}{s}\left(\frac{1 - B_s^{1/2}}{1 + B_s^{1/2}}\right)$$

the hermitian metric on L and g_s with $g_s^2 = H_{1,s}^{-1}$ is the complex gauge transformation giving rise to an exact solution $(A_{1,s}, \Phi_{1,s})$ of the self-duality equations.

Moreover, after the change in coordinates

$$\zeta = e^{iz},\ idz = \frac{d\zeta}{\zeta}$$

which describes the conformal mapping of the cylinder $\mathcal{C}$ to the punctured unit disk, one sees that

$$A_{1,s} = O\left(|\zeta|^s\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{d\zeta}{\zeta} - \frac{d\bar{\zeta}}{\bar{\zeta}} \right), \ \Phi_{1,s} = \left(1 + O\left(|\zeta|^s\right)\right) \begin{pmatrix} 0 & \frac{s}{2} \\ \frac{s}{2} & 0 \end{pmatrix} \frac{d\zeta}{i\zeta}.$$

Therefore, after a unitary change of frame, the Higgs field $\Phi_{1,s}$ is asymptotic to the model Higgs field $\Phi_s^{\mathrm{mod}} = \begin{pmatrix} \frac{s}{2} & 0 \\ 0 & -\frac{s}{2} \end{pmatrix} \frac{d\zeta}{i\zeta}$, while the connection $A_{1,s}$ is asymptotic to the trivial flat connection.

In conclusion, the *model solution* to the SL(2,ℝ)-Hitchin equations we will be considering is described by

$$A^{\mathrm{mod}} = 0, \ \Phi^{\mathrm{mod}} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \frac{dz}{z}$$

over a punctured disk with z-coordinates around the puncture with the condition that $C \in \mathbb{R}$ with $C \neq 0$, and that the meromorphic quadratic differential $q := \det \Phi^{\mathrm{mod}}$ has at least one simple zero. That this is indeed the generic case, is discussed in [29].

3.2 Weighted Sobolev spaces

In order to develop the necessary analytic arguments for the gluing construction later on, we need to define Sobolev spaces. Let X be a compact Riemann surface and $D := \{p_1, \ldots, p_s\}$ be a collection of s-many distinct points on X. Moreover, let (E, h) be a hermitian vector bundle on E. Choose an initial pair $\left(A^{\mathrm{mod}}, \Phi^{\mathrm{mod}}\right)$ on E, such that in some unitary trivialization of E around each point $p \in D$, the pair coincides with the local model from §3.1. Of course, on the interior of each region $X \setminus \{p\}$ the pair $\left(A^{\mathrm{mod}}, \Phi^{\mathrm{mod}}\right)$ need not satisfy the Hitchin equations.

For fixed local coordinates z around each point $p \in D$, such that $z(p) = 0$, define r to be a positive function which coincides with $|z|$ around the puncture. Using the singular measure $r^{-1} dr d\theta$ and a fixed weight $\delta > 0$ define *weighted L^2-Sobolev spaces*:

$$L^2_\delta = \left\{ f \in L^2\left(r dr d\theta\right) \middle| \frac{f}{r^{\delta+1}} \in L^2 \right\}$$

and

$$H^k_\delta = \left\{ u, \nabla^j u \in L^2_\delta\left(r dr\right), 0 \leq j \leq k \right\}.$$

The *Sobolev space with k-derivatives in* L^2 is defined as:

$$L^{k,2}_{\delta}=\left\{f,\frac{\nabla^j f}{r^{k-j}}\in L^2_{\delta}\left(rdrd\theta\right),\,0\le j\le k\right\}$$

where ∇ is the covariant derivative associated to a fixed background unitary connection on E. We are interested in deformations of A and Φ such that the curvature of the connection $D=A+\Phi+\Phi^*$ remains $O\left(r^{-2+\delta}\right)$, that is, slightly better than L^1. We can then define *global Sobolev spaces on* X as the spaces of admissible deformations of the model unitary connection and the model Higgs field $\left(A^{\text{ mod}},\Phi^{\text{ mod}}\right)$ as:

$$\mathcal{A}=\left\{A^{\text{ mod}}+\alpha\,\middle|\alpha\in H^{1,2}_{-2+\delta}\left(\Omega^1\otimes\mathfrak{su}\left(E\right)\right)\right\}$$

and

$$\mathcal{B}=\left\{\Phi^{\text{ mod}}+\varphi\,\middle|\varphi\in H^{1,2}_{-2+\delta}\left(\Omega^{1,0}\otimes\operatorname{End}\left(E\right)\right)\right\}$$

The space of unitary gauge transformations

$$\mathcal{G}=\left\{g\in\mathrm{U}\left(E\right),\,g^{-1}dg\in L^{1,2}_{-2+\delta}\right\}$$

acts on $\mathcal{A}$ and $\mathcal{B}$ as follows

$$g^*\left(A,\Phi\right)=\left(g^{-1}Ag+g^{-1}dg,g^{-1}\Phi g\right)$$

for a pair $(A,\Phi)\in\mathcal{A}\times\mathcal{B}$.

These considerations allow us to introduce the moduli space of solutions which are close to the model solution over a punctured Riemann surface $X^{\times}:=X-D$ for some fixed parameter $C\in\mathbb{R}$:

$$\mathcal{M}\left(X^{\times}\right)=\frac{\left\{(A,\Phi)\in\mathcal{A}\times\mathcal{B}\,|(A,\Phi)\text{ satisfies the Hitchin equations}\right\}}{\mathcal{G}}$$

This moduli space was explicitly constructed by H. Konno in [28] as a hyperkähler quotient.

3.3 Approximate solutions of the SL(2,$\mathbb{R}$)-Hitchin equations

In §3.2 we have seen that a point in the moduli space $\mathcal{M}\left(X^{\times}\right)$ differs from a model pair $\left(A^{\text{ mod}},\Phi^{\text{ mod}}\right)$ by some element in $H^1_{-2+\delta}$. The following result by O. Biquard and P. Boalch shows that (A,Φ) is asymptotically close to the model in a much stronger sense:

Lemma 3.3.1. *Lemma 5.3 in [4]. For each point $p \in D$ let $\left(A_p^{\mathrm{mod}}, \Phi_p^{\mathrm{mod}}\right)$ be a model pair as was defined in §3.1. If $(A, \Phi) \in \mathcal{M}(X^\times)$, then there exists a unitary gauge transformation $g \in \mathcal{G}$ such that in a neighborhood of each point $p \in D$ it is*

$$g^*(A, \Phi) = \left(A_p^{\mathrm{mod}}, \Phi_p^{\mathrm{mod}}\right) + O\left(r^{-1+\delta}\right)$$

for a positive constant δ.

The decay described in this lemma can be further improved by showing that in a suitable complex gauge transformation the point (A, Φ) coincides precisely with the model near each puncture in D. With respect to the *singular measure* $r^{-1}drd\vartheta$ on $\mathbb{C}$, we first introduce the Hilbert spaces

$$L^2_{-1+\delta}\left(r^{-1}drd\vartheta\right) = \left\{u \in L^2(\mathbb{D}) \,\middle|\, r^{-\delta}u \in L^2\left(r^{-1}drd\vartheta\right)\right\}$$

$$H^k_{-1+\delta}\left(r^{-1}drd\vartheta\right) = \left\{u \in L^2(\mathbb{D}) \,\middle|\, (r\partial r)^j \partial_\vartheta^l u \in L^2_{-1+\delta}\left(r^{-1}drd\vartheta\right), 0 \le j + l \le k\right\}$$

for $\mathbb{D} = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$ the punctured unit disk. We then have the following result by J. Swoboda:

Lemma 3.3.2. *Lemma 3.2 in [42]. Let $(A, \Phi) \in \mathcal{M}(X^\times)$ and let δ be the constant provided by Lemma 3.3.1. Fix another constant $0 < \delta' < \min\left\{\frac{1}{2}, \delta\right\}$. Then there is a complex gauge transformation $g = \exp(\gamma) \in \mathcal{G}^c$ with $\gamma \in H^2_{-1+\delta'}(r^{-1}drd\vartheta)$, such that $g^*(A, \Phi)$ coincides with $\left(A_p^{\mathrm{mod}}, \Phi_p^{\mathrm{mod}}\right)$ in a sufficiently small neighborhood of the point p, for each $p \in D$.*

We shall now use this complex gauge transformation as well as a smooth cut-off function to obtain an approximate solution to the SL(2,$\mathbb{R}$)-Hitchin equations. For the fixed local coordinates z around each puncture p and the positive function r coinciding with $|z|$ around the puncture, fix a constant $0 < R < 1$ and choose a smooth cut-off function $\chi_R : [0, \infty) \to [0, 1]$ with $\operatorname{supp}\chi \subseteq [0, R]$ and $\chi_R(r) = 1$ for $r \le \frac{3R}{4}$. We impose the further requirement on the growth rate of this cut-off function:

$$|r\partial_r \chi_R| + \left|(r\partial r)^2 \chi_R\right| \le C \tag{3.1}$$

for some constant C not depending on R.

The map $x \mapsto \chi_R(r(x)) : X^\times \to \mathbb{R}$ gives rise to a smooth cut-off function on the punctured surface $X^\times$ which by a slight abuse of notation we shall still denote by χ_R. We may use this function χ_R to glue the two pairs (A, Φ) and $\left(A_p^{\mathrm{mod}}, \Phi_p^{\mathrm{mod}}\right)$ into an *approximate solution*

$$\left(A_R^{app}, \Phi_R^{app}\right) := \exp(\chi_R \gamma)^* (A, \Phi).$$

The pair $(A_R^{app}, \Phi_R^{app})$ is a smooth pair and is by construction an exact solution of the Hitchin equations away from each punctured neighborhood $\mathcal{U}_p$, while it coincides with the model pair $\left(A_p^{\mathrm{mod}}, \Phi_p^{\mathrm{mod}}\right)$ near each puncture. More precisely, we have:

$$(A_R^{app}, \Phi_R^{app}) = \begin{cases} (A, \Phi), \text{ over } X \backslash \bigcup\limits_{p \in D} \left\{ z \in \mathcal{U}_p \,\middle|\, \frac{3R}{4} \leq |z| \leq R \right\} \\ \left(A_p^{\mathrm{mod}}, \Phi_p^{\mathrm{mod}}\right), \text{ over } \left\{ z \in \mathcal{U}_p \,\middle|\, 0 < |z| \leq \frac{3R}{4} \right\}, \text{ for each } p \in D \end{cases}$$

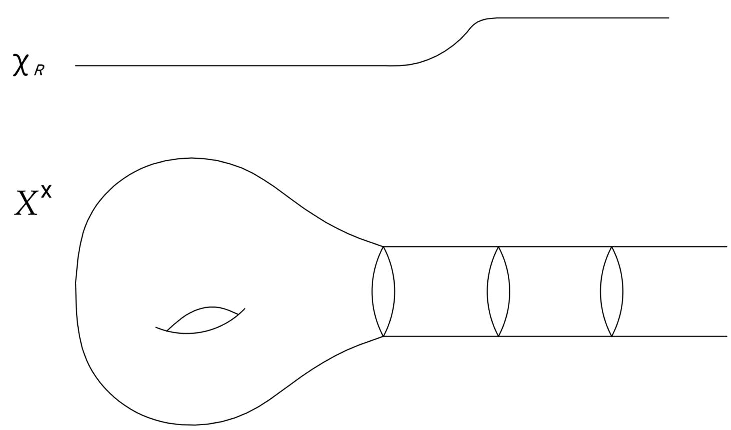

Figure 3.1: Constructing an approximate solution over the punctured surface $X^\times$.

Since $(A_R^{app}, \Phi_R^{app})$ is complex gauge equivalent to an exact solution (A, Φ) of the Hitchin equations, it does still *satisfy the second equation*, in other words it holds that $\bar{\partial}_{A_R^{app}} \Phi_R^{app} = 0$. Indeed, for $\tilde{g} := \exp(\chi_R \gamma)$, we defined $(A_R^{app}, \Phi_R^{app}) = \tilde{g}^* (A, \Phi) = (\tilde{g}^{-1} A \tilde{g} + \tilde{g}^{-1} d\tilde{g}, \tilde{g}^{-1} \Phi \tilde{g})$ and (A, Φ) is an exact solution, thus in particular

$$0 = \bar{\partial}_A \Phi = \bar{\partial} \Phi + \left[A^{0,1} \wedge \Phi\right]$$

We may now check

$$\begin{aligned} \bar{\partial}_{A_R^{app}} \Phi_R^{app} &= \bar{\partial} \Phi_R^{app} + \left[(A_R^{app})^{0,1} \wedge \Phi_R^{app} \right] \\ &= \bar{\partial} \left(\tilde{g}^{-1} \Phi \tilde{g} \right) + \left[\left(\tilde{g}^{-1} A^{0,1} \tilde{g} + \tilde{g}^{-1} \bar{\partial} \tilde{g} \right) \wedge \tilde{g}^{-1} \Phi \tilde{g} \right] \\ &= \bar{\partial} \left(\tilde{g}^{-1} \Phi \tilde{g} \right) + \tilde{g}^{-1} \left[A^{0,1} \wedge \Phi \right] \tilde{g} + \tilde{g}^{-1} \left(\bar{\partial} \tilde{g} \right) \tilde{g}^{-1} \Phi \tilde{g} - \tilde{g}^{-1} \Phi \bar{\partial} \tilde{g} \\ &= \bar{\partial} \left(\tilde{g}^{-1} \Phi \tilde{g} \right) + \tilde{g}^{-1} \left(-\bar{\partial} \Phi \right) \tilde{g} + \tilde{g}^{-1} \left(\bar{\partial} \tilde{g} \right) \tilde{g}^{-1} \Phi \tilde{g} - \tilde{g}^{-1} \Phi \bar{\partial} \tilde{g} \\ &= \bar{\partial} \left(\tilde{g}^{-1} \right) \Phi \tilde{g} + \tilde{g}^{-1} \left(\bar{\partial} \tilde{g} \right) \tilde{g}^{-1} \Phi \tilde{g} = 0, \end{aligned}$$

using the identity $\left(\bar{\partial} \tilde{g} \right) \tilde{g}^{-1} + \tilde{g} \bar{\partial} \left(\tilde{g}^{-1} \right) = 0$.

Moreover, Lemma 3.3.2 as well as the Assumption (3.1) we made on the growth rate of the bump function χ_R provide us with a good estimate of the error up to which $(A_R^{app}, \Phi_R^{app})$ satisfies the first equation:

Lemma 3.3.3. *Let $\delta' > 0$ be as in Lemma 3.3.2 and fix some further constant $0 < \delta'' < \delta'$. The approximate solution $(A_R^{app}, \Phi_R^{app})$ to the parameter $0 < R < 1$ satisfies*

$$\left\| *F^{\perp}_{A_R^{app}} + * \left[\Phi_R^{app} \wedge (\Phi_R^{app})^*\right] \right\|_{C^0(X^\times)} \leq C R^{\delta''}$$

for some constant $C = C(\delta', \delta'')$ which does not depend on R.

Proof. See [42] Lemma 3.5. □

3.4 Gluing over a complex connected sum

3.4.1 Set up

We will now use the approximate solutions from §3.3 in order to obtain an approximate solution by gluing parabolic Higgs bundles. Let X_1 be a closed Riemann surface of genus g_1 and $D_1 = \{p_1, \ldots, p_s\}$ a collection of distinct points on X_1. Let $(E_1, \Phi_1) \to X_1$ be a parabolic stable SL(2,$\mathbb{R}$)-Higgs bundle. Then there exists an adapted Hermitian metric h_1, such that (D_{h_1}, Φ_1) is a solution to the equations, with $D_{h_1} = \nabla\left(\bar{\partial}_1, h_1\right)$ the associated Chern connection.

As we have seen in §3.3, there exists a complex gauge transformation $g_1 = \exp(\gamma_1)$, such that $g_1^*(D_{h_1}, \Phi_1)$ is asymptotically close to a model solution $\left(A_{1,p}^{\text{mod}}, \Phi_{1,p}^{\text{mod}}\right)$ near the puncture p, for each $p \in D_1$. Choose a trivialization τ over a neighborhood $\mathcal{U}_p \subset X_1$ so that $(D_{h_1})^\tau$ denotes the connection matrix and let χ_1 be a smooth bump function on $\mathcal{U}_p$ with the assumptions made in §3.3, so that we may define $\tilde{g}_1 = \exp(\chi_1 \gamma_1)$ and take the approximate solution over X_1

$$(A_1^{app}, \Phi_1^{app}) = \tilde{g}_1^*(D_{h_1}, \Phi_1) = \begin{cases} (D_{h_1}, \Phi_1), & \text{away from the points in the divisor } D_1 \\ \left(A_{1,p}^{\text{mod}}, \Phi_{1,p}^{\text{mod}}\right), & \text{near the point } p, \text{ for each } p \in D_1 \end{cases}$$

The connection A_1^{app} is given, in that same trivialization, by the connection matrix $\chi_1(D_{h_1})^\tau$. The fact that $\tilde{g}_1$ is a complex gauge transformation may cause the holonomy over the bump region not to be real, so a priori we are considering this pair as *SL(2,$\mathbb{C}$)-data*.

We wish to obtain an approximate Sp(4,$\mathbb{C}$)-pair by extending the SL(2,$\mathbb{C}$)-data via an embedding

$$\phi : \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$$

and its extension $\phi : \mathrm{SL}(2,\mathbb{C}) \hookrightarrow \mathrm{Sp}(4,\mathbb{C})$. For the Cartan decompositions

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2) \oplus \mathfrak{m}(\mathrm{SL}(2,\mathbb{R}))$$
$$\mathfrak{sp}(4,\mathbb{R}) = \mathfrak{u}(2) \oplus \mathfrak{m}(\mathrm{Sp}(4,\mathbb{R}))$$

their complexifications respectively read

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}}(\mathrm{SL}(2,\mathbb{R}))$$
$$\mathfrak{sp}(4,\mathbb{C}) = \mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}}(\mathrm{Sp}(4,\mathbb{R}))$$

Assume now that copies of a maximal compact subgroup of SL(2,$\mathbb{R}$) are mapped via ϕ into copies of a maximal compact subgroup of Sp(4,$\mathbb{R}$). Then, since $\mathrm{SO}(2)^{\mathbb{C}} = \mathrm{SO}(2,\mathbb{C})$ and $\mathrm{U}(2)^{\mathbb{C}} = \mathrm{GL}(2,\mathbb{C})$, the embedding ϕ describes an embedding $\mathrm{SO}(2,\mathbb{C}) \hookrightarrow \mathrm{GL}(2,\mathbb{C})$ and so we may use its infinitesimal deformation $\phi_* : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sp}(4,\mathbb{C})$ to extend SL(2,$\mathbb{C}$)-data to Sp(4,$\mathbb{C}$)-data as follows:

We have constructed a pair $(A_1^{app}, \Phi_1^{app})$, where A_1^{app} is a unitary connection on a principal $H^{\mathbb{C}} = \mathrm{SO}(2,\mathbb{C})$-bundle $P_{\mathrm{SO}(2,\mathbb{C})}$ over X_1. Consider the principal $\mathrm{GL}(2,\mathbb{C})$-bundle $Q_{\mathrm{GL}(2,\mathbb{C})}$ by extension of structure group through the homomorphism ϕ:

$$Q_{\mathrm{GL}(2,\mathbb{C})} := P_{\mathrm{SO}(2,\mathbb{C})} \times_{\phi|_{\mathrm{SO}(2,\mathbb{C})}} \mathrm{GL}(2,\mathbb{C})$$

This principal bundle $Q_{\mathrm{GL}(2,\mathbb{C})}$ can be equipped with a connection form obtained by extension of structure group through this same homomorphism. Now, since ϕ_* respects the adjoint action of SL(2,$\mathbb{C}$) on $\mathfrak{sl}(2,\mathbb{C})$, we have an induced homomorphism of vector bundles from the adjoint bundle $\mathrm{ad}\left(P_{\mathrm{SO}(2,\mathbb{C})}\right)$ to $\mathrm{ad}\left(Q_{\mathrm{GL}(2,\mathbb{C})}\right)$. We may obtain now a curvature 2-form with values in $\mathrm{ad}\left(Q_{\mathrm{GL}(2,\mathbb{C})}\right)$ by composing the curvature form for A_1^{app} with this induced homomorphism on the adjoint bundles. This is the curvature form for the extended $\mathrm{GL}(2,\mathbb{C})$-connection on $Q_{\mathrm{GL}(2,\mathbb{C})}$ (see [35], §5.4, 5.5 for further details).

We shall denote the Sp(4,$\mathbb{C}$)-pair obtained by extension through ϕ by (A_l, Φ_l), with the curvature of the connection denoted by

$$F_{A_l} \in \Omega^2\left(\mathbb{R}^2; \mathrm{ad}\left(Q_{\mathrm{GL}(2,\mathbb{C})}\right)\right)$$

and with the Higgs field Φ_l given by

$$\Phi_l = \phi_* \left|_{\mathfrak{m}^{\mathbb{C}}(\mathrm{SL}(2,\mathbb{C}))} \left(\Phi_1^{app}\right)\right.$$

Assume, moreover, that the norm of the infinitesimal deformation ϕ_* satisfies a Lipschitz condition, in other words it holds that

$$\|\phi_*(M)\|_{\mathfrak{sp}(4,\mathbb{C})} \leq C\|M\|_{\mathfrak{sl}(2,\mathbb{C})}$$

for $M \in \mathfrak{sl}(2,\mathbb{C})$. In fact, the norms considered above are equivalent to the C_0-norm, since $\mathfrak{gl}(n)$ is finite dimensional, hence all norms are equivalent and induce the same topology. Restricting these norms to $\mathfrak{so}(2,\mathbb{C})$ and $\mathfrak{m}^{\mathbb{C}}(\mathrm{SL}(2,\mathbb{R}))$ respectively, we may deduce that the error in curvature is still described by the inequality

$$\left\| *F^{\perp}_{A_l^{app}} + *\left[\Phi_l^{app} \wedge \left(\Phi_l^{app}\right)^*\right] \right\|_{C^0} \leq k_l R^{\delta''}$$

for a (different) real constant k_l, which still does not depend on the parameter $R > 0$.

In summary, using an embedding $\phi : \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$ with the properties described above, we may extend the approximate solution $(A_1^{app}, \Phi_1^{app})$ to take an approximate $\mathrm{Sp}(4,\mathbb{C})$-pair (A_l, Φ_l) over X_1, which agrees with a model solution $\left(A_{l,p}^{\mathrm{mod}}, \Phi_{l,p}^{\mathrm{mod}}\right)$ over an annulus Ω_1^p around each puncture $p \in D_1$; the model solution $\left(A_{l,p}^{\mathrm{mod}}, \Phi_{l,p}^{\mathrm{mod}}\right)$ is the extension via ϕ of the model $\left(A_{1,p}^{\mathrm{mod}}, \Phi_{1,p}^{\mathrm{mod}}\right)$ in $\mathrm{SL}(2,\mathbb{R})$. The pair (A_l, Φ_l) lives in the holomorphic principal $\mathrm{GL}(2,\mathbb{C})$-bundle obtained by extension of structure group via ϕ, which we shall keep denoting as $\left(E_1 = \left(\mathbb{E}_1, \bar{\partial}_1\right), h_1\right)$ to ease notation.

Repeating the above considerations for another closed Riemann surface X_2 of genus g_2 and $D_2 = \{q_1, \ldots, q_s\}$ a collection of s-many distinct points of X_2, we obtain an approximate $\mathrm{Sp}(4,\mathbb{C})$-pair (A_r, Φ_r) over X_2, which agrees with a model solution $\left(A_{r,q}^{\mathrm{mod}}, \Phi_{r,q}^{\mathrm{mod}}\right)$ over an annulus Ω_2^q around each puncture $q \in D_2$. This pair lives on the holomorphic principal $\mathrm{GL}(2,\mathbb{C})$-bundle obtained by extension of structure group via another appropriate embedding $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$; let this hermitian bundle be denoted by $\left(E_2 = \left(\mathbb{E}_2, \bar{\partial}_2\right), h_2\right)$.

3.4.2 Gluing of the Riemann surfaces

We begin with a classical result from complex analysis and conformal geometry:

Theorem 3.4.1 (Schottky's Theorem on Conformal Mappings between Annuli)**.** *An annulus $\mathbb{A}_1 = \{z \in \mathbb{C}\,|r_1 < |z| < R_1\}$ can be mapped conformally onto the annulus $\mathbb{A}_2 =$*

$\{z \in \mathbb{C} \,|r_2 < |z| < R_2\}$ *if and only if* $\frac{R_1}{r_1} = \frac{R_2}{r_2}$. *Moreover, every conformal map* $f : \mathbb{A}_1 \to \mathbb{A}_2$ *takes the form* $f(z) = \lambda z$ *or* $f(z) = \frac{\lambda}{z}$, *where* $\lambda \in \mathbb{C}$ *with* $|\lambda| = \frac{r_2}{r_1}$ *or* $|\lambda| = r_2 R_1$ *respectively.*

Proof. See p. 35 in [2]. $\square$

Let us consider the Möbius transformation $f_\lambda : \mathbb{A}_1 \to \mathbb{A}_2$ with $f_\lambda(z) = \frac{\lambda}{z}$, where $\lambda \in \mathbb{C}$ with $|\lambda| = r_2 R_1 = r_1 R_2$. This is a conformal biholomorphism (equivalently bijective, angle-preserving and orientation-preserving) between the two annuli and the continuous extension of the function $z \mapsto |f_\lambda(z)|$ to the closure of $\mathbb{A}_1$ reverses the order of the boundary components. Indeed

- for $|z| = r_1$: $|f_\lambda(z)| = \frac{|\lambda|}{|z|} = \frac{r_2 R_1}{r_1} = \frac{r_1 R_2}{r_1} = R_2$.
- for $|z| = R_1$: $|f_\lambda(z)| = \frac{|\lambda|}{|z|} = \frac{r_2 R_1}{R_1} = r_2$.

Let two compact Riemann surfaces X_1, X_2 of respective genera g_1, g_2. Choose points $p \in X_1$, $q \in X_2$ and local charts around these points $\psi_i : U_i \to \Delta(0, \varepsilon_i)$ on X_i, for $i = 1, 2$. Now fix positive real numbers $r_i < R_i < \varepsilon_i$ such that the following two conditions are satisfied:

- $\psi_i^{-1}\left(\overline{\Delta(0, R_i)}\right) \cap U_j \neq \varnothing$, for every $U_j \neq U_i$ from the complex atlas of X_i. In other words, we are considering an annulus around each of the p and q contained entirely in the neighborhood of a single chart.
- $\frac{R_2}{r_2} = \frac{R_1}{r_1}$

Now set

$$X_i^* = X_i \backslash \psi_i^{-1}\left(\overline{\Delta(0, r_i)}\right)$$

Finally, choose the biholomorphism $f_\lambda : \mathbb{A}_1 \to \mathbb{A}_2$ described in the previous subsection. This biholomorphism is used to glue the two Riemann surfaces X_1, X_2 along the inverse image of the annuli $\mathbb{A}_1, \mathbb{A}_2$ on the surfaces, using the biholomorphism

$$g_\lambda : \Omega_1 = \psi_1^{-1}(\mathbb{A}_1) \to \Omega_2 = \psi_2^{-1}(\mathbb{A}_2)$$

with $g_\lambda = \psi_2^{-1} \circ f_\lambda \circ \psi_1$.

Define $X_\lambda = X_1 \#_\lambda X_2 = X_1^* \coprod X_2^* / \sim$, where the gluing of Ω_1 and Ω_2 is performed through the equivalence relation which identifies $y \in \Omega_1$ with $w \in \Omega_2$ iff $w = g_\lambda(y)$. For collections of s-many distinct points D_1 on X_1 and D_2 on X_2, this procedure is assumed to be taking place for annuli around each pair of points (p, q) for $p \in D_1$ and $q \in D_2$.

The manifold X_λ is endowed with a complex structure inherited from the complex structures of X_1 and X_2: Indeed, if $\mathcal{A}_1, \mathcal{A}_2$ are complex atlases for X_1, X_2, then $\mathcal{A}_1 \left|_{X_1^*} \bigcup \mathcal{A}_2 \right|_{X_2^*}$ is an atlas for X_λ, since we have chosen the gluing region not to overlap between two different charts on each side. On the glued region Ω, there are two charts $\left(\Omega_1, \psi_1 |_{\Omega_1}\right), \left(\Omega_2, \psi_2 |_{\Omega_2}\right)$, whereas $\varphi_{12} = \psi_1 \circ \psi_2^{-1} : \psi_2 \left(\Omega_1 \cap \Omega_2\right) \to \psi_1 \left(\Omega_1 \cap \Omega_2\right)$ is actually $\varphi_{12} \equiv f_\lambda : \mathbb{A}_1 \to \mathbb{A}_2$.

If X_1, X_2 are orientable and orientations are chosen for both, since f_λ is orientation preserving we obtain a natural orientation on the connected sum $X_1 \# X_2$ which coincides with the given ones on X_1^* and X_2^*.

Therefore, $X_\# = X_1 \# X_2$ is a Riemann surface of genus $g_1 + g_2 + s - 1$, the *complex connected sum*, where g_i is the genus of the X_i and s is the number of points in D_1 and D_2. Its complex structure however is heavily dependent on the parameters p_i, q_i, λ.

Gluing of hermitian metrics. Suppose further that the Riemann surfaces considered are equipped with a hermitian metric on their tangent bundle (X_1, h_1), (X_2, h_2) which are flat over neighborhoods around the points p_i containing the annuli Ω_1 and Ω_2. Consider an equivalent complex atlas $\mathcal{A} = (\{(U_\alpha, \psi_\alpha)\})$ and let $\{\rho_\beta\}$ be a partition of unity subordinate to a covering $\{V_\beta\}$ of the complex connected sum X_λ, such that $V_\beta \subset U_\alpha$.

We have a hermitian inner product on each V_β:

- h_1^x, over each $x \in X_1 \backslash \Omega_1$.
- h_2^x, over each $x \in X_2 \backslash \Omega_2$.
- h_Ω^x, over each $x \in \Omega := \Omega_1 \sim \ \Omega_2$ considering the cylinder equipped with a flat metric.

We may now use the partition of unity to glue all those together to a global hermitian metric over the complex connected sum:

$$h^x\left(u, v\right) = \sum_\beta \rho_\beta\left(x\right) \cdot h_\beta^x\left(u, v\right)$$

since any positive linear combination of positive definite hermitian products of $\mathbb{C}$ is again positive definite and hermitian.

3.4.3 Gluing of the bundles

For the Riemann surfaces X_1, X_2 as considered in §3.4.1, their connected sum $X_\# = X_1 \# X_2$ is constructed by gluing annuli around the points p_i of D_1, with annuli around the points q_i

of D_2, as described in §3.4.2. Moreover, for the pairs (A_l, Φ_l) and (A_r, Φ_r) defined in §3.4.1 we make the following important assumption:

Assumption 3.4.2. *The model solutions satisfy* $\left(A_{l,p}^{\text{mod}}, \Phi_{l,p}^{\text{mod}}\right) = -\left(A_{r,q}^{\text{mod}}, \Phi_{r,q}^{\text{mod}}\right)$ *for each pair of points* (p, q).

Given this assumption, now notice that for the bundles $(\mathbb{E}_1, \nabla_l := A_l + \Phi_l + \Phi_l^*)$ and $(\mathbb{E}_2, \nabla_r := A_r + \Phi_r + \Phi_r^*)$, the model flat connections will coincide. Let $\nabla := \nabla_l = -\nabla_r$ denote this flat connection over the annuli; we can then fix an identification of these flat bundles over the annuli to get a new bundle $\mathbb{E}_{\#}$ as follows:

Let Ω_1 be the annulus on X_1 for any point $p \in D_1$ and pick coordinates z around p with $z(p) = 0$. Let $V_1 \cup V_2$ an open covering of Ω_1, with $V_1 \cap V_2$ having two connected components, say $(V_1 \cap V_2)^+$ and $(V_1 \cap V_2)^-$. For a loop γ in Ω_1 around p take transition functions

$$g_1^z(x) = \begin{cases} 1, & z \in (V_1 \cap V_2)^- \\ \operatorname{hol}(\gamma, \nabla_l), & z \in (V_1 \cap V_2)^+ \end{cases}$$

Similarly, let Ω_2 be the annulus on X_2 for any point $q \in D_2$ and pick coordinates w around q with $w(q) = 0$. For a loop δ in Ω_2 around q take transition functions

$$g_2^w(x) = \begin{cases} \operatorname{hol}(\delta, \nabla_r), & w \in (V_1 \cap V_2)^- \\ 1, & w \in (V_1 \cap V_2)^+ \end{cases}$$

Using an orientation reversing isometry to glue the annuli Ω_1 and Ω_2 in constructing the connected sum, the region $(V_1 \cap V_2)^+$ of Ω_1 is glued together with the region $(V_1 \cap V_2)^-$ of Ω_2. The gluing of the Riemann surfaces is realized along the curve $zw = \lambda$, thus we have

$$\frac{dz}{z} = -\frac{dw}{w}$$

on the annuli. Now from Assumption 3.4.2, $\nabla_l = -\nabla_r$, and so there is defined a 1-cocycle on $\Omega := \Omega_1 \sim \Omega_2$ by $g(s) := g_1(z) = g_2\left(\frac{\lambda}{z}\right)$, since $w = \frac{\lambda}{z}$ for a point $s \in \Omega$. This is repeated for each pair of points (p, q). We may use this identification of the cocycles to define a bundle isomorphism $\mathbb{E}_1|_{\Omega_1} \xrightarrow{\simeq} \mathbb{E}_2|_{\Omega_2}$ and use this isomorphism to glue the bundles over Ω for every pair (p, q) to define the connected sum bundle $\mathbb{E}_1 \# \mathbb{E}_2$.

Remark 3.4.3. We can alternatively glue the bundles by picking a globally trivial frame on each side, flat with respect to the unitary connection A but not for ∇. Indeed for such a frame for A_l and A_r glue $(\Omega_1 \times \mathbb{C}^2) \coprod (\Omega_2 \times \mathbb{C}^2)$ under the identification map $(z, u) \mapsto (w, v)$ with $w = \frac{\lambda}{z}$ and $u = v$.

3.4.4 Gluing the connections and hermitian metrics

The pairs $(A_l, \Phi_l), (A_r, \Phi_r)$ agree over neighborhoods around the points in the divisors D_1 and D_2, with $A_l = A_r = 0$ and with $\Phi_l(z) = -\Phi_r(w)$, thus there is a suitable frame for ∇ over which the hermitian metrics are both described by the identity matrix and so they are constant in particular. Set $\left(A_{p,q}^{\text{mod}}, \Phi_{p,q}^{\text{mod}}\right) := \left(A_{l,p}^{\text{mod}}, \Phi_{l,p}^{\text{mod}}\right) = -\left(A_{r,q}^{\text{mod}}, \Phi_{r,q}^{\text{mod}}\right)$. We can glue the pairs $(A_l, \Phi_l), (A_r, \Phi_r)$ together to get an approximate solution of the Sp(4,$\mathbb{R}$)-Hitchin equations:

$$(A_R^{app}, \Phi_R^{app}) := \begin{cases} (A_l, \Phi_l), & \text{over } X_1 \backslash X_2 \\ \left(A_{p,q}^{\text{mod}}, \Phi_{p,q}^{\text{mod}}\right), & \text{over } \Omega \text{ around each pair of points } (p,q), \\ (A_r, \Phi_r) & \text{over } X_2 \backslash X_1 \end{cases}$$

considered on the bundle $(\mathbb{E}_1 \# \mathbb{E}_2, h_\#)$ over the complex connected sum $X_\# := X_1 \# X_2$.

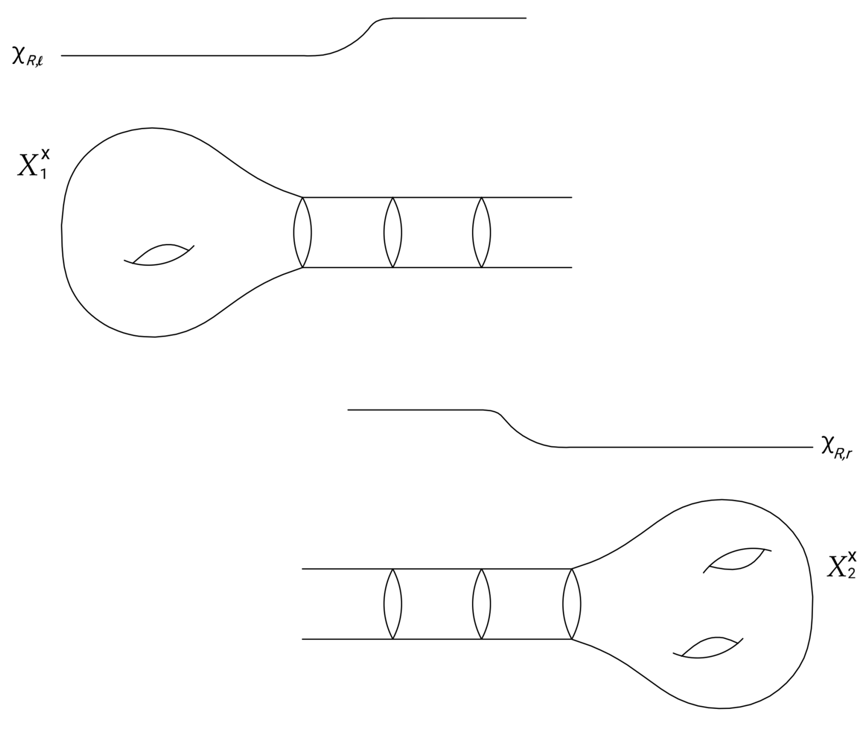

Figure 3.2: Constructing approximate solutions over $X_1^\times$ and $X_2^\times$.

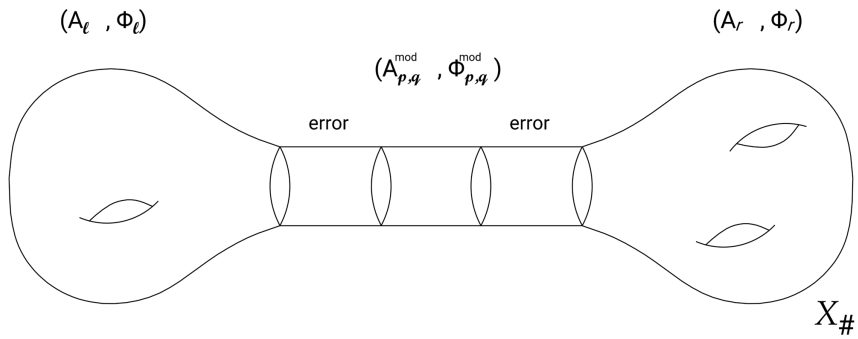

Figure 3.3: $\left(A_R^{app}, \Phi_R^{app}\right)$ over the complex connected sum $X_\#$.

By construction, $(A_R^{app}, \Phi_R^{app})$ is a smooth pair on $X_\#$, complex gauge equivalent to an exact solution of the Hitchin equations by a smooth gauge transformation defined over all of $X_\#$. It satisfies the second equation, while the first equation is satisfied up to an error which we have good control of:

Lemma 3.4.4. *The approximate solution* $(A_R^{app}, \Phi_R^{app})$ *to the parameter* $0 < R < 1$ *satisfies*

$$\left\| *F_{A_R^{app}} + *\left[\Phi_R^{app}, -\tau\left(\Phi_R^{app}\right)\right] \right\|_{C^0(X^\times)} \leq C R^{\delta''}$$

for some constants $\delta'' > 0$ *and* $C = C\left(\delta''\right)$*, which do not depend on* R*.*

Proof. Follows from Lemma 3.3.3; take $C := \max\{C_l, C_r\}$, for C_l, C_r the constants appearing in the bound of the error for the approximate solutions constructed over each of the Riemann surfaces X_1 and X_2. □

3.4.5 The representations ϕ_{irr} and ψ

In this subsection, we see that the Assumption 3.4.2 we made for the model pairs can be achieved by taking particular representations from SL(2,ℝ) into Sp(4,ℝ).

The irreducible representation ϕ_{irr} : SL(2,ℝ) $\hookrightarrow$ Sp(4,ℝ). Let $(A_1^{app}, \Phi_1^{app})$ over X_1 be the approximate SL(2,ℂ)-pair in parameter $R > 0$, as was constructed in §3.3, which agrees with the model pair

$$A_1^{\mathrm{mod}} = 0, \; \Phi_1^{\mathrm{mod}} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} \frac{dz}{z}$$

for $C \in \mathbb{R}$, over an annulus in z-coordinates around a point $p \in D_1$.

The embedding ϕ_{irr} considered in §1.4 extends to give an embedding $\phi_{irr} : \mathrm{SL}(2,\mathbb{C}) \hookrightarrow \mathrm{Sp}(4,\mathbb{C})$. For the Lie algebra of SL(2,ℂ), $\mathfrak{sl}\,(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a,b,c \in \mathbb{C} \right\}$, we may use a Cartan basis for the Lie algebra to determine the infinitesimal deformation, $\phi_{irr*} : \mathfrak{sl}\,(2,\mathbb{C}) \to \mathfrak{sp}\,(4,\mathbb{C})$ with

$$\phi_{irr*}\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \begin{pmatrix} 3a & -\sqrt{3}b & 0 & 0 \\ -\sqrt{3}c & a & 0 & 2b \\ 0 & 0 & -3a & \sqrt{3}c \\ 0 & 2c & \sqrt{3}b & -a \end{pmatrix}$$

We now notice that $\phi_{irr}\,(\mathrm{SO}(2))$ lies in a copy of $\mathrm{U}(2) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$, that is

$$\mathrm{U}(2) \cong \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A^T A + B^T B = I_2,\ A^T B - B^T A = 0 \right\}.$$

In other words, copies of a maximal compact subgroup of SL(2,ℝ) are mapped into copies of a maximal compact subgroup of Sp(4,ℝ). Furthermore, one can check that for $A \in \mathfrak{sl}\,(2,\mathbb{C})$:

$$\|\phi_{irr*}\,(A)\|_{\mathfrak{sp}(4,\mathbb{C})} = 10\|A\|_{\mathfrak{sl}(2,\mathbb{C})}$$

As was described in §3.4.1, ϕ_{irr} can be used to extend SL(2,ℂ)-data to Sp(4,ℂ)-data (A_l, Φ_l), where in this case, it is $A_l = 0$ and

$$\Phi_l = \phi_{irr*}\big|_{\mathfrak{m}^{\mathbb{C}}(\mathrm{SL}(2,\mathbb{C}))}\,(\Phi_1^{app}) = \begin{pmatrix} 3C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & -3C & 0 \\ 0 & 0 & 0 & -C \end{pmatrix} \frac{dz}{z}$$

over the annulus on X_1 in z-coordinates around the point p.

The representation ψ : SL(2,ℝ) × SL(2,ℝ) ↪ Sp(4,ℝ). Let $\left(A_{2,1}^{app}, \Phi_{2,1}^{app}\right), \left(A_{2,2}^{app}, \Phi_{2,2}^{app}\right)$ over X_2 be two approximate SL(2,ℂ)-pairs in parameter $R > 0$, as constructed in §3.3 which agree respectively with the model pairs

$$A_{2,1}^{\mathrm{mod}} = 0,\ \Phi_{2,1}^{\mathrm{mod}} = \begin{pmatrix} -3C & 0 \\ 0 & 3C \end{pmatrix} \frac{dz}{z} \quad \text{and} \quad A_{2,2}^{\mathrm{mod}} = 0,\ \Phi_{2,2}^{\mathrm{mod}} = \begin{pmatrix} -C & 0 \\ 0 & C \end{pmatrix} \frac{dz}{z}$$

for the same real parameter $C \in \mathbb{R}$ considered in defining the pair $(A_1^{app}, \Phi_1^{app})$ over X_1 above, over an annulus in w-coordinates around a point $q \in D_2$.

We extend SL(2,$\mathbb{C}$) × SL(2,$\mathbb{C}$)-data into Sp(4,$\mathbb{C}$) using the homomorphism ψ from §1.4. Take the extension of the embedding ψ into SL(2,$\mathbb{C}$) × SL(2,$\mathbb{C}$), and now the infinitesimal deformation of this homomorphism is given by $\psi_* : \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{sp}(4, \mathbb{C})$ with

$$\psi_* \left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} e & f \\ g & -e \end{pmatrix} \right) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & e & 0 & f \\ c & 0 & -a & 0 \\ 0 & g & 0 & -e \end{pmatrix}$$

We may still check that $\psi\,(\mathrm{SO}(2) \times \mathrm{SO}(2))$ is a copy of U(2). On the other hand, a norm on the space $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ is given by

$$\psi\,(A, B) = \|A\| + \|B\|$$

and again since this is a finite dimensional space, all norms are equivalent to this one. Thus, we compute

$$\|\psi_*\,(A, B)\|_{\mathfrak{sp}(4,\mathbb{C})} = \|(A, B)\|_{\mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})} = \|A\|_{\mathfrak{sl}(2,\mathbb{C})} + \|B\|_{\mathfrak{sl}(2,\mathbb{C})}$$

and so the map ψ_* at the level of Lie algebras is an isometry. Therefore, ψ extends to give an embedding ψ : SO(2,$\mathbb{C}$) × SO(2,$\mathbb{C}$) $\hookrightarrow \mathrm{GL}\,(2, \mathbb{C})$, and so we may use the infinitesimal deformation ψ_* to extend the SL(2,$\mathbb{C}$) × SL(2,$\mathbb{C}$)-data $\left(\left(A_{2,1}^{app}, \Phi_{2,1}^{app}\right), \left(A_{2,2}^{app}, \Phi_{2,2}^{app}\right)\right)$ to an Sp(4,$\mathbb{C}$)-pair (A_r, Φ_r), with $A_r = 0$ and Higgs field Φ_r given by

$$\Phi_r = \psi_* \big|_{\mathfrak{m}^{\mathbb{C}}(\mathrm{SL}(2,\mathbb{R})) \times \mathfrak{m}^{\mathbb{C}}(\mathrm{SL}(2,\mathbb{R}))} \left(\Phi_{2,1}^{app}, \Phi_{2,2}^{app}\right) = \begin{pmatrix} -3C & 0 & 0 & 0 \\ 0 & -C & 0 & 0 \\ 0 & 0 & 3C & 0 \\ 0 & 0 & 0 & C \end{pmatrix} \frac{dz}{z}$$

over the annulus on X_2 in w-coordinates around the point q.

3.5 Perturbing an approximate solution to an exact solution

3.5.1 The contraction mapping argument

A standard method for correcting an approximate solution to an exact solution of gauge-theoretic equations is by using the linearization of a relevant elliptic operator. This set of ideas was first developed by C. Taubes in [43] in the case of instantons over 4-manifolds (see also §7.2 of [16] for a gluing construction of instantons over a connected sum of 4-manifolds). These techniques have been adapted to develop grafting procedures for several other cases of solutions of gauge-theoretic equations; see for instance [24] for a gluing construction for the Nahm pole solutions to the Kapustin-Witten equations over $\mathbb{R}^3 \times (0, +\infty)$. Describing the linearization of a relevant elliptic operator is critical in these techniques. In the Higgs bundle setting, the linearization of the Hitchin operator was described in [29] and furthermore in [42] for solutions to the SL(2,$\mathbb{C}$)-self-duality equations over a noded surface. We are going to use this analytic machinery to correct our approximate solution to an exact solution over the complex connected sum of Riemann surfaces. We begin by summarizing the strategy to be followed; further details can be found in the above mentioned references.

For the complex connected sum $X_\#$ consider the nonlinear G-Hitchin operator at a pair $(A, \Phi) \in \Omega^1\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \oplus \Omega^{1,0}\left(X_\#, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$:

$$\mathcal{H}(A, \Phi) = \left(F(A) - [\Phi, \tau(\Phi)], \bar{\partial}_A \Phi\right)$$

Moreover, consider the orbit map

$$\gamma \mapsto \mathcal{O}_{(A,\Phi)}(\gamma) = g^*(A, \Phi) = \left(g^* A, g^{-1}\Phi g\right)$$

for $g = \exp(\gamma)$ and $\gamma \in \Omega^0(X_\#, E_H(\mathfrak{h}))$, where $H \subset G$ is a maximal compact subgroup.

Therefore, correcting the approximate solution $(A_R^{app}, \Phi_R^{app})$ to an exact solution of the G-Hitchin equations accounts to finding a point γ in the complex gauge orbit of $(A_R^{app}, \Phi_R^{app})$, for which $\mathcal{H}(g^*(A_R^{app}, \Phi_R^{app})) = 0$. However, since we have seen that the second equation is satisfied by the pair $(A_R^{app}, \Phi_R^{app})$ and since the condition $\bar{\partial}_A \Phi = 0$ is preserved under the action of $\mathcal{G}_H$, we actually seek for a solution γ to the following equation

$$\mathcal{F}_R(\gamma) := pr_1 \circ \mathcal{H} \circ \mathcal{O}_{\left(A_R^{app}, \Phi_R^{app}\right)}(\exp(\gamma)) = 0$$

For a Taylor series expansion of this operator

$$\mathcal{F}_R(\gamma) = pr_1\mathcal{H}\left(A_R^{app}, \Phi_R^{app}\right) + L_{\left(A_R^{app}, \Phi_R^{app}\right)}(\gamma) + Q_R(\gamma)$$

where Q_R includes the quadratic and higher order terms in γ, we can then see that $\mathcal{F}_R(\gamma) = 0$ if and only if γ is a fixed point of the map:

$$\begin{aligned} T : H_B^2(X_\#) &\to H_B^2(X_\#) \\ \gamma &\mapsto -G_R\left(\mathcal{H}\left(A_R^{app}, \Phi_R^{app}\right) + Q_R(\gamma)\right) \end{aligned}$$

where we denoted $G_R := L^{-1}_{\left(A_R^{app}, \Phi_R^{app}\right)}$.

The problem then reduces to showing that the mapping T is a contraction of the open ball B_{ρ_R} of radius ρ_R in $H_R^2(X_\#)$, since then from Banach's fixed point theorem there will exist a unique γ such that $T(\gamma) = \gamma$, i.e. such that $\mathcal{F}_R(\gamma) = 0$. In particular, one needs to show that:

1. T is a contraction defined on B_{ρ_R} for some ρ_R, and

2. T maps B_{ρ_R} to B_{ρ_R}

In order to perform the above described contraction mapping argument, we need to show the following:

i The linearized operator at the approximate solution $L_{\left(A_R^{app}, \Phi_R^{app}\right)}$ is invertible.

ii There is an upper bound for the inverse operator $G_R = L^{-1}_{\left(A_R^{app}, \Phi_R^{app}\right)}$ as an operator $L^2(r^{-1}drd\theta) \to L^2(r^{-1}drd\theta)$.

iii There is an upper bound for the inverse operator $G_R = L^{-1}_{\left(A_R^{app}, \Phi_R^{app}\right)}$ also when viewed as an operator $L^2(r^{-1}drd\theta) \to H_B^2(X_\#, r^{-1}drd\theta)$.

iv We can control a Lipschitz constant for Q_R, i.e. there exists a constant $C > 0$ such that

$$\|Q_R(\gamma_1) - Q_R(\gamma_0)\|_{L^2} \leq C\rho\|\gamma_1 - \gamma_0\|_{H_B^2}$$

for all $0 < \rho \leq 1$ and $\gamma_0, \gamma_1 \in B_\rho$, the closed ball of radius ρ around 0 in $H_B^2(X_\#)$.

3.5.2 The Linearization operator $L_{(A,\Phi)}$

We first need to characterize the linearization operator $L_{(A,\Phi)}$ in general, before considering this for the particular approximate pair $\left(A_R^{app}, \Phi_R^{app}\right)$ that we have constructed. The differential of the G-Hitchin operator at a pair $(A,\Phi)\in\Omega^1\left(X_\#,E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right)\oplus\Omega^{1,0}\left(X_\#,E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$ is described by

$$D\mathcal{H}\begin{pmatrix}\dot{A}\\ \dot{\Phi}\end{pmatrix}=\begin{pmatrix}d_A & [\Phi,-\tau(\cdot)]+[\cdot,-\tau(\Phi)]\\ [\cdot,\Phi] & \bar{\partial}_A\end{pmatrix}\begin{pmatrix}\dot{A}\\ \dot{\Phi}\end{pmatrix}$$

Moreover, the differential at $g=Id$ of the orbit map $\mathcal{O}_{(A,\Phi)}$ is

$$\Lambda_{(A,\Phi)}\gamma=\left(\bar{\partial}_A\gamma-\partial_A\gamma^*,[\Phi,\gamma]\right)$$

and so when $\gamma\in\Omega^0\left(X_\#,E_H(\mathfrak{h})\right)$:

$$\Lambda_{(A,\Phi)}\gamma=\left(\bar{\partial}_A\gamma-\partial_A\gamma,[\Phi,\gamma]\right)$$

Therefore,

$$\left(D\mathcal{H}\circ\Lambda_{(A,\Phi)}\right)(\gamma)=\begin{pmatrix}\left(\partial_A\bar{\partial}_A-\bar{\partial}_A\partial_A\right)\gamma+[\Phi,-\tau([\Phi,\gamma])+[[\Phi,\gamma],-\tau(\Phi)]]\\ \left[\bar{\partial}_A\gamma-\partial_A\gamma,\Phi\right]+\bar{\partial}_A[\Phi,\gamma]\end{pmatrix}$$

Now, take

$$\begin{aligned}D\mathcal{F}(\gamma):&=D\left(pr_1\circ\mathcal{H}\circ\mathcal{O}_{(A,\Phi)}\right)(\gamma)=D\mathcal{H}\circ\Lambda_{(A,\Phi)}(\gamma)\\ &=\left(\partial_A\bar{\partial}_A-\bar{\partial}_A\partial_A\right)\gamma+[\Phi,-\tau([\Phi,\gamma])+[[\Phi,\gamma],-\tau(\Phi)]]\end{aligned}$$

and consider the operator $M_\Phi:\Omega^0\left(X_\#,E_H(\mathfrak{h})\right)\to\Omega^0\left(X_\#,E_H(\mathfrak{h})\right)$ defined by

$$M_\Phi\gamma:=-[\Phi,[\tau(\Phi),\gamma]]+[\tau(\Phi),[\Phi,\gamma]]$$

for $\Phi\in\Omega^1\left(X_\#,E_{H^{\mathbb{C}}}\left(\mathfrak{m}^{\mathbb{C}}\right)\right)$. Then from the identities

$$\begin{aligned}2\bar{\partial}_A\partial_A&=F(A)-i*\Delta_A\\ 2\partial_A\bar{\partial}_A&=F(A)+i*\Delta_A\\ [\Phi,\tau([\Phi,\gamma])]&=-[\Phi,[\tau(\Phi),\gamma]]\end{aligned}$$

we may deduce that $(i*\Delta_A+M_\Phi)(\gamma)=D\mathcal{F}(\gamma)$. (For the first two identities see [33], Propositions 1.421, 1.422; the third identity is derived by direct calculations). Now define

$$L_{(A,\Phi)}:=\Delta_A-i*M_\Phi:\Omega^0\left(X_\#,iE_H(\mathfrak{h})\right)\to\Omega^0\left(X_\#,iE_H(\mathfrak{h})\right)$$

The following lemma first observed by C. Simpson in [39] provides that the linearization operator $L_{(A,\Phi)}$ is *nonnegative*. The proof given here is a modification of the proof of the analogous statement for the case of SL(2,$\mathbb{C}$) given in [29].

Lemma 3.5.1. *For* $\gamma \in \Omega^0(X_\#, E_H(\mathfrak{h}))$

$$\langle -i * M_\Phi \gamma, \gamma \rangle = 4 |[\Phi, \gamma]|^2 \geq 0$$

Proof. Fixing a local holomorphic coordinate z, write $\Phi = \varphi dz$ and $\tau(\Phi) = -\varphi^* d\bar{z}$. Then $[\tau(\Phi), [\Phi, \gamma]] = [\varphi^*, [\varphi, \gamma]]\, dz \wedge d\bar{z}$ and $-[\Phi, [\tau(\Phi), \gamma]] = [\varphi, [\varphi^*, \gamma]]\, dz \wedge d\bar{z}$. Altogether, we may write

$$M_\Phi \gamma = ([\varphi^*, [\varphi, \gamma]] + [\varphi, [\varphi^*, \gamma]])\, dz \wedge d\bar{z}$$

The compact real form $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ induces an ad-invariant inner product on $\mathfrak{g}^{\mathbb{C}}$, thus we get $\langle [\varphi^*, [\varphi, \gamma]], \gamma \rangle = |[\varphi, \gamma]|^2$ as well as $\langle [\varphi, [\varphi^*, \gamma]], \gamma \rangle = |[\varphi^*, \gamma]|^2 = |[\varphi, \gamma]|^2$. Finally, since $2i * 1 = -dz \wedge d\bar{z}$, we get $\langle M_\Phi \gamma, i * \gamma \rangle = |[\varphi, \gamma]|^2 |dz \wedge d\bar{z}|^2 = 4|[\varphi, \gamma]|^2$. □

The following corollary is now immediate:

Corollary 3.5.2. *If* $\gamma \in \Omega^0(X_\#, E_H(\mathfrak{h}))$*, then*

$$\left\langle L_{(A,\Phi)} \gamma, \gamma \right\rangle_{L^2} = \|d_A \gamma\|^2_{L^2} + 4 \|[\Phi, \gamma]\|^2_{L^2} \geq 0$$

In particular, $L_{(A,\Phi)}\gamma = 0$ *if and only if* $d_A \gamma = [\Phi, \gamma] = 0$.

3.6 Cylindrical Dirac-type operators and the Cappell-Lee-Miller gluing theorem

A very useful method when dealing with surgery problems in gauge theory over manifolds with very long necks involves the study of the space of eigenfunctions corresponding to small eigenvalues (low eigensolutions) of a self-adjoint Dirac type operator on such a manifold (see [13], [34], [49]). For our purposes we will make use of the Cappell-Lee-Miller gluing theorem from [13] and its generalization to small perturbations of constant coefficient operators due to L. Nicolaescu in [34]. In the latter article, a family of manifolds M_T for $T_0 \leq T \leq \infty$ is considered, each containing a long cylindrical neck of length $\sim T = |\log R|$, obtained by gluing of two disjoint manifolds $M_T^\pm$ along the boundaries of a pair of cylindrical ends. A self-adjoint first-order Dirac-type operator $\mathfrak{D}_T$ is then considered on a hermitian vector bundle over each manifold M_T.

The Cappell-Lee-Miller gluing theorem asserts that under suitable assumptions, the operator $\mathfrak{D}_T$ admits two types of eigenvalues, namely those of order of decay $O\left(T^{-1}\right)$ (large eigenvalues) and those of order of decay $o\left(T^{-1}\right)$ (small eigenvalues). For $T \to \infty$, the subspace of L^2 spanned by the eigenvectors to small eigenvalues is "parameterized" by the kernel of the limiting operator $\mathfrak{D}_\infty$. This way, the Dirac operator $\mathfrak{D}_T$ has no small eigenvalues, if the limiting operator $\mathfrak{D}_\infty$ is invertible.

We may obtain the invertibility of $L_{\left(A_R^{app}, \Phi_R^{app}\right)}$ by showing that an appropriate self-adjoint Dirac-type operator has no small eigenvalues. Note that a punctured neighborhood on a Riemann surface can be also thought of, using a cylindrical coordinate transformation, as a half cylinder attached to the surface, and also an annulus in the real parameter R can be thought of as a finite tube of length $\sim T = |\log R|$. Thus, the gluing of two punctured Riemann surfaces as we described it in §3.4.2 can be thought of as the gluing of two Riemann surfaces with cylindrical ends to get a smooth surface with a finite number of long Euclidean cylinders of length $2\left|\log R\right|$, one for each $p \in \mathfrak{p}$. This is the set-up also considered in [42].

3.6.1 Cylindrical structures over cylindrical manifolds

In this section we include the necessary background for applying the Cappell-Lee-Miller theorem for $\mathbb{Z}_2$-graded Dirac-type operators on cylindrical vector bundles, following largely [42] and [34]; further details can be found in these articles.

Definition 3.6.1. A *cylindrical* $(n+1)$-manifold is an oriented Riemannian $(n+1)$-manifold $\left(\hat{N}, \hat{g}\right)$ with a cylindrical end modeled by $\mathbb{R}_+ \times N$, where (N, g) is an oriented compact Riemannian n-manifold. In other words, the complement of an open precompact subset of $\hat{N}$ is isometric in an orientation preserving fashion to the cylinder $\mathbb{R}_+ \times N$.

Definition 3.6.2. Let $\pi : \mathbb{R}_+ \times N \to N$ denote the canonical projection and τ the outgoing longitudinal coordinate along the neck. A *cylindrical structure* on a vector bundle $\hat{E} \to \hat{N}$ consists of a vector bundle $E \to N$ and a bundle isomorphism

$$\hat{\vartheta} : \hat{E}\left|_{\mathbb{R}_+ \times N} \right. \to \pi^* E$$

We will use the notation $E := \partial_\infty \hat{E}$. A *cylindrical vector bundle* will be a vector bundle together with a cylindrical structure $\left(\hat{\vartheta}, E\right)$. Moreover, the metric $\hat{g}$ is described by $\hat{g} = d\tau^2 \oplus g$ along the cylindrical end.

The cotangent bundle $T^*\hat{N}$ has a natural cylindrical structure such that

$$\partial_\infty T^*\hat{N} \cong \mathbb{R}\left\langle d\tau \right\rangle \oplus T^*N$$

Definition 3.6.3. A section $\hat{u}$ of a cylindrical vector bundle $\left(\hat{E}, \hat{\vartheta}, E\right)$ will be called *cylindrical* if there exists a section u of $\partial_\infty \hat{E}$ such that along the neck

$$\hat{\vartheta}\hat{u} = \pi^* u$$

We shall simply write $\hat{u} = \pi^* u$ and $u := \partial_\infty \hat{u}$.

For any cylindrical vector bundle $\left(\hat{E}, \hat{\vartheta}, E\right)$ there exists a canonical first order partial differential operator ∂_τ acting on sections over the cylindrical end $\hat{E}\left|_{\mathbb{R}_+ \times N}\right.$. It is uniquely determined by the conditions

1. $\partial_\tau \left(\hat{f}\hat{u}\right) = \frac{d\hat{f}}{d\tau}\hat{u} + \hat{f}\partial_\tau \hat{u}$, for every $\hat{f} \in C^\infty \left(\mathbb{R}_+ \times N\right)$ and $\hat{u} \in \hat{E}\left|_{\mathbb{R}_+ \times N}\right.$

2. $\partial_\tau \hat{v} = 0$ for any cylindrical section $\hat{v}$ of $\hat{E}\left|_{\mathbb{R}_+ \times N}\right.$.

Thus, the family of cylindrical vector bundles over a given cylindrical manifold defines a category. The vector bundles we will be considering are of the particular type described below:

Definition 3.6.4. A cylindrical hermitian vector bundle $\left(\hat{E}, \hat{H}\right)$ will be called $\mathbb{Z}_2$-*graded* if

1. The cylindrical vector bundle $\hat{E}$ splits into the orthogonal sum $\hat{E} = \hat{E}^+ \oplus \hat{E}^-$ of cylindrical vector bundles, and

2. The hermitian metric $\hat{H}$ on $\hat{E}$ is along the cylindrical end of the form $\hat{H} = \pi^* H$ for some hermitian metric H on E.

Moreover, $\hat{E}$ carries a Clifford structure and let $G : E^+ \to E^-$ denote the bundle isomorphism given by the Clifford multiplication by $d\tau$.

Definition 3.6.5. A cylindrical partial differential operator $\hat{L} : \hat{E} \to \hat{F}$ between cylindrical bundles is called a *first order partial differential operator* if along the neck $[T, \infty) \times N$ with $T \gg 0$ it can be written as

$$\hat{L} = G\partial_\tau + L$$

where $L : C^\infty(E) \to C^\infty(E)$ is a first order partial differential operator, $E = \hat{E}\left|_N\right.$, $F = \hat{F}\left|_N\right.$ and $G : E \to F$ is a bundle morphism. We also denote $L := \partial_\infty \hat{L}$.

We lastly define the family of differential operators we will be considering:

Definition 3.6.6. Let $\hat{E} \to \hat{N}$ be a $\mathbb{Z}_2$-graded cylindrical hermitian vector bundle. A first order partial differential operator $\mathfrak{D} : C^\infty\left(\hat{E}\right) \to C^\infty\left(\hat{E}\right)$ is called a *$\mathbb{Z}_2$-graded cylindrical Dirac-type operator* if with respect to the $\mathbb{Z}_2$-grading of $\hat{E}$, it takes the form

$$\mathfrak{D} = \begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix}$$

such that along the cylindrical end $\mathcal{D} = G\,(d\tau - D)$ for a self-adjoint Dirac-type operator $D : C^\infty(E^+) \to C^\infty(E^+)$.

Recall that the Dirac-type condition asserts that the square D^2 has the same principal symbols as a Laplacian. D is independent of the longitudinal coordinate τ along the necks.

For our purposes, we will need to use the perturbed operator $\mathfrak{D}+\mathfrak{B} = \begin{pmatrix} 0 & \mathcal{D}+B \\ \mathcal{D}^*+B^* & 0 \end{pmatrix}$, where B is an *exponentially decaying operator* of order 0; that means there exists a pair of constants $C, \lambda > 0$ for which

$$\sup\left\{|B(x)|\,|x \in [\tau, \tau+1] \times N\right\} \leq Ce^{-\lambda|\tau|}$$

for all $\tau \in \mathbb{R}^+$.

3.6.2 The Cappell-Lee-Miller gluing theorem for $\mathbb{Z}_2$-graded cylindrical Dirac-type operators

We now describe the version of the Cappell-Lee-Miller theorem that we are going to use. Let $\left(\hat{N}_i, \hat{g}_i\right)$ for $i = 1, 2$ be two oriented Riemannian manifolds with cylindrical ends, where if τ denotes the outgoing longitudinal coordinate on the cylinder $(0, \infty) \times N_1$, then $-\tau < 0$ denotes the longitudinal coordinate on $(-\infty, 0) \times N_2$. Let also $\hat{E}_i \to \hat{N}_i$ be a pair of $\mathbb{Z}_2$-graded cylindrical hermitian vector bundles over the manifolds $\hat{N}_i$, and let $\mathfrak{D}_i$ be $\mathbb{Z}_2$-graded cylindrical Dirac-type operators for self-adjoint Dirac-type operators $\mathcal{D}_i$, $i = 1, 2$. We further impose the following assumptions:

1. There exists an orientation reversing isometry $\varphi : (N_1, g_1) \to (N_2, g_2)$ between the manifolds, as well as an isometry $\gamma : E_1 \to E_2$ of the hermitian vector bundles covering φ and respecting the gradings.

2. The operators $\mathcal{D}_i$ are of the form $\mathcal{D}_i = G_i\,(\partial_\tau - D_i)$ along the cylindrical ends, and $G_1 + G_2 = L_1 - L_2 = 0$.

We can then use the orientation preserving diffeomorphism φ to obtain for each $T > 0$ the manifold N_T by attaching the region $\hat{N}_1 \backslash (T+1, \infty) \times N_1$ to the region $\hat{N}_2 \backslash (-\infty, -T-1) \times N_2$ using the orientation preserving identification

$$\begin{aligned} [T+1, T+2] \times N_1 &\to [-T-2, -T-1] \times N_2 \\ (\tau, x) &\mapsto (\tau - 2T - 3, \varphi(x)) \end{aligned}$$

The $\mathbb{Z}_2$-graded cylindrical hermitian vector bundles $\hat{E}_i$ can be similarly glued together providing a $\mathbb{Z}_2$-graded hermitian vector bundle $E_T = E_T^+ \oplus E_T^-$ over the manifold N_T. Moreover, the cylindrical operators $\mathfrak{D}_i$ combine to give a $\mathbb{Z}_2$-graded Dirac-type operator $\mathfrak{D}_T$ on the bundle E_T. For a pair of perturbed operators, we can also obtain a perturbed Dirac-type operator defined on the bundle E_T; let us still denote this by $\mathfrak{D}_T$ and write such an operator as

$$\mathfrak{D}_T = \begin{pmatrix} 0 & \mathcal{D}_T^* \\ \mathcal{D}_T & 0 \end{pmatrix}$$

Consider also $\mathfrak{D}_{i,\infty} := \mathfrak{D}_i + \mathfrak{B}_i$ for $i = 1, 2$ and write

$$\mathfrak{D}_{i,\infty} = \begin{pmatrix} 0 & \mathcal{D}_{i,\infty}^* \\ \mathcal{D}_{i,\infty} & 0 \end{pmatrix}$$

We are going to need one last piece of notation to introduce:

Definition 3.6.7. Let $\hat{e}$ a cylindrical vector bundle over the cylindrical manifold $\hat{N}$. We define the *extended L^2 space* $L^2_{ext}\left(\hat{N}, \hat{E}\right)$ as the space of all sections $\hat{u}$ of $\hat{E}$, such that there exists an L^2 section u_∞ of E satisfying

$$\hat{u} - \pi^* u_\infty \in L^2(N, E)$$

The section u_∞ is uniquely determined by $\hat{u}$, thus the so-called *asymptotic trace map* is well-defined

$$\begin{aligned} \partial_\infty : L^2_{\text{ext}}\left(\hat{N}, \hat{E}\right) &\to L^2(N, E) \\ \hat{u} &\mapsto u_\infty \end{aligned}$$

The following theorem is the version of the Cappell-Lee-Miller gluing theorem, which we are going to apply. For a proof see [34], §5.B:

Theorem 3.6.8. *Let $\mathfrak{D}_{i,\infty}$ be a pair of $\mathbb{Z}_2$-graded Dirac-type operators on the cylindrical vector bundles $\hat{E}_i \to \hat{N}_i$ for $i = 1, 2$ as was defined above. Suppose that the kernel $K_i^+ \subseteq$*

$L^2_{ext}\left(\hat{N}_i, \hat{E}_i\right)$ *of the operator* $\mathcal{D}_{i,\infty}$ *is trivial for* $i = 1, 2$. *Then there exist a* $T_0 > 0$ *and a constant* $C > 0$ *such that the operator* $\mathcal{D}_T^*\mathcal{D}_T$ *is bijective for all* $T > T_0$ *and admits a bounded inverse* $(\mathcal{D}_T^*\mathcal{D}_T)^{-1} : L^2\left(N_T, E_T^+\right) \to L^2\left(N_T, E_T^+\right)$ *with*

$$\left\|(\mathcal{D}_T^*\mathcal{D}_T)^{-1}\right\|_{\mathcal{L}(L^2,L^2)} \leq CT^2.$$

3.7 The linearization operator for an approximate solution

3.7.1 The elliptic complex over the complex connected sum

Into our setting, we have already noted that the complex connected sum Riemann surface $X_\#$ can be thought of as a closed surface with a finite number of long Euclidean cylinders of length $2\left|\log R\right|$. The connected sum bundle can be also thought of as a cylindrical vector bundle over $X_\#$. For our approximate solution $(A_R^{app}, \Phi_R^{app})$ constructed over $X_\#$ with $0 < R < 1$ and $T = -\log R$, consider the elliptic complex:

$$\begin{aligned} 0 \to \Omega^0\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) &\xrightarrow{L_{1,T}} \Omega^1\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \oplus \Omega^{1,0}\left(X_\#, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \\ &\xrightarrow{L_{2,T}} \Omega^2\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \oplus \Omega^2\left(X_\#, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \to 0 \end{aligned}$$

where

$$L_{1,T}\gamma = \left(d_{A_R^{app}}\gamma, [\Phi_R^{app}, \gamma]\right)$$

is the linearization of the complex gauge group action and

$$L_{2,T}(\alpha, \varphi) = D\mathcal{H}(\alpha, \varphi) = \begin{pmatrix} d_{A_R^{app}}\alpha + [\Phi_R^{app}, -\tau(\varphi)] + [\varphi, -\tau(\Phi_R^{app})] \\ \bar{\partial}_{A_R^{app}}\varphi + [\varphi, \Phi_R^{app}] \end{pmatrix}$$

is the differential of the Hitchin operator considered in §3.5.2.

Note that in general it does not hold that $L_{2,T}L_{1,T} = \left[F_{A_R^{app}}, \gamma\right] + \left[[\Phi_R^{app}, -\tau(\Phi_R^{app})], \gamma\right] = 0$, since $(A_R^{app}, \Phi_R^{app})$ need not be an exact solution. Decomposing $\Omega^*\left(X_\#, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$ into forms of even, respectively odd total degree, we may introduce the $\mathbb{Z}_2$-graded Dirac-type operator

$$\mathfrak{D}_T := \begin{pmatrix} 0 & L_{1,T}^* + L_{2,T} \\ L_{1,T} + L_{2,T}^* & 0 \end{pmatrix}$$

on the closed surface $X_\#$.

As $R \searrow 0$, the curve $X_\#$ degenerates to a noded surface $X_\#^\times$ (equivalently the cylindrical neck of $X_\#$ extends infinitely). For the cut-off functions χ_R that we considered in obtaining the approximate pair $(A_R^{app}, \Phi_R^{app})$, their support will tend to be empty as $R \searrow 0$, i.e. the "error regions" disappear along with the neck N, thus $(A_R^{app}, \Phi_R^{app}) \to (A_0, \Phi_0)$ uniformly on compact subsets with

$$(A_0^{app}, \Phi_0^{app}) = \begin{cases} (A_l, \Phi_l), & X_l \backslash N \\ (A_r, \Phi_r), & X_r \backslash N \end{cases}$$

an exact solution with the holonomy of the associated flat connection in G.

For $T = \infty$ the elliptic complex for the exact solution $(A_0^{app}, \Phi_0^{app})$ gives rise to the Dirac-type operator

$$\mathfrak{D}_\infty = \begin{pmatrix} 0 & L_1^* + L_2 \\ L_1 + L_2^* & 0 \end{pmatrix}$$

We now describe the map $L_1 + L_2^*$ more closely. Using the Hodge $*$-operator we can identify

$$\Omega^2\left(X_\#^\times, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \cong \Omega^0\left(X_\#^\times, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \text{ and } \Omega^2\left(X_\#^\times, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \cong \Omega^0\left(X_\#^\times, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$$

as well as $\Omega^1\left(X_\#^\times, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \cong \Omega^{0,1}\left(X_\#, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$ via the projection $A \mapsto \pi^{0,1}A$. We further identify

$$(\gamma_1, \gamma_2) \in \Omega^0\left(X_\#^\times, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right) \oplus \Omega^0\left(X_\#^\times, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right)$$

with $\psi_1 = \gamma_1 + i\gamma_2 \in \Omega^0\left(X_\#^\times, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right)$. The operator $L_1 + L_2^*$ can be now expressed as the map

$$\begin{aligned} L_1 + L_2^* : \Omega^0\left(X_\#^\times, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \oplus \Omega^0\left(X_\#^\times, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right) &\to \Omega^{0,1}\left(X_\#^\times, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \oplus \Omega^{1,0}\left(X_\#^\times, E_H\left(\mathfrak{g}^{\mathbb{C}}\right)\right) \\ (\psi_1, \psi_2) &\mapsto \begin{pmatrix} \bar{\partial}_{A_0^{app}}\psi_1 + [\psi_2, -\tau\left(\Phi_0^{app}\right)] \\ \partial_{A_0^{app}\psi_2} + [\psi_1, \Phi_0^{app}] \end{pmatrix} \end{aligned}$$

3.7.2 $\mathfrak{D}_\infty$ is an exponentially small perturbation of a cylindrical operator

Consider the operator $\hat{\mathfrak{D}}_\infty := \begin{pmatrix} 0 & \hat{L}_1^* + \hat{L}_2 \\ \hat{L}_1 + \hat{L}_2^* & 0 \end{pmatrix}$ arising similarly from the elliptic complex for some model solution $\left(A^{\text{ mod}}, \Phi^{\text{ mod}}\right)$ replacing $(A_0^{app}, \Phi_0^{app})$, and for which

$$\left(A^{\text{ mod}}, \Phi^{\text{ mod}}\right) = \left(0, \varphi\frac{dz}{z}\right)$$

along each cylindrical neck. The operator $\hat{\mathfrak{D}}_\infty$ is in fact cylindrical. Indeed, introducing the complex coordinate $\zeta = \tau + i\theta$, we have the identities $d\tau = -\frac{dr}{r}$, $d\theta = -d\theta$, $\frac{dz}{z} = -d\zeta$, and

$\frac{d\bar{z}}{\bar{z}} = -d\bar{\zeta}$. Hence the operator $\hat{L}_1 + \hat{L}_2^*$ (as well as the operator $\hat{L}_1^* + \hat{L}_2$ similarly) can be written as a cylindrical differential operator $\hat{L}_1 + \hat{L}_2^* : \frac{\sqrt{2}}{2} G \left(\partial_\tau - D\right)$ with

$$\left(\psi_1, \psi_2\right) \mapsto \frac{1}{2}\begin{pmatrix} \partial_\tau \psi_1 d\bar{\zeta} \\ \partial_\tau \psi_2 d\zeta \end{pmatrix} - \begin{pmatrix} \left(\frac{i}{2}\partial_\theta \psi_1 + \left[\psi_2, \tau\left(\varphi\right)\right]\right) d\bar{\zeta} \\ \left(-\frac{i}{2}\partial_\theta \psi_2 - \left[\psi_2, \varphi\right]\right) d\zeta \end{pmatrix}$$

where

$$D\left(\psi_1, \psi_2\right) := 2\begin{pmatrix} \frac{i}{2}\partial_\theta \psi_1 + \left[\psi_2, \tau\left(\varphi\right)\right] \\ -\frac{i}{2}\partial_\theta \psi_2 - \left[\psi_2, \varphi\right] \end{pmatrix}$$

and $G = \left(\psi_1, \psi_2\right) = \frac{\sqrt{2}}{2}\left(\psi_1 d\bar{\zeta}, \psi_2 d\zeta\right)$ denotes *Clifford multiplication* by $d\tau$.

The following proposition asserts that the operator $\mathfrak{D}_\infty$ is an exponentially small perturbation of $\hat{\mathfrak{D}}_\infty$:

Proposition 3.7.1. *The operator $L_1 + L_2^*$ can be written as $L_1 + L_2^* = \hat{L}_1 + \hat{L}_2^* + B$, where B is an exponentially decaying operator of order 0, in the sense made precise in §3.6.1.*

Proof. By construction of the approximate solution, Lemma 3.3.1 provides that we can express

$$\left(A_R^{app}, \Phi_R^{app}\right) = \left(A^{\text{ mod}}, \Phi^{\text{ mod}}\right) + \left(0, \varphi_1 \frac{dz}{z}\right)$$

for $\varphi_1 \in C_\delta^0$ for some $\delta > 0$. Therefore, for the operator

$$B\left(\psi_1, \psi_2\right) = \begin{pmatrix} -\left[\psi_2, \tau\left(\varphi_1\right)\right] d\bar{\zeta} \\ \left[\psi_2, \varphi_1\right] d\zeta \end{pmatrix}$$

it holds precisely that $\sup\left\{|B|\right\} \leq Ce^{-\lambda|t|}$, for every $t \in \mathbb{R}^+$. $\square$

3.7.3 The space $\ker\left(L_1 + L_2^*\right) \cap L^2_{\text{ext}}\left(X_\#^\times\right)$ is trivial

We now restrict to the case $G = \mathrm{Sp}(4,\mathbb{R})$ in order to study the space $\ker\left(L_1 + L_2^*\right) \cap L^2_{\text{ext}}\left(X_\#^\times\right)$ for the operator $\mathfrak{D}_\infty$ more closely. We are also taking here into consideration the particular model Higgs field we picked for the $G = \mathrm{Sp}(4,\mathbb{R})$-Hitchin equations coming from the embeddings ϕ_{irr} and ψ from §3.4.5. In other words, we fix

$$\varphi \equiv \varphi^{\text{ mod}} = \begin{pmatrix} 3C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & -3C & 0 \\ 0 & 0 & 0 & -C \end{pmatrix}$$

Moreover, the compact real form on φ in this case is $\tau(\varphi) = -\varphi^*$. We have the following:

Proposition 3.7.2. *Let $(\psi_1, \psi_2) \in \ker(L_1 + L_2^*) \cap L^2_{ext}\left(X^{\times}_{\#}\right)$. Then its asymptotic trace is described by*

$$\partial_\infty(\psi_1, \psi_2) = \left(\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & -d_1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & -d_2 \end{pmatrix}\right)$$

for constants $a_i, d_i \in \mathbb{C}$, for $i = 1, 2$.

Proof. By [34], p. 169, the space of asymptotic traces of $\ker(L_1 + L_2^*)$ is a subspace of $\ker D$ with D as defined in §3.7.2. We will check that the elements of the latter have the asserted form. Consider the Fourier decomposition $(\psi_1, \psi_2) = \left(\sum_{j\in\mathbb{Z}} \psi_{1,j} e^{ij\vartheta}, \sum_{j\in\mathbb{Z}} \psi_{2,j} e^{ij\vartheta}\right)$ where

$$\psi_{i,j} \in \mathfrak{sp}(4, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \middle| A, B, C \in M_{2\times 2}(\mathbb{C});\; B^T = B, C^T = C \right\},$$

Then the equation $D(\psi_1, \psi_2) = 0$ is equivalent to the system of linear equations

$$\begin{pmatrix} -\frac{j}{2}\psi_{1,j} - [\varphi^*, \psi_{2,j}] \\ \frac{j}{2}\psi_{2,j} - [\varphi, \psi_{1,j}] \end{pmatrix} = 0 \tag{3.2}$$

for $j \in \mathbb{Z}$. Since the Higgs field φ is diagonal, the operator D acts invariantly on diagonal, respectively off-diagonal endomorphisms. It therefore suffices to consider these two cases separately.

Case 1. Let $(\psi_{1,j}, \psi_{2,j}) = \left(\begin{pmatrix} a_{1,j} & 0 & 0 & 0 \\ 0 & d_{1,j} & 0 & 0 \\ 0 & 0 & -a_{1,j} & 0 \\ 0 & 0 & 0 & -d_{1,j} \end{pmatrix}, \begin{pmatrix} a_{2,j} & 0 & 0 & 0 \\ 0 & d_{2,j} & 0 & 0 \\ 0 & 0 & -a_{2,j} & 0 \\ 0 & 0 & 0 & -d_{2,j} \end{pmatrix}\right)$, with $a_{i,j}, d_{i,j} \in \mathbb{C}$ for $i = 1, 2$. Then Equation (3.2) is equivalent to the pair of equations

$$\frac{j}{2}\begin{pmatrix} a_{1,j} & 0 & 0 & 0 \\ 0 & d_{1,j} & 0 & 0 \\ 0 & 0 & -a_{1,j} & 0 \\ 0 & 0 & 0 & -d_{1,j} \end{pmatrix} = \mathbb{O}, \text{ for } i = 1, 2$$

thus the system has a non-trivial solution if and only if $j = 0$. In other words, $\psi_1 = \psi_{1,0}$ and $\psi_2 = \psi_{2,0}$ are of the asserted form.

Case 2. Let now $(\psi_{1,j}, \psi_{2,j}) = \left(\begin{pmatrix} 0 & b_{1,j} & e_{1,j} & f_{1,j} \\ c_{1,j} & 0 & f_{1,j} & g_{1,j} \\ k_{1,j} & l_{1,j} & 0 & -c_{1,j} \\ l_{1,j} & m_{1,j} & -b_{1,j} & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_{2,j} & e_{2,j} & f_{2,j} \\ c_{2,j} & 0 & f_{2,j} & g_{2,j} \\ k_{2,j} & l_{2,j} & 0 & -c_{2,j} \\ l_{2,j} & m_{2,j} & -b_{2,j} & 0 \end{pmatrix} \right)$ with all entries in $\mathbb{C}$. Then Equation (3.2) reads as the pair of equations

$$\frac{j}{2} \begin{pmatrix} 0 & b_{1,j} & e_{1,j} & f_{1,j} \\ c_{1,j} & 0 & f_{1,j} & g_{1,j} \\ k_{1,j} & l_{1,j} & 0 & -c_{1,j} \\ l_{1,j} & m_{1,j} & -b_{1,j} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2b_{2,j}\bar{C} & -6e_{2,j}\bar{C} & -4f_{2,j}\bar{C} \\ 2c_{2,j}\bar{C} & 0 & -4f_{2,j}\bar{C} & -2g_{2,j}\bar{C} \\ 6k_{2,j}\bar{C} & 4l_{2,j}\bar{C} & 0 & -2c_{2,j}\bar{C} \\ 4l_{2,j}\bar{C} & 2m_{2,j}\bar{C} & 2b_{2,j}\bar{C} & 0 \end{pmatrix}$$

and

$$-\frac{j}{2} \begin{pmatrix} 0 & b_{2,j} & e_{2,j} & f_{2,j} \\ c_{2,j} & 0 & f_{2,j} & g_{2,j} \\ k_{2,j} & l_{2,j} & 0 & -c_{2,j} \\ l_{2,j} & m_{2,j} & -b_{2,j} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2b_{1,j}C & -6e_{1,j}C & -4f_{1,j}C \\ 2c_{1,j}C & 0 & -4f_{1,j}C & -2g_{1,j}C \\ 6k_{1,j}C & 4l_{1,j}C & 0 & -2c_{1,j}C \\ 4l_{1,j}C & 2m_{1,j}C & 2b_{1,j}C & 0 \end{pmatrix}$$

This pair of equations is then equivalent to the equation

$$\begin{pmatrix} \frac{j}{2} & 2\bar{C} \\ -2C & \frac{j}{2} \end{pmatrix} \begin{pmatrix} b_{1,j} \\ b_{2,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.3}$$

and seven more similar equations for the $c_{i,j}, e_{i,j}, f_{i,j}, g_{i,j}, k_{i,j}, l_{i,j}, m_{i,j}$, $i = 1, 2$. Since $C \neq 0$, we have that the determinant of the 2×2 matrix in Equation (3.3) is $\left(\frac{j}{2}\right)^2 + 4C\bar{C} > 0$, and so this system has no non-trivial solution for $(b_{1,j}, b_{2,j})$; the same is true for the rest seven equations. Therefore, there are no non-trivial off-diagonal elements in $\ker D$ and so the only non-trivial elements are of the asserted form in the proposition. ☐

Lemma 3.7.3. *Suppose* $(\psi_1, \psi_2) \in \ker(L_1 + L_2^*) \cap L^2_{ext}\left(X^\times_\#\right)$*. Then*

$$d_{A_0^{app}}\psi_i = [\psi_i, \Phi_0^{app}] = \left[\psi_i, (\Phi_0^{app})^*\right] = 0$$

for $i = 1, 2$.

Proof. The proof is similar to the one for the case when $G = \mathrm{SL}(2,\mathbb{C})$. We adapt these steps here to the case $G = \mathrm{Sp}(4,\mathbb{R})$ for the reader's convenience. For a more detailed description, see [42], Lemma 3.11, Step 1.

By definition of the operator $(L_1 + L_2^*)$, an element (ψ_1, ψ_2) lies in the kernel of this operator if and only if it is a solution to the system

$$\begin{cases} 0 = \bar{\partial}_{A_0^{app}}\psi_1 + \left[\psi_2, (\Phi_0^{app})^*\right] \\ 0 = \partial_{A_0^{app}}\psi_2 + \left[\psi_1, \Phi_0^{app}\right] \end{cases} \tag{3.4}$$

Differentiate the first equation and use that $\partial_{A_0^{app}}(\Phi_0^{app})^* = 0$ to imply that

$$\begin{aligned} 0 &= \partial_{A_0^{app}}\bar{\partial}_{A_0^{app}}\psi_1 - \left[\partial_{A_0^{app}}\psi_2, (\Phi_0^{app})^*\right] \\ &= \partial_{A_0^{app}}\bar{\partial}_{A_0^{app}}\psi_1 + \left[\left[\psi_1, \Phi_0^{app}\right], (\Phi_0^{app})^*\right] \end{aligned}$$

From this it follows that

$$\begin{aligned} \partial\left\langle\bar{\partial}_{A_0^{app}}\psi_1, \psi_1\right\rangle &= \left\langle\partial_{A_0^{app}}\bar{\partial}_{A_0^{app}}\psi_1, \psi_1\right\rangle - \left\langle\bar{\partial}_{A_0^{app}}\psi_1, \bar{\partial}_{A_0^{app}}\psi_1\right\rangle \\ &= -\left|\left[\psi_1, \Phi_0^{app}\right]\right|^2 - \left|\bar{\partial}_{A_0^{app}}\psi_1\right|^2 \end{aligned}$$

and similarly

$$\bar{\partial}\left\langle\partial_{A_0^{app}}\psi_1, \psi_1\right\rangle = -\left|\left[\psi_1, (\Phi_0^{app})^*\right]\right|^2 - \left|\partial_{A_0^{app}}\psi_1\right|^2$$

Now let $X_S := X_\#^\times \setminus \bigcup_{p\in\mathfrak{p}} \mathrm{C}_p(S)$, where for $S > 0$ we denote by $\mathrm{C}_p(S)$ the subcylinders of points $(\tau, \vartheta) \in \mathrm{C}_p(0)$ with $\tau \geq S$. From Stokes' theorem it follows that

$$\int_{X_S} \partial\left\langle\bar{\partial}_{A_0^{app}}\psi_1, \psi_1\right\rangle + \bar{\partial}\left\langle\partial_{A_0^{app}}\psi_1, \psi_1\right\rangle = \int_{\partial X_S} \left\langle d_{A_0^{app}}\psi_1, \psi_1\right\rangle$$

Letting $S \to \infty$, $\psi_1|_{\tau=S}$ L^2-converges to its asymptotic trace $\partial_\infty\psi_1 \in \Omega^0(S^1, \mathfrak{sp}(4, \mathbb{C}))$, which by the previous lemma is of the form

$$\psi_1(\infty) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & -d_1 \end{pmatrix}$$

for $a_1, d_1 \in \mathbb{C}$. Therefore, $d_{A_0^{app}}(\partial_\infty\psi_1(\infty)) = 0$ and so

$$\int_{X_\#^\times} \partial\left\langle\bar{\partial}_{A_0^{app}}\psi_1, \psi_1\right\rangle + \bar{\partial}\left\langle\partial_{A_0^{app}}\psi_1, \psi_1\right\rangle = \lim_{S\to\infty} \int_{\partial X_S} \left\langle d_{A_0^{app}}\psi_1, \psi_1\right\rangle = 0$$

This implies that $\bar{\partial}_{A_0^{app}}\psi_1 = \partial_{A_0^{app}}\psi_1 = \left[\psi_1, \Phi_0^{app}\right] = \left[\psi_1, (\Phi_0^{app})^*\right] = 0$.

We may as well derive that $\bar{\partial}_{A_0^{app}}\psi_2 = \partial_{A_0^{app}}\psi_2 = [\psi_2, \Phi_0^{app}] = \left[\psi_2, (\Phi_0^{app})^*\right] = 0$ by taking the hermitian adjoint of Equation (3.4) and repeating the same arguments for the solution $(A_0^{app}, -\Phi_0^{app})$. $\square$

Proposition 3.7.4. *The operator $L_1 + L_2^*$ considered as a densely defined operator on $L^2_{ext}\left(X_\#^\times\right)$ has trivial kernel.*

Proof. Let $(\psi_1, \psi_2) \in \ker\left(L_1 + L_2^*\right) \cap L^2_{\mathrm{ext}}\left(X_\#^\times\right)$. From Lemma 3.7.3 we have:

$$d_{A_0^{app}}\psi_i = [\psi_i, \Phi_0^{app}] = \left[\psi_i, (\Phi_0^{app})^*\right] = 0$$

for $i = 1, 2$. We show that $\psi_1 = 0$ by showing that $\gamma := \psi_1 + \psi_1^* \in \Omega^0\left(X_\#^\times, \mathfrak{u}(2)\right)$ and $\delta := i\left(\psi_1 - \psi_1^*\right) \in \Omega^0\left(X_\#^\times, \mathfrak{u}(2)\right)$ both vanish. Choosing a holomorphic coordinate z centered at the node of $X_\#^\times$, the Higgs field Φ_0^{app} in our exact solution is written

$$\Phi_0^{app} = \varphi \frac{dz}{z}$$

with $\varphi \in \mathfrak{m}^{\mathbb{C}}\left(\mathrm{Sp}(4,\mathbb{R})\right) = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \middle| A, B \in \mathcal{M}_2(\mathbb{C}) \text{ with } A^T = A,\ B^T = B \right\}$. We get that $d|\gamma|^2 = 2\left\langle d_{A_0^{app}}\gamma, \gamma\right\rangle = 0$, i.e. $|\gamma|$ is constant on $X_\#^\times$, as well as that $\gamma(x) \in \ker M_{\varphi(x)}$ for all $x \in X_\#^\times$, since it is in general $M_\Phi\gamma = [\Phi, \tau([\Phi, \gamma])] + [\tau(\Phi), [\Phi, \gamma]]$.

Now, this $\gamma(x) \in \mathfrak{u}(2)$ is hermitian. It has orthogonal eigenvectors for distinct eigenvalues, but even if there are degenerate eigenvalues, it is still possible to find an orthonormal basis of $\mathbb{C}^4$ consisting of four eigenvectors of $\gamma(x)$, thus $\mathbb{C}^4 = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_4}$, where λ_i the eigenvalues of $\gamma(x)$. Assuming that $\gamma(x)$ is non-zero, since $[\varphi(x), \gamma(x)] = 0$ it follows that $\varphi(x)$ preserves the eigenspaces of $\gamma(x)$ for all $x \in X_\#^\times$ and so $\langle \varphi(x)v, \varphi(x)w \rangle = \langle v, w \rangle$ for $v, w \in \mathbb{C}^4$. In other words, $\varphi(x)$ ought to be an isometry with respect to the usual norm in $\mathbb{C}^4$. Equivalently, $\varphi(x)$ is unitary for all $x \in X_\#^\times$. However, for a zero x_0 of $\det\Phi = \det\tilde{\varphi}(x_0)\frac{dz^2}{z^2}$ chosen on the left hand side surface X_l of $X_\#^\times$ we see that

$$\varphi(x_0) = \phi_{irr*}\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ -\sqrt{3}z & 0 & 0 & 2 \\ 0 & 0 & 0 & \sqrt{3}z \\ 0 & 2z & \sqrt{3} & 0 \end{pmatrix}$$

which is not unitary. Therefore, $\gamma = 0$.
That δ vanishes, as well as $\psi_2 = 0$, is proven similarly. $\square$

3.7.4 Upper bound for $L_{(A_R^{app},\Phi_R^{app})}$ in $H^2\left(X_\#^\times\right)$

Define the operator

$$\mathcal{D}_T := L_{1,T} + L_{2,T}^*$$

The following proposition is an immediate consequence of the Cappell-Lee-Miller theorem (Theorem 3.6.8) for this operator $\mathcal{D}_T$ using the fact that the kernel of the limiting operator $L_1 + L_2^*$ is trivial on $L^2_{\text{ext}}\left(X_\#^\times\right)$, as was shown in §3.7.3.

Proposition 3.7.5. *There exist constants $T_0 > 0$ and $C > 0$ such that the operator $\mathcal{D}_T^*\mathcal{D}_T$ is bijective for all $T > T_0$ and its inverse $\left(\mathcal{D}_T^*\mathcal{D}_T\right)^{-1} : L^2\left(X_\#\right) \to L^2\left(X_\#\right)$ satisfies*

$$\left\|\left(\mathcal{D}_T^*\mathcal{D}_T\right)^{-1}\right\|_{\mathcal{L}(L^2,L^2)} \leq CT^2.$$

We are finally in position to imply the existence of the inverse operator $G_R = L^{-1}_{\left(A_R^{app},\Phi_R^{app}\right)}$: $L^2\left(X_\#\right) \to L^2\left(X_\#\right)$ and provide an upper bound for its norm, by adapting the analogous proof from [42] into our case. We first need the following:

Corollary 3.7.6. *There exist constants $T_0 > 0$ and $C > 0$ such that for all $T > T_0$ and $\gamma \in \Omega^0\left(X_\#, E_H\left(\mathfrak{h}\right)\right)$ it holds that*

$$\left\|L_{1,T}^*L_{1,T}\gamma\right\|_{L^2\left(X_\#\right)} \geq CT^{-2}\|\gamma\|_{L^2\left(X_\#\right)}$$

Proof. The previous proposition provides the existence of constants $T_0 > 0$ and $C > 0$ such that for all $T > T_0$ and $\gamma \in \Omega^0\left(X_\#, E_H\left(\mathfrak{h}\right)\right)$:

$$\left\|\left(\mathcal{D}_T^*\mathcal{D}_T\right)^{-1}\gamma\right\|_{L^2\left(X_\#\right)} \leq CT^2\|\gamma\|_{L^2\left(X_\#\right)}$$

and thus

$$\left\|\mathcal{D}_T^*\mathcal{D}_T\gamma\right\|_{L^2\left(X_\#\right)} \geq CT^{-2}\|\gamma\|_{L^2\left(X_\#\right)}$$

According to the definition of $\mathcal{D}_T$ we have

$$\begin{aligned}\mathcal{D}_T^*\mathcal{D}_T &= \left(L_{1,T} + L_{2,T}^*\right)^*\left(L_{1,T} + L_{2,T}^*\right)\\ &= L_{1,T}^*L_{1,T} + L_{2,T}L_{1,T} + L_{1,T}^*L_{2,T}^* + L_{2,T}L_{2,T}^*\end{aligned}$$

as well as $L_{2,T}L_{1,T}\gamma = \left[F_{A_R^{app}}, \gamma\right] + \left[\left[\Phi_R^{app}, -\tau\left(\Phi_R^{app}\right)\right], \gamma\right]$, for sections $\gamma \in \Omega^0\left(X_\#, E_H\left(\mathfrak{h}\right)\right)$.

For parameter $T = -\log R$, Lemma 3.4.4 provides the estimate

$$\begin{aligned} \|L_{2,T}L_{1,T}\gamma\|_{L^2(X_{\#})} &\leq C_1 R^{\delta''}\|\gamma\|_{L^2(X_{\#})} \\ &= C_1 e^{-\delta'' T}\|\gamma\|_{L^2(X_{\#})} \end{aligned}$$

for T-independent constants $C_1, \delta'' > 0$.
Remember that the operator $\mathcal{D}_T^*\mathcal{D}_T$ acts on forms of even total degree. Now, decomposing forms of even total degree into forms of degree zero and degree two, for a 0-form γ we may write $\gamma = \gamma + 0$ and thus is

$$L_{1,T}^* L_{1,T}\gamma = \mathcal{D}_T^*\mathcal{D}_T\gamma - L_{2,T}L_{1,T}\gamma$$

The triangle inequality now provides that

$$\begin{aligned} \left\|L_{1,T}^* L_{1,T}\gamma\right\|_{L^2(X_{\#})} &\geq \|\mathcal{D}_T^*\mathcal{D}_T\gamma\|_{L^2(X_{\#})} - \|L_{2,T}L_{1,T}\gamma\|_{L^2(X_{\#})} \\ &\geq CT^{-2}\|\gamma\|_{L^2(X_{\#})} - C_1 e^{-\delta'' T}\|\gamma\|_{L^2(X_{\#})}, \end{aligned}$$

which in turn for sufficiently large T implies the desired inequality. □

Proposition 3.7.7. *There exist constants $R_0 > 0$ and $C > 0$, such that for all sufficiently small $0 < R < R_0$ the operator $L_{\left(A_R^{app}, \Phi_R^{app}\right)}$ is invertible and satisfies the estimate*

$$\|G_R\gamma\|_{L^2(X_{\#})} \leq C|\log R|^2\|\gamma\|_{L^2(X_{\#})}$$

Proof. It suffices to show the statement for the unitarily equivalent operator (which we shall still denote by $L_{\left(A_R^{app}, \Phi_R^{app}\right)}$) acting on the space $\Omega^0\left(X_{\#}, E_H\left(\mathfrak{h}\right)\right)$ defined after conjugation by the map $\gamma \mapsto i\gamma$. From Corollary 3.5.2 it follows for all $\gamma \in \Omega^0\left(X_{\#}, E_H\left(\mathfrak{h}\right)\right)$ that

$$\left\langle \left(L_{\left(A_R^{app}, \Phi_R^{app}\right)} - L_{1,T}^* L_{1,T}\right)\gamma, \gamma \right\rangle = 3\|[\Phi_R^{app}, \gamma]\|^2 \geq 0$$

Consequently, $L_{\left(A_R^{app}, \Phi_R^{app}\right)} - L_{1,T}^* L_{1,T}$ is a nonnegative operator. Furthermore, from the previous Corollary we obtain the estimate:

$$\left\|L_{\left(A_R^{app}, \Phi_R^{app}\right)}\gamma\right\|_{L^2(X_{\#})} \geq \left\|L_{1,T}^* L_{1,T}\gamma\right\|_{L^2(X_{\#})} \geq CT^{-2}\|\gamma\|_{L^2(X_{\#})}$$

Therefore, the operator $L_{\left(A_R^{app}, \Phi_R^{app}\right)}$ is strictly positive, and so invertible, and the norm of its inverse is bounded above by the inverse of the smallest eigenvalue of $L_{\left(A_R^{app}, \Phi_R^{app}\right)}$, thus providing the statement of the proposition. □

This upper bound for the inverse operator G_R is valid also when G_R is viewed as an operator $L^2(X_\#, r^{-1}drd\theta) \to H^2_B(X_\#, r^{-1}drd\theta)$, where $H^2_B(X_\#)$ is the Banach space defined by:

$$H^2_B(X_\#) := \left\{\gamma \in L^2(X_\#) \,\middle|\, \nabla_B\gamma, \nabla^2_B\gamma \in L^2(X_\#)\right\}$$

The proof of this statement readily adapts from the proof of Proposition 3.14 and Corollary 3.15 in [42]; we refer the interested reader to this article for the details.

3.7.5 Lipschitz constants for Q_R

The last step before being able to apply the contraction mapping argument described in §3.5.1 is to control the quadratic and higher order terms Q_R in the Taylor series expansion of $\mathcal{F}_R$.

The orbit map for any Higgs pair (A, Φ) and any $g = \exp(\gamma)$ with $\gamma \in \Omega^0\left(X_\#, E_H\left(\mathfrak{h}^{\mathbb{C}}\right)\right)$ is given by

$$\mathcal{O}_{(A,\Phi)}(\gamma) = g^*(A, \Phi) = \left(A + g^{-1}\left(\bar{\partial}_A g\right) - (\partial_A g)\, g^{-1}, g^{-1}\Phi g\right)$$

thus

$$\begin{aligned}\exp(\gamma)^* A &= A + \left(\bar{\partial}_A - \partial_A\right)\gamma + R_A(\gamma)\\ \exp(-\gamma)\,\Phi\exp(\gamma) &= \Phi + [\Phi, \gamma] + R_\Phi(\gamma)\end{aligned}$$

where these reminder terms are

$$R_A(\gamma) = \exp(-\gamma)\left(\bar{\partial}_A \exp(\gamma)\right) - (\partial_A \exp(\gamma))\exp(-\gamma) - \left(\bar{\partial}_A - \partial_A\right)\gamma$$

$$R_\Phi(\gamma) = \exp(-\gamma)\,\Phi\exp(\gamma) - [\Phi, \gamma] - \Phi$$

The Taylor series expansion of the operator $\mathcal{F}_R$ is then

$$\mathcal{F}_R(\exp(\gamma)) = \mathrm{pr}_1(\mathcal{H}_R(A, \Phi)) + L_R\gamma + Q_R\gamma$$

with

$$\begin{aligned}Q_R(\gamma) := {} & d_A(R_A(\gamma)) + [\Phi^*, R_\Phi(\gamma)] + [\Phi, R_\Phi(\gamma)^*]\\ & + \frac{1}{2}\left[\left(\left(\bar{\partial}_A - \partial_A\right)\gamma + R_A(\gamma)\right), \left(\left(\bar{\partial}_A - \partial_A\right)\gamma + R_A(\gamma)\right)\right]\\ & + \left[([\Phi, \gamma] + R_\Phi(\gamma)), ([\Phi, \gamma] + R_\Phi(\gamma))^*\right]\end{aligned}$$

Lemma 3.7.8. *In the above, let* $(A, \Phi) \equiv (A_R^{app}, \Phi_R^{app})$*. Then there exists a constant* $C > 0$ *such that*

$$\|Q_R(\gamma_1) - Q_R(\gamma_0)\|_{L^2(X_\#)} \leq Cr\|\gamma_1 - \gamma_0\|_{H_B^2(X_\#)}$$

for all $0 < r \leq 1$ *and* $\gamma_0, \gamma_1 \in B_r$*, the closed ball of radius* r *around 0 in* $H_B^2(X_\#)$*.*

Proof. see [42], Lemma 4.1. □

3.8 Gluing theorems

The necessary prerequisites are now in place in order to apply the contraction mapping argument described in §3.5.1 and correct the approximate solution constructed into an exact solution of the Sp(4,ℝ)-Hitchin equations.

Theorem 3.8.1. *There exists a constant* $0 < R_0 < 1$*, and for every* $0 < R < R_0$ *there exist a constant* $\sigma_R > 0$ *and a unique section* $\gamma \in H_B^2(X_\#, \mathfrak{u}(2))$ *satisfying* $\|\gamma\|_{H_B^2(X_\#)} \leq \sigma_R$*, so that for* $g = \exp(\gamma)$*:*

$$(A_\#, \Phi_\#) = g^*(A_R^{app}, \Phi_R^{app})$$

is an exact solution of the Sp(4,ℝ)-Hitchin equations over the closed surface $X_\#$*.*

Proof. We show that for $\sigma_R > 0$ sufficiently small, the operator T from §3.5.1 defined by $T(\gamma) = -G_R(\mathcal{H}((A_R^{app}, \Phi_R^{app})) + Q_R(\gamma))$ is a contraction of B_{σ_R}, the open ball of radius σ_R. From Proposition 3.7.7 and Lemma 3.7.8 we get

$$\begin{aligned}\|T(\gamma_1 - \gamma_0)\|_{H_B^2(X_\#)} &= \|G_R(Q_R(\gamma_1) - Q_R(\gamma_0))\|_{H_B^2(X_\#)} \\ &\leq C(\log R)^2\|Q_R(\gamma_1) - Q_R(\gamma_0)\|_{L^2(X_\#)} \\ &\leq C(\log R)^2\sigma_R\|\gamma_1 - \gamma_0\|_{H_B^2(X_\#)}\end{aligned}$$

Let $\varepsilon > 0$ and set $\sigma_R := C^{-1}|\log R|^{-2-\varepsilon}$. Then for all $0 < R < e^{-1}$ it follows that $C(\log R)^2\sigma_R < 1$ and therefore T is a contraction on the ball of radius σ_R.
Furthermore, since $Q_R(0) = 0$, using again Proposition 3.7.7 and Lemma 3.7.8 we have

$$\begin{aligned}\|T(0)\|_{H_B^2(X_\#)} &= \|G_R(\mathrm{pr}_1(\mathcal{H}_R(A_R^{app}, \Phi_R^{app})))\|_{H_B^2(X_\#)} \\ &\leq C(\log R)^2\|\mathrm{pr}_1(\mathcal{H}_R(A_R^{app}, \Phi_R^{app}))\|_{L^2(X_\#)} \\ &\leq C(\log R)^2 R^{\delta''}\end{aligned}$$

Thus, when R_0 is chosen to be sufficiently small, then $\|T(0)\|_{H^2_B(X_\#)} < \frac{1}{10}\sigma_R$, for all $0 < R < R_0$ and for the above choice of σ_R; thus the ball B_{σ_R} is mapped to itself by T. □

Remark 3.8.2. The analytic arguments developed in the preceding sections provide also that the Main Theorem 1.1 in [42] also holds for solutions to the Sp(4,$\mathbb{R}$)-Hitchin equations. In particular, we have the following:

Corollary 3.8.3. *Let (Σ, J_0) be a Riemann surface with nodes at a finite collection of points $\mathfrak{p} \subset \Sigma$. Let (A_0, Φ_0) be a solution to the Sp(4,$\mathbb{R}$)-Hitchin equations with logarithmic singularities at $\mathfrak{p}$, which is obtained from a solution to the SL(2,$\mathbb{R}$)-Hitchin equations via an embedding ρ : SL(2,$\mathbb{R}$) $\hookrightarrow$ Sp(4,$\mathbb{R}$) that maps a copy of a maximal compact subgroup of SL(2,$\mathbb{R}$) into a maximal compact subgroup of Sp(4,$\mathbb{R}$). Suppose that there is a model solution near those nodes which is of the form described in §3.1. Let (Σ, J_i) be a sequence of smooth Riemann surfaces converging uniformly to (Σ, J_0). Then, for every sufficiently large $i \in \mathbb{N}$, there exists a smooth solution (A_i, Φ_i) on (Σ, J_i), such that $(A_i, \Phi_i) \to (A_0, \Phi_0)$ as $i \to \infty$, uniformly on compact subsets of $\Sigma \backslash \mathfrak{p}$.*

Theorem 3.8.1 now implies that for $\bar{\partial} := A^{0,1}_\#$, the Higgs bundle $\left(E_\# := \left(\mathbb{E}_\#, \bar{\partial}\right), \Phi_\#\right)$ is a polystable Sp(4,$\mathbb{R}$)-Higgs bundle over the complex connected sum $X_\#$. Collecting the steps from all sections in this chapter, we now have our main result:

Theorem 3.8.4. *Let X_1 be a closed Riemann surface of genus g_1 and $D_1 = \{p_1, \ldots, p_s\}$ be a collection of s-many distinct points on X_1. Consider respectively a closed Riemann surface X_2 of genus g_2 and a collection of also s-many distinct points $D_2 = \{q_1, \ldots, q_s\}$ on X_2. Let $(E_1, \Phi_1) \to X_1$ and $(E_2, \Phi_2) \to X_2$ be parabolic stable Sp(4,$\mathbb{R}$)-Higgs bundles with corresponding solutions to the Hitchin equations (A_1, Φ_1) and (A_2, Φ_2). Assume that these solutions agree with model solutions $\left(A^{\text{mod}}_{1,p_i}, \Phi^{\text{mod}}_{1,p_i}\right)$ and $\left(A^{\text{mod}}_{2,q_j}, \Phi^{\text{mod}}_{2,q_j}\right)$ near the points $p_i \in D_1$ and $q_j \in D_2$, and that the model solutions satisfy $\left(A^{\text{mod}}_{1,p_i}, \Phi^{\text{mod}}_{1,p_i}\right) = -\left(A^{\text{mod}}_{2,q_j}, \Phi^{\text{mod}}_{2,q_j}\right)$, for s-many possible pairs of points (p_i, q_j). Then there is a polystable Sp(4,$\mathbb{R}$)-Higgs bundle $(E_\#, \Phi_\#) \to X_\#$, constructed over the complex connected sum of Riemann surfaces $X_\# = X_1 \# X_2$, which agrees with the initial data over $X_\# \backslash X_1$ and $X_\# \backslash X_2$.*

Remark 3.8.5. In §3.4.5 we checked that for the particular parabolic Sp(4,$\mathbb{R}$)-Higgs bundles arising from representations ϕ_{irr} and ψ, the main assumption in the theorem does apply.

Definition 3.8.6. We call an Sp(4,$\mathbb{R}$)-Higgs bundle constructed by the procedure developed in this chapter, a *hybrid Sp(4,$\mathbb{R}$)-Higgs bundle.*

CHAPTER 4

TOPOLOGICAL INVARIANTS

So far we were able to construct a polystable Higgs bundle over a complex connected sum of Riemann surfaces by gluing stable parabolic Higgs bundles over Riemann surfaces with a divisor. We are now dealing with the problem of identifying the connected component of the moduli space a hybrid Higgs bundle lies, given a choice of stable parabolic ingredients to glue. For this, we need to look at how do the Higgs bundle topological invariants behave under the complex connected sum operation. As an application, we see that under the right initial choices for the gluing data, we can find model Higgs bundles in the exceptional components of the maximal Sp(4,$\mathbb{R}$)-Higgs bundle moduli space; these models are described by the hybrid Higgs bundles of Chapter 3. More importantly, this allows for the first time a comparison between the Higgs bundle invariants and the topological invariants for Anosov representations established by O. Guichard and A. Wienhard in [22].

4.1 Degree of a connected sum bundle

Let X_1 and X_2 be closed Riemann surfaces with divisors D_1 and D_2 of s-many distinct points on each, and let V_1, V_2 be two parabolic principal $H^{\mathbb{C}}$-bundles over X_1, X_2 respectively. Assume that the underlying smooth bundles $\mathbb{V}_1, \mathbb{V}_2$ come equipped with adapted hermitian metrics h_1, h_2. In Chapter 3 we described the construction of the smooth hermitian bundle $(\mathbb{V}_1 \# \mathbb{V}_2, h_\#)$ over the complex connected sum $X_\#$ of X_1 and X_2. The hermitian metric $h_\#$ coincides with h_1 and h_2 in a neighborhood of $X_1 \backslash \Omega$ and $X_2 \backslash \Omega$ respectively, where Ω is the neck region in the connected sum construction. Next, we equipped this hermitian bundle with a holomorphic structure obtained through the arguments in §3.5-3.8. We have the following:

Proposition 4.1.1. *Let $X_\# = X_1 \# X_2$ be the complex connected sum of two closed Riemann surfaces X_1 and X_2 with divisors D_1 and D_2 of s-many distinct points on each surface, and let V_1, V_2 be parabolic principal $H^{\mathbb{C}}$-bundles over X_1 and X_2 respectively. For a parabolic subgroup $P \subset H^{\mathbb{C}}$, a holomorphic reduction σ of the structure group of E from $H^{\mathbb{C}}$ to P and* *an antidominant character χ of P, the following identity holds:*

$$\deg\left(V_1\#V_2\right)(\sigma,\chi) = par\deg_{\alpha_1}\left(V_1\right)(\sigma,\chi) + par\deg_{\alpha_2}\left(V_2\right)(\sigma,\chi)$$

Proof. Consider smooth metrics $\hbar_1, \hbar_2$ on the principal $H^{\mathbb{C}}$-bundles V_1, V_2 defined over X_1 and X_2, which coincide with the adapted metrics h_1, h_2 on $X_1\backslash D_1$, $X_2\backslash D_2$ respectively.
For $v > 0$, let $X_{i,v} := \{x \in X_i \,|\, d(x,D) \geq e^{-v}\}$ and $B_{i,v} := X_i\backslash X_{i,v}$, for $i = 1,2$. For a holomorphic reduction σ and an antidominant character χ, the metrics $\hbar_i, h_i$ induce metrics $\hbar_{i,L}, h_{i,L}$ on $\left(V_i\right)_{\sigma,L}$ with curvature $F_{h_i,L}$ and $F_{\hbar_i,L}$ respectively. Similarly, the smooth metric $h_\#$ on $V_1\#V_2$ induces a metric $h_{\#,L}$ on $\left(V_1\#V_2\right)_{\sigma,L}$ with curvature $F_{h_\#,L}$. We now have:

$$\deg\left(V_1\#V_2\right)(\sigma,\chi) = \frac{\sqrt{-1}}{2\pi}\int_{X_\#}\left\langle F_{h_\#,L}, s_\sigma\right\rangle$$
$$= \frac{\sqrt{-1}}{2\pi}\int_{X_{1,v}}\left\langle F_{h_1,L}, s_\sigma\right\rangle + \frac{\sqrt{-1}}{2\pi}\int_{X_{2,v}}\left\langle F_{h_2,L}, s_\sigma\right\rangle + \frac{\sqrt{-1}}{2\pi}\int_{X_\#\backslash(X_{1,v}\cup X_{2,v})}\left\langle F_{h_\#,L}, s_\sigma\right\rangle$$

Now notice:

$$\frac{\sqrt{-1}}{2\pi}\int_{X_{1,v}}\left\langle F_{h_1,L}, s_\sigma\right\rangle = \frac{\sqrt{-1}}{2\pi}\int_{X_1}\left\langle F_{\hbar_1,L}, s_\sigma\right\rangle - \frac{\sqrt{-1}}{2\pi}\int_{B_{1,v}}\left\langle F_{h_1,L}, s_\sigma\right\rangle$$

and

$$\frac{\sqrt{-1}}{2\pi}\int_{X_1}\left\langle F_{\hbar_1,L}, s_\sigma\right\rangle = \deg\left(\mathbb{V}_1\right)(\sigma,\chi);$$

similarly for the integral over $X_{2,v}$. Therefore, for every $v > 0$:

$$\deg\left(V_1\#V_2\right)(\sigma,\chi) = \deg\left(V_1\right)(\sigma,\chi) - \frac{\sqrt{-1}}{2\pi}\int_{B_{1,v}}\left\langle F_{h_1,L}, s_\sigma\right\rangle + \deg\left(V_2\right)(\sigma,\chi)$$
$$- \frac{\sqrt{-1}}{2\pi}\int_{B_{2,v}}\left\langle F_{h_2,L}, s_\sigma\right\rangle + \frac{\sqrt{-1}}{2\pi}\int_{X_\#\backslash(X_{1,v}\cup X_{2,v})}\left\langle F_{h_\#,L}, s_\sigma\right\rangle$$

Passing to the limit as $v \to +\infty$, the last integral vanishes, while each integral over $B_{i,v}$ for $i = 1,2$ converges to the local term measuring the contribution of the parabolic structure in the definition of the parabolic degree (see Lemma 2.10 in [5]). The desired identity now follows. $\square$

Proposition 4.1.1 implies in particular that the complex connected sum of *maximal* parabolic

Sp(4,$\mathbb{R}$)-Higgs bundles is a *maximal* (non-parabolic) Sp(4,$\mathbb{R}$)-Higgs bundle. This is the analogue in the language of Higgs bundles of the additivity property for the Toledo invariant from the point of view of fundamental group representations (Proposition 1.4.5).

4.2 Model Higgs bundles in the exceptional components of $\mathcal{M}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$

4.2.1 Model maximal parabolic $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundles.

Let X be a compact Riemann surface of genus g and let the divisor $D := \{x_1, \dots, x_s\}$ of s-many distinct points on X, assuming that $2g - 2 + 2s > 0$. Fix a square root of the canonical bundle, that is, a line bundle $L \to X$, such that $L^2 = K$ and consider

$$E = (L \otimes \iota)^* \oplus L$$

where $\iota = \mathcal{O}_X(D)$ is the line bundle over the divisor D. Assign a parabolic structure on E given by a trivial flag $E_{x_i} \supset \{0\}$ and weight $\frac{1}{2}$ for every $x_i \in D$. Moreover, for any $q \in H^0(X, K^2 \otimes \iota)$, let

$$\theta(q) = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \in H^0(X, \mathrm{End}(E) \otimes K \otimes \iota)$$

be the parabolic Higgs field on the parabolic bundle E. The authors in [7] show that the pair $(E, \theta(q))$ is a parabolic stable Higgs bundle of parabolic degree zero. Under the non-abelian Hodge correspondence for non-compact curves, there is a tame harmonic metric on the bundle E. Moreover, it is shown in [7] that parabolic Higgs bundles of the type $(E, \theta(q))$ defined above, are in 1-1 correspondence with Fuchsian representations of n-punctured Riemann surfaces. This also implies that the holonomy of the flat connection on X corresponding to $(E, \theta(q))$ is contained (after conjugation) in SL(2,$\mathbb{R}$).

As was done in the non-parabolic case [9], we shall use embeddings of SL(2,$\mathbb{R}$) into Sp(4,$\mathbb{R}$), in order to obtain model parabolic Sp(4,$\mathbb{R}$)-Higgs bundles:

Example 4.2.1. Consider the parabolic Sp(4,$\mathbb{R}$)-Higgs bundle (V_1, β_1, γ_1) which is induced by the embedding through ϕ_{irr} from §1.4 of the model parabolic SL(2,$\mathbb{R}$)-Higgs bundle $(E, \theta(q))$. Under the preceding terminology, the bundle $V_1 \to X_1$ is then described as:

$$V_1 = \left(L^3 \otimes \iota\right) \oplus (L \otimes \iota)^*$$

and it comes equipped with a parabolic structure defined by a trivial flag $(V_1)_{x_i} \supset \{0\}$ and weight $\frac{1}{2}$ for every $x_i \in D$.

Moreover, V_1 can be expressed as $V_1 = N_1 \oplus N_1^* K$. Indeed, for $N_1 = L^3 \otimes \iota$ we see that:

$$N_1^* K = \left(L^3 \otimes \iota\right)^* \otimes K = L^{-3} \otimes \xi \otimes L^2 = (L \otimes \iota)^*$$

It can be checked that this is a parabolic *stable* Sp(4,ℝ)-Higgs bundle. Also notice that

$$\begin{aligned} par \deg V_1 &= par \deg \left(L^3 \otimes \iota\right) + par \deg (L \otimes \iota)^* \\ &= 3g - 3 + s + \frac{s}{2} + 1 - g - s + \frac{s}{2} = 2g - 2 + s. \end{aligned}$$

Therefore, $(V_1, \beta_1, \gamma_1) \in \mathcal{M}_{par}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$ is a *model maximal parabolic Sp(4,ℝ)-Higgs bundle*.

Example 4.2.2. Consider the parabolic Sp(4,ℝ)-Higgs bundle (V_2, β_2, γ_2) which is induced by the embedding through ϕ_Δ from §1.4 of the model parabolic SL(2,ℝ)-Higgs bundle $(E, \theta(q))$. Under the preceding terminology, the bundle $V_2 \to X$ is then described as:

$$V_2 = L \oplus L$$

and it comes equipped with a parabolic structure defined by a trivial flag $(V_2)_{x_i} \supset \{0\}$ and weight $\frac{1}{2}$ for every $x_i \in D$.

Moreover, V_2 can be expressed as $V_2 = N_2 \oplus N_2^* K$. Indeed, for $N_2 = L$ we see that:

$$N_2^* K = L^{-1} \otimes K = L$$

It can be checked that this is a parabolic *stable* Sp(4,ℝ)-Higgs bundle. Also notice that

$$par \deg V_2 = 2par \deg L = 2\left(g - 1 + \frac{s}{2}\right) = 2g - 2 + s$$

Therefore, $(V_2, \beta_2, \gamma_2) \in \mathcal{M}_{par}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$ is a *model maximal parabolic Sp(4,ℝ)-Higgs bundle*.

In light of Proposition 4.1.1 we now derive that the polystable hybrid Sp(4,ℝ)-Higgs bundle constructed, $\left(V_\#, \Phi_\#, h_\#, \bar{\partial}\right)$, is *maximal*:

Proposition 4.2.3. *The hybrid Higgs bundle* $\left(V_\#, \Phi_\#, h_\#, \bar{\partial}\right)$ *constructed by gluing the maximal parabolic Higgs bundles* (V_1, β_1, γ_1) *and* (V_2, β_2, γ_2) *described above is maximal, i.e.* $\deg(V_\#) = 2(g_1 + g_2 + s - 1) - 2 = 2g - 2$*, where* g *is the genus of the Riemann surface* $X_\#$*, the connected sum of the* s*-punctured Riemann surfaces* X_1 *and* X_2*.*

4.2.2 Gluing the Cayley partners.

Let $X_1 = \{\phi_{ij}^1\}$ (resp. $X_2 = \{\phi_{ij}^2\}$) the holomorphic transition functions defining the Riemann surface X_1 (resp. X_2), with respect to an atlas $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). Then $(\phi_{ij}^1)'$ is nowhere zero. Set

$$t_{ij}^1 := (\phi_{ij}^1)' \circ \phi_i^1 \,|_{U_i \cap U_j}$$

and now these define the tangent bundle $T_{X_1} = \{t_{ij}^1\}$. Since $\frac{1}{t_{ij}^1}$ is well-defined, we now get:

$$K_{X_1} = T_{X_1}^* = \left\{ l_{ij}^1 := \left(t_{ij}^1\right)^{-1} \right\}$$

and similarly for the Riemann surface X_2

$$K_{X_2} = T_{X_2}^* = \left\{ l_{ij}^2 := \left(t_{ij}^2\right)^{-1} \right\}$$

The transition functions ϕ_{ij}^1, ϕ_{ij}^2 from the atlas $\mathcal{A} = \mathcal{A}_1 \left|_{X_1^*} \cup \mathcal{A}_2 \right|_{X_2^*}$ of $X_\#$ must agree on the gluing region, the annulus Ω. Thus, $l_{ij}^1(x) = l_{ij}^2(x)$ over $x \in \Omega$. Considering a cover $V_1 \cup V_2$ of Ω, we can define a line bundle isomorphism $\tilde{l} : V_1 \cap V_2 \to \mathbb{C}^*$ and now the 1-cocycles $l_{ij}^1, l_{ij}^2, \tilde{l}$ define the connected sum canonical bundle

$$K_{X_\#} := K_{X_1} \# K_{X_2}$$

Now, take the maximal parabolic model (V_1, β_1, γ_1) described in the previous section. Fix another square root M_1 of the canonical line bundle K_{X_1}. Now, define:

$$\begin{aligned} W_1 = V_1^* \otimes M_1 &= \left[\left(L_1^3 \otimes \iota\right) \oplus (L_1 \otimes \iota)^*\right]^* \otimes M_1 \\ &= \left[(L_1 \otimes \iota) \oplus \left(L_1^{-3} \otimes \xi\right)\right] \otimes M_1 = (K_{X_1} \otimes \iota) \oplus \left(K_{X_1}^* \otimes \xi\right) \end{aligned}$$

i.e. W_1 is of the form $\mathcal{L} \oplus \mathcal{L}^*$ for $\mathcal{L} := K_{X_1} \otimes \iota$ and also the map $\gamma_1 \otimes I_{M_1^*} : W_1^* \to W_1$ is an isomorphism, which comes from the fact that γ_1 is, as follows of the proof of the Milnor-Wood inequality in the parabolic case.

Therefore, the bundle $W_1 \to X_1$ is determined by an $\mathrm{O}(2)$-cocycle $\{w_{\alpha\beta}^1\}$ with

$$\det \{w_{\alpha\beta}^1\} = 1.$$

Similarly, for the maximal parabolic model (V_2, β_2, γ_2) we fix another square root M_2 of the canonical line bundle K_{X_2} and define:

$$W_2 = V_2^* \otimes M_2 = (L_2 \oplus L_2)^* \otimes M_2 = L_2^* M_2 \oplus L_2^* M_2$$

i.e. W_2 is of the form $\mathcal{L} \oplus \mathcal{L}^*$ for $\mathcal{L} := \mathcal{O}$.

Therefore, the bundle $W_2 \to X_2$ is determined by an $\mathrm{O}(2)$-cocycle $\left\{w_{\alpha\beta}^2\right\}$ with

$$\det \left\{w_{\alpha\beta}^2\right\} = 1.$$

As was done in §3.4.3, let 1-cocycles around each puncture $x_i \in D$ for the bundles $\mathbb{W}_1, \mathbb{W}_2$ over the annulus $\Omega \equiv \Omega_1 \sim \Omega_2$

$$\begin{aligned} w_i : U_1 \cap U_2 &\to \mathrm{GL}(4, \mathbb{C}) \\ x &\mapsto g_i^{-1}(x) \cdot m_i(x) \end{aligned}$$

while $\{m_i(x)\} = M_i$. At this point, we are using the 1-cocycles that define the connected sum canonical bundle $K_{X_\#}$.

For an induced hermitian metric on $\mathbb{W}_1$, using the Gram-Schmidt process one can obtain an orthonormal local frame over Ω_1, such that the associated 1-cocycle $\tilde{w}_1$ is $\mathrm{SO}(2)$-valued. We may use the isomorphism $\mathbb{W}_1 |_{\Omega_1} \xrightarrow{\cong} \mathbb{W}_2 |_{\Omega_2}$ induced by the two isomorphisms between the V_i and M_i described before, to glue the bundles over Ω subordinate to the covering $U_1 \cup U_2$. For the 1-cocycle over the connected sum bundle $\mathbb{W}_1 \# \mathbb{W}_2$ we also have:

$$\det \left\{w_{\alpha\beta}^\#\right\} = 1$$

Thus, the first Stiefel-Whitney class $w_1(W_\#)$ vanishes, and so $V_\# = N_\# \oplus N_\#^* K_{X_\#}$ with $N_\# = N_1 \# N_2$. Moreover, this provides that the Cayley partner $W_\#$ of $V_\#$ decomposes as $W_\# = L_\# \oplus L_\#^{-1}$ for some line bundle $L_\#$. We thus have established the following:

Proposition 4.2.4. *The hybrid Higgs bundle $(V_\#, \Phi_\#)$ constructed by gluing the maximal parabolic Higgs bundles (V_1, β_1, γ_1) and (V_2, β_2, γ_2) of §4.2.1 is maximal with a corresponding Cayley partner $W_\#$ for which $w_1(W_\#) = 0$ and $W_\# = L_\# \oplus L_\#^{-1}$, for some line bundle $L_\#$ over $X_\#$.*

Remark 4.2.5. Compare this result to Proposition 5.9 in [22], where an analogous property for the Stiefel-Whitney classes of a hybrid representation was established.

The degree of this line bundle $L_\#$ fully determines the connected component a hybrid Higgs bundle will lie:

Proposition 4.2.6. *For the line bundle $L_{\#}$ appearing in the decomposition $W_{\#} = L_{\#} \oplus L_{\#}^{-1}$ of the Cayley partner, it is*

$$\deg\left(L_{\#}\right) = par \deg K_{X_1} \otimes \iota_1$$

where $\iota_1 = \mathcal{O}_{X_1}\left(D_1\right)$.

Proof. The identity of Proposition 4.1.1 now applies to provide the computation of the degree for the bundle $N_{\#}$ appearing in the decomposition $V_{\#} = N_{\#} \oplus N_{\#}^{*} K_{X_{\#}}$:

$$\begin{aligned}
\deg\left(N_{\#}\right) &= par \deg\left(L_1^3 \otimes \iota_1\right) + par \deg\left(L_2\right) \\
&= 3\left(g_1 - 1\right) + s + \frac{s}{2} + g_2 - 1 + \frac{s}{2} \\
&= g + 2g_1 - 3 + s
\end{aligned}$$

where $g := g_1 + g_2 + s - 1$ is the genus of $X_{\#}$.
Considering $N_{\#} \otimes L_0^{-\frac{1}{2}}$ for some $L_0 = K_{\#}^{\frac{1}{2}}$ now gives

$$\begin{aligned}
\deg\left(N_{\#} \otimes L_0^{-\frac{1}{2}}\right) &= g + 2g_1 - 3 + s + 1 - g \\
&= 2g_1 + s - 2 \\
&= -\chi\left(\Sigma_1\right) = par \deg K_{X_1} \otimes \iota_1
\end{aligned}$$

where $\iota_1 = \mathcal{O}_{X_1}\left(D_1\right)$. $\square$

Therefore, we have constructed a holomorphic vector bundle $V_{\#} \to X_{\#}$ with $\deg\left(V_{\#}\right) = 2g - 2$ and $V_{\#} = N_{\#} \oplus N_{\#}^{*} K_{X_{\#}}$ with $\deg\left(N_{\#} \otimes L_0^{-\frac{1}{2}}\right) = 2g_1 - 2 + s$, which is *odd* (resp. *even*) whenever s is odd (resp. even). The contraction mapping argument of §3.5-3.8 will provide a holomorphic structure $\bar{\partial}$ with respect to which $\left(V_{\#}, \bar{\partial}\right)$ is a polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle. The numerical information we already have for the topological invariants of $V_{\#}$ is preserved and it identifies the connected component of the maximal moduli space the tuple $\left(V_{\#}, \Phi, h_{\#}, \bar{\partial}\right)$ will lie.

Remarks 4.2.7.

1. The component a hybrid Higgs bundle lies depends on the genera and the number of points in the divisors of the initial Riemann surfaces X_1 and X_2 in the construction; there are no extra parameters arising from the deformation of stable parabolic data to model data near these points, or the perturbation argument to correct the approximate solution to an exact solution.

2. The gluing of two parabolic Higgs bundles of the same type as the model $\left(V_1, \beta_1, \gamma_1\right)$ from Example 4.2.1 implies that $\deg\left(N_{\#}\right) = 3g - 3$. On the other hand, the gluing of

two parabolic Higgs bundles of the same type as (V_2, β_2, γ_2) from Example 4.2.2 implies that $\deg(N_\#) = g - 1$, as expected.

3. As was described in §1.5, for a hybrid representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(4,\mathbb{R})$ there is a well defined Euler class with values $e(\varepsilon \otimes \rho, L_+) = -\chi(\Sigma_l)[\Sigma] \in H^2(T^1\Sigma, \mathbb{Z})$. In addition to Proposition 1.5.5, which describes a relation between the Stiefel-Whitney classes for maximal Sp(4,$\mathbb{R}$)-Higgs bundles and the Stiefel-Whitney classes for Sp(4,$\mathbb{R}$)-representations, we deduce that in the case of Riemann surfaces with $s = 1$ point in the divisors, the degree $\deg(L_\#)$ of the underlying bundle $L_\#$ in the decomposition of the Cayley partner $W_\# = L_\# \oplus L_\#^{-1}$ of a hybrid Sp(4,$\mathbb{R}$)-Higgs bundle *equals* the Euler class $e(\varepsilon \otimes \rho, L_+)$ for the hybrid representation, although these invariants live naturally in different cohomology groups.

In conclusion, since $1 \leq g_1 \leq g_1 + g_2 - 1$, it follows that

$$s \leq \deg\left(N_\# \otimes L_0^{-\frac{1}{2}}\right) \leq 2g - s - 2$$

with s an integer between 1 and $g - 1$. Therefore, the hybrid Higgs bundles constructed are modeling *all* exceptional $2g - 3$ connected components of $\mathcal{M}^{\max}(X, \mathrm{Sp}(4,\mathbb{R}))$. These components are fully distinguished by the degree of the line bundle $L_\#$ for the hybrid Higgs bundle constructed by gluing.

REFERENCES

[1] M. Aparicio Arroyo. The geometry of $\mathrm{SO}(p,q)$-Higgs bundles, Ph.D. thesis, Universidad de Salamanca, Consejo Superior de Investigaciones Científicas, 2009.

[2] K. Astala, T. Iwaniec, G. Martin, and J. Onninen. Schottky's theorem on conformal mappings between annuli: a play of derivatives and integrals. *Complex analysis and dynamical systems III*, 35-39, *Contemp. Math.* **455**, *Israel Math. Conf. Proc.*, Amer. Math. Soc., Providence, RI, 2008.

[3] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A* **308** (1983), no. 1505, 523-615.

[4] O. Biquard and P. Boalch. Wild non-abelian Hodge theory on curves. *Compos. Math.* **140** (2004), no. 1, 179-204.

[5] O. Biquard, O. García-Prada and I. Mundet i Riera. Parabolic Higgs bundles and representations of the fundamental group of a punctured surface into a real group. arXiv:1510.04207.

[6] I. Biswas. Parabolic ample bundles. *Math. Ann.* **307** (1997), 511-529.

[7] I. Biswas, P. Arés-Gastesi and S. Govindarajan. Parabolic Higgs bundles and Teichmüller spaces for punctured surfaces. *Trans. Amer. Math. Soc.* **349** (1997), no. 4, 1551-1560.

[8] H. U. Boden and K. Yokogawa. Moduli spaces of parabolic Higgs bundles and $K(D)$ pairs over smooth curves: I. *Int. J. Math.* **7** (1996), 573-598.

[9] S. B. Bradlow, O. García-Prada and P. B. Gothen. Deformations of maximal representations in $\mathrm{Sp}(4,\mathbb{R})$. *Q. J. Math.* **63** (2012), no. 4, 795-843.

[10] S. B. Bradlow, O. García-Prada and P. B. Gothen. Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. *Geom. Dedicata* **122** (2006), 185-213.

[11] M. Burger, A. Iozzi, F. Labourie and A. Wienhard. Maximal representations of surface groups: Symplectic Anosov structures. *Pure Appl. Math. Q.* **1** (2005), no. 3, 543-590.

[12] M. Burger, A. Iozzi and A. Wienhard. Surface group representations with maximal Toledo invariant. *Ann. of Math. (2)* **172** (2010), no. 1, 517-566.

[13] S. Cappell, R. Lee and E. Miller. Self-adjoint elliptic operators and manifold decompositions. Part I: Low eigenmodes and stretching. *Comm. Pure Appl. Math.* **49** (1996), 825-866.

[14] K. Corlette. Flat G-bundles with canonical metrics. *J. Diff. Geom.* **28** (1988), 361-382.

[15] S. K. Donaldson. Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc.* (3) **55** (1987), 127-131.

[16] S. Donaldson and P. Kronheimer. The geometry of four-manifolds. *Oxford Math. Monographs*, Oxford Science Publications, 1990.

[17] O. García-Prada, P. B. Gothen and I. Mundet i Riera. Higgs bundles and surface group representations in the real symplectic group. *J. Topol.* **6** (2013), no. 1, 64-118.

[18] O. García-Prada, P. B. Gothen and I. Mundet i Riera. The Hitchin-Kobayashi correspondence, Higgs pairs and surface group representations. arXiv:0909.4487.

[19] O. García-Prada, P. B. Gothen and V. Muñoz. Betti numbers of the moduli space of rank 3 parabolic Higgs bundles. *Mem. Amer. Math. Soc.* **187** (2007), no. 879, viii+80 pp.

[20] W. M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. Math.* **54** (1984), no. 2, 200-225.

[21] P. B. Gothen. Components of spaces of representations and stable triples. *Topology* **40** (2001), no. 4, 823-850.

[22] O. Guichard and A. Wienhard. Topological invariants of Anosov representations. *J. Topol.* **3** (2010), no. 3, 578-642.

[23] T. Hartnick and A. Ott. Milnor-Wood type inequalities for Higgs bundles. arXiv:1105.4323.

[24] S. He. A gluing theorem for the Kapustin-Witten equations with a Nahm pole. arXiv:1707.06182.

[25] N. J. Hitchin. Lie groups and Teichmüller space. *Topology* **31** (1992), no. 3, 449-473.

[26] N. J. Hitchin. The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* (3) **55** (1987), 59-126.

[27] S. Kim and G. Wilkin. Analytic convergence of harmonic metrics for parabolic Higgs bundles. *J. Geom. Phys.* **127** (2018), 55-67.

[28] H. Konno. Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface. *J. Math. Soc. Japan* **45** (1993), no. 2, 253-276.

[29] R. Mazzeo, J. Swoboda, H. Weiss and F. Witt. Ends of the moduli space of Higgs bundles. *Duke Math. J.* **165** (2016), no. 12, 2227-2271.

[30] V. B. Mehta and C. H. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Math. Ann.* **248** (1980), no. 3, 205-239.

[31] J. Milnor. On the existence of a connection with curvature zero. *Comment. Math. Helv.* **32** (1958), 215-223.

[32] D. Mumford. Theta characteristics of an algebraic curve. *Ann. Sci. École Norm. Sup.* **4** (1971), no. 4, 181-192.

[33] L. Nicolaescu. Notes on Seiberg-Witten Theory. *Graduate Texts in Mathematics* **28**, American Mathematical Society, 2000.

[34] L. Nicolaescu. On the Cappell-Lee-Miller gluing theorem. *Pacif. J. of Math.* **206** (2002), no. 1, 159-185.

[35] S. Ramanan. Global calculus. *Graduate Texts in Mathematics* **65**, American Mathematical Society, 2005.

[36] A. H. W. Schmitt. Geometric invariant theory and decorated principal bundles, *Zürich Lectures in Advanced Mathematics*, European Mathematical Society, 2008.

[37] A. H. W. Schmitt. Moduli for decorated tuples of sheaves and representation spaces for quivers, *Proc. Indian Acad. Sci. Math. Sci.* **115** (2005), no. 1, 15-49.

[38] C. H. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Bul. of the AMS* **83** (1977), no. 1, 124-126.

[39] C. T. Simpson. Constructing variations of Hodge structures using Yang-Mills theory and applications to uniformization. *J. Amer. Math. Soc.* **1** (1988), 867-918.

[40] C. T. Simpson. Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.* **3** (1990), no. 3, 713-770.

[41] C. T. Simpson. Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5-95.

[42] J. Swoboda. Moduli spaces of Higgs bundles on degenerating Riemann surfaces. *Adv. Math.* **322** (2017), 637-681.

[43] C. H. Taubes. Self-dual Yang-Mills connections over non self-dual 4-manifolds, *J. Diff. Geom.*, **17** (1982), 139-170.

[44] V. G. Turaev. A cocycle of the symplectic first Chern class and Maslov indices. (Russian) *Funktsional. Anal. i Prilozhen.* **18** (1984), no. 1, 43-48.

[45] M. Wolf. Infinite energy harmonic maps and degeneration of hyperbolic surfaces in moduli space. *J. Diff. Geom.* **33** (1991), 487-539.

[46] J. W. Wood. Bundles with totally disconnected structure group. *Comment. Math. Helv.* **46** (1971), 257-273.

[47] K. Yokogawa. Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves. *J. Math. Kyoto Univ.* **33** (1993), 451-504.

[48] K. Yokogawa. Infinitesimal deformation of parabolic Higgs sheaves. *Int. J. Math.* **6** (1995), 125-148.

[49] T. Yoshida. Floer homology and splittings of manifolds. *Ann. of Math.* **134** (1991), 277-324.