STATE ESTIMATION OF SWITCHED NONLINEAR SYSTEMS AND SYSTEMS WITH BOUNDED INPUTS: ENTROPY AND BIT RATES

BY

HUSSEIN M. SIBAI

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

Adviser:

Associate Professor Sayan Mitra

ABSTRACT

State estimation is a fundamental problem when monitoring and controlling dynamical systems. Engineering systems interconnect sensing and computing devices over shared bandwidth-limited channels, and therefore, estimation algorithms should strive to use bandwidth optimally. Often, the dynamics of these systems are affected by external factors. In certain cases, these factors would lead the system to switch between different modes. In other cases, they would affect the dynamics of the system continuously in time without leading to explicit mode transitions. In this thesis, we present two notions of entropy for state estimation of nonlinear switched and non-autonomous dynamical systems as lower bounds on the average number of bits needed to be sent from the sensors to the estimators to estimate the states with deterministic (worst case) error bounds. Our approach relies on the notion of topological entropy and uses techniques from control under limited information. Since the computation of these entropies is hard in general, we compute corresponding upper bounds. Additionally, we design a state estimation algorithm for switched systems when their modes cannot be observed. We show that the average bit rate used by the algorithm is optimal in the sense that the efficiency gap is within an additive constant from the gap between the entropy of the considered system and its computed upper-bound. Finally, we apply our theory and algorithms to linear and nonlinear models of systems such as a glycemic index for diabetic patients, a controller of a Harrier jet and a Pendulum.

To my parents, for their love and support.

ACKNOWLEDGMENTS

First and foremost, no words can show how much I am grateful and thankful to my parents. They dedicated their life to our well-being and happiness. Nothing would have been possible without them. I would like to thank my brothers: Ali El Hadi for being the kind-hearted and caring big brother, Ahmad for being a best friend too, Jamal for being amazingly lovely and helpful, and my sister-in-law Zahraa for being the sister we always wanted and for bringing beautiful little Fatima to our family.

I would also like to thank my advisor Sayan Mitra for his guidance, advice, help and support during the past years. This thesis would not have been accomplished without him. His door was always open to me. He changed the way I think, speak and write to being more rigorous, clear and precise. He encouraged and endorsed individual creativity as well as collaborative thinking. I want to thank him for not just being an academic advisor but for being a life mentor too. Also, I would like to thank Professor Daniel Liberzon for his plentiful reviews, suggestions and feedback which had a major impact on the ideas and methods presented in this thesis.

Moreover, I would like to thank my friends Ahmed Fawaz and Mohammad Nouriddine for being the go-to friends if I am hungry, bored, stuck on any sort of a question, or just tired. They have been extremely supportive and helpful making my graduate student life much easier. I also want to thank my friends: Nabil Ramlawi, Hussein Hazimeh, Hasan Dbouk, Ali Kanj, Farah Hariri, Pamela Tannous, Tarek Sakakini, Mariam Fawaz, Lina Farouk, Ali Kotb and many more from the Lebanese and Arab community at UIUC for their invaluable companionship.

Furthermore, I am proud of being part of a great research group. Among the group, I want to thank first Nicole Chan for being a great friend and colleague. I would also like to thank Ritwika Ghosh, Mathew Potok, Chuchu Fan and Bolon Qi for the lively discussions in the office.

Finally, I want to thank the Air Force Office of Scientific Research (AFOSR) for supporting this work through a research grant (FA9550-12-1-0336).

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Motivation

Contemporary engineering systems interconnect sensing and computing devices over a shared communication channel for monitoring and control. For example, more than 70 embedded computing units communicate over a shared 1 Mbps CAN bus in cars [1]. Many machines, conveyor belts, and robotic manipulators need to be monitored in warehouses and factory floors—again over a shared network backbone [2]. Additionally, one of the major problems in controlling a platoon of underwater vehicles is the limited bandwidth channels [3]. Such bandwidth constraints call for optimal allocation of network resources for estimation and detection.

This thesis deals with monitoring continuous time switched dynamical systems and continuous time dynamical systems with bounded inputs with optimal usage of network resources. The key problem is to estimate the state of the system from a small number of bits coming from quantized sensor measurements (see Figure 1.1). This is the *state estimation* problem. The related problem of *mode detection* arises when the plant dynamics itself is unknown or changing.

In the stochastic setting, Kalman and particle filtering are used for solving these problems, in some cases using neural networks (see, for example [4, 5, 6]). Our approach relies on the theory of topological entropy for dynamical systems. The measure-theoretic notion of entropy plays a central role in information theory, estimation and detection. In the theory of dynamical systems, the analogous topological notion of entropy plays a fundamental role in describing the rate of growth of uncertainty about system state ([7, 8, 9, 10, 11, 12]). It also relates to the rate at which information about the system should be collected for state estimation. Drawing this connection, the

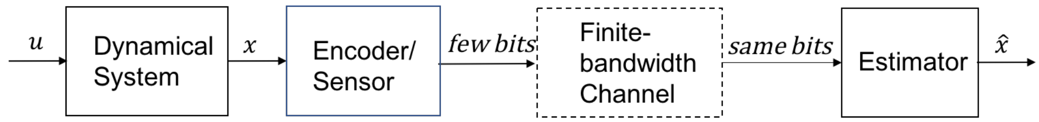

Figure 1.1: Block diagram showing the flow of information from a dynamical system to the sensor to the estimator.

notion of *estimation entropy* has been defined in [13, 14, 15] for nonlinear systems. For a dynamical system of the form $\dot{x}(t) = f(x(t))$, roughly, it is the minimum bit rate needed by the estimator to construct state estimates from quantized measurements that converge to the actual state of the system at a desired exponential rate of α. Estimation entropy is in general hard to compute exactly, but can be upper-bounded by $(C + \alpha)n/\ln 2$, where n is the dimension of system and C is either the Lipschitz constant L of f [13] or an upper-bound on the matrix measure of the Jacobian of f [14]. In [13], an algorithm for state estimation is given which uses an average bit rate of $(L + \alpha)n/\ln 2$. This is optimal in the sense that the efficiency gap of the algorithm is no more than the gap between estimation entropy and its upper-bound. In this thesis, we extend this notion of entropy to the case where the plant model is a nonlinear switched system (Chapter 3) or a nonlinear system with bounded inputs (Chapter 4). We solve similar problems to those solved in [13, 14] for these generalized systems.

In switched and non-autonomous systems, the dynamics of the system are changing over time because of uncontrollable and probably unobservable external factors. This results in a larger uncertainty in an estimate of the system state than that of an autonomous system. In the case of a switched system, if the sensor and estimator are uncertain about the model, and the possible models lead to sufficiently different dynamics, the estimator cannot accurately estimate the state. Hence, in that case, the state estimation and model detection problems should be solved simultaneously to decrease the uncertainty in the state estimate. Similarly, for non-autonomous systems, the dynamics are changing over time; hence, unless the sensor and estimator have a sufficiently accurate estimate of the input, the estimator cannot decrease the uncertainty in the state estimate.

Defining entropy for state estimation requires first specifying an upper bound on the estimation error. For switched systems, we require the error to be upper bounded by a specific constant for a specific amount of time after a mode change (switch) and then decrease exponentially at a specific rate till the next one. For this to be feasible, one may constrain the switches to be spaced in time, as we will see later in the thesis. For systems with bounded inputs, we require the error to be bounded by a constant all the time. A system with bounded input is more general than a switched one since the input signal can have discontinuities and can vary between them. In contrast, in a switched system, the input (switching) signal stays constant between discontinuities.

1.2 Contributions

In this thesis, our main contributions are as follows:

- We define modified notions of topological entropy for state estimation of two types of continuous time dynamical systems: switched nonlinear systems and non-autonomous systems with bounded inputs.
- We prove that these notions lower bound the bit rate needed to estimate the state of the systems up to the predefined bound on the estimation error.
- Since computing the actual values of the entropies is hard, we compute upper bounds in terms of systems' parameters such as their Lipschtiz constants and the constants representing the bounds on the estimation error.
- We present a state estimation algorithm of switched nonlinear systems with bit rate that is close to the computed upper bound on entropy.
- We show the results of experiments where we applied our algorithm to estimate the state of two switched linear and nonlinear systems.

1.3 Related Work

There is a significant body of work on computing the bit rates needed for different control tasks for different types of dynamical systems. In this section, we present a quick overview of several works on the topic.

The problem of m^{th} moment stabilization of an infinite dimensional discrete-time linear time-variant dynamical systems over a noiseless limited-bandwidth feedback channel was discussed in [16] by Nair and Evans. They considered coders with infinite memory and did adaptive quantization to avoid the non-controllability issue that may arise if the quantization map is fixed over time. After that, they presented a tight lower bound on the data rate of a noiseless feedback channel for mean square stabilization of a finite-dimensional stochastic linear system in terms of its unstable eigenvalues. Then, they showed a necessary and sufficient lower bound on the minimal data rate of a noiseless limited-bandwidth feedback channel for m^{th} moment exponential stabilization of a discrete-time linear time-invariant (LTI) system in [17].

In [18], Brocket and Liberzon discussed stabilization of linear dynamical systems using limited bit rate feedback channel. There, they introduced the idea of using adaptive quantization and showed its ability to asymptotically stabilize systems that are stabilizable by linear time-invariant feedback. This could not have been done using traditional fixed quantization.

In their book [19], Ishii and Francis discussed stabilizing distributed control systems using limited-bandwidth network from a hybrid systems theory point of view. They also discussed control of linear systems using limited-bandwidth feedback channel with time delays.

In [20], Matveev and Savkin tackled the problem of stabilizing discrete-time partially observed time-invariant linear system using a noisy limited-bandwidth feedback channel. They showed that the system is stabilizable iff the sum of the logarithms of the absolute values of the unstable eigenvalues is smaller than the classic Shannon capacity of the feedback channel. Additionally, in [21], they discussed the state estimation problem under the same conditions. They reached a similar result: the state can be estimated with probability as high as needed iff the logarithm of the absolute value of the determinant of the unstable part of the dynamics matrix of the system is smaller than the capacity of the channel. If it is larger, they proved that the estimation error would diverge with high probability. Moreover, in [22], they presented necessary and sufficient conditions for stabilizability using multiple noisy limited-bandwidth channels with different capacities and time delays corresponding to different sensors.

Tatikonda, in his PhD thesis [23], investigated centralized control of a distributed system consisting of discrete stochastic systems while using noisy limited bandwidth channels with delays.

Metric entropy of dynamical systems was first introduced by Kolmogorov in 1958 [24, 25] driven by Shannon's pioneering work in 1948 [26]. After that, topological entropy of dynamical systems was introduced by Adler, Konheim and McAndrew in [27].

In [28], Nair et al. presented the notion of topological feedback entropy (TFE) of discrete topological dynamical systems based on the cardinality of open covers in the state space and showed that it is the minimum data rate in the feedback loop needed to keep the state in a compact region. Moreover, they presented the notion of local TFE (LTFE) at a given point and showed that it, under some stabilizability conditions, lower bounds the data rate needed for local uniform asymptotic stability of the system. Finally, they showed that the LTFE is equal to the sum of the unstable eigenvalues of the Jacobian at that point. Our case differs in several aspects: (a) we are tackling the state estimation problem rather than the control one, (b) our definition of entropy relies on spanning sets of trajectories rather than open covers, and (c) we consider continuous time dynamical systems rather than discrete ones.

In [10], Colonius and Kawan defined the notion of invariance entropy of continuous time control systems and showed that it is equal to the minimum data rate needed in the feedback channel to keep the state in a compact set K. Their notion of entropy depends on the cardinality of the set of open loop control inputs needed to keep the system in K for a finite amount of time $T > 0$ starting from any initial state in K. It is equal to the rate of exponential growth of the cardinality of that set as T goes to infinity. They provide lower and upper bounds on entropy. Finally, they show that invariance entropy for linear systems, as in [28], is equal to the sum of the real values of the unstable eigenvalues of the system matrix. In [29], Colonuis and Kawan showed that these two entropy definitions are in fact equivalent.

In [30], Colonius presented the notion of exponential stabilization entropy of continuous control systems. It represents the exponential growth of the number of control signals needed to exponentially stabilize the system over a finite interval $[0, T]$ as T go to infinity. Due the fact that there is no finite number of control signals that can exponentially stabilize a linear system from any initial state, he used a relaxed version of exponential stability. Finally, he showed that the entropy represents an upper bound on the minimal bit rate needed to achieve stability.

In [31], Savkin extended the notion of topological entropy of open-loop discrete time uncertain dynamical systems and discussed its relation to their observability and optimal control under limited bit rate constraints. He provided inequalities relating the bit rate needed for optimal control with the topological entropy of the system. Finally, he computed that entropy for some classes of linear dynamical systems.

The problem of state estimation of continuous-time time-variant linear dynamical systems over a noiseless continuous finite-bandwidth channel was tackled by Savkin and Peterson in [32]. They proposed a recursive coder-decoder scheme for such systems.

In [13, 14, 15], Liberzon and Mitra defined estimation entropy for continuous autonomous dynamical systems to lower bound the bit rate needed to achieve exponentially converging estimates of their states over a limited-bandwidth channel. Their entropy definition represents the rate of exponential growth of the size of a representative sample of the system's trajectories over a finite interval of time. They computed upper and lower bounds on entropy in terms of the system's Lipschitz constant or the matrix measure of its Jacobian. Finally, they presented state estimation and model detection algorithms with bit rates equal to the computed upper bound on entropy. Our work in this thesis is mainly an extension of their work.

Most of the entropy results are for autonomous systems, those that have no input or disturbance. Non-autonomous systems provide more challenges for the computation of bounds of topological entropy [33] and [34].

In [35], Rungger and Zamani presented the notion of invariance feedback entropy for uncertain discrete-time dynamical systems. They showed that it is a tight lower bound on the bit rate needed in the feedback channel for the controller to be able to maintain a subset of the state space invariant. They showed that the entropies of controlled invariant topological systems with upper semi-continuous transition function and finite systems are finite.

The first attempt to tackle the problem of topological entropy for switched systems was by J. Schmidt in his master's thesis [12]. He defined a notion of topological entropy for linear switched systems while having some solvability assumptions on the Lie algebra generated by the matrices of the individual systems. He computed upper and lower bounds on the entropy based on the eigenvalues of the individual systems and their average times of activation.

Schmidt's work differs from ours in several aspects: (a) we consider general nonlinear modes instead of linear ones, (b) we do not assume solvability of the Lie algebra of the modes (in case of the linear ones), and (c) we consider exponential convergence of the error after a while after the switch instead of constant upper bound on error.

A new definition of estimation entropy of stochastic hybrid systems along with an upper bound was derived by Awan and Zamani in [36]. They considered switches that are modeled as Poisson processes while the dynamics are modeled as stochastic differential equations.

CHAPTER 2

PRELIMINARIES

In this chapter, we provide a list of concepts and definitions that we will use in the following chapters.

2.1 General Mathematical Definitions

Vector norms and covers For a real vector $v \in \mathbb{R}^n$, we denote by $\|v\|$ the infinity norm of the vector and by v^T the transpose of v. $B(v, \delta)$ is a δ-ball—closed hypercube of radius δ—centered at v. For a hyperrectangle $S \subseteq \mathbb{R}^n$ and $\delta > 0$, $grid(S, \delta)$ is a collection of 2δ-separated points along axis parallel planes such that the δ-balls around these points cover S. In that case, we say that the grid is of size δ. For a compact set $S \subset \mathbb{R}^n$, $diam(S) = \max_{x_1, x_2 \in S} \|x_1 - x_2\|$ denotes the diameter of S. We denote by $[a; b]$ the set of integers in $\mathbb{Z}$ that belong to the interval $[a, b]$. For a matrix A, $\lambda_{max}(A)$ denotes the largest eigenvalue of A. Note that for any positive definite matrix A, $\lambda_{max}(A) \leq |||A|||$, where $|||\cdot|||$ is any matrix norm. For a finite set S, we denote by $|S|$ the cardinality of S.

Class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions A function $f : [0, \infty) \to [0, \infty)$ is a class $\mathcal{K}$ function if it is continuous, strictly increasing and $f(0) = 0$. It is a class $\mathcal{K}_\infty$ function if it is a class $\mathcal{K}$ function and goes to infinity at infinity.

2.2 Switched Systems

A *switched system* is a standard way for describing control systems with several different modes (see, for example, the book [37]). Suppose we are given a family $f_p, p \in [N]$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assuming that the functions f_p are Lipschitz continuous with Lipschitz constant L_p, the above gives rise to a family of dynamical system modes:

$$\dot{x} = f_p(x), p \in [N] \tag{2.1}$$

evolving on $\mathbb{R}^n$. If the mode $p \in [N]$ is known, then the solution of the differential equation is the function $\xi_p : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. If in addition the initial state x_0 is known, then for any point in time t the state $\xi_p(x_0, t)$ can be approximated using numerical integration. However, for the state estimation problem we are interested in, both the initial state and the mode are unknown.

The time varying mode is modeled as a *switching signal.* This is a piecewise constant function $\sigma : [0, \infty) \to [N]$ which specifies at each time instant t, the index $\sigma(t) \in [N]$ of the function from the family (2.1) that is currently being followed. The points of discontinuity in σ are called *switching times.* Thus, the switched system with a time-dependent switching signal σ can be described by:

$$\dot{x} = f_\sigma(x). \tag{2.2}$$

For a fixed switching signal σ the solution of the above switched system is defined in the standard way and denoted by the function $\xi_\sigma : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. Moreover, $f_{\sigma(t)}$ is Lipschitz continuous with Lipschitz constant $L = \max_{p \in [N]} L_p$.

The switching signal σ models the adversary, the environment, or a controller changing the underlying mode of the system. In general, it may have arbitrary discontinuities; however, to prove stability, or in our case, correctness of state estimation, typically one assumes bounds on switching speed [37, 38, 39].

2.2.1 Dwell-times and reachable sets

A switching signal σ has a *minimum dwell time* $T_d > 0$ if at least T_d time units elapses between consecutive switches. We denote $\Sigma(T_d)$ the family of switching signals with minimum dwell-time T_d switching between the N modes. Moreover, we define $Reach(\Sigma, K)$ to be the set of *reachable states* by System (2.2) with any $\sigma \in \Sigma(T_d)$ from the compact initial set K. More formally,

$$Reach(\Sigma, K) = \{x \in \mathbb{R}^n \mid \exists\ \sigma \in \Sigma(T_d),\ x_0 \in K,\ t \in [0, \infty) : \xi_\sigma(x_0, t) = x\}.$$

2.2.2 Separation between modes

Maximum separation Later on, we will need a bound on the error in state estimates when the system evolves according to two *different* dynamics, from the same state. To this end we introduce the function:

$$d(t) := \max_{p \in [N]} \sup_{\substack{\sigma \in \Sigma(T_d), \\ x_0 \in K, \\ s \geq T_d}} \max_{u \in [0,t]} \|\xi_p(\xi_\sigma(x_0, s), u) - \xi_\sigma(x_0, s+u)\|. \tag{2.3}$$

In addition, σ should not have a switch between $s - T_d$ and $s + t$.

Minimum separation In order for an algorithm to distinguish two modes $p, r \in [N]$, $p \neq r$, it is necessary for the solutions generated by the two modes to be separable in some sense. The following notion of *exponential separation* is proposed in [13]. For $L_s, T_s > 0$ we say that the two modes $p, r \in [N]$ are (L_s, T_s)-*exponentially separated* if there exists a constant $\epsilon_{\min} > 0$ such that for any $\varepsilon \leq \epsilon_{\min}$, for any two nearby initial states $x_1, x_2 \in \mathbb{R}^n$ with $\|x_1 - x_2\| \leq \varepsilon$,

$$\|\xi_p(x_1, T_s) - \xi_r(x_2, T_s)\| > \varepsilon e^{L_s T_s}.$$

That is, trajectories separate out exponentially if they start from a sufficiently small neighborhood. The exponential separation holds if, for example, (1) the two vector fields have a positive separation angle, and (2) at least one of them has a positive velocity. It is believed that this property is generic in the sense that it holds for almost all pairs of modes. For example, it was proven in Proposition 8 in [15] that if for all $x \in D$, where D is some compact set in $\mathbb{R}^n$, $f_p(x) \neq f_r(x)$, then the two modes are exponentially separated over D for small enough T_s and arbitrary L_s.

2.3 Dynamical Systems with Bounded Inputs

We consider in Chapter 4 a dynamical system of the form:

$$\dot{x} = f(x, u), \tag{2.4}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The function f is globally Lipschitz with Lipschitz constants L_x and L_u with respect to the first and the second argument, respectively. Furthermore, we assume that f has piecewise-continuous Jacobian matrices $J_x = \frac{\partial f(x,u)}{\partial x}$ and $J_u = \frac{\partial f(x,u)}{\partial u}$, with respect to the first and second argument, respectively. Once an initial state x_0 and a piecewise-continuous input function u are fixed, the solution exists and is unique. We denote it by $\xi_{x_0,u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$.

In the following section, we define the type of input signals that we consider in Chapter 4.

2.3.1 Bounded input signals

Let U be a compact set in $\mathbb{R}^m$ and $u_{max} := \max_{q \in U} \|q\|$. We denote by $\mathcal{U}$ the set of all piecewise-right-continuous functions that map $\mathbb{R}_{\geq 0}$ to U. Let u be a function in $\mathcal{U}$. Then, for all $t \in \mathbb{R}_{\geq 0}$, define the following:

$$u(t^+) = \lim_{s \to t^+} u(s)$$
$$u(t^-) = \lim_{s \to t^-} u(s).$$

If t is a point of discontinuity, we let $u(t) = u(t^+)$. Note that there exists an $\eta \in [0, u_{max}]$ and $\mu \geq 0$ s.t. for all $u \in \mathcal{U}$, t and $\tau \geq 0$,

$$\|u(t + \tau) - u(t)\| \leq \mu\tau + \eta. \tag{2.5}$$

For example, with $\eta = 2u_{max}$, the bound is satisfied for any $\mu \geq 0$. In that case, the change in u is only constrained by the bound on its norm. In other words, it can have frequent points of discontinuity (jumps), in a short interval, each with a difference between the before and after values being as large as having a norm of $2u_{max}$. However, knowing that u cannot vary much, i.e. having few points of discontinuity or small gradient, can be expressed by setting μ and η to smaller values. η restricts the maximum norm of a jump and μ restricts the number of large jumps in a short interval. Geometrically, the constraint means that for any $u \in \mathcal{U}$, and any $t \in \mathbb{R}_{\geq 0}$, and any $\tau > 0$, $u(t+\tau)$ should belong to the truncated m-dimensional cone with initial radius η and slope μ as shown in Figure 2.1.

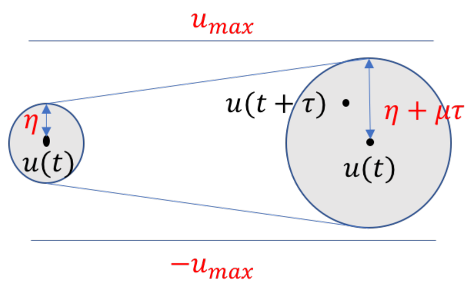

Figure 2.1: Constraints on the variation of u.

This constraint is similar to Assumption 1 in [40] which was made on the variation of the system matrix of a time-varying linear dynamical system to relate its stability conditions to those of a switched linear dynamical system with slow switching. Also, it is similar to the slow switching assumption made by Hesphana and Morse in [41] to prove stability of switched systems with stable subsystems.

To compute bounds on entropy, we will need to bound the distance between trajectories in terms of the distance between the input signals and the distance between the initial states. The following section defines functions for this purpose.

2.3.2 Input-to-state discrepancy functions

We use a modified version of the definition of local input-to-state discrepancy introduced in [42] in order to upper bound the distance between two trajectories. We relax their definition to include systems with piece-wise continuous Jacobians and piece-wise continuous input signals (rather than continuous ones).

Definition 1 (Local IS Discrepancy). For System (2.4), a function $V : \mathcal{X}^2 \rightarrow \mathbb{R}_{\geq 0}$ is a *local input to state discrepancy function* over a set $\mathcal{X} \subset \mathbb{R}^n$ and a time interval $[t_0, t_1] \subset \mathbb{R}_{\geq 0}$ if:

(i) there exist class-$\mathcal{K}$ functions $\bar{\alpha}, \underline{\alpha}$ such that for any $x, x' \in K$, $\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \bar{\alpha}(\|x - x'\|)$, and

(ii) there exist a class-$\mathcal{K}$ function in the first argument, $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class-$\mathcal{K}$ function, $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that for any $x_0, x_0' \in \mathcal{X}$ and $u, u' \in \mathcal{U}$, if $\xi_{x_0,u}(t), \xi_{x_0',u'}(t) \in \mathcal{X}$ for all $t \in [t_0, t_1]$, then for all $t \in [t_0, t_1]$, $V(\xi_{x_0',u'}(t), \xi_{x_0,u}(t)) \leq$

$$\beta(\|x_0 - x_0'\|, t - t_0) + \int_{t_0}^{t} \gamma(\|u(s) - u'(s)\|) ds. \tag{2.6}$$

The local discrepancy function V together with β and γ give the sensitivity of the solutions of the system to changes in the initial state and the input. The functions $\bar{\alpha}, \underline{\alpha}, \beta, \gamma$ are sometimes called witnesses of the local IS discrepancy V. Techniques for computing local discrepancy functions have been presented in [42].

CHAPTER 3

ESTIMATION ENTROPY AND ESTIMATION ALGORITHM FOR SYSTEMS WITH UNKNOWN SWITCHING

3.1 Introduction

In this chapter, we consider the state estimation of switched nonlinear dynamical system of the form $\dot{x} = f_\sigma(x)$, defined formally in Section 2.2, where the switches between N modes are brought about by a switching signal $\sigma : \mathbb{R}_{\geq 0} \to [N]$ unknown to both the sensor and estimator. The dynamics of each mode $\dot{x} = f_p(x)$, $p \in [N]$, where $[N]$ is the set of integers from 0 to $N-1$, could capture, for example, uncertainties in the plant, different operating regimes, nominal and failure dynamics, and parameter values.

Since the mode information is not available to the estimator, exponential convergence of state estimates may be impossible immediately after a mode switch. Thus, we relax the notion of estimation entropy of [13] by allowing a period of time $\tau > 0$ following a mode switch, during which the estimation error is only bounded by a constant ε; and thereafter the error decays exponentially at a rate α as in [13]. On the other hand, we assume a maximal separation between the modes to show that for a small enough T_e—determined by the minimum dwell time T_d of σ, the error parameters ε and τ, and the maximal difference in the dynamics of the different modes (see Proposition 1)—the estimation entropy is upper-bounded by $\frac{(L+\alpha)n}{\ln 2} + \min\{\frac{\log N}{T_e}, \frac{1}{T_d}(N + \log(\frac{T_d}{T_e}))\}$. Here L is the largest of the Lipschitz constants of all f_p's.

We present an algorithm for state estimation for switched systems. The interdependence of the uncertainties in the state and the mode requires this algorithm to simultaneously solve the estimation and mode detection problems: Unless a mode $f_p, p \in [N]$ is detected, it may be impossible to get exponentially converging estimates, and unless an accurate enough estimate for the state is known, it may not be possible to distinguish between two candidate modes.

Our algorithm starts by keeping track of $\hat{N}$ possible modes of the switched system, where $\hat{N}$ is a parameter between 1 and N, and falsifies the wrong ones as more information coming from the sampled states. If at a given iteration the actual mode of the system is the only tracked mode, then, owing to a shrinking quantized measurement strategy, the state estimate converges at the desired exponential rate. If the actual mode is not tracked, then the actual state of the system may *escape* the constructed state estimate bounds. In this case, the algorithm expands the estimate and captures the state. When a mode switch happens, there may be a burst of escapes, but we prove that if the rate of switches is slow enough and the modes are different enough, then the correct mode is detected, and thereafter, the state estimate converges exponentially.

We establish worst case estimation error bounds and time bounds on mode detection. We also show that the average bit rate used is within $\frac{1+\log \hat{N}}{T_p} - \min\{\frac{\log N}{T_e}, \frac{1}{T_d}(N + \log(\frac{T_d}{T_e}))\}$ from the upper bound on the entropy, i.e. the upper bound on the optimal bit-rate, where T_p is the sampling time of the algorithm. We present preliminary experimental results on applying the algorithm to linear and nonlinear switched systems, and discuss the implications of the choice of the key parameter $\hat{N}$.

3.2 State Estimation, Bit-rate, and Entropy

Let us fix throughout this chapter a compact set K of possible initial states of System (2.2), the family of all switching signals with minimal dwell time $T_d > 0$: $\Sigma(T_d)$, two estimation accuracy related constants $\varepsilon, \alpha > 0$ and a time constant τ ($\tau \leq T_d$). In this chapter, we will assume that $d(t)$ exists for all $t \leq \tau$. This condition can be checked, for example, if the reach set $Reach(\Sigma, K)$ is compact. Moreover, we assume (without loss of generality) that the modes are mutually (L, T_p)-exponentially separated (see Remark 1). Also, ϵ_{min} (see Section 2.2.2) is assumed to be global for all pairs of the exponentially separated modes.

Remark 1 (Similar modes) If there are two modes $p, r \in [N]$ such that for all $x \in Reach(\Sigma, K)$ and for all $t \in [0, T_p]$, $\|\xi_p(x_1, t) - \xi_r(x_2, t)\|_\infty \leq \|x_1 - x_2\|_\infty e^{Lt}$, then they will not be exponentially separated. However, although they will *not* be distinguished by the algorithm presented in this chapter, this does *not* influence the correctness of the state estimation. An example would be modes that are exponentially stable, with convergence rate larger than α, to a common equilibrium point.

In our setup, a sensor has access to the actual current state of the system $\xi_\sigma(x_0, t)$ and not the switching signal σ, and it needs to send bits across a bandwidth-constrained channel such that for any initial state $x_0 \in K$ and for any (unknown) switching signal $\sigma \in \Sigma(T_d)$, the estimator would be able to construct a function $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, where for all $j \geq 0$ and for all $t \in [s_j, s_{j+1})$,

$$\|z(t) - \xi_\sigma(x_0, t)\| \leq \begin{cases} \varepsilon & t \in [s_j, s_j + \tau), \\ \varepsilon e^{-\alpha(t-(s_j+\tau))} & \textit{otherwise}, \end{cases} \tag{3.1}$$

where $s_0 = 0, s_1, \ldots$ are the switching times in σ. The norm in inequality (3.1) can be arbitrary. We call such a function $z(.)$ an $(\varepsilon, \alpha, \tau)$-*approximation* of $\xi_\sigma(x_0, \cdot)$. The second bound gives the ideal behavior in which the estimate converges to the actual trajectory $\xi_\sigma(x_0, \cdot)$ exponentially at the rate α as in [13] and [10]. The first condition allows a "lenient" period of duration τ, during which the error is bounded by ε.

A finite set of functions $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_M\}$ from $[0, T]$ to $\mathbb{R}^n$ is $(T, \varepsilon, \alpha, \tau)$-*approximating* if for every initial state $x \in K$ and every switching signal $\sigma \in \Sigma(T_d)$ there exists some $\hat{x}_i \in \hat{X}$ such that for all $t \in [0, T]$, $\hat{x}_i$ is an $(\varepsilon, \alpha, \tau)$-approximating function for $\xi_\sigma(x_0, t)$. Note that $\hat{X}$ also depends on K, T_d and the dynamics of the N modes, but we are suppressing these parameters for brevity.

Let $s_{\text{est}}(T, \varepsilon, \alpha, \tau)$ denote the minimal cardinality of such a $(T, \varepsilon, \alpha, \tau)$-approximating set. The *estimation entropy* of the system is defined as

$$h_{\text{est}}(\varepsilon, \alpha, \tau) := \limsup_{T \to \infty} \frac{1}{T} \log s_{\text{est}}(T, \varepsilon, \alpha, \tau). \tag{3.2}$$

Intuitively, since s_{est} corresponds to the minimal number of functions needed to approximate the state with desired accuracy, h_{est} is the minimum average number of bits needed to identify these approximating functions. The $\limsup$ extracts the base-2 exponential growth rate of s_{est} with time.

Then, s_{est} corresponds to the number of different quantization points needed to identify the trajectories, and h_{est} gives a measure of the long-term bit rate needed for communicating sensor measurements to the estimator.

Notice that we do not take the limit as ε goes to zero in the definition of estimation entropy in contrast to that in [13]. That is because we do not expect the entropy to stay finite as ε approaches zero because of the possible significant difference in the dynamics between the modes. In that case, exponentially increasing the number of bits sent may be needed to keep the estimation error within ε after a change of dynamics (switch between modes) as ε goes to zero. The upper bound on entropy we derive in the following section indeed approaches infinity as ε approaches zero.

3.2.1 Entropy upper bound

In this section, we will establish an upper-bound on the estimation entropy h_{est} for switched systems. First, we fix a time horizon $T > 0$ and prove an upper bound on s_{est} using an inductive construction of approximating functions. The following proposition will be used in the proof.

Proposition 1 *Given the minimum dwell time $T_d > 0$, and the estimation accuracy constants ε and $\alpha > 0$, there exists $T_e \in (0, \tau]$ such that $d(T_e) \leq \varepsilon(1 - e^{-\alpha(T_d - T_e)})$.*

Recall that

$$d(t) := \max_{p \in [N]} \sup_{\substack{\sigma \in \Sigma(T_d),\\ x_0 \in K,\\ s \geq T_d}} \max_{u \in [0,t]} \|\xi_p(\xi_\sigma(x_0, s), u) - \xi_\sigma(x_0, s + u)\|.$$

Then, it is a monotonically increasing continuous function for $t \geq 0$ and equal to zero at $t = 0$. Moreover, the right-hand side of the inequality increases as T_e decreases. Therefore, we can always find a T_e small enough that satisfies the inequality.

Let us fix a trajectory $\xi_\sigma(x_0, \cdot)$ of System (2.2). We define an inductive procedure that constructs a corresponding approximating function $z(\cdot)$ whose pseudo code is presented in Algorithm 1. It follows that the set of all functions that can be computed by this procedure is a $(T, \varepsilon, \alpha, \tau)$-*approximating set*. Then, the cardinality of the set of all functions that can be computed by this procedure gives us an upper bound.

Let $s_0 = 0, s_1, \ldots$ be the sequence of switching times in the switching signal σ generating $\xi_\sigma(x_0, \cdot)$. The approximating function $z(\cdot)$ is constructed in time steps of size T_e ($T_e \leq \tau$), where T_e is a constant that satisfies the inequality in Proposition 1. As we will see later, the upper bound on entropy is inversely proportional to T_e. Hence, the larger the T_e that we fix, the tighter the upper bound we get. We start by choosing an open cover C_0 of K with balls of radii $\varepsilon e^{-(L+\alpha)T_e}$ (Line 4). Let q_0 be the center of a ball that contains x_0. We construct $z(t) := \xi_{\sigma(0)}(q_0, t)$ for $t \in [0, T_e]$. Since $\sigma(t) = \sigma(0)$ for $t \in [0, T_e)$ (recall, $T_d \geq \tau \geq T_e$), the estimation error over that interval would be $\|z(t) - \xi_\sigma(x_0, t)\| \leq e^{Lt}\|x_0 - q_0\| \leq e^{Lt}\varepsilon e^{-(L+\alpha)T_e} \leq \varepsilon e^{-\alpha t}$ (by Bellman-Grownall inequality).

Algorithm 1 Construction of $(\varepsilon, \alpha, \tau)$-approximating function.

1: input: T,T_e, ε
2: $S_0 \leftarrow K$;
3: $R_0 \leftarrow \varepsilon e^{-(L+\alpha)T_e}$;
4: $C_0 \leftarrow$ $grid(S_0, R_0)$;
5: $i \leftarrow 0$;
6: **while** $i \leq \lfloor \frac{T}{T_e} \rfloor$ **do**
7:     $x_i \leftarrow \xi_\sigma(x_0, iT_e)$;
8:     $q_i \leftarrow quantize(x_i, C_i)$;
9:     $z_i \leftarrow \xi_{\sigma(iT_e)}(q_i, \cdot)$;
10:     i++; ▷ parameters for next iteration
11:     **if** $\exists j \in \mathbb{N}$ s.t. $s_j \in ((i-1)T_e, iT_e)$ **then**
12:         $R_i \leftarrow R_{i-1}e^{-\alpha T_e} + d(T_e)$;
13:     **else**
14:         $R_i \leftarrow R_{i-1}e^{-\alpha T_e}$;
15:     **end if**
16:     $S_i \leftarrow B(z_{i-1}(T_e^-), R_i)$;
17:     $C_i \leftarrow grid(S_i, R_i e^{-(L+\alpha)T_e})$;
18:     wait(T_e);
19: **end while**
20: output: $\{z_i : 0 \leq i \leq \lfloor \frac{T}{T_e} \rfloor\}$

Next, for each iteration $1 \leq i \leq \lfloor \frac{T}{T_e} \rfloor$, we compute an n-dimensional ball over-approximating the reachable set of states at $t = iT_e$ given q_{i-1}, a bound on $\|x_{i-1} - q_{i-1}\|$ (which is $R_{i-1}e^{-(L+\alpha)T_e}$) and $\sigma((i-1)T_e)$. Then, we construct a grid with a predefined resolution over that ball. Next, we quantize the actual state at $t = iT_e$ with respect to the grid to get q_i (Line 8). After that, we compute the trajectory which results from running the actual mode at $t = iT_e$ over the time interval $(iT_e, (i+1)T_e]$ starting from q_i (Line 9). Finally, we bound the difference between the actual trajectory $\xi_\sigma(x_0, \cdot)$ and the constructed one $z(\cdot)$ and we prove that the ball computed at the $(i+1)^{th}$ iteration does contain the actual state at $t = (i+1)T_e$.

Formally, let s_j be the time of the last switch before iT_e. We construct C_i to be an open cover of $B(z(iT_e), R_i)$, where

$$R_i = \begin{cases} R_{i-1}e^{-\alpha T_e} + d(T_e) & \text{if } s_j \in ((i-1)T_e, iT_e), \\ R_{i-1}e^{-\alpha T_e} & \text{otherwise,} \end{cases}$$

and $R_0 = \varepsilon$, with balls of radii equal to $r_i = R_i e^{-(L+\alpha)T_e}$ (Lines 11 through 17). Then, we let q_i to be any of the centers of the balls in C_i that contain $\xi_\sigma(x_0, iT_e)$. Note that $\xi_\sigma(x_0, T_e) \in B(z(T_e), R_1)$. Next, we construct $z(t) := \xi_{\sigma(iT_e)}(q_i, t - iT_e)$ for $t \in (iT_e, (i+1)T_e]$.

Lemma 1 *The output $z(\cdot)$ of Algorithm 1 is an $(\varepsilon, \alpha, \tau)$-approximating function of $\xi_\sigma(x_0, \cdot)$.*

Proof Consider an iteration $i \geq 0$ and let s_{j+1} be the first switch at or after iT_e. Based on where s_{j+1} falls with respect to the interval $[iT_e, (i+1)T_e]$, there are two cases here: (a) $s_{j+1} = iT_e$ or $s_{j+1} \geq (i+1)T_e$ and (b) $s_{j+1} \in (iT_e, (i+1)T_e)$. For (a),

$$\begin{aligned}
&\|\xi_\sigma(x_0, t) - z(t)\| = \|\xi_\sigma(x_0, t) - \xi_{\sigma(iT_e)}(q_i, t - iT_e)\| \\
&= \|\xi_{\sigma(iT_e)}(\xi_\sigma(x_0, iT_e), t - iT_e) - \xi_{\sigma(iT_e)}(q_i, t - iT_e)\| \\
&\qquad [\text{since } \sigma(t) = \sigma(iT_e) \text{ for } t \in [iT_e, (i+1)T_e)] \\
&\leq e^{L_{\sigma(iT_e)}(t - iT_e)} \|\xi_\sigma(x_0, iT_e) - q_i\| \\
&\qquad [\text{Bellman-Gronwall inequality}] \\
&\leq e^{L(t - iT_e)} r_i \\
&\qquad [L_{\sigma(iT_e)} \leq L; \text{ by the definition of } q_i \in C_i] \\
&= e^{L(t - iT_e)} R_i e^{-(L+\alpha)T_e} \\
&\qquad [\text{substituting } r_i] \\
&\leq e^{L(t - iT_e)} R_i e^{-(L+\alpha)(t - iT_e)} \\
&\qquad [\text{since } t - iT_e \leq T_e] \\
&= R_i e^{-\alpha(t - iT_e)}.
\end{aligned}$$

For (b), we can repeat the same steps of part (a) for any $t \in (iT_e, s_{j+1})$ to get $\|z(t) - \xi_\sigma(x_0, t)\| \leq R_i e^{-\alpha(t-iT_e)}$. After the switch at s_{j+1}, that is, for any $t \in [s_{j+1}, (i+1)T_e]$,

$$
\begin{aligned}
&\|\xi_\sigma(x_0, t) - z(t)\| = \|\xi_\sigma(x_0, t) - \xi_{\sigma(iT_e)}(q_i, t - iT_e)\| \\
&= \|\xi_\sigma(\xi_\sigma(x_0, s_{j+1}), t - s_{j+1}) - \xi_{\sigma(iT_e)}(q_i, t - iT_e)\| \\
&\leq \|\xi_\sigma(\xi_\sigma(x_0, s_{j+1}), t - s_{j+1}) - \xi_\sigma(q_i, t - iT_e)\| \\
&\qquad + \|\xi_\sigma(q_i, t - iT_e) - \xi_{\sigma(iT_e)}(q_i, t - iT_e)\| \\
&\qquad\qquad \text{[by triangular inequality]} \\
&\leq e^{L(t-s_{j+1})}\|\xi_\sigma(x_0, s_{j+1}) - \xi_\sigma(q_i, s_{j+1} - iT_e)\| \\
&\qquad + \|\int_0^{t-iT_e} (f_\sigma(\xi_\sigma(q_i, t')) - f_{\sigma(iT_e)}(\xi_{\sigma(iT_e)}(q_i, t')))dt'\| \\
&\qquad\qquad \text{[by Bellman-Gronwall inequality]} \\
&\leq e^{L(t-s_{j+1})} e^{L(s_{j+1}-iT_e)}\|\xi_\sigma(x_0, iT_e) - q_i\| + d(t - iT_e) \\
&\qquad\qquad \text{[using the definition of } d(\cdot)\text{]} \\
&\leq e^{L(t-iT_e)} R_i e^{-(L+\alpha)T_e} + d(t - iT_e) \leq R_i e^{-\alpha(t-iT_e)} + d(T_e),
\end{aligned}
$$

where the last line follows from substituting $\|\xi_\sigma(x_0, iT_e) - q_i\|$ with r_i's value and from the fact that $d(t)$ is an increasing function. In both cases, $\xi_\sigma(x_0, (i+1)T_e) \in B(z((i+1)T_e), R_{i+1})$. Now we want to prove that $z(\cdot)$ is an approximation function to $\xi_\sigma(x_0, \cdot)$. First, note that $R_i = \varepsilon e^{-\alpha i T_e}$ for all i before the first switch s_1. Hence, $\|z(t) - \xi_\sigma(x_0, t)\| \leq \varepsilon e^{-\alpha t}$ for all $t \in [0, s_1]$ by part (a) above. Therefore, $z(\cdot)$ satisfies inequality (3.1) between time 0 and s_1. Next, we let $i_1 = \lceil s_1/T_e \rceil$ (the first iteration after the first switch). We know from the previous argument that $R_{i_1} \leq \varepsilon e^{-\alpha i_1 T_e} + d(T_e) \leq \varepsilon e^{-\alpha T_d} + d(T_e)$. Thus, $R_{i_1} \leq \varepsilon$ by our choice of T_e that satisfies the inequality in Proposition 1. Then, $\|z(t) - \xi_\sigma(x_0, t)\| \leq \varepsilon e^{-\alpha t} + d(t - s_1) \leq \varepsilon e^{-\alpha T_d} + d(T_e) \leq \varepsilon$ for $t \in [s_1, i_1 T_e]$ by part (b) above. Moreover, since $T_e \leq \tau$, $z(\cdot)$ satisfies the first part of inequality (3.1) for $t \in [s_1, i_1 T_e]$. Now, the same argument made before for $t \in [0, s_1]$ can be repeated for the time interval $t \in [i_1 T_e, s_2]$ which has a size greater than or equal to $T_d - T_e$. Finally, by induction on all switches, $z(\cdot)$ satisfies the properties in (3.1). Therefore, $z(\cdot)$ is an approximating function to $\xi_\sigma(x_0, \cdot)$.

Lemma 2 *$s_{est}(T, \varepsilon, \alpha, \tau)$ is upper-bounded by $\#C_0 N (HN)^{\lfloor T/T_e \rfloor + 1}$, where $H =$*

$\lceil e^{(L+\alpha)T_e} \rceil^n$ *and* $\#C_0$ *is the cardinality of* C_0.

Proof We count the number of functions that can be computed by the above procedure. First, note that a function $z(.)$ is defined by the quantization points and the modes chosen at multiples of T_e. Moreover, the cardinality of C_0, $\#C_0$, is upper bounded by $\lceil \frac{diam(K)}{2\varepsilon e^{-(L+\alpha)T_e}} \rceil^n$, where $diam(K)$ is the diameter of K. Moreover, for any $i \geq 1$, the cardinality of C_i is upper bounded by $\lceil \frac{R_i}{R_i e^{-(L+\alpha)T_e}} \rceil^n = \lceil e^{(L+\alpha)T_e} \rceil^n = H$, which is independent of R_i. At each iteration $0 \leq i \leq \lfloor T/T_e \rfloor$, Algorithm 1 chooses one from the N modes and a quantization point in the cover C_i (see Lines 8 and 9). We can conclude that the number of functions that can be computed using Algorithm 1 is upper bounded by $(\#C_0)H^{\lfloor T/T_e \rfloor} N^{\lfloor T/T_e \rfloor + 1}$.

The following lemma presents a more accurate method to count the approximating functions that can be constructed by the procedure in the proof which leads to a tighter upper bound in general.

Lemma 3 $s_{est}(T, \varepsilon, \alpha, \tau)$ *is upper-bounded by* $\#C_0 N H^{T/T_e} (\frac{T_d}{T_e} e^N)^{T/T_d}$, *where* $H = \lceil e^{(L+\alpha)T_e} \rceil^n$ *and* $\#C_0$ *is the cardinality of* C_0.

Proof As in the previous lemma, we count the number of approximating functions that can be computed by Algorithm 1 to upper bound s_{est}. First, note that all switches between two sampling times would affect the computations in Algorithm 1 in the same way. Thus, they are indistinguishable in that sense. Moreover, no two switches can happen within T_d time units. Then, the maximum number of switches that can happen in a time interval of size T is $\lfloor \frac{T}{T_d} \rfloor + 1$; remember that we are assuming that the first switch s_0 is at $t = 0$. At the sampling time following each of these switches, Algorithm 1 should choose one from N possible modes. Also, there are $\lfloor \frac{T}{T_e} \rfloor$ intervals when these switches can happen. Then, any switching signal $\sigma \in \Sigma(T_d)$ is mapped to one of $N \sum_{i=0}^{k_1} \binom{k_2}{i} N^i$ switching signals by Algorithm 1, where $k_1 = \lfloor \frac{T}{T_d} \rfloor$ and $k_2 = \lfloor \frac{T}{T_e} \rfloor$. To compute a simple upper bound on this sum, we multiply it first by $\left(\frac{k_1}{k_2}\right)^{k_1}$ to get:

$$\begin{aligned}
(\frac{k_1}{k_2})^{k_1}\sum_{i=0}^{k_1}\binom{k_2}{i}N^i &= \sum_{i=0}^{k_1}\binom{k_2}{i}(\frac{k_1}{k_2})^{k_1}N^i \le \sum_{i=0}^{k_1}\binom{k_2}{i}(\frac{k_1}{k_2})^{i}N^i \\
&\qquad [\text{since } \tfrac{k_1}{k_2} \le 1 \text{ (remember } T_e \le T_d)] \\
&\le \sum_{i=0}^{k_2}\binom{k_2}{i}(\frac{k_1}{k_2})^{i}N^i < \sum_{i=0}^{\infty}\binom{k_2}{i}(\frac{k_1}{k_2})^{i}N^i \\
&\qquad [\text{again since } \tfrac{k_1}{k_2} \le 1 \text{ and } k_2 < \infty] \\
&= \left(1+\frac{Nk_1}{k_2}\right)^{k_2} \le e^{Nk_1}.
\end{aligned}$$

Hence, $N\sum_{i=0}^{k_1}\binom{k_2}{i}N^i \le N\left(\frac{k_2}{k_1}\right)^{k_1}e^{Nk_1}$. And assuming without loss of generality that $\frac{T}{T_e}$ and $\frac{T}{T_d}$ are integers, then $\frac{k_2}{k_1} = \frac{T_d}{T_e}$. So, $N\sum_{i=0}^{k_1}\binom{k_2}{i}N^i \le N\left(\frac{T_d}{T_e}\right)^{k_1}e^{Nk_1}$. From now on, following the same steps taken in the proof of Lemma 2 by multiplying this bound on the number of switching signals with the product of the cardinalities of the grids C_i's over all the iterations before T, one can conclude that the number of functions that can be computed by Algorithm 1 is upper bounded by $\#C_0 N H^{T/T_e}(\frac{T_d}{T_e}e^N)^{T/T_d}$.

Theorem 1 $h_{est}(\varepsilon,\alpha,\tau) \le \frac{(L+\alpha)n}{\ln 2} + \min\{\frac{\log N}{T_e}, \frac{1}{T_d}(N+\log(\frac{T_d}{T_e}))\}$, *where* T_e *is as chosen in Section 3.2.1.*

Proof This proof is along the lines of the proof of Proposition 2 in [13].

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\log s_{\text{est}}(T,\varepsilon,\alpha,\tau) \\
&\le \limsup_{T\to\infty}\frac{1}{T}\log\left((\#C_0)(HN)^{\lfloor\frac{T}{T_e}\rfloor+1}\right) \\
&\qquad\quad [\text{using Lemma 2}] \\
&\le \limsup_{T\to\infty}\frac{1}{T}\log\#C_0 \\
&\qquad + \limsup_{T\to\infty}\frac{1+{}^{T_e}\!/_T}{T_e}(\log\lceil e^{(L+\alpha)T_e}\rceil^n + \log N) \\
&\le \frac{(L+\alpha)n}{\ln 2} + \frac{\log N}{T_e}.
\end{aligned}$$

The last step follows from the fact that $\limsup_{T\to\infty}\frac{1}{T}\log\#C_0 = \limsup_{T\to\infty}$

$\frac{1}{T}\log\lceil\frac{diam(K)}{2\varepsilon e^{-(L+\alpha)T_e}}\rceil^n = 0$, since the term inside the log is finite. However, if we use the bound in Lemma 3 in the second step instead of Lemma 2 we get $h_{est}(\varepsilon,\alpha,\tau) \leq \frac{(L+\alpha)n}{\ln 2} + \frac{1}{T_d}(N + \log(\frac{T_d}{T_e}))$. Hence the theorem.

Corollary 1 *If $N = 1$, we get the previous bound on entropy $\frac{(L+\alpha)n}{\ln 2}$ given in [13].*

Remark 2 (Relationship between parameters) Recall that T_e should satisfy $d(T_e) \leq \varepsilon(1 - e^{-\alpha(T_d - T_e)})$ and $T_e \leq \tau$. Hence, larger values of the parameters ε or τ allow T_e to be larger. This decreases the upper bound on entropy. However, having a larger α may increase or decrease the upper bound since while it decreases the second term by allowing a larger T_e, it increases the first term in the entropy bound in Theorem 1.

3.2.2 Relation between entropy and the bit rate of estimation algorithms

In this section, we show that there is no state estimation algorithm for System (2.2) that uses bit rates smaller than its estimation entropy. First, let us define state estimation algorithms given ε, τ, and $\alpha > 0$:

Definition 2 A state estimation algorithm for System (2.2) with a fixed bit rate is a pair of functions $(\mathcal{S}, \mathcal{E})$, where $\mathcal{S} : \mathbb{R}^n \times Q_s \to \Gamma \times Q_s$, $\mathcal{E} : \Gamma \times Q_e \to ([0, T_p] \to \mathbb{R}^n) \times Q_e)$, T_p is the sampling time, Γ is an alphabet with N symbols, for some $N \in \mathbb{N}$, and Q_s and Q_e are the sets of internal states of the sensor $\mathcal{S}$ and estimator $\mathcal{E}$, respectively. $\mathcal{S}$ runs at the sensor side and $\mathcal{E}$ on the estimator one. $\mathcal{S}$ samples the state of the system each T_p time units and sends to $\mathcal{E}$ a symbol from Γ representing an estimate of the state at the corresponding sampling time. Finally, $\mathcal{E}$ maps the received symbol to an $(\varepsilon, \tau, \alpha)$-approximating function of the trajectory for the next T_p time units.

Now, let us define the bit rate of the algorithm:

$$b_r(\varepsilon, \tau, \alpha) := \limsup_{T\to\infty} \frac{1}{T} \sum_{i=0}^{\lfloor T/T_p \rfloor} \log N = \limsup_{j\to\infty} \frac{1}{jT_p} \sum_{i=0}^{j} \log N = \frac{\log N}{T_p}. \quad (3.3)$$

Proposition 2 *There is no state estimation algorithm for System (2.4) with a fixed bit rate smaller than its estimation entropy.*

Proof The proof is similar to the proof of Proposition 2 in [43]. For the sake of contradiction, assume that there exists such an algorithm with a bit rate smaller than $h_{\text{est}}(\varepsilon, \alpha, \tau)$. Then, for a sufficiently large T', we should have $\frac{(l+1)\log N}{T'} < \frac{1}{T'} \log s_{\text{est}}(T', \varepsilon, \alpha, \tau)$, where $l = \lfloor T'/T_p \rfloor$. Hence, we get the inequality $N^{l+1} < s_{\text{est}}(T', \varepsilon, \alpha, \tau)$. However, N^{l+1} is the number of possible sequences of symbols of length $l+1$ that can be sent by the sensor over $l+1$ iterations. There are $l+1$ instead of l iterations over the interval $[0, T']$ since the sensor starts sending the codewords at $t = 0$ s. Hence, the number of functions that can be constructed by the estimator is upper bounded by N^{l+1}. Moreover, for any given trajectory of the system, the output of the estimator is a corresponding $(\varepsilon, \alpha, \tau)$-approximating function over the interval $[0, T']$. This is true since the estimator should be able to construct an $(\varepsilon, \alpha, \tau)$-approximating function for the corresponding trajectory of the system over the interval $[0, (l+1)T_p)$ given the codewords sent by the sensor in the first $l+1$ iterations. Hence, the set of functions that can be constructed by the estimator defines a $(T', \varepsilon, \alpha, \tau)$-approximating set. But, $s_{\text{est}}(T', \varepsilon, \alpha, \tau)$ is the minimal cardinality of such a set. Therefore, the set of functions that can be constructed by the algorithm defines a $(T', \varepsilon, \alpha, \tau)$-approximating set which has a cardinality smaller than s_{est}, the supposed minimal one.

3.3 State Estimation Algorithm

We consider a setup where a sensor is sampling the state of the switched system each T_p time units without being able to sense the mode. It sends a quantized version of the state along with other few bits over a communication channel to the estimator. In turn, the estimator needs to compute $(\varepsilon, \alpha, \tau)$-approximating function of the trajectory of the system using the measurements received from the sensor (see Figure 3.1).

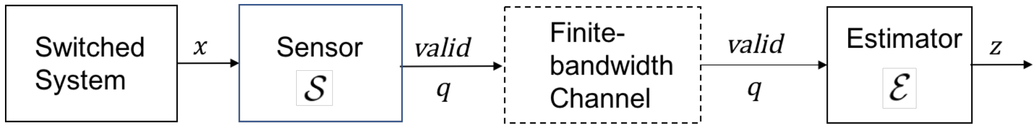

Figure 3.1: Block diagram showing the flow of information from the switched system to the sensor to the estimator.

3.3.1 Estimation algorithm overview

First, we briefly discuss the basic principle of constant bit-rate state estimation for a single dynamical system (see for example [13]). In this case, the system evolves as $\dot{x} = f_p(x)$, for a given $p \in [N]$, $x_0 \in K$, and there is no uncertainty about the mode. Suppose at a given time t the estimator has somehow computed a certain estimate for the state of the system, say represented by a hypercube S. In the absence of any new measurement information, the uncertainty in a state estimate or the size of S blows up exponentially with time as $e^{L_p t}$, where L_p is the Lipschitz constant of f_p. In order to obtain the required exponentially shrinking state estimates, i.e., S shrinking as $e^{-\alpha t}$, the sensor has to send new measurements to the estimator.

One strategy is for the sensor to send information every $T_p > 0$ time units as follows: it partitions S, which has a radius r, into a grid with cells of radii $re^{-(L_p+\alpha)T_p}$, makes a quantized measurement of the state of the system $\xi(x_0, t)$ according to this grid and sends a few bits to the estimator so that the algorithm running at the estimator can identify the correct cell in which the state resides (see Figure 3.1). At this point, the uncertainty in the state reduces by a factor of $e^{(L_p+\alpha)T_p}$ so that after T_p time units when the uncertainty grows by a multiple of $e^{L_p T_p}$ there is still a net reduction in uncertainty by a factor of $e^{\alpha T_p}$. It can also be seen that the number of bits the sensor needs to send (for identifying one grid cell out of $e^{(L_p+\alpha)T_p n}$) is $O(n(L_p+\alpha)T_p)$ and this gives the average bit rate of ${}^{n(L_p+\alpha)}/_{\ln 2}$.

Algorithm 2 which runs on the sensor side extends this strategy to work with switched systems. The basic idea is to track a number $(1 \le \hat{N} \le N)$ of possible modes that the system could be in, and run the above algorithm of quantization-based estimation, for each of these $\hat{N}$ modes. The set of tracked modes is stored in the vector m. A mode $m_i[r]$, $r \in [\hat{N}]$, is valid ($valid_i[r] = 1$) if the current state $\xi_\sigma(x_0, iT_p)$ is contained in the corresponding state estimate $S_i[r]$ at Line 9 and $m_i[r] \neq -1$. However, it is possible that none of the $\hat{N}$

tracked modes are valid. In particular, the mode may switch and the state may evolve to fall outside of the estimates of the tracked modes or it may be that none of the $\hat{N}$ tracked modes in m_i is the actual mode of the system over $[(i-1)T_p, iT_p]$. In this scenario where none of the modes are valid, the state is said to have *escaped* (Line 15). In the case of an escape, the algorithm replaces all modes from the vector m and considers a new set of modes from $[N]$. If the rate of actual mode switches is slow enough (Lemma 5) then it is guaranteed to include the actual mode of the system in m before the next switch. And once the actual mode is tracked in m, the estimation error converges exponentially.

In the above description of the algorithm, we suggested that each tracked mode $m_i[r]$ maintains its own corresponding state estimate $S_i[r]$ and quantization grid $C_i[r]$. This not only uses excessive memory, but also implies that $\hat{N}$ different quantized measurements of the state has to be sent by the sensor. In Algorithm 2, at any iteration $i \geq 1$, only a single state estimate S_i is maintained, a single grid C_i is computed according to which a single measurement is sent by the sensor. That is S_i and C_i are actually $S_i[mode_{i-1}]$ and $C_i[mode_{i-1}]$, where $mode_{i-1}$ is some $r \in [\hat{N}]$ agreed on between the sensor and the estimator. In our case we consider it the valid mode with the minimum index in m_i (Line 11). In order to check the validity of the other tracked modes in m_i, the actual state is shifted with vectors which are computed according to the dynamics of these modes. That is, $v_i[r]$ represents the center of hypercube $S_i[r]$ which is the state estimate of the system corresponding to the dynamics $\dot{x} = f_{m_i[r]}(x)$. To check if $x_i \in S_i[r]$, x_i is shifted with the vector $v_i[mode_{i-1}] - v_i[r]$ and then checked if it belongs to S_i.

If there is an escape at a certain iteration, S_i is constructed as a hyperrectangle centered at $v_i[mode_{i-1}]$ with radius δ_i plus $d(T_p)$. Recall that δ_i is the radius used for computing S_i assuming that there is no escape (Line 34) and $d(T_p)$ is the additional factor that capture maximum deviation between two trajectories of two different modes in $[N]$ starting from the same state in $Reach(\Sigma, K)$, the reachable states by (2.2), and running for T_p seconds. Next, q_i will be the quantization of x_i with respect to the new C_i computed in Line 19.

The NextMode() function cycles through all the $[N]$ modes in the following two-phase fashion. For a sequence of N calls in phase I, it returns the modes in $[N]$ in some arbitrary order. Then, it returns -1 for the next $\hat{N} - 1$ calls in Phase II and then goes back to Phase I. Phase I is used by the estimation algorithm to cycle through all the modes fairly in discovering the actual mode after a switch. Phase II is used to keep the actual mode as the only mode tracked in m_i while the rest of m_i is equal to -1.

Estimator side algorithm On the estimator side, an algorithm similar to Algorithm 2 is executed with small changes: instead of taking x_i as input (Line 7), q_i, a quantized version of x_i, and the $valid_i$ vector are taken. Hence, the estimator knows if $x_i \in S_i[r]$ or not for a certain $r \in [N]$ by examining the $valid_i$ vector sent from the sensor. In addition, Line 8 is replaced by "true". Finally, Lines 8 to 10, Line 20 and Line 22 are omitted. These lines only compute values which are sent by the sensor.

Reading the pseudo-code $B(x_c, r_c)$ defines an over approximation of the initial set K as a hypercube of radius r_c centered at x_c. The **input** x_i (Line 7) executed at time t reads the current state of the system $\xi_\sigma(x_0, iT_p)$ into the program variable x_i. In the next line $x_i \in S_i[r]$ is assumed to be computed by checking if $x_i + (v_i[mode_{i-1}] - v_i[r]) \in S_i$ if $i \geq 1$ and $x_i + (v_i[0] - v_i[r]) \in S_i$ if $i = 0$. In Line 11, the minimum index of a valid mode is assigned to $mode_i$ but this could be any arbitrary choice. It is set to $\perp$ if there is no valid mode.

Comparison with upper bound construction Algorithm 2 is similar to Algorithm 1 which was used for the construction of an approximating function in the proof of the upper bound in Section 3.2.1. However, the mode is known to the sensor at the sampling times in Algorithm 1 while it is not assumed to be known in Algorithm 2. Thus, the construction used in the upper bound knows the iterations where the switch happens, which enables Algorithm 1 to increase the size of the ball representing the state estimate in the iteration following a switch. However, because it is assumed that the mode is not known, the algorithm needs to wait till the state x_i leaves the state estimate S_i to know that a switch happened or that a mode considered in m_i is different from the actual mode. That fact requires the additional assumption that the modes are exponentially separated to bound the number of iterations needed for the state to leave a state estimate constructed based on a wrong mode, which in turn requires Algorithm 2 to sample faster ($T_p \leq T_e$)

and track several modes in parallel to figure out the actual mode and upper-bound the error by ε between a switch and its following τ time units.

3.4 Analysis of Estimation Algorithm

In this section, we prove a sequence of error bounds on the state estimate for different cases that arise from considering a mode which is different from the actual mode over a time interval of size T_p. Then, in Section 3.4.2 we establish bounds on the maximum number of possible escapes between switches. Theorem 2 in Section 3.4.3 uses these results together with an upper bound on the speed of mode switches to give detailed bounds on the state estimation error. Finally, in Section 3.4.4 we analyze the average bit rate and compare it to the upper bound on h_{est} defined in Theorem 1.

Notations We fix all the parameters of the algorithm including the sampling period T_p and the mode window size $\hat{N}$. We also fix a particular (unknown) initial state $x_0 \in K$ and a particular (unknown) switching signal σ for the system described by Equation (2.2). This defines a particular solution $\xi_\sigma(x_0, \cdot)$ of the switched system and the sequence of states $\xi_\sigma(x_0, T_p), \xi(x_0, 2T_p), \ldots,$ sampled by Algorithm 2 which runs on the sensor side. We abbreviate $\xi_\sigma(x_0, iT_p)$ as x_i and the quantized measurement of x_i that is sent by the sensor as q_i. Moreover, δ_i, S_i, C_i, etc., denote the valuations of the variables δ, S, C, etc., at Line 22 in the i^{th} iteration of the algorithm. However, the modes in m_{i+1} are the modes considered over the interval $(iT_p, (i+1)T_p]$. The switching times in σ are denoted by $s_0 = 0, s_1, \ldots$ For a given switching time s_j, we define $last(j) := \lfloor s_j/T_p \rfloor$ and $next(j) := \lceil s_j/T_p \rceil$ as the last iterations of the algorithm before the j^{th} switch and the first iteration after the j^{th} switch respectively.

Recall that an escape occurs when the state of the system $\xi(x_0, iT_p)$ is not in any of the state estimates $S_i[r]$'s at Line 9, i.e., it occurs when the **else** branch in Line 15 is taken.

Algorithm 2 Procedure for estimating the state of a switched system (sensor side).

input: T_p, α, δ_0, $K \subset B(x_c, r_c)$, $\hat{N}$
$m_0 \leftarrow \langle 0, 1, \ldots \hat{N} - 1 \rangle$;
$S_0 \leftarrow B(x_c, r_c)$;
$C_0 \leftarrow grid(S_0, \delta_0 e^{-(L+\alpha)T_p})$;
$mode_0 \leftarrow 0$; $i \leftarrow 0$;
while true **do** ▷ i^{th} iteration
  input x_i;
  for $r \in [\hat{N}]$ **do**
    $valid_i[r] \leftarrow [x_i \in S_i[r]$ *and* $m_i[r] \neq -1]$;
  end for
  $mode_i \leftarrow \min\{r \mid valid_i[r]\}$;
  $escape \leftarrow mode_i \neq \perp$;
  if not $escape$ **then** ▷ no escape
    $q_i \leftarrow quantize(x_i, C_i[mode_i])$;
  else ▷ escape
    $mode_i \leftarrow mode_{i-1}$;
    $\delta_i \leftarrow d(T_p) + \delta_i$;
    $S_i \leftarrow B(z_i(T_p), \delta_i)$;
    $C_i \leftarrow grid(S_i, \delta_i e^{-(L+\alpha)T_p})$;
    $q_i \leftarrow quantize(x_i, C_i[mode_i])$;
  end if
  send $\langle q_i, valid_i \rangle$;
  i++; ▷ parameters for next iteration
  $m_i \leftarrow m_{i-1}$;
  for $r \in [\hat{N}]$ **do**
    if $escape$ or (not $valid_{i-1}[r]$ and $m_i[r] \neq -1$) **then**
      $m_i[r] \leftarrow NextMode()$;
    end if
    if $m_i[r] \neq -1$ **then**
      $v_i[r] \leftarrow \xi_{m_i[r]}(q_{i-1}, T_p)$;
    end if
  end for
  $\delta_i \leftarrow e^{-\alpha T_p} \delta_{i-1}$;
  $S_i \leftarrow B(v_i[mode_{i-1}], \delta_i)$;
  $C_i \leftarrow grid(S_i, \delta_i e^{-(L+\alpha)T_p})$;
  $z_i(.) \leftarrow \xi_{m_i[mode_{i-1}]}(q_{i-1}, \cdot)$;
  wait(T_p);
end while

3.4.1 Error bounds across a single iteration

In this section, we establish how the error in state estimation, $\|\xi_\sigma(x_0,t) - z(t)\|_\infty$, evolves over a single iteration of the algorithm, that is, over $t \in [iT_p, (i+1)T_p]$. The estimate $z(t)$ over $[iT_p, (i+1)T_p]$ is $\xi_{m_{i+1}}[r](q_i, \cdot)$ for some r, and therefore, we track the error by bounding $\|\xi_\sigma(x_0,t) - \xi_{m_{i+1}[r]}(q_i,t)\|_\infty$, for all $r \in [\hat{N}]$ with $m_{i+1}[r] \neq -1$.

There are several sub-cases to consider based on (a) whether there is a switch, and (b) whether the tracked mode $m_{i+1}[r]$ matches the actual mode at a given time, over the considered interval between the iterations. For each of these cases, we establish a bound on $\|\xi_\sigma(x_0,t) - z(t)\|_\infty$ using (a) Bellman-Gronwall inequality to bound $\|\xi_u(x,t) - \xi_u(x',t)\|_\infty$, and (b) triangular inequality to bound $\|\xi_u(x,t) - \xi_p(x',t)\|_\infty$, where $u \neq p \in [\hat{N}]$ and $x \neq x' \in \mathbb{R}^n$. Recall that $T_p \leq \tau \leq T_d$, so no more than one switch can occur between iT_p and (i+1)T_p.

Each of the following propositions covers one of the above cases. Proposition 3 considers the case when there is a switch between iT_p and $(i+1)T_p$, the considered mode $m_{i+1}[r]$ is the same as the actual mode $\sigma(iT_p)$ at $t = iT_p$, and there exists a state estimate $S_i[p]$ that contains the actual state $\xi_\sigma(x_0, iT_p)$ at $t = iT_p$. It shows that the estimate converges exponentially until the switch, and after that it accumulates an additive factor of $d(T_p)$.

Proposition 3 *Fix an iteration i, a switching time $s_j \in (iT_p, (i+1)T_p)$, and an index $r \in [\hat{N}]$. If $m_{i+1}[r] = \sigma(iT_p)$ and $x_i \in S_i[p]$ for some $p \in [\hat{N}]$, then for all $t \in [iT_p, (i+1)T_p]$,* $\|\xi_\sigma(x_0,t) - \xi_{m_{i+1}[r]}(q_i, t - iT_p)\|_\infty \leq$

$$\begin{cases} \delta_i e^{-\alpha(t-iT_p)} & \text{if } t < s_j \quad (3.4) \\ d(T_p) + \delta_i e^{-\alpha(t-iT_p)} & \text{otherwise.} \quad (3.5) \end{cases}$$

Proof For (3.4), $\|x_i - q_i\|_\infty \leq \delta_i e^{-(L+\alpha)T_p}$ since $x_i \in S_i[p]$ for some $p \in [\hat{N}]$

and the boxes in $C_i[p]$ are of radii $\delta_i e^{-(L+\alpha)T_p}$. Then,

$$
\begin{aligned}
&\|\xi_\sigma(x_0,t) - \xi_{m_{i+1}[r]}(q_i, t-iT_p)\|_\infty \\
&= \|\xi_\sigma(x_i, t-iT_p) - \xi_{m_{i+1}[r]}(q_i, t-iT_p)\|_\infty \\
&\qquad\qquad\qquad [\text{since } \xi_\sigma(x_0,t) = \xi_\sigma(\xi_\sigma(x_0, iT_p), t-iT_p)] \\
&= \|\xi_{m_{i+1}[r]}(x_i, t-iT_p) - \xi_{m_{i+1}[r]}(q_i, t-iT_p)\|_\infty \\
&\qquad\qquad\qquad [\sigma(iT_p) = m_{i+1}[r]] \\
&\leq e^{L_{m_{i+1}[r]}(t-iT_p)} \|x_i - q_i\|_\infty \\
&\qquad\qquad\qquad [\text{Bellman-Gronwall inequality}] \\
&\leq \delta_i e^{L_{m_{i+1}[r]}(t-iT_p)} e^{-(L+\alpha)T_p} \\
&\qquad\qquad\qquad [q_i \text{ is quantization of } x_i] \\
&\leq \delta_i e^{-\alpha(t-iT_p)}.
\end{aligned}
$$

The last inequality follows because $L_{m_{i+1}[r]} \leq L$ and $t - iT_p \leq T_p$. For (3.5), we assume without loss of generality that $m_{i+1}[r] = \sigma(t) = 1$ for $t \in [iT_p, s_j)$, $\sigma(t) = 2$ for $t \in [s_j, (i+1)T_p]$. Then, $\|\xi_\sigma(x_0,t) - \xi_{m_{i+1}[r]}(q_i, t-iT_p)\|_\infty$

$$
\begin{aligned}
&= \|\xi_2(\xi_1(x_0, s_j), t-s_j) - \xi_1(\xi_1(q_i, s_j - iT_p), t-s_j)\|_\infty \\
&\leq \|\xi_2(\xi_1(x_0, s_j), t-s_j) - \xi_1(\xi_1(x_0, s_j), t-s_j)\|_\infty \\
&\qquad + \|\xi_1(\xi_1(x_0, s_j), t-s_j) - \xi_1(\xi_1(q_i, s_j - iT_p), t-s_j)\|_\infty \\
&\qquad\qquad\qquad [\text{by triangle inequality}] \\
&\leq \| \int_0^{t-s_j} (f_2(\xi_2(\xi_1(x_0, s_j), t')) - f_1(\xi_1(\xi_1(x_0, s_j), t')))dt' \| \\
&\qquad + \|\xi_1(\xi_1(x_0, s_j), t-s_j) - \xi_1(\xi_1(q_i, s_j - iT_p), t-s_j)\|_\infty
\end{aligned}
$$

$$
\begin{aligned}
&\leq d(t-s_j) + e^{L_1(t-iT_p)} \|x_i - q_i\|_\infty \\
&\qquad\qquad\qquad [\text{by Bellman-Gronwall inequality}] \\
&\leq d(T_p) + \delta_i e^{-\alpha(t-iT_p)}.
\end{aligned}
$$

The next proposition holds under the same conditions as Proposition 3 except that the considered mode $m_{i+1}[r]$ matches the mode of the switched system $\sigma((i+1)T_p)$ at $t = (i+1)T_p$ iteration, but it is not the same as $\sigma(iT_p)$. The proof of (3.6) is analogous to the proof of (3.5).

Proposition 4 *Fix an iteration i, a switching time $s_j \in (iT_p, (i+1)T_p)$, and an index $r \in [\hat{N}]$. If $m_{i+1}[r] \neq \sigma(iT_p)$, $m_{i+1}[r] = \sigma((i+1)T_p)$ and $x_i \in S_i[p]$ for some $p \in [\hat{N}]$, then, for all $t \in [iT_p, (i+1)T_p]$,*
$\|\xi_\sigma(x_0, t) - \xi_{m_{i+1}[r]}(q_i, t - iT_p)\|_\infty \leq$

$$\begin{cases} d(T_p) + \delta_i e^{-\alpha(t - iT_p)} & \text{if } t < s_j \quad (3.6) \\ 2d(T_p) + \delta_i e^{-\alpha(t - iT_p)} & \text{otherwise.} \quad (3.7) \end{cases}$$

Proof For (3.7), $\|\xi_\sigma(x_0, t) - \xi_{m_{i+1}[r]}(q_i, t - iT_p)\|_\infty$

$$\begin{aligned} &\leq \|\xi_\sigma(x_0, t) - \xi_{\sigma(iT_p)}(x_i, t - iT_p)\|_\infty \\ &\quad + \|\xi_{\sigma(iT_p)}(x_i, t - iT_p) - \xi_{m_{i+1}[r]}(q_i, t - iT_p)\|_\infty \\ &\qquad \text{[by triangle inequality]} \\ &\leq \|\xi_\sigma(\xi_\sigma(x_0, s_j), t - s_j) - \xi_{\sigma(iT_p)}(\xi_\sigma(x_0, s_j), t - s_j)\|_\infty \\ &\quad + \|\xi_{\sigma(iT_p)}(x_i, t - iT_p) - \xi_{m_{i+1}[r]}(q_i, t - iT_p)\|_\infty \end{aligned}$$

$$\begin{aligned} &\leq d(t - s_j) + d(T_p) + \delta_i e^{-\alpha(t - iT_p)} \\ &\qquad \text{[by similar argument to (3.5)]} \\ &\leq 2d(T_p) + \delta_i e^{-\alpha(t - iT_p)}. \end{aligned}$$

Proposition 5 also holds under the same conditions as Proposition 3 except that the considered mode $m_{i+1}[r]$, the actual mode $\sigma(iT_p)$ at the i^{th} iteration and $\sigma((i+1)T_p)$ at the $(i+1)^{st}$ iteration are all distinct. Inequality (3.8) is the same as (3.6). Also, the proof of (3.9) is analogous to the proof of (3.7).

Proposition 5 *Fix an iteration i, a switching time $s_j \in (iT_p, (i+1)T_p)$, and an index $r \in [\hat{N}]$. If $m_{i+1}[r] \neq \sigma(iT_p)$, $m_{i+1}[r] \neq \sigma((i+1)T_p)$, $m_{i+1} \neq -1$ and $x_i \in S_i[p]$ for some $p \in [\hat{N}]$, then, for all $t \in [iT_p, (i+1)T_p]$,*
$\|\xi_\sigma(x_0, t) - \xi_{m_{i+1}[r]}(q_i, t - iT_p)\|_\infty \leq$

$$\begin{cases} d(T_p) + \delta_i e^{-\alpha(t - iT_p)} & \text{if } t < s_j \quad (3.8) \\ 2d(T_p) + \delta_i e^{-\alpha(t - iT_p)} & \text{otherwise.} \quad (3.9) \end{cases}$$

From the above propositions, it follows immediately that if there is no switch between the i^{th} and the $(i+1)^{st}$ iteration, then the bounds given by inequalities (3.4), (3.6) and (3.8) will continue to hold for the entire period between the iterations.

The following assumption will be used to prove several intermediate results about the estimation algorithm detecting the right mode and estimation bounds. Then, in Lemma 5 in Section 3.4.2, we will establish a lower bound on the dwell-time T_d which guarantees this assumption.

Assumption 1 For each switching time s_j other than $s_0 = 0$, let $i = last(j)$. Then, there exists $r \in [\hat{N}]$ where $m_{i+1}[r]$ is the actual mode of the system $\sigma(iT_p)$ and $m_{i+1}[p] = -1$ for all $p \neq r$ and $\delta_i \leq \min\{\delta_0, \epsilon_{min}\}$.

Proposition 6 *Under Assumption 1, for each i there exists $r \in [\hat{N}]$ with $x_i \in S_i[r]$.*

Proof If there is an escape at iteration i, then the state x_i is not in any of the $S_i[r]$'s at Line 9; however, it is still guaranteed to be in all the expanded (corrected) estimates $S_i[r]$'s computed at Line 18 based on δ_i and $d(T_p)$. That is because, under Assumption 1, inequalities (3.7) and (3.9) in Propositions 4 an 5 are not relevant (they are useful for analyzing the error bounds for faster switching signals). Therefore, Line 17 takes care of the worst case scenario in the estimation error over a single iteration.

3.4.2 Bounding escapes between switches

Proposition 7 upper bounds the number of escapes that can happen between two consecutive switches to $\lceil N/\hat{N} \rceil$.

Proposition 7 *Under Assumption 1, the maximum number of escapes between two consecutive switches is $\lceil N/\hat{N} \rceil$.*

Proof First, note that at an escape, all the $\hat{N}$ invalid modes are dropped from the vector m_i and new candidate modes are added fairly by the *NextMode*() function. Hence, all the N modes would have been considered after $\lceil N/\hat{N} \rceil$ escapes. Thus, the correct mode $\sigma(t)$ would have been in m at some iteration i. Then, let $m_{i+1}[r] = \sigma(iT_p)$. Second, we know that $x_i \in S_i[p]$ for some $p \in [\hat{N}]$ by Proposition 6. Therefore, we can apply the estimation error bound given by (3.4) in Proposition 3 to conclude that in the next iteration $valid_i[r]$ will be set to 1 until a new switch occurs. Thus, there will be no more escapes till the next switch.

Because of the exponential separation property, we can show that if the dwell time of the switching signal is large enough, then after some maximum number of iterations after a switch, the actual mode $\sigma(t)$ still remains unchanged and the size of the state estimate S_i will be small enough to the point that all incorrect modes in m_i will be invalidated. We define $i_{inv}(\delta)$ to be an upper bound on the number of iterations needed to invalidate a mode when the current radius of the ball representing the state estimate S is δ. Let us define: for any $\delta > 0$,

$$i_{inv}(\delta) := \max\{\lceil \frac{1}{\alpha T_p} \ln(\frac{\delta}{\epsilon_{min}}) - \frac{L}{\alpha} \rceil, 1\}.$$

Proposition 8 *Under Assumption 1, if at a given iteration $i \geq 0$, $-1 \neq m_{i+1}[r] \neq \sigma(t)$, then $m_{i+1}[r]$ will be replaced with a different mode after a maximum of $i_{inv}(\delta_i)$ iterations.*

Proof Let $c = \lceil \frac{1}{\alpha T_p} \ln(\frac{\delta_i}{\epsilon_{min}}) - \frac{L+\alpha}{\alpha} \rceil$. First, note that until $m_{i+1}[r]$ is replaced, δ_i will be decreasing by a $e^{\alpha T_p}$ factor in each iteration (because there is no escape if it is not replaced). Then, $\delta_{i+c} e^{-(L+\alpha)T_p} = \delta_i e^{-((i+c)-i)\alpha T_p} e^{-(L+\alpha)T_p} < \epsilon_{min}$. Thus, by the exponential separation property:

$$\begin{aligned}
&\|\xi_\sigma(x_i, (c+1)T_p) - \xi_{m_{i+c+1}[r]}(q_{i+c}, T_p)\|_\infty \\
&= \|\xi_\sigma(x_{i+c}, T_p) - \xi_{m_{i+c+1}[r]}(q_{i+c}, T_p)\|_\infty > \delta_{i+c} e^{-(L+\alpha)T_p} e^{LT_p} \\
&= \delta_{i+c+1}. \qquad \text{[computed at Line 33]}
\end{aligned}$$

Thus, the actual state will not belong to $S_{i+c+1}[r]$ computed at Line 34 and $m_{i+c+2}[r] \neq m_{i+c+1}[r]$. We upper bound the radius δ_i of the state estimate S_i at iteration i with

$$\delta_{max} := \max_{i \in [1, \lceil N/\hat{N} \rceil]} \{\delta_0 e^{-i\alpha T_p} + d(T_p) \frac{1 - e^{-i\alpha T_p}}{1 - e^{-\alpha T_p}}\}.$$

Note that the first term decays geometrically with i and the second term increases, and the max value could be attained somewhere in the middle.

Proposition 9 *Under Assumption 1, $\delta_i \leq \delta_{max}$ for all i.*

Proof The radius δ_i of S_i decreases between two escapes and possibly increase at an escape. Therefore, the maximum of δ_i would be achieved if some number of escapes (less than or equal to $\lceil N/\hat{N} \rceil$) happened in consecutive iterations immediately after a switch. Assumption 1 is used to make sure that $\delta_i \leq \delta_0$ at $i = last(j)$. The following definitions and two lemmas are used to compute the minimum dwell-time that suffices for Assumption 1 to be true. The following i_{det} represents the maximum number of iterations needed after a switch for the actual mode to be detected, all other modes to be invalidated and $\delta_i \leq \epsilon_{min}$.

$$
\begin{aligned}
i_{det} &:= \sum_{i=1}^{\lceil N/\hat{N} \rceil} i_{inv}\big(\delta_0 e^{-i\alpha T_p} + d(T_p) \sum_{j=0}^{i-1} e^{-j\alpha T_p}\big) + 2 \\
&\leq \lceil \frac{N}{\hat{N}} \rceil i_{inv}(\delta_{max}) + 2.
\end{aligned}
$$

Lemma 4 *Under Assumption 1, after a maximum of i_{det} iterations of any switch s_j, $m_{i+1}[r] = \sigma(t)$, for some $r \in [\hat{N}]$, $m_{i+1}[u] = -1$ for all $u \neq r$ and $\delta_i \leq \epsilon_{min}$.*

Proof (sketch) After a switch, the only mode considered in m_i will no longer be the correct mode. In the worst case, $\sigma(t)$ will be considered in the last set of modes m_{i+1}. Each set of modes m_{i+1} needs a maximum of $i_{inv}(\delta_i)$ iterations to be invalidated. Moreover, there is a maximum of $\lceil N/\hat{N} \rceil$ escapes. The first escape will happen after a maximum of 2 iterations after the switch to invalidate $m_{i+1}[r]$ by the exponential separation assumption since $\delta_i \leq \epsilon_{min}$ before the switch. Since i_{inv} is monotonically increasing w.r.t δ, we summed the values of i_{inv} when evaluated on the $\lceil N/\hat{N} \rceil$ maximum possible values of δ_i. The last $i_{inv}(\delta_{max})$ in i_{det} is to invalidate all wrong modes (and replace them with -1) and keep the actual one in m_i. It will also make $\delta_i \leq \epsilon_{min}$ by the definition of $i_{inv}(\delta_{max})$. Finally, we define the following to upper bound the number of iterations, with no escapes, needed to decrease δ_i from ϵ_{min} to less than δ_0:

$$
i_{est} := \max(\lceil \frac{1}{\alpha T_p} \ln(\frac{\epsilon_{min}}{\delta_0}) \rceil, 0).
$$

Lemma 5 *If the minimum dwell-time of σ is greater than $(i_{det} + i_{est} + 1)T_p$, then Assumption 1 is true.*

Proof Lemma 4 holds between $s_0 = 0$ and s_1 given the minimum dwell time and the fact that $\epsilon_{min}\, e^{-\alpha T_p (i_{est})} \leq \delta_0$ without Assumption 1. Then, the argument holds inductively for the rest of the intervals.

3.4.3 Estimation error

Combining the above, we derive bounds on the estimation error in Theorem 2. It shows that after a switch, the algorithm will be in four possible "phases". The estimation error will increase in the first few iterations after a switch where escapes occur, until the correct mode is found in m, and thereafter, the estimate converges exponentially, provided the dwell time is large enough.

Let the iterations of the algorithm when escapes occur between two consecutive switches s_j and s_{j+1} be numbered $w_1, \dots w_k$. Fixing j we avoid indexing the w's and k with j.

Theorem 2 *If σ has dwell time $T_d \geq (i_{det} + i_{est} + 1)T_p$, then for any $t \in [s_j, s_{j+1})$, the estimation error*
$\|\xi_\sigma(x_0, t) - z(t)\|_\infty \leq$

$$\begin{cases} d(T_p) + \delta_0 e^{-\alpha(t - last(j)T_p)} & \text{if } t \in [s_j, w_1 T_p] & (3.10)\\ d(T_p) + \delta_{w_h} e^{-\alpha(t - w_h T_p)} & \text{if } \exists\, h \in \{1, \dots, k\}, t \in [w_h T_p, w_{h+1} T_p] & (3.11)\\ d(T_p) + \delta_{w_k} e^{-\alpha(t - w_k T_p)} & \text{if } t \in [w_k T_p, (w_k + i_{inv}(\delta_{w_k}))T_p] & (3.12)\\ \delta_{w_k} e^{-\alpha(t - w_k T_p)} & \text{otherwise.} & (3.13) \end{cases}$$

Proof We start by proving (3.10): By Lemma 5, $\delta_{last(j)} \leq \epsilon_{min}$, $\delta_{last(j)} \leq \delta_0$ and $z(t) = \xi_\sigma(q_{last(j)}, t - last(j)T_p)$ for $t \in [last(j)T_p, s_j)$. Then, by inequality (3.5) in Proposition 3 , the inequality is satisfied for $t \in [s_j, next(j)T_p]$. Moreover, if w_1, the first escape after s_j, was not at $next(j)$ then it will be at $next(j)+1$, since, by the exponential separation property, $\|z(t) - \xi_\sigma(x_0, t)\| \geq \delta_0 e^{LT_p}$, so $w_1 = next(j) + 1$. If that is the case, then the inequality holds for $t \in [next(j)T_p, (next(j)+1)T_p]$ as a result of inequality (3.6) in Proposition 4 and the fact that $\delta_{next(j)} \leq \delta_0 e^{-\alpha T_p} \leq \delta_0$.

Inequalities (3.11) and (3.12) have similar proofs as (3.10) but instead of δ_0 we have δ_{w_h}. Inequality (3.13) follows from the fact that at $t = (w_k + i_{inv}(\delta_{w_k})T_p)$ there is $r \in [\hat{N}]$ with $m[r] = \sigma(s_j)$ and $m[p] = -1$ for $p \neq r$, and the repeated application of inequality (3.4) in proposition 3.

Corollary 2 summarizes the error bounds in Theorem 2.

Corollary 2 *Under the assumptions of Lemma 5, consider the time between the two consecutive switches s_j and s_{j+1}. Then, for all $t \in [s_j, s_{j+1})$,*
$\|\xi_\sigma(x_0, t) - z(t)\|_\infty \leq$

$$\begin{cases} \delta_{max} + d(T_p) & t \in [s_j, w_k T_p] \quad (3.14) \\ \delta_{w_k} e^{-\alpha(t - w_k T_p)} & \textit{otherwise.} \quad (3.15) \end{cases}$$

Thus, for a given ε, τ and α defined as for Theorem 1, we can choose δ_0, T_p and $\hat{N}$ to control the variables i_{det}, $d(T_p)$ and δ_{max} so as to achieve the inequalities in (3.1).

3.4.4 Optimal network usage

We show that the estimation algorithm uses network bandwidth optimally in the following sense: An analysis similar to that of Proposition 4 of [13] shows that the average bit rate used by our algorithm is $^{(L+\alpha)n}/_{\ln 2} + ^{\hat{N}}/_{T_p}$. The sensor needs to send (a) q_i: the quantization of x_i with respect to one of the $\hat{N}$ $S_i[r]$'s and (b) the $valid_i$ bit vector: for each $r \in [\hat{N}]$ one bit indicating whether or not x_i belongs to $S_i[r]$. The quantized state q_0 requires $\#C_0 = \lceil \frac{diam(K)}{2\delta_0 e^{-(L+\alpha)T_p}} \rceil^n$ bits to be sent. For $i \geq 1$, the number of bits required to represent q_i is $\#C_i = \lceil \frac{\delta_i}{\delta_i e^{-(L+\alpha)T_p}} \rceil^n = \lceil e^{(L+\alpha)T_p} \rceil^n$. Hence, the average bit rate used by the algorithm is $b_r(\varepsilon, \alpha, T_p) = \lim_{i\to\infty} {}^1/_{T_p} \log(\#C_i \hat{N}) = \frac{(L+\alpha)n}{\ln 2} + \frac{\hat{N}}{T_p}$.

Theorem 3 *Average bit rate of Algorithm 2 is* $\frac{(L+\alpha)}{\ln 2} + \frac{\hat{N}}{T_p}$.
Hence, it follows that the bit-rate used by the estimation algorithm is larger than the upper bound on the estimation entropy by at most $\frac{\hat{N}}{T_p} - \min\{\frac{\log N}{T_e}, \frac{1}{T_d} (N + \log(\frac{T_d}{T_e}))\}$ bits. Therefore, the efficiency gap between the bit-rate used by our algorithm and the bit rate (h_{est}) used by the best possible algorithm is at most $\frac{\hat{N}}{T_p} - \min\{\frac{\log N}{T_e}, \frac{1}{T_d}(N + \log(\frac{T_d}{T_e}))\}$ bits more than the gap between h_{est} and its upper-bound. The unobservability of the switching signal and the switching times contributes to the gap.

3.5 Experiments

We implemented Algorithm 2 and experimented on two switched systems.[1] We used Python 2.7 and ODEint package to generate the trajectories. The running time of each iteration of the algorithm is $O(n + N)$, assuming $O(1)$

[1]Code available at: `https://github.com/HusseinSibai/SwitchedSystemsStateEstimation`

time computation of trajectories. In practice, it took milliseconds on a laptop with 2 GHz Intel Core i7 processor, which suggests that the algorithm can be used in real-time.

Switched linear systems In a switched linear system, the dynamics of all the modes are of the form: $\dot{x} = A_p x + B_p u$. We present estimation of a five dimensional switched linear system with five modes. For each $p \in [5] = \{0, \ldots 4\}$ the matrix A_p and the column vector B_p are generated randomly, and the input u is also a random constant. In the presented results, the settling time for the first mode is 11.89 and the others are unstable. The maximum Lipschitz constant was $L = 28.28$. We worked with switching signals that satisfy Assumption 1. We chose the following parameters $\alpha = 1$, $T_p = 0.1$ s, $\varepsilon = 2$ and $\hat{N} = 2$. Two state components of the system are shown in Figure 3.2 (a). Observe that the state estimates (yellow and blue) enlarge after escapes and that the state and the mode eventually converge to the correct values. $d(T_p)$ was approximated at each escape by computing the distance between all possible pairs of modes starting from the actual state of the system (can be replaced with the estimated state) at the time of that escape. It was around 2. The bit rate used here is $(L+\alpha)n/\ln 2+\hat{N}/T_p = 231$ bps. The maximum time needed to detect the correct mode is 2.2 s and the maximum radius of a bounding box δ was around 3. So, if $\tau \geq 2.2$ s and $\varepsilon \geq 5$, the parameters of the algorithm in this experiment satisfy the properties in (3.1).

Nonlinear glycemic index model Estimating the blood glucose level is an important problem for administering controlling insulin for diabetes patients given [44]. We consider a polynomial switched system model of plasma glucose concentration.[2] The model has nine modes representing different control inputs. The state consists of three variables: G, I and X. In this model, the switching between different modes is brought about by certain threshold based rules depending on the state variables. In the span of 150 s of each execution, 6 switches happened. Although Assumption 1 was not always satisfied, it was still able to do state estimation. The Lipschitz constant of each of the modes is estimated through sampling. The parameters of the

[2] Switched system benchmark available from: `https://ths.rwth-aachen.de/research/projects/hypro/glycemic-control/`

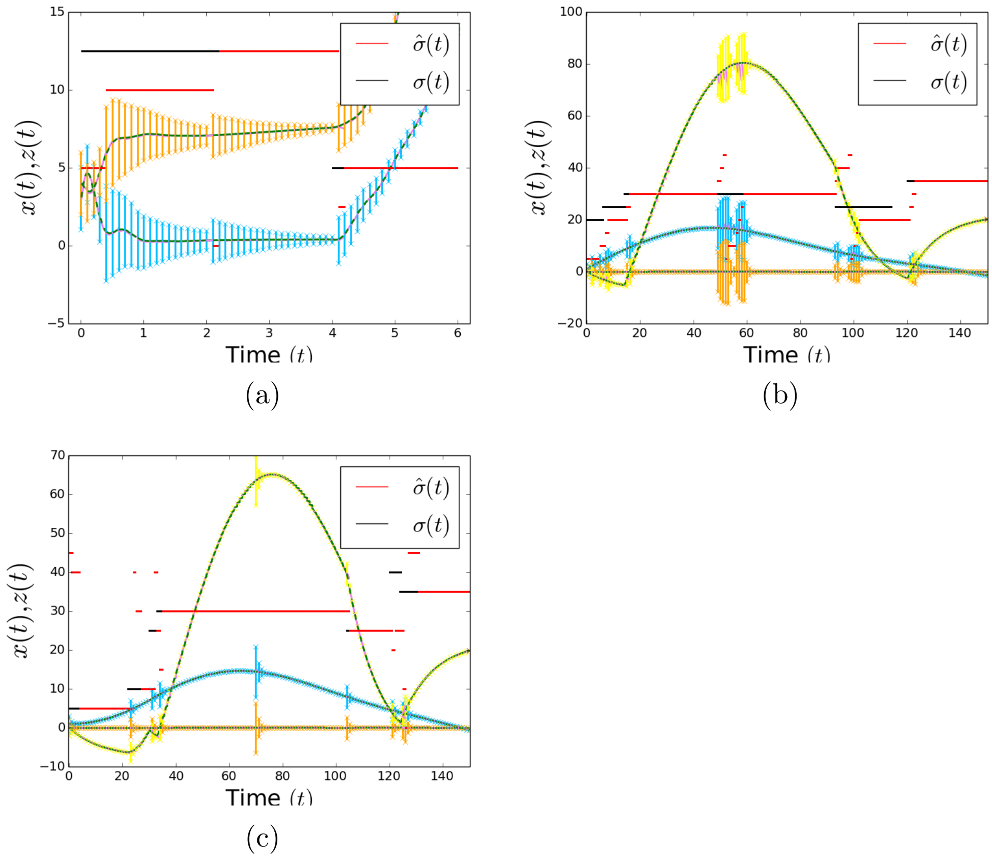

Figure 3.2: Execution of estimation algorithm. Actual mode (black), mode estimate (red), the values of the other variables are shown by the continuous plots. The vertical cut lines show the error estimates (δ) on those variables. (a) Linear five-dimensional system, Glycemic nonlinear control system, (b) $\hat{N} = 1$ and (c) $\hat{N} = 9$. Figure with $\hat{N} = 9$ has much less escapes than that with $\hat{N} = 1$.

algorithm are chosen as $\alpha = 1$ and $T_p = 1$ s. For each value of $\hat{N} \in [1, 9]$, 100 initial states x_0 are drawn randomly and the algorithm is executed on the resulting solutions $\xi_\sigma(x_0, \cdot)$. Two sample executions are shown in Figure 3.2 and the average results are shown in Table 3.1.

As the number of modes tracked $\hat{N}$ increases, as expected, the number of escapes decreases. In fact, beyond $\hat{N} = 5$, the marginal benefit to sending more bits is small as far as the worst case error estimate (δ_{max}) is concerned. In practice, the choice for d_0, $\hat{N}$ and T_p should be chosen to satisfy the convergence parameters specified.

Table 3.1: Average results of Glycemic index model experiments

$\hat{N}$	δ_{max}	Escapes
1	14.17	25
2	12.97	12.92
3	12.3	8.95
4	10.16	6.95
5	9.67	6.38
6	10.12	6.5
7	9.67	6.06
8	9.66	6.0
9	9.59	5.81

3.6 Conclusion

We have presented an algorithm for state estimation of switched nonlinear systems with finite number of modes and unobservable switching signal using quantized measurements with optimality guarantees on the number of bits needed to be sent from the sensor to the estimator. These results suggest several future research directions including extensions to hybrid models with partially known switching structure and developing lower-bounds on estimation entropy.

CHAPTER 4

STATE ESTIMATION OF NONLINEAR SYSTEMS WITH BOUNDED INPUTS: ENTROPY AND BIT RATES

4.1 Introduction

In this chapter, we study the problem of estimating the state of nonlinear dynamical systems with unknown, possibly discontinuous, inputs. This is a much more challenging problem than that of autonomous dynamical systems studied in [13], because even if the uncertainty about the state can be made to decrease over time using sensor measurements, the uncertainty about the input may not decrease. The input can change arbitrarily with few constraints and the continuous effect of the uncertain input prevents the uncertainty about the state from going to zero. We contend this using a weaker notion of estimation, akin to that in [32], that only requires the error to be bounded by a constant $\varepsilon > 0$ instead of exponentially decaying to zero.

We show that there is no state estimation algorithm with a bit rate smaller than the entropy. For the purpose of computing an upper bound, we use a corrected version of a previous result in [42] to upper bound the sensitivity of a trajectory of a nonlinear system to changes in the initial state and in the input signal. Then, we present a procedure that, given sampled states of a trajectory and corresponding sampled values of an input signal, constructs a function that estimates the trajectory. This procedure is of independent interest, as it can be used as an estimation algorithm if the unknown input signal can be sampled. We count the number of trajectories that can be constructed by this procedure for different initial states and input signals, up to a time bound T. The rate of growth of this number gives an upper bound on entropy.

The upper bound is presented in terms of the state and the input dimensions n and m, global bounds on the norm of the Jacobian matrices of the vector field with respect to the state and the input, M_x and M_u, the upper bound on the norm of the input u_{max}, and two constants μ and η that represent how much the input signal is allowed to vary over time. Roughly, η upper bounds the size of the jumps in the input signal and μ constrains the number of large jumps in a short amount of time. We show that if the upper bound on the input norm goes to zero, we recover the upper bound on estimation entropy $\frac{nL_x}{\ln 2}$ computed in [13] for α equal to zero. The entropy upper bound increases logarithmically with u_{max}, quadratically with η, and as $\mu^{2/3}$ with μ, when ε is small. The bound also increases as $O(\varepsilon^{-2})$ as the allowed estimation error ε decreases.

Finally, we compute an upper bound on entropy of systems with linear inputs. We present a better way to compute the sensitivity of the system with respect to changes in the initial state and in the input signal. We show how our results can be used to get sufficient estimation bit rates for two examples.

The chapter is organized as follows: we start by defining the entropy for systems with inputs, which were described in Section 2.3.1, in Section 4.2. Then, we compute the upper bound on entropy for general nonlinear systems in Section 4.3. After that, we compute a new upper bound on entropy for systems where the input affects the dynamics linearly in Section 4.4. Finally, we discuss the results and suggest future directions in Section 4.5.

4.2 Entropy Definition

Let us fix throughout this chapter a compact set K of possible initial states of System (2.4), a bound on the input norm u_{max} and two constants bounding the variation of the input signal μ and η as in Section 2.3.1. This will in turn lead to a specific corresponding set of possible input signals $\mathcal{U}$. We fix the constant $\varepsilon > 0$, which will bound the norm of the estimation error, too.

Given a time bound $T > 0$, initial state $x_0 \in K$ and an input signal $u \in \mathcal{U}$, we say that a function $z : [0, T] \to \mathbb{R}^n$ is ε-*approximating* for the trajectory $\xi_{x_0,u}$ over the interval $[0, T]$, if

$$\|z(t) - \xi_{x_0,u}(t)\| \leq \varepsilon, \tag{4.1}$$

for all $t \in [0, T]$. We say that a set of functions $Z := \{z_i \mid z_i : [0, T] \to \mathbb{R}^n\}$

is (T, ε, K)-approximating for system (2.4), if for every $x_0 \in K$ and $u \in \mathcal{U}$, there exists an ε-approximating function $z_i \in Z$ for the trajectory $\xi_{x_0,u}$ over $[0, T]$. The minimal cardinality of such a set is denoted by $s_{\text{est}}(T, \varepsilon, K)$.

The entropy of System (2.4) is defined as follows:

$$h_{\text{est}}(\varepsilon, K) = \limsup_{T \to \infty} \frac{1}{T} \log s_{\text{est}}(T, \varepsilon, K). \tag{4.2}$$

The entropy $h_{\text{est}}(\varepsilon, K)$ represents the exponential growth in the number of distinguishable trajectories of the system. Hence, it also represents the bit rate need to be sent by the sensor so that the estimator can construct a "good" estimate of the state.

Notice that we do not take the limit as ε goes to zero in the definition of entropy in contrast to the majority of the literature where the limsup as ε goes to zero is taken after taking the limsup as T goes to infinity [31, 10, 13, 14, 7]. That is because we do not expect the entropy to stay finite as ε approaches zero because of the unknown and possibly fast varying input. The upper bounds on entropy we derive in the following sections actually approach infinity as ε approaches zero.

4.2.1 Relation between entropy and the bit rate of estimation algorithms

In this section, we show that there is no state estimation algorithm for System (2.4) that uses bit rate smaller than its entropy as we did in Section 3.2.2 where we showed a similar result for switched system. First, let us define state estimation algorithms given an estimation error bound $\varepsilon > 0$. It is the same as Definition 2 with the only difference being that the estimator $\mathcal{E}$ should output an ε-approximating function for the system trajectory instead of an $(\varepsilon, \tau, \alpha)$-approximating one as there is no switching signal in this case.

Definition 3 A state estimation algorithm for System (2.4) with a fixed bit rate is a pair of functions $(\mathcal{S}, \mathcal{E})$, where $\mathcal{S} : \mathbb{R}^n \times Q_s \to \Gamma \times Q_s$, $\mathcal{E} : \Gamma \times Q_e \to ([0, T_p] \to \mathbb{R}^n) \times Q_e)$, T_p is the sampling time, Γ is an alphabet with N symbols, for some $N \in \mathbb{N}$, and Q_s and Q_e are the sets of internal states of the sensor $\mathcal{S}$ and estimator $\mathcal{E}$, respectively. $\mathcal{S}$ runs at the sensor side and $\mathcal{E}$ on the estimator one. $\mathcal{S}$ samples the state of the system each T_p time units and sends to $\mathcal{E}$ a symbol from Γ representing an estimate of the state at the corresponding sampling time. Finally, $\mathcal{E}$ maps the received symbol to an ε-approximating function of the trajectory for the next T_p time units.

Now, let us define the bit rate of the algorithm. It is the same as that in equation (3.3) with the only difference being that it is parameterized with ε only (in addition to K) instead of being also parametrized with α and τ.

$$b_r(\varepsilon, K) := \limsup_{T\to\infty} \frac{1}{T} \sum_{i=0}^{\lfloor T/T_p \rfloor} \log N = \limsup_{j\to\infty} \frac{1}{jT_p} \sum_{i=0}^{j} \log N = \frac{\log N}{T_p}. \tag{4.3}$$

Proposition 10 *There is no state estimation algorithm for System (2.4) with a fixed bit rate smaller than its entropy.*

Proof The proof is similar to the proof of Proposition 2 in [43] and Proposition 2 in Chapter 3. For the sake of contradiction, assume that there exists such an algorithm with a bit rate smaller than $h_{\text{est}}(\varepsilon, K)$. Recall that $h_{\text{est}}(\varepsilon, K) = \limsup_{T\to\infty} 1/T \log s_{\text{est}}(T, \varepsilon, K)$. Then, for a sufficiently large T', we should have $\frac{(l+1)\log N}{T'} < \frac{1}{T'} \log s_{\text{est}}(T', \varepsilon, K)$, where $l = \lfloor T'/T_p \rfloor$. Hence, we get the inequality $N^{l+1} < s_{\text{est}}(T', \varepsilon, K)$. However, N^{l+1} is the number of possible sequences of symbols of length $l + 1$ that can be sent by the sensor over $l + 1$ iterations. There are $l + 1$ instead of l iterations over the interval $[0, T']$ since the sensor starts sending the codewords at $t = 0$ s. Hence, the number of functions that can be constructed by the estimator is upper bounded by N^{l+1}. Moreover, for any given trajectory of the system, the output of the estimator is a corresponding ε-approximating function over the interval $[0, T']$. This is true since the estimator should be able to construct an ε-approximating function for the corresponding trajectory of the system over the interval $[0, (l + 1)T_p)$ given the codewords sent by the sensor in the first $l + 1$ iterations. Hence, the set of functions that can be constructed by the estimator defines a (K, ε, T')-approximating set. But, $s_{\text{est}}(T', \varepsilon, K)$ is the minimal cardinality of such a set. Therefore, the set of functions that can be constructed by the algorithm defines a (T', ε, K)-approximating set which has a cardinality smaller than s_{est}, the supposed minimal one.

4.3 Entropy Upper Bound and Algorithm

In this section, we derive an upper bound on the entropy of System (2.4) in terms of its parameters and the required bound on the estimation error, ε. To do that, we will need to first upper bound the distance between any two trajectories of System (2.4) in terms of the distance between the initial states and the distance between the input signals. This will be done in Section 4.3.1. Then, in Section 4.3.2, we will describe a procedure that, given $\varepsilon > 0$, a time bound $T > 0$, an initial state $x_0 \in K$ and an input $u \in \mathcal{U}$, constructs an ε-approximating function for the trajectory $\xi_{x_0,u}$ over the interval $[0, T]$. We will count the number of functions that can be produced by this procedure for any fixed ε and T (and varying $x_0 \in K$ and $u \in \mathcal{U}$) to upper bound the cardinality of the minimal approximating set. This will be used to derive the upper bound on entropy in Section 4.3.3.

4.3.1 Input-to-State Discrepancy Function Construction

Here we correct and use a method for construction of local input-to-state discrepancy function (with proof in the Appendix). This is a straight-forward generalization of Lemma 15 of [42] to handle systems with piece-wise continuous inputs and Jacobian matrices of f (instead of continuous ones).

Lemma 6 *The function $V(x, x') := \|x - x'\|^2$ is a local IS discrepancy for System (2.4) over any compact set $\mathcal{X} \subset \mathbb{R}^n$ and interval $[t_0, t_1] \subseteq \mathbb{R}_{\geq 0}$, with*

$$\beta(y, t - t_0) := e^{2a(t-t_0)} y^2 \ \textit{and}\ \gamma(y) := b^2 e^{2a(t_1 - t_0)} y^2,$$

where

$$t \in [t_0, t_1], a := \sup_{\substack{t \in [t_0,t_1] \\ u \in \mathcal{U}, x \in \mathcal{X}}} \lambda_{max}\Big(\frac{J_x + J_x^T}{2}\Big) + \frac{1}{2} \ \textit{and}\ b := \sup_{\substack{t \in [t_0,t_1] \\ u \in \mathcal{U}, x \in \mathcal{X}}} |||J_u|||. \tag{4.4}$$

Since f is globally Lipschitz continuous in both arguments, one can infer that a and b are finite over all the input and state spaces. We will denote a global upper bound on a by M_x and on b by M_u. An example of such bounds is presented in the following proposition, with the proof being in the Appendix.

Proposition 11 *For any time interval $[t_0, t_1] \subset \mathbb{R}_{\geq 0}$ and compact set $\mathcal{X} \subset \mathbb{R}^n$, $a \leq nL'_x + \frac{1}{2}$ and $b \leq m\sqrt{m}L'_u$, where L'_x and L'_u are the Lipschitz constants of f with respect to each coordinate of the state and the input respectively.*

Therefore, for any $\tau > 0$, $t \in [0, \tau]$, $x_0, x'_0 \in \mathbb{R}^n$, and $u, u' \in \mathcal{U}$, the distance between the trajectories of $\xi_{x_0,u}$ and $\xi_{x'_0,u'}$, $\|\xi_{x_0,u}(t) - \xi_{x'_0,u'}(t)\|^2$, is upper bounded by:

$$e^{2M_x t}\|x_0 - x'_0\|^2 + M_u^2 e^{2M_x \tau} \int_0^\tau \|u(s) - u'(s)\|^2 ds. \tag{4.5}$$

Further if f has a continuous Jacobian, one can get tighter local bounds on a and b that depend on the set of input functions $\mathcal{U}$, the compact set $\mathcal{X}$, and the interval $[t_0, t_1]$.

4.3.2 Approximating set construction

Let us fix $\varepsilon > 0$ throughout this section. We will describe a procedure (Algorithm 3) that, given a time bound $T > 0$, an initial state $x_0 \in K$ and an input signal $u \in \mathcal{U}$, constructs an ε-approximating function for the trajectory $\xi_{x,u}$ over the time interval $[0, T]$. It follows that the set of functions that can possibly be constructed by that procedure for different $x_0 \in K$ and $u \in \mathcal{U}$ is a (T, ε, K)-approximating set for System (2.4). An upper bound on its cardinality will give an upper bound on entropy in the next section.

The procedure (Algorithm 3) is parameterized by a time horizon $T > 0$, a sampling period $T_p > 0$, two quantization constants δ_x and $\delta_u > 0$. The procedure also uses the initial set K, the input set U, and particular initial state $x_0 \in K$ and input $u \in \mathcal{U}$ for system (2.4). The output is a piece-wise continuous function $z : [0, T] \to \mathbb{R}^n$ that is constructed over each $[iT_p, (i+1)T_p)$ interval for $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$. Later we will infer several constraints on the parameters such that the output z is indeed an ε-approximating function for the given trajectory $\xi_{x_0,u}$.

Initially, S_0 is set to be the initial set K. $C_{x,0}$ is a grid of size δ_x over K and C_u is a grid of size δ_u over U. At the i^{th} iteration, $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$, x_i stores the

Algorithm 3 Construction of ε-approximating function.

1: input: T,T_p, δ_x, δ_u
2: $S_0 \leftarrow K$;
3: $C_{x,0} \leftarrow \ grid(S_0, \delta_x)$;
4: $C_u \leftarrow \ grid(U, \delta_u)$;
5: $i \leftarrow 0$;
6: **while** $i \leq \lfloor \frac{T}{T_p} \rfloor$ **do**
7: $\quad x_i \leftarrow \xi_{x_0,u}(iT_p)$;
8: $\quad q_{x,i} \leftarrow quantize(x_i, C_{x,i})$;
9: $\quad q_{u,i} \leftarrow quantize(u(iT_p), C_u)$;
10: $\quad z_i \leftarrow \xi_{q_{x,i},q_{u,i}}$;
11: $\quad$ i++; $\triangleright$ parameters for next iteration
12: $\quad S_i \leftarrow B(z_{i-1}({T_p}^-), \varepsilon)$;
13: $\quad C_{x,i} \leftarrow grid(S_i, \delta_x)$;
14: $\quad$ wait(T_p);
15: **end while**
16: output: $\{z_i : 0 \leq i \leq \lfloor \frac{T}{T_p} \rfloor\}$

value $\xi_{x_0,u}(iT_p)$. Then, $q_{x,i}$ is set to be the quantization of x_i with respect to $C_{x,i}$. Similarly, $q_{u,i}$ is set to be the quantization of $u(iT_p)$ with respect to C_u. With slight abuse of notation, we will also denote the function of time that maps the interval $[0, T_p)$ to $q_{u,i}$ by $q_{u,i}$, as in line 10, for example. The variable z_i stores the trajectory that results from running System (2.4) starting from initial state $q_{x,i}$, with input signal $q_{u,i}$, and running for T_p time units. After that, i is incremented by 1 and the next iteration variables S_i and $C_{x,i}$ are initialized. Finally, the procedure outputs the concatenation of the z_i's, for all $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$ that is denoted later by the function $z : [0, T] \to \mathbb{R}^n$.

In the following lemma, we show that if the parameters of the procedure T_p, δ_x and δ_u, are small enough, then the output is an ε-approximating function for $\xi_{x_0,u}$.

Lemma 7 *Fix $\varepsilon > 0$ and a constant $k \in (0, 1)$. Then, choose the parameters T_p, δ_x, and δ_u, such that:*

1. $\varepsilon\sqrt{k} \geq \delta_x e^{M_x T_p}$, *and*
2. $\varepsilon\sqrt{(1-k)} \geq M_u e^{M_x T_p} \sqrt{\frac{1}{3}\mu^2 {T_p}^3 + (\delta_u + \eta)\mu {T_p}^2 + (\delta_u + \eta)^2 T_p}$.

Then, for any $x_0 \in K$ and $u \in \mathcal{U}$, for all $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$, and for all $t \in [iT_p, (i+1)T_p)$, *(i)* $x_i \in S_i$,

(ii) $\|z_i(t - iT_p) - \xi_{x_i,u_i}(t - iT_p)\| \le \varepsilon$,

where $u_i(t) := u(iT_p + t)$, *the* i^{th} *piece of the input signal of size* T_p.

Proof

First, fix $t \in [0, T]$ and let $i = \lfloor \frac{t}{T_p} \rfloor$. Then,

$$
\begin{aligned}
&\|\xi_{x_i,u_i}(t - iT_p) - \xi_{q_{x,i},q_{u,i}}(t - iT_p)\|^2 \\
&\le \|x_i - q_{x,i}\|^2 e^{2M_x(t-iT_p)} + M_u^2 e^{2M_xT_p} \int_{iT_p}^{t} \|u(s) - q_{u,i}\|^2 ds \\
&\qquad\qquad\qquad \text{[by (4.5)]} \\
&\le \|x_i - q_{x,i}\|^2 e^{2M_x(t-iT_p)} \\
&\quad + M_u^2 e^{2M_xT_p} \int_{iT_p}^{t} \big(\|u_i(0) - q_{u,i}\| + \|u(s) - u_i(0)\|\big)^2 ds \\
&\qquad\qquad\qquad \text{[by triangular inequality]} \\
&\le \delta_x^2 e^{2M_x(t-iT_p)} \\
&\quad + M_u^2 e^{2M_xT_p} \int_{iT_p}^{t} \big(\delta_u^2 + 2\delta_u\|u(s) - u_i(0)\| + \|u(s) - u_i(0)\|^2\big) ds, \qquad (4.6)
\end{aligned}
$$

where the last inequality follows from the fact that $\|u(iT_p) - q_{u,i}\| \le \delta_u$, $\|x_i - q_{x,i}\| \le \delta_x$. But, we know from (2.5) that there exist μ and η such that for all $u \in \mathcal{U}$, $\|u(s) - u(iT_p)\| \le \mu(s - iT_p) + \eta$. Hence, $\int_{iT_p}^{t} \|u(s) - u_i(0)\| ds \le \int_{iT_p}^{t} (\mu(s - iT_p) + \eta) ds = \frac{\mu}{2}(t - iT_p)^2 + \eta(t - iT_p) \le \frac{\mu}{2} {T_p}^2 + \eta T_p$, since $t - iT_p \le T_p$. Similarly, $\int_{iT_p}^{t} \|u(s) - u_i(0)\|^2 ds \le \int_{iT_p}^{t} (\mu^2(s - iT_p)^2 + 2\mu\eta(s - iT_p) + \eta^2) ds \le \frac{1}{3}\mu^2 {T_p}^3 + \mu\eta {T_p}^2 + \eta^2 T_p$. Substituting this in (4.6) leads to:

$$
\begin{aligned}
&\|\xi_{x_i,u_i}(t - iT_p) - \xi_{q_{x,i},q_{u,i}}(t - iT_p)\|^2 \\
&\le \delta_x^2 e^{2M_xT_p} + M_u^2 e^{2M_xT_p} \big(\frac{1}{3}\mu^2 {T_p}^3 + (\delta_u + \eta)\mu {T_p}^2 + (\delta_u + \eta)^2 T_p\big) \\
&\le k\varepsilon^2 + (1 - k)\varepsilon^2 = \varepsilon^2,
\end{aligned}
$$

where the last inequality follows by substituting δ_x, δ_u and T_p by their upper bounds stated in the statement of the lemma. Hence, for any $t \in [0, T]$, for $i = \lfloor \frac{t}{T_p} \rfloor$, $\|z_i(t - iT_p) - \xi_{x_i,u_i}(t)\| \le \varepsilon$. Therefore, for all $i \in [1; \lfloor \frac{T}{T_p} \rfloor]$ and $t \in [0, T]$, $x_i \in B(z_{i-1}(T_p), \varepsilon) = S_i$.

Corollary 3 *Under the same conditions of Lemma 7, for all $t \in [0, T]$,*

$$\|z(t) - \xi_{x_0,u}(t)\| \leq \varepsilon. \tag{4.7}$$

Now that we proved that, for a given trajectory $\xi_{x_0,u}$, the output of Algorithm 3 is an ε-approximating function, one can conclude that the set of all functions that can be constructed by Algorithm 3 for any input trajectory $\xi_{x_0,u}$, where $x_0 \in K$ and $u \in \mathcal{U}$, is a (T, ε, K)-approximating set. Therefore, in the following lemma, we will compute an upper bound on the number of these functions to obtain upper bound on $s_{\text{est}}(T, \varepsilon, K)$.

Before stating the lemma, note that whenever we choose k, we let $\delta_x = \varepsilon\sqrt{k}e^{-M_x T_p}$ from now on, in order to simplify the presentation.

Lemma 8 *For fixed $T \geq 0$, $k \in (0, 1)$, and δ_u and T_p that satisfy the conditions of Lemma 7, the number of functions that can be constructed by Algorithm 3 for all possible $x_0 \in K$ and $u \in \mathcal{U}$, is upper bounded by:*

$$\begin{aligned}
&|C_{x,0}|(|C_{x,1}||C_u|)^{\lfloor \frac{T}{T_p} \rfloor + 1} \\
&\leq \lceil \frac{diam(K)}{2\varepsilon\sqrt{k}e^{-M_x T_p}} \rceil^n \left(\lceil \frac{1}{\sqrt{k}} e^{M_x T_p} \rceil^n \lceil \frac{u_{max}}{\delta_u} \rceil^m \right)^{\left(\lfloor \frac{T}{T_p} \rfloor + 1\right)}.
\end{aligned}$$

Proof To construct an ε-approximating function for a given trajectory $\xi_{x,u}$, at an iteration $i \in [0; \lfloor \frac{t}{T_p} \rfloor]$, Algorithm 3 picks one point in $C_{x,i}$ and picks one point in C_u for each of the $\lfloor T/T_p \rfloor + 1$ iterations. Hence, the number of different outputs that it can produce is upper bounded by:

$$|C_u|^{\lfloor T/T_p \rfloor + 1} \prod_{i=0}^{\lfloor T/T_p \rfloor} |C_{x,i}|. \tag{4.8}$$

Now, note that $K \subseteq B(v_c, diam(K))$, for some $v_c \in \mathbb{R}^n$. Hence, in each of the n dimensions in the state space, we should partition a segment of length $diam(K)$ to smaller segments of size $2\delta_x = 2k\varepsilon e^{-M_x T_p}$ to construct the grid $C_{x,0}$. Then, $|C_{x,0}| \leq \lceil \frac{diam(K)}{2\sqrt{k}\varepsilon e^{-M_x T_p}} \rceil^n$. Similarly, for all $i > 0$, $S_i = B(z_{i-1}(T_p{}^-), \varepsilon)$. Hence, $|C_{x,i}| \leq \lceil \frac{2\varepsilon}{2\sqrt{k}\varepsilon e^{-M_x T_p}} \rceil^n = \lceil \frac{1}{\sqrt{k}} e^{M_x T_p} \rceil^n$, since $diam(S_i) = 2\varepsilon$. In each of the m dimensions, $u(t)$ is bounded between $-u_{max}$ and u_{max}. Hence, $diam(U) = 2u_{max}$ and $|C_u| \leq \lceil \frac{u_{max}}{\delta_u} \rceil^m$. Substituting these values in (4.8) leads to the upper bound in the lemma.

4.3.3 Entropy upper bound

The following proposition gives an upper bound on the entropy of system (2.4) in terms of k, T_p and δ_u. This form provides an an intermediate level bound where the parameters of Algorithm 3 directly appear in its expression, before providing the more complex upper bound that depends directly on the system parameters. It shows the effect of our choice of the parameters of Algorithm 3. It will also help us recover the bound on estimation entropy of systems with no inputs in [13] in Corollary 4. Moreover, it provides insights about the choice of parameters that simplify the expression of the bound.

Proposition 12 *Fix $k \in (0,1)$. If T_p, δ_x and δ_u satisfy the conditions in Lemma 7, then the entropy $h_{est}(\varepsilon, K)$ of system (2.4) is upper bounded by:*

$$\frac{nM_x}{\sqrt{k}\ln 2} + \frac{n}{T_p}\log(1+\sqrt{k}e^{-M_xT_p}) + \frac{m}{T_p}\log\lceil\frac{u_{max}}{\delta_u}\rceil.$$

PROOF We substitute the upper bound on the cardinality of the minimal approximating set obtained in the previous section in Definition (4.2) to get:

$$\begin{aligned}
h_{\text{est}}(\varepsilon, K) &= \limsup_{T\to\infty}\frac{1}{T}\log s_{\text{est}}(T,\varepsilon) \\
&\leq \limsup_{T\to\infty}\frac{1}{T}\log|C_{x,0}|(|C_{x,1}||C_u|)^{\lfloor\frac{T}{T_p}\rfloor+1} \\
&\qquad\qquad [\text{by Lemma 8}] \\
&\leq \limsup_{T\to\infty}\frac{1}{T}\log(\lceil\frac{1}{\sqrt{k}}e^{M_xT_p}\rceil^n\lceil\frac{u_{max}}{\delta_u}\rceil^m)^{(\lfloor\frac{T}{T_p}\rfloor+1)} \\
&\qquad\qquad [|C_{x,0}| \text{ is constant}] \\
&= \limsup_{T\to\infty}\frac{1+{}^{T_p}/_{T}}{T_p}n\log\lceil\frac{1}{\sqrt{k}}e^{M_xT_p}\rceil \\
&\qquad + \limsup_{T\to\infty}\frac{1+{}^{T_p}/_{T}}{T_p}m\log\lceil\frac{u_{max}}{\delta_u}\rceil \\
&\leq \frac{n}{T_p}\log\lceil\frac{1}{\sqrt{k}}e^{M_xT_p}\rceil + \frac{m}{T_p}\log\lceil\frac{u_{max}}{\delta_u}\rceil \\
&\leq \frac{nM_x}{\sqrt{k}\ln 2} + \frac{n}{T_p}\log(1+\sqrt{k}e^{-M_xT_p}) + \frac{m}{T_p}\log\lceil\frac{u_{max}}{\delta_u}\rceil.
\end{aligned}$$

We show that if the bound on the input norm is negligible, we recover the upper bound on estimation entropy of $\frac{nL_x}{\ln 2}$ derived in [13] with the only difference being the replacement of L_x by M_x (which is upper bounded by

$nL_x + 1/2$).

Corollary 4 *Given any* $\varepsilon > 0$, $\lim\limits_{u_{max}\to 0} h(\varepsilon, K) \leq \frac{nM_x}{\ln 2}$.

Proof (Sketch) First, recall that setting η to $2u_{max}$ and μ to zero satisfies (2.5). We will fix them to these values in this proof. Let k be approximately equal to 1. Moreover, fix δ_u to be equal to u_{max}. Doing this will set the last term ($\log\lceil\frac{u_{max}}{\delta_u}\rceil$) in the bound in Proposition 12 to zero. Recall that we also fixed δ_x to be equal to $k\varepsilon e^{-M_x T_p}$. Now, observe that there exists a $T_p > 0$ that would satisfy the conditions in Lemma 7. Hence, by Proposition 12, $h_{\text{est}}(\varepsilon, K) \leq \frac{nM_x}{\sqrt{k}\ln 2} + \frac{n}{T_p}\log(1 + \sqrt{k}e^{-M_x T_p})$. Moreover, as u_{max} decreases to zero, η and δ_u go to zero. Hence, the conditions of Lemma 7 become satisfied with larger values of T_p. This would result in a negligible second term in the bound which in turn results in an upper bound of $\frac{nM_x}{\sqrt{k}\ln 2}$ which is almost $\frac{nM_x}{\ln 2}$.

The following proposition presents an upper bound on the entropy of System (2.4). We assume, without loss of generality, that μ and $\eta > 0$. That is not a restrictive choice since, for a given μ and η that satisfy (2.5), any larger values would still satisfy it.

Proposition 13 *Fix* $\varepsilon > 0$ *and let*

$$\rho(k, \delta_u) = \big(\frac{\delta_u + \eta}{\mu}\big)\Bigg(-1 + \sqrt[3]{1 + \big(\frac{\varepsilon}{M_u e}\big)^2 \frac{3\mu(1-k)}{(\delta_u + \eta)^3}}\Bigg). \tag{4.9}$$

Then, the entropy of system (2.4) is upper bounded by:

$$\frac{nM_x}{\sqrt{k}\ln 2} + \frac{1}{\min\{\rho(k, \delta_u), 1/M_x\}}\big(n\log(1 + \sqrt{k}) + m\log\lceil\frac{u_{max}}{\delta_u}\rceil\big).$$

For example, k *and* δ_u *can be* $1/2$ *and* η, *respectively.*

Proof To prove this result, it is sufficient to show that assigning T_p to $\rho(k, \delta_u)$ if it is smaller than $1/M_x$, and to $1/M_x$ otherwise, satisfies condition (2) in Lemma 7. Then, the result will follow from plugging in this value in Proposition 12. First, assume that T_p is less than or equal to $1/M_x$. Then, $e^{M_x T_p}$ in condition (2) in Lemma 7 can be upper bounded by e. Thus, the truth value of condition (2) is equal to that of the following 3$^{\text{rd}}$ order polynomial inequality:

$$T_p{}^3 + 3\big(\frac{\delta_u+\eta}{\mu}\big)T_p{}^2 + 3\big(\frac{\delta_u+\eta}{\mu}\big)^2 T_p - 3(1-k)\big(\frac{\varepsilon}{\mu M_u e}\big)^2 \leq 0. \tag{4.10}$$

The only real root of the polynomial on the LHS is:

$$\big(\frac{\delta_u+\eta}{\mu}\big)\Bigg(-1+\sqrt[3]{1+\big(\frac{\varepsilon}{M_u e}\big)^2\frac{3\mu(1-k)}{(\delta_u+\eta)^3}}\Bigg) = \rho(k,\delta_u). \tag{4.11}$$

Thus, we set T_p to $\rho(k,\delta_u)$, as it is the largest value that satisfies the needed condition. If $\rho(k,\delta_u) > \frac{1}{M_x}$, assigning T_p to $\frac{1}{M_x}$ would still satisfy the conditions of Lemma 7, hence the bound.
The following corollary gives a more concise upper bound if ε is small enough with respect to the other parameters.

Corollary 5 *Let $\nu_1 := \big(\frac{\varepsilon}{M_u e}\big)^2\frac{3\mu(1-k)}{(\delta_u+\eta)^3}$. If $\nu_1 \leq 1$, then the entropy of system (2.4) is upper bounded by:*

$$\frac{nM_x}{\sqrt{k}\ln 2} + \frac{1}{\min\big\{\big(\frac{\delta_u+\eta}{\mu}\big)\frac{\nu_1}{3}(1-\frac{\nu_1}{3}), \frac{1}{M_x}\big\}}\big(n\log(1+\sqrt{k}) + m\log\lceil\frac{u_{max}}{\delta_u}\rceil\big). \tag{4.12}$$

PROOF Since $\nu_1 \leq 1$, $\sqrt[3]{1+\nu_1} > 1+\frac{\nu_1}{3}-\frac{\nu_1^2}{9}$. Then, $\rho(k,\delta_u)$ is lower bounded by:

$$\big(\frac{\delta_u+\eta}{\mu}\big)\Big(-1+1+\frac{\nu_1}{3}-\frac{\nu_1^2}{9}\Big) \geq \big(\frac{\delta_u+\eta}{\mu}\big)\frac{\nu_1}{3}(1-\frac{\nu_1}{3}). \tag{4.13}$$

Thus, if we set T_p to $\min\{\frac{4\nu_1}{3}(1-\frac{\nu_1}{3}), \frac{1}{M_x}\}$, we get $e^{M_x T_p} \leq e$. Moreover, one can easily check that this assignment satisfies the conditions of Lemma 7. If we substitute this value in Proposition 12, we get the corollary.

If the input signal is Lipschitz continuous with Lipschitz constant L_v, then for all $t \geq 0$ and $\tau > 0$, $\|u(t+\tau)-u(t)\| \leq L_v\tau$. This leads to the following corollary.

Corollary 6 *If f is Lipschitz continuous in both arguments with Lipschitz constants L_x and $L_u > 0$ respectively, and the input signal u is Lipschitz continuous with Lipschitz constant L_v, its entropy have the same upper bounds as in Proposition 13 and corollary 5 with μ replaced by L_v and η by zero.*

An example: Harrier jet

We study the Harrier "jump jet" model from [45]. The dynamics of the system is given by:

$$\begin{aligned}
&\dot{x}_1 = x_2;\ \dot{x}_2 = -g\sin\theta_1 - \frac{c}{m'}x_2 + \frac{u_1}{m'}\cos\theta_1 - \frac{u_2}{m'}\sin\theta_1\\
&\dot{y}_1 = y_2;\ \dot{y}_2 = g(\cos\theta_1 - 1) - \frac{c}{m'}y_2 + \frac{u_1}{m'}\sin\theta_1 + \frac{u_2}{m'}\cos\theta_1\\
&\dot{\theta}_1 = \theta_2;\ \dot{\theta}_2 = \frac{r}{J}u_1,
\end{aligned}$$

where (x_1, y_1, θ_1) are the position and the orientation of the center of mass of the aircraft in the vertical plane, and (x_2, y_2, θ_2) are the corresponding time derivatives. The mass of the aircraft is m', the moment of inertia is J, the gravitational constant is g, and the damping coefficient is c. The Harrier uses maneuvering thrusters for vertical take-off and landing. The inputs u_1 and u_2 are the force vectors generated by the main downward thruster and the maneuvering thrusters.

To compute the upper bound on entropy, we need to find the parameters M_x, M_u, u_{max}, μ, and η for the system. To compute M_x, we compute the Lipschitz constant of f with respect to each of the coordinates in the state vector. Then, we use Proposition 11, to get $M_x = nL'_x + \frac{1}{2}$. To compute the Lipschitz constant, we compute the partial derivative of f with respect to each coordinate and use an upper bound on the infinity norm of each of the resulting vectors. We get $L'_x = g + 2\frac{u_{max}}{m'}$ to be the maximum of these norms, and thus $M_x = 6g + 12\frac{u_{max}}{m'} + \frac{1}{2}$. We get $M_u = 2\sqrt{2}L'_u$ in a similar manner.

Fixing $u_{max} = 50$, $m' = 100$, $g = 9.81$, $r = 5$, and $J = 50$, we get $M_x = 83.36$ and $M_u = 0.2828$. Moreover, we choose $\mu = 10$ and $\eta = 20$ and the estimate accuracy $\varepsilon = 0.5$. Therefore, if we choose k to be equal to $\frac{1}{2}$ and δ_u to be equal to η (i.e. 20), then $\nu_1 = 9.915 \times 10^{-5}$, $\big(\frac{\delta_u+\eta}{\mu}\big)\frac{\nu_1}{3}(1 - \frac{\nu_1}{3}) = 1.32 \times 10^{-4} \leq \frac{1}{M_x} = 0.012 \leq 1$. Then, using Corollary 5, we get $h_{\text{est}}(0.5, K) \leq 60017$ bps. We get the same upper bound if we instead use Proposition 13.

4.3.4 Entropy upper bound discussion

In this section we discuss how the bounds in Proposition 13 and Corollary 5 vary as different system parameters vary.

1. The upper bounds in Proposition 13 and Corollary 5 increase quadratically with η. That is expected as larger jumps in the input signal would lead to a higher uncertainty in the system's state.

2. As μ increases, the bound in Corollary 5 will increase in the order of $\frac{1}{1-O(\mu)}$ while the bound in Proposition 13 will increase as $O(\mu^{2/3})$. That is, if all the parameters are treated as constants.

3. The bounds in both the proposition and the corollary increase logarithmically in u_{max}. This means that the growth in the uncertainty in the state estimate because of the increase in the bound on the input is at least exponentially slower than the growth caused by its faster variation.

4. Finally, as ε goes to zero, the upper bound in Proposition 13 grows as $\Omega(\varepsilon^{-2/3})$ and that of Corollary 5 as $\Omega(\varepsilon^{-2})$.

4.4 Systems with Linear Inputs

In this section, we provide better bounds on entropy than that of Proposition 13 for systems where the input signal affects the dynamics linearly. Formally, we consider dynamical systems of the form:

$$\dot{x} = f(x) + u, \tag{4.14}$$

where the initial state $x_0 \in K$ and $u \in \mathcal{U}$, as before.

We will show in the next section a new IS discrepancy function designed to utilize the linear relation between the input and the state dynamics of the system. Then, in the following section, we will use Algorithm 3 to construct ε-approximating functions for the trajectories of this system while utilizing the new IS discrepancy function. After that, we will show that the number of functions that can be constructed by the modified algorithm is the same as that of Lemma 8 in terms of its parameters δ_x, δ_u and T_p. However, larger values of these parameters would suffice to get ε-approximating function. Finally, we will compute the new upper bound on entropy and present an example to show the difference between the two bounds.

4.4.1 Input-to-state discrepancy function construction for systems with linear inputs

We will show that we can get a tighter upper bound on the distance between two different trajectories than that of (4.5). Basically, for any two initial states $x_0, x'_0 \in K$, two input signals $u, u' \in \mathcal{U}$, and for all $t \in \mathbb{R}_{\geq 0}$,

$$
\begin{aligned}
&\|\xi_{x_0,u}(t) - \xi_{x'_0,u'}(t)\| \\
&= \|x_0 + \int_0^t \big(f(\xi_{x_0,u}(s)) + u(s)\big)ds \\
&\qquad - x'_0 - \int_0^t \big(f(\xi_{x'_0,u'}(s)) + u'(s)\big)ds\| \\
&\leq \|x_0 - x'_0\| + \int_0^t \|f(\xi_{x_0,u}(s)) - f(\xi_{x'_0,u'}(s))\|ds \\
&\qquad + \|\int_0^t \big(u(s) - u'(s)\big)ds\| \\
&\qquad\qquad \text{[by triangular inequality]} \\
&\leq \|x_0 - x'_0\| + \int_0^t L_x\|\xi_{x_0,u}(s) - \xi_{x'_0,u'}(s)\|ds \\
&\qquad + \|\int_0^t \big(u(s) - u'(s)\big)ds\| \\
&\qquad\qquad \text{[by the Lipschitz continuity of } f\text{]} \\
&\leq \|x_0 - x'_0\| + \int_0^t L_x\|\xi_{x_0,u}(s) - \xi_{x'_0,u'}(s)\|ds \\
&\qquad + \int_0^t \|u(s) - u'(s)\|ds \\
&\leq \big(\|x_0 - x'_0\| + \int_0^t \|u(s) - u'(s)\|ds\big)e^{L_x t},
\end{aligned} \tag{4.15}
$$

where the last inequality follows from the Bellman-Gronwall inequality. Notice that we have a linear discrepancy function instead of the quadratic one we got in (4.5). This means that the sensitivity of this system with respect to changes in the input is smaller than that of nonlinear systems in general.

4.4.2 Approximating set construction

Let us fix $\varepsilon > 0$ throughout this section. To construct an ε-approximating function for a given trajectory, we use Algorithm 3 again. The following lemma is similar to Lemma 7 as it specifies the conditions that the values of δ_x, δ_u, and T_p should satisfy in order for the output of Algorithm 3 to be an ε-approximating for the input trajectory.

Lemma 9 *If $\delta_x e^{L_x T_p} \leq k\varepsilon$ and $T_p(\frac{\mu T_p}{2} + \eta + \delta_u)e^{L_x T_p} \leq (1-k)\varepsilon$, then, for any $x_0 \in K$ and $u \in \mathcal{U}$, for all $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$, and for all $t \in [iT_p, (i+1)T_p)$,*

(i) $x_i \in S_i$,

(ii) $\|z_i(t - iT_p) - \xi_{x_i,u_i}(t - iT_p)\| \leq \varepsilon$,

where z is the output of Algorithm 3.

Proof Fix $x_0 \in K$ and $u \in \mathcal{U}$ and let $t' = t - iT_p$. Then, by (4.15),

$$
\begin{aligned}
&\|z_i(t) - \xi_{x_i,u_i}(t)\| \\
&\leq \big(\|x_i - q_{x,i}\| + \int_0^{t'} \|u_i(s) - q_{u,i}\| ds\big)e^{L_x t'} \\
&\qquad\qquad \text{[by (4.15)]} \\
&\leq \big(\|x_i - q_{x,i}\| + \int_0^{t'} \big(\|u_i(s) - u_i(0)\| + \|u_i(0) - q_{u,i}\|\big)ds\big)e^{L_x t'} \\
&\qquad\qquad \text{[by triangular inequality]} \\
&\leq \big(\|x_0 - q_{x,i}\| + \int_0^{t'} \|u_i(s) - u_i(0)\| ds + t'\delta_u\big)e^{L_x t'}, \\
&\qquad\qquad \text{[since } \|x_0 - q_{x,i}\| \leq \delta_x,\ \|u_i(0) - q_{u,i}\| \leq \delta_u\text{]} \\
&\leq \big(\|x_0 - q_{x,i}\| + \int_0^{t'} (\mu s + \eta) ds + t'\delta_u\big)e^{L_x t'} \\
&\qquad\qquad \text{[by (2.5)]} \\
&\leq \big(\delta_x + T_p\delta_u + T_p(\frac{\mu T_p}{2} + \eta)\big)e^{L_x T_p} \\
&\qquad\qquad \text{[since } t' \leq T_p\text{]} \\
&\leq k\varepsilon + (1-k)\varepsilon = \varepsilon,
\end{aligned} \tag{4.16}
$$

where the last inequality follows from the assumption in the Lemma on T_p, δ_x and δ_u. Hence, for all $i \in [0, \lfloor \frac{T}{T_p} \rfloor]$ and $t \in [iT_p, (i+1)T_p]$, $x_i \in S_i$ and $\|\xi_{x_i,u_i}(t) - \xi_{q_{x,i},q_{u,i}}(t)\| \leq \varepsilon$.

Corollary 7 *Under the same conditions of Lemma 9, the output z of Algorithm 3 is an ε-approximating function of the corresponding trajectory of* *system (4.14). Moreover, since we are still using Algorithm 3 to construct the approximating function, we have the same upper bound on entropy of system (4.14) as in Proposition 12 in terms of the new values k, δ_u and T_p that satisfy the new constraints.*

4.4.3 Entropy upper bound on systems with linear inputs

It follows from the last corollary in the previous sections that we can substitute the upper bounds on the parameters δ_u, δ_x and T_p assumed in Lemma 9 to get the new upper bound. This is shown in the following proposition.

Proposition 14 *Fix $\varepsilon > 0$ and let*

$$\rho(k, \delta_u) = \big(\frac{\eta + \delta_u}{\mu}\big)\Big(-1 + \sqrt{1 + \frac{2\mu(1-k)\varepsilon}{(\eta+\delta_u)^2}}\Big). \tag{4.17}$$

Then, the entropy of system (2.4) is upper bounded by:

$$\frac{nM_x}{k \ln 2} + \frac{1}{\min\{\rho(k,\delta_u), {}^1\!/\!{}_{L_x}\}}\big(n \log(1+k) + m \log\lceil \frac{u_{max}}{\delta_u} \rceil\big). \tag{4.18}$$

For example, k and δ_u can be $1/2$ and η, respectively.

Proof This proof is almost the same as that of that of Proposition 13. Let us assume first that $T_p \leq 1/L_x$, then $e^{L_x T_p}$ is upper bounded by e. In that case, to get a value of T_p that satisfies the condition of Lemma 9, we solve the following polynomial inequality:

$$\frac{\mu {T_p}^2}{2} + T_p(\eta + \delta_u) - (1-k)\varepsilon \leq 0, \tag{4.19}$$

which has the following roots:

$$\big(\frac{\eta + \delta_u}{\mu}\big)\Big(-1 \pm \sqrt{1 + \frac{2\mu(1-k)\varepsilon}{(\eta+\delta_u)^2}}\Big). \tag{4.20}$$

First, note that the smaller root is negative. Thus, assigning T_p to any value between zero and the larger root, $\rho(k, \delta_u)$ would satisfy the conditions of Lemma 9. Hence, if $\rho(k, \delta_u) \leq 1/L_x$, and we assign T_p to it, we get the first bound in the proposition. If $\rho(k, \delta_u) > 1/L_x$, assigning T_p to $1/L_x$ would still satisfy the conditions of Lemma 9. Hence, we get the second part of the bound.

As before, we can get a more concise bound if ε is small enough with respect to the other parameters. This is shown in the following corollary.

Corollary 8 *Let* $\nu_2 = \frac{2\mu(1-k)\varepsilon}{(\eta+\delta_u)^2}$. *If* $\nu_2 \leq 1$, *then the entropy of system (4.14) is upper bounded by:*

$$\frac{nM_x}{k \ln 2} + \frac{1}{\min\left\{\left(\frac{\delta_u+\eta}{\mu}\right)\frac{\nu_2}{2}(1-\frac{\nu_2}{4}), \frac{1}{L_x}\right\}}\left(n\log(1+k) + m\log\lceil\frac{u_{max}}{\delta_u}\rceil\right). \quad (4.21)$$

Proof Since $\sqrt{1+\nu_2} \geq 1 + \frac{\nu_2}{2} - \frac{\nu_2^2}{8}$ if $\nu \leq 1$, the larger root is lower bounded by:

$$\begin{aligned} &\left(\frac{\eta+\delta_u}{\mu}\right)\left(-1+1+\frac{\nu_2}{2}-\frac{\nu_2^2}{8}\right) \\ &= \left(\frac{\eta+\delta_u}{\mu}\frac{\nu_2}{2}\right)\left(1-\frac{\nu_2}{4}\right). \end{aligned} \quad (4.22)$$

Setting T_p to this value in the conditions of Lemma 9 shows that they are satisfied. Moreover, substituting these values instead of $\rho(k, \delta_u)$ in the bound of Proposition 14 results in the bound.

In the following, we show how to compute the derived upper bound for a standard example in the dynamical systems literature and compare the values of the two upper bounds that we can get for the same example.

A second example: Pendulum

Consider a pendulum system:

$$\dot{x}_1 = x_2;\ \dot{x}_2 = -\frac{Mgl}{I}\sin x_1 + \frac{u}{I},$$

where I is the moment of inertia of the pendulum around the pivot point, u is its input from a DC motor, x_1 is the angular position (with respect to y-axis), x_2 is the angular speed, and l is the length, and M is the mass.

Consider the case when $\frac{Mgl}{I} = 0.98$, $I = 1$, $u_{max} = 2$, $\mu = 0.1$ and $\eta = 1$.

Jacobians of f are:

$$J_x = \begin{bmatrix} 0 & 1 \\ -\frac{Mgl}{I}\cos x_1 & 0 \end{bmatrix} \; J_u = \begin{bmatrix} 0 & 0 \\ 0 & 1/I \end{bmatrix}.$$

Hence, $\|\|J_x\|\|_\infty = 1$ and thus $M_x = \frac{3}{2}$, $\|\|J_u\|\| = \lambda_{max}(J_u^T J_u) = I^2 = 0.96$, and $M_u = 0.96$. We shall compute the entropy bounds for estimation accuracy $\varepsilon = 0.01$.

Hence, if we choose k and δ_u to be equal to $1/2$ and η respectively and use the bound of Proposition 13 or Corollary 5, we get $\nu_1 = 2.75 \times 10^{-7}$ and $\left(\frac{\delta_u+\eta}{\mu}\right)\frac{\nu_1}{3}(1 - \frac{\nu_1}{3}) = 1.836 \times 10^{-6} \leq \frac{1}{M_x} = 0.667$, which means $h_{\text{est}}(0.01, K) \leq$ 1385442 bps. Since the input linearly affects the dynamics, we can also use Proposition 14, which gives $\nu_2 = 2.5 \times 10^{-4}$ and $\left(\frac{\delta_u+\eta}{\mu}\right)\frac{\nu_2}{2}(1 - \frac{\nu_2}{4}) = 5 \times 10^{-3}$ and hence $h_{\text{est}}(0.01, K) \leq 515$ bps. As we can see from this example, the linear bound can be much tighter than that of the nonlinear one.

4.5 Conclusion

We presented a modified notion of topological entropy as a lower bound on the needed bit rate of a communication channel between a sensor and an estimator to construct an estimate of the state of a nonlinear dynamical system with inputs. We computed an upper bound on entropy that is split into two cases based on how large the estimation error is allowed to be and discussed how the different systems parameters affect it. We showed that we recover the upper bound on estimation entropy of autonomous systems in [13] as the bound on the input decreases to zero. We showed an example of computing the upper bound for a Harrier jet. Then, we presented a new upper bound for systems with linear inputs. We showed the difference with the previous upper bound in a pendulum example. In the future, we plan to apply this theory to get bounds on the entropy of switched systems with bounded and average dwell times and to apply it to a network of dynamical systems. Moreover, we plan to extend this theory to switched systems with inputs.

CHAPTER 5

CONCLUSION AND FUTURE WORK

In this thesis, in Chapter 3, we presented first a new notion of estimation entropy for switched nonlinear systems with finite number of modes and unknown switching signals with known minimal dwell time which represents a lower bound on the number of bits needed to describe the behavior of the system on finite intervals, in the limit as the size of the interval goes to infinity. Additionally, we presented an upper bound on the entropy. Then, we presented a state estimation algorithm for switched nonlinear systems using quantized measurements with optimality guarantees on the number of bits needed to be sent from the sensor to the estimator. Finally, we showed the result of applying the algorithm to a few example linear and nonlinear switched systems. In Chapter 4, we first presented a modified notion of topological entropy of dynamical systems with bounded input which lower bounds the number of bits needed to estimate its state up to a specified constant error. Then, we presented an upper bound on the entropy. Finally, we presented an upper bound on the entropy when the input linearly affects the system dynamics.

These results suggest several future research directions. First, we plan to compute lower bounds on the estimation entropy for both kinds of systems. Second, we plan to design state estimation algorithms for systems with inputs with optimality guarantees on the data rate used. Third, we want to extend the work to switched systems with inputs and hybrid systems with resets and guards. Finally, we plan to investigate more the necessity of our finiteness assumption on $d(t)$.

APPENDIX A

PROOF OF LEMMAS

This generalization of the mean-value theorem is used in the construction of the local IS discrepancy functions in [42] restricted to time-invariant systems rather than general time variant ones.

Proposition 15 *For any differentiable $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, for any $x, x' \in \mathbb{R}^n$, any $u, u' \in \mathbb{R}^m$:*

$$\begin{aligned} f(x', u') - f(x, u) = &\left(\int_0^1 J_x(x + (x' - x)s, u')ds \right)(x' - x) \\ &+ \left(\int_0^1 J_u(x, u + (u' - u)\tau)d\tau \right)(u' - u). \end{aligned}$$

Lemma 6 *The function $V(x, x') := \|x - x'\|^2$ is a local IS discrepancy for System (2.4) over any compact set $\mathcal{X} \subset \mathbb{R}^n$ and interval $[t_0, t_1] \subseteq \mathbb{R}_{\geq 0}$, with*

$$\beta(y, t - t_0) := e^{2a(t-t_0)}y^2 \text{ and } \gamma(y) := b^2 e^{2a(t_1-t_0)}y^2,$$

where

$$t \in [t_0, t_1], a := \sup_{\substack{t \in [t_0,t_1] \\ u \in \mathcal{U}, x \in \mathcal{X}}} \lambda_{max}\left(\frac{J_x + J_x^T}{2}\right) + \frac{1}{2} \text{ and } b := \sup_{\substack{t \in [t_0,t_1] \\ u \in \mathcal{U}, x \in \mathcal{X}}} \vert\vert\vert J_u \vert\vert\vert. \tag{4.4}$$

Proof Let x and $x' \in K$, and u and $u' \in \mathcal{U}$. Define $y(t) = \xi_{x',u'}(t) - \xi_{x,u}(t)$

and $v(t) = u'(t) - u(t)$. For a $t \in \mathbb{R}_{\geq 0}$, using proposition (15), we have

$$\begin{aligned}
\dot{y}(t) &= f(\xi_{x',u'}(t), u'(t)) - f(\xi_{x,u}(t), u(t)), \\
&= \Big(\int_0^1 J_x(\xi_{x,u}(t) + sy(t), u'(t))ds\Big)y(t) \\
&\qquad + \Big(\int_0^1 J_u(\xi_{x,u}(t), u(t) + v(t)\tau)d\tau\Big)v(t). \qquad \text{(A.1)}
\end{aligned}$$

We write $J_x(\xi_{x,u}(t)+sy(t), u'(t))$ as $J_x(t,s)$ or simply J_x when the dependence on t and s is clear from context. Similarly, $J_u(\xi_{x,u}(t), u(t) + v(t)\tau)$ is written as $J_u(t,\tau)$ or J_u. Then, differentiating $\|y(t)\|^2$ with respect to t leads to:

$$\begin{aligned}
\frac{d}{dt}\|y(t)\|^2 &= \frac{d}{dt}(y(t)^T y(t)) = \dot{y}(t)^T y(t) + y(t)^T \dot{y}(t) \\
&= y(t)^T\big(\int_0^1 (J_x^T + J_x)ds\big)y(t) + v(t)^T\big(\int_0^1 J_u^T d\tau\big)y(t) \\
&\qquad + y(t)^T\big(\int_0^1 J_u d\tau\big)v(t) \\
&\qquad\qquad [\text{substituting } \dot{y}(t) \text{ with (A.1)}] \\
&\leq y(t)^T\big(\int_0^1 (J_x^T + J_x)ds\big)y(t) + y(t)^T y(t) \\
&\qquad + \big(\big(\int_0^1 J_u d\tau\big)v(t)\big)^T\big(\big(\int_0^1 J_u d\tau\big)v(t)\big), \qquad \text{(A.2)}
\end{aligned}$$

where the inequality follows from the fact that for all $w, z \in \mathbb{R}^n$, $w^T z + z^T w \leq w^T w + z^T z$, since $0 \leq (z-w)^T(z-w) = z^T z - w^T z - z^T w + w^T w$. Let $\lambda_J(\mathcal{X}) = \sup_{x \in \mathcal{X}} \lambda_{max}(\frac{J_x + J_x^T}{2})$ be the upper bound of the eigenvalues of the symmetric part of J_x over $\mathcal{X}$, so $J_x + J_x^T \preceq 2\lambda_J(K)I$. Thus, (A.2) becomes:

$$\begin{aligned}
\frac{d}{dt}\|y(t)\|^2 &\leq (2\lambda_J(\mathcal{X}) + 1)\|y(t)\|^2 + \|\big(\int_0^1 J_u d\tau\big)v(t)\|^2 \\
&\leq 2a\|y(t)\|^2 + (b\|v(t)\|)^2,
\end{aligned}$$

for $t \in [t_0, t_1]$. Integrating both sides of the above inequality from t_0 to t and using Bellman-Gronwall inequality results in:

$$\|y(t)\|^2 \leq e^{2at}\big(\|y(0)\|^2 + \int_0^t (b\|v(\tau)\|)^2 d\tau\big).$$

Proposition 11 *For any time interval* $[t_0, t_1] \subset \mathbb{R}_{\geq 0}$ *and compact set* $\mathcal{X} \subset \mathbb{R}^n$, $a \leq nL'_x + \frac{1}{2}$ *and* $b \leq m\sqrt{m}L'_u$, *where* L'_x *and* L'_u *are the Lipschitz constants of* f *with respect to each coordinate of the state and the input respectively.*

Proof First, J_u and J_x exist since f is differentiable in both arguments. Second, note that $\lVert\!| J_u |\!\rVert \leq \sqrt{m}\lVert\!| J_u |\!\rVert_\infty$, where $\lVert\!| J_u |\!\rVert_\infty = \max_{i\in[n]} \sum_{j=1}^{m} |(J_x)_{i,j}|$, and $(J_u)_{i,j}$ is the entry in the i^{th} row and j^{th} column of J_u. Moreover, since for all $i \in [n], j \in [m]$, $|(J_u)_{i,j}| \leq L'_u$, by Lipschitz continuity of f with respect to u, then $\lVert\!| J_u |\!\rVert_\infty \leq mL'_u$. Hence, $\lVert\!| J_u |\!\rVert \leq m\sqrt{m}L'_u$. Similarly, one can prove that $\lVert\!| J_x |\!\rVert_\infty \leq nL'_x$, since the number of columns is n instead of m. Therefore,

$$
\begin{aligned}
&a \leq \lVert\!| \frac{J_x + J_x^T}{2} |\!\rVert_\infty + \frac{1}{2} \leq \frac{\lVert\!| J_x |\!\rVert_\infty + \lVert\!| J_x^T |\!\rVert_\infty}{2} + \frac{1}{2} \leq nL'_x + \frac{1}{2}, \text{ and,}\\
&b \leq m\sqrt{m}L'_u.
\end{aligned}
$$

REFERENCES

[1] Bosch, "CAN specification version 1.0," Robert Bosch Gmbh, Postfach 50, Stuttgart, Tech. Rep., September 1991.

[2] C. Steiner, "Bot in the delivery:kiva systems," *Forbes Magazine*, March 2009, http://www.forbes.com/forbes/2009/0316/040_bot_time_saves_nine.html.

[3] D. J. Stilwell and B. E. Bishop, "Platoons of underwater vehicles," *IEEE Control Systems*, vol. 20, no. 6, pp. 45–52, Dec 2000.

[4] E. A. Wan and A. T. Nelson, "Dual Kalman filtering methods for nonlinear prediction, smoothing and estimation," in *NIPS 9, Denver, CO, USA, December 2-5, 1996*, 1996, pp. 793–799.

[5] R. Wilson and L. Finkel, "A neural implementation of the Kalman filter," in *NIPS 22*, Y. Bengio, D. Schuurmans, J. D. Lafferty, C. K. I. Williams, and A. Culotta, Eds. Curran Associates, Inc., 2009, pp. 2062–2070. [Online]. Available: http://papers.nips.cc/paper/3665-a-neural-implementation-of-the-kalman-filter.pdf

[6] R. Van Der Merwe, A. Doucet, N. De Freitas, and E. Wan, "The unscented particle filter," in *NIPS*, vol. 2000, 2000, pp. 584–590.

[7] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[8] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[9] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: tightest attainable bounds," *Systems Control Lett.*, vol. 41, pp. 49–56, 2000.

[10] F. Colonius and C. Kawan, "Invariance entropy for control systems," *SIAM J. Control Optim.*, vol. 48, pp. 1701–1721, 2009.

[11] G. N. Nair, "A nonstochastic information theory for communication and state estimation," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1497–1510, 2013.

[12] J. Schmidt, "Topological entropy bounds for switched linear systems with lie structure," M.S. thesis, Electrical and Computer Engineering from the University of Illinois at Urbana-Champaign, 2016.

[13] D. Liberzon and S. Mitra, "Entropy and minimal data rates for state estimation and model detections," in *HSCC 2016*. Vienna: ACM, April 2016.

[14] D. Liberzon and S. Mitra, "Entropy notions for state estimation and model detection with finite-data-rate measurements," in *CDC*, Las Vegas, 2016.

[15] D. Liberzon and S. Mitra, "Entropy and minimal bit rates for state estimation and model detection," *IEEE Transactions on Automatic Control*, vol. PP, no. 99, pp. 1–1, 2017.

[16] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: tightest attainable bounds," *Systems & Control Letters*, vol. 41, no. 1, pp. 49 – 56, 2000. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0167691100000372

[17] G. N. Nair and R. J. Evans, "Mean square stabilisability of stochastic linear systems with data rate constraints," in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, vol. 2, Dec 2002, pp. 1632–1637 vol.2.

[18] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279–1289, Jul 2000.

[19] H. Ishii and B. A. Francis, *Limited Data Rate in Control Systems with Networks*. Springer-Verlag Berlin Heidelberg, 2002.

[20] A. S. Matveev and A. V. Savkin, "An analogue of shannon information theory for networked control systems. stabilization via a noisy discrete channel," in *2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No.04CH37601)*, vol. 4, Dec 2004, pp. 4491–4496 Vol.4.

[21] A. S. Matveev and A. V. Savkin, "An analogue of shannon information theory for networked control systems: State estimation via a noisy discrete channel," in *2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No.04CH37601)*, vol. 4, Dec 2004, pp. 4485–4490 Vol.4.

[22] A. S. Matveev and A. V. Savkin, "Multirate stabilization of linear multiple sensor systems via limited capacity communication channels," *SIAM Journal on Control and Optimization*, vol. 44, no. 2, pp. 584–617, 2005. [Online]. Available: https://doi.org/10.1137/S0363012902419965

[23] S. C. Tatikonda, "Control under communication constraints," Ph.D. dissertation, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, June 2000.

[24] A. N. Kolmogorov, "A new metric invariant of transient dynamical systems and automorphisms in lebesgue spaces," *Dokl. Akad. Nauk SSSR (N.S.)*, vol. 119, no. 5, pp. 861–864, 1958.

[25] A. N. Kolmogorov, "Entropy per unit time as a metric invariant of automorphisms," *Dokl. Akad. Nauk SSSR (N.S.)*, vol. 124, no. 54, pp. 754–755, 1959.

[26] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, no. 3, pp. 379–423, 1948. [Online]. Available: http://dx.doi.org/10.1002/j.1538-7305.1948.tb01338.x

[27] R. L. Adler, A. G. Konheim, and M. McAndrew, "Topological entropy," *Transactions of the American Mathematical Society*, vol. 114, no. 2, pp. 309–319, 1965.

[28] G. N. Nair, R. J. Evans, I. M. Y. Mareels, and W. Moran, "Topological feedback entropy and nonlinear stabilization," *IEEE Trans. Automat. Control*, vol. 49, pp. 1585–1597, 2004.

[29] F. Colonius, C. Kawan, and G. Nair, "A note on topological feedback entropy and invariance entropy," *Systems Control Lett.*, vol. 62, pp. 377–381, 2013.

[30] F. Colonius, "Minimal bit rates and entropy for exponential stabilization," *SIAM J. Control Optim.*, vol. 50, pp. 2988–3010, 2012.

[31] A. V. Savkin, "Analysis and synthesis of networked control systems: Topological entropy, observability, robustness and optimal control," *Automatica*, vol. 42, pp. 51–62, 2006.

[32] A. V. Savkin and I. R. Petersen, "Set-valued state estimation via a limited capacity communication channel," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 676–680, April 2003.

[33] C. Kawan and Y. Latushkin, "Some results on the entropy of non-autonomous dynamicalsystems," *Dynamical Systems*, vol. 31, no. 3, pp. 251–279, 2016. [Online]. Available: http://dx.doi.org/10.1080/14689367.2015.1111299

[34] S. Kolyada and L. Snoha, "Topological entropy of nonautonomous dynamical systems," *Random and Computational Dynamics*, vol. 4, no. 2, p. 205, 1996.

[35] M. Rungger and M. Zamani, "Invariance feedback entropy of nondeterministic control systems," in *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control*, ser. HSCC '17. New York, NY, USA: ACM, 2017. [Online]. Available: http://doi.acm.org/10.1145/3049797.3049801 pp. 91–100.

[36] A. U. Awan and M. Zamani, "On a notion of estimation entropy for stochastic hybrid systems," in *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, Sept 2016, pp. 780–785.

[37] D. Liberzon, *Switching in Systems and Control*, ser. Systems and Control: Foundations and Applications. Boston: Birkhauser, June 2003.

[38] J. Hespanha and A. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of 38th IEEE Conference on Decision and Control*, 1999, pp. 2655–2660.

[39] G. Yang and D. Liberzon, "Stabilizing a switched linear system with disturbance by sampled-data quantized feedback," in *American Control Conference (ACC), 2015*. IEEE, 2015, pp. 2193–2198.

[40] X. Gao, D. Liberzon, J. Liu, and T. Basar, "Connections between stability conditions for slowly time-varying and switched linear systems," in *2015 54th IEEE Conference on Decision and Control (CDC)*, Dec 2015, pp. 2329–2334.

[41] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No.99CH36304)*, vol. 3, 1999, pp. 2655–2660 vol.3.

[42] Z. Huang, C. Fan, and S. Mitra, "Bounded invariant verification for time-delayed nonlinear networked dynamical systems," *Nonlinear Analysis: Hybrid Systems*, vol. 23, pp. 211 – 229, 2017. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1751570X16300279

[43] H. Sibai and S. Mitra, "Optimal data rate for state estimation of switched nonlinear systems," in *HSCC 2017*. Pittsburgh, PA: ACM, April 2017.

[44] S. M. Furler, M. Biomed E, E. W. Kraegen, R. H. Smallwood, and D. J. Chisholm, "Blood glucose control by intermittent loop closure in the basal mode: Computer simulation studies with a diabetic model," *Diabetes Care*, vol. 8, no. 6, pp. 553–561, 1985. [Online]. Available: http://care.diabetesjournals.org/content/8/6/553

[45] K. J. Astrom and R. M. Murray, *Feedback Systems: An Introduction for Scientists and Engineers.* Princeton, NJ, USA: Princeton University Press, 2008.