

GENERIC BEHAVIOR OF A MEASURE PRESERVING
TRANSFORMATION

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

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Abstract

We study two different problems: generic behavior of a measure preserving transformation and extending partial isometries of a compact metric space. In Chapter 1, we consider a result of Del Junco–Lemańczyk [4] which states that a generic measure preserving transformation satisfies a certain orthogonality conditions, and a result of Solecki [17] which states that every continuous unitary representations of $L^0(X, \mathbb{T})$ is a direct sum of action by multiplication on measure spaces $(X^{|\kappa|}, \lambda_\kappa)$ where κ is an increasing finite sequence of non-zero integers. The orthogonality conditions introduced by Del Junco–Lemańczyk motivates a condition, which we denote by the DL-condition, on continuous unitary representations of $L^0(X, \mathbb{T})$. We show that the probabilistic (in terms of category) statement of the DL-condition translates to some deterministic orthogonality conditions on the measures λ_κ . Also, we show a certain notion of disjointness for generic functions in $L^0(\mathbb{T})$ and a similar orthogonality conditions to the result of Del Junco–Lemańczyk for a generic unitary operator on a Hilbert space H .

In Chapter 2, we show that for every $\epsilon > 0$, every compact metric space X can be extended to another compact metric space, Y , such that every partial isometry of X extends to an isometry of Y with ϵ -distortion. Furthermore, we show that the problem of extending partial isometries of a compact metric space, X , to isometries of another compact metric space, $X \subseteq Y$, is equivalent to extending partial isometries of X to certain functions in $\text{Homeo}(Y)$ that look like isometries from the point of view of X .

Acknowledgments

I would like to express my appreciation to many people who helped me during my studies and made this thesis possible. I am thankful to the committee members Prof. van den Dries, Prof. Solecki, Prof. Hieronymi, and Prof. Tserunyan for kindly accepting to be on the thesis committee and for their helpful remarks and professional advices. I am also grateful to the kind staff at the math department especially Marci and Paula for their help over the years.

Especially, I am thankful to my knowledgeable advisor Prof. Solecki. I am grateful for Slawek's guidance and for his teachings that greatly broaden my understanding of mathematics. He was a phenomenal mentor and was always helpful through my graduate studies. I very much appreciate many fruitful discussions and numerous kind advices.

I am glad that I had the fortune of working with very friendly and brilliant mathematicians during my studies. I am thankful to Prof. Pourmahdian, Prof. Kiani, Prof. Honari, and Prof. Khosravi who introduced me to advanced mathematics during my undergraduate studies. I would also like to thank Prof. Pourmahdian and Prof. Kiani for their trust and confidence in me that greatly helped me to advance my studies.

Many thanks to my kind friends in Urbana Iman, Amir, Taha and Mohammad. Your support and friendship was instrumental to enhance enjoys of life and ease the hardships on the way. I would like to especially thank Taha for his useful insights and advices that greatly helped me over the years.

Last but not least, I am thankful to my family for their great support. Many thanks to my siblings Fateme, Reza, Farzad and Farshad. Spending time with you was always joyful and your kindness made me energized and thankful. I would also like to acknowledge the significant role that my parents played in my academic and non-academic life. My mother introduced me to and made me interested in mathematics. Her kindness through the years has made me a better person and I am very much thankful for all her efforts. I am also grateful to my father for his support and guidance. I am especially thankful for many discussions and his always objective opinions through the years that shaped my way of thinking.

Finally, I am grateful to God for many blessings in my life and his help in times of need.

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Chapter 1

Generic behavior of a measure preserving transformation

1.1 Introduction

We call a homomorphism $\Pi : \Gamma \rightarrow G$ a **representation** of Γ in G where Γ is a countable group and G is a Polish group. Recall that a topological group G is Polish if its topology is separable and is induced by a complete metric. Although some of the earlier results that we mention here are true in a more general setting, we study generic behavior of a representation of $\Gamma = \mathbb{Z}$ in $G = \text{Aut}(X, \mu)$. That is, we are interested in the behavior of $\overline{\langle T \rangle}$, the closure of the group generated by T , for a generic $T \in \text{Aut}(X, \mu)$. In particular, does there exist a topological group, G , such that for a generic $T \in \text{Aut}(X, \mu)$, $\overline{\langle T \rangle}$ is isomorphic to G . There is a conjecture by Glasner and

Weiss which states that for a generic measure preserving transformation, T ,

$$\overline{\langle T \rangle} \cong L^0(\lambda, \mathbb{T})$$

where λ is the Lebesgue measure. In particular, for a generic measure preserving transformation, T , $\overline{\langle T \rangle}$ is Lévy (note that $L^0(\lambda, \mathbb{T})$ is Lévy).

In the following, we mention some of the well-known results about generic behavior of $T \in \text{Aut}(X, \mu)$. Mellaray–Tsankov [14, Theorem 1.4] proved that for a generic measure preserving transformation T , $\overline{\langle T \rangle}$ is extremely amenable. Recall that a Polish group is called **extremely amenable** if every continuous action of the group on a compact Hausdorff space has a fixed point and note that every Lévy group is extremely amenable. Furthermore, they showed a generalization of the result that for a generic measure preserving transformation, T , $C(T) = \overline{\langle T \rangle}$ where

$$C(T) = \{S \in \text{Aut}(X, \mu) : ST = TS\}.$$

This was originally proved by Chacon–Schwartzbauer [1] and another proof was presented by King [13] who showed that the conclusion holds for a rank-1 transformation (note that a generic measure preserving transformation has rank-1). Glasner–Weiss [5, Theorem 5.2] proved that for a generic measure preserving transformation, $T \in \text{Aut}(X, \mathbb{B}, \mu)$, the action of $\overline{\langle T \rangle}$ on the measure space (X, \mathbb{B}, μ) is whirly. It is noteworthy that actions of a Lévy group are whirly ([5, Theorem 3.11]).

There is evidence to suggest that the conjecture by Glasner and Weiss might be true. For instance, Mellaray–Tsankov [14, Theorem 4.4] considered generic behavior of a unitary operator on a Hilbert space. They showed that

in $\mathcal{U}(H)$, the unitary group of a Hilbert space H with the strong topology, for a generic $u \in \mathcal{U}(H)$

$$\overline{\langle u \rangle} \cong L^0(\mathbb{T}).$$

Furthermore, Solecki [16, Corollary 2] showed that for a generic measure preserving transformation T , $\overline{\langle T \rangle}$ is a continuous homomorphic image of a closed linear subspace of $L^0(\lambda, \mathbb{R})$, where λ is the Lebesgue measure on the set of real numbers.

Del Junco–Lemańczyk [4] proved certain orthogonality conditions for a generic measure preserving transformation. More precisely, they showed that for a generic $T \in \text{Aut}(X, \mu)$, for every $k(1), k(2), \dots, k(l) \in \mathbb{Z}^+$ and $k'(1), k'(2), \dots, k'(l') \in \mathbb{Z}^+$, the convolutions

$$\sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}} \text{ and } \sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l')}}$$

where σ_{T^k} is the maximal spectral type of T^k , are mutually singular, provided that $(k(1), \dots, k(l))$ is not a rearrangement of $(k'(1), \dots, k'(l'))$. Assuming the conjecture by Glasner and Weiss, for a generic $T \in \text{Aut}(\mu)$ we can define a unitary representation of $L^0(\mathbb{T})$ by

$$U_T : L^0(\mathbb{T}) \simeq \overline{\langle T \rangle} \hookrightarrow \text{Aut}(\mu) \subseteq \mathcal{U}(L^2(X, \mu)) \quad \text{where} \quad U_T(f) = f \circ T^{-1}.$$

This suggests some orthogonality conditions for a unitary representation of $L^0(\mathbb{T})$ similar to the orthogonality conditions introduced by Del Junco–Lemańczyk [4]. Specifically, we say a continuous unitary representation of $L^0(\mu, \mathbb{T})$,

$$\Phi : L^0(\mu, \mathbb{T}) \rightarrow \mathcal{U}(H)$$

satisfies the DL-condition if for a generic measurable function $f \in L^0(\mu, \mathbb{T})$, for every $k(1), \dots, k(l) \in \mathbb{Z}^+$, $k'(1), \dots, k'(l') \in \mathbb{Z}^+$, the convolutions

$$\sigma_{\Phi(f)^{k(1)}} * \dots * \sigma_{\Phi(f)^{k(l)}} \text{ and } \sigma_{\Phi(f)^{k'(1)}} * \dots * \sigma_{\Phi(f)^{k'(l')}}$$

are mutually singular, provided that $(k(1), \dots, k(l))$ is not a rearrangement of $(k'(1), \dots, k'(l'))$.

This raises the question that what kind of unitary representations of $L^0(X, \mu, \mathbb{T})$ satisfy the DL-condition. Solecki [17, Theorem 2.1] showed that every continuous unitary representation of $L^0(X, \mu, \mathbb{T})$ is a direct sum of action by multiplication on measure spaces $(X^{|\kappa|}, \lambda_\kappa)$ where κ is an increasing finite sequence of non-zero integers and λ_κ is a finite measure on $X^{|\kappa|}$ whose marginal measures are absolutely continuous with respect to μ . Based on the measures λ_κ , we identify unitary representations of $L^0(X, \mu, \mathbb{T})$ that satisfy the DL-condition. More precisely, in Section 1.3, we prove some notion of disjointness for a generic $f \in L^0(\mathbb{T})$ and in Section 1.4 and 1.7, we use the result of Section 1.3 to show that a unitary representation of $L^0(X, \mu, \mathbb{T})$, $\Phi = \bigoplus \sigma(\kappa, \lambda_\kappa)$, satisfies the DL-condition if and only if we have (see Definition 1.2.1 and Definition 1.2.2)

$$\lambda_{\kappa_1} \times \lambda_{\kappa_2} \times \dots \times \lambda_{\kappa_l} \perp r(\lambda_{\kappa'_1} \times \lambda_{\kappa'_2} \times \dots \times \lambda_{\kappa'_{l'}})$$

for every $(\kappa_1, \kappa_2, \dots, \kappa_l)$, $(\kappa'_1, \kappa'_2, \dots, \kappa'_{l'})$, and $r \in S_t$ where S_t is the permutation group of $\{1, 2, \dots, t\}$, such that

$$k(1)\kappa_1 + k(2)\kappa_2 + \dots + k(l)\kappa_l = r(k'(1)\kappa'_1 + k'(2)\kappa'_2 + \dots + k'(l')\kappa'_{l'})$$

for some non-zero integers $(k(1), k(2), \dots, k(l))$ and $(k'(1), k'(2), \dots, k'(l'))$,

provided that one is not a rearrangement of the other. Note that this equivalence translates the "probabilistic" statement of the DL-condition to a "deterministic" condition on the measures λ_κ .

In Section 1.8, we use the above equivalence to show that the orthogonality conditions proved by Del Junco–Lemańczyk [4] also hold for a generic $u \in \mathcal{U}(H)$, that is, the convolutions

$$\sigma_{u^{k(1)}} * \cdots * \sigma_{u^{k(l)}} \text{ and } \sigma_{u^{k'(1)}} * \cdots * \sigma_{u^{k'(l')}}$$

are mutually singular, provided that $(k(1), k(2), \dots, k(l))$ is not a rearrangement of $(k'(1), k'(2), \dots, k'(l'))$.

1.2 Preliminaries

In this section, we introduce the notation that we use in this chapter. By a standard Borel measure space (X, μ) we mean a standard Borel space, X , equipped with a non-atomic Borel probability measure on X , μ . All such spaces are Borel isomorphic to $([0, 1], \lambda)$, where λ is the Lebesgue measure on the Borel subsets of $[0, 1]$. We denote by $\text{Aut}(X, \mu)$, and sometimes just $\text{Aut}(\mu)$ when X is understood, the group of Borel automorphisms of X which preserve the measure μ , that is for every $T \in \text{Aut}(X, \mu)$ and A , a Borel subset of X , A , $T(A)$, and $T^{-1}(A)$ have the same measure. In which we identify two Borel automorphisms if they coincide μ -almost everywhere, that is they coincide on a full measure subset of X . When we talk about Borel subsets of (X, μ) , we usually consider them up to null sets.

There are two fundamental topologies on $\text{Aut}(X, \mu)$, the weak and the

uniform topology. In this paper, we consider $\text{Aut}(X, \mu)$ with the weak topology. Denote by MALG_μ the measure algebra of μ , that is, the algebra of Borel subsets of X , modulo null sets. It is a Polish Boolean algebra under the topology given by the complete metric

$$d(A, B) = d_\mu(A, B) = \mu(A \Delta B),$$

where Δ is the symmetric difference. The weak topology is the topology generated by the functions

$$T \mapsto T(A), \quad A \in \text{MALG}_\mu.$$

With the weak topology, which we denote by w , $(\text{Aut}(X, \mu), w)$ is a Polish topological group. A compatible left-invariant metric is given by

$$\delta_w(S, T) = \sum 2^{-n} \mu(S(A_n) \Delta T(A_n)),$$

where A_n is a dense subset of MALG_μ .

Let (X, μ) be a standard Borel measure space and G be a topological group. By $L^0(X, \mu, G)$ we denote the topological group of all μ -equivalence classes of μ -measurable functions with values in G . We consider $L^0(X, \mu, G)$ with the pointwise multiplication and the convergence in measure topology. Furthermore, we assume that the measure μ is non-atomic. For a tuple $\kappa = (k_1, k_2, \dots, k_m)$ in \mathbb{Z} and a function f in $L^0(X, \mu, G)$, we define the

function $f^\kappa : X^m \rightarrow G$ by

$$f^\kappa(x_1, x_2, \dots, x_m) = \prod_{i=1}^m (f(x_i))^{k_i}. \quad (1.1)$$

We consider $L^0(X, \mu, \mathbb{T})$, where \mathbb{T} is the unit circle (also denoted by S^1 in the literature). When (X, μ) is understood, we denote this group by $L^0(\mathbb{T})$. Moreover, for $A_1, \dots, A_k \subseteq X$ and $f_1, \dots, f_k \in L^0(\mu, \mathbb{T})$, we define

$$f_1(A_1) \cdots f_k(A_k) := \{f_1(x_1) \cdots f_k(x_k) : \text{for every } x_1 \in A_1, \dots, x_k \in A_k\}.$$

We denote the set of Borel probability measures on a standard Borel space, X , by $\mathbb{P}(X)$. The topology on $\mathbb{P}(X)$ is generated by the following basic open sets: let $\Phi : X \rightarrow \mathbb{R}$ be a bounded and continuous function, $\epsilon > 0$, and r be a real number. Then,

$$U = \{\mu \in \mathbb{P}(X) : \left| \int_{\mathbb{T}} \Phi(z) d\mu - r \right| < \epsilon\}$$

is a basic open set. $\mathbb{P}(X)$ with this topology is a Polish space.

Furthermore, S_m denotes the permutation group of $\{1, 2, \dots, m\}$.

Definition 1.2.1. *Let $r \in S_n$, then r induces three functions, all of which denoted by the same operator name, r :*

1. *a function $r : X^n \rightarrow X^n$ which sends (x_1, \dots, x_n) to $(x_{r(1)}, \dots, x_{r(n)})$,*
2. *a function $r : \mathbb{P}(X^n) \rightarrow \mathbb{P}(X^n)$ which sends $\mu \in \mathbb{P}(X)$ to the measure obtained by permuting coordinates of X^n with respect to r ,*
3. *a function $r : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ which sends the tuple $(k(1), k(2), \dots, k(n))$ to $(k(r(1)), k(r(2)), \dots, k(r(n)))$.*

Definition 1.2.2. Let $\kappa = (k(1), k(2), \dots, k(l))$, $\kappa' = (k'(1), k'(2), \dots, k'(l'))$ be two sequences of integer numbers. We define:

1. $\kappa + \kappa' = (k(1), k(2), \dots, k(l), k'(1), k'(2), \dots, k'(l'))$,
2. for $d \in \mathbb{Z}$, $d\kappa = (dk(1), dk(2), \dots, dk(l))$.

For a Polish space X , we say a property, P , holds for comeagerly many $x \in X$ or P holds for a generic $x \in X$ if the set of points in X with property P is comeager, that is, contains the intersection of countably many dense open subsets of X .

Definition 1.2.3. Let X, Y be two Polish spaces. We call a map $f : X \rightarrow Y$ **category preserving** if inverse image of a comeager subset of Y is comeager in X .

Let H be a separable complex Hilbert space. By $\mathcal{U}(H)$ we denote the group of all unitary operators on H . We consider $\mathcal{U}(H)$ with the strong topology, that is pointwise convergence. This topology coincides with the weak topology on $\mathcal{U}(H)$ defined by

$$T_k \rightarrow_w T \text{ if for all } x, y \in H, \langle T_k(x), y \rangle \rightarrow \langle T(x), y \rangle.$$

Note that $\mathcal{U}(H)$ with the strong topology is a Polish group. Let u be a unitary operator on H and $h \in H$. Then, there exists a unique measure (up to mutual absolute continuity), σ_h , on the circle \mathbb{T} such that for every integer number k

$$\langle u^k(h), h \rangle = \int_{\mathbb{T}} z^k d\sigma_h.$$

This measure is known as the spectral measure of the vector h . Let $C(h)$ be the closure the set of all finite linear combinations of $\{u^k(h)\}_{k=1}^{\infty}$. Then,

there are $\{h_i\}_{i=1}^\infty$ in H such that

1. $C(h_i) \perp C(h_j)$ for $i \neq j$ and $H = \bigoplus C(h_i)$,
2. $\sigma_{h_{i+1}} \ll \sigma_{h_i}$ for every positive integer number i ,
3. such sequence of measures, $\{\sigma_{h_i}\}_{i=1}^\infty$, is unique up to mutual absolute continuity.

The maximal spectral type of u is the largest measure among $\{\sigma_{h_i}\}_{i=1}^\infty$, that is, σ_{h_1} . Note that the maximal spectral type of u is unique up to mutual absolute continuity. We denote the maximal spectral type of u by σ_u . We encourage the reader to review Cornfeld–Fomin–Sinai [3] appendix 2, for more information on the maximal spectral type.

1.3 Generic behavior of a function in $L^0(X, \mu, \mathbb{T})$

Let (X, μ) be a Borel probability measure. If $\alpha, \beta \in \mathbb{T}$ are independent over \mathbb{Q} , then non-zero powers of α and β are distinct, that is, for a generic constant function in $L^0(\mathbb{T})$ non-zero powers are distinct. In this section we show that a similar property holds for a generic function in $L^0(\mathbb{T})$. We use the property to prove the main theorem in the next section.

We need to introduce the following definitions before stating the main theorem of this section. In the following definition, we measure disjointness between functions in $L^0(\mathbb{T})$.

Definition 1.3.1. *Let μ be a Borel probability measure on a standard Borel space X , and*

$$(f_1, \dots, f_m, g_1, \dots, g_n) \in L^0(\mu, \mathbb{T})^{m+n}.$$

We define

$$\begin{aligned} \rho_{m,n}(f_1, \dots, f_m, g_1, \dots, g_n) \\ = \inf\{\mu(A) : f_1(X \setminus A) \cdots f_m(X \setminus A) \cap g_1(X \setminus A) \cdots g_n(X \setminus A) = \emptyset\}. \end{aligned}$$

Note that if $(f_1, \dots, f_m, g_1, \dots, g_n)$ and $(f'_1, \dots, f'_m, g'_1, \dots, g'_n)$ represent the same member of $L^0(\mu, \mathbb{T})^{m+n}$, then there exists $B \subseteq X$ such that $\mu(B) = 0$,

$$f_i \upharpoonright (X \setminus B) = f'_i \upharpoonright (X \setminus B)$$

for every $1 \leq i \leq m$, and

$$g_j \upharpoonright (X \setminus B) = g'_j \upharpoonright (X \setminus B)$$

for every $1 \leq j \leq n$. Therefore, the value of

$$\rho_{m,n}(f_1, \dots, f_m, g_1, \dots, g_n)$$

depends only on the class of $(f_1, \dots, f_m, g_1, \dots, g_n)$ in $L^0(\mu, \mathbb{T})^{m+n}$. Furthermore, when

$$\rho_{m,n}(f_1, \dots, f_m, g_1, \dots, g_n) = 0,$$

we use the notation

$$f_1 \cdots f_m \cap g_1 \cdots g_n \approx \emptyset.$$

Lemma 1.3.2. *Let μ be a Borel probability measure on a standard Borel space X , and*

$$(f_1, \dots, f_m, g_1, \dots, g_n) \in L^0(\mu, \mathbb{T})^{m+n}.$$

If

$$\rho_{m,n}(f_1, \dots, f_m, g_1, \dots, g_n) = 0,$$

then there exists $A \subseteq X$ with $\mu(A) = 0$ such that

$$f_1(X \setminus A) \cdots f_m(X \setminus A) \cap g_1(X \setminus A) \cdots g_n(X \setminus A) = \emptyset.$$

Proof. If

$$\rho_{m,n}(f_1, \dots, f_m, g_1, \dots, g_n) = 0,$$

then for every natural number k we can find a Borel subset of X , A_k , such that $\mu(A_k) \leq \frac{1}{2^k}$ and

$$f_1(X \setminus A_k) \cdots f_m(X \setminus A_k) \cap g_1(X \setminus A_k) \cdots g_n(X \setminus A_k) = \emptyset.$$

Let

$$A = \bigcap_{i=1}^{\infty} \bigcup_{k \geq i} A_k.$$

Then, $\mu(A) = 0$ and for every $x_1, \dots, x_m, y_1, \dots, y_n \notin A$, there exists a natural number k such that $x_1, \dots, x_m, y_1, \dots, y_n \notin A_k$. Thus,

$$f_1(X \setminus A) \cdots f_m(X \setminus A) \cap g_1(X \setminus A) \cdots g_n(X \setminus A) = \emptyset.$$

□

Theorem 1.3.3. *Let μ be a non-atomic Borel probability measure on a standard Borel space X . Then, for every $m, n \in \mathbb{N}$, and sequences $1 \leq k_1, \dots, k_p \leq m$, $1 \leq l_1, \dots, l_q \leq n$*

$$E = \{(f_1, \dots, f_m, g_1, \dots, g_n) \in L^0(\mu, \mathbb{T})^{m+n} : f_{k_1} \cdots f_{k_p} \cap g_{l_1} \cdots g_{l_q} \approx \emptyset\}$$

is comeager.

We need the following lemma and proposition to prove Theorem 1.3.3.

Lemma 1.3.4. *Let μ be a non-atomic Borel probability measure on a standard Borel space X . Then, for every $m, n \in \mathbb{N}$, and sequences of natural numbers $1 \leq k_1, \dots, k_p \leq m$, $1 \leq l_1, \dots, l_q \leq n$*

$$\{(f_1, \dots, f_m, g_1, \dots, g_n) \in L^0(\mu, \mathbb{T})^{m+n} : f_1, \dots, f_m, g_1, \dots, g_n \text{ are} \\ \text{finite step functions and } f_{k_1} \cdots f_{k_p} \cap g_{l_1} \cdots g_{l_q} \approx \emptyset\}$$

is dense.

Proof. We prove the lemma for $(m, n) = (2, 1)$ and the general statement follows with a similar argument. Set $\rho = \rho_{2,1}$ and let $(f_0, g_0, h_0) \in L^0(\mu, \mathbb{T})^3$. We can arbitrarily closely approximate (f_0, g_0) with (f, g) where f, g are finite step functions. Since the range of f and the range of g are finite, $f(X)g(X)$ is finite. Therefore, we can find h so that h is a finite step function, h is arbitrarily close to h_0 , and $f(X)g(X) \cap h(X) = \emptyset$. \square

Proposition 1.3.5. *Let μ be a non-atomic Borel probability measure on a standard Borel space X . Then, for every $k, m, n \in \mathbb{N}$, and sequences $1 \leq k_1, \dots, k_p \leq m$, $1 \leq l_1, \dots, l_q \leq n$*

$$E_k = \{(f_1, \dots, f_m, g_1, \dots, g_n) \in L^0(\mu, \mathbb{T})^{m+n} : \\ \rho_{p,q}(f_{k_1}, \dots, f_{k_p}, g_{l_1}, \dots, g_{l_q}) \geq \frac{1}{k}\}$$

is NWD.

Proof. We prove the proposition for $(m, n) = (2, 1)$ and the general statement follows with a similar argument. Set $\rho = \rho_{2,1}$ and let $U \subseteq L^0(\mu, \mathbb{T})^3$ be

an arbitrary open subset. By Lemma 2.2.1, we can find finite step functions $(f_0, g_0, h_0) \in U$ such that $\rho(f_0, g_0, h_0) = 0$. We show that ρ is continuous at (f_0, g_0, h_0) , that is, for every $\epsilon > 0$ there is an open neighborhood of (f_0, g_0, h_0) , V_ϵ , such that for every $(f, g, h) \in V_\epsilon$ we have $\rho(f, g, h) < \epsilon$. Fix $\epsilon > 0$, we define

$$V_\epsilon = \{(f, g, h) \in L^0(\mu, \mathbb{T})^3 : \int (|f - f_0| + |g - g_0| + |h - h_0|) d\mu < \epsilon^2\}.$$

Note that V_ϵ is an open subset of $L^0(\mu, \mathbb{T})^3$ and $\{V_{\frac{1}{n}}\}_{n=1}^\infty$ is a basis for open neighborhoods of (f_0, g_0, h_0) in $L^0(\mu, \mathbb{T})^3$. For $(f, g, h) \in V$, we define

$$A = \{x : |f(x) - f_0(x)| > \epsilon\},$$

$$B = \{x : |g(x) - g_0(x)| > \epsilon\},$$

$$C = \{x : |h(x) - h_0(x)| > \epsilon\}.$$

Note that since $(f, g, h) \in V_\epsilon$,

$$\mu(A \cup B \cup C) < \epsilon.$$

For $x, y, z \in X \setminus (A \cup B \cup C)$, assuming ϵ is small enough, we have

$$\begin{aligned} |f(x)g(y) - h(z)| &\geq |f_0(x)g_0(y) - h_0(z)| + |h(z) - h_0(z)| + \\ &\quad + |(f(x) - f_0(x))g(y)| + |f_0(x)(g(y) - g_0(y))| \\ &\geq |f_0(x)g_0(y) - h_0(z)| - |f(x) - f_0(x)| - \\ &\quad - |g(y) - g_0(y)| - |h(z) - h_0(z)| \\ &\geq |f_0(x)g_0(y) - h_0(z)| - 3\epsilon > 0. \end{aligned}$$

Note that since $\rho(f_0, g_0, h_0) = 0$, $f_0(x)g_0(y)$ and $h_0(z)$ are distinct and if ϵ is small enough, then

$$|f_0(x)g_0(y) - h_0(z)| - 3\epsilon > 0.$$

Therefore,

$$f\left(X \setminus (A \cup B \cup C)\right)g\left(X \setminus (A \cup B \cup C)\right) \cap h\left(X \setminus (A \cup B \cup C)\right) = \emptyset$$

and

$$\mu(A \cup B \cup C) < \epsilon.$$

Hence, putting the above two equations together we get that for every $(f, g, h) \in V_\epsilon$

$$\rho(f, g, h) < \epsilon.$$

Assuming ϵ is small enough, $V_\epsilon \subseteq U$ since $\{V_{\frac{1}{n}}\}_{n=1}^\infty$ is a basis for open neighborhoods of (f_0, g_0, h_0) , and ρ is less than $\frac{1}{k}$ on V_ϵ . Hence, E_k is not dense in U . Since U is an arbitrary open subset of $L^0(\mu, \mathbb{T})^3$, E_k is NWD. \square

Proof of Theorem 1.3.3. Theorem follows from Proposition 1.3.5 since

$$E^c = \bigcup_{k=1}^\infty E_k. \quad \square$$

It is noteworthy that a similar argument can be repeated to prove the following generalization of Theorem 1.3.3 for non-discrete Polish groups. Note that if a Polish space G is non-discrete, then every non-empty open subset of G is infinite (uncountable).

Theorem 1.3.6. *Let μ be a non-atomic Borel probability measure on a*

standard Borel space X and G be a non-discrete Polish group. Then, for every $m, n \in \mathbb{N}$, and sequences of natural numbers $1 \leq k_1, \dots, k_p \leq m$, $1 \leq l_1, \dots, l_q \leq n$

$$E = \{(f_1, \dots, f_m, g_1, \dots, g_n) \in L^0(\mu, G)^{m+n} : f_{k_1} \cdots f_{k_p} \cap g_{l_1} \cdots g_{l_q} \approx \emptyset\}$$

is comeager.

Definition 1.3.7. Let $\kappa = (u_1, \dots, u_t)$, $\kappa' = (v_1, \dots, v_{t'})$ be tuples in $\mathbb{Z} \setminus \{0\}$, and $R \subseteq S_t$. Recall definition of f^κ from (1.1). We say f^κ and $f^{\kappa'}$ are **almost R -disjoint** if there exists $A \subseteq X$ such that $\mu(A) = 0$ and for all

$$x_1, \dots, x_t, y_1, \dots, y_{t'} \in X \setminus A$$

where x_1, \dots, x_t are pairwise distinct and $y_1, \dots, y_{t'}$ are pairwise distinct, we have

$$f^\kappa(x_1, \dots, x_t) = f^{\kappa'}(y_1, \dots, y_{t'}) \Rightarrow (y_1, \dots, y_{t'}) = r(x_1, \dots, x_t)$$

for some $r \in R$. Note that if $t \neq t'$ (or $R = \emptyset$), then

$$f^\kappa(x_1, \dots, x_t) \neq f^{\kappa'}(y_1, \dots, y_{t'})$$

given that there is no $r \in S_t$ (or $r \in R$) such that

$$(y_1, \dots, y_{t'}) = r(x_1, \dots, x_t).$$

In particular, if $\kappa = \kappa'$, we say f^κ is **almost R -to-one**.

Theorem 1.3.8. Let (X, μ) be a Borel measure space where μ is a finite

non-atomic measure. Let $\kappa = (u_1, \dots, u_t)$, $\kappa' = (v_1, \dots, v_{t'})$ be tuples in $\mathbb{Z} \setminus \{0\}$ and

$$R = \{r \in S_t : (v_1, \dots, v_{t'}) = r(u_1, \dots, u_t)\}.$$

Then, for comeagerly many $f \in L^0(X, \mu, \mathbb{T})$, f^κ and $f^{\kappa'}$ are almost R -disjoint.

Proof. Let

$$\begin{aligned} \Omega = \{(\mathbf{x}, \mathbf{y}) \in X^t \times X^{t'} : & \text{coordinates of } \mathbf{x} \text{ are pairwise distinct} \\ & \text{and coordinates of } \mathbf{y} \text{ are pairwise distinct}\}. \end{aligned}$$

Given $\mathbf{x} = (x_1, \dots, x_t)$ and $\mathbf{y} = (y_1, \dots, y_{t'})$ with $(\mathbf{x}, \mathbf{y}) \in \Omega$, there are unique $\mathbf{i} = (i_1 < \dots < i_w)$ and $\mathbf{j} = (j_1 < \dots < j_w)$ for some natural number w such that

$$\{x_i : i \leq t\} \cap \{y_j : j \leq t'\} = \{x_{i_1}, \dots, x_{i_w}\} = \{y_{j_1}, \dots, y_{j_w}\}.$$

Let $r \in S_w$ be the unique permutation such that

$$(y_{j_1}, \dots, y_{j_w}) = r(x_{i_1}, \dots, x_{i_w}),$$

that is, $x_{i_k} = y_{j_{r(k)}}$ for every $1 \leq k \leq w$. Let $m(\mathbf{x}, \mathbf{y}) = (\mathbf{i}, \mathbf{j}, r)$. We define

$$P_{\mathbf{i}, \mathbf{j}, r} := \{(\mathbf{x}, \mathbf{y}) \in \Omega : m(\mathbf{x}, \mathbf{y}) = (\mathbf{i}, \mathbf{j}, r)\}.$$

Then,

$$\Omega = \bigcup_{\mathbf{i}, \mathbf{j}, r} P_{\mathbf{i}, \mathbf{j}, r}.$$

Since there are finitely many such sets, it is enough to show that given $\mathbf{i}, \mathbf{j}, r$, for comeagerly many $f \in L^0(\mathbb{T})$ there exists $A \subseteq X$ with $\mu(A) = 0$ such that for all $(\mathbf{x}, \mathbf{y}) \in P_{\mathbf{i}, \mathbf{j}, r}$ with $x_i, y_j \notin A$ for every $1 \leq i \leq t, 1 \leq j \leq t'$,

$$\text{if } f^\kappa(x_1, \dots, x_t) = f^{\kappa'}(y_1, \dots, y_{t'}), \text{ then } (y_1, \dots, y_{t'}) = r(x_1, \dots, x_t).$$

Fix $\mathbf{i} = (i_1 < \dots < i_w)$, $\mathbf{j} = (j_1 < \dots < j_w)$, and r . We have

$$P_{\mathbf{i}, \mathbf{j}, r} = \bigcup_{U_1, \dots, U_t, V_1, \dots, V_{t'}} \{(\mathbf{x}, \mathbf{y}) \in \prod_{i=1}^t U_i \times \prod_{j=1}^{t'} V_j : m(\mathbf{x}, \mathbf{y}) = (\mathbf{i}, \mathbf{j}, r)\},$$

where U_i , $1 \leq i \leq t$, V_j , $j \in \{1, \dots, t'\} \setminus \{j_1, \dots, j_w\}$, are pairwise disjoint basic open subsets of X and $U_{i_k} = V_{j_{r(k)}}$ for every $1 \leq k \leq w$. Since there are only countably many different choices for $U_1, \dots, U_t, V_1, \dots, V_{t'}$, it is enough to show that if

$$C = \{(\mathbf{x}, \mathbf{y}) \in \prod_{i=1}^t U_i \times \prod_{j=1}^{t'} V_j : m(\mathbf{x}, \mathbf{y}) = (\mathbf{i}, \mathbf{j}, r)\},$$

then for comeagerly many $f \in L^0(\mathbb{T})$ there exists $A \subseteq X$ with $\mu(A) = 0$ such that for all $(\mathbf{x}, \mathbf{y}) \in C$ with $x_i, y_j \notin A$ for every $1 \leq i \leq t, 1 \leq j \leq t'$,

$$\text{if } f^\kappa(x_1, \dots, x_t) = f^{\kappa'}(y_1, \dots, y_{t'}), \text{ then } (y_1, \dots, y_{t'}) = r(x_1, \dots, x_t).$$

Fix C with sequences U_i , $1 \leq i \leq t$, V_j , $1 \leq j \leq t'$, as above. We may

assume that $U_i, V_j, 1 \leq i \leq t, 1 \leq j \leq t'$, have non-zero measure since otherwise, we can take A to be the union of all basic open subsets of X with measure 0. In this case, there is no $(\mathbf{x}, \mathbf{y}) \in C$ with $x_i, y_j \notin A$ for every $1 \leq i \leq t, 1 \leq j \leq t'$.

Consider the map

$$\Psi : L^0(\mathbb{T}) \rightarrow \prod_{i=1}^t L^0(U_i, \frac{\mu}{\mu(U_i)}, \mathbb{T}) \times \prod_{j=1}^{t'-w} L^0(V_{l_j}, \frac{\mu}{\mu(V_{l_j})}, \mathbb{T})$$

defined by

$$\Psi(f) = (f \upharpoonright U_1, \dots, f \upharpoonright U_t, f \upharpoonright V_{l_1}, \dots, f \upharpoonright V_{l_{t'-w}})$$

where

$$\{l_1, \dots, l_{t'-w}\} = \{1, \dots, t'\} \setminus \{j_1, \dots, j_w\}.$$

The map Ψ is open since for each open subset $U \subseteq X$ with $\mu(U) > 0$, $f \rightarrow f \upharpoonright U$ is open and $U_i, V_j, 1 \leq i \leq t, j \in \{1, \dots, t'\} \setminus \{j_1, \dots, j_w\}$, are pairwise disjoint. Therefore, Ψ is category preserving. Let

$$\Psi(f) = (f_1, \dots, f_t, g_{l_1}, \dots, g_{l_{t'-w}})$$

and

$$(x_1, \dots, x_t, y_1, \dots, y_{t'}) \in C.$$

We have

$$f^\kappa(x_1, \dots, x_t) = f_1(x_1)^{u_1} f_2(x_2)^{u_2} \cdots f_t(x_t)^{u_t},$$

$$f^{\kappa'}(y_1, \dots, y_{t'}) = \prod_{k=1}^{t'-w} g_{l_k}(y_{l_k})^{v_{l_k}} \cdot \prod_{k=1}^w f_{i_{r^{-1}(k)}}(x_{i_{r^{-1}(k)}})^{v_{j_k}}.$$

If $w \neq t$ or in the case of $w = t$, $(v_1, \dots, v_{t'}) \neq r(u_1, u_2, \dots, u_t)$, then by Theorem 1.3.3, for comeagerly many $f \in L^0(\mathbb{T})$,

$$f_1^{u_1} f_2^{u_2} \cdots f_t^{u_t} \cap \prod_{k=1}^{t'-w} g_{l_k}^{v_{l_k}} \cdot \prod_{k=1}^w f_{i_{r^{-1}(k)}}^{v_{j_k}} \approx \emptyset.$$

Note that since U_i , $1 \leq i \leq t$, V_j , $1 \leq j \leq t'$, have non-zero measures, $L^0(U_i, \mathbb{T})$, $1 \leq i \leq t$, and $L^0(V_j, \mathbb{T})$, $1 \leq j \leq t'$, are isomorphic to $L^0(X, \mathbb{T})$ as topological groups. Therefore, Theorem 1.3.3 can be applied to

$$\prod_{i=1}^t L^0(U_i, \mathbb{T}) \times \prod_{j=1}^{t'-w} L^0(V_j, \mathbb{T}) \cong L^0(X, \mathbb{T})^{t+t'-w}.$$

By Lemma 1.3.2, there exists $A \subseteq X$ with $\mu(A) = 0$ such that for all $(\mathbf{x}, \mathbf{y}) \in C$ with $x_i, y_j \notin A$ for every $1 \leq i \leq t, 1 \leq j \leq t'$,

$$f^\kappa(x_1, \dots, x_t) = f^{\kappa'}(y_1, \dots, y_{t'}) \Rightarrow (y_1, \dots, y_{t'}) = r(x_1, \dots, x_t). \quad \square$$

With a similar argument one can prove the following generalization of Theorem 1.3.8 for non-discrete Polish groups.

Theorem 1.3.9. *Let μ be a non-atomic Borel probability measure on a standard Borel space X and G be a non-discrete Polish group. Let $\kappa =$*

$(u_1, \dots, u_t), \kappa' = (v_1, \dots, v_{t'})$ be tuples in $\mathbb{Z} \setminus \{0\}$ and

$$R = \{r \in S_t : (v_1, \dots, v_{t'}) = r(u_1, \dots, u_t)\}.$$

Then, for comeagerly many $f \in L^0(\mu, G)$, f^κ and $f^{\kappa'}$ are almost R -disjoint.

1.4 The DL-condition and the statement of the main theorem

Generic behavior of a measure preserving transformation is of interest in Ergodic Theory. For example, the following papers are devoted to study generic behavior of a measure preserving transformation: Del Junco–Lemańczyk [4], Glasner–Weiss [5], King [12]. Of particular interest is characterization of $\overline{\langle T \rangle}$ for a generic $T \in L^0(\mathbb{T})$. More precisely, does there exists a topological group G such that $\overline{\langle T \rangle}$ is isomorphic to G for a generic $T \in L^0(\mathbb{T})$. The following conjecture is due to Glasner and Weiss.

Conjecture 1.4.1 (Glasner–Weiss). *For a generic measure preserving transformation, T ,*

$$\overline{\langle T \rangle} \cong L^0(\lambda, \mathbb{T})$$

where λ is the Lebesgue measure. In particular, for a generic measure preserving transformation, T , $\overline{\langle T \rangle}$ is Lévy.

Del Junco and Lemańczyk [4] proved a generic property of measure preserving transformations. They showed that for a generic $T \in \text{Aut}(\mu)$, maximal spectral types of powers of T satisfy certain orthogonality conditions.

Theorem 1.4.2 (Del Junco–Lemańczyk). *For a generic $T \in \text{Aut}(\mu)$, we have*

$$\begin{aligned} & \text{if } k(1), k(2), \dots, k(l) \in \mathbb{Z}^+, \ k'(1), k'(2), \dots, k'(l') \in \mathbb{Z}^+, \text{ then} \\ & \sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l')}} \end{aligned} \quad (\text{D})$$

provided that $(k(1), \dots, k(l))$ is not a rearrangement of $(k'(1), \dots, k'(l'))$.

Assuming the conjecture, for a generic $T \in \text{Aut}(\mu)$, we can define a unitary representation of $L^0(\mathbb{T})$ by

$$\Phi : L^0(\mathbb{T}) \cong \overline{\langle T \rangle} \hookrightarrow \text{Aut}(\mu) \subseteq \mathcal{U}(L^2(X, \mu)).$$

Note that $\text{Aut}(\mu)$ can be viewed as a subset of $\mathcal{U}(L^2(X, \mu))$ by identifying $T \in \text{Aut}(\mu)$ with $U_T \in \mathcal{U}(H)$ where

$$U_T(f) = f \circ T^{-1}.$$

The orthogonality conditions from Theorem 1.4.2 motivates the following definition.

Definition 1.4.3. *Fix a non-atomic Borel probability measure μ on a standard Borel space X . We say that a continuous unitary representation of $L^0(\mu, \mathbb{T})$,*

$$\Phi : L^0(\mu, \mathbb{T}) \rightarrow \mathcal{U}(H)$$

*satisfies the **DL-condition** if there is a dense G_δ subset $G \subseteq L^0(\mu, \mathbb{T})$ such that, for each $f \in G$ and $k(1), k(2), \dots, k(l) \in \mathbb{Z}^+, \ k'(1), k'(2), \dots, k'(l') \in$*

\mathbb{Z}^+ , the convolutions

$$\sigma_{\Phi(f)^{k(1)}} * \cdots * \sigma_{\Phi(f)^{k(l)}} \text{ and } \sigma_{\Phi(f)^{k'(1)}} * \cdots * \sigma_{\Phi(f)^{k'(l')}}.$$

are mutually singular, provided that there does not exist $r \in S_l$ such that

$$(k'(1), k'(2), \dots, k'(l')) = r(k(1), k(2), \dots, k(l)).$$

Solecki [17] proved that a continuous unitary representation of $L^0(\mu, \mathbb{T})$ can be written as a direct sum of unitary representations of $L^0(\mu, \mathbb{T})$ of the following form: Assume that we are given a sequence $\kappa = (k(1), \dots, k(n))$ of elements of $\mathbb{Z} \setminus \{0\}$ with

$$k(1) \leq k(2) \leq \cdots \leq k(n).$$

Assume we have a finite Borel measure λ on X^n whose marginal measures are absolutely continuous with respect to μ , that is, for $i \leq n$

$$(\pi_i)_*(\lambda) \ll \mu \tag{1.2}$$

where π_i is the projection on the i -th coordinate. With this set of data we associate the following representation of $L^0(\mu, \mathbb{T})$ on $L^2(\lambda, \mathbb{C})$

$$L^0(\mu, \mathbb{T}) \ni f \rightarrow U_f \in \mathbb{U}(L^2(\lambda, \mathbb{C}))$$

where for $h \in L^2(\lambda, \mathbb{C})$

$$U_f(h) = \left(\prod_{i \leq n} (f \circ \pi_i)^{k(i)} \right) h.$$

This representation is denoted by $\sigma(\kappa, \lambda)$. Furthermore, we consider the following additional condition on a finite measure λ as above, for $1 \leq i < j \leq n$

$$\lambda(\{(x_1, x_2, \dots, x_n) \in X^n : x_i = x_j\}) = 0. \quad (1.3)$$

Let S be the set of all sequences $\kappa = (k(1), k(2), \dots, k(n))$ of elements of $\mathbb{Z} \setminus \{0\}$ such that $k(1) \leq k(2) \leq \dots \leq k(n)$. The natural number n is called the length of κ and we denote it by $|\kappa|$.

Theorem 1.4.4 (Solecki). *Let Φ be a continuous unitary representation of $L^0(\mu, \mathbb{T})$ on a separable complex Hilbert space H . Consider H_0 , the orthogonal complement of*

$$\{v \in H : \forall f \in L^0(\mu, \mathbb{T}) \ \Phi(f)(v) = v\}.$$

For $\kappa \in S$ and $i \in \mathbb{N}$ there exist finite Borel measures λ_κ^i on $X^{|\kappa|}$ with properties 1.2, 1.3, and

$$\lambda_\kappa^j \ll \lambda_\kappa^i \text{ for } i < j \quad (1.4)$$

such that the representation Φ restricted to H_0 is the direct sum of the representations $\sigma(\kappa, \lambda_\kappa^i)$ with $\kappa \in S$ and $i \in \mathbb{N}$.

Furthermore, these measures, λ_κ^i , can be chosen in such a way that for every $\kappa = (k(1), \dots, k(n))$, $m \in \mathbb{N}$, and $1 \leq i < j \leq n$ with $k(i) = k(j)$

$$\lambda_\kappa^m \{(x_1, x_2, \dots, x_n) \in X^n : x_j <_X x_i\} = 0. \quad (1.5)$$

Here $<_X$ is a linear order on X with the property that the order topology it generates is compact, second countable and the Borel sets with respect to this topology coincide with the Borel sets on X . Assuming (1.5), measures

λ_κ^i obtained from Theorem 1.4.4 are unique up to mutual absolute continuity.

In the following, we show that the DL-condition for a continuous representation of $L^0(\mathbb{T})$ is equivalent to some orthogonality conditions on the measures, λ_κ^1 .

Theorem 1.4.5. *Let $\Phi = \bigoplus \sigma(\kappa, \lambda_\kappa^i)$ be a continuous unitary representation of $L^0(\mu, \mathbb{T})$. Then, Φ satisfies the DL-condition iff we have*

$$\lambda_{\kappa_1}^1 \times \lambda_{\kappa_2}^1 \times \cdots \times \lambda_{\kappa_l}^1 \perp r(\lambda_{\kappa'_1}^1 \times \lambda_{\kappa'_2}^1 \times \cdots \times \lambda_{\kappa'_{l'}}^1) \quad (1.6)$$

for every $(\kappa_1, \kappa_2, \dots, \kappa_l)$, $(\kappa'_1, \kappa'_2, \dots, \kappa'_{l'})$, and $r \in S_t$ such that

$$k(1)\kappa_1 + k(2)\kappa_2 + \cdots + k(l)\kappa_l = r(k'(1)\kappa'_1 + k'(2)\kappa'_2 + \cdots + k'(l')\kappa'_{l'})$$

for some non-zero integer numbers $(k(1), \dots, k(l))$ and $(k'(1), \dots, k'(l'))$, provided that there does not exist $s \in S_l$ such that

$$(k'(1), k'(2), \dots, k'(l')) = s(k(1), k(2), \dots, k(l)).$$

1.5 A simplified proof of the result by Del Junco and Lemańczyk (Theorem 1.4.2)

By a standard Borel measure space (X, μ) we mean a standard Borel space, X , equipped with a non-atomic Borel probability measure on X , μ . All such spaces are Borel isomorphic to $([0, 1], \lambda)$, where λ is the Lebesgue measure on the Borel subsets of $[0, 1]$. We denote by $\text{Aut}(X, \mu)$, and sometimes just $\text{Aut}(\mu)$ when X is understood, the group of Borel automorphisms of X

which preserve the measure μ , that is for every $T \in \text{Aut}(X, \mu)$ and A , a Borel subset of X , A , $T(A)$, and $T^{-1}(A)$ have the same measure. In which we identify two Borel automorphisms if they coincide μ -almost everywhere, that is they coincide on a subset of X with full measure.

Definition 1.5.1. *Let $\alpha_1, \alpha_2, \dots, \alpha_K \in [0, 1]$ and T be a measure preserving automorphism of $([0, 1], \lambda)$. T is $(\alpha_1, \alpha_2, \dots, \alpha_K)$ -**weakly mixing** if there is a sequence $\{n_i\}$ such that for each $k \in \{1, 2, \dots, K\}$ and A, B Borel subsets of $[0, 1]$*

$$\lim_{i \rightarrow \infty} \lambda(T^{kn_i} A \cap B) = \alpha_k \lambda(A) \lambda(B) + (1 - \alpha_k) \lambda(A \cap B).$$

We say T is $(\alpha_1, \alpha_2, \dots, \alpha_K)$ -weakly mixing along $\{n_i\}$.

As mentioned in [4], Theorem 1.4.2 follows from the following lemmas and proposition. It is noteworthy that Katok [9] and Stepin [19] showed that for a generic $T \in \text{Aut}(\mu)$, different convolution powers of the maximal spectral type of T are mutually singular. Moreover, Choksi and Nadkarni [2] showed that for a generic $T \in \text{Aut}(\mu)$, the maximal spectral type of different powers of T are mutually singular. Many ideas from prior studies like [2], [9], and [19] are the cornerstone of the proof of Theorem 1.4.2 and the following lemmas are inspired by those ideas. The main contribution is Proposition 1.5.6, the construction of a single $(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_K)$ -weakly mixing transformation.

Lemma 1.5.2 (Katok–Stepin). *T is $(1 - \alpha)$ -weakly mixing along $\{n_i\}$ if and only if for each $f \in L_1(\sigma_T)$, where σ_T is the maximal spectral type of T , we have*

$$\lim_{i \rightarrow \infty} \int_{\mathbb{T}} z^{n_i} f(z) d\sigma_T(z) = \alpha \int_{\mathbb{T}} f(z) d\sigma_T(z).$$

Lemma 1.5.3. *Suppose that T is $(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_K)$ -weakly mixing with $\alpha_1, \alpha_2, \dots, \alpha_K \in (0, 1)$ and $0 < k(1), k(2), \dots, k(l) \leq K$, and let $\sigma = \sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}}$. Then for every $f \in L_1(\sigma)$, we have*

$$\lim_{i \rightarrow \infty} \int_{\mathbb{T}} z^{n_i} f(z) d\sigma(z) = \alpha_{k(1)} \alpha_{k(2)} \dots \alpha_{k(l)} \int_{\mathbb{T}} f(z) d\sigma(z).$$

Lemma 1.5.4. *Suppose that T is $(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_K)$ -weakly mixing with $\alpha_1, \alpha_2, \dots, \alpha_K \in (0, 1)$ and $\{\log \alpha_1, \log \alpha_2, \dots, \log \alpha_K\}$ is linearly independent over \mathbb{Q} . Then T satisfies the following finite version of D*

$$\begin{aligned} \text{if } 0 < k(1), k(2), \dots, k(l), k'(1), k'(2), \dots, k'(l') \leq K, \text{ then} \\ \sigma_{T^{k(1)}} * \dots * \sigma_{T^{k(l)}} \perp \sigma_{T^{k'(1)}} * \dots * \sigma_{T^{k'(l')}} \end{aligned} \quad (1.7)$$

provided that $(k(1), \dots, k(l))$ is not a rearrangement of $(k'(1), \dots, k'(l'))$.

Lemma 1.5.5. *For every $K > 0$ the set of $T \in \text{Aut}(\mu)$ satisfying (1.7) is a G_δ subset of $\text{Aut}(\mu)$.*

Proposition 1.5.6. *Suppose $\beta_1, \beta_2, \dots, \beta_K > 0$ and $\sum_{j=1}^K \beta_j = 1$. For $1 \leq k \leq K$, let $\alpha_k = \sum_{j|k} \beta_j$. Then, there is a $T \in \text{Aut}(\mu)$ that is $(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_K)$ -weakly mixing.*

Lemma 1.5.7. *For every $K > 0$ there exists $\alpha_1, \alpha_2, \dots, \alpha_K \in (0, 1)$ and $T \in \text{Aut}(\mu)$ such that $\{\log \alpha_1, \log \alpha_2, \dots, \log \alpha_K\}$ is linearly independent over \mathbb{Q} and T is $(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_K)$ -weakly mixing.*

Proof of Theorem 1.4.2. Since condition (D) is the intersection of the conditions (1.7) for every $K > 0$, it is enough to show that condition (1.7) is a dense G_δ . By Lemma 1.5.5 condition (1.7) is G_δ . It remains to show that it is also dense. In light of the conjugacy lemma ([4], p. 77), for every $K > 0$

we need to construct a T which satisfies condition (1.7). This is provided by Lemmas 1.5.4, 1.5.7, and Proposition 1.5.6. \square

Here we present a simplified proof of the main ingredient of the proof of Theorem 1.4.2.

Proof of Proposition 1.5.6. Using cutting and stacking, we construct such measure preserving transformation, T , on $([0, d], \lambda)$ where λ is the Lebesgue measure and $d > 0$ is a real number. We start with the interval $[0, 1]$ and the partial measure preserving transformation with the empty set as its domain. At each step, we enlarge the interval by adding some spacers and extend the partial measure preserving transformation to a larger portion of the new interval so that by taking the limit we get a measure preserving transformation on the interval $[0, d]$ for some positive real number d . Suppose after n steps we have $\mathbf{a}_n = 012 \cdots h - 1$, that is, the n th tower has h levels I_0, I_1, \dots, I_{h-1} and the partial measure preserving transformation sends I_i to I_{i+1} by translation for every $0 \leq i \leq h - 2$. We introduce a new sequence $\mathbf{a}_{n+1} = s_1 s_2 \cdots s_t$ that is consist of a large number of copies $\mathbf{a}_n = 012 \cdots h - 1$ and some symbol x , which represent a spacer that enlarges the interval, between different copies of \mathbf{a}_n . If there are R many copies of \mathbf{a}_n , then we are dividing each level at stage n , I_i , into R many levels with the same length in stage $n + 1$. Moreover, we require the new spacer levels, symbol x , to have the length equal to $\frac{1}{R}$ of the length of a level in stage n . Now the partial measure preserving transformation at stage $n + 1$, sends level s_i to level s_{i+1} by translation. One can check that the partial measure preserving transformation at stage $n + 1$ extends the partial measure preserving transformation at stage n and if the portion of spacers is small

enough then in the limit the length of the interval will converge to some $d > 0$.

We use the above strategy to define the desired transformation T . Let

$$\mathbf{c} = \underbrace{\mathbf{a}\mathbf{a}\cdots\mathbf{a}}_{C \text{ times}} = \mathbf{a}^C$$

where C is a large number to be specified later. For $0 \leq s \leq h-1$, we define

$$\mathbf{c}_s = x^s \mathbf{c} x^{h-s-1},$$

that is, we are adding $h-1$ many new spacers. Note that \mathbf{c}_s has

$$\omega = |\mathbf{c}_s| = Ch + h - 1$$

levels. For $1 \leq k \leq K$ and $(i_0, i_1, \dots, i_{k-1}) \in \{0, 1, \dots, h-1\}^k$, we define

$$\mathbf{d}_{i_0 i_1 \dots i_{k-1}} = \mathbf{c}_{i_0} \mathbf{c}_{i_1} \cdots \mathbf{c}_{i_{k-1}}.$$

Let $\{\tau_0^{(k)}, \dots, \tau_{z_k-1}^{(k)}\}$ be an enumeration of $\{0, 1, \dots, h-1\}^k$. We define

$$\mathbf{b}_k = \underbrace{\mathbf{d}_{\tau_0^{(k)}} \mathbf{d}_{\tau_0^{(k)}} \cdots \mathbf{d}_{\tau_0^{(k)}}}_{L_k \text{ times}} \underbrace{\mathbf{d}_{\tau_1^{(k)}} \mathbf{d}_{\tau_1^{(k)}} \cdots \mathbf{d}_{\tau_1^{(k)}}}_{L_k \text{ times}} \cdots \underbrace{\mathbf{d}_{\tau_{z_k-1}^{(k)}} \mathbf{d}_{\tau_{z_k-1}^{(k)}} \cdots \mathbf{d}_{\tau_{z_k-1}^{(k)}}}_{L_k \text{ times}}$$

where L_k is a large number to be specified later. Moreover, we define

$$\mathbf{b} = \mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_K.$$

Note that all quantities like L_k, \mathbf{b}_k are computed at stage n but we are

omitting the subscript n , that is,

$$\mathbf{a}_n = \mathbf{a} \quad , \quad \mathbf{a}_{n+1} = \mathbf{b}.$$

Now we have to show that T , defined as above, is indeed $(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_K)$ -mixing.

We show that the measure preserving transformation defined above, T , is $(1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_K)$ -mixing along $\{\omega_n\}_{n=1}^\infty$ where

$$\omega_n = C_n h_n + h_n - 1.$$

WLOG, we may assume that $d = 1$, that is, T is defined on the interval $[0, 1]$. It suffices to show that for $1 \leq k \leq K$ and A, B , Borel subsets of $[0, 1]$, we have

$$\lim_{n \rightarrow \infty} \lambda(T^{k\omega_n} A \cap B) = (1 - \alpha_k) \lambda(A) + \alpha_k (A \cap B).$$

Let E be a Borel subset of $[0, 1]$. At stage n , we define

$$E_l^{(n)} = \begin{cases} I_l^{(n)} & \text{if } E \cap I_l^{(n)} \neq \emptyset, \\ \emptyset & \text{if } E \cap I_l^{(n)} = \emptyset. \end{cases}$$

We have

$$\lambda(E) = \lim_{n \rightarrow \infty} \sum_{l=0}^{h_n-1} \lambda(E_l^{(n)}).$$

Thus, for large enough n , we have

$$\lambda(E) \approx \sum_{l=0}^{h_n-1} \lambda(E_l^{(n)}),$$

that is, we can approximate E by a union of some intervals of the form $I_l^{(n)}$. Therefore, we may assume that A, B are unions of some intervals of the form $I_l^{(n)}$.

Fix n to be a large enough number. Omitting the subscript n , we have

$$\begin{aligned}
\lambda(T^{k\omega} A \cap B) &= \lambda\left(\left(\bigcup_{l=0}^{h-1} T^{k\omega} A_l\right) \cap \left(\bigcup_{l=0}^{h-1} B_l\right)\right) = \sum_{0 \leq l, l' \leq h-1} \lambda(T^{k\omega} A_l \cap B_{l'}) \\
&= \sum_{j=1}^K \sum_{0 \leq l, l' \leq h-1} \lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j) \\
&= \sum_{j|k} \sum_{l, l'} \lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j) + \sum_{j \nmid k} \sum_{l, l'} \lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j).
\end{aligned}$$

It is enough to show that

$$\lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j) \approx \begin{cases} \beta_j \lambda(A_l \cap B_{l'}) & \text{if } j \mid k, \\ \beta_j \lambda(A_l) \lambda(B_{l'}) & \text{if } j \nmid k \end{cases} \quad (1.8)$$

since by assuming (1.8), we have

$$\begin{aligned}
\lambda(T^{k\omega} A \cap B) &= \sum_{j|k} \sum_{l, l'} \lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j) + \sum_{j \nmid k} \sum_{l, l'} \lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j) \\
&\approx \sum_{j|k} \sum_{l, l'} \beta_j \lambda(A_l \cap B_{l'}) + \sum_{j \nmid k} \sum_{l, l'} \beta_j \lambda(A_l) \lambda(B_{l'}) \\
&= \sum_{j|k} \beta_j \sum_l \lambda(A_l \cap B_l) + \sum_{j \nmid k} \beta_j \left(\sum_l \lambda(A_l)\right) \left(\sum_l \lambda(B_l)\right) \\
&= \left(\sum_{j|k} \beta_j\right) \lambda(A \cap B) + \left(\sum_{j \nmid k} \beta_j\right) \lambda(A) \lambda(B) \\
&= \alpha_k \lambda(A \cap B) + (1 - \alpha_k) \lambda(A) \lambda(B).
\end{aligned}$$

For every $j \in \{1, 2, \dots, K\}$ by taking L_j to be large enough, we may assume

$$T^r(\underbrace{\mathbf{d}_\tau \mathbf{d}_\tau \cdots \mathbf{d}_\tau}_{L_j \text{ times}}) \approx \underbrace{\mathbf{d}_\tau \mathbf{d}_\tau \cdots \mathbf{d}_\tau}_{L_j \text{ times}}$$

for every $\tau \in \{0, 1, \dots, h-1\}^j$ and $r \in \{1, \dots, K\omega\}$. Furthermore, we choose L_1, L_2, \dots, L_K so that $\lambda(\mathbf{b}_j) \approx \beta_j$ for every $1 \leq j \leq K$.

Fix $j \in \{1, 2, \dots, K\}$. If $j \mid k$, then $j\omega \mid k\omega$ and $T^{k\omega}$ sends level l of \mathbf{d}_τ to level l of another copy of \mathbf{d}_τ , therefore

$$\lambda(T^{k\omega} I_l \cap I_{l'} \cap \mathbf{d}_\tau) \approx \frac{\delta_{l,l'}}{h} \lambda(\mathbf{d}_\tau)$$

and

$$\lambda(T^{k\omega} I_l \cap I_{l'} \cap \mathbf{b}_j) \approx \frac{\delta_{l,l'}}{h} \sum_{\tau} \lambda(\mathbf{d}_\tau) = \frac{\lambda(\mathbf{b}_j)}{h} \delta_{l,l'} \approx \frac{\beta_j}{h} \delta_{l,l'}.$$

Therefore, considering that $h\lambda(I_l) \approx 1$ for every $0 \leq l \leq h-1$ and large enough n , we have

$$\begin{aligned} \lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j) &= \frac{\lambda(A_l \cap B_{l'})}{\lambda(I_l)} \delta_{l,l'} \cdot \lambda(T^{k\omega} I_l \cap I_{l'} \cap \mathbf{b}_j) \\ &\approx \beta_j \lambda(A_l \cap B_{l'}) \end{aligned}$$

Now if $j \nmid k$ and $\tau = (i_0, i_1, \dots, i_{j-1})$, then $T^{k\omega}$ sends level l of \mathbf{c}_{i_m} in \mathbf{d}_τ to level $l + i_m - i_{m'}$ in $\mathbf{c}_{i_{m'}}$ where

$$m' \equiv m + k \pmod{j}$$

and $m' \in \{0, 1, \dots, j-1\}$. Since all elements of $\{0, 1, \dots, h-1\}^j$ appear L_j many times in the construction of \mathbf{b}_j , distribution of $i_m - i_{m'}$ modulo h is uniform. Therefore, considering that $T^{k\omega}(\mathbf{b}_j) \approx \mathbf{b}_j$, $T^{k\omega}$ sends level l of \mathbf{b}_j

to level l' of \mathbf{b}_j with probability almost $\frac{1}{h}$. Thus,

$$\lambda(T^{k\omega} I_l \cap I_{l'} \cap \mathbf{b}_j) \approx \frac{1}{h} \lambda(\mathbf{b}_j) \approx \frac{\beta_j}{h^2}.$$

Therefore, we have

$$\begin{aligned} \lambda(T^{k\omega} A_l \cap B_{l'} \cap \mathbf{b}_j) &= \frac{\lambda(A_l) \lambda(B_{l'})}{\lambda(I_l) \lambda(I_{l'})} \lambda(T^{k\omega} I_l \cap I_{l'} \cap \mathbf{b}_j) \\ &\approx \beta_j \lambda(A_l) \lambda(B_{l'}). \end{aligned} \quad \square$$

1.6 Maximal spectral type of $\bigoplus \sigma(\kappa, \lambda_\kappa^i)$

For every $f \in L^0(\mu, \mathbb{T})$ and $\kappa = (k_1, k_2, \dots, k_n)$, let $\mu_{f, \kappa}$ and h_κ be the maximal spectral type and the corresponding vector of $\sigma(\kappa, \lambda_\kappa)(f)$. We have

$$\int z^n d\mu_{f, \kappa} = \langle \sigma(\kappa, \lambda_\kappa)(f) h, h \rangle = \int f^{n\kappa}(x) \|h(x)\|^2 d\lambda_\kappa.$$

Let λ'_κ be a measure on $X^{|\kappa|}$ such that for a Borel subset of $X^{|\kappa|}$, A , we have

$$\lambda'_\kappa(A) = \int \|h(x)\|^2 d\lambda_\kappa.$$

Then, λ'_κ is absolutely continuous with respect to λ_κ and

$$\int f^{n\kappa}(x) \|h(x)\|^2 d\lambda_\kappa = \int f^{n\kappa}(x) d\lambda'_\kappa.$$

By the definition of the Lebesgue integral, we have

$$\begin{aligned}
\int \operatorname{Re}(f^{n\kappa}(x)) d\lambda'_\kappa(x) &= \sup \left\{ \sum_{i=1}^m b_i \lambda'_\kappa(B_i) : m \in \mathbb{N}, B_i \cap B_j = \emptyset \text{ for} \right. \\
&\text{every distinct } 1 \leq i, j \leq m, (f^\kappa)^{-1}(f^\kappa(B_i)) = B_i \text{ for every } 1 \leq i \leq m, \\
&\left. \bigcup_{i=1}^m B_i = X^{|\kappa|}, \text{ and } b_i \leq \inf_{x \in B_i} \operatorname{Re}(f^{n\kappa}(x)) \right\} \\
&= \sup \left\{ \sum_{i=1}^m a_i \mu_{f,\kappa}^*(A_i) : m \in \mathbb{N}, A_i \cap A_j = \emptyset \text{ for} \right. \\
&\text{every distinct } 1 \leq i, j \leq m, \bigcup_{i=1}^m A_i = \mathbb{T}, a_i \leq \inf_{x \in A_i} \operatorname{Re}(z^n), \text{ and} \\
&\left. \mu_{f,\kappa}^*(A_i) = \lambda'_\kappa((f^\kappa)^{-1}(A_i)) \right\} \\
&= \int \operatorname{Re}(z^n) d\mu_{f,\kappa}^*.
\end{aligned}$$

Similarly, we can show that

$$\int \operatorname{Im}(f^{n\kappa}(x)) d\lambda'_\kappa(x) = \int \operatorname{Im}(z^n) d\mu_{f,\kappa}^*$$

and therefore,

$$\int f^{n\kappa}(x) d\lambda'_\kappa(x) = \int z^n d\mu_{f,\kappa}^*.$$

Hence, $\mu_{f,\kappa}$ is equivalent to $\mu_{f,\kappa}^*$, that is, $\mu_{f,\kappa}$ is equivalent to the push-forward measure of λ'_κ under f^κ . Similar computation shows that the spectral measure of $h = 1$ is the push-forward measure of λ_κ under f^κ . Since $\mu_{f,\kappa}$ is the maximal spectral type of $U_{f,\kappa}$ and λ'_κ is absolutely continuous with respect to λ_κ , $\mu_{f,\kappa}$ is equivalent to the push-forward measure of λ_κ under f^κ .

Proposition 1.6.1. *Let $\Phi = \bigoplus \sigma(\kappa, \lambda_\kappa)$. The maximal spectral type of*

$\Phi(f)$ is equivalent to $\sum_{\kappa} \alpha_{\kappa} \mu_{f,\kappa}$ where $\mu_{f,\kappa}$ is the maximal spectral type of $\sigma(\kappa, \lambda_{\kappa})(f)$ and $0 < \alpha_{\kappa} < 1$ is chosen so that $\sum_{\kappa} \alpha_{\kappa} \mu_{f,\kappa}$ is finite.

Proof. Let $\mu_{f,\kappa}$ and h_{κ} be the maximal spectral type and corresponding vector of $\sigma(\kappa, \lambda_{\kappa})(f)$. Let $0 < \alpha_{\kappa} < 1$ be such that $\sum_{\kappa} \alpha_{\kappa} \mu_{f,\kappa}$ is finite. For $h = (\sqrt{\alpha_{\kappa}} h_{\kappa})_{\kappa} \in \bigoplus X^{|\kappa|}$, we have

$$\langle \Phi(f)^n(h), h \rangle = \sum_{\kappa} \langle \sigma(\kappa, \lambda_{\kappa})(f)(\sqrt{\alpha_{\kappa}} h_{\kappa}), \sqrt{\alpha_{\kappa}} h_{\kappa} \rangle = \sum_{\kappa} \int \alpha_{\kappa} z^n d\mu_{f,\kappa}.$$

Thus, spectral measure of h , μ_h , is equivalent to

$$\sum_{\kappa} \alpha_{\kappa} \mu_{f,\kappa}.$$

Since each $\mu_{f,\kappa}$ is the maximal spectral type of $\sigma(\kappa, \lambda_{\kappa})(f)$, μ_h is the maximal spectral type of $\Phi(f)$. \square

Proposition 1.6.2. *Let X be a Borel measurable space and $\lambda_{\kappa}, \lambda'_{\kappa}$ be Borel measures on $X^{|\kappa|}, X^{|\kappa'|}$, respectively. Assume $\mu_{f,\kappa}$ and $\mu_{f,\kappa'}$ are push-forward measures of λ_{κ} and λ'_{κ} , respectively, under f^{κ} and $f^{\kappa'}$. Then, $\mu_{f,\kappa} * \mu_{f,\kappa'}$ is the push-forward measure of $\lambda_{\kappa} \times \lambda_{\kappa'}$ under $f^{\kappa \cdot \kappa'}$.*

Proof. Let $E \subseteq \mathbb{T}$. We have

$$\begin{aligned}
\mu_{f,\kappa} * \mu_{f,\kappa'}(E) &= \int_{\mathbb{T}} \mu_{f,\kappa}(y^{-1}E) d\mu_{f,\kappa'}(y) = \int_{\mathbb{T}} \lambda_{\kappa}((f^{\kappa})^{-1}(y^{-1}E)) d\mu_{f,\kappa'}(y) \\
&= \sup \left\{ \sum_{i=1}^m a_i \mu_{f,\kappa'}(A_i) : m \in \mathbb{N}, A_i \cap A_j = \emptyset \text{ for every distinct } \right. \\
&\quad \left. 1 \leq i, j \leq m, \bigcup_{i=1}^m A_i = \mathbb{T}, \text{ and } a_i \leq \inf_{y \in A_i} \lambda_{\kappa}((f^{\kappa})^{-1}(y^{-1}E)) \right\} \\
&= \sup \left\{ \sum_{i=1}^m b_i \lambda_{\kappa'}(B_i) : m \in \mathbb{N}, B_i \cap B_j = \emptyset \text{ for every distinct } \right. \\
&\quad \left. 1 \leq i, j \leq m, \bigcup_{i=1}^m B_i = X^{|\kappa'|}, \text{ and } b_i \leq \inf_{y \in B_i} \lambda_{\kappa}((f^{\kappa})^{-1}(f^{-\kappa'}(y)E)) \right\} \\
&= (\lambda_{\kappa} \times \lambda_{\kappa'}) \{(x, y) \in X^{\kappa} \times X^{\kappa'} : f^{\kappa}(x) f^{\kappa'}(y) \in E\}. \quad \square
\end{aligned}$$

1.7 Proof of the main theorem (Theorem 1.4.5)

We prove the theorem under the assumption that for every κ , at most one of the measures, $\lambda_{\kappa} = \lambda_{\kappa}^1$, is non-zero. Then, we show that the general statement follows consequently.

Assume that $\Phi = \bigoplus \sigma(\kappa, \lambda_{\kappa})$ satisfies the DL-condition. We will show that Φ satisfies (1.6). Let $(\kappa_1, \kappa_2, \dots, \kappa_l)$, $(\kappa'_1, \kappa'_2, \dots, \kappa'_{l'})$, and $r \in S_t$ be such that $(t = |\kappa_1| + \dots + |\kappa_l| = |\kappa'_1| + \dots + |\kappa'_{l'}|)$

$$k(1)\kappa_1 + k(2)\kappa_2 + \dots + k(l)\kappa_l = r \left(k'(1)\kappa'_1 + k'(2)\kappa'_2 + \dots + k'(l')\kappa'_{l'} \right)$$

for some non-zero integer numbers $(k(1), \dots, k(l))$ and $(k'(1), \dots, k'(l'))$, provided that one is not a rearrangement of the other. For $f \in L^0(\mathbb{T})$, we

define $h, h' : X^t \rightarrow \mathbb{T}$ as follows:

$$\begin{aligned} h(x_1, \dots, x_t) &= f^{k(1)\kappa_1 + k(2)\kappa_2 + \dots + k(l)\kappa_l}(x_1, \dots, x_t), \\ h'(x_1, \dots, x_t) &= f^{k'(1)\kappa'_1 + k'(2)\kappa'_2 + \dots + k'(l')\kappa'_{l'}}(x_1, \dots, x_t). \end{aligned}$$

By the DL-condition, for comeagerly many $f \in L^0(\mathbb{T})$ we have

$$\sigma_{\Phi(f)^{k(1)}} * \dots * \sigma_{\Phi(f)^{k(l)}} \perp \sigma_{\Phi(f)^{k'(1)}} * \dots * \sigma_{\Phi(f)^{k'(l')}}.$$

From the previous section, the maximal spectral type of $\Phi(f)$ is equivalent to $\sum_{\kappa} \alpha_{\kappa} \mu_{f, \kappa}$ where $\mu_{f, \kappa}$ is the maximal spectral type of $\sigma(\kappa, \lambda_{\kappa})(f)$ and $0 < \alpha_{\kappa} < 1$ is chosen so that $\sum_{\kappa} \alpha_{\kappa} \mu_{f, \kappa}$ is finite. Therefore, for comeagerly many $f \in L^0(\mathbb{T})$

$$\mu_{f, k(1)\kappa_1} * \dots * \mu_{f, k(l)\kappa_l} \perp \mu_{f, k'(1)\kappa'_1} * \dots * \mu_{f, k'(l')\kappa'_{l'}}. \quad (1.9)$$

Furthermore, $\mu_{f, k(1)\kappa_1} * \dots * \mu_{f, k(l)\kappa_l}$ is the push-forward measure of $\lambda_{\kappa_1} \times \dots \times \lambda_{\kappa_l}$ under h , and $\mu_{f, k'(1)\kappa'_1} * \dots * \mu_{f, k'(l')\kappa'_{l'}}$ is the push-forward measure of $\lambda_{\kappa'_1} \times \dots \times \lambda_{\kappa'_{l'}}$ under h' . Since $h = r(h')$, equation (1.9) indicates that the push-forward measure of $\lambda_{\kappa_1} \times \dots \times \lambda_{\kappa_l}$ and $r(\lambda_{\kappa'_1} \times \dots \times \lambda_{\kappa'_{l'}})$ under the same function, $h = r(h')$, are orthogonal to each other. Hence,

$$\lambda_{\kappa_1} \times \lambda_{\kappa_2} \times \dots \times \lambda_{\kappa_l} \perp r(\lambda_{\kappa'_1} \times \lambda_{\kappa'_2} \times \dots \times \lambda_{\kappa'_{l'}}).$$

Note that the same argument can be used to prove the general statement.

Now assume Φ satisfies (1.6), we show that Φ also satisfies the DL-

condition. By (1.6),

$$\lambda_{\kappa_1} \times \lambda_{\kappa_2} \times \cdots \times \lambda_{\kappa_l} \perp r(\lambda_{\kappa'_1} \times \lambda_{\kappa'_2} \times \cdots \times \lambda_{\kappa'_{l'}})$$

for every $(\kappa_1, \kappa_2, \dots, \kappa_l)$, $(\kappa'_1, \kappa'_2, \dots, \kappa'_{l'})$, and $r \in S_t$ so that

$$k(1)\kappa_1 + k(2)\kappa_2 + \cdots + k(l)\kappa_l = r(k'(1)\kappa'_1 + k'(2)\kappa'_2 + \cdots + k'(l')\kappa'_{l'})$$

for some non-zero integer numbers $(k(1), \dots, k(l))$ and $(k'(1), \dots, k'(l'))$, provided that one is not a rearrangement of the other. We have

$$\sigma_{\Phi(f)^{k(1)}} * \cdots * \sigma_{\Phi(f)^{k(l)}} \perp \sigma_{\Phi(f)^{k'(1)}} * \cdots * \sigma_{\Phi(f)^{k'(l')}})$$

if and only if

$$\mu_{f,k(1)\kappa_1} * \cdots * \mu_{f,k(l)\kappa_l} \perp \mu_{f,k'(1)\kappa'_1} * \cdots * \mu_{f,k'(l')\kappa'_{l'}}$$

for every $(\kappa_1, \kappa_2, \dots, \kappa_l)$ and $(\kappa'_1, \kappa'_2, \dots, \kappa'_{l'})$.

Fix $(\kappa_1, \kappa_2, \dots, \kappa_l)$ and $(\kappa'_1, \kappa'_2, \dots, \kappa'_{l'})$. We define $h : X^t \rightarrow \mathbb{T}$ and $h' : X^{t'} \rightarrow \mathbb{T}$ as follows

$$\begin{aligned} h(x_1, \dots, x_t) &= f^{k(1)\kappa_1 + k(2)\kappa_2 + \cdots + k(l)\kappa_l}(x_1, \dots, x_t), \\ h'(x_1, \dots, x_{t'}) &= f^{k'(1)\kappa'_1 + k'(2)\kappa'_2 + \cdots + k'(l')\kappa'_{l'}}(x_1, \dots, x_{t'}) \end{aligned}$$

where

$$\begin{aligned} t &= |\kappa_1| + |\kappa_2| + \cdots + |\kappa_l|, \\ t' &= |\kappa'_1| + |\kappa'_2| + \cdots + |\kappa'_{l'}|. \end{aligned}$$

Then, $\mu_{f,k(1)\kappa_1} * \cdots * \mu_{f,k(l)\kappa_l}$ and $\mu_{f,k'(1)\kappa'_1} * \cdots * \mu_{f,k'(l')\kappa'_{l'}}$ are push-forward measures of $\lambda_{\kappa_1} \times \cdots \times \lambda_{\kappa_l}$ and $\lambda_{\kappa'_1} \times \cdots \times \lambda_{\kappa'_{l'}}$ under h and h' , respectively. Assume there is $r_0 \in S_t$ such that (in particular $t = t'$)

$$k(1)\kappa_1 + k(2)\kappa_2 + \cdots + k(l)\kappa_l = r_0(k'(1)\kappa'_1 + k'(2)\kappa'_2 + \cdots + k'(l')\kappa'_{l'}).$$

We define

$$R = \{r \in S_t : k(1)\kappa_1 + \cdots + k(l)\kappa_l = r(k(1)\kappa_1 + \cdots + k(l)\kappa_l)\}.$$

Since Φ satisfies (1.6), for every $r \in R$, we have

$$\lambda_{\kappa_1} \times \lambda_{\kappa_2} \times \cdots \times \lambda_{\kappa_l} \perp rr_0(\lambda_{\kappa'_1} \times \lambda_{\kappa'_2} \times \cdots \times \lambda_{\kappa'_{l'}}).$$

Therefore, we can find $F, G \subseteq X^t$ such that

$$\begin{aligned} \lambda_{\kappa_1} \times \lambda_{\kappa_2} \times \cdots \times \lambda_{\kappa_l}(F) &= 1, \\ rr_0(\lambda_{\kappa'_1} \times \lambda_{\kappa'_2} \times \cdots \times \lambda_{\kappa'_{l'}})(r(G)) &= 1 \text{ for every } r \in R, \end{aligned}$$

and $F \cap r(G) = \emptyset$ for every $r \in R$. By Theorem 1.3.8, for comeagerly many $f \in L^0(\mathbb{T})$ the push-forward measures of $\lambda_{\kappa_1} \times \cdots \times \lambda_{\kappa_l}$ and $r_0(\lambda_{\kappa'_1} \times \cdots \times \lambda_{\kappa'_{l'}})$ under $f^{k(1)\kappa_1 + \cdots + k(l)\kappa_l}$ are perpendicular to each other since the latter holds for every $f \in L^0(\mathbb{T})$ such that $f^{k(1)\kappa_1 + \cdots + k(l)\kappa_l}$ is almost R -to-one. Therefore,

$$\mu_{f,k(1)\kappa_1} * \cdots * \mu_{f,k(l)\kappa_l} \perp \mu_{f,k'(1)\kappa'_1} * \cdots * \mu_{f,k'(l')\kappa'_{l'}}.$$

If there does not exist such $r_0 \in S_t$, by Theorem 1.3.8, for comeagerly many

$$f \in L^0(\mathbb{T})$$

$$f^{k(1)\kappa_1+\dots+k(l)\kappa_l} \cap f^{k'(1)\kappa'_1+\dots+k'(l')\kappa'_{l'}} \approx \emptyset$$

Thus, for comeagerly many $f \in L^0(\mathbb{T})$ the push-forward measures of $\lambda_{\kappa_1} \times \dots \times \lambda_{\kappa_l}$ and $\lambda_{\kappa'_1} \times \dots \times \lambda_{\kappa'_{l'}}$ under $f^{k(1)\kappa_1+\dots+k(l)\kappa_l}$ and $f^{k'(1)\kappa'_1+\dots+k'(l')\kappa'_{l'}}$, respectively, are perpendicular to each other. Hence,

$$\mu_{f,k(1)\kappa_1} * \dots * \mu_{f,k(l)\kappa_l} \perp \mu_{f,k'(1)\kappa'_1} * \dots * \mu_{f,k'(l')\kappa'_{l'}}. \quad (1.10)$$

Note that the general statement follows from (1.10) since for Borel measures

$$\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n, \mu'_1, \dots, \mu'_m, \nu'_1, \dots, \nu'_n$$

where $\mu'_i \ll \mu_i$ and $\nu'_j \ll \nu_j$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

$$\text{if } \mu_1 * \dots * \mu_m \perp \nu_1 * \dots * \nu_n, \text{ then } \mu'_1 * \dots * \mu'_m \perp \nu'_1 * \dots * \nu'_n.$$

1.8 Generic behavior of a unitary transformation

Let H be a separable infinite dimensional Hilbert space and $\psi \in H$ be a vector of length 1. Melleray–Tsankov [14] proved that for a generic $u \in \mathcal{U}(H)$, $\overline{\langle u \rangle}$ is isomorphic to $L^0(\mathbb{T})$.

Furthermore, they showed that for a generic $u \in \mathcal{U}(H)$ the representation of $L^0(\mathbb{T})$ obtained from $\overline{\langle u \rangle} \cong L^0(\mathbb{T})$ has only one non-zero measure, namely, $\lambda_1^1 = \mu_u$. In the following, we use this result of Melleray–Tsankov [14] and Theorem 1.4.5 to show orthogonality conditions for a generic $u \in \mathcal{U}(H)$ analogous to orthogonality conditions in Theorem 1.4.2.

Corollary 1.8.1. *Let H be a separable infinite dimensional Hilbert space. Then, for a generic $u \in \mathcal{U}(H)$, the convolutions*

$$\sigma_{u^{k(1)}} * \cdots * \sigma_{u^{k(l)}} \text{ and } \sigma_{u^{k'(1)}} * \cdots * \sigma_{u^{k'(l')}}$$

are mutually singular, provided that there does not exist $r \in S_l$ such that

$$(k'(1), k'(2), \dots, k'(l')) = r(k(1), k(2), \dots, k(l)).$$

We need the following theorem to prove Corollary 1.8.1.

Theorem 1.8.2. *Let H be a separable infinite dimensional Hilbert space and A be a subset of $\mathcal{U}(H)$ with the Baire property. Then, A is comeager (meager, respectively) iff $A \cap \overline{\langle u \rangle}$ is comeager (meager, respectively) in $\overline{\langle u \rangle}$ for comeagerly many $u \in \mathcal{U}(H)$.*

Solecki [16, Lemma 3] proved a similar statement for $A \subseteq \text{Aut}(\mu)$ with the Baire property. Assuming the following two lemmas (Lemma 1.8.3, 1.8.4), the same proof can be repeated to prove Theorem 1.8.2.

Lemma 1.8.3. *Let H be a separable infinite dimensional Hilbert space. For a non-zero integer number, n , we define*

$$f_n : \mathcal{U}(H) \rightarrow \mathcal{U}(H)$$

such that

$$f_n(u) = u^n$$

Then, f_n is category preserving for every $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Since $f_{-n} = f_n \circ f_{-1}$ and f_{-1} is a homeomorphism and therefore

category preserving, we may assume $n \in \mathbb{N}$. It is enough to show that if $A \subseteq \mathcal{U}(H)$ is dense and open, then $f_n^{-1}(A)$ is dense. Let

$$W = \{u \in \mathcal{U}(H) : \forall 1 \leq i \leq k \ \|u(Z_i) - u_0(Z_i)\| < \epsilon\}$$

for some $u_0 \in \mathcal{U}(H)$, $Z_i \in H$ for $i = 1, 2, \dots, k$, and $\epsilon > 0$. We have to find $\tilde{u} \in W$ such that $\tilde{u}^n \in A$. Let

$$H_0 = \text{span}\{Z_1, \dots, Z_k, u_0(Z_1), \dots, u_0(Z_k)\}.$$

By modifying u_0 on $\text{span}\{Z_1, \dots, Z_k\}^\perp$, we may assume that $u_0(H_0) = H_0$ and $u_0 \upharpoonright H_0^\perp = id$. Since $u_0 \upharpoonright H_0$ is a unitary operator of H_0 , it is diagonalizable. Thus, we can find an orthonormal basis for H_0 , $\{h_1, \dots, h_m\}$, and eigenvalues $\lambda_1, \dots, \lambda_m$ such that

$$u_0(h_i) = \lambda_i h_i \quad \text{for } i = 1, 2, \dots, m.$$

Furthermore, we may assume that $\{(\lambda_1^{ni}, \dots, \lambda_m^{ni})\}_{i=1}^\infty$ is dense in \mathbb{T}^m . Let $l \in \mathbb{N}$ be such that

$$\sum_{i=1}^m \|\lambda_i^{nl} - \lambda_i\| \ll \epsilon.$$

Then, $\|u_0^{nl} - u_0\| \ll \epsilon$. Since A is open and dense, there exists $u_1 \in A$ and a finite dimensional subspace $H_0 \subseteq H_1$ such that $u_1(H_1) = H_1$ and

$$\|u_1(h_i) - u_0^n(h_i)\| \ll \epsilon \quad \text{for } i = 1, 2, \dots, m.$$

Let $\{h'_i\}_{i=1}^\infty$ be an orthonormal basis for H where for every $i \in \mathbb{N}$

$$u_1(h'_i) = \lambda'_i h'_i.$$

Note that since u_1 is unitary, it is possible to find such basis for H . Moreover, by modifying u_1 on H_1^\perp , we may assume that $H_0 \subseteq \text{span}\{h'_1, \dots, h'_{nm}\}$ and for every $i \in \mathbb{N}$

$$\lambda'_{ni-n+1} = \dots = \lambda'_{ni}.$$

We define $\tilde{u} \in \mathcal{U}(H)$ such that

$$\tilde{u}(h'_{an+b}) = \begin{cases} u_1^l(h'_{an+b+1}) & \text{if } 1 \leq b \leq n-1, \\ u_1^{1-(n-1)l}(h'_{an+1}) & \text{if } b = n. \end{cases}.$$

Then, $\tilde{u}^n = u_1$. Moreover, for every $1 \leq i \leq k$

$$\|u_1(Z_i) - u_0^n(Z_i)\| \ll \epsilon \text{ and } \|u_0(Z_i) - u_0^{nl}(Z_i)\| \ll \epsilon.$$

Therefore, for every $i = 1, 2, \dots, k$

$$\|u_1^l(Z_i) - u_0(Z_i)\| \ll \epsilon \text{ and } \|u_1^{1-(n-1)l}(Z_i) - u_0(Z_i)\| \ll \epsilon.$$

Thus, for every $i = 1, 2, \dots, k$

$$\|\tilde{u}(Z_i) - u_0(Z_i)\| < \epsilon,$$

that is, $\tilde{u} \in W$. □

Lemma 1.8.4. *Let H be a separable infinite dimensional Hilbert space.*

Then,

$$\{u \in \mathcal{U}(H) : \{vuv^{-1} : v \in \mathcal{U}(H)\} \text{ is dense}\}$$

is dense.

Proof. We say a unitary operator $u \in \mathcal{U}(H)$ is diagonalizable if there is a basis of H , $\{h_i\}_{i=1}^\infty$, and eigenvalues, $\{\lambda_i\}_{i=1}^\infty$, such that for every $i \in \mathbb{N}$

$$u(h_i) = \lambda_i h_i.$$

Let

$$D = \{u \in \mathcal{U}(H) : u \text{ is diagonalizable and for every } N \in \mathbb{N}, \\ \{\lambda_i\}_{i=N}^\infty \text{ is dense in } \mathbb{T}\}.$$

We claim that D is dense and for every $u \in D$

$$\{vuv^{-1} : v \in \mathcal{U}(H)\}$$

is dense. Let

$$W = \{u \in \mathcal{U}(H) : \forall 1 \leq i \leq k \ \|u(Z_i) - u_0(Z_i)\| < \epsilon\}$$

for some $u_0 \in \mathcal{U}(H)$, $Z_i \in H$ for $i = 1, 2, \dots, k$, and $\epsilon > 0$. Let

$$H_0 = \text{span}\{Z_1, \dots, Z_k, u_0(Z_1), \dots, u_0(Z_k)\}.$$

By modifying u_0 on $\text{span}\{Z_1, \dots, Z_k\}^\perp$, we may assume that $u_0(H_0) = H_0$. By modifying u_0 on H_0^\perp , we can find $u \in W \cap D$. Furthermore, assume $\lambda_1, \dots, \lambda_m$ are eigenvalues of $u_0 \upharpoonright H_0$, h_1, \dots, h_m are corresponding eigenvectors of $u_0 \upharpoonright H_0$, and $u \in D$. Then, there is an orthonormal set of

vectors, $\{h'_1, \dots, h'_m\}$, such that for every $i = 1, 2, \dots, m$

$$u(h'_i) = \lambda'_i h'_i$$

where $|\lambda_i - \lambda'_i| \ll \epsilon$. Then, there is a unitary operator, v , that sends (h_1, \dots, h_m) to (h'_1, \dots, h'_m) . Then, $v^{-1}uv \in W$. \square

Proof of Theorem 1.8.2. Let $A \subseteq \mathcal{U}(H)$ be a comeager subset. Let $B \subseteq A$ be a dense G_δ subset. Define

$$B_\infty = \bigcap_{n \in \mathbb{Z} \setminus \{0\}} f_n^{-1}(B).$$

By Lemma 1.8.3, B_∞ is comeager. Let C be the set of unitary operators $u \in B_\infty$ where $\{u^n : n \in \mathbb{Z}\}$ is not discrete. Since the set of $u \in \mathcal{U}(H)$ where $\{u^n : n \in \mathbb{Z}\}$ is not discrete is comeager, C is comeager. For every $u \in C$,

$$\{u^n : n \in \mathbb{Z} \setminus \{0\}\} \subseteq B$$

is dense in $\overline{\langle u \rangle}$. Therefore, for every $u \in C$, $B \cap \overline{\langle u \rangle}$ is dense G_δ in $\overline{\langle u \rangle}$. Hence, for every $u \in C$, $A \cap \overline{\langle u \rangle}$ is comeager in $\overline{\langle u \rangle}$.

To complete the proof, it is enough to show that if A is non-meager, then there is a non-meager set of $u \in \mathcal{U}(H)$ such that $A \cap \overline{\langle u \rangle}$ is non-meager. Since A is non-meager, there is an open subset $W \subseteq \mathcal{U}(H)$ such that A is comeager in W . By Lemma 1.8.4, there exists $u_0 \in W$ such that

$$\{vu_0v^{-1} : v \in \mathcal{U}(H)\}$$

is dense. Therefore, we can find a sequence of unitary operators on H , $\{v_i\}_{i=1}^\infty$, such that $\Omega = \bigcup_{i=1}^\infty v_i W v_i^{-1}$ is comeager. Thus, there are comea-

gerly many $u \in \mathcal{U}(H)$ such that $\Omega \cap \overline{\langle u \rangle}$ is comeager in $\overline{\langle u \rangle}$. Hence, there is a natural number, m , and a non-meager set of unitary operators u on H such that $v_m W v_m^{-1}$ is non-meager in $\overline{\langle u \rangle}$. For every such u , A is non-meager in $\overline{\langle v_m^{-1} u v_m \rangle}$. \square

Proof of Corollary 1.8.1. By [14, Theorem 4.4], for a generic $u \in \mathcal{U}(H)$, the representation of $L^0(\mu, \mathbb{T})$ obtained by $\overline{\langle u \rangle} \cong L^0(\mu, \mathbb{T})$ is equal to $\sigma(1, \mu_u)$, that is, the representation has only one non-zero measure, namely, $\lambda_1^1 = \mu_u$. Therefore, by Theorem 1.4.5, the representation satisfies the DL-condition. Thus, for a generic $u \in \mathcal{U}(H)$, for a generic $v \in \overline{\langle u \rangle}$, we have: for every $k(1), k(2), \dots, k(l) \in \mathbb{Z}^+$, $k'(1), k'(2), \dots, k'(l') \in \mathbb{Z}^+$, the convolutions

$$\sigma_{v^{k(1)}} * \dots * \sigma_{v^{k(l)}} \text{ and } \sigma_{v^{k'(1)}} * \dots * \sigma_{v^{k'(l')}}$$

are mutually singular, provided that there does not exist $r \in S_l$ such that

$$(k'(1), k'(2), \dots, k'(l')) = r(k(1), k(2), \dots, k(l)).$$

Fix $k(1), k(2), \dots, k(l) \in \mathbb{Z}^+$, $k'(1), k'(2), \dots, k'(l') \in \mathbb{Z}^+$, where there does not exist $r \in S_l$ such that

$$(k'(1), k'(2), \dots, k'(l')) = r(k(1), k(2), \dots, k(l)).$$

Let

$$G = \{u \in \mathcal{U}(H) : \sigma_{u^{k(1)}} * \dots * \sigma_{u^{k(l)}} \perp \sigma_{u^{k'(1)}} * \dots * \sigma_{u^{k'(l')}}\}.$$

Since G has the Baire property, by Theorem 1.8.2, G is a comeager subset of $\mathcal{U}(H)$. \square

Chapter 2

Extending partial isometries

2.1 Introduction

The second problem on the famous Scottish book asks whether it is possible to define an isometry-invariant finitely additive measure on a compact metric space.

Definition 2.1.1. *Let X be a metric space and suppose $E \subseteq X$. E is **paradoxical** if for some positive integers m, n there are pairwise disjoint subsets*

$$A_1, \dots, A_m, B_1, \dots, B_n \subseteq E$$

and

$$A'_1, \dots, A'_m, B'_1, \dots, B'_n \subseteq E$$

such that A_i, A'_i and B_j, B'_j are congruent and $E = \bigcup A'_i = \bigcup B'_j$.

Tarski showed that if a group G acts on a metric space, X , then there exists a G -invariant Borel probability measure on X iff X is not paradoxical. Note that the set of partial isometries of a compact metric space generally is

not a group and therefore we can not apply the above equivalence directly. Now, it is natural to ask whether it is possible to extend a compact metric space, X , to another compact metric space, Y , such that every partial isometry of X extends to an isometry of Y .

Let C_1, C_2 be two structures in a given finite relational language \mathcal{L} . A **partial isomorphism** from C_1 into C_2 is an isomorphism of a substructure of C_1 onto a substructure of C_2 .

Definition 2.1.2. *Let \mathcal{C} be a class of \mathcal{L} -structures (containing both finite and infinite structures). \mathcal{C} is said to have **the extension property for partial automorphisms (EPPA)** if whenever C_1 and C_2 are structures in \mathcal{C} , C_1 is finite, C_1 is a substructure of C_2 , and every partial automorphism of C_1 extends to an automorphism of C_2 , then there exist a finite structure C_3 in \mathcal{C} which extends C_1 and every partial automorphism of C_1 extends to an automorphism of C_3 .*

Hrushovski [8] was one of the first papers to consider the question of whether a certain class of structures has the EPPA. More precisely, he showed that the class of simple graphs has the EPPA, that is, every finite graph G can be extended to another finite graph, H , such that every partial isomorphism of G extends to an isomorphism of H . Herwig–Lascar [7] generalized the result of Hrushovski to finite relational structures.

Definition 2.1.3. *If M is an \mathcal{L} -structure and \mathcal{T} a set of \mathcal{L} -structures, we say that M is **\mathcal{T} -free** if there is no structure $T \in \mathcal{T}$ and weak homomorphism $h : T \xrightarrow{w} M$.*

Theorem 2.1.4 (Herwig–Lascar). *Let \mathcal{L} be a finite relational language and \mathcal{T} a finite set of finite \mathcal{L} -structures. Then the class of \mathcal{T} -free \mathcal{L} -structures has the EPPA.*

Solecki [18] considered the class of metric spaces. He used Theorem 2.1.4 to show that the class of metric spaces has the EPPA.

Theorem 2.1.5 (Solecki). *Let X be a finite metric space. There exists a finite metric space Y such that X isometrically embeds into Y and every partial isometry of X extends to a full isometry of Y .*

In Section 2.2, we present an elementary proof of Theorem 2.1.5 for a special class of finite metric spaces. The proof is of interest because it is independent of Theorem 2.1.4. In Section 2.3, we consider the extension problem in the class compact metric spaces and we show that for every compact metric space, X , and every $\epsilon > 0$ there exists a compact metric space, Y , which extends X and every partial isometry of X extends to an isometry of Y with ϵ distortion. In Section 2.4, we generalize the idea of the construction in Section 2.2 to show that a compact metric space X can be extended to another compact metric space, Y , such that every partial isometry of X extends to an isometry of Y if and only if there exists a compact metric space Y that extends X and a compact group $G \subseteq \text{Homeo}(Y)$ such that every partial isometry of X , p , extends to $p^* \in G$ with the property that for every $x \in \text{dom}(p)$ and $y \in Y$

$$d(x, y) = d(p^*(x), p^*(y)).$$

2.2 Special class of finite metric spaces

Solecki [18] proved that every finite metric space X can be extended to another metric space, Y , such that every partial isometry of X extends to an isometry of Y . He used the general result of Herwig–Lascar [7] which states that the class of τ -free finite relational structures has EPPA. In this

section, we present a direct proof of the same result for a finite metric space, (X, d) , with the property that the summation of the least two distances in X is greater or equal to the maximum distance in X , that is, if d_{\max} is the maximum distance in X , then for every

$$x_1, x_2, x_3, x_4 \in X$$

where $x_1 \neq x_2$ and $x_3 \neq x_4$ we have

$$d(x_1, x_2) + d(x_3, x_4) \geq d_{\max}.$$

In particular, this extends the result of Hurshovski [8] for finite graphs. We break the proof into two parts. In Lemma 2.2.1, we prove that every such finite metric space, (X, d) , can be extended to another finite metric space, (Y, ρ) , such that every partial isometry of X , p , extends to $p^* \in \text{Sym}(Y)$ with the property that if $x \in X$ and $y \in Y$, then

$$\rho(x, y) = \rho(p^*(x), p^*(y)).$$

Lemma 2.2.1. *Let (X, d) be a finite metric space with the property that the summation of the least two distances in X is greater or equal to the maximum distance in X . There exists a finite metric space, (Y, ρ) , such that (Y, ρ) extends (X, d) and for every partial isometry of X , p , there exists $p^* \in \text{Sym}(Y)$ that extends p and if*

$$\text{dom}(p) = \{y_1, \dots, y_k\}, \text{ range}(p) = \{y'_1, \dots, y'_k\}, y'_i = p(y_i) \text{ for all } 1 \leq i \leq k,$$

then for every d_1, \dots, d_k ,

$$\{y \in Y : \rho(y, y'_i) = d_i, 1 \leq i \leq k\} = p^* \{y \in Y : \rho(y, y_i) = d_i, 1 \leq i \leq k\}.$$

Furthermore, we can choose p^* such that $(p^*)^{-1} = (p^{-1})^*$.

Proof. Let $X = \{x_1, \dots, x_n\}$. Since the conclusion is clear for $n \leq 2$, we may assume $n \geq 3$. Define G a weighted graph as follows:

1. $V(G)$, the set of vertices of G , is equal to

$$\{(d_{ij_1}, \dots, d_{ij_n}) : 1 \leq i \leq n, (j_1, \dots, j_n) \text{ is a permutation of } (1, \dots, n)\},$$

2. for every $1 \leq i, k \leq n$ and every permutation of $(1, \dots, n)$, τ , if

$$\tau \neq id \text{ or } i \neq k,$$

then

$$\begin{aligned} ((d_{k1}, \dots, d_{kn}), (d_{i\tau(1)}, \dots, d_{i\tau(n)})) &\in E(G) \text{ with} \\ w((d_{k1}, \dots, d_{kn}), (d_{i\tau(1)}, \dots, d_{i\tau(n)})) &= d(x_i, x_{\tau(k)}) + \epsilon \delta_{i, \tau(k)} \end{aligned}$$

where $\delta_{i, \tau(k)}$ is the Kronecker delta function and $\epsilon > 0$ is smaller than any non-zero distance in X , that is, for every distinct $1 \leq i, j \leq n$

$$\epsilon < d(x_i, x_j),$$

3. for every $1 \leq k \leq n$

$((d_{k1}, \dots, d_{kn}), (d_{k1}, \dots, d_{kn})) \in E(G)$ with

$$w((d_{k1}, \dots, d_{kn}), (d_{k1}, \dots, d_{kn})) = 0.$$

4. points (2) and (3) describe all edges of G .

For every $y, y' \in G$, we say $(y_i)_{i=0}^k$ is a path of length k from y to y' if

$$y_0 = y, y_k = y', \text{ and } (y_i, y_{i+1}) \in E(G) \text{ for all } i \in \{0, 1, \dots, k-1\}$$

For a path $(y_i)_{i=0}^k$, we define its weight to be $\sum_{i=0}^{k-1} w(y_i, y_{i+1})$. Let (Y, ρ) be the metric space obtained by considering G with the path metric, that is,

$$\rho(y, y') = \inf(\{\sum_{i=0}^{k-1} w(y_i, y_{i+1}) : (y_i)_{i=0}^k \text{ is a path from } y \text{ to } y'\} \cup \{1\}).$$

We claim that (Y, ρ) has the desired properties. Since

$$x_i \mapsto (d_{i1}, \dots, d_{in})$$

induces an isometric copy of X in Y , it is enough to show that for every partial isometry of X , p , with

$$\text{dom}(p) = \{y_1, \dots, y_k\}, \text{ range}(p) = \{y'_1, \dots, y'_k\}, y'_i = p(y_i) \text{ for all } 1 \leq i \leq k,$$

and for every d_1, \dots, d_k , the number of points in Y with distances d_1, \dots, d_k from y'_1, \dots, y'_k , respectively, is the same as the number of points in Y with distances d_1, \dots, d_k from y_1, \dots, y_k , respectively.

Fix d_1, \dots, d_k . We prove the property by considering the following cases:

1. $d_l = 0$ for some $1 \leq l \leq k$. In this case, there is at most one point in Y , y_l , with distances d_1, \dots, d_k from y_1, \dots, y_k , respectively. Furthermore, y_l has distances d_1, \dots, d_k from y_1, \dots, y_k , respectively, if and only if $y'_l = p(y_l)$ has distances d_1, \dots, d_k from y'_1, \dots, y'_k , respectively. Thus, the number of points with distances d_1, \dots, d_k from y_1, \dots, y_k is equal to the number of points with distances d_1, \dots, d_k from y'_1, \dots, y'_k .
2. $d_1, \dots, d_k > \epsilon$. Let r_i be the number of permutations of $(1, \dots, n)$, τ , such that

$$d_m = d(x_i, x_{\tau(i_m)}) \text{ for every } 1 \leq m \leq k.$$

Then, there are r_i many points in Y of the form $(d_{i\tau(1)}, \dots, d_{i\tau(n)})$ with distances d_1, \dots, d_k from y_1, \dots, y_k , respectively. The number of points in Y of the form $(d_{i\tau(1)}, \dots, d_{i\tau(n)})$ with distances d_1, \dots, d_k from y'_1, \dots, y'_k , respectively, is equal to the number of permutations of $(1, \dots, n)$, τ , such that

$$d_m = d(x_i, x_{\tau(j_m)}) \text{ for every } 1 \leq m \leq k,$$

which is also equal to r_i .

3. $d_1, \dots, d_k > 0$ and $d_l = \epsilon$ for some $1 \leq l \leq k$. Note that such l is unique. Let r_i be the number of permutations of $(1, \dots, n)$, τ , such that $\tau(i) = i_l$ and

$$d_m = d(x_i, x_{\tau(i_m)}) \text{ for every } m \in \{1, \dots, l-1, l+1, \dots, k\}.$$

Then, the number of points in Y of the form $(d_{i\tau(1)}, \dots, d_{i\tau(n)})$ with distances d_1, \dots, d_k from y_1, \dots, y_k , respectively, is equal to the num-

ber of non-identity permutations of $(1, \dots, n)$, τ , such that $\tau(i) = i_l$ and

$$d_m = d(x_i, x_{\tau(i_m)}) \text{ for every } m \in \{1, \dots, l-1, l+1, \dots, k\}.$$

Moreover, the number of points in Y of the form $(d_{i\tau(1)}, \dots, d_{i\tau(n)})$ with distances d_1, \dots, d_k from y'_1, \dots, y'_k , respectively, is equal to the number of non-identity permutations of $(1, \dots, n)$, τ , such that $\tau(i) = j_l$ and

$$d_m = d(x_i, x_{\tau(j_m)}) \text{ for every } m \in \{1, \dots, l-1, l+1, \dots, k\}.$$

If $i \neq i_l$ and $i \neq j_l$, then the number of points in Y of the form $(d_{i\tau(1)}, \dots, d_{i\tau(n)})$ with distances d_1, \dots, d_k from y_1, \dots, y_k , respectively, and the number of points in Y of the form $(d_{i\tau(1)}, \dots, d_{i\tau(n)})$ with distances d_1, \dots, d_k from y'_1, \dots, y'_k , respectively, are equal to r_i .

Assume $i = i_l$ or $i = j_l$. The number of points in Y of the form $(d_{i_l\tau(1)}, \dots, d_{i_l\tau(n)})$ or $(d_{j_l\tau(1)}, \dots, d_{j_l\tau(n)})$ with distances d_1, \dots, d_k from y_1, \dots, y_k , respectively, is equal to the number of non-identity permutations of $(1, \dots, n)$, τ , such that $\tau(i_l) = i_l$ and

$$d_m = d(x_{i_l}, x_{\tau(i_m)}) \text{ for every } m \in \{1, \dots, l-1, l+1, \dots, k\},$$

or $\tau(j_l) = i_l$ and

$$d_m = d(x_{j_l}, x_{\tau(i_m)}) \text{ for every } m \in \{1, \dots, l-1, l+1, \dots, k\}.$$

This is equal to the number of non-identity permutations of $(1, \dots, n)$,

τ , such that $\tau(j_l) = j_l$ and

$$d_m = d(x_{j_l}, x_{\tau(j_m)}) \text{ for every } m \in \{1, \dots, l-1, l+1, \dots, k\},$$

or $\tau(i_l) = j_l$ and

$$d_m = d(x_{i_l}, x_{\tau(j_m)}) \text{ for every } m \in \{1, \dots, l-1, l+1, \dots, k\}.$$

Finally, we observe that this number is equal to the number of points in Y of the form $(d_{j_l\tau(1)}, \dots, d_{j_l\tau(n)})$ or $(d_{i_l\tau(1)}, \dots, d_{i_l\tau(n)})$ with distances d_1, \dots, d_k from y'_1, \dots, y'_k , respectively. \square

In the following theorem, we use Lemma 2.2.1 to show that the class of finite metric spaces, X , with the property that the summation of the least two distances in X is greater or equal to the maximum distance in X , has the EPPA.

Theorem 2.2.2. *Let (X, d) be a finite metric space such that there exists a finite metric space, (Y, d') , where (Y, d') extends (X, d) and for every partial isometry of X , p , there exists $\tilde{p} \in \text{Sym}(Y)$ that extends p and if*

$$\text{dom}(p) = \{y_1, \dots, y_k\}, \text{ range}(p) = \{y'_1, \dots, y'_k\}, y'_i = p(y_i) \text{ for all } 1 \leq i \leq k,$$

then for every d_1, \dots, d_k ,

$$\{y \in Y : d'(y, y'_i) = d_i, 1 \leq i \leq k\} = \tilde{p}\{y \in Y : d'(y, y_i) = d_i, 1 \leq i \leq k\}.$$

Then, there exists a finite metric space, (Z, ρ) , such that (Z, ρ) extends (X, d) and for every partial isometry of X , p , there exists an isometry of Z , p^ , that extends p .*

Proof. Let P be the set of partial isometries of X and $H = Y \times G$ where

$$G = \langle \{\tilde{p} : p \in P\} \rangle.$$

We consider H as a graph with a weight function $w : E(H) \rightarrow \mathbb{R}^{\geq 0}$ as follows:

1. $V(H) = Y \times G$,

2. if $y, y' \in Y$ then

$$\forall g \in G : ((y, g), (y', g)) \in E(H) \text{ and } w(((y, g), (y', g)))) = d'(y, y'),$$

3. if $y \in \text{range}(p)$ for some $p \in P$, then

$$\forall g \in G : ((y, g), (p_i^{-1}(y), g\tilde{p}_i)) \in E(H) \text{ and}$$

$$w(((y, g), (p_i^{-1}(y), g\tilde{p}_i)))) = 0,$$

4. points (2) and (3) describe all edges of H .

For every $z, z' \in H$, we say $(z_i)_{i=0}^k$ is a path of length k if

$$z_0 = z, \quad z_k = z'.$$

Furthermore, we say $(z_i)_{i=0}^k$ is a proper path from z to z' if

$$z_0 = z, \quad z_k = z', \text{ and } (z_i, z_{i+1}) \in E(H) \text{ for all } i \in \{0, 1, \dots, k-1\}.$$

For a path $(z_i)_{i=0}^k$, we define its weight to be $\sum_{i=0}^{k-1} w(z_i, z_{i+1})$ with the con-

vention that if (z_i, z_{i+1}) is not an edge, then

$$w(z_i, z_{i+1}) = 0.$$

We define (Z, ρ) to be the metric space obtained by considering H with the path metric, that is,

$$\rho(z, z') = \inf\{\sum_{i=0}^{k-1} w(z_i, z_{i+1}) : (z_i)_{i=0}^k \text{ is a proper path from } z \text{ to } z'\}.$$

We claim (Z, ρ) has the desired properties. One can show that ρ is a metric and (Z, ρ) is finite. Furthermore, if we assume $Y \times \{e\}$ is an isometric copy of Y inside Z , then $p^* : Z \rightarrow Z$ defined by

$$p^*((y, g)) = (y, \tilde{p}g)$$

is an isometry of Z which extends p . Note that if $(y_i, g_i)_{i=0}^k$ is a proper path from (y, g) to (y', g') then for every $p \in P$, $(y_i, \tilde{p}g_i)_{i=0}^k$ is a proper path from $(y, \tilde{p}g)$ to $(y', \tilde{p}g')$ with the same weight.

It remains to show that $Y \times \{e\} \subseteq Z$ is an isometric copy of Y . It suffices to show that the function $f : Y \rightarrow Z$ defined by

$$f(y) = (y, e)$$

is an isometry. By definition of ρ , we have

$$\forall y, y' \in Y : \rho((y, e), (y', e)) \leq d'(y, y').$$

We claim that for every $y, y' \in Y$ a proper path from (y, e) to (y', e) has a

weight greater than or equal to $d'(y, y')$. Let $(z_i = (y_i, g_i))_{i=0}^k$ be a proper path from (y, e) to (y', e) . We define $(h_i)_{i=0}^{k-1}$ such that for every $0 \leq i \leq k-1$

$$g_{i+1} = g_i h_i.$$

Note that h_i is in $\{\tilde{p} : p \in P\} \cup \{e\}$. If (z_j, z_{j+1}, z_{j+2}) , for some j , be such that $g_j = g_{j+1} = g_{j+2}$, then by replacing $(z_i)_{i=0}^k$ with

$$(z_0, z_1, \dots, z_j, z_{j+2}, \dots, z_k)$$

we get another proper path with a smaller or equal weight. Thus, we may assume

$$\text{for every } 0 \leq j \leq k-2 : g_j \neq g_{j+1} \text{ or } g_{j+1} \neq g_{j+2}. \quad (2.1)$$

Furthermore, by repeating some points in the proper path if necessary, we may assume k is an odd number and for every $0 \leq i \leq k-1$, h_i is identity (non-identity) if i is an even (odd) number. Since $z_0, z_k \in Y \times \{e\}$, we have

$$h_1 h_3 \dots h_{k-2} = e.$$

Note that since $h_1 \neq e$, $k \geq 5$. For $(y, g) \in Z$ and $p \in P$, we define

$$\tilde{p}(y, g) = (\tilde{p}^{-1}(y), g\tilde{p}).$$

Then,

$$(z_0, z_1, \dots, z_{k-5}, h_{k-4}^{-1}(z_{k-2}), z_{k-2}, z_{k-1}, z_k)$$

is a path from (y, e) to (y', e) with a smaller or equal weight since

$$d'(y_{k-4}, h_{k-4}(y_{k-2})) = d'(h_{k-4}^{-1}(y_{k-4}), y_{k-2}).$$

Note that if $h_{k-4} = \tilde{p}$ for some $p \in P$, then y_{k-4} is in $\text{dom}(p)$ and for every $x \in \text{dom}(p)$ and $y \in Y$, we have

$$d'(x, y) = d'(\tilde{p}(x), \tilde{p}(y)).$$

By induction, one can show that for every $1 \leq i \leq \frac{k-3}{2}$

$$(z_0, z_1, \dots, z_{k-2i-3}, h_{k-2i-2}^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), h_{k-2i}^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), \dots \\ \dots, h_{k-4}^{-1}(z_{k-2}), z_{k-2}, z_{k-1}, z_k)$$

is a path from (y, e) to (y', e) with a smaller or equal weight. In particular, if $i = \frac{k-3}{2}$,

$$(z_0, h_1^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), h_3^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), \dots, h_{k-4}^{-1}(z_{k-2}), z_{k-2}, z_{k-1}, z_k)$$

is a path from (y, e) to (y', e) with smaller or equal weight. Since

$$h_1^{-1} \cdots h_{k-4}^{-1}(z_{k-2}) = (h_1 \cdots h_{k-4}(y_{k-2}), e) = (h_{k-2}^{-1}(y_{k-2}), e) = z_{k-1},$$

the weight of this path is

$$d'(y_0, y_{k-1}) + d'(y_{k-1}, y_k) \geq d'(y_0, y_k). \quad \square$$

2.3 Extending partial isometries with ϵ -distorsion

Definition 2.3.1. *Let X, Y be metric spaces such that $X \subseteq Y$, p be a partial isometry of X , and q be a full isometry of Y . For $\epsilon > 0$, we say q **extends** p **with ϵ distortion** if*

$$\forall x \in \text{dom}(p) \ (d_Y(p(x), q(x)) < \epsilon).$$

Pestov [15] proved EPPA with ϵ distortion for the class of metric spaces, that is, for every $\epsilon > 0$ and every finite metric space, X , there exists another finite metric space, Y , such that every partial isometry of X extends to a full isometry of Y with ϵ -distortion.

We extend the result of Pestov for finite metric spaces to compact metric spaces, that is,

Theorem 2.3.2. *Let (X, d) be a compact metric space. For every $\epsilon > 0$ there exists a compact metric space, (Y, d') , such that Y extends X and for every partial isometry of X , p , there exists a full isometry of Y , q , so that:*

$$\forall x \in \text{dom}(p) \ (d'(p(x), q(x)) < \epsilon).$$

Let (X, d) be a metric space. We define $\mathbb{P}(X)$ to be the set of maximal partial isometries of X . Note that $\mathbb{P}(X)$ is a closed subset of $K(X)$, compact subsets of X . Therefore, we can define the Hausdorff metric, ρ_H , on $\mathbb{P}(X)$, that is

$$p, q \in \mathbb{P}_X : \ \rho_H(p, q) = \sup_{x \in \text{dom}(p)} \inf_{y \in \text{dom}(q)} \{d(x, y) + d(p(x), q(y))\}.$$

First, we prove a weaker version of theorem 2.3.2.

Proposition 2.3.3. *Let (X, d) be a compact metric space and p_1, p_2, \dots, p_n be partial isometries of X . For every $\epsilon > 0$, there exists a compact metric space, (Y, d') , which extends X and each p_i extends to a full isometry of Y , q_i , with ϵ distortion, that is,*

$$\forall x \in \text{dom}(p_i) \ d'(p_i(x), q_i(x)) < \epsilon.$$

Proof. WLOG, we may assume $\text{diam}(X) = 1$. Let F_n be the free group with n generators, $\{a_1, a_2, \dots, a_n\}$. Let $M \in \mathbb{N}$ be such that $M\epsilon > 1$. By Hall's theorem, there exists $H \trianglelefteq F_n$ such that $[F_n : H] < \infty$ and

$$H \cap \{e_1 e_2 \dots e_{2M} \neq e : e_i \in \{a_1, a_2, \dots, a_n, e\} \text{ for } 1 \leq i \leq 2M\} = \emptyset.$$

Let $Y = X \times F$ where $F = F_n/H$. We define a graph, G , and a weight function $w : E(G) \rightarrow \mathbb{R}^{\geq 0}$ as follows:

1. $V(G) = Y$,
2. if $x, x' \in X$ then

for every $a \in F : ((x, a), (x', a)) \in E(G)$ and

$$w(((x, a), (x', a)))) = d_X(x, x'),$$

3. if $x \in \text{range}(p_i)$ for some $1 \leq i \leq n$ then

for every $\omega \in F : ((x, \omega), (p_i^{-1}(x), \omega a_i)) \in E(G)$ and

$$w(((x, \omega), (p_i^{-1}(x), \omega a_i)))) = \epsilon$$

where by a_i we mean $a_i H \in F_n/H$,

4. points (2) and (3) describe all edges of G .

For every $y, y' \in Y$, we say $(y_i)_{i=0}^k$ is a path of length k from y to y' if

$$y_0 = y, y_k = y', \text{ and } (y_i, y_{i+1}) \in E(G) \text{ for all } i \in \{0, 1, \dots, k-1\}$$

For a path $(y_i)_{i=0}^k$, we define its weight to be $\sum_{i=0}^{k-1} w(y_i, y_{i+1})$. We define a metric, d' , on Y as follows

$$d'(y, y') = \inf(\{\sum_{i=0}^{k-1} w(y_i, y_{i+1}) : (y_i)_{i=0}^k \text{ is a path from } y \text{ to } y'\} \cup \{1\})$$

We claim (Y, d') has the desired properties. It is easy to see that d' is actually a metric and (Y, d') is compact. Also, if we assume $X \times \{e\}$ is an isometric copy of X inside Y , then $q_i : Y \rightarrow Y$ defined by

$$q_i((x, \omega)) = (x, a_i \omega) \text{ for all } i \in \{1, 2, \dots, n\}$$

is the desired isometry of Y which extends p_i with at most ϵ distortion. Note that if $(x_i, \omega_i)_{i=0}^k$ is a path from (x, ω) to (x', ω) then $(x_i, a_j \omega_i)_{i=0}^k$ is a path from $(x, a_j \omega)$ to $(x', a_j \omega)$ for every $1 \leq j \leq n$. Furthermore, all such paths have the same length.

It remains to show that $X \times \{e\} \subseteq Y$ is an isometric copy of X . It suffices to show that the function $f : X \rightarrow Y$ defined by $f(x) = (x, e)$ is an isometry. By definition of d' , we have

$$\forall x, x' \in X \quad d'((x, e), (x', e)) \leq d_X(x, x').$$

We claim that for every $x, x' \in X$ a path from (x, e) to (x', e) has weight greater than or equal to $d_X(x, y)$. Let $(y_i)_{i=0}^k$ be a path from (x, e) to (x', e) ,

if (y_j, y_{j+1}, y_{j+2}) for some j be such that all of them belong to $X \times \{\omega\}$ for some $\omega \in F$ then by replacing $(y_i)_{i=0}^k$ with $(y_0, y_1, \dots, y_j, y_{j+2}, \dots, y_k)$ we get another path with smaller weight. Thus, we can assume for a path from (x, e) to (x', e) , $(y_i)_{i=0}^k$, we have

$$\forall 0 \leq j \leq k-2 : w(y_j, y_{j+1}) \text{ or } w(y_{j+1}, y_{j+2}) = \epsilon.$$

There are two cases:

1. k (length of the path from (x, e) to (x', e)) is greater than or equal to $2M$. In this case, we have

$$\sum_{i=0}^{k-1} w(y_i, y_{i+1}) \geq M\epsilon > 1 \geq d_X(x, x')$$

2. $k < 2M$. In this case we define $(b_i)_{i=0}^{k-1}$ such that:

$$\forall 0 \leq i \leq k-1 \text{ if } y_i \in X \times \{\omega_i\} \text{ then } y_{i+1} \in X \times \{\omega_i b_i\}.$$

Note that b_i is in $\{a_1, a_2, \dots, a_n, e\}$ for every $0 \leq i \leq k-1$. We know that $y_0, y_k \in X \times \{e\}$, thus we have

$$b_0 b_1 \dots b_{k-1} = e.$$

By definition of F , this means that $b_0 b_1 \dots b_{k-1} = e$ in F_n . If $k > 1$, then we can find $0 \leq i < j \leq k-1$ such that $b_i = b_j^{-1}$ and $b_l = e$ for some $i < l < j$. Therefore, replacing $(y_i)_{i=0}^k$ with

$$(y_0, y_1, \dots, y_{i-1}, e, y_{j+1}, \dots, y_k)$$

will give us another path from x to x' with smaller or equal weight.

Therefore, we may assume $k = 1$. In the case of $k = 1$, the conclusion is clear.

Thus, we have

$$d'((x, e), (x', e)) = d(x, x'). \quad \square$$

proof of theorem 2.3.2. Fix $\epsilon > 0$. Since $(\mathbb{P}(X), \rho_H)$ is a compact metric space, we can find

$$\{p_1, p_2, \dots, p_n\} \subseteq \mathbb{P}(X)$$

such that for every $p \in \mathbb{P}(X)$ there exists $1 \leq i \leq n$ such that $\rho_H(p, p_i) < \frac{\epsilon}{4}$. By proposition 2.3.3, we can find a compact metric space, (Y, d') , such that it extends (X, d) and each p_i extends to a full isometry of Y , q_i , with ϵ distortion. We claim Y has the desired properties. Let $p \in \mathbb{P}_X$, then $\rho(p, p_i) < \epsilon$ for some $i \in \{1, 2, \dots, n\}$. It is easy to see that

$$\forall x \in \text{dom}(p) \ (d'(p(x), q_i(x)) < \epsilon). \quad \square$$

2.4 Compact metric spaces

In this section, we show that a compact metric space, (X, d) , can be extended to another compact metric space, (Z, ρ) , such that every partial isometry of X , p , extends to an isometry of Z , p^* , if and only if there exists a compact metric space, (Y, d') , and a compact subgroup $G \leq \text{Homeo}(Y)$ such that (Y, d') extends (X, d) and every partial isometry of X , p , extends to $\tilde{p} \in G$ such that if $x \in X$ and $y \in Y$, then

$$d'(x, y) = d'(\tilde{p}(x), \tilde{p}(y)).$$

We consider $\text{Homeo}(Y)$ with the supremum norm, ψ , that is, for $f, g \in \text{Homeo}(Y)$

$$\psi(f, g) = \sup\{d'(f(y), g(y)) : y \in Y\}.$$

This metric is right invariant and on a compact subgroup $G \leq \text{Homeo}(Y)$, we can define an invariant metric, σ , on G which is equivalent to ψ . This metric, σ , can be defined as follows

$$\text{for every } f, g \in \text{Homeo}(Y) : \sigma(f, g) = \sup\{\psi(hf, hg) : h \in \text{Homeo}(Y)\}.$$

Theorem 2.4.1. *Let (X, d) be a compact metric space. Then, there exists a compact metric space (Y, d') , and a compact subgroup $G \leq \text{Homeo}(Y)$ such that (Y, d') extends (X, d) and every partial isometry of X , p , extends to $\tilde{p} \in G$ with the property that for every $x \in X$ and $y \in Y$*

$$d'(x, y) = d'(\tilde{p}(x), \tilde{p}(y))$$

if and only if (X, d) can be extended to another compact metric space, (Z, ρ) , such that every partial isometry of X , p , extends to an isometry of Z , p^ .*

Proof. It is enough to show that the first part implies the second part. Let \mathbb{P} be the set of partial isometries of X and $Z = Y \times G$. We define a weighted graph H with a weight function $w : E(H) \rightarrow \mathbb{R}^{\geq 0}$ as follows:

1. $V(H) = Z$,
2. if $y, y' \in Y$ then

$$\forall g \in G : ((y, g), (y', g)) \in E(H) \text{ and } w(((y, g), (y', g))) = d'(y, y'),$$

3. if $y \in \text{range}(p)$ for some $p \in \mathbb{P}$, then

for every $g \in G : ((y, g), (\tilde{p}^{-1}(y), g\tilde{p})) \in E(H)$ and

$$w(((y, g), (p_i^{-1}(y), g\tilde{p}))) = 0,$$

4. if $y \in Y$ and $g \in G$, then for every $h \in G$ we define a jump to be an edge of the form

$$((y, h), (y, hg)) \in E(H) \text{ and } w(((y, h), (y, hg))) = \sigma(g, e),$$

5. points (2), (3), and (4) describe all edges of H .

For every $z, z' \in Z$, we say $(z_i)_{i=0}^k$ is a path of length k from z to z' if

$$z_0 = z, \quad z_k = z'.$$

Furthermore, we say $(z_i)_{i=0}^k$ is a proper path from z to z' if

$$z_0 = z, \quad z_k = z', \text{ and } (z_i, z_{i+1}) \in E(H) \text{ for all } i \in \{0, 1, \dots, k-1\}.$$

For a path $(z_i)_{i=0}^k$, we define its weight to be $\sum_{i=0}^{k-1} w(z_i, z_{i+1})$ with the convention that if (z_i, z_{i+1}) is not an edge of H , then

$$w(z_i, z_{i+1}) = 0.$$

We define a metric, ρ , on Z as follows

$$\rho(z, z') = \inf \{ \sum_{i=0}^{k-1} w(z_i, z_{i+1}) : (z_i)_{i=0}^k \text{ is a proper path from } z \text{ to } z' \}.$$

We claim (Z, ρ) has the desired properties. One can show that ρ is a metric and (Z, ρ) is compact. Furthermore, if we assume $Y \times \{e\}$ is an isometric copy of Y inside Z , then $p^* : Z \rightarrow Z$ defined by

$$p^*((y, g)) = (y, \tilde{p}g)$$

is an isometry of Z which extends p . Note that if $(y_i, g_i)_{i=0}^k$ is a proper path from (y, g) to (y', g') then for every $p \in P$, $(y_i, \tilde{p}g_i)_{i=0}^k$ is a proper path from $(y, \tilde{p}g)$ to $(y', \tilde{p}g')$ with the same weight.

It remains to show that $Y \times \{e\} \subseteq Z$ is an isometric copy of Y . It suffices to show that the function $f : Y \rightarrow Z$ defined by

$$f(y) = (y, e)$$

is an isometry. By definition of ρ , we have

$$\forall y, y' \in Y : \rho((y, e), (y', e)) \leq d'(y, y').$$

We claim that for every $y, y' \in Y$ a proper path from (y, e) to (y', e) has a weight greater than or equal to $d'(y, y')$. Let $(z_i = (y_i, g_i))_{i=0}^k$ be a proper path from (y, e) to (y', e) . We define $(h_i)_{i=0}^{k-1}$ such that for every $0 \leq i \leq k-1$

$$g_{i+1} = g_i h_i.$$

If (z_j, z_{j+1}, z_{j+2}) , for some j , be such that $g_j = g_{j+1} = g_{j+2}$, then by replacing $(z_i)_{i=0}^k$ with

$$(z_0, z_1, \dots, z_j, z_{j+2}, \dots, z_k)$$

we get another proper path with a smaller or equal weight. Thus, we may assume

$$\text{for every } 0 \leq j \leq k-2 : g_j \neq g_{j+1} \text{ or } g_{j+1} \neq g_{j+2}. \quad (2.2)$$

Furthermore, by repeating some points in the proper path if necessary, we may assume k is an odd number and for every $0 \leq i \leq k-1$, h_i is identity (non-identity) if i is an even (odd) number. There are two cases:

1. there are no jumps in the proper path. In this case, h_i is in

$$\{\tilde{p} : p \in P\} \cup \{e\}$$

for every $0 \leq i \leq k-1$. Since $z_0, z_k \in Y \times \{e\}$, we have

$$h_1 h_3 \dots h_{k-2} = e.$$

Note that since $h_1 \neq e$, $k \geq 5$. For $(y, g) \in Z$ and $p \in \mathbb{P}$, we define

$$\tilde{p}(y, g) = (\tilde{p}^{-1}(y), g\tilde{p}).$$

Then,

$$(z_0, z_1, \dots, z_{k-5}, h_{k-4}^{-1}(z_{k-2}), z_{k-2}, z_{k-1}, z_k)$$

is a path from (y, e) to (y', e) with a smaller or equal weight since

$$d'(y_{k-4}, h_{k-4}(y_{k-2})) = d'(h_{k-4}^{-1}(y_{k-4}), y_{k-2}).$$

Note that if $h_{k-4} = \tilde{p}$ for some $p \in \mathbb{P}$, then y_{k-4} is in $\text{dom}(p)$ and for

every $x \in \text{dom}(p)$ and $y \in Y$, we have

$$d'(x, y) = d'(\tilde{p}(x), \tilde{p}(y)).$$

By induction, one can show that for every $1 \leq i \leq \frac{k-3}{2}$

$$(z_0, z_1, \dots, z_{k-2i-3}, h_{k-2i-2}^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), h_{k-2i}^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), \\ \dots, h_{k-4}^{-1}(z_{k-2}), z_{k-2}, z_{k-1}, z_k)$$

is a path from (y, e) to (y', e) with a smaller or equal weight. In particular, if $i = \frac{k-3}{2}$,

$$(z_0, h_1^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), h_3^{-1} \cdots h_{k-4}^{-1}(z_{k-2}), \\ \dots, h_{k-4}^{-1}(z_{k-2}), z_{k-2}, z_{k-1}, z_k)$$

is a path from (y, e) to (y', e) with smaller or equal weight. Since

$$h_1^{-1} \cdots h_{k-4}^{-1}(z_{k-2}) = (h_1 \cdots h_{k-4}(y_{k-2}), e) = (h_{k-2}^{-1}(y_{k-2}), e) = z_{k-1},$$

the weight of this path is

$$d'(y_0, y_{k-1}) + d'(y_{k-1}, y_k) \geq d'(y_0, y_k).$$

2. there is at least one jump in the proper path. Assume l is the smallest number such that (z_l, z_{l+1}) is a jump. By replacing $(z_i = (y_i, g_i))_{i=0}^k$

with

$$((y_0, e), (y_0, g_l h_l g_l^{-1}), (y_1, g_l h_l g_l^{-1} g_1), \dots, (y_l, g_l h_l g_l^{-1} g_l) = z_{l+1}, \\ z_{l+2}, \dots, z_k)$$

we get another proper path with the same length which starts with a jump. Note that since σ is invariant we have

$$\sigma(h_l, e) = \sigma(g_l h_l g_l^{-1}, e)$$

and therefore the new proper path has the same length as the previous one. Hence, we may assume all the jumps occur at the beginning of the proper path and since σ is a metric, we may further assume that there is only one jump. Moreover, by replacing the proper path with

$$(z_0, z_0, z_1, z_2, \dots, z_k)$$

we may assume k is an odd number and for every $0 \leq i \leq k-1$, h_i is identity (non-identity) if i is an even (odd) number. Similar to the previous case, we can show that

$$(z_0, z_0, z_1, h_3^{-1} \dots h_{k-4}^{-1}(z_{k-2}) = h_1(z_{k-1}), h_5^{-1} \dots h_{k-4}^{-1}(z_{k-2}), \dots \\ \dots, h_{k-4}^{-1}(z_{k-2}), z_{k-2}, z_{k-1}, z_k)$$

is a path from (y, e) to (y', e) with smaller or equal weight. The weight

of this path is

$$\begin{aligned}\sigma(h_1, e) + d'(y_0, h_1^{-1}(y_{k-1})) + d'(y_{k-1}, y_k) &\leq d'(y_0, y_{k-1}) + d'(y_{k-1}, y_k) \\ &\leq d'(y_0, y_k). \quad \square\end{aligned}$$

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