SMOOTHING ESTIMATES FOR NON COMMUTATIVE SPACES

BY

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DISSERTATION

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Abstract

In the first part of this thesis, we follow Varopoulos's perspective to establish the noncommutaive Sobolev inequaties (namely, Hardy-Littlewood-Sobolev inequalites), and extend the Sobolev embedding from noncommutative L_p spaces to general Orlicz function spaces related with Cowling and Meda's work. Also we will show some examples to illustract the relation between the Orlicz function, dispersive estimate on semigroup T_t and general resolvent formula on the generator A of the semigroup (i.e. $Ax = \lim_{t\to 0} \frac{T_t x - x}{t}$). And we prove a borderline case of noncommutaive Sobolev inequality, namely the noncommutative Trudinger Moser's inequality.

The focus of the second part of the thesis is the completely bounded version of noncommutative Sobolev inequalities. We prove a cb version of the Sobolev inequality for noncommutative L_p spaces. As a tool, we further develop a general embedding theory for von Neumann algebra, continuing the work for [JP10]. Finally we prove the cb version of Varopolous's theorem and provide some examples and applications.

The third part of the thesis proves the existence of abstract Strichartz estimates on $\mathcal{R}_\theta$ for operators that satisfies ultracontractivity and energy estimate. And we show the abstract Strichartz estimates are applicable to the Schrödinger equation problem on quantum Euclidean spaces $\mathcal{R}_\theta^n$.

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List of Symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathcal{B}(\mathcal{H})$	the space of bounded operators on a Hilbert space $\mathcal{H}$
$\mathbb{F}_n$	free group of n generators
$\mathbb{F}_\infty$	free group of countably infinite generators
$C_r^*(G)$	the reduced group C*-algebra of a group G
$C^*(G)$	the full group C*-algebra of a group G
$\mathcal{L}(A, B)$	linear operators from a space A to B
$(T_t)_{t\geq 0}$	noncommutative symmetric Markov semigroup
$C(\mathbb{T})$	the space of continuous functions on the torus
M_n	the algebra of $n \times n$ matrices
$\mathcal{A}_\theta$	the 2-dimensional rotation algebra associated to θ
Θ	$d \times d$ skew symmetric matrix with entries in $[0, 1)$
$\mathcal{A}_\Theta^d$	the d-dimensional rotation algebra associated to Θ
$\mathcal{R}_\Theta$	the rotation von Neumann algebra associated to Θ

Chapter 1

Introduction

Singular integral theory is a fundamental and important topic in harmonic analysis. Real variable methods of singular integral for higher dimension were original by A. P. Calderón and A. Zygmund [CZ52] in the 1950s. Singular integral operators of convolution type commute with translation on $\mathbb{R}^n$ and $\mathbb{T}^n$. Interpolation method, Poisson integrals and the Hardy-Littlewood maximal function are the main classical techniques. We are mostly concerned with the following kernel:

$$k(x) = |x|^{\alpha-n} \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}, 0 < \alpha < n. \tag{1.0.1}$$

The singular integral operator I_α is formally given by

$$I_\alpha(g)(x) = \frac{\gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)} \int_{\mathbb{R}^n} k(x-y)g(y)dy,$$

which is called a fractional integral. If $1/p - 1/q = \alpha/n$ and $1 < p < n/\alpha$, then

$$\|I_\alpha f\|_q \lesssim \|f\|_p. \tag{HLS}$$

This estimate is called Hardy-Littlewood-Sobolev inequality, proved by Hardy, Littlewood [HL28], [HL30] and Sobolev [Sob38], dating back to 1920's. In the pioneering work of N. Varopoulos in [Var85a] and E. B. Davis[Cha84], they tried to put the HLS inequality into the setting of general Markovian semigroups [VSCC08]. Then the HLS inequality starts to be widely used in heat kernel estimates in many different areas. Recalling N. Varopoulos's paper[Var85a], he identified the equivalence between heat kernel estimates and Sobolev inequality in a more abstract context. Indeed, Varopoulos proved, given a measure space (Ω, μ), T_t is a symmetric Markovian semigroup with dimension n in the sense, i.e.

$$\|T_t f\|_\infty \lesssim t^{-n/2}\|f\|_1, \ t > 0$$

if and only if

$$\|I_\alpha(f)\|_q \lesssim \|f\|_p, \qquad I_\alpha f(x) = \int_0^\infty T_t f(x) t^{\alpha/2-1} dt,$$

with $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Moreover, for the symmetric Markovian semigroup, it's well-known that $T_t = e^{-tA}$, where A is a positive self-adjoint operator on $L^2(\Omega, \mu)$ (See [Var85b] for more information). Then $I_\alpha = c(\alpha)A^{-\alpha/2}$ where $c(\alpha)$ is a constant depending on α. In the classical case of the heat diffusion semigroup on $\mathbb{R}^n$ $(n \geq 3), I_\alpha$ is the standard fractional intergral by the convolution kernel $k(x)$ in (1.0.1). The Varopoulos dimension of the semigroup has been quite influential in many different areas of mathematics like probability theory, statistical mechanics or differential geometry. See the survey paper [Gro14] for more details in this topic.

It's interesting to understand the HLS inequality between the abstract semigroup theory and the fractional integral (namely, resolvent formula) in the noncommutative setting. Many contributions [JPPP17],[JM10] represent the starting point of Varopoulos dimension of abstract semigroup in the noncommutative context. For example, Junge, Palazuelos, Parcet and Perrin[JPPP17] proved that for the free Poisson semgiroup acting on the free group $\mathbb{F}_\infty$, the ultracontractivity bounds are invariant with the number of generators.

$$\|T_t : L_1(\mathbb{F}_\infty) \to L_\infty(\mathbb{F}_\infty)\| \leq t^{-3/2}$$

which is a known consequence of Haagerup's inequality for homogeneous polynomials [Haa78]. Junge and Mei [JM10] predict the Sobolev embedding results based on the Varopoulos dimension on the von Neumann algebra.

In the first part of this thesis, we prove the noncommutative HLS inequalities and extend to the noncommutative Orlicz function spaces. Recall that Cowling and Meda [CM93] showed that $\{T_t\}$ is ϕ-ultracontractive, i.e.

$$\|T_t f\|_\infty \lesssim \phi(t)^{-1}\|f\|_1, \forall f \in L_1(\Omega), t > 0,$$

if and only if the generator A has the Sobolev embedding properties(HLS), namely,

$$\|\psi(A)^{-\alpha} f\|_q \lesssim \|f\|_p,$$

whenever $1 < p < q < \infty, \alpha = \frac{1}{p} - \frac{1}{q}$ on the measure space. Xiao [Xio16] extended the ϕ-ultracontractivity of $\{T_t\}$ and $\psi(A)^{-\alpha}$ into the noncommutative setting. However, his work heavily rely on complex analysis, holomorphic functions, multiplier operator theory and noncommutaive Lorentz space. We investigate some basic notions and discover that Sobobev embedding can be extended to Orlicz function spaces with suitable connection between ϕ and modified version of ψ. Indeed we define g and h is p-related if (1) g is increasing, continuous, positive and p-convex, with satisfying Δ_2-condition; (2) h is decreasing and invertible; (3) $xg(x)h(x)^{1-p}$ is increasing. And we define $\psi_{g,h}(w) = 2wh^{-1}(w)g(h^{-1}(w))$ as the generator function of g and h. In this section, thanks to [JX07] on maximal inequalitiy for the semigroup on von Neumann algebra, we use the "optimal" splitting point techinque of a Peter-Paul inequality to get the resullt as follows:

Theorem 1.1. *Let the semigroup (T_t) be a symmetric Markovian semigroup with the generator operator A on a von Neumann algebra $\mathcal{M}$. Assume the function g and h are p-related by the following conditions:*

(i) the semigroup $\{T_t\}$satisfies the ordered h-p-contractivity, i.e. $T_t f \leq h(t)\|f\|_p 1, \quad for \quad f \in L_p^+(\mathcal{M})$;

(ii) $\mathcal{G}(A)(x) = \int_0^\infty T_t x g(t) dt$.

Let $\Phi(w)$ be the inverse function of $\psi_{g,h}(w^{1/p})$. Then we have the following embedding:

$$\|\mathcal{G}(A) : L_p(\mathcal{M}) \longrightarrow L_\Phi(\mathcal{M})\| \leq C(p, \alpha), \qquad 1 < p < \infty.$$

In the second part of this thesis, we investigate the completely bounded HLS (cb-HLS) inequality over the archetypal algebras of noncommutative geometry: quantum forms of euclidean spaces $\mathcal{R}_\theta$ and tori $\mathcal{A}_\theta$. It's very interesting to ask whether the completely bounded HLS holds when the semigroup has the cb varopoulos dimension. i.e.

$$\|T_t : L_1(\mathcal{M}) \to L_\infty(\mathcal{M})\|_{cb} \lesssim t^{-n/2} \xrightarrow{?} \|\Delta^{-\alpha} : L_2(\mathcal{M}) \to L_q(\mathcal{M})\|_{cb} \leq C(\alpha, q). \tag{1.0.2}$$

for two reasons. The first reason is that some semigroup on $\mathcal{R}_\theta^n$ (and $\mathcal{A}_\theta^n$) has the cb-Varopoulos dimension. Indeed, a crucial point, as in abelian algebras, is to identify kernels over $\mathcal{R}_\theta^n \bar{\otimes} (\mathcal{R}_\theta^n)^{op}$, where the op-structure (reversed product law) is used in the second copy. This is justified by the important map

$$\pi_\theta : L_\infty(\mathbb{R}^n) \to \mathcal{R}_\theta \bar{\otimes} \mathcal{R}_\theta^{op},$$
$$\exp(2\pi i \langle \xi, \cdot \rangle) \mapsto \lambda_\theta(\xi) \otimes \lambda_\theta(\xi)^*,$$

which extends to a normal *-homomorphism, for which the op-structure is strickly necessary. We apply Effros and Ruan's theorem [ER00] on $\mathcal{R}_\theta \bar{\otimes} \mathcal{R}_\theta^{op}$ to the heat semigroup $T_t = e^{-t\Delta}$, where Δ is the Laplacian operator. We prove that

$$\|T_t : L_1(\mathcal{R}_\theta^n) \to L_\infty(\mathcal{R}_\theta^n)\|_{cb} \lesssim t^{-n/2}.$$

Assume the cb-HLS inequality (1.0.2) holds, then it implies that

$$\|(|\Delta|)^{-\alpha} : L_2(\mathcal{R}_\theta^n) \to L_q(\mathcal{R}_\theta^n)\|_{cb} \leq C(\alpha, q)$$

The second reason is that the cb-Sobolev embedding on $\mathcal{R}_\theta$ strengthens the classical vector-valued Sobolev embedding when $\theta = 0$ and $n = 1$. Let us denote x as $(A^{-\alpha} f)$ and recall the norms of an operator in $L_p(S_q)$ and $S_p(L_q)$ as follows:

$$\|x\|_{L_p(S_q)} = \sup_{\|\alpha\|_{2r}, \|\beta\|_{2r}=1} (\int \|\alpha x \beta\|_p^p)^{1/p} \leq (\int \sup_{\|\alpha\|_{2r}, \|\beta\|_{2r}=1} \|\alpha x \beta\|_p^p)^{1/p} = \|x\|_{S_p(L_q)}, \quad \frac{1}{r} = |\frac{1}{p} - \frac{1}{q}|.$$

Therefore there exists a completely contraction from $S_p(L_q(\mathbb{R}))$ to $L_p(\mathbb{R}, S_q)$. The cb Sobolev inequality reduces the vector-valued Sobolev inequality, but the opposite direction is not. Therefore the cb-HLS inequality is more general than the vector-valued one. In chapter 3, we prove (1.0.2) in an abstract setting described as follows:

Theorem 1.2. *Suppose $\{T_t\}$ is a strongly continuous semigroup of normal selfadjoint subunital completely positive maps on some semifinite von Neumann algebra $\mathcal{M}$. The following are equivalent:*

(1) there exist $1 < p < q < \infty$ with $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ such that

$$\|A^{-\alpha} : L_p(\mathcal{M}) \longrightarrow L_q(\mathcal{M})\|_{cb} \leq c_1;$$

(2) the semigroup $\{T_t\}$ satisfies

$$\|T_t : L_1(\mathcal{M}) \longrightarrow L_\infty(\mathcal{M})\|_{cb} \leq c_2 t^{-n/2}.$$

We want to emphasize that the proof of Theorem 1.2 develops a new tool to prove cb-boundness for maps on the semifinite von Neumann algebra $\mathcal{M}$. The cb-norm estimates of maps between noncommutative $L^p(\mathcal{M})$ spaces have achieved a rapid and considerable progress in recent years. For example, the quantum Fourier multipliers is easier to prove by transference methods stablished in [GPJP17],[CXY13],[Ric16]. It's known from [Bou86][Wei01] that the UMD-property implies the boundedness of all invariant singular integrals or standard multiplier operators under some regularity assumption. However, the tools for cb-boundness from L_p to L_q are very rare. A. Harcharras [Har99] dealed with completely bounded version of Fourier multipliers on L^p and Schur multipliers on the Scatten class S^p when L^p and S^p are viewed as operator spaces by using subsets of $\mathbb{Z}$ enjoying the noncommutative $\Lambda(p)$-property. In our work, the main obstacle that we deal with is how to completely isomorphically embed a noncommutative L_p space into the sum of two weighted spaces L_2, L_q with $2 < q < p$. We use Junge and Xu's machinery [JX07] as the main ingredient. In chapter 3, we prove a representation theorm as follows:

Theorem 1.3. *Let $2 < p < q$. Then there is a map $u : L_2(\mathcal{M}) \cap L_p(\mathcal{M}) \to L_2(\mathcal{M} \otimes \mathbb{B}(\ell_2))$ such that for any operator $x \in L_p(\mathcal{M}) \cap L_0^{sa}(\mathcal{M})$,*

$$u(x) \lesssim D_q a D_q + D_2 b D_2, a \in L_2(\mathcal{M} \otimes \mathbb{B}(\ell_2)), b \in L_q(\mathcal{M} \otimes \mathbb{B}(\ell_2)).$$

Here D_q and D_2 are two multipliers in $\mathcal{M} \otimes \mathbb{B}(\ell_2)$.

We discover that we can use an explicite decomposition of $A^{-\alpha}$. We define two matrix-valued singular integral operators Φ_θ^{up} and Φ_θ^{down} in a concrete form, derived from the split point in the classical proof of HLS inequality. Following the scheme of Theorem 1.3 for $A^{-\alpha} f$, we show that by a deterministic split point, $\Phi_\theta^{down} f$ is an operator in $L_2(\mathcal{M} \otimes \mathbb{B}(\ell_2), w_1)$ and $\Phi_\theta^{up} f$ is an operator in $L_q(\mathcal{M} \otimes \mathbb{B}(\ell_2), w_2)$ with a suitable choice of weights (w_1, w_2) from interpolation theory.

In the third part of this thesis, we investigate the abstract Strichartz estimates theory in noncommutative space. Strichartz estimates are a family of inequalities for linear dipersive partial differential equations. These inequalities establish size and decay of solutions in mixed norm Lebesgue spaces. In [Seg76], the author investigates the linear Klein-Gordon equation. In the pioneering paper [S$^+$77], Strichartz builds the connection between space-time estimate and the restriction theorem of Tomas and Stein. See [LS95],[Kap89], [MSS93], [GV95], [Sog95] for many known Strichartz wave equations. See [GV92] [Yaj87] for Strichartz results for the Schrödinger equation. In [KT98], Keel and Tao showed an abstract Strichartz estimates for a family of operators with Varopoulos dimension and boundedness on some Hilbert space. In chapter 5, we adapt their proof and [Tag08] by defining some bilinear operators. Next we show that the existence of abstract Strichartz estimates for $\mathcal{R}_\theta^n$. Indeed the Schrödinger operator $T_t = e^{-it\Delta}$ satisfies the dispersive estimate proved in chapter 3. And Schrödinger operator has the energe estimate followed from Plansherel's estimates. Then we apply the Strichartz estimates on $\mathcal{R}_\theta^n$ for the Schrödinger equations.

The thesis is organized as follows. In Chapter 2, we assume the reader has basic knowledge on C*-algebras and von Neumann algebra theory. We give a brief introduction on interpolation theory, operator spaces and completely bounded maps, noncommutative L^p spaces and vector-valued noncommutative L_p spaces, abstract semigroup theory.

In chapter 3, we prove the Sobolev inequalities in the noncommutative setting. Moreover we extend the Sobolev embedding from noncommutaive L_p spaces to Orlicz function spaces with the Luxemburg-Nakano norm for $1 < p < \infty$. We study three examples to show the existence of such Orlicz function spaces. Then we establish a normal representation from $\mathbb{R}^n$ to $\mathcal{R}_\theta \bar{\otimes} \mathcal{R}_\theta^{op}$ which provide the main ingredient to prove the ultracontractivity. Then we introduce the notion of Trudinger-Moser's inequality, which concerns the borderline case of Sobolev inequality when p leads to infinity. We show some analytic embedding of $L_p(\mathcal{R}_\theta^n)$ and prove a noncommutative version of Trudinger-Moser's inequality on $\mathcal{R}_\theta^n$.

In chapter 4, we prove the cb Sobolev inequalities in the noncommutaive setting. We introduce the notions of "weighted truncated resolvent operator" and show norm estimates of these maps. Then we prove the cb Sobolev inequlity for Schatten p classes S_p with embedding theory for discrete space. Then we explore the L_p embedding theory for general von Neumann algebra. We introdcuce the notion of homogeneous space K and give some certain regularity condition on the pair of weights. We show that, the fundamental sequences with a mild regularity assumption, completely determine the operator space structure of K. We find a canonical representation of the homogeneous space K in terms of weighted row and column spaces. Then we prove the cb-sobolev inequalities for semifinte von Neumann algebra and the cb-version of Varopolous's theorem and provide some examples.

In chapter 5, we adapt the proof of abstract Strichartz estimates in Keel and Tao's paper [KT98], with notations from [Tag08]. In this section, we replace (T_t) by the group unitaries $(e^{it\Delta})$. With the ultracontractivity of $\{T_t\}$ proved in chapter 3, we prove the existence of Strichartz estimates on the quantum euclidean space $\mathcal{R}_\theta^n$.

Chapter 2

Preliminaries

2.1 Interpolation

In this section, we recall some basic knowledge of interpolation theory. Our main reference is from the thesis paper [Tag08].

2.1.1 Complex interpolation

We first recall the definition of complex interpolation for Banach spaces. Let (E_0, E_1) be a compatible couple of complex Banach spaces. Recall that

$$E_0 + E_1 = \{a_0, +a_1 : a_0 \in E_0, a_1 \in E_1\}$$

with norm

$$\|a\|_{E_0+E_1} = \inf\{\|a_0\|_{E_0} + \|a_1\|_{E_1} : a = a_0 + a_1, a_0 \in E_0, a_1 \in E_1\}$$

(see [[Tri95], 1.2.1] for more details).

Let $\mathcal{F}(E_0, E_1)$ be the family of all functions $f : S \to E_0 + E_1$ satisfying the following conditions:

- f is continous on S and analytic in the interior of S;
- $f(k + it) \in E_k$ for all $t \in \mathbb{R}$ and the function $t \to f(k + it)$ is continuous from $\mathbb{R}$ to E_k for $k = 0$ and $k = 1$;
- $\lim_{|t|\to\infty} \|f(k + it)\|_{E_k} = 0$ for $k = 0$ and $k = 1$.

When the family $\mathcal{F}(E_0, E_1)$ is equipped with the norm

$$\|f\|_{\mathcal{F}(E_0,E_1)} = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{E_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{E_1}\},$$

$\mathcal{F}(E_0, E_1)$ is a Banach space.

Definition 2.1. For $0 < \theta < 1$ the complex interpolation space $E_\theta = (E_0, E_1)_\theta$ is defined as the space of all those $x \in E_0 + E_1$ for which there exists $f \in \mathcal{F}(E_0, E_1)$ such that $f(\theta) = x$, with the norm

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{F}(E_0,E_1)} : f(\theta) = x, f \in \mathcal{F}(E_0, E_1)\}.$$

Remark 2.2. The complex interpolation space E_θ becomes a Banach space. Thanks to maximum principle, E_θ is isomorphic to the quotient of $\mathcal{F}(E_0, E_1)$.

Theorem 2.3 ([Tri95], Theorem 1.9.3)**.** *(E_0, E_1) is an interpolation couple. Then*

(i) $(E_0, E_1)_\theta = (E_1, E_0)_{1-\theta}$ *holds for* $0 \le \theta \le 1$,

(ii) $E_0 \subset E_1$ *implies that*

$$E_0 \subset (E_0, E_1)_{\theta_0} \subset (E_0, E_1)_{\theta_1} \subset E_1$$

where $0 < \theta_0 < \theta_1 < 1$,

(iii) $(E_0, E_0)_\theta = E_0$ *if* $0 < \theta < 1$, *and*

(iv) $E_0 \cap E_1$ *is dense in* $(E_0, E_1)_\theta$.

Moreover, suppose that $0 < \theta < 1, B = (B_0, B_1)_\theta$ *and* $C = (C_0, C_1)_\theta$. *If S is a linear operator from B_0 to C_0 and from B_1 to C_1, i.e.*

$$\|Sb_0\|_{C_0} \le M_0\|b_0\|_{B_0},\ \|Sb_1\|_{C_1} \le M_1\|b_1\|_{B_1}, \forall b_0 \in B_0, b_1 \in B_1,$$

then S is a bounded linear operator from the interpolation space B to the interpolation space C satisfying

$$\|Sb\|_C \le M_0^{1-\theta} M_1^{\theta} \|b\|_B \qquad \forall b \in B.$$

In Chapter 3, we will often use the bilinear version below.

Theorem 2.4 ([BL12], Theorem 4.4.1)**.** *Suppose that the pairs $(A_0, A_1), (B_0, B_1)$ and (C_0, C_1) are Banach interpolation couples. Assume that $S : A_0 \cap A_1 \times B_0 \cap B_1 \to C_0 \cap C_1$ is bilinear and that for every (a, b) in $A_0 \cap A_1 \times B_0 \cap B_1$ the inequalities*

$$\|S(a, b)\|_{C_0} \le M_0\|a\|_{A_0}\|b\|_{B_0}, \|S(a, b)\|_{C_1} \le M_1\|a\|_{A_1}\|b\|_{B_1}$$

holds. If $0<\theta<1$ and $0<q\le\infty$ then S extends uniquely to a bilinear mapping from $(A_0,A_1)_\theta\times(B_0,B_1)_\theta$ to $(C_0,C_1)_\theta$ with norm at most $M_0^{1-\theta}M_1^\theta$.

2.1.2 Real interpolation

Let E_0, E_1 be Banach spaces.We assume the interpolation couple (E_0, E_1) is embeded into some larger Banach space A.

Definition 2.5. Let the real interpolation spaces $(E_0,E_1)_{\theta,q}$ for $0<\theta<1, 1\le q\le\infty$ be via the norm

$$\|a\|_{(E_0,E_1)_{\theta,q}} = \Big(\int_0^\infty (t^{-\theta}K(t,a))^q \frac{dt}{t}\Big)^{1/q},$$

where

$$K(t,a) = \inf_{a=a_0+a_1} \|a_0\|_{E_0} + t\|a_1\|_{E_1}.$$

We list some basic properties of real interpolation spaces frequently used in this paper.

Theorem 2.6 ([BL12], Theorem 3.1.2 and 3.4.1)**.** *Suppose that (E_0,E_1) is a Banach interpolation couple, $0<\theta<1$ and $1\le q\le\infty$. Then the following properties hold:*

(i) $(E_0,E_1)_{\theta,q} = (E_1,E_0)_{1-\theta,q}$ with equal norms,

(ii) if $1\le q\le r\le\infty$ then

$$(E_0,E_1)_{\theta,1}\subset(E_0,E_1)_{\theta,q}\subset(E_0,E_1)_{\theta,r}\subset(E_0,E_1)_{\theta,\infty},$$

(iii) $(E_0,E_0)_\theta = E_0$ with equivalent norms, and

(iv) if E_0 and E_1 are complete then so is $(E_0,E_1)_{\theta,q}$.

Moreover, suppose that $0<\theta<1, B=(B_0,B_1)_{\theta,q}$ and $C=(C_0,C_1)_{\theta,q}$. If S is a linear operator such that

$$\|Sb_0\|_{C_0}\le M_0\|b_0\|_{B_0},\ \|Sb_1\|_{C_1}\le M_1\|b_1\|_{B_1}, \forall b_0\in B_0, b_1\in B_1,$$

then S is a bounded linear operator from B to C satisfying

$$\|Sb\|_C\le M_0^{1-\theta}M_1^\theta\|b\|_B \qquad \forall b\in B.$$

We also use some bilinear results for real interpolation spaces as follows.

Theorem 2.7 ([BL12],pp 76-77)**.** *Suppose that the pairs $(A_0, A_1), (B_0, B_1)$ and (C_0, C_1) are Banach interpolation couples.*

(i) Suppose that for every (a, b) in $A_0 \cap A_1 \times B_0 \cap B_1$ the inequalities

$$\|S(a,b)\|_{C_0} \leq M_0\|a\|_{A_0}\|b\|_{B_0}, \|S(a,b)\|_{C_1} \leq M_1\|a\|_{A_1}\|b\|_{B_1}$$

holds. If $0 < \theta < 1$ and $1/r + 1 = 1/p + 1/q$ with $1 \leq r \leq \infty$, then S extends uniquely to a bilinear mapping from $(A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q}$ to $(C_0, C_1)_{\theta,r}$ with norm at most $M_0^{1-\theta}M_1^{\theta}$.

(ii) Suppose that the bilinear operator S acts as a bounded transformation as indicated below:

$$S : A_0 \times B_0 \to C_0,$$

$$S : A_0 \times B_1 \to C_1,$$

$$S : A_1 \times B_0 \to C_1.$$

If $\theta_0, \theta_1 \in (0, 1)$ and $p, q, r \in [1, \infty]$ such that $1 \leq 1/p + 1/q$ and $\theta_0 + \theta_1 < 1$, then S also acts as a bounded transformation in the following way:

$$S : (A_0, A_1)_{\theta_0,pr} \times (B_0, B_1)_{\theta_1,qr} \to (C_0, C_1)_{\theta_0+\theta_1,r}.$$

2.1.3 Interpolation of L^p spaces by real and complex method

Complex interpolation of L^p spaces gives us the desired result as follows:

Theorem 2.8 ([BL12], Theorem 5.1.2)**.** *Suppose that (χ, μ) is a measure space, (E_0, E_1) is a Banach interpolation couple, $p_0, p_1 \in [1, \infty]$ and $0 < \theta < 1$. If $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $E_\theta = (E_0, E_1)_\theta$ then*

$$(L^{p_0}(\chi; E_0), L^{p_1}(\chi; E_1))_\theta = L^p(\chi; E_\theta).$$

If $p_i = \infty$ for some $i \in \{1, 2\}$, then L^{p_i} must be replaced with the space L^∞_0 of bounded functions with compact support.

In general, real interpolation of L^p spaces gives Lorentz spaces rather that L^p spaces, since there is an extra interpolation parameter. We first give the definition of Lorentz spaces as follows:

Definition 2.9. Suppose that (χ, μ) is a measure space, E is a Banach space and $1 < p < \infty$. If $1 \leq q < \infty$

then the Lorentz space $L^{p,q}(\chi)$ is given by

$$L^{p,q}(\chi;E)=\{F\in L^1(\chi;E)+L^\infty(\chi;E):\|F\|_{L^{p,q}(\chi;E)}<\infty\},$$

where

$$\|F\|_{L^{p,q}(\chi;E)}=\Big(\int_0^\infty (t^{1/p}F^*(t))^q\frac{dt}{t}\Big)^{1/q}$$

and F^* is the measure-preserving rearrangement function of F (see [[Tri95], 1.18.6] for further details).

Theorem 2.10. *Suppose that χ is a measure space, E is a Banach space, $1\le p_0<p_1\le\infty, p_0<q\le\infty$ and $0<\theta<1$. If $1/p=(1-\theta)/p_0+\theta/p_1$ then*

$$(L^{p_0}(\chi;E),L^{p_1}(\chi;E))_{\theta,q}=L^{p,q}(\chi;E)\ \textit{with equivalent norms.}$$

Lorentz spaces are equivalent to L_p space in some conditions.

Lemma 2.11 ([BL12], p.8)**.** *Suppose that χ is a measure space and E is a Banach space.*

(i) If $1\le r_1<r_2\le\infty$ and $1<p<\infty$, then $L^{p,r_1}(\chi;E)\subset L^{p,r_2}(\chi;E)$.

(ii) If $1\le p\le\infty$ then $L^{p,p}(\chi;E)=L^p(\chi;E)$ with equal norms.

Theorem 2.12 ([BL12],p.130)**.** *Suppose that χ is a measure space, $p_0,p_1\in[1,\infty),\theta\in(0,1)$ and $1/p=(1-\theta)/p_0+\theta/p_1$. If (E_0,E_1) is a Banach interpolation couple then*

$$\Big(L^{p_0}(\chi;E_0),L^{p_1}(\chi,E_1)\Big)_{\theta,p}=L^p(\chi;(E_0,E_1)_{\theta,p}).$$

Remark 2.13. We will need the interpolation space identities $(L_t^2L_x^{p_0},L_t^2L_x^{p_1})_{\theta,2}=L_t^2L_x^{p,2}$ whenever $p_0\neq p_1$ and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{1}{p_1}$ in next chapter.

We shall also use interpolation results for weighted Lebesgue sequence spaces in Chapter 5. Let $s\in\mathbb{R}$ and $1<q<\infty, l_q^s$ denote the space of all scalar-valued sequence $\{a_j\}_{j\in\mathbb{Z}}$ such that

$$\|\{a_j\}_{j\in\mathbb{Z}}\|_{l_q^s}=\Big(\sum 2^{js}|a_j|^q\Big)^{1/q}<\infty. \tag{2.1.1}$$

If $q=\infty$ then the norm is defined by $\|\{a_j\}_{j\in\mathbb{Z}}\|_{l_\infty^s}=\sup 2^{js}|a_j|$.

Theorem 2.14 ([BL12], Theorem 5.6.1)**.** *Assume that $0<q_0\le\infty, 0<q_1\le\infty, 0<\theta<1$ and $s_0\neq s_1$. If $0<q\le\infty$ then*

$$(l_{q_0}^{s_0},l_{q_1}^{s_1})_{\theta,q}=l_q^s$$

where $s = (1-\theta)s_0 + \theta s_1$.

Remark 2.15. In chapter 3, we will use the special case $(l_\infty^{s_0}, l_\infty^{s_1})_{\theta,1} = l_1^s$ with $s = (1-\theta)s_0 + \theta s_1$.

2.2 Operator spaces and completely bounded maps

We introduce some basic concepts from operator space theory on those aspects which are useful for this work, and our mainly standard references are from [Xu08].

Definition 2.16. An operator space E is a complex Banach space together with a sequence of matrix norms $\|\cdot\|_k$ on $M_k[E] = M_k \otimes E$ satisfying the following conditions:

- $(R_1) : \|v \oplus w\|_{k+l} = \max\{\|v\|_k, \|w\|_l\}$ and
- $(R_2) : \|\alpha w \beta\|_k = \|\alpha\|\|w\|_l\|\beta\|$

for all $v \in M_k[E], w \in M_l[E], \alpha \in M_{k,l}\beta \in M_{l,k}$. (R_1) and (R_2) above are usually called Ruan's axioms.

The following theorem is proved in [Rua88].

Theorem 2.17 (Ruan's characterization)**.** *Let E be a vector space. Assume that each $M_n(E)$ is equipped with a norm $\|\cdot\|_n$. If these norm $\|\cdot\|_n$ satisfy Ruan's axioms (R_1) and (R_2), then there are Hilbert space H and a linear map $J : E \to \mathbb{B}(\mathcal{H})$ such that*

$$J_n = id_{M_n} \otimes J : M_n(E) \to M_n(\mathbb{B}(\mathcal{H})) \text{ is isometric for every } n.$$

In other words, the sequence $(\|\cdot\|_n)$ comes from the operator space structure of E given by the embedding $J : E \to \mathbb{B}(\mathcal{H})$.

Definition 2.18. An operator space E is called homogeneous if every bounded map on E is c.b. and $\|u\|_{cb} = \|u\|$. E is called Hilbertian if E is isometric to a Hilbert space.

Definition 2.19. Given operator spaces E and F and a linear map $T : E \longrightarrow F$, is said to be completely bounded if

$$\|T\|_{cb} = \sup_n \|T_n\| < \infty,$$

here $T_n : M_n[E] \to M_n[W]$ denotes the linear map by $T_n(v) = (\mathrm{id}_n \times T)(v) = (T(v_{ij}))_{i,j}$. We say that T is completely contractive if $\|T\|_{cb} \leq 1$. Moreover, T is said to be a completely isomorphism (resp. completely isometry) if each map T_k is an isomorphism (resp. an isometry).

Definition 2.20. Given an operator space E, we define the dual operator space E^* by means of acceptable matrix norms

$$M_n[E^*] = CB(E, M_n), n \geq 1.$$

It is not difficult to see that $\|T^*\|_{cb} = \|T\|_{cb}$ for every $T : E \to F$, denotes the adjoint map of T.

Theorem 2.21 (Haagerup-Paulsen-Wittstock factorization)**.** *Let $E \subset \mathbb{B}(\mathcal{H})$ and $F \subset \mathbb{B}(\mathcal{K})$be two operator spaces. Let $u : E \to F$ be a c.b. map. Then there are a Hilbert space $\tilde{\mathcal{H}}$, a representation $\pi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\tilde{\mathcal{H}})$ and two bounded operators $a, b \in \mathbb{B}(\mathcal{K}, \tilde{\mathcal{H}})$ such that*

$$u(x) = b^* \pi(x) a, \forall x \in E.$$

Namely, $u = L_{b^*} \circ R_a \circ \pi|_E$. Moreover, $\|u\|_{cb} = \inf\{\|a\|\|b\|\}$ where the infimum is taken over all factorizations of u as above.

2.3 Noncommutative L^p spaces

2.3.1 Definitions and Propositions

In this section, we mainly follows the notations of [JLMX06]. Let us take a brief presentation of noncommutative L^p spaces associated with a trace. We mainly refer the reader to [Ter81] and [PX03] for further information on these spaces.

$\mathcal{M}$ is a semifinite von Neumman algebra with a normal semifinite faithful (n.s.f) trace τ. We use $\mathcal{M}_+$ to represent the positive part of $\mathcal{M}$. Let $\mathcal{S}_+$ be the set of all $x \in \mathcal{M}_+$ whose support projection have a finite trace. Then any $x \in \mathcal{S}_+$ has a finite trace. Let $\mathcal{S} \subset \mathcal{M}$ be the linear span of $\mathcal{S}_+$, then $\mathcal{S}$ is a w*-dense *-subalgebra of $\mathcal{M}$.

Definition 2.22. Let $0 < p < \infty$. For any $x \in \mathcal{S}$, the operator $|x|^p$ belongs to $\mathcal{S}_+$ and we define

$$\|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}, x \in \mathcal{S}.$$

Here $|x| = (x^*x)^{\frac{1}{2}}$ denotes the modulus of x. It turns out that $\|x\|_p$ is a norm on $\mathcal{S}$ if $p \geq 1$, and a p-norm if $p < 1$. By definition, the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$ is the completion of $(\mathcal{S}, \|x\|_p)$, denoted as $L_p(\mathcal{M})$.

Remark 2.23. For convience, we let $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with its operator norm. $L^p(\mathcal{M}) \cap \mathcal{M}$ is dense in $L^p(\mathcal{M})$ for $1 \leq p \leq \infty$. $\mathcal{M} = L_\infty(\Omega, \mu)$ and Schatten classes S_p are two simple examples.

Assume that $\mathcal{M} \subset \mathbb{B}(\mathcal{H})$ acts on some Hilbert space $\mathcal{H}$. We want to have a description of the elements of $L^p(\mathcal{M})$ as (possibly unbounded) operators on $\mathcal{H}$. Let $\mathcal{M}' \subset \mathbb{B}(\mathcal{H})$ denote the commutant of $\mathcal{M}$.

We recall the **noncommutative Hölder inequality**. If $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then

$$\|xy\|_r \leq \|x\|_p \|y\|_q, x \in L^p(\mathcal{M}), y \in L^q(\mathcal{M}). \tag{H}$$

Conversely for any $z \in L^r(\mathcal{M})$,there exist $x \in L^p(\mathcal{M})$ and $y \in L^q(\mathcal{M})$ such that $z = xy$, and $\|z\|_r = \|x\|_p \|y\|_q$.

Thanks to Hölder inequality, we get an isometric isomorphism $L^p(\mathcal{M})^* = L^{p'}(\mathcal{M})$ with $\frac{1}{p} + \frac{1}{p'} = 1, 1 \leq p < \infty$. We can include the case $p = \infty$ by identifying $L^1(\mathcal{M})$ with the (unique) predual $\mathcal{M}_*$ of $\mathcal{M}$.

Another important property of noncommutative L^p-spaces is that they form an **interpolation** scale(see [Ter82]), i.e.

$$[L^\infty(\mathcal{M}), L^1(\mathcal{M})]_{\frac{1}{p}} = L^p(\mathcal{M}), 1 \leq p \leq \infty,$$

where $[,]_\theta$ means the complex interpolation method. We refer to the survey paper [PX03] for more information and historical references on noncommuative L_p spaces.

2.3.2 Weighted noncommutative L_p spaces

We mainly follow the notations in [JRS05]. Let ϕ be a normal faithful semifinite weight on $\mathcal{M}$. Consider the one-parameter modular automorphism σ_t^ϕ (associated with ϕ) initiated by Haagerup [Haa79]. Then the semifinite von Neumann algebra $\tilde{\mathcal{M}} := \mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ will have an induced trace τ with a dual action θ. And the trace and action have the relation $\tau \circ \theta_s = e^{-s}\tau$ for all $s \in \mathbb{R}$.

Because of the one-parameter group, $\mathcal{M}$ now can be considered as a θ-invariant von Neumann subalgebra $L_\infty(\mathcal{M})$ of $\tilde{\mathcal{M}}$. As for $L_p(\mathcal{M}, \phi)(1 \leq p < \infty)$, it can be identified as the space of all (unbounded) τ-measurable operators affiliated with $\tilde{\mathcal{M}}$ such that $\theta_s(T) = e^{-\frac{s}{p}} T$ for all $s \in \mathbb{R}$. Following [Ter81], we have the follow theorem,

Theorem 2.24. *There is a one-to-one correspondence between bounded (positive) linear functionals $\varphi \in \mathcal{M}_*$ and τ-measurable (positive self-adjoint) operators $h_\varphi \in L_1(\mathcal{M}, \phi)$ under the connection given by*

$$\hat{\varphi}(\tilde{x}) = \tau(h_\varphi \tilde{x}), \tilde{x} \in \tilde{\mathcal{M}},$$

where $\hat{\varphi}$ is the so-called dual weight *for φ*

This correspondence actually extends to all of $\mathcal{M}_*$ and $L_1(\mathcal{M}, \varphi)$. We set the "tracial" linear functional

$tr = tr_{\mathcal{M}} : L_1(\mathcal{M}, \phi) \to \mathbb{C}$ by

$$tr(h_\varphi) = \varphi(1), \quad \text{satisfying} \quad tr(h_\varphi x) = tr(x h_\varphi) = \varphi(x), \quad \varphi \in \mathcal{M}_*^+, x \in \mathcal{M}.$$

Given any $h \in L_p(\mathcal{M})$, we have the polar decomposition $h = w|h|$. Here $|h|$ is the positive operator in $L_p(\mathcal{M})^+$ and w is the partial isometry contained in $\mathcal{M}$. We call $s_l(h) = ww^*$ the *left support* of h (respectively the projection $s_r(h) = w^*w$, *right support* of h). Thanks to the above therem, we naturally define a new norm on $L_p(\mathcal{M}, \phi)$

Definition 2.25. For $h \in L_p(\mathcal{M}, \phi)$, we define

$$\|h\|_p = tr(|h|^p)^{1/p} = \varphi(1)^{1/p}$$

if $\varphi \in \mathcal{M}_*$ corresponds to $|h|^p \in L_1(\mathcal{M}, \phi)^+$.

With this norm, it is easy to see that $L_1(\mathcal{M}, \varphi)$ is isometrically and orderly isomorphic to $\mathcal{M}_*$. By isometry, L_p-space constructed above is actually independent of n.f.s weight chosed on $\mathcal{M}$.

If $h \in L_p(\mathcal{M})^+, h^p$ is a positive operator in $L_1(\mathcal{M})^+$. So h^p is h_φ for a positive $\varphi \in \mathcal{M}_*^+$. Therefore, we may identify h with $\varphi^{1/p}$, i.e. $h = \varphi^{1/p}$. See [Con00, Yam92, She] for more details.

2.4 Vector-valued noncommutative L_p spaces

2.4.1 Vector-valued schatten classes

We want to define Schatten classes with values in operator spaces. Our main reference for this section is from [Pis03]. Let E be an operator space, $S_\infty[E]$ be $S_\infty \otimes_{\min} E$ and $S_1[E]$ be $S_1 \hat{\otimes} E$ (projective operator space tensor product). We define $S_p[E]$ by interpolation for any $1 < p < \infty$, i.e. $S_p[E] = (S_\infty[E], S_1[E])_{1/p}$. The elements of $S_p[E]$ are often represented as infinite matrices with entries in E. The following theorems in [Pis03] are very useful:

Theorem 2.26. *Let* $1 \le p < \infty$.

1) *Any* $x = (x_{ij}) \in S_p[E]$ *admits a factorization* $x = ayb$ *with* $a, b \in S_{2p}$ *and* $y \in S_\infty[E]$. *Here the product is the usual matrix product. Moreover, we have*

$$\|x\|_{S_p[E]} = \inf_{x=ayb} \{\|a\|_{2p} \|y\|_{S_\infty[E]} \|b\|_{2p}\}.$$

2) Conversely, for any $x = (x_{ij}) \in S_\infty[E]$

$$\|x\|_{S_\infty[E]} = \sup\{\|axb\|_{S_p[E]} : a, b \in S_{2p}, \|a\|_{2p} \leq 1, \|b\|_{2p} \leq 1.\}$$

Corollary 2.27. *Let E and F be two operator spaces. Let $1 \leq p < \infty$. Then a linear map $u : E \to F$ is c.b. iff*

$$\sup_n \|I_{S_p^n} \otimes u : S_p^n[E] \to S_p^n[F]\| < \infty;$$

Moreover, in this case the supremum above is equal to $\|u\|_{cb}$.

Proposition 2.28. *For the case operator space $E = S_q^d$ for some $1 \leq q \leq \infty$. Given $1 \leq p, q \leq \infty$ and $\frac{1}{r} = |\frac{1}{p} - \frac{1}{q}|$, we have*

1) If $p \leq q, \|x\|_{S_p^n[S_q^d]} = \inf\{\|a\|_{S_{2r}^n}\|y\|_{S_q^{nd}}\|b\|_{S_{2r}^n}\}$, where the infimum runs over all representations $X = (a \otimes Id_d)y(b \otimes Id_d)$ with $a, b \in M_n$ and $y \in M_n \otimes M_d$.

2) If $p \geq q, \|x\|_{S_p^n[S_q^d]} = \sup\{\|(a \otimes Id_d)x(b \otimes Id_d)\|_{S_q^{nd}} : a, b \in S_{2r}^n$ with $\|a\|, \|b\| \leq 1\}$

2.4.2 Column and Row p-spaces

Recall that the column and row spaces, C and R, are the first column and row subspaces of S_∞. Now let E be an operator space. We denote by $C_p[E]$ (resp. $R_p[E]$) the closure of $C_p \otimes E$ (resp. $R_p \otimes E$) in $S_p[E]$. For any finite sequence $(x_k) \in E$,

$$\|\sum_k x_k \otimes e_k\|_{C_p[E]} = \|(\sum_k x_k^* x_k)^{1/2}\|_{L_p(E)}$$

where (e_k) denotes the canonical basis of C_p. More generally, if $a_k \in C_p$, then

$$\|\sum_k x_k \otimes a_k\|_{C_p[E]} = \|(\sum_{k,j} \langle a_j, a_k \rangle x_k^* x_j)^{1/2}\|_{L_p(E)}.$$

We also have a similar description for $R_p[E]$.

In general, for a Hilbert space $\mathcal{H}$ and $1 \leq p \leq \infty$, we define the Schatten p-class $S_p(\mathbb{C}, \mathcal{H})$ (resp.$S_p(\bar{\mathcal{H}}, \mathbb{C})$) as $\mathcal{H}_p^c$ (resp.$\mathcal{H}_p^r$). When $\mathcal{H}$ is separable and infinite dimensional, $\mathcal{H}_p^c$ and $\mathcal{H}_p^r$ are respectively C_p and R_p above. If dim$\mathcal{H}$ = n $< \infty$, we set $\mathcal{H}_p^c = C_p^n$ and $\mathcal{H}_p^r = R_p^n$. We call $\mathcal{H}_p^c$ (resp.$\mathcal{H}_p^r$) is a p-column (resp. p-row) space.

2.5 Semigroups

The relationship between Littlewood-Paley theory and symmetric contraction semigroups with additional hypotheses was investigated by Stein [Ste16] in 1970s. Then more attention has been turn to the application of abstract semigroups on noncommutative L_p-spaces. The authors [JX02] and [JX07] use diffusion semigroup to prove noncommutative maximal inequalities on L_p-spaces for the first time. [Mei08] and [JX02] provide the definition of BMO and tent spaces associated with semigroups. [JM10] studied noncommutative Riesz transform.

Without special explanation, we always assume that $\{T_t\}$ is a semigroup of completely positive maps on a finite von Neumann algebra $\mathcal{M}$ satisfying the following the assumptions

(i) Every T_t is a normal completely positive maps on $\mathcal{M}$ such that $\|T_t(1)\| \leq 1$;

(ii) Every T_t is selfadjoint with respect to the trace τ, i.e. $\tau(T_t(x)y) = \tau(xT_t(y))$;

(iii) The family (T_t) is strongly continuous, i.e. $\lim_{t\to 0} T_t x = x$ with respect to the strong operator topology in $\mathcal{M}$ for $x \in \mathcal{M}$.

Remark 2.29. The first two conditions imply that $\tau(T_t x) = \tau(x)$ for all x, so T_t is faithful and contractive on $L_1(\mathcal{M})$. By interpolation technique, T_t can be extend to a contraction on $L_p(\mathcal{M})$ for $1 \leq p < \infty$ and satifies $\lim_{t\to 0} T_t x = x$ in $L_p(\mathcal{M})$ for any $x \in L_p(\mathcal{M})$.

Let us recall that the dommain $dom_p(A)$ of the generator A(formally depending on p) is the set of all $x \in L_p(\mathcal{M})$ such that

$$Ax := \lim_{t\to 0} \frac{T_t x - x}{t}$$

We calll that this semigroup admits an infinitesimal generator A. On the other hand, the resolvent formula

$$A^{-\alpha} = \Gamma(\alpha)^{-1} \int_0^\infty T_t t^{\alpha-1} dt \text{ for } \alpha > 0.$$

We can easily get the associativity property of $A^{-\alpha}$ by functional calculus. But we want to show an alternative proof of it to illustrate the reason that there is a coefficient $\Gamma(\alpha)^{-1}$ in this formula.

Proposition 2.30. *Given arbitrary two positive numbers α_q and γ with $\alpha = \alpha_q + \gamma$,*

$$A^{-\alpha} = A^{-\alpha_q} A^{-\gamma}.$$

Proof. Since Beta function B(x, y) and gamma $\Gamma(x)$ have the relationship $B(\alpha_q, \gamma) = \frac{\Gamma(\alpha_q)\Gamma(\gamma)}{\Gamma(\alpha_q+\gamma)}$, we get

$$\begin{aligned} A^{-\alpha_q}A^{-\gamma}(x) &= \Gamma(\alpha_q)^{-1}\int_0^\infty T_t(A^{-\gamma}(x))t^{\alpha_q-1}dt = \Gamma(\alpha_q)^{-1}\int_0^\infty T_t\Big(\int_0^\infty T_s(x)s^{\gamma-1}ds\Big)t^{\alpha_q-1}dt \\ &= (\Gamma(\alpha_q)\Gamma(\gamma))^{-1}\iint T_tT_sxs^{\gamma-1}t^{\alpha_q-1}dsdt = (\Gamma(\alpha_q)\Gamma(\gamma))^{-1}\iint T_{t+s}xs^{\gamma-1}t^{\alpha_q-1}dsdt \\ &= (\Gamma(\alpha_q)\Gamma(\gamma))^{-1}\iint T_mx(m-t)^{\gamma-1}t^{\alpha_q-1}dmdt \\ &= (\Gamma(\alpha_q)\Gamma(\gamma))^{-1}\int T_mx\big(\int_0^m (m-t)^{\gamma-1}t^{\alpha_q-1}dt\big)dm \\ &= (\Gamma(\alpha_q)\Gamma(\gamma))^{-1}\int T_mx\big(\int_0^1 m^\alpha(1-s)^{\gamma-1}s^{\alpha_q-1}dt\big)dm \\ &= (\Gamma(\alpha_q)\Gamma(\gamma))^{-1}\int T_mxm^{\alpha-1}B(\alpha_q,\gamma)dm = \Gamma(\alpha)^{-1}\int T_mxm^{\alpha-1}dm = A^{-\alpha}(x) \end{aligned}$$ ☐

Definition 2.31. A standard semigroup (T_t) on a finite von Neumann algebra $\mathcal{M}$ admits a *Markov dilation* if there exists a larger finite von Neumann algebra $\mathcal{N}$, an increasing filtration $(\mathcal{N}_s)_{s\geq 0}$ with conditional expectation $\mathcal{N}_s = \mathbb{E}_s(\mathcal{N})$ and trace preserving $*$-homomorphisms $\pi_s : \mathcal{M} \to \mathcal{N}$ such that $\pi_s(\mathcal{M}) \subset \mathcal{N}_s$ and $\mathcal{E}_s(\pi_t(x)) = \pi_s(T_{t-s}x)), 0 \leq s < t < \infty, x \in \mathcal{M}$.

In [JM10], the authors proved that every semigroup of completely positive unital selfadjoint maps on a finite von Neumann algebra admits a Markov dilation.

Definition 2.32. Let $\mathcal{M}$ be a von Neumann algebra. A semigroup (T_t) is *completely bounded* ultracontractive if it satisfies

$$\|T_t : L_2(\mathcal{M}) \to \mathcal{M}\|_{cb} \leq Ct^{-\frac{n}{4}}, \qquad (R_n^2)$$

$$\|T_t : L_1(\mathcal{M}) \to \mathcal{M}\|_{cb} \leq Ct^{-\frac{n}{2}}. \qquad (R_n)$$

If any of these holds, we say the semigroup $\{T_t\}$ has the cb-Varopoulos dimension n.

Remark 2.33. In general, the Varopoulos dimension can be any constant. For example,

(i) For the classical case, i.e. $\mathbb{R}^n$, it's known that the heat semigroup $T_t = e^{-t\Delta}$ has the classical(not cb) Varopoulos dimension n. Here the n is exactly the dimension of the space. And the poisson semigroup has the Varopoulos dimension n/2.

(ii) Free Poisson semigroup acting on the group algebra of the free group $\mathbb{F}_\infty$, uniformly in the number of generators, has the following

$$\|T_t : L_1(\mathbb{F}_\infty) \to L_\infty(\mathbb{F}_\infty)\| \leq t^{-3/2}$$

which is a known consequence of Haagerup's inequality for homogeneous polynomials [Haa78] (See [JPPP17] for more details).

(iii) More examples of semigroup can be found in the Application and Examples section of Chapter 4.

We may also define the family of condition on the semigroup T_t

$$\|T_t : L_p(\mathcal{M}) \to L_q(\mathcal{M})\|_{cb} \le Ct^{-\frac{n}{2}(1/p-1/q)}., 1 \le p \le q \le \infty. \tag{R_n^{pq}}$$

However, (R_n^{pq}) is equivalent to (R_n^2) by the following property, proved in Lemma 1.1.2 in [JM10]

Proposition 2.34. *Let (T_t) be a selfadjoint family of operators, uniformly bounded on $L_p(\mathcal{M})$. Then (R_n^{pq}) holds for one pair $1 \le p < q \le \infty$ if and only if it holds for all $1 \le p \le q \le \infty$.*

We have an explicit form for T_t as follows:

Proposition 2.35. *Let (T_t) be a selfadjoint family of operators, uniformly bounded on $L_p(\mathcal{N})$. Then there exists a map u_t such that*

$$T_t(y_1^* y_2) = u_t(y_1)^* u_t(y_2), \forall y_1, y_2.$$

Moreover, let $u_\alpha(y)(t) = t^{(\alpha-1)/2} u_t(y)$. Then $A^{-\alpha}(y^*y) = u_\alpha(y)^* u_\alpha(y)$ for $y \in \mathcal{N}$.

Proof. Let us recall the GNS-presentation of a completely positive map. Let $T : \mathcal{N} \longrightarrow \mathcal{N}$ a completely positive mapping. Then $\mathcal{N} \otimes \mathcal{N}$ with inner product defined by setting

$$\langle a \otimes b, x \otimes y \rangle = b^* T(a^* x) y$$

is a semi-Hilbert $\mathcal{N}$-$\mathcal{N}$-module in a natural way. Denote $\mathrm{Null}(\mathcal{N} \otimes \mathcal{N}) = \{x \in E : \langle x, x \rangle = 0\}$. Setting $E = \mathcal{N} \otimes \mathcal{N} / \mathrm{Null}_{\mathcal{N} \otimes \mathcal{N}}$ and $\xi = 1 \otimes 1 + \mathrm{Null}_{\mathcal{N} \otimes \mathcal{N}} \in E$, we have $T(a) = \langle \xi, a\xi \rangle$. The pair (E, ξ) is called the GNS-presentation of T. Indeed, the Hilbertian module $\overline{N \otimes_T N} = \rho C \bar{\otimes} N$. Therefore $u_t : x \mapsto (x \otimes_N 1)$ is the map satisfying $\langle y \otimes 1, x \otimes 1 \rangle_N = T_t(y^* x)$. $u_t(y_1)^* u_t(y_2) = T_t(y_1^* y_2)$ be the map obtained from the GNS construction of T_t. Then let $u_\alpha(y)(t) = t^{(\alpha-1)/2} u_t(y)$ satisfying

$$u_\alpha(y)^* u_\alpha(y) = \int T_t(y^* y) t^{\alpha - 1} dt = A^{-\alpha}(y^* y).$$

2.6 Quantum Euclidean spaces and Noncommutative tori

Our main reference for this section is from [GPJP17]. Given an integer $n \geq 1$, fix an anti-symmetric $\mathbb{R}$-valued $n\times n$ matrix Θ. We define A_Θ as the universal C*-algebra generated b a family $u_1(s), u_2(s), \cdots, u_n(s)$ of one-parameter unitary groups in $s \in \mathbb{R}^n$ which are strongly continuous and satisfy the Θ-commutation relations below

$$u_j(s)u_k(t) = e^{2\pi i\Theta_{jk}st}u_k(t)u_j(s).$$

If $\Theta = 0$ and by Stone's theorem we may take $u_j(s) = \exp(2\pi i s\langle e_j, \cdot\rangle)$ and A_Θ is the space of bounded continuous functions $\mathbb{R}^n \to \mathbb{C}$. In general given $\xi \in \mathbb{R}^n$ we shall extensively use the unitaries $\lambda_\theta(\xi) = u_1(\xi_1)u_2(\xi)\cdots u_n(\xi_n)$ and we define E_Θ as the closure in A_Θ of $\lambda_\Theta(L_1(\mathbb{R}^n))$ with

$$\lambda_\Theta(f) = \int_{\mathbb{R}^n} f(\xi)\lambda_\Theta(\xi)d\xi.$$

If $\Theta = 0$, we find $E_\Theta = C_0(\mathbb{R}^n)$.

Proposition 2.36. *Given any Θ, we easily see that*

(1) $\lambda_\Theta(\xi)^* = e^{2\pi i\sum_{j>k}\Theta_{jk}\xi_j\xi_k}\lambda_\Theta(-\xi)$,

(2) $\lambda_\Theta(\xi)\lambda_\Theta(\eta) = e^{2\pi i\langle\xi,\Theta\eta\rangle}\lambda_\Theta(\eta)\lambda_\Theta(\xi) = e^{2\pi i\sum_{j>k}\Theta_{jk}\xi_j\eta_k}\lambda_\Theta(\xi+\eta)$,

(3) $\lambda_\Theta(f_1)\lambda_\Theta(f_2) = \lambda_\Theta(f_1 *_\Theta f_2)$ *with Θ-convolution given by*

$$f_1 *_\Theta f_2(\xi) = \int_{\mathbb{R}^n} f_1(\xi-\eta)f_2(\eta)e^{2\pi i\sum_{j>k}\Theta_{jk}(\xi_j-\eta_j)\eta_k}d\eta.$$

Definition 2.37. For any smooth and integrable function $f : \mathbb{R}^n \to \mathbb{C}$, let

$$\tau_\Theta(\lambda_\Theta(f)) = \tau_\Theta(\int_{\mathbb{R}^n} f(\xi)\lambda_\Theta(\xi)d\xi) = f(0).$$

τ_Θ extends to a n.f.s trace on E_Θ. Let $\mathcal{R}_\Theta = E_\Theta''$ be the von Neumann algebra generated by E_Θ in the GNS representation of τ_Θ. We obtain $\mathcal{R}_\Theta = L_\infty(\mathbb{R}^n)$ for $\Theta = 0$. In general, we call the Θ-deformation $\mathcal{R}_\Theta$ a *quantum Euclidean space.*

Proposition 2.38. *The following results hold:*

(i) If $n = 2$ and $\Theta \neq 0$, we have

$$E_\Theta \simeq C_0(\mathbb{R}) \rtimes \mathbb{R}.$$

In the case, the crossed product action is given by $\mathbb{R}$-translation.

(ii) Let $\tilde{\Theta}$ denote the $(n-1)\times(n-1)$ upper left corner of $\Theta \in M_n(\mathbb{R})$. Then there exists a continuous group action $\beta_{n-1} : \mathbb{R} \to Aut(E_{\tilde{\Theta}})$ satisfying

$$E_\Theta = E_{\tilde{\Theta}} \rtimes_{\beta_{n-1}} \mathbb{R}.$$

(iii) τ_Θ extends to a n.f.s trace on $\mathcal{R}_\Theta$, and the action β_{n-1} is trace preserving on $(\mathcal{R}_{\tilde{\Theta}}, \tau_{\tilde{\Theta}})$. Induction on n and iteration give

$$\mathcal{R}_\Theta \simeq \mathcal{R}_{\tilde{\Theta}} \rtimes_{\beta_{n-1}} \mathbb{R},$$
$$\mathcal{R}_\Theta \simeq \Big((L_\infty(\mathbb{R}) \rtimes_{\beta_1} \mathbb{R}) \cdots \rtimes_{\beta_{n-1}} \mathbb{R}\Big).$$

Let us now recall the weak-* continuous map

$$\sigma_\Theta : \lambda_\Theta(\xi) \mapsto \exp_\xi \otimes \lambda_\Theta(\xi),$$

where $\exp_\xi$ stands for the character $x \mapsto \exp(2\pi i\langle x, \xi\rangle)$ in $L_\infty(\mathbb{R}^n)$.

Corollary 2.39. *$\sigma : \mathcal{R}_\Theta \to L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{R}_\Theta$ is a normal injective *-homomorphism.*

Let us consider the linear map π_Θ, determined by

$$\pi_\Theta : \exp_\xi \mapsto \lambda_\Theta(\xi) \otimes \lambda_\Theta(\xi)^*.$$

As an illustration, recall that for $\Theta = 0$ we may expect to get the following identity for any Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$

$$\pi_0(f)(x,y) = \pi_0\Big(\int_{\mathbb{R}^n} \hat{f}(\xi)\exp_\xi d\xi\Big)(x,y) = \int_{\mathbb{R}^n} \hat{f}(\xi)\exp_\xi(x-y)d\xi = f(x-y).$$

Lemma 2.40. *π_Θ extends to a normal *-homomorphism*

$$\pi_\Theta : L_\infty(\mathbb{R}^n) \to \mathcal{R}_\Theta\bar{\otimes}\mathcal{R}_\Theta^{op} \textit{satisfying } (\sigma_\Theta \otimes id_{\mathcal{R}_\Theta^{op}}) \circ \pi_\Theta = (id_{\mathbb{R}^n} \otimes \pi_\Theta) \circ \triangle_{\mathbb{R}^n}.$$

where $\triangle_{\mathbb{R}^n}(\exp_\xi) = \exp_\xi \otimes \exp_\xi$ is the comultiplication map in $\mathbb{R}^n$. Moveover, the map π_Θ also extends to a completely isometry $\pi_\Theta : L_2^c(\mathbb{R}^n) \to L_2^c(\mathcal{R}_\Theta)\bar{\otimes}\mathcal{R}_\Theta^{op}$.

2.6.1 BMO space theory

We mainly follow the notations in [JX$^+$03].

Definition 2.41. Let $2 < q \leq \infty$.

(i) We define $L_q^c MO(\mathcal{M})$ (mean oscillation in L^q in the column sense) as the space of all martingale difference sequences (d_k) in L_q such that the sequence $x = (x_n)_{n\geq 0}$ defined by $x_n = \sum_{k=1}^n d_k$ satifies

$$\|x\|^2_{L_c^q MO(\mathcal{M})} = \sup_{m\geq 0} \| \sup_{0\leq n\leq m} E_n(|x_m - x_{n-1}|^2)\|_{q/2} < \infty$$

Note that

$$E_n(|x_m - x_{n-1}|^2) = E_n\Big(\sum_{k=n}^m |d_k|^2\Big).$$

(ii) Define $L_q^r MO(\mathcal{M})$ as the space of all x such that $x^* \in L_q^c MO(\mathcal{M})$, equipped with the norm

$$\|x\|_{L_q^r MO(\mathcal{M})} = \|x^*\|_{L_q^c MO(\mathcal{M})}$$

(iii) Finally,

$$L_q MO(\mathcal{M}) = L_q^c MO(\mathcal{M}) \cap L_q^r MO(\mathcal{M})$$

equipped with the intersection norm

$$\|x\|_{L_q MO(\mathcal{M})} = \max\{\|x\|_{L_q^r MO(\mathcal{M})}, \|x\|_{L_q^c MO(\mathcal{M})}\}.$$

Remark 2.42. If $q = \infty$, all these spaces $L^\infty MO(\mathcal{M}) = BMO(\mathcal{M}), L_c^\infty MO(\mathcal{M}) = BMO_c(\mathcal{M}), L_r^\infty MO(\mathcal{M}) = BMO_r(\mathcal{M})$ coincide [Pis93] and [JX$^+$03].

Define $\mathcal{H}_c^p(\mathcal{M})$[resp.$\mathcal{H}_r^p(\mathcal{M})$] to be the space of all L^p-martingales x with respect to the filtration $(\mathcal{M}_n)_{n\geq 0}$ such that $dx \in L^p(\mathcal{M}; \ell_c^2)$[resp.$dx \in L^p(\mathcal{M}; \ell_r^2)$], and set

$$\|x\|_{\mathcal{H}_c^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M};\ell_c^2)} \text{ and } \|x\|_{\mathcal{H}_r^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M};\ell_r^2)}.$$

Proposition 2.43. *For faithful von Neumann algebra $\mathcal{M}$, $p \geq 2$,we have the following properties:*

(i) $[BMO, L_p^c MO]_\theta = L_q^c MO$, *here* $\frac{1-\theta}{p} = \frac{1}{q}$

(ii) $L_p^c MO \approx \mathcal{H}_p^c$

(iii) $\mathcal{H}_p^c \cap H_p^r = L_p$

In general, for hyperfinite von Neumann algebras, we have to do some modification to derive the non-faithful state. In [JX+03], the authors consider to use the increasing filtration of w^*-closed *-subalgebras $\{\mathcal{M}_n\}_{n\geq 0}$ of $\mathcal{M}$ such that $\cup_{n\geq 0}\mathcal{M}_n$ is w^*-dense in $\mathcal{M}$. Let (e_n) be an increasing sequence of projections converging to 1 in $\mathcal{M}$, and $\mathcal{D}$ be the subalgebras of $\mathcal{M}$ generated by the e_n's. Let $f_n = e_n - e_{n-1}$ (with $e_{-1} = 0$). Clearly (f_n) is a sequence of orthogonal projections which sum to 1. Set $\mathcal{D}_n = \mathbb{C}f_n$ and define $T_n : \mathcal{M} \to \mathcal{D}_n$ by $T_n(x) = \langle xf_n, f_n\rangle f_n$. Note that T_n is the normal conditional expectation from M onto $\mathcal{D}_n$. We denote

$$q_n(x) = \sum_{0\leq k\leq n} T_k(x), x \in \mathcal{M}.$$

Define $\widetilde{M}_n$ as the von Neumann subalgebra generated by $\mathcal{M}_n$ and $\mathcal{D}$. Then there is a faithful normal conditional expectation $\tilde{\varepsilon}_n$ from $\mathcal{M}$ onto $\widetilde{\mathcal{M}}_n$, i.e.

$$\tilde{\varepsilon}_n(x) = \varepsilon_n(x) + (1 - q_n)(x), x \in \mathcal{M}$$

$$\begin{bmatrix} 0 & x_{1n} & 0 \\ x_{n1} & x_{nn} & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

All these mappings extend to contractions and converge on $L^p(M)$. Let $d_0 = \varepsilon_0, \tilde{d}_0 = \tilde{\varepsilon}_0$ and $d_n = \varepsilon_n - \varepsilon_{n-1}, \tilde{d}_n = \tilde{\varepsilon}_n - \tilde{\varepsilon}_{n-1}$ for $n \geq 1$. Note that $dx_n = d_n x$ for all n. Then for any $x \in L^p(M), (d_n x)_n[$ resp.$(\tilde{d}_n x)_n]$ is the martingale difference sequence with respect to $(\mathcal{M}_n)[$ resp.$(\widetilde{\mathcal{M}}_n)]$. We have the following easily checked relations between these martingale differences

$$\tilde{d}_n x = \begin{cases} d_0 x + \sum\limits_{k\geq 1} T_k x, & n = 0 \\ d_n x - T_n x, & n \geq 1. \end{cases}$$

Proposition 2.44. *The bmo norm has the following properties:*

(i) $\|x\|_{\widetilde{bmo}} = \sup\limits_k \|T_k x\| + \sup\limits_n \|\widetilde{E}_n(\sum_{k>n} |d_k|^2)\|^{1/2}$

(ii) $\|x\|_{\widetilde{bmo}_c} \leq \sup\limits_k \|q_k x(1 - q_{k-1})\|, \qquad \|x\|_{\widetilde{bmo}_r} \leq \sup\limits_k \|(1 - q_{k-1})xq_k\|;$

(iii) $\|x\|_{\widetilde{bmo}} \leq \sup\limits_k \{\|q_k x(1 - q_{k-1})\|, \|(1 - q_{k-1})xq_k\|\}$

Proof. For (i), note that $\tilde{d}_n x = d_n x - T_n x, n \geq 1$. Then by triangle inequality we obtain,

$$\|x\|_{\widetilde{bmo}} = \sup_n \|\widetilde{E}_n(\sum_{k>n} |\tilde{d}_k|^2)\|^{1/2} \leq \sup_k \|T_k x\| + \sup_n \|\widetilde{E}_n(\sum_{k>n} (d_k x)^2)\|^{1/2}.$$

For (ii), we decompose $d_k = d_k^c + d_k^r$.

$$\|\widetilde{E}_n(\sum_{k>n} d_k^c x (d_k^c x)^*)\|^{1/2} = \|\widetilde{E}_n| \sum_{k>n} d_k^c x \otimes e_{1k}|^2\|^{1/2} = \|q_n x(1-q_{n-1})x^* q_n\|^{1/2}$$

$$= \|q_n x(1-q_{n-1})\|.$$

$$\begin{bmatrix} 0 & x_{1n} & \vdots & \vdots & \vdots \\ 0 & x_{nn} & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Similarly we get

$$\|\widetilde{E}_n(\sum_{k>n} (d_k^r x)^* d_k^r x)\|^{1/2} = \|\widetilde{E}_n| \sum_{k>n} d_k^r x \otimes e_{k1}|^2\|^{1/2} = \|(1-q_{n-1})x q_n x^*(1-q_{n-1})\|^{1/2}$$

$$= \|(1-q_{n-1})x q_n\|.$$

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ x_{n1} & x_{nn} & 0 & \cdots \\ \vdots & \vdots & 0 & \cdots \end{bmatrix}$$

Part (iii) follows from (i) and (ii) immediately. $\square$

Chapter 3

Smoothing estimates on von Neumann algebras

3.1 Generalized Sobolev inequality with Orlicz function

In 1985, N.Varopoulos [Var85a] discovered the equivalence between heat kernel bounds and Sobolev inequality in an abstract settings. He proved that, for any fixed $n>2$ and a complete noncompact Riemannian manifold $\mathcal{M}$, the equivalence between *Varopoulos dimension* n of a symmetric Markovian semigroup T_t in the sense, i.e.

$$\|T_t f\|_\infty \lesssim t^{-n/2}\|f\|_1, \ t > 0$$

and the Sobolev inequality

$$\|f\|^2_{2n/(n-2)} \leq S \int_M |\nabla f|^2 d\mu, f \in C_c^\infty(M).$$

For further developments in this direction, see the following references, e.g. [CKS87], [Gri94],[LS97], [RS97], [SC09]. In [CM93], Cowling and Meda extended the ultracontractivity of semigroup $(T_t)_{t>0}$ to ϕ-ultracontractivity on σ-measure space $\mathcal{M}$ (i.e. $(T_t)_{t>0}$ satisfies the condition $\|T_t f\|_\infty \leq C\phi(t)^{-1}\|f\|_1$ for all f in $L^1(\mathcal{M})$ and all t in $\mathbb{R}^+$). They prove that $\{T_t\}$ is ϕ-ultracontrative if and only if the infinitesimal generator A has Sobolev embedding properties, namely, $\|\phi(A)^{-\alpha}\|_q \lesssim \|f\|_p$ for all f in $L^p(\mathcal{M})$, whenever $1 < p < q < \infty$ and $\alpha = 1/p - 1/q$. More recently, M.Junge and T.Mei [JM10] predicted that given a semigroup (T_t) of normal self-adjoint contractions with $\|T_t : L_1^0(\mathcal{N}) \to L_\infty(\mathcal{N})\| \leq Ct^{-\frac{n}{2}}$ on some semifinite von Neumann algebra $\mathcal{N}$.

$$\|A^{-z} : L_p^0(\mathcal{N}) \to L_q^0(\mathcal{N})\| \leq C(\alpha) \quad z \in \mathbb{C}, \alpha = Re(z)$$

holds for all $1 < p < q < \infty$ such that $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$. Then Xiao [Xio16] proved their theorem in the noncommutative setting. Their proof heavily relied on complex analysis and multiplier theory. Inspired by the classical proof, we want to prove Sobolev inequalities by real analysis techinques. We also establish the connection between ϕ-ultracontrativity of the semigroup $\{T_t\}$ and Orlicz function spaces on the semifinite von Neumann algebra $\mathcal{M}$.

Orlicz spaces generalize Lebesgue spaces. Recall notations of Orlicz spaces in [CL17]. *An Orlicz function* is a convex function $\varphi : [0, \infty) \to [0, \infty]$ satisfying $\varphi(0) = 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$, which is neither identically zero nor infinite valued on all of $(0, \infty)$, and which is left continuous at $b_\varphi = \sup\{u>0 : \varphi(u)<\infty\}$. It's worth pointing out that any Orlicz function must also be increasing, and continuous on $[0, b_\varphi]$. An Orlicz function is *p-convex*, i.e. for all $0 < \lambda \leq 1$, $g(\lambda x) \leq C_g \lambda^p g(x)$. An Orlicz function g satisfies Δ_2*-condition* if there exists a constant C_g such that $g(2t) \leq C_g g(t), \forall t \in \mathbb{R}^+$.

Let $L^0(X, \Sigma, m)$ be the space of measurable function s on some $\sigma-$finite measure space (X, Σ, m). *The Orlicz space* $L^\varphi(X, \Sigma, m)$ *associated with* φ is defined to be the set

$$L_\varphi = \{f \in L^0 : \varphi(\lambda|f|) \in L^1 \text{ for some } \lambda = \lambda(f)>0\}.$$

This space turns out to be a linear subspace of L^0 which becomes a Banach space when equipped with the so-called *Luxemburg-Nakano norm*

$$\|f\|_\varphi = \inf\{\lambda>0 : \|\varphi(|f|/\lambda)\| \leq 1\}.$$

Let φ be a given Orlicz function. In the context of semifinite von Neumann algebras $\mathcal{M}$ equipped with an f.n.s. trace τ, the space of all $\tau-$measurable operators $\overline{\mathcal{M}}$(equipped with the topology of convergence in measure) plays the role of L^0. In the specific casew where φ is a so-called Young's function (i.e., a map $\varphi : \mathbb{R} \to [0, \infty]$) having the properties of Orlicz function with additional symmetry $\varphi(x) = \varphi(-x)$

$$L_\varphi(\mathcal{M}, \tau) = \{f \in \overline{\mathcal{M}} : \tau(\varphi(\lambda|f|))<\infty \text{ for some } \lambda = \lambda(f)>0\}$$

with the Luxemburg-Nakano norm

$$\|f\|_\varphi = \inf\{\lambda>0 : \tau(\varphi(|f|/\lambda)) \leq 1\}.$$

See [Lux55], [KR], [LZ56], [Kun90] for further development in this direction.

Proposition 3.1. *For any two elements $x, y \in L^0(\mathcal{M})$ and φ is an Orlicz function,*

(i) If $0 < x \leq y$, $\|x\|_\varphi \leq \|y\|_\varphi$;

(ii) If $x \leq y$ and $y \in L_\varphi(\mathcal{M})$, $x \in L_\varphi(\mathcal{M})$.

Proof. (i) According to [FK86], for any two measurable elements x, y in $L^0(\mathcal{M})$, if $x \leq y$, their reagrrangement functions has the inequality $u_s(x) \leq u_s(y)$. Therefore, we obtain $\tau(\varphi(\frac{|x|}{\lambda})) = \int \varphi(\mu_s(\frac{|x|}{\lambda}))ds \leq \int \varphi(\mu_s(\frac{|y|}{\lambda}))ds = \tau(\varphi(\frac{|y|}{\lambda}))$. Then by the definition, we get $\|x\|_\varphi \leq \|y\|_\varphi$. (ii) follows from [DDPSS98]. □

In our context, we borrow the concept "related pair" from [Cow83], but with a totally different definition as follows:

Definition 3.2. A pair of functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{C}$ is *p-related* $(p > 1)$ if

(i) g is a p-convex Orlicz function satisfying the Δ_2-condition;

(ii) h is decreasing and invertible when $p > 1$.

If g and h are p-related, we call $\psi_{g,h}(w) := wh^{-1}(w)g(h^{-1}(w))$ *the generator function* of g and h. Using the definition of Orlicz function, we can easily obtain the following:

Proposition 3.3. *Assume a pair (g, h) is p-related, the following holds*

(1) g is increasing, continuous, and positive function;

(2) $xg(x)h(x)^{1-p}$ is increasing.

Proposition 3.4. *A function Φ has Δ_2-condition if and only if the function $\frac{\Phi(x)}{x}$ is increasing.*

Proof. By induction for all $0 < \lambda < 1, \Phi(\lambda x) \lesssim \lambda\Phi(x)$. Assume $y < x$ and let $\lambda = \frac{y}{x}$, then we can get $\Phi(y) \leq \frac{y}{x}\Phi(x)$ i.e. $\frac{\Phi(y)}{y} \leq \frac{\Phi(x)}{x}$. For the other direction, let $y = 2x$ and we get the Δ_2-condition immediately. □

The following Lemma motivates us our definition of the generator function $\psi_{g,h}$.

Lemma 3.5. *Assume g and h are p-related and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows:*

$$\phi(w) := \inf_b \Big(w \int_0^b g(t)dt + \int_b^\infty g(t)h(t)dt \Big)$$

Then

$$\phi(w) \leq \psi_{g,h}(w)$$

Proof. Denote the function $\mathcal{F}(w, b) = w \int_0^b g(t)dt + \int_b^\infty g(t)h(t)dt$. Therefore, $\phi(w) = \inf_b \mathcal{F}(w, b)$. Since $\mathcal{F}(w, b)$ is an integrable formula with respect to variable b. Then we can differentiate the function $\mathcal{F}(w, b)$ with the variable b.

$$\frac{\partial \mathcal{F}(w, b)}{\partial b} = wg(b) - g(b)h(b) = g(b)\Big(w - h(b)\Big).$$

The function admits the global minimum value when $w - h(\tilde{b}) = 0, i.e.\ \tilde{b} = h^{-1}(w)$. Therefore $\phi(w) = \mathcal{F}(w, h^{-1}(w))$. Indeed,

$$\begin{aligned}\phi(w) = \mathcal{F}(w, \tilde{b}) &\leq wg(\tilde{b})\tilde{b} + \tilde{b}g(\tilde{b})h(\tilde{b}) = wg(h^{-1}(w))h^{-1}(w) + h^{-1}(w)g(h^{-1}(w))w \\ &= 2wg(h^{-1}(w))h^{-1}(w) \leq \psi_{gh}(w)\end{aligned}$$

□

Lemma 3.6. *Assume g and h are p-related and $\psi_{g,h}$ is their generator function. We define $\Phi_{g,h}$ as*

$$x^p = \Phi_{g,h} \circ \psi_{g,h}(x).$$

(i) $\Phi_{g,h}$ has Δ_2-condition if and only if $f(x) = xg(x)h(x)^{1-p}$ is increasing.

(ii) Therefore, $\Phi_{g,h}$ is an Orlicz function.

Proof. (i) According to Proposition 3.4, $\Phi_{g,h}$ satisfies Δ_2-condition if and only if $\Phi_{g,h}(x)/x$ is an increasing function. Then we have the following equivalences:

(i) $\Phi_{g,h}(x)/x$ increasing

(ii) $\Phi^{-1}_{g,h}(x)/x$ decreasing

(iii) $\Phi^{-1}_{g,h}(x^p)/x^p$ decreasing

According to the definition, we know $\Phi^{-1}_{g,h}(x^p) = wg(h^{-1}(w))h^{-1}(w)$. Therefore, $\frac{wg(h^{-1}(w))h^{-1}(w)}{w^p}$ is decreasing if and only if $\frac{bg(b)h(b)}{h(b)^p} = bg(b)h(b)^{1-p}$ is increasing. (ii) It immediately holds by the definition of that g and h are p-related and (i). □

Remark 3.7. Assume $f(x) = xg(x)h(x)^{1-p}$ is increasing, the connection between monotonicity of f and p-convexity of g can be illustrated as follows: given any positive numbers b, c with $b \leq c$, there exists a number $\lambda = \frac{b}{c} < 1, b = \lambda c$. By monotonicity of f, we obtain $h(\lambda c)^{1-p}g(\lambda c)\lambda c \leq h(c)^{1-p}g(c)c$. After simplifying, we have

$$\Big(\frac{h(c)}{h(\lambda c)}\Big)^{1-p} \cdot \frac{g(c)}{g(\lambda c)} \geq \lambda.$$

Lemma 3.8. *For any positive operator w in $L^0(\mathcal{M})$, there exists an operator $\tilde{w} \in \mathcal{M}$, such that*

$$w \leq \tilde{w} \leq 2w. \tag{3.1.1}$$

Proof. For any positive operator in $\mathcal{M}$, we get the spectral decomposation $0 \leq w = \int t dE_t$. Define the spectral projection $e_k := 1_{[2^k, 2^{k+1})}(w)$, we obtain

$$w \leq \sum_k 2^{k+1} e_k \leq 2 \sum_k 2^k e_k \leq 2w$$

Let $w_k := 2^{k+1}$ and $\tilde{w} := \sum_k w_k e_k$, we get (3.1.1). $\square$

Lemma 3.9. *For a sequence of self-adjoint projects $\{e_k\}$ in $\mathcal{M}$ with $\sum e_k = 1$. The representation $\pi : \mathcal{M} \to \mathcal{M} \otimes \mathbb{B}(\ell_2)$ as $\pi(x) = \sum e_k x e_j \otimes e_{kj}$ is a $*$-homomorphism. Here e_{kj} is the elementary basis of $\mathbb{B}(\ell_2)$.*

Proof. For any two elements $x, y \in \mathcal{M}$, we get

$$\pi(x)\pi(y) = \Big(\sum e_k x e_l \otimes e_{kl}\Big)\Big(\sum e_l y e_j \otimes e_{lj}\Big) = \sum (\sum_l e_k x e_l y e_j) \otimes e_{kj} = \sum e_k x y e_j \otimes e_{kj} = \pi(xy)$$

$\square$

We refter the reader to semigroup theory in the chapter of preliminaries.

Theorem 3.10. *Let the semigroup (T_t) be a family of self-adjoint positive contraction maps, with the generator operator A on a von Neumann algebra $\mathcal{M}$. Assume the pair (g, h) is p-related with the following conditions:*

(i) the semigroup $\{T_t\}$ satisfies the ordered h-p-contractivity, i.e. $T_t f \leq h(t)\|f\|_p 1$ for $f \in L_p^+(\mathcal{M})$;

(ii) $\mathcal{G}(A)(x) = \int_0^\infty T_t x g(t) dt$.

Let $\Phi(w)$ be the inverse function of $\psi_{gh}(w^{1/p})$. We have the following resolvent estimate:

$$\|\mathcal{G}(A) : L_p(\mathcal{M}) \to L_\Phi(\mathcal{M})\| \leq C_{pq}(\alpha) \quad 1 < p < \infty.$$

Proof. Thanks to [JX07], for a sequence of self-adjoint positive contraction semigroups $\{T_t\}$, we have the maximal inequality: there exists an operator W such that $T_t(x) \leq W$. By Lemma 3.8, there exists a family of basis $\{e_j\}$ such that $[e_k T_t(x) e_j]_{k,j} \leq [e_k W e_j]_{k,j} = [w_{kj}]_{k,j}$. Let $\{b_j\}$ be abstract positive numbers. Following from Lemma 3.9, we decompose the map $\mathcal{G}(A)(x)$ into four parts with the family of basis $\{e_j\}$:

$$I_{k,j}^{1,1}(x) = \int_0^\infty 1_{[0,b_k]}(t) e_k T_t x e_j 1_{[0,b_j]}(t) g(t) dt \ \text{ and } A_{11} = \sum_{k,j} I_{k,j}^{1,1}$$
$$I_{k,j}^{1,2}(x) = \int_0^\infty 1_{[0,b_k]}(t) e_k T_t x e_j 1_{[b_j,\infty)}(t) g(t) dt \ \text{ and } A_{12} = \sum_{k,j} I_{k,j}^{1,2}$$

$$I_{k,j}^{2,1}(x) = \int_0^\infty 1_{[b_k,\infty)}(t) e_k T_t x e_j 1_{[0,b_j]}(t) g(t) dt \text{ and } A_{21} = \sum_{k,j} I_{k,j}^{2,1}$$
$$I_{k,j}^{2,2}(x) = \int_0^\infty 1_{[b_k,\infty)}(t) e_k T_t x e_j 1_{[b_j,\infty)}(t) g(t) dt \text{ and } A_{22} = \sum_{k,j} I_{k,j}^{2,2}.$$

Therefore the integral function can be reformulated by $\mathcal{G}(A)(x) = A_{11} + A_{12} + A_{21} + A_{22}$. Next we define three operators $V_1, V_2 \in L_2^c(\mathbb{R})\bar{\otimes}\ell_2^c\bar{\otimes}\mathcal{M}$, $J \in L^\infty(\mathbb{R}) \otimes \mathcal{M}$ as follows:

$$V_1 = \sum_j \sqrt{g}|1_{[0,b_j]}\rangle^c \otimes e_{j1} \otimes e_{jj}, V_2 = \sum_j \sqrt{g}|1_{[b_j,\infty}\rangle^c \otimes e_{j1} \otimes e_{jj}$$
$$J(t) = \pi(T_t x) = \sum_{k,j} e_k T_t(x) e_j \otimes e_{k,j}.$$

Then we have $A_{11} = V_1^* J V_1 A_{12} = V_1^* J V_2, A_{21} = V_2^* J V_1, A_{22} = V_2^* J V_2$. Using 2×2 matrix, we get

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} V_1^* & 0 \\ V_2^* & 0 \end{bmatrix} \times \begin{bmatrix} J & J \\ J & J \end{bmatrix} \times \begin{bmatrix} V_1 & V_2 \\ 0 & 0 \end{bmatrix}$$

Observe that A is a positive definite matrix. Then the anti-diagonal part can be dominated by the diagonal part as

$$A_{12} + A_{21} \leq A_{11} + A_{22}.$$

Therefore, $\mathcal{G}(A)(x) \leq 2(A_{11} + A_{22})$.

By Lemma 3.9, π is a *-homomorphism. Observe that $J = \pi(T_t(x)) \leq \pi(W) = J_w$. Therefore, we obtain

$$A_{11} = V_1^* J V_1 \leq V_1^* J_w V_1 = \sum_k \int_0^{b_k} w_k g(t) dt e_{kk}$$

For the estimate of A_{22}, we use the ordered h-p-contractivity of the semigroup:

$$A_{22} = V_2^* J V_2 \int \sum_{k,j} (1 - b_k(t)) e_k T_t f e_j (1 - b_j(t)) g_\alpha(t) dt \leq \sum_k \int_{b_k}^\infty g(t) h(t) \|f\|_p dt e_{kk}.$$

Assume $\|f\|_p \leq 1$, we take the summation of the two parts

$$\begin{aligned} A_{11} + A_{22} &\leq \sum_k \Big(w_k \int_0^{b_k} g_\alpha(t) dt + \|f\|_p \int_{b_k}^\infty g_\alpha(t) h(t) dt \Big) e_{kk} \\ &\leq \sum_k \phi(w_k) e_{kk} \leq \sum_k \psi_{gh}(w_k) e_{kk} \leq \psi_{gh}(W). \end{aligned}$$

Let $\psi^p_{gh}(w) := \psi_{gh}(w^{\frac{1}{p}})$. Taking the above inequality with (3.2.3), we have

$$\mathcal{G}_\alpha(A)f \lesssim A_{11} + A_{22} \leq \psi_{gh}(W) = \psi^p_{gh}(W^p).$$

Let $x = G_\alpha(A)f$ and $y = \psi^p_{gh}(W^p)$. We denote $\Phi = (\psi^p_{gh})^{-1}$. Then $\Phi(y) = \Phi(\psi^p_{gh}(W^p)) = W^p \in L_1(\mathcal{M})$. By Lemma 3.6, we know Φ is an Orlicz function space. Thus by the definition of the Orlicz function, $y \in L_\Phi(\mathcal{M})$. Then by Proposation 3.1 with $x \leq y$, $x \in L_\Phi(\mathcal{M})$ i.e. $G_\alpha(A)f \in L_\Phi(\mathcal{M})$. □

As an important example of Theorem 3.10, let $g_\alpha(t) = t^{\alpha-1}, h(t) = t^{-n_1}$. Then $\mathcal{G}_\alpha(A) = A^{-\alpha}$ and T_t satisfies the Varopoulos dimension n_1. And we get

$$\psi^p_{gh}(w) = \frac{\|f\|_p^{\frac{\alpha}{n_1}}}{n_1 - \alpha} w^{\frac{1}{p} - \frac{\alpha}{n_1 p}},\ \Phi(w) = \frac{n_1 - \alpha}{\|f\|_p^{\frac{\alpha}{n_1}}} w^{\frac{n_1 p}{n_1 - \alpha}}.$$

Then let $\frac{n_1 p}{n_1 - \alpha} = q$, i.e. $\alpha = n_1(1 - \frac{p}{q})$, then $\Phi(w) = Cw^q$. Therefore the Orlicz function space $L_\Phi(\mathcal{M})$ is $L_q(\mathcal{M})$. Then we have the following noncommutative HLS inequality:

Theorem 3.11. *If the symmetric Markovian semigroup (T_t) generated by operator A on a von neumann algebra $\mathcal{M}$ satisfies the Varopoulos dimension n. Then we have the Sobolev embedding:*

$$\|A^{-\alpha} : L_p(\mathcal{M}) \to L_q(\mathcal{M})\| \leq C_{pq}(\alpha), \quad \alpha = n_1(1 - \frac{p}{q}), 1 < p < q < \infty.$$

Corollary 3.12. *If the semigroup (T_t) generated by operator A on a von Neumann algebra M satisfies these conditions in Theorem 3.10. Let $g_\alpha(t) = t^{\alpha-1}, h(t) = \begin{cases} t^{-n_1} & t \leq 1 \\ t^{-n_2} & t > 1 \end{cases}$. We obtain the following embedding*

$$\|\mathcal{G}_\alpha(A) : L_p(\mathcal{M}) \longrightarrow L_{q_1}(\mathcal{M}) + L_{q_2}(\mathcal{M})\| \leq C_{pq}(\alpha),\ \alpha = n_1(1 - \frac{p}{q_1}) = n_2(1 - \frac{p}{q_2}).$$

Proof. According to the assertions of g and h, we can calculate that

$$\varphi(w) = \begin{cases} \frac{\|f\|_p^{\frac{\alpha}{n_1}}}{n_1 - \alpha} w^{1 - \frac{\alpha}{n_1}} & w \geq 1 \\ \frac{\|f\|_p^{\frac{\alpha}{n_2}}}{n_2 - \alpha} w^{1 - \frac{\alpha}{n_2}} & w < 1 \end{cases}$$

Therefore,

$$\psi^p_{gh}(w) = \begin{cases} \frac{\|f\|_p^{\frac{\alpha}{n_1}}}{n_1-\alpha} w^{\frac{1}{p}(1-\frac{\alpha}{n_1})} & w \geq 1 \\ \frac{\|f\|_p^{\frac{\alpha}{n_2}}}{n_2-\alpha} w^{\frac{1}{p}(1-\frac{\alpha}{n_2})} & w<1 \end{cases}, \Phi(w) = \begin{cases} \frac{n_1-\alpha}{\|f\|_p^{\frac{\alpha}{n_1}}} w^{\frac{n_1 p}{n_1-\alpha}} & w \geq 1 \\ \frac{n_2-\alpha}{\|f\|_p^{\frac{\alpha}{n_2}}} w^{\frac{n_2 p}{n_2-\alpha}} & w<1 \end{cases}$$

If $\frac{n_1 p}{n_1-\alpha} = q_1, \frac{n_2 p}{n_2-\alpha} = q_2, i.e. \alpha = n_1(1-\frac{p}{q_1}) = n_2(1-\frac{p}{q_2})$, we have

$$\Phi(w) = \begin{cases} \frac{n_1-\alpha}{\|f\|_p^{\frac{\alpha}{n_1}}} w^{q_1} & w \geq 1 \\ \frac{n_2-\alpha}{\|f\|_p^{\frac{\alpha}{n_2}}} w^{q_2} & w<1 \end{cases}$$

Recall that the measure $\mu_t = \inf\{\|X(1-p)\|, \tau(X) \leq t\}$ in [FK86], we get

$$tr(\Phi(x)) \leq 1 \Leftrightarrow \int_0^\infty \Phi(\mu_t)dt = \int_{\mu_t \leq 1} u_t^{q_1} dt + \int_{\mu_t \geq 1} u_t^{q_2} dt.$$

If we truncate any element $X \in L_\Phi$ as $X = X_1 \cdot 1_{[0,t]} + X_2 \cdot 1_{[t,\infty)}, X_1 \in L_{q_1}, X_2 \in L_{q_2}$. Then $X \in L_\Phi \Leftrightarrow X \in L_{q_1} + L_{q_2}$. □

Remark 3.13. (i) For any von Neumann algebra $\mathcal{M}$ with only $\tau(\cdot) \leq 1$ part. Therefore, it can be improved to be $\mathcal{G}_\alpha(A) : L_p(\mathcal{M}) \longrightarrow L_{q_1}(\mathcal{M}) + L_{q_2}(\mathcal{M}) \cong L_{q_1}(\mathcal{M})$.

(ii) For discrete von Neumann algebra S_p, $\mathcal{G}_\alpha(A) : S_p \longrightarrow S_{q_1} + S_{q_2} \cong S_{q_2}$.

Corollary 3.14. *If the semigroup (T_t) generated by operator A on a von neumann algebra M satisfies these assumptions in Theorem 3.10. Let* $g_\alpha(t) = \begin{cases} t^{\alpha_1 - 1} & t \leq 1 \\ t^{\alpha_2-1} & t>1 \end{cases}$, $h(t) = t^{-n_1}$. *We have the following inequality*

$$\|\mathcal{G}_\alpha(A) : L_p(\mathcal{M}) \to L_{q_1}(\mathcal{M}) + L_{q_2}(\mathcal{M})\| \leq C_{pq}(\alpha_1, \alpha_2),\ n_1 = \frac{\alpha_1 q_1}{q_1 - p} = \frac{\alpha_2 q_2}{q_2 - p}$$

Proof. According to the assertions of g and h, we can calculate that

$$\psi_{gh}(w) = \begin{cases} \frac{\|f\|_p^{\frac{\alpha}{n_1}}}{n_1-\alpha_1} w^{1-\frac{\alpha_1}{n_1}} & w \geq 1 \\ \frac{\|f\|_p^{\frac{\alpha}{n_1}}}{n_1-\alpha_2} w^{1-\frac{\alpha_2}{n_1}} & w<1 \end{cases}$$

Therefore,

$$\psi^p_{gh}(w)=\begin{cases}\frac{\|f\|_p^{\frac{\alpha}{n_1}}}{n_1-\alpha_1}w^{\frac{1}{p}(1-\frac{\alpha_1}{n_1})} & w\geq 1\\ \frac{\|f\|_p^{\frac{\alpha_2}{n_1}}}{n_1-\alpha_2}w^{\frac{1}{p}(1-\frac{\alpha_2}{n_1})} & w<1\end{cases},\Phi(w)=\begin{cases}\frac{n_1-\alpha_1}{\|f\|_p^{\frac{\alpha_1}{n_1}}}w^{\frac{n_1p}{n_1-\alpha_1}} & w\geq 1\\ \frac{n_1-\alpha_2}{\|f\|_p^{\frac{\alpha_2}{n_1}}}w^{\frac{n_1p}{n_1-\alpha_2}} & w<1\end{cases}$$

If $\frac{n_1p}{n_1-\alpha_1}=q_1,\frac{n_1p}{n_1-\alpha_2}=q_2,i.e.n_1=\frac{\alpha_1q_1}{q_1-p}=\frac{\alpha_2q_2}{q_2-p}$, we have

$$\Phi(w)=\begin{cases}\frac{n_1-\alpha_1}{\|f\|_p^{\frac{\alpha_1}{n_1}}}w^{q_1} & w\geq 1\\ \frac{n_1-\alpha_2}{\|f\|_p^{\frac{\alpha_2}{n_1}}}w^{q_2} & w<1\end{cases}$$

Repeating the same techique in the above Corollary, we take the measure $\mu_t=\inf\{\|X(1-p)\|,\tau(X)\leq t\}$

$$tr(\Phi(x))\leq 1\Leftrightarrow\int_0^\infty\Phi(\mu_t)dt=\int_{\mu_t\leq 1}u_t^{q_1}dt+\int_{\mu_t\geq 1}u_t^{q_2}dt.$$

If we truncate any element $X\in L_\Phi$ as $X=X_1\cdot 1_{[0,t]}+X_2\cdot 1_{[t,\infty)},X_1\in L_{q_1},X_2\in L_{q_2}$. Then we get$X\in L_\Phi\Leftrightarrow X\in L_{q_1}+L_{q_2}$. □

In the remainning part of this section, we want to apply the above results to quantum Euclidean spaces $\mathcal{R}_\theta$. A crucial point, as in abelian algebras, is to identify kernels over $\mathcal{R}^n_\theta\bar{\otimes}(\mathcal{R}^n_\theta)^{op}$, where the op-structure (reversed product law) is used in the second copy. This is justified by the important map

$$\pi_\theta:L_\infty(\mathbb{R}^n)\to\mathcal{R}_\theta\bar{\otimes}\mathcal{R}_\theta^{op},$$
$$\exp(2\pi i\langle\xi,\cdot\rangle)\mapsto\lambda_\theta(\xi)\otimes\lambda_\theta(\xi)^*,$$

which extends to a normal *-homomorphism, for which the op-structure is strickly necessary. We refer [GPJP17] for the below Lemma.

Lemma 3.15. *Fix $n\in N$ and σ. Then there exists a normal $*$-endomorphism $\pi:L_\infty(\mathbb{R}^n)\to\mathcal{R}_\theta\bar{\otimes}\mathcal{R}_\theta^{op}$ satisfying*

$$\pi(\lambda_0(\xi))=\lambda_\theta(\xi)\otimes\lambda_\theta(\xi)^*.$$

Moreover, π and flip$\circ\pi$ extend to an isometries from $L_2^c(\mathbb{R}^n)\to L_2^c(\mathcal{R}_\theta)\bar{\otimes}\mathcal{R}_\theta$.

Let us recall $CB(L_1(\mathcal{R}_\theta),\mathcal{R}_\theta)=\mathcal{R}_\theta\bar{\otimes}\mathcal{R}_\theta^{op}$ in [ER00]. Then by a combination of this equality and Lemma 3.15, we have the following result:

Theorem 3.16. *Let $m\in L_\infty(\mathcal{R}^n)$ and define the multiplier map $T_m(\lambda_\theta(h))=\int m(\xi)h(\xi)\lambda_\theta(\xi)d\xi$ on $\mathcal{R}^n_\theta$,* *then we get*

$$\|T_m : L_1(\mathcal{R}_\theta^n) \to L_\infty(\mathcal{R}_\theta^n)\|_{cb} \le \|m\|_{L_\infty(\mathbb{R}^n)}.$$

Example 3.17. The following operators have the Varopolus dimension n on $\mathcal{R}_\theta^n$:

(i) For Schordöinger evolution operator $T_t = e^{it\triangle}$, the action on $\mathbb{R}^n$ is as follows:

$$e^{it\triangle} \cdot f(x) := \frac{1}{(2\pi it)^{n/2}} \int_{R^n} e^{-\frac{|x-y|^2}{2it}} f(y)dy.$$

Denote the multiplier $m_t(\xi) := e^{-it|\xi|^2}$, then $\|T_{m_t}\|_{cb} \le ct^{-\frac{n}{2}}$.

(ii) For heat semigroup $T_t = e^{-t\triangle}$, the action on $\mathbb{R}^n$ is as follows:

$$e^{-t\triangle} \cdot f(x) := \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Denote the multiplier $m_t(\xi) := e^{-t|\xi|^2}$, then $\|T_{m_t}\|_{cb} \le ct^{-\frac{n}{2}}$.

Lemma 3.18. *Given a fixed ε_1, for any $p \ge 2$*

$$\sup_{\epsilon \le \varepsilon_1} \|A^{-\varepsilon} : L_p(\mathcal{R}_\theta^n) \longrightarrow L_p(\mathcal{R}_\theta^n)\| \le C(p)$$

Proof. According to Cowling's theorem [Cow83], the operator A^{iu} is bounded on $L^p(\mathcal{R}_\theta^n)$. Therefore, we have

$$\|A^{iu} f\|_p \le C_0(p)(1 + |u|^3 \log^2(|u|))^{|1/p-1/2|} \exp(\pi|1/p - 1/2||u|)\|f\|_p.$$

Applying this for some $2 < p_1 < \infty$,

$$\|A^{-is} : L_{p_1}(\mathcal{R}_\theta^n) \longrightarrow L_{p_1}(\mathcal{R}_\theta^n)\| \le p_1 c_1 e^{\frac{|s|}{2}}.$$

On the other hand when p = 2, by Plancherel's theorem, we get

$$\|A^{-is} : L_2(\mathcal{R}_\theta^n) \longrightarrow L_2(\mathcal{R}_\theta^n)\| \le c_2.$$

By the complex interpolation in [Pis93], let $x \in L_p(\mathcal{R}_\theta), X(\theta) = x, \theta \neq 0$.

$$\|X(it)\|_{p_1} \le 1, \|X(1 + it)\|_2 \le 1, \text{here } \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{2}.$$

Let f(z) $= A^{-z\epsilon/\theta}X(z)e^{r(z-\theta)^2}$ and $r \geq \frac{|\epsilon|}{2\theta}$,

$$\|f(it)\|_{p_1} \leq \|A^{-it\epsilon/\theta}\|e^{r(\theta^2-t^2)}\|x(it)\|_{p_1} \leq c_1p_1e^{\frac{|\epsilon|}{2\theta}t+r(\theta^2-t^2)} \leq c_1p_1e^{r(\theta^2+t-t^2)}$$
$$\leq c_1p_1e^{r(\theta^2+\frac{1}{4}-(t-\frac{1}{2})^2)} \leq c_1p_1e^{r(\theta^2+\frac{1}{4})} \leq Cp_1.$$

By a similar calculation, we get $\|f(1+it)\|_2 \leq C_2$. Then we conclude that

$$\sup_{\varepsilon\leq\varepsilon_1}\|A^{-\varepsilon} : L_p(\mathcal{R}_\theta^n) \longrightarrow L_p(\mathcal{R}_\theta^n)\| \leq C(p).$$

□

Corollary 3.19. *If the symmetric Markovian semigroup (T_t) generated by operator A on a von neumann algebra $\mathcal{R}_\theta^n$ satisfies the Varopoulos dimension n, then Sobolev embedding holds with $\alpha = \frac{n}{2p}$, i.e.*

$$\|A^{-\alpha} : L_p(\mathcal{R}_\theta^n) \longrightarrow L_q(\mathcal{R}_\theta^n)\| \leq C_{pq}(\alpha).$$

Proof. The generator operator can be written as $A^{-\frac{n}{2p}} = A^{-(\frac{n}{2p}-\frac{n}{2q})}A^{-\frac{n}{2q}}$. We choose the q satisfying the condition $\frac{n}{2q} \leq \epsilon_1$. Here the ϵ_1 is the one in the above Lemma 3.18. Then for $\forall f \in L_p(\mathcal{R}_\theta^n)$,

$$\|A^{-\frac{n}{2q}}f\|_p \leq C(p)\|f\|_p.$$

Then for $\alpha' = \frac{n}{2}(\frac{1}{p}-\frac{1}{q})$. by Theorem 3.11. we get

$$\|A^{-\alpha'}(A^{-\frac{n}{2q}}f)\|_q \leq C_{pq}\|A^{-\frac{n}{2q}}f\|_p \leq C_{pq}C(p)\|f\|_p.$$

Therefore we get $\|A^{-\frac{n}{2p}}f\|_q \leq C\|f\|_p$. □

3.2 A noncommutative version of Moser's inequality

In the section, we follow the notations from [FK86]. Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal semifinite faithfull trace. Let $\widetilde{\mathcal{M}}$ be the set of the closed, densely defined operators on $\mathcal{H}$ affiliated with $\mathcal{M}$. Then for any positive element $x \in \widetilde{M}_+$, we take its spectrum decomposation $x = \int_0^\infty t de_x(t)$. Then for any subset $E \subset \mathbb{R}$, we define $v_x(E) = tr(e_x(E))$. Observe that v_x is a Borel measure on $\mathbb{R}$ and tr(x):=$\int_0^\infty t dv_x(t)$ is a faithful extension of tr to $\widetilde{M}_+$. Define

$$\overline{M} := \{x \in \widetilde{M} : tr(e_{|x|}(t,\infty)) < \infty \text{ for some } t > 0\}.$$

Then $\overline{M}$ equipped with strong sense operations and with the topology of convergence in measure, becomes a topological *-algebra, called the algebra of tr-measurable operators.

Remark 3.20. When $\mathcal{M} = L^\infty(X,m)$ and tr(f)= $\int f dm$, then $\widetilde{\mathcal{M}}$ is the *-algebra of m-measurable functions that are finite m-a.e. and $\overline{\mathcal{M}}$ is the *-subalgebra of $\widetilde{\mathcal{M}}$ consisting of functions that are bounded except on a set of finite m-measure.

Definition 3.21. Let $a \in \overline{\mathcal{M}}$ and for all $t \geq 0$, we define

(i) $\lambda_a(t) := tr(e_{|a|(t,\infty)})$, the distribution function of the operator a with respect to the trace tr,

(ii) $\mu_a(t) := \inf\{s \geq 0 : \lambda_a(s) \leq t\}, t > 0$, the non-increasing rearrangement of the operator a with respect to the trace tr,

(iii) $K_a(t) := \frac{1}{t}\int_0^t \mu_a(r)dr$.

Proposition 3.22 ([FK86]). *Let a, b be two operators in $\overline{\mathcal{M}}$, then*

(i) $tr(|a|) = \int_0^\infty \mu_a(t)$

(ii) $\mu_a(t) = \inf\{\|ap\| : p \in Proj(\mathcal{M}), tr(p^\perp) \leq t\}$

(iii) $K_a(t) = \frac{1}{t}\sup\limits_{tr(p)\leq t} tr(|a|p)$

(iv) $K_{a+b}(t) \leq K_a(t) + K_b(t)$

Recall the definition of the convolution of functions f and g on $\mathbb{R}^n$, i.e. $T_g(f)(x) = \int g(x-y)f(y)dy,\ h := T_g(f)$. Similarly, we have an analogy of convolutiion in quantum Eculidean space $\mathcal{R}_\theta$ as follow:

Definition 3.23. The convolution in $\mathcal{R}_\theta$ is defined as

$$T_g^\theta(f) = g *_\theta F = \int_{\mathbb{R}^n} \lambda_\theta(\xi)\hat{g}(\xi)f(\xi)d\xi, \quad F = \lambda_\theta(f).$$

Remark 3.24. When $\theta = 0$, we get $g * \lambda_0(\xi) = \int \exp(i\langle \xi, y-x\rangle)g(x)dx = \lambda_0(\xi)(y)\int \exp(-\langle \xi, x\rangle)g(x)dx = \lambda_0(\xi)\hat{g}(\xi)$.

Lemma 3.25. *There exists a normal, injective $*$-homomophism $\sigma : \mathcal{R}_\theta^n \to \mathcal{R}_\theta^n \otimes L_\infty(\mathbb{R}^n)$ such that* *(i)* $\sigma(\lambda_\theta(\xi)) = \lambda_\theta(\xi) \otimes \lambda_0(\xi)$,

(ii) $\sigma(T_g^\theta(f)) = (Id \otimes T_g)(\sigma(\lambda_\theta(f)))$

Proof. We refer to [GPJP15], for the fact that σ, defined by (i), extends to a normal and injective $*$-homomophism. Thus it suffices to calculate the second part:

$$\begin{aligned}\sigma\left[g *_\theta \lambda_\theta(f)\right] =&\sigma(\int_{\mathbb{R}^n} \lambda_\theta(\xi)\hat{g}(\xi)f(\xi)d\xi) = \int_{\mathbb{R}^n} \lambda_\theta(\xi) \otimes \lambda_0(\xi)f(\xi)\hat{g}(\xi)d\xi \\ =&(Id \otimes T_g^0)(\int \lambda_\theta(\xi) \otimes \lambda_0(\xi)f(\xi)d\xi) = (Id \otimes T_g^0)(\sigma(F)).\end{aligned}$$

□

Problem 3.26. It remains open whether there exists some $\sigma : (\mathcal{R}_\theta, tr_\theta) \to (\mathcal{R}_\theta, \mathrm{tr}_\theta) \otimes L^\infty(\mathbb{R}^n, tr_0)$ to become a trace preserving *-homomorphism. It is, however, true for $\mathcal{A}_\theta$.

Lemma 3.27. *Let $\mathcal{R}_\theta$ be a quantum Euclidean space, then*

(i) $L_2(\mathbb{R}^n, tr) \cong L_2(\mathcal{R}_\theta, tr_\theta)$ *by the map* $f \mapsto \lambda_\theta(f)$*;*

(ii) $L_2^c(\mathcal{R}_\theta) \hookrightarrow_\sigma L_\infty(\mathbb{R}^d, L_2^c(\mathcal{R}_\theta))$ *by the map* $f \mapsto \sigma(\lambda_\theta(f))$ *is isometric;*

(iii) $L_2^c(\mathcal{R}_\theta) \hookrightarrow_\sigma L_\infty(\mathcal{R}_\theta, L_2^c(\mathbb{R}^d))$ *by the map* $f \mapsto \sigma(\lambda_\theta(f))$ *is isometric;*

(iv) $L_1(\mathcal{R}_\theta) \hookrightarrow_\sigma L_\infty(\mathbb{R}^d,\ L_1(\mathcal{R}_\theta^d)), L_1(\mathcal{R}_\theta) \hookrightarrow_\sigma L_\infty(\mathcal{R}_\theta^d, L_1(\mathbb{R}^d))$ *by the map* $f \mapsto \sigma(\lambda_\theta(f))$ *are isometric.*

Proof. (i) By Plancherel's theorem, it turns out to be trivial.

(ii) For a function $f \in L_2(\mathcal{R}_\theta)$. We have $f = \int \tilde{f}(\xi)\lambda_\theta(\xi)d\xi, \|f\|_2 = \|\tilde{f}\|_2$. In particular, given a Schwartz function $f \in \mathcal{S} \cap L_2(\mathcal{R}_\theta)$. Let $X = \int \hat{f}(\xi)\lambda_\theta(\xi) \otimes \lambda_0(\xi)d\xi$. By using the fact $tr_\theta(\lambda_\theta(\xi)^*\lambda_\theta(f)) = f(\xi)$, we get

$$Id_{L^\infty(\mathbb{R}^d)} \otimes tr_\theta(X^*X) = \int \overline{\hat{f}(\xi)}tr_\theta[\int \lambda_\theta(\xi)^*\lambda_\theta(\xi')\hat{f}(\xi')d\xi']d\xi = \int |\overline{\hat{f}(\xi)}|^2 d\xi.$$

Since the Schwartz function space $\mathcal{S}$ is dense in $L_2(\mathcal{R}_\theta)$, then we can extend the operator to $L_2(\mathcal{R}_\theta)$.

(iii) Let X be defined as in (ii). Now we take the trace of the $L_\infty(\mathbb{R}^n)$ part as follows:

$$(tr_0 \otimes Id_{\mathcal{R}_\theta})(X^*X) = \int \overline{\hat{f}(\xi)}\lambda_\theta(\xi)^*[tr_0(\lambda_0(\xi)^* \int \hat{f}(\xi')\lambda_\theta(\xi') \otimes \lambda_0(\xi')d\xi']d\xi$$

Next we take an arbitrary smooth function $\phi \in (\mathcal{R}_\theta)_*$,

$$\phi[tr_0(\lambda_0(\xi)^* \int \hat{f}(\xi')\lambda_\theta(\xi') \otimes \lambda_0(\xi')d\xi'] = tr_0(\lambda_0(\xi)^* \int \hat{f}(\xi')\phi(\lambda_\theta(\xi'))\lambda_0(\xi')d\xi' = \hat{f}(\xi)\phi(\lambda_\theta(\xi))$$

Therefore $tr_0(\lambda_0(\xi)^* \int \hat{f}(\xi')\lambda_\theta(\xi')\otimes\lambda_0(\xi')d\xi' = \hat{f}(\xi)\lambda_\theta(\xi)\cdot 1$ with respect to the weak operator topology. Then we get

$$tr_0(X^*X) = \int \overline{\hat{f}(\xi)}\hat{f}(\xi)\lambda_\theta(\xi)^*\lambda_\theta(\xi)d\xi = \int |\hat{f}(\xi)|^2 d\xi = \|f\|_2^2 \cdot 1$$

(iv) Analogously we can get the map $L_2(\mathcal{R}_\theta) \to L_\infty(\mathbb{R}^d, L_2^r(\mathcal{R}_\theta))$ is isometric from (iii). By [JP07] we know

$$L_\infty(\mathbb{R}^d, L_2^r(\mathcal{R}_\theta^d)) \cdot L_\infty(\mathbb{R}^d, L_2^c(\mathcal{R}_\theta^d)) \subset L_\infty(\mathbb{R}^d, L_1(\mathcal{R}_\theta^d)),$$

we get the cb-isomorphism $L_1(\mathcal{R}_\theta) \hookrightarrow L_\infty(\mathbb{R}^d, L_1(\mathcal{R}_\theta^d))$. The cb-isomorphism $L_1(\mathcal{R}_\theta) \hookrightarrow L_\infty(\mathcal{R}_\theta^d, L_1(\mathbb{R}^d))$ is proved similarily. □

Let's recall the notation of weighted noncommutative L_p spaces in Chapter 2, we have the following:

Lemma 3.28. *Let $F = \int \hat{F}(\xi)\lambda_\theta(\xi)d\xi$, $\hat{F} \in \mathcal{S}$ and $\sigma : (\mathcal{R}_\theta, tr_\theta) \to (\mathcal{R}_\theta, tr_\theta) \otimes L^\infty(\mathbb{R}^n, tr_0)$ as $\sigma(F) = \int \hat{F}(\xi)\lambda_\theta(\xi) \otimes \lambda_0(\xi)d\xi$. For any arbitrary function $\phi \in (\mathcal{R}_\theta)_*$*

(i) $(\phi \otimes tr_0)\sigma(F) = tr_\theta(F)$

(ii) Denote ω as the associated weight with respect to $\phi \otimes tr_0$, then we have

$$\|Id \otimes T_g : L_1(\mathcal{R}_\theta, w) \to L_1(\mathcal{R}_\theta, w)\| \leq \|g\|_1.$$

Proof. (i) By using Lemma 3.27, we observe that $Id \otimes tr_0(\sigma(F)) = 1 \cdot \int \hat{F}(\xi)d\xi$. Then for any function $\phi \in (\mathcal{R}_\theta)_*$, we get $(\phi \otimes tr_0)\sigma(F) = tr_\theta(F)$.

(ii) For any function $F \in L_1(\mathcal{R}_\theta), \omega(\sigma(F)) = tr_\theta(F)$ and $\sigma(F *_\theta g) = \sigma(F) *_0 g$. This gives

$$\omega((Id \otimes T_g)(\sigma(F))) = \omega(\sigma(F *_\theta g)) = tr_\theta(F *_\theta g)$$

In fact, the weight ω is associated with a density D, for which given an arbitrary positive $x \geq 0$,

$$\|x\|_{L^1(\omega)} = tr(D^{1/2}xD^{1/2}) = \omega(x)$$

For a completely positive map $T_g : L_1(\mathcal{R}_\theta) \to L_1(\mathcal{R}_\theta), \sigma(T_g(\lambda_\theta(F)^*\lambda_\theta(F)) = (Id \otimes T_g)\sigma(\lambda_\theta(F)^*\lambda_\theta(F))$. Thus we obtain

$$\|T_g(x)\|_{L^1(tr_\theta)} = \|\sigma(T_g(x))\|_{L^1(\omega)} = \|Id \otimes T_g(\sigma(x))\|_{L^1(\omega)} = \|D^{1/2} Id \otimes T_g(x) D^{1/2}\|$$

$$\leq \|1 \otimes D^{1/2} x D^{1/2}\|_{L^1} \|Tg\| \leq \|\sigma(x)\|_{L^1(w)} \|g\|_1 \qquad \square$$

Proposition 3.29. *Let a be an element in $L^1(\mathcal{R}_\theta) \cap L^\infty(\mathcal{R}_\theta), g$ be a scalar-valued function on $\mathbb{R}^d$ and h be the convolution function, i.e. $h = g *_\theta a = T_g^\theta(a)$. The convolution map has the following properties:*

(i) $\|h\|_\infty \leq \|a\|_\infty \|g\|_1$

(ii) $\|h\|_1 \leq \|a\|_1 \|g\|_1$

(iii) $\|h\|_\infty \leq \|a\|_1 \|g\|_\infty$

Proof. (i) By Lemma 3.28 we know

$$\|\sigma(T_g^\theta(a))\|_\infty = \|Id \otimes T_g(\sigma(\lambda_\theta(a)))\|_\infty \leq \|Id \otimes T_g\| \cdot \|a\|_\infty = \|g\|_1 \|a\|_\infty$$

(iii) It follows from Lemma 3.28 $T_g : L_1(\mathcal{R}^n) \to L_1(\mathcal{R}^n)$ is bounded by $\|g\|_1$.

(iii) By using Lemma 3.27, we know that $L_1(\mathcal{R}_\theta^n) \hookrightarrow L_\infty(\mathcal{R}_\theta, L^1(R^n))$. Therefore, given an arbitrary operator $a \in \mathcal{R}_\theta^d$, we have

$$\sigma(T_g(a))(z) = Id \otimes T_g(\sigma(\lambda_\theta(a)))(z) = \int g(z-y)\sigma(\lambda_\theta(a))(y)dy$$

Define $\varphi(f) = \int g(z-y)f(y)dy, \forall f \in L^1(\mathcal{R}_\theta^n)$. Then we know $\sigma(T_g(a))(z) = Id \otimes \varphi(\lambda_\theta(a))$. Hence we get

$$\begin{aligned} \|\sigma(T_g(a))(z)\|_\infty &= \|Id \otimes \varphi(\lambda_\theta(a))\|_\infty \leq \|Id \otimes \varphi\|_{cp} \|(\lambda_\theta(a))\|_\infty \\ &= \|\varphi\|_{cp} \|a\|_{\mathcal{R}_\theta(L^1(\mathbb{R}^d))} = \|g\|_\infty \|a\|_{\mathcal{R}_\theta(L^1(\mathbb{R}^d))} \end{aligned} \qquad \square$$

These three inequalities above are necessary in order to develop many basic properties for convolutions. We want to investigate whether $\mathcal{R}_\theta$ has the analogous properties for convolution operatos mentioned in [O$^+$63]. Thanks to the functional calculus on $\mathcal{R}_\theta$, the next Lemma follows from [O$^+$63] immediately. For the convenience of the reader, we show the sketch of the proof in this section.

Lemma 3.30. *Suppose the convolution function $h = a *_\theta g$, where the operator a has $tr(supp(a)) \leq \gamma, \|a\|_\infty \leq \alpha$ and g is a scalar-valued function. Then for all $t > 0$,*

$$K_h(t) \leq \alpha\gamma K_g(\gamma), \;\; K_h(t) \leq \alpha\gamma K_g(t)$$

Proof. Let $s > 0$, we define a function as follows:

$$g_s(z) = \begin{cases} g(z) & \text{if } |g(z)| \le s \\ s \text{ sgn } g(z) & \text{otherwise.} \end{cases}$$

Observe that $g^s(z) = g(z) - g_s(z), h = a *_\theta g = a *_\theta (g_s + g^s) = a *_\theta g_s + a *_\theta g^s := h_1 + h_2$. By Proposation 3.29, we have the following three inequalities:

$$\|h_2\|_\infty \le \|a\|_\infty \|g^s\|_1 \le \alpha \int_s^\infty \mu(|g|>y)dy$$

$$\|h_1\|_\infty \le \|a\|_1 \|g^s\|_\infty \le \alpha x s$$

$$\|h_2\|_1 \le \|a\|_1 \|g^s\|_1 \le \alpha\gamma \int_s^\infty \mu(|g|>y)dy$$

Taking $s = \mu_g(x)$

$$K_h(t) = \frac{1}{t}\int_0^t \mu_h(z)dz \le \|h_1\|_\infty + \|h_2\|_\infty \le \alpha\gamma\mu_g(\gamma) + \alpha \int_{g^*(\gamma)}^\infty \mu(|g|>y)dy \le \alpha\gamma K_g(\gamma)$$

Taking $s = \mu_g(t)$ yields

$$\begin{aligned} tK_h(t) = \int_0^t \mu_h(z)dz \le \int_0^t \mu_{h_1}(z)dz + \int_0^t \mu_{h_2}(z)dz \le t\|h_1\|_\infty + \int_0^\infty \mu_{h_2}(z)dz = t\|h_1\|_\infty + \|h_2\|_1 \\ \le t\alpha\gamma\mu_g(t) + \alpha\gamma \int_{\mu_g(t)}^\infty \mu(|g|>y)dy \le \alpha\gamma(t\mu_g(t) + \int_{\mu_g(t)}^\infty \mu(|g|>y)dy) \le \alpha\gamma t K_g(t) \end{aligned}$$

Dividing by t, we complete our assertion. □

Lemma 3.31. *(Basic lemma on convolution operators) For arbitrary self adjoint* $a \in L_\infty(\mathcal{R}_\theta, tr_\theta) \cap L_1(\mathcal{R}_\theta, tr_\theta)$ *and* $h = a *_\theta g$*. Then for any* $t > 0$

$$K_h(t) \le tK_a(t)K_g(t) + \int_t^\infty \mu_a(s)\mu_g(s)ds$$

Proof. By definition we know $\mu_a([t,\infty) = tr_\theta(e_{|a|}(t,\infty))$, therefore μ_a is a measure on $\mathbb{R}$. Set $f(z) = z$. Then there exists a normal $*-$homomorphism $\pi_a(f) = f(a) = a$. By functional calculus, (f, μ_a) and (a, tr) have the same distribution, i.e. $tr_\theta(\phi(a)) = \int \phi(f)d\mu_a(f)$, for any measurable function ϕ.

Given an arbitrary ϵ, there exists a doubly infinite sequence $\{y_n\}_{-\infty}^\infty$ such that $y_0 = u_f(t), y_n \le y_{n+1}, \lim_{n\to\infty} y_n = \infty, \lim_{n\to-\infty} y_n = 0$.

Decompose $f(z) = \sum_{-\infty}^{\infty} f_n(z)$. where

$$f_n(z) = \begin{cases} 0 \ \text{ if } \ |f(z)| \leq y_{n-1} \\ z - y_{n-1} * \text{sgn} f(z) \quad \text{if } \ y_{n-1} < |f(z)| \leq y_n \\ (y_n - y_{n-1}) * \text{sgn} f(z) \ \text{ if } \ y_n < |f(z)| \end{cases}$$

For each f_n, we denote $F_n := \pi_a(f_n) \in \mathcal{R}_\theta$. Then they have the following properties:

(i) $\text{supp}(f_n) \subset E_n := \{z : |f(z)| > y_{n-1}\} \Longrightarrow \|F_n\|_\infty = \|f_n\|_\infty, \tau(\text{supp}F_n) = \mu(|f_n| > 0)$;

(ii) $\mu(E_n) = \mu(|f| > y_{n-1}) \Longrightarrow \tau(\text{supp}F_n) \leq \mu(|f| > y_{n-1})$;

(iii) $\|f_n(z)\|_\infty \leq y_n - y_{n-1} \Longrightarrow \|F_n\|_\infty \leq y_n - y_{n-1}$

(iv) $\|f_n\|_1 \leq (y_n - y_{n-1})\mu(|f| > y_{n-1}) \Longrightarrow \|F_n\|_1 \leq (y_n - y_{n-1})\mu(|f| > y_{n-1})$

With these properties, we get

$$h = a *_\theta g = (\sum_{-\infty}^{\infty} F_n, g) = (\sum_{n=-\infty}^{0} F_n) * g + (\sum_{n=1}^{\infty} F_n) * g := h_1 + h_2$$

By Proposition 3.22, $K_h(t) \leq K_{h_1}(t) + K_{h_2}(t)$. To evaluate $K_{h_2}(t)$ we use Lemma 3.30,

$$\begin{aligned} K_{h_2}(t) &\leq \sum_{n=1}^{\infty} K_{\pi_a(f_n) *_\theta g} \leq \sum_{n=1}^{\infty} \|F_n\|_\infty \tau_\theta(\text{supp}F_n) K_g(t) \leq \sum_{n=1}^{\infty} (y_n - y_{n-1})\mu_a(|f| > y_{n-1}) K_g(t) \\ &\leq (1+\epsilon)(\int_{\mu_a^*(t)} f^*(\xi) d\xi) K_g(t) = (1+\epsilon)(\int_{f^*(t)} \mu_a(y) dy) K_g(t) \end{aligned}$$

The series on the last second line is an infinite Riemann sum for the integral $\int_{f^*(t)}^{\infty} \mu(|f| > y) dy$ using a proper choice of the sequence $\{y_n\}$. □

Lemma 3.32. *[Ada88] Let a(s,t) be a non-negative measurable function on* $(-\infty, \infty) \times [0, +\infty)$ *such that*

$$\alpha(s,t) \leq 1, \ 0 < s < t, \tag{3.2.1}$$

$$\sup_{t>0} (\int_{-\infty}^{0} + \int_{t}^{\infty} \alpha(s,t)^{p'} ds)^{1/p'} = b < \infty. \tag{3.2.2}$$

Then there is a constant $c_0 = c_0(p, b)$ *such that for* $\phi \geq 0$,

$$\int_{-\infty}^{\infty} \phi(s)^p ds \leq 1, \tag{3.2.3}$$

then

$$\int_0^\infty e^{-F(t)}dt \le c_0,\ \ F(t) = t - (\int_{-\infty}^{\infty} \alpha(s,t)\phi(s)ds)^{p'}$$

Theorem 3.33. *For* $1<p<\infty$*, there is a constant* $c_0 = c_0(p)$*, such that for any self adjoint element* $a \in L^p(\mathcal{R}_\theta)$ *with* $tr(supp(a)) \le \gamma$*,*

$$\int_0^\gamma \left(exp(C|\frac{\mu_h}{\|a\|_p}|^{p'})\right) \le c_0\gamma,\ \ for\ h = I_\beta *_\theta a \in L^1(\mathcal{R}_\theta),$$

where $\beta = \frac{n}{p}$ *and* $I_\beta(x) = |x|^{\beta-n}$ *is the Riesz potential of order* β*.*

Proof. Set $h(x) = I_{n/p} *_\theta a(x)$ for $a \ge 0$. Let $g(x) = |x|^{\beta-n}$ with $\beta = n/p$,

$$u_g(t) = C(t^{-1})^{1/p'} \text{and}\ \ K_g(t) = p \cdot u_g(t).$$

Then by Lemma 3.31, we can write

$$\begin{aligned}\mu_h(t) \le K_h(t) &\le tK_a(t)K_g(t) + \int_t^\infty u_a(s)u_g(s)ds \\ &= C(pt^{-1/p'}\int_0^t u_a(s)ds + \int_t^\gamma u_a(s)s^{-1/p'}ds).\end{aligned}$$

Denote $G(t) = pt^{-1/p'}\int_0^t u_a(s)ds + \int_t^\gamma u_a(s)s^{-1/p'}ds$. Then the above inequality is as follow

$$\mu_h(t) \le CG(t)$$

Then replacing C_1 with $C^{-p'}$, we obtain

$$C_1\mu_h(t)^{p'} \le G(t)^{p'}$$

Next, we change variables by setting $\phi(s) = \frac{\gamma^{1/p}}{\|a\|_p} \cdot u_a(\gamma e^{-s}) \cdot e^{-s/p}$. Then we obtain

$$\alpha(s,t) = \begin{cases} 1 & 0<s<t \\ pe^{(t-s)/p'} & t<s<\infty \\ 0 & -\infty<s \le 0 \end{cases}$$

The construction of $\alpha(s,t)$ satifies the assumption (5.1.1) and (3.2.2) and $\phi(s)$ satisfies (3.2.3).

$$
\begin{aligned}
F(t) &= t - \Big(\int_0^\infty \alpha(s,t)\phi(s)ds\Big)^{p'} \\
&= t - \Big(\frac{\gamma^{1/p}}{\|a\|_p}\big(\int_0^t \cdot u_a(\gamma e^{-s})e^{-s/p}ds + \int_t^\gamma u_a(\gamma e^{-s})e^{t/p'}e^{-s}ds\big)\Big)^{p'} \\
&= t - \Big(\frac{\gamma^{1/p}}{\|a\|_p}\big(\int_{\gamma e^{-t}}^\gamma (\frac{\gamma}{x})^{1/p'}u_a(x)dx + pe^{t/p'}\int_0^{\gamma e^{-t}} u_a(x)dx\big)\Big)^{p'} \\
&= t - \Big(\frac{\gamma^{1/p}}{\|a\|_p}\big(pe^{t/p'}\int_0^{\gamma e^{-t}} u_a(x)dx + \gamma^{1/p'}\int_{\gamma e^{-t}}^\gamma x^{-1/p'}u_a(x)dx\big)\Big)^{p'} \\
&= t - \Big(\frac{1}{\|a\|_p}\big(p(\gamma e^{-t})^{-1/p'}\int_0^{\gamma e^{-t}} u_a(x)dx + \int_{\gamma e^{-t}}^\gamma x^{-1/p'}u_a(x)dx\big)\Big)^{p'} \\
&= t - \Big(\frac{G(\gamma e^{-t})}{\|a\|_p}\Big)^{p'}
\end{aligned}
$$

Then by Lemma 3.32, we have

$$
\int_0^\infty \phi(s)^p ds \le 1 \ \text{ implies } \ \int_0^\infty e^{-F(t)}dt \le c_0,
$$

and

$$
\int_0^\infty e^{-F(t)}dt = \int_0^\infty e^{\left(\frac{G(\gamma e^{-t})}{\|a\|_p}\right)^{p'}} \cdot e^{-t}dt = \frac{1}{\gamma}\int_0^\gamma e^{\left(\frac{G(x)}{\|a\|_p}\right)^{p'}} dx \le c_0
$$

Since

$$
tr\Big(e^{Ca(x)^{p'}}\Big) = \int_0^\gamma e^{C\mu_a(x)^{p'}}dx, \ tr\Big(a(x)^p\Big) = \int_0^\gamma \mu_a(x)^p dx
$$

Then we have

$$
\int_0^\gamma e^{C_1(\frac{\mu_h}{\|a\|_p})^{p'}}dx \le \int_0^\gamma e^{(\frac{G(x)}{\|a\|_p})^{p'}}dx \le c_0\gamma.
$$

Then we finish the proof. □

Let $\Phi(a) = \exp\Big(C|\frac{I_\beta *_\theta a}{\|a\|_p}|^{p'} - I_\beta *_\theta a\Big), a \in L_p(\mathcal{R}_\theta) \cap L_1(\mathcal{R}_\theta)$ with $tr(supp(a)) < \infty$. Then according to this Theorem above, we obtain $tr(\Phi(a)) \lesssim 1$, which means $a \in L_\Phi(\mathcal{R}_\theta)$. Now we define the space $L_p^{00}(\mathcal{R}_\theta) = \{a \in L_p(\mathcal{R}_\theta) | tr(supp(a)) < \infty\}$ and the $(L_\Phi + L_\infty)(\mathcal{R}_\theta)$ space with the norm $\|x\| = \max(\|x\|_\Phi, \|x\|_\infty)$.

Corollary 3.34. *Let $I_\beta(x) = |x|^{\beta-n}$ be the Riesz potential of order β. Then the map*

$$
I_\beta : L_p^{00}(\mathcal{R}_\theta) \to L_\Phi + L_\infty(\mathcal{R}_\theta)
$$

is bounded. Here $\Phi(x) = \exp\left(C|x|^{p'}\right) - 1$.

Proof. Assume the element a in $L_p(\mathcal{R}_\theta)$ has $\|a\|_p = 1$ and h is the convolution element of the resolvent operator, i.e. $h = I_\beta(a)$. Thanks to the monotonicity of μ_h, we get

$$\frac{\gamma}{2} e^{C_1(\mu_h(\gamma))^{p'}} \leq \int_{\gamma/2}^{\gamma} e^{C_1(\mu_h(x))^{p'}} dx \leq c_0(p)(\gamma).$$

Therefore, we know $\mu_h(\gamma) \leq C_2(p)$. Then we decompose the μ_h as follows:

$$\mu_h = 1_{[0,\gamma)}\mu_h + I_{[\gamma,\infty)}\mu_h := \mu_h^1 + \mu_h^2.$$

According to Theorem 3.33, we know $\int_0^\gamma (e^{C_1(\mu_h^1(x))^{p'}} - 1)dx \leq (c_0(p) - 1)\gamma \leq c_0(p)'\gamma$. Thus $\mu_h^1 \in L_\Phi$ where $\Phi(\mu_h) = \exp\left(C|\mu_h|^{p'}\right) - 1$. And $\|\mu_h^2\|_\infty = \max(\mu_h^2) = \mu_h^2(\gamma) \leq C_2(p)$. Thus $\mu_h^2 \in L_\infty$. Hence $\mu_h \in L_\Phi + L_\infty(\mathbb{R}^n)$. By functional calculus, we obtain our assertion of the resolvent operator I_β. □

Recall the noncommutative Lorentz space $L_{p,q}(\mathcal{M}, \tau)$ with the norm $\|x\|_{p,q}$ defined as follows:

$$\|x\|_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p}\mu_x(t))^q \frac{dt}{t}\right)^{1/q}, & q < \infty, \\ \sup_t t^{1/p}\mu_x(t), & q = \infty. \end{cases}$$

Now for any element a in $L_{q,\infty}$, we choose a family of elements a_k, where we can decompose

$$\mu_a = \sum_{k=0}^{\infty} 1_{[2^{k-1},2^k]}\mu_a \text{ with respect to } a = \sum_k a_k.$$

Then from the above Corollary 3.34, we have

$$\begin{aligned} \|I_\beta(a)\|_{L_\Phi + L_\infty} &\leq \sum_k \|I_{a_k}\|_{L_\Phi + L_\infty} \leq \sum 2^k (\int_{2^k}^{2^{k+1}} \mu_a(x)^p dx)^{1/p} \leq (\int_0^1 \mu_a(x)^p dx)^{\frac{1}{p}} + \sum \mu_a(2^k) 2^{k(1+1/p)} \\ &\leq (\int_0^1 \mu_a(x)^p dx)^{\frac{1}{p}} + \int_1^\infty x^{1/p}\mu_a(x)dx \leq (\int_0^1 \mu_a(x)^p dx)^{\frac{1}{p}} + \int_1^\infty x^{1/p} x^{-q} dx. \end{aligned}$$

When $1/p - q + 1$ is strictly less than zero, the last term above is bounded. Therefore we imply the result below:

Corollary 3.35. *The resolvent operator* I_β

$$I_\beta : L_p \cap L_{q,\infty} \to L_\Phi + L_\infty,$$

is bounded when $1/p - q + 1 < 0$*. Here* $\Phi(x) = \exp\left(C|x|^{p'}\right) - 1$.

Chapter 4

Completely bounded Sobolev inequality

Our main goal in this chapter is to prove the complete version of Varapoulos' Theorem, i.e. we replace the operator norm by the *completely bounded norm*. This makes sense because, thanks to interpolation, Pisier found a suitable definition of noncommutative L_p spaces as operator spaces, [Pis93] and [Pis03]. Quite surprisingly, there are natural examples of semigroups which admit Sobolev inequalities in the bounded, but not in the completely bounded sense. However, we show that the archetypical examples from noncommutative geometry, noncommutative tori with finitely many generators satisfy the assumption on completely bounded heat kernels. Moreover, the easiest examples of noncommutative, noncompact spaces, the higher dimensional Moyal planes, i.e. the quantum Euclidean spaces from [GPJP15], also satisfy cb-heat kernel estimates. The complete version of Varopoulos' theorem is interesting for two reasons. First, for commutative spaces such as $\mathbb{R}^n$, $\mathbb{T}^n$ or compact Riemanian manifolds, the heat kernel estimates from L_1 to L_∞ *are automatically completely bounded*. However, estimates for the cb-norm of the resolvent are *strict improvements* of the classical estimates by Hardy-Littlewood and Sobolev. Indeed, by positivity the classical HLS inequality implies that for a matrix valued function

$$\|id \otimes A^{-\alpha}(f)\|_{L_q(\mathbb{R}^n, S_p^m)} \leq c_{pq}\Big(\int_{\mathbb{R}^n} \|f(x)\|_{S_p^m}^p dx\Big)^{1/p}, \tag{4.0.1}$$

whereas the cb-norm implies that $id \otimes A^\alpha(f) = (a \otimes 1)F(b \otimes 1)$ and

$$\|a\|_{S_{2r}^m} \|F\|_{L_q(\mathbb{R}^n; S_q^m)} \|b\|_{S_{2r}^m} \leq c_{pq}\Big(\int_{\mathbb{R}^n} \|f(x)\|_{S_p^m}^p dx\Big)^{1/p}.$$

From this algebraic factorization a simple applications of Hölder's inequality for matrices easily implies (4.0.1). Secondly, cb-estimates for operators from L_p to L_q for $p \neq q$ are very rare. A notable excpetion to this rule are results on completely bounded version of Fourier multipliers and Schur multipliers on the Scatten class p-classes obtained by A. Harcharras [Har99]. In her work strong algebraic tools for subset of integers enjoying the noncommutative $\Lambda(p)$-property are used. The theory of estimates for the cb-norm estimates of maps from $L^p(\mathcal{M})$ to $L^p(\mathcal{N})$ has made considerable progress in recent years. For example, estimates for quantum Fourier multipliers can easily proved by variations of the transference methods stablished in [GPJP17],[CXY13],[Ric16]. It's known from [Bou86][Wei01] that the UMD-property implies the boundedness of all invariant singular integrals or standard multiplier operators under some regularity assumption, for further results using BMO-techniques see [JM12, JM10, JMP13]. However, in our proof we have to develop abstract interpolation tools which allow us to the apply the standard 'divide and concquer principle from analysis'. The main difficulty lies, of course, in the lack of pointwise estimates which are replaced by sophisticated matrix decomposition of operators inspired by the classical proof.

Finally, we also discovered an efficient method to prove complete heat kernel estimates. Starting from Effros-Ruan's theorem $CB(N_*, M) = N\bar{\otimes}M$, we observe that it suffices to estimate the noncommutative 'heat kernel' in $R^n_\theta\bar{\otimes}(R^n_\theta)^{op}$ for the quantum euclidean spaces R^n_θ of dimension n. It is convenient to think of R^n_θ as the von Neumann algebras with a trace generated by strongly continuous one parameter group $u_k(t)$, $k = 1, ..., n$ such that

$$u_k(t)u_k(s) \leq e^{i\theta_{kj}st}u_k(s)u_j(t).$$

Recall also that A^{op} is the C^*-algebra obtained from inverting the order in the multiplication. Since $L_1(R^n_\theta)^* = R^n_\theta$ holds for the duality bracket given by the trace, the kernel estimates follow from a simple use of the *-homomorphism

$$\begin{aligned}
&\pi_\theta : L_\infty(\mathbb{R}^n) \to \mathbb{R}_\theta\bar{\otimes}\mathbb{R}^{op}_\theta, \\
&\exp(i\langle\xi, \cdot\rangle) \mapsto u_1(\xi_1)\cdots u_n(\xi_n) \otimes (u_1(\xi_1)\cdots u_n(\xi_n))^*.
\end{aligned}$$

Then $\pi_\theta(\frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}})$ turn out to be the 'correct' kernel for the 'integral operator' implementing the heat semigroup. Hence the cb-estimates follows immediately from the explicit knowledge of the commutative heat kernel.

4.1 Matrixed resolvent estimates

Because of the failure of the noncommutative analogue of the usual pointwise resolvent formula in the classical proof, we investigate the matrix decomposation of the matrixed valued resolvent formula. We first introduce the weighted upper and lower resolvent operators. Then we use the bmo-norm instead of L^∞, and L^2 norm for other cases to investigate some estimate properties of these two operators.

4.1.1 BMO estimates

Definition 4.1. Assume $\{R_t\}$ satisfies the condition (R_n^2) and $\alpha < \frac{n}{4}$, we define

$$R_{k,j}(x) = 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} \int 1_{[2^k,\infty)} 1_{[2^j,\infty)} R_t x t^{\alpha-1} dt$$

Thanks to $\alpha < \frac{n}{4}$, this is well-defined. Observe that $\|R_{k,j} : \mathcal{X} \to \mathcal{N}\|_{cb} < \infty$, because $\|R_{k,j}(x)\| = 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} \| \int_{\max(2^j,2^k)} R_t x t^{\alpha-1} dt\| \leq_{cb} 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} \left(\int_{\max(2^j,2^k)} t^{\alpha-1-\frac{n}{4}} \right) \cdot \|x\| \leq_{cb} c\|x\|$.

Definition 4.2. Given a semigroup T_t such that (R_n^2) holds and $\alpha < \frac{n}{4}$, we define the weighted upper resolvent operator $\Phi^{\mathrm{up}}_{\theta,\alpha}$ by

$$\Phi^{\mathrm{up}}_{\theta,\alpha}(x) = \sum e_{kj} \otimes 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} \int 1_{[2^k,\infty)} 1_{[2^j,\infty)} T_t x t^{\alpha-1} dt = \sum e_{kj} \otimes R_{k,j}(x).$$

Similarly, the weighted lower resolvent operator $\Phi^{\mathrm{lower}}_{\theta,\alpha}$ is defined as

$$\Phi^{\mathrm{lower}}_{\theta,\alpha}(x) = \sum e_{kj} \otimes 2^{-\frac{k(1-\theta)}{2}} 2^{-\frac{j(1-\theta)}{2}} \int 1_{[0,2^k)} 1_{[0,2^j)} T_t x t^{\alpha-1} dt.$$

Lemma 4.3. *Let $\theta + \sigma = \frac{n}{4}$, then the upper resolvent operator*

$$\Phi^{up}_{\theta,\sigma} : \ell_2 \to bmo(B(l_2))$$

is completely bounded.

Proof. We observe that $d_k(x) = d_k^r(x) + d_k^c(x)$ given the following two terms

$$d_k^r(x) = \sum_{j\leq k} e_{k,j} \otimes R_{k,j}(x), d_k^c(x) = \sum_{j\leq k} e_{j,k} \otimes R_{j,k}(x)$$

Note that

$$d_k^r(x)^* d_k^r(x) = \sum_{m,n\leq k} e_{m,n} \otimes R^*_{k,m} R_{k,n}$$

$$d_k^r(x) d_k^r(x)^* = e_{k,k} \otimes (\sum_{j,k} R_{k,j} R^*_{k,j})$$

By the definition of BMO's norm, we have

$$\|x\|_{L^c_{BMO}} = \sup_v \|\sum_{k\geq v} \mathbb{E}_v(d_k^* d_k)\|$$

$$\|x\|_{L^r_{BMO}} = \sup_v \|\sum_{k\geq v} \mathbb{E}_v(d_k d_k^*)\|$$

When $k \geq v$, we have

$$\mathbb{E}_v(d_k^r(x)^* d_k^r(x)) = \sum_{m,n\leq v} e_{m,n} \otimes R_{k,m}^* R_{k,n}$$

$$\mathbb{E}_v(d_k^r(x) d_k^r(x)^* = \begin{cases} e_{vv} \otimes \sum_{j\leq v} R_{v,j} R_{v,j}^* & k = v \\ 0 & k < v \end{cases}$$

Summing over all $k > v$, we get

$$\sum_{k\geq v} \mathbb{E}_v(d_k^r(x)^* d_k^r(x)) = \sum_{k\geq v} \Big(\sum_{m,n\leq v} e_{m,n} \otimes R_{k,m}^* R_{k,n} \Big) = \sum_{m,n\leq v\leq k} e_{m,n} \otimes R_{k,m}^* R_{k,n}$$

$$\sum_{k\geq v} \mathbb{E}_v(d_k^r(x) d_k^r(x)^*) = \sum_{k\geq v} e_{v,v} \otimes \sum_{j\leq v} R_{v,j} R_{v,j}^*$$

We mainly focus on the scalar case, i.e.

$$\alpha_{k,j} := \|R_{k,j}\| \leq 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} \int_{2^k}^{\infty} t^{\alpha-1} t^{-\frac{n}{4}} dt = 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} 2^{k(\alpha-\frac{n}{4})} := \beta_k \gamma_j,$$

here $\beta_k = 2^{k(\theta/2+\alpha-n/4)}, \gamma_j = 2^{j\theta/2}$.

$$\|(\alpha_{k,j})_{k,j}\| \leq (\sum_{k\geq v} \beta_k^2)^{1/2} (\sum_{j\leq v} \gamma_j^2)^{1/2} \leq C_1 2^{v(\theta/2+\alpha-n/4)} 2^{v\theta/2} = C_1 2^{v(\theta+\alpha-n/4)} = C_1.$$

Therefore, we get

$$\|\sum d_k^r(x)\|^2_{L^r_{bmo}} \leq \sup_v \sum_{j\leq v} \|R_{vj} R_{vj}^*\| \leq C_1 2^{\theta+\alpha-n/4} = C_1,$$

$$\|\sum d_k^r(x)\|^2_{L^c_{bmo}} \leq \sup_v \|\sum_{m,n\leq v\leq k} e_{m,n} \otimes R_{k,m}^* R_{k,n}\|$$

$$= \sup_v \|\sum_{m,n\leq v} e_{m,n} \otimes (\sum_{k\geq v} R_{km}^* R_{kn})\|$$

$$= \sup_{v} \| \sum_{n\le v\le k} e_{kn} \otimes R_{kn} \|^2 \le C_2$$

Similarily, we get the $\max\{\| \sum d_k^c(x)\|^2_{L^c_{bmo}}, \| \sum d_k^c(x)\|^2_{L^r_{bmo}}\} \le \max\{C_1, C_2\}$. We finish the proof of the scalar case. The matrix-valued version follows from $M_d(bmo(\mathcal{N})) = bmo(M_d \otimes \mathcal{N})$, where now M_d is incorporated in the constant functions. Replacing by a matrix of elements in $B(\ell^2)$, the same argument applies and gives the cb-version. $\square$

4.1.2 L^2 estimates

With the concrete resolvent formula of these weighted trucated operators $\Phi^{\text{up}}_{\theta,\alpha}, \Phi^{\text{lower}}_{\theta,\alpha}$, we prove the *completely boundness* of these maps from noncommutative $L_2(\mathcal{M})$ to $L_2(\mathcal{M})$ spaces by functional calucalus. Then by interpolation theory between bmo estimates and L_2 estimates, we show the cb-norm of the map $\Phi^{\text{up}}_{\theta,\alpha}$ from noncommutative $L_2(\mathcal{M})$ to $L_q(B(\ell_2)\otimes\mathcal{M})$.

Lemma 4.4. *Let $\{T_t\}$ be a semigroup of normal selfadjoint contractions such that (R_n^2) holds. then for any $x \in L_2(\mathcal{N})$, the following holds:*

(i) $\|\Phi^{up}_{\theta,\alpha}(x)\|_2 \le_{cb} \|A^{-\alpha-\theta}x\|_2$,

(ii) $\|\Phi^{lower}_{\theta,\alpha}(x)\|_2 \le_{cb} \|A^{-\alpha+1-\theta}x\|_2$.

Proof. (i) We decompose $\Phi^{\text{up}}_{\theta,\alpha}(x)$ into two parts

$$\Phi^{\text{up}}_{\theta,\alpha}(x) = \sum_{k\le j} e_{kj} \otimes 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} \int 1_{[2^k,\infty)} 1_{[2^j,\infty)} T_t x t^{\alpha-1} dt$$
$$+ \sum_{k> j} e_{kj} \otimes 2^{\frac{k\theta}{2}} 2^{\frac{j\theta}{2}} \int 1_{[2^k,\infty)} 1_{[2^j,\infty)} T_t x t^{\alpha-1} dt$$

By symmetry, it suffcies to estimate the part

$$\|\Phi^{\text{up}}_{\theta,\alpha}(x)\|_2^2 \le 2 \sum_{k\le j} 2^{k\theta} 2^{j\theta} \| \int_{2^j}^{\infty} T_t x t^{\alpha-1} \|_2^2$$

For the right side of the above inquality, we have

$$\sum_{k\le j} 2^{k\theta} 2^{j\theta} \| \int_{2^j}^{\infty} T_t x t^{\alpha-1} \|_2^2 \le \langle x, \int_{-\infty}^{\infty} \Big(\sum_j 2^{2j\theta} \int_{2^j}^{\infty} \int_{2^j}^{\infty} e^{-\lambda t} e^{-\lambda s} t^{\alpha-1} s^{\alpha-1} dt ds \Big) d\mu_x(\lambda) \rangle$$

Now it remains to estimate the integral part in the above inequality via some calculus

$$\begin{aligned}
&\sum_j 2^{2j\theta} \int_{2^j}^{\infty} \int_{2^j}^{\infty} e^{-\lambda t} e^{-\lambda s} t^{\alpha-1} s^{\alpha-1} dt ds \\
&= \iint \sum_j \min(t,s)^{2\theta} 1_{[j \le \min(\log t, \log s)]} e^{-\lambda t} e^{-\lambda s} t^{\alpha-1} s^{\alpha-1} dt ds \\
&\le \int_0^{\infty} e^{-\lambda s} s^{\alpha-1} s^{2\theta} \Big(\int_s^{\infty} t^{\alpha-1} e^{-\lambda t} dt \Big) ds
\end{aligned}$$

We recall that for $\int_s^{\infty} t^{\alpha-1} e^{-\lambda t} dt \le \frac{1}{\lambda} s^{\alpha-1} e^{-\lambda s}$ for all $\lambda \neq 0$. Hence we have

$$\begin{aligned}
&\sum_j 2^{2j\theta} \int_{2^j}^{\infty} \int_{2^j}^{\infty} e^{-\lambda t} e^{-\lambda s} t^{\alpha-1} s^{\alpha-1} dt ds \\
&\le \int_0^{\infty} e^{-\lambda s} s^{\alpha-1} s^{2\theta} \Big(\frac{1}{\lambda} s^{\alpha-1} e^{-\lambda s} \Big) ds \le \frac{2}{\lambda} \int_0^{\infty} e^{-2\lambda s} s^{2\alpha-2+2\theta} ds \\
&= \frac{2^{2-2\alpha-2\theta}}{\lambda^{2\alpha+2\theta}} \int_0^{\infty} e^{-u} u^{2\alpha-2+2\theta} du \le C \lambda^{-2\alpha-2\theta}
\end{aligned}$$

Therefore we have

$$\|\Phi^{\mathrm{up}}_{\theta,\alpha}(x)\|_2^2 = \langle x, (\Phi^{\mathrm{up}}_{\theta,\alpha})^* \Phi^{\mathrm{up}}_{\theta,\alpha} x \rangle \le C \langle x, \int \lambda^{-2\alpha-2\theta} d\mu_x(\lambda) \rangle / \tag{4.1.1}$$

(ii) Similarily, we start the proof for $\Phi^{\mathrm{lower}}_{\theta,\alpha}(x)$ as follows:

$$\|\Phi^{\mathrm{lower}}_{\theta,\alpha}(x)\|_2^2 \le 2 \sum_{k \le j} 2^{-k(1-\theta)} 2^{-j(1-\theta)} \| \int_0^{2^k} T_t x t^{\alpha-1} \|_2^2$$

Then we use agin the spectral measure and get the analogous inquality of (4.1.1):

$$\|\Phi^{\mathrm{lower}}_{\theta,\alpha}(x)\|_2^2 = \langle x, (\Phi^{\mathrm{lower}}_{\theta,\alpha})^* \Phi^{\mathrm{lower}}_{\theta,\alpha} x \rangle \le C_1 \langle x, \int \lambda^{-2\alpha-2\theta+2} d\mu_x(\lambda) \rangle.$$

Theorem 4.5. *Let $\{T_t\}$ be a strongly continuous semigroup with normal selfadjoint completely positive maps such that (R_n^2) holds. Then*

$$\|\Phi^{up}_{\theta_q,\alpha_q} x\|_q \le_{cb} \||A^{-\gamma_q} x\|_2 \tag{4.1.2}$$

here $\alpha_q + \theta_q = (1 - \frac{2}{q}) \frac{n}{4} + \gamma_q$.

Proof. In the following section, we will use complex interpolation theory for a function defined as follow

$$F(z) = \sum_{k,j} e_{kj} \otimes 2^{\frac{k}{2}[(1-z)\theta_0 + z\theta_1]} 2^{\frac{j}{2}[(1-z)\theta_0 + z\theta_1]} \int 1_{[2^\sigma,\infty)} 1_{[2^\sigma,\infty)} T_t(A^{-\gamma_1 z}) t^{(1-z)\alpha_0 + z\alpha_1 - 1} dt$$

We have the boundary conditions

$$\|F(it) : \ell_2 \longrightarrow bmo\|_{\mathrm{cb}} < \infty \text{ and } \|F(1+it) : \ell_2 \longrightarrow \ell_2\|_{\mathrm{cb}} < \infty$$

By the desired Stein interpolation theorem in [Ste56], we obtain

$$\|F(2/p) : \ell_2 \longrightarrow \ell_q\|_{\mathrm{cb}} < \infty$$

Then the parameters have identities:

$$\theta_0 + \alpha_0 - \frac{n}{4} = 0 \text{ and } \gamma_1 = \alpha_1 + \theta_1$$

Take $z_q = \frac{2}{q}$, we have

$$F(z_q) = \sum e_{kj} \otimes 2^{k\theta_q/2} 2^{j\theta_q/2} \int 1_{[2^{\sigma_q},\infty)} 1_{[2^{\sigma_q},\infty)} T_t(A^{-\gamma_q}) t^{\alpha_q - 1} dt$$

Then $\gamma_q = \frac{2}{q}\gamma_1, \alpha_q = (1-\frac{2}{q})\alpha_0 + \frac{2}{q}\alpha_1, \theta_q = (1-\frac{2}{q})\theta_0 + \frac{2}{q}\theta_1$ And $\alpha_q = \frac{n}{2}(\frac{1}{2} - \frac{1}{q})$. Similarly, we have the following equations

$$\begin{cases} \theta_0 + \alpha_0 - \frac{n}{4} = 0, \\ \gamma_1 = \alpha_1 + \theta_1, \\ \gamma_q = \frac{2}{q}\gamma_1, \\ \alpha_q = (1-\frac{2}{q})\alpha_0 + \frac{2}{q}\alpha_1, \\ \theta_q = (1-\frac{2}{q})\theta_0 + \frac{2}{q}\theta_1, \end{cases}$$

After simplifying the above equations, we have

$$\alpha_q + \theta_q = (1 - \frac{2}{q}) \cdot \frac{n}{4} + \gamma_q$$

4.2 Embedding for discrete noncommutative spaces

In this chapter, we investigate the cb-version of the HLS inequality happens to $S_p(\mathcal{H})$. As usual, we shall write $R_n = B(l_n^2, \mathbb{C})$ and $C_n = B(\mathbb{C}, l_n^2)$ to denote the row and column Hilbert spaces over ℓ_n^2. Both spaces embed isometrically in $B(\ell_n^2)$. Hence, they admit a natural o.s.s. As usual, we shall write $L_2^r(\mathcal{M}) = B(L_2(\mathcal{M}), \mathbb{C})$ and $L_2^c(\mathcal{M}) = B(\mathbb{C}, L_2(\mathcal{M}))$ to denote the row and column Hilbert spaces over $L_2(M)$. According to the interpolation in [JP10], we obtain $L_p^r(\mathcal{M})$ and $L_p^c(\mathcal{M})$. Let $a_1, a_2, \ldots, a_n \in L_p(\mathcal{M})$ and $b_1, b_2, \ldots, b_n \in L_p(\mathcal{M})$. Using that the Haagerup tensor product commutes with complex interpolation, it is not difficult to check that the following identities hold

$$\|\sum_{k=1}^n a_k \otimes e_{1k}\|_{L_p^r(\mathcal{M})\otimes_h R_n} = \|\Big(\sum_{k=1}^n a_k a_k^*\Big)^{1/2}\|_{L_p(\mathcal{M})},$$
$$\|\sum_{k=1}^n e_{k1} \otimes b_k\|_{C_n\otimes_h L_p^c(\mathcal{M})} = \|\Big(\sum_{k=1}^n b_k^* b_k\Big)^{1/2}\|_{L_p(\mathcal{M})},$$

See [JP10] for more details.

4.2.1 Discrete case

Given a Hilbert space $\mathcal{H}$ and $1 \le p \le \infty$, we denote by $\mathcal{H}^{C_p}$(resp. $\mathcal{H}^{C_p}$) the Schatten p-class $S_p(\mathbb{C}, \mathcal{H})$ (resp, $S_p(\bar{\mathcal{H}}, \mathbb{C})$) equipped with its natural operator space structure. $\mathcal{H}^{C_p}$(resp. $\mathcal{H}^{C_p}$) can be naturally viewed as the column (resp. row) subspace of the Schatten class $S_p(\mathcal{H})$ (resp. $S_p(\bar{H})$). When $\mathcal{H}$ is a separable and infinite dimensional, $\mathcal{H}^{C_p}$ and $\mathcal{H}^{R_p}$ are respectively C_p and R_p from above. $\mathcal{H}^{C_p}$ and $\mathcal{H}^{R_p}$ are respectively the p-column and p-row spaces associated with $\mathcal{H}$. In the following, we consider the case when $\mathcal{H} = \ell_n^2 \otimes \ell^2(w))$. For $1 \le p \le \infty$, we define the weighted $L_p(w)$ as follows:

$$L_p(w) = \{x \in L_0(\mathcal{M}) : wx + xw \in L_p(\mathcal{M})\} \text{ and } \|x\|_p = \|wx + xw\|_p.$$

Lemma 4.6. *The natural embedding*

$$i_\alpha : C_q^n \hookrightarrow (\ell_n^2 \otimes \ell^2(w_\theta))^{C_{q_1}} + (\ell_n^2 \otimes \ell^2(w_\sigma))^{C_2} \text{ as } i_\alpha(x) = x \otimes 1$$

is a completely bounded isomorphism with weights defined as

$$w_\theta = 2^\theta, w_\sigma = 2^\sigma, \text{ and } \frac{1}{p} = \frac{\theta}{\theta+\sigma}\cdot\frac{1}{q_1} + \frac{\sigma}{\theta+\sigma}\cdot\frac{1}{2}$$

Proof. According to [Pis93], we know

$$\left\|\sum_{j=1}^{n} e_{j1}\otimes e_j\right\|_{C_n(H)} = \inf_{k_0} \|\sum_{k\le k_0}\sum_{j=1}^{n} e_{j1}\otimes (e_{1j}\otimes e_k)\|$$

For $x=\sum_{k=1}^{n} e_{k1}\otimes e_k = a_k+b_k$ where

$$a_k=\begin{cases}\sum_{j=1}^{n} e_{j1}\otimes e_j & k\ge k_0\\ 0 & k<k_0\end{cases},\quad b_k=\begin{cases}0 & k\ge k_0\\ \sum_{j=1}^{n} e_{j1}\otimes e_j & k<k_0\end{cases}$$

Denote $a=\sum a_k, b=\sum b_k$, for $C[C_{q_1}]=[CC,CR]_{\frac{1}{q_1}}=[S_2,S_\infty]_{\frac{1}{q_1}}=S_p$, here $\frac{1}{p}=(1-\frac{1}{q_1})\frac{1}{2}$.

$$\begin{aligned}
\|a\|_{C_n(C_{q_1})} &= \|1_n\|_{S_p^n}\Big(\sum_{k\le k_0} 2^{k\theta}\Big)^{1/2} = n^{\frac{1}{p}}2^{\frac{k_0\theta}{2}}\\
\|b\|_{C_2(l_2\otimes l_2;2^{-k\sigma})} &= \|1_n\|_{S_2^n}\Big(\sum_{k\ge k_0} 2^{-k\sigma}\Big)^{1/2} = \sqrt{n}2^{-k_0\sigma}\\
\|x\| &= \inf_{x=a+b}(\|a\|_{C_n(C_{q_1})}+\|b\|_{C_2(l_2\otimes l_2;2^{-k\sigma})}) = \inf_{k_0}(n^{\frac{1}{p}}2^{\frac{k_0\theta}{2}}+\sqrt{n}2^{-k_0\sigma})
\end{aligned}$$

Consider the function $f(k_0)=n^{\frac{1}{p}}2^{\frac{k_0\theta}{2}}+\sqrt{n}2^{-k_0\sigma}$ with its derivative function $f'(k_0)=\frac{\theta}{2}n^{\frac{1}{p}}2^{\frac{k_0\theta}{2}}-\frac{\sigma}{2}\sqrt{n}2^{-k_0\sigma}$. The critical point appears whenever the equation $f'(k_0)=0$ holds, then we will have the following equation

$$\frac{\theta}{\sigma}=n^{\frac{1}{2}-\frac{1}{p}}2^{-\frac{k_0}{2}(\sigma+\theta)}$$

Then we know

$$\min f(k_0)=(1+\frac{\theta}{\sigma})n^{\frac{1}{p}}2^{\frac{k_0\theta}{2}}=(1+\frac{\theta}{\sigma})n^{\frac{1}{p}}\Big(\frac{\sigma}{\theta}\Big)^{\frac{\theta}{\theta+\sigma}}n^{\frac{1}{2q_1}(\frac{\theta}{\theta+\sigma})}=c(\theta,\sigma)n^{\frac{1}{p}+\frac{1}{2q_1}(\frac{\theta}{\theta+\sigma})}$$

Since we know $\|\sum e_{j1}\otimes e_j\|_{C_n(C_q)}=n^{\frac{1}{2}(1-\frac{1}{q})}$, then we have the following identities

$$\frac{1}{2}\frac{\theta}{\theta+\sigma}+\frac{1}{q_1}\frac{\sigma}{\theta+\sigma}=\frac{1}{q}.$$

Remark 4.7. In general, we assume that $\theta+\sigma=1$. Then we find the standard interpolation pair, i.e. $\frac{1}{p}=\frac{1-\theta}{2}+\frac{\theta}{q_1}$.

Theorem 4.8. *Let $\{T_t\}$ be a strongly continuous semigroup of normal selfadjoint completely positive maps* *such that* (R_n^2) *holds for* S_p. *Then*

$$\|A^{-\alpha} : \ell_2 \longrightarrow \ell_q\|_{cb} \le C,$$

here $\alpha = \frac{n}{2}(\frac{1}{2} - \frac{1}{q})$.

Proof. For a finite matrix element x with entries in dom$(A^{-\gamma})$, recall (4.1.2) and Lemma 4.4

$$\|\Phi^{\text{up}}_{\theta,\alpha_q} x\|_{S_2^m(S_{q_1})} \le \|A^{-\gamma} x\|_{S_2^m(S_2)}$$
$$\|\Phi^{\text{lower}}_{\theta,\alpha_q}(x)\|_{S_2^m(S_2)} \le_{cb} \|A^{-\alpha_q+1-\theta} x\|_{S_2^m(S_2)}$$

with $\alpha_q + \theta = (1 - \frac{2}{q_1}) \cdot \frac{n}{4} + \gamma$.

From Lemma 4.6 [1] , we have

$$\begin{aligned}\|A^{-\alpha_q} x\|_{S_2^m(S_q)} &\le 2(\|\Phi^{\text{up}}_{\theta,\alpha_q} x\|_{S_2^m(S_{q_1})} + \|\Phi^{\text{lower}}_{\sigma,\alpha_q}(x)\|_{S_2^m(S_2)}) \\ &\le 2(\|A^{-\gamma} x\|_{S_2^m(S_2)} + \|A^{-\alpha_q+1-\theta} x\|_{S_2^m(S_2)}).\end{aligned}$$

Then let $\gamma = \alpha_q - (1-\theta)$, we have

$$\|A^{-\alpha_q} x\|_{S_2^m(S_q)} \le 4\|A^{-\gamma} x\|_{S_2^m(S_2)}$$

with the identities

$$\begin{cases} \gamma = \alpha_q - (1-\theta) \\ \alpha_q + \theta = (1 - \frac{2}{q_1}) \cdot \frac{n}{4} + \gamma \\ \frac{1-\theta}{2} + \frac{\theta}{q_1} = \frac{1}{q} \end{cases}$$

Then we will have

$$\alpha_q - \gamma = \frac{n}{2}(\frac{1}{2} - \frac{1}{q})$$

Replacing $A^{\gamma} x$ with x,

$$\|A^{-(\alpha_q - \gamma)} x\|_q \le_{cb} 2\|x\|_2,$$

Since $\alpha = \alpha_q - \gamma$, we deduce the assertion for $0 < \alpha < 1$. For the case $\alpha \ge 1$, we interpolate between

$$\|A^{-1} : L_2 \to L_q\| \le c_1$$

[1] Lemma 4.6 is not enough to induce (4.2.1). More details can be found in Theorem 4.26.

$$\|A^{-i\theta} : L_{p_1} \to L_{p_1}\| \le c_2.$$

We obtain

$$\|A^{-2/q} : L_q \to L_p\| \le C(q).$$

We choose the parameter with the condition $\frac{2}{q} = \frac{n}{2}(\frac{1}{q} - \frac{1}{p})$, i.e. $\frac{1}{p} = \frac{1}{q}(1 - \frac{4}{n})$. Then we choose a geometric sequence (p_k) with $\frac{1}{p_{k+1}} = \frac{1}{p_k}(1 - \frac{4}{n})(n > 4), p_1 = p$ and

$$\|A^{-2/p_k} : L_{p_k} \to L_{p_{k+1}}\| \le C_{p_k}.$$

Therefore we show that for any $\alpha = 2/q + \sum_k^m 2/p_k = n/2(1/q - 1/p_{m+1})$, we have

$$\|A^{-\alpha} : L_q \to L_{p_{m+1}}\| \le C(q, p_{m+1})$$

When the sequence approaches to infinity with $n > 4$, p_{m+1} approaches to infinity as well and α can be a number greater than 1. Therefore, we finish the assertion when $\alpha > 1$ and $n > 4$. $\square$

4.2.2 General case

In this section, we explore the L_p embedding theory for general von Neumann algebras. We introdcuce the notion of homogeneous space and give some certain regularity condition on the pair of weights. We show that, the fundamental sequences with a mild regularity assumption, completely determine the operator space structure of a homogeneous space. We find a canonical representation of the homogeneous space in terms of weighted row and column spaces. Then we prove the cb-Sobolev inequalities for semifinte von Neumann algebra and the cb-version of Varopolouss theorem and provide some examples. Let us recall the following space mentioned in [JX10]:

Definition 4.9. Let (Ω, m) be a measure space and $2 \le p < q \le \infty, (\mu, \nu)$ is a pair of weights on Ω(i.e., a nonnegative measurable functions)

$$\int min(\mu, \nu) dm < \infty. \tag{4.2.1}$$

Given a semifinite von Neumann algebra $\mathcal{M}$, define a norm as

$$\|x\|_K = \inf_{x=a(w)+b(w) \text{ a.s.}} \left(\|a\|_{L_{2q}^{C_{2q}}(\mathcal{M}\otimes B(\mathbb{C}, L_2(\mu,m)))} + \|b\|_{L_{2p}^{C_{2p}}(\mathcal{M}\otimes B(\mathbb{C}, L_2(\nu,m)))} \right)$$

and the space

$$K^{q,p}_{\mathcal{M}}(\mu,\nu,m) = \{x \in L_1(\mathcal{M}) \cap \mathcal{M} | \|x\|_K < \infty\}$$

Proposition 4.10. *Let $\mu, \nu, (\Omega, m), \mathcal{M}, K^{q,p}_{\mathcal{M}}(\mu,\nu)$ defined as above:*

1. *If there exists a map α such that $\|\alpha : L_2(\Omega, m, \mu) \to L_2(\Omega, m_1, \mu)\| \leq \lambda$ and $\|\alpha : L_2(\Omega, m, \nu) \to L_2(\Omega, m_1, \nu_1)\| \leq \lambda$, then $\|id : K^{q,p}_{\mathcal{M}}(\mu,\nu) \to K^{q,p}_{\mathcal{M}}(\mu_1,\nu_1)\|_{cb} \leq \lambda$.*

2. *For two pairs of measures (μ,ν) and (μ_1,ν_1), if they are equivalent in the following sense $\lambda\mu_1 \leq \mu \leq \lambda^{-1}\mu_1$ and $\lambda\nu_1 \leq \nu \leq \lambda^{-1}\nu_1$ then $K^{q,p}_{\mathcal{M}}(\mu,\nu) \approx_{cb} K^{q,p}_{\mathcal{M}}(\mu_1,\nu_1)$.*

Lemma 4.11. *Assume μ, ν are Σ-measurable, $K^{q,p}_{\mathcal{M}}(\mu,\nu,m) = K^{q,p}_{\mathcal{M}}(\mu,\nu,m_\Sigma)$*

Proof. By the definition of the norm, $K^{q,p}_{\mathcal{M}}(\mu,\nu,m) \geq K^{q,p}_{\mathcal{M}}(\mu,\nu,m_\Sigma)$ since there are fewer Σ-measurable functions. On the other hand, there exists a conditional expectation such that $\|E : L_2(\mu, m) \to L_2(\Omega, \Sigma)\| \leq 1$ and $\|\alpha : L_2(\Omega, m) \to L_2(\nu, \Sigma)\| \leq 1$. Then by Proposition 4.10, we know that $\|id : K^{q,p}_{\mathcal{M}}(\mu,\nu) \to K^{q,p}_{\mathcal{M}}(\mu_1,\nu_1)\|_{cb} \leq 1$. □

It is sometimes convenient to work with the discrete analogue of $K^{q,p}_{\mathcal{M}}(\mu,\nu)$, i.e, when (μ,ν) are two weights on $\mathbb{Z}$. Then the two weights μ and ν become two positive sequeseces $(\mu(j))_{j\geq 1}$ and $(\nu(j))_{j\geq 1}$ satisfying the following weight condition

$$\sum_j \min(\mu(j), \nu(j)) < \infty. \tag{4.2.2}$$

By standard arguments it is easy to transfer the continuous case to the discrete one and vice versa. More details can be found in [JX10]. However, we need more monotonicity property of the canonical weights. Therefore, we show the construction as follows:

In the discrete case, we decompose the space $\mathbb{Z}$ into $\mathbb{Z}_+$ and $\mathbb{Z}_-$ as follows

$$\mathbb{Z}_+ := \{k \in \mathbb{Z} : \nu(k) < \mu(k)\} \text{ and } \mathbb{Z}_- := \{k \in \mathbb{Z} : \nu(k) \geq \mu(k)\}$$

And we decompose our spaces as follows:

$$K^{q,p}_{\mathcal{M},+}(\mu,\nu) = \{x \in L_1(\mathcal{M}) \cap \mathcal{M} | x = a(k) + b(k), k \in \mathbb{Z}_+ \\ \|a\|_{L^{C_{2q}}_{2q}(\mathcal{M}\otimes B(\mathbb{C}, l_2(\mathbb{Z}_+,\mu)))} + \|b\|_{L^{C_{2p}}_{2p}(\mathcal{M}\otimes B(\mathbb{C}, l_2(\mathbb{Z}_+,\mu)))} < \infty\}$$

and

$$K^{q,p}_{\mathcal{M},-}(\mu,\nu) = \{x \in L_1(\mathcal{M}) \cap \mathcal{M} | x = a(k) + b(k)\ k \in \mathbb{Z}_-,$$

$$\|a\|_{L^{C_{2q}}_{2q}(\mathcal{M}\otimes B(\mathbb{C},l_2(\mathbb{Z}_-,\mu)))} + \|b\|_{L^{C_{2p}}_{2p}(\mathcal{M}\otimes B(\mathbb{C},l_2(\mathbb{Z}_-,\nu)))} < \infty\}$$

Proposition 4.12. *For a pair of weights on $\Omega = [0,T]$ satisfying the condition (4.2.2), then there exists two functions $f^+ \geq 1, f^- \geq 1$ on Ω, such that*

$$K^{q,p}_{\mathcal{M}}(\mu,\nu) = K^{q,p}_{\mathcal{M},+}(f^+\lambda,\lambda) \bigcap K^{q,p}_{\mathcal{M},-}(\lambda, f^-\lambda).$$

Here λ means Lebesgue measure.

Proof. Since $\mathbb{Z} = \mathbb{Z}_+ \bigcup \mathbb{Z}_-, \emptyset = \mathbb{Z}_+ \bigcap \mathbb{Z}_-$, then

$$\max\{\|x\|_{K^+_{\mathcal{M}}}, \|x\|_{K^-_{\mathcal{M}}}\} \leq \|x\|_{K_{\mathcal{M}}} \leq \|x\|_{K^+_{\mathcal{M}}} + \|x\|_{K^-_{\mathcal{M}}} \leq 2\max\{\|x\|_{K^+_{\mathcal{M}}}, \|x\|_{K^-_{\mathcal{M}}}\}.$$

Since the norms are equivalent, then we get

$$K^{q,p}_{\mathcal{M}}(\mu,\nu) = K^{q,p}_{\mathcal{M},+}(\mu,\nu) \bigcap K^{q,p}_{\mathcal{M},-}(\mu,\nu).$$

For the domain $\Omega = [0,T]$, we divide it into disjoint intervals as $I_1, I_2, \cdots$, with $|I_k| = \nu(k), T = \sum_{k\geq 0} \nu(k)$. And we define the Randon Nikodym derivative $F := \sum_k \frac{\mu(k)}{\nu(k)} 1_{I_k}$ on $[0,T]$. Therefore given the measure space $(\mathbb{Z}_+, \mu, \nu)$, there exists a pair of weights $(F\nu, \nu)$ on $[0,T]$ satisfying (4.2.1). Then we can find a rearrangement decreasing function $f^+ := F^*$ with respect to the Lebesgue measure λ. Therefore according to Lemma 4.11,

$$K^{q,p}_{\mathcal{M},+}(\mu,\nu) = K^{q,p}_{\mathcal{M},+}(f^+\lambda,\lambda).$$

Similarly, on the domain $\Omega = [0,\tilde{T}]$ where we decompose into disjoint intervals as $\tilde{I}_1, \tilde{I}_2, \cdots, |\tilde{I}_k| = \mu(k), \tilde{T} = \sum_{k\leq 0} \mu(k)$, we can find the rearrangement decreasing function f^- on $\mathbb{Z}_-$, and we get

$$K^{q,p}_{\mathcal{M},-}(\mu,\nu) = K^{q,p}_{\mathcal{M},-}(\lambda, f^-\lambda) \text{ with completely equivalent norms.}$$

Therefore we get the result. $\square$

Remark 4.13. We call two functions $f_1 \sim f_2$ is equivalent in the following sense

$$\frac{1}{c} f_1(\lambda s) \leq f_2(s) \leq c f_1(\frac{s}{\lambda}),$$

then

$$K_M^+(f_1\lambda,\lambda) = K_M^+(f_2\lambda,\lambda) \text{ with completely equivalent norms.}$$

We should call the reader's attention to the fact that the complex interpoolation space is a subspace of the direct sum of two Hilbert spaces. Therefore we need recall the following decomposition for Hilbertian operator spaces in [Xu06].

Theorem 4.14. *Let X_0, X_1 be 1-homogeneous 1-Hilbertian operator spaces. Let $X = X_0 \otimes_2 X_1$ and $S \subset X$ be a closed subspace. Then there are closed subspaces $Y_0, Z_0 \subset X_0, Y_1, Z_1 \subset X_1$ and an injective closed densely defined operator T from Z_0 to Z_1 with dense range such that*

$$S = Y_0 \oplus_2 Y_1 \oplus_2 G(T) \textit{ with completely equivalent norms.}$$

Here $G(T) = \{(x, Tx) : x \in \mathrm{Dom}(T)\}$, where Dom(T) stands for the domain of T. Let $H_0 = L_2(\Omega, u_0), H_1 = L_2(\Omega, u_0), \xi = (g, h) \in H_0 \oplus H_1$ and $H = \{(x, fx)|x \in \mathrm{dom}(T)\}, f \in L_0(\Omega, u_0)$. Recall the norm defined in $H_0 \oplus H_1/H$

$$\|(g,h)\|^2_{H_0\oplus H_1/H} = \inf_{v\in L_2} \|(g-v)\|_2^2 + \|h - fv\|_2^2.$$

As a technical tool, we need to define the interpolation space as follows:

Definition 4.15. Let $H \subset H_0 \oplus H_1$ and $\xi \in H_0 \oplus H_1/G(T)$, denote the space

$$\chi_{\xi,H}(\mathcal{M}) = \{\xi \otimes x | x \in \mathcal{M} \cap L_1(\mathcal{M})\} \subset \mathcal{M} \otimes \Big(H_0 \oplus H_1/H\Big)$$

and

$$\chi_\xi^{q,p}(\mathcal{M}) = \{\xi \otimes x | x \in \mathcal{M} \cap L_1(\mathcal{M})\} \subset L_{2q}^{C_{2q}}(\mathcal{M} \otimes H_0^r) \oplus L_{2p}^{C_{2p}}(\mathcal{M} \otimes H_1^r)/L_1(\mathcal{M}) \cap \mathcal{M} \otimes H$$

with the norm defined as

$$\|x \otimes \xi\| = \inf_{\xi\otimes x = a+b} \|a\|_{L_{2q}^{C_{2q}}(\mathcal{M}\otimes H_0^r))} + \|b\|_{L_{2p}^{C_{2p}}(\mathcal{M}\otimes H_1^r)}$$

Lemma 4.16. *There exists a pair of weights (μ, ν) satisfying the condition* (4.2.1) *such that*

$$\chi^{q,p}_{\xi,G(T)}(\mathcal{M}) = K^{q,p}_{\mathcal{M}}(\mu, \nu, m)$$

Proof. By the polar decomposition $T = u|T|$ and using homogeneity, we can assume $H_0 = H_1 = L_2(m)$ and T is a positive operator. Therefore we can assume there exists a function $f \in L_2(m)$, such that T is the multiplication operator $M_f : L_2(m) \to L_2(m)$. Let $\xi \in L_2(m) \oplus L_2(m)/G(T)$

$$\begin{aligned}\|x \otimes \xi\| &= \inf_{(y,fy)\in\mathcal{M}\otimes G(M_f)} \|x \otimes (\xi_1, \xi_2) + (y, fy)\| \\ &= \inf_{y\in\mathcal{M}\otimes L_2(m)} \|x \otimes \xi_1 + y\|_{L_{2q}^{C_{2q}}(\mathcal{M},B(\mathbb{C},L_2(m)))} + \|x \otimes \xi_2 + fy\|_{L_{2p}^{C_{2p}}(\mathcal{M},B(\mathbb{C},L_2(m)))}\end{aligned}$$

Denote $a := x \otimes \xi_1 + y$ and $b := -\xi_2 + fy$. Then we have the following observation

$$af + b = x \otimes (f\xi_1 - \xi_2) := x \otimes h$$

Since $\xi \in H_0 \otimes H_1/G(T)$, we can assume ξ is not equivalent to 0. Therefore $h = f\xi_1 - \xi_2$ is not 0. Replacing h with $h \cdot 1_{h>0}$, we can assume h is a positive function and strictly greater than 0 (Otherwise we multiply by $-1_{h<0}$). Thus we get

$$a' + b' = \frac{af}{h} + \frac{b}{h} = x \otimes 1.$$

This implies

$$\begin{aligned}\|x \otimes \xi\| &= \inf_{x\otimes 1=a'+b'} \|a'\|_{L_{2q}^{C_{2q}}(\mathcal{M},B(\mathbb{C},L_2(\sqrt{\frac{h}{f}}m)))} + \|b'\|_{L_{2p}^{C_{2p}}(\mathcal{M},B(\mathbb{C},L_2(\sqrt{h}m)))} \\ &= \|x\|_{K_{\mathcal{M}}^{q,p}(\sqrt{\frac{h}{f}},\sqrt{h},m)}\end{aligned}$$

□

Definition 4.17. A function φ satisfies the Δ_2 condition, if there exist positive constants c, d and α with $\alpha > 1$ such that

$$c(\frac{t}{s})^\alpha \le \frac{\varphi(s)}{\varphi(t)} \text{ and } \frac{\varphi(s)}{\varphi(2s)} \le d, \forall t \ge s \ge 1.$$

By normalization of the domain, we can assume the measure space as $\Omega = [0, 1]$. Given a continuous decreasing positive function f defined on $[0, 1]$, we define

$$\varphi_f(t) := \int_0^1 \min(t, f(s))ds$$

Lemma 4.18. *If the function $\varphi_f(t)$ satifies the Δ_2 condition, then*

$$f^{-1}(t) \sim \frac{\varphi_f(t)}{t}.$$

Proof. Decomposing the interval into two parts yields

$$\varphi_f(t) = tf^{-1}(t) + \int_{f^{-1}(t)}^{1} f(s)ds \geq tf^{-1}(t).$$

Therefore we get that $\frac{\varphi_f(t)}{t} \geq f^{-1}(t)$. Moreover, if we differentiate the above decomposition form, we obtain $\varphi_f(t)' = f^{-1}(t)$. Now we assume $\delta^\alpha \leq \frac{1}{2}$

$$\begin{aligned}\varphi_f(t) = \int_0^t \varphi_f(t)'ds &= \int_0^{\delta t} \varphi_f(t)'ds + \int_{\delta t}^{t} \varphi_f(t)'ds \\ &\leq \varphi_f(\delta t) + \varphi_f(\delta t)'(1-\delta)t \leq \frac{1}{2}\varphi_f(t) + \varphi_f(\delta t)'t.\end{aligned}$$

Therefore we get

$$\frac{\varphi_f(t)}{t} \leq 2\varphi_f(\delta t)' \leq 2f^{-1}(\delta t).$$

Now let us consider a special case $\mathcal{M} = B(l_2)$. We will often identify the canonical bases (e_{k1}) of C and (e_{1k}) of R with (e_k) of l_2. And we have

$$\|\sum_{k=1}^{n} e_k \otimes e_k\|_{C_n \otimes R_s^n} = n^{\frac{1}{2s}} \text{ and } \|\sum_{k=1}^{n} e_k \otimes e_k\|_{R_s^n \otimes R} = n^{\frac{1}{2s'}} \tag{4.2.3}$$

Lemma 4.19. *The fundament sequences have the following equivalence*

$$\begin{aligned}\varphi_c(n) &:= \Big\| \sum_1^n e_{k1} \otimes e_{1k} \Big\|^2_{K^{q,p}_{B(l_2)}} \sim n^{1/q}\varphi_{f^+}(n^{1/p-1/q}), \\ \varphi_r(n) &:= \Big\| \sum_1^n e_k \otimes e_{1k} \Big\|^2_{K^{q,p}_{B(l_2)} \otimes R_n} \sim n^{1-1/p}\varphi_{f^-}(n^{1/p-1/q}).\end{aligned}$$

Proof. Denote $x = \sum_k e_{k1} \otimes e_k$, we get

$$\begin{aligned}\|x\|_{C_n \otimes K_{B(l_2)}} &= \inf_{1=a(k)+b(k)} \|\sum_k a(k) \otimes e_{1k}\sqrt{\mu(k)}\|_{C_n \otimes C_q^n} + \|\sum_k b(k) \otimes e_{1k}\sqrt{\nu(k)}\|_{C_n \otimes C_p^n} \\ &= \inf_A \big(\sum_{k \in A} \mu(k)\big)^{1/2} \|1\|_{C_n \otimes C_q^n} + \big(\sum_{k \in A^c} \nu(k)\big)^{1/2} \|1\|_{C_n \otimes C_p^n}\end{aligned}$$

By (4.2.3), we get

$$\|x\| \sim \inf_{A \subset \Omega} \big(\mu(A)n^{1/q} + \nu(A^c)n^{1/p}\big)^{1/2} = n^{1/2q} \inf_{A \subset \Omega} \big(\mu(A) + \nu(A^c)n^{1/p-1/q}\big)^{1/2}$$

Denote

$$\hat{\varphi}_c(t) := \inf_{A\subset\Omega} \mu(A) + t\nu(A^c) \text{ and } \hat{\varphi}_r(t) := \inf_{A\subset\Omega} t\mu(A) + \nu(A^c). \tag{4.2.4}$$

Then we obtain

$$\Big\| \sum_1^n e_{k1} \otimes e_k \Big\|^2_{K^{q,r}_{B(l_2)}(\mu,\nu)} = n^{1/q} \hat{\varphi}_c(n^{1/p-1/q}).$$

By the decomposition

$$\begin{aligned}\hat{\varphi}_c(t) &= \inf_{|A|=t} \mu(A) + t\nu(A^c)\\ &= \inf_{|A|=t} \mu(A\cap\mathbb{Z}_+) + \mu(A\cap\mathbb{Z}_-) + t\nu(A^c\cap\mathbb{Z}_+) + t\nu(A^c\cap\mathbb{Z}_-))\end{aligned}$$

Since $t \geq 1$ and $\nu \geq \mu$ on $\mathbb{Z}_-$, $\mu(A\cap\mathbb{Z}_-) + t\nu(A^c\cap\mathbb{Z}_-) \geq \mu(A\cap\mathbb{Z}_-) + \mu(A^c\cap\mathbb{Z}_-) = \mu(\mathbb{Z}_-)$. Let f^+ be the Random-Nikodym derivative in Proposition 4.12 and denote $A_t = \{s | f^+(s) \leq t\}$, thanks to $\mu \geq \nu$ on $\mathbb{Z}_+$

$$\mu(A\cap\mathbb{Z}_+) + t\nu(A^c\cap\mathbb{Z}_+) \geq \int_0^1 \min(t, f^+(s))ds = \varphi_{f^+}(t)$$

Therefore, we get

$$\varphi_{f^+}(t) \leq \hat{\varphi}_c(t) \leq \varphi_{f^+}(t) + \mu(\mathbb{Z}_-).$$

i.e.

$$\hat{\varphi}_c(t) \sim \varphi_{f^+}(t)$$

Analogously we denote $y = \sum_k e_k \otimes e_{1k}$, then we have

$$\|y\|_{K_{B(l_2)}\otimes_h R_n} = \inf_A n^{1/2q'} \mu(A)^{1/2} + n^{1/2p'} \nu(A^c)^{1/2}.$$

Therefore we get

$$\Big\| \sum_1^n e_{1k} \otimes e_{1k} \Big\|^2_{K^{q,p}_{B(l_2)}(\mu,\nu)\otimes R_n} \sim n^{1-1/p} \varphi_{f^-}(n^{1/p-1/q})$$

Proposition 4.20. *Let $\varphi_c, \tilde{\varphi}_c$ be as in* (4.2.4) *and both satisfy Δ_2 condition, then*

$$\varphi_c \sim \tilde{\varphi}_c \Longleftrightarrow f^+ \sim \tilde{f}^+$$

The same results hold for φ_r and f^-.

Proof. By Lemma 4.18, we know $f(t) \sim \frac{\varphi_f(t)}{t}$. Then by Lemma 3.5 in [JX10], we get the conclusion. ∎

Theorem 4.21. *If $K^{q,p}_{B(l_2)}(\mu,\nu) = K^{q,p}_{B(l_2)}(\tilde{\mu},\tilde{\nu})$ and φ_c, φ_r satisfying Δ_2, then*

$$K^{q,p}_M(\mu,\nu) = K^{q,p}_M(\tilde{\mu},\tilde{\nu}) \text{ (completely equivalent norm).}$$

Proof. According to the definition of φ_c, φ_r,

$$\varphi_c(n) = \Big\|\sum_1^n e_{k1}\otimes e_k\Big\|^2_{C_n K^{q,p}_{B(l_2)}(\mu,\nu)} \text{ and } \tilde{\varphi}_c(n) = \Big\|\sum_1^n e_{k1}\otimes e_k\Big\|^2_{C_n K^{q,p}_{B(l_2)}(\tilde{\mu},\tilde{\nu})}$$

Then we obtain $\varphi_c \sim \tilde{\varphi}_c$. Then Proposition 4.20 implies $f^+ \sim \tilde{f}^+$. Analogously we get $f^- \sim \tilde{f}^-$. Then by Proposition 4.12, we finish the proof. □

Theorem 4.22. *For a von Neumann algebra $\mathcal{M}$,*

$$L^{C_{2s}}_{2s}(\mathcal{M}) \approx \chi_{\xi,H}(\mathcal{M}) \text{ (completely equivalent norm)}$$

here $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$.

Proof. Thanks to reiteration theorem for the complex interpolation method, we get

$$L^{C_{2s}}_{2s} = \Big[L^{C_{2q}}_{2q}, L^{C_{2p}}_{2p}\Big]_\theta = \Big\{f(\theta)\Big| f\big|_{\partial_0} \in L_\infty(L^{C_{2q}}_{2q}), f\big|_{\partial_1} \in L_\infty(L^{C_{2p}}_{2p}), \text{ f is analytic}\Big\}.$$

Then by definition of $\chi_{\xi,H}$, we observe that

$$L^{C_{2s}}_{2s}(\mathcal{M}) \subset \chi_{\xi,H}(\mathcal{M}) \text{ with constant 2.}$$

We interpolate between these two following spaces:

$$q = \infty : \mathcal{M}\bar{\otimes}L^\infty(\partial_0) \longrightarrow \mathcal{M}\bar{\otimes}L^r_2(\partial_0,\mu_\theta)$$
$$q = 2 : L^c_2(\mathcal{M})\bar{\otimes}L_\infty \longrightarrow L^c_2(\mathcal{M})\bar{\otimes}L^c_2(\partial_0,\mu_\theta)$$

We obtain

$$f\big|_{\partial_0} \in L^\infty(L^{C_{2q}}_{2q}(\mathcal{M})) \longrightarrow L^{C_{2q}}_{2q}(\mathcal{M}\otimes B(\mathbb{C}, L_2(\partial_0,\mu_\theta)))$$

Similarily we get

$$f\big|_{\partial_1} \in L^\infty(L^{C_{2p}}_{2p}(\mathcal{M})) \longrightarrow L^{C_{2p}}_{2p}(\mathcal{M}\otimes B(\mathbb{C}, L_2(\partial_1,\mu_\theta))).$$

Then it remains to prove the opposite direction inclusion. Recall that

$$\begin{aligned}\|x\|^2_{M_m(L_{2s}^{C_{2s}})} &= \|x^*x\|_{M_m(L_s)} \\ &= \sup_{a\in S_{2s}^m} \|(a\otimes 1)x^*x(a\otimes 1)\|_{L_s(M_m\otimes\mathcal{M})} = \sup_{a\in S_{2s}^m} \|x(a\otimes 1)\|_{L_{2s}(M_m\otimes\mathcal{M})}\end{aligned}$$

By Hölder's inequality with $\frac{1}{2}=\frac{1}{2s}+\frac{1}{v}$, we get

$$\|x\|^2_{M_m(L_{2s}^{C_{2s}})} = \sup_{\|\alpha\|_v\le 1, \|a\|_{2s}\le 1} \|\alpha x a\otimes 1\|_{L_2(M_m\otimes\mathcal{M})}$$

Assume $x\in \chi_{\xi,H}$, this means there exists an analytic function $f:\Omega\to\mathcal{M}\cap L_1(\mathcal{M}), s.t. f(\theta)=x$. Fix $\alpha, a\ge 0, \frac{1}{s}=\frac{1-\theta}{q}+\frac{\theta}{p}, \frac{1}{v}=\frac{1-\theta}{v_0}+\frac{1}{v_1}$, define

$$g(z)=\alpha^{v(\frac{1-z}{v_0}+\frac{z}{v_1})}, h(z)=a^{s(\frac{1-z}{q}+\frac{z}{p})}.$$

Then we define

$$F(z):=\alpha(z)f(z)[a(z)\otimes 1]$$

Thanks to interpolation theory in [BL12], we know

$$\|F(\theta)\|^2_{L_2(M_m\otimes\mathcal{M})}\le \int \|F(it)\|^2_{L_2(M_m\otimes\mathcal{M})}d\mu_\theta(it)+\int \|F(1+it)\|^2_{L_2(M_m\otimes\mathcal{M})}d\mu_\theta(1+it).$$

For $F(z)=\alpha^{z(\frac{1}{v_1}-\frac{1}{v_0})}\alpha^{\frac{v}{v_0}}f(z)a^{\frac{s}{q}}a^{z(\frac{1}{p}-\frac{1}{q})}\otimes 1$

$$\|F(it)\|_2=\|\alpha^{\frac{v}{v_0}}f(it)a^{\frac{s}{q}}\otimes 1\|_2$$

Similarly,

$$\|F(1+it)\|_2=\|\alpha^{\frac{v}{v_1}}f(it)a^{\frac{s}{p}}\otimes 1\|_2$$

The function $\alpha^{\frac{v}{v_0}}f(it)a^{\frac{s}{q}}\in L_2(M_m\otimes\mathcal{M}\otimes B(\mathbb{C},L_2(\mu_\theta|_{\partial_0})))$.

$$\begin{aligned}\|F(it)\|_2^2 &\le tr\big[\alpha^{\frac{2v}{v_0}}\int f(it)a^{\frac{s}{q}}a^{\frac{s}{q}}f(it)^*\big]d\mu(t) \\ &\le \|\alpha\|_{L_{2v}}\cdot\|f(it)a^{\frac{s}{q}}\|_{L_{2q}(\mathcal{M}\otimes B(\mathbb{C},L^2(\partial_0)))}\end{aligned}$$

Take $\tilde{a} = a^{\frac{s}{q}} \in S_{2q}^m$. By assumption, $f(it) \in M_m L_{2q}(\mathcal{M} \otimes L_2^r)$. Then $\int f(it)\tilde{a}\tilde{a}^* f(it)^* d\mu(t) \in L_q(M_m \otimes \mathcal{M})$ Therefore,

$$\|F(i\cdot)\|_2 \leq \|f(i\cdot)\|_{L_{2q}(M_m \otimes \mathcal{M})} \tag{4.2.5}$$

Analogously, we get

$$\|F(1+i\cdot)\|_2 \leq \|f(1+i\cdot)\|_{L_{2p}(M_m \otimes \mathcal{M})} \tag{4.2.6}$$

Plugging (4.2.5) and (4.2.6) into (4.2.2), we have

$$\|x\|^2_{M_m(L_{2p}^{C_{2p}})} \leq \|f(i\cdot)\|^2_{L_{2q}(M_m \otimes \mathcal{M})} + \|f(1+i\cdot)\|^2_{L_{2p}(M_m \otimes \mathcal{M})}$$

Therefore, we obtain the missing inclusion. $\square$

Theorem 4.23. *For a semifinite von Neumann algebra $\mathcal{M}$ and $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$,*

$$L_{2s}^{C_{2s}}(\mathcal{M}) = K_{\mathcal{M},H}^{q,p}(2^\theta, 2^{1-\theta}, \lambda) \text{ (completely equivalent norms)}$$

Proof. Thanks to Theorem 4.22 and Lemma 4.16, for arbitrary von Neumann $\mathcal{M}$, we obtain

$$L_{2s}^{C_{2s}}(\mathcal{M}) = \chi_{\xi,H}(\mathcal{M}) = K_{\mathcal{M},H}^{q,p}(\mu,\nu). \tag{4.2.7}$$

With Theorem 4.21 for $B(l_2)$

$$\chi_{\xi,H}(B(l_2)) = K_{B(l_2)}^{q,p}(2^\theta, 2^{1-\theta}, \lambda)$$

we get

$$K_{\mathcal{M}}^{q,p}(\mu,\nu) = K_{\mathcal{M}}^{q,p}(2^\theta, 2^{1-\theta}, \lambda) \tag{4.2.8}$$

Combining (4.2.7) and (4.2.8), we obtain the result. $\square$

With the help of conditional L_p spaces mentioned in [JP10], Theorem 4.23 implies the following result:

Corollary 4.24. *For arbitrary semifinite von Neumann algebra $\mathcal{M}$ and $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$, ,*

$$L_s(\mathcal{M}) = K_{\mathcal{M}}^{q,p}(2^\theta, 2^{1-\theta}, \lambda)^* \otimes_h K_{\mathcal{M}}^{q,p}(2^\theta, 2^{1-\theta}, \lambda)$$

Proposition 4.25. *The Scatten class S_p has the following embedding*

$$S_p \hookrightarrow_{cb} (l_2^{C_{q_1}}(l_2^n \otimes l_2, w_\theta) + l_2^{C_2}(l_2^n \otimes l_2)w_\sigma) \otimes_h (l_2^{R_{q_1}}(l_2^n \otimes l_2, w_\theta) + l_2^{R_2}(l_2^n \otimes l_2)w_\sigma)$$

Recall the resolvent formula

$$A^{-\alpha}x = \Gamma(\alpha)^{-1}\int_0^\infty T_t t^{\alpha-1}dt \text{ for } \alpha > 0.$$

Given a family of positive semigroup T_t, we want to prove

$$\|A^{-\alpha} : L_2(\mathcal{M}) \longrightarrow L_q(\mathcal{M})\|_{\mathrm{cb}} < \infty$$

holds for any semifinite von Neumann algebra $\mathcal{M}$. Because the generator $A^{-\alpha}$ is completely positive, $A^{-\alpha}(x^*y) = u_\alpha(x)^*u_\alpha(y)$. In our context, we need to use an explicite decomposition of $A^{-\alpha}$.

Theorem 4.26. *Let $\{T_t\}$ be a strongly continuous semigroup of normal selfadjoint subunital completely positive maps such that (R_n^2) holds for $L_p(\mathcal{M})$. Then*

(i) $\|A^{-\alpha_q}x\|_s \leq_{cb} \|\Phi^{up}_{\theta,\alpha_q}x\|_q + \|\Phi^{lower}_{1-\theta,\alpha_q}(x)\|_2$

(ii) $\|A^{-\alpha} : L_2(\mathcal{N}) \longrightarrow L_s(\mathcal{N})\|_{cb} \leq C$

with $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{2}, \alpha = \alpha_q + \gamma = \frac{n}{2}(\frac{1}{2} - \frac{1}{s})$ *and* $\alpha + \theta = (1 - \frac{2}{q})\frac{n}{4} + \gamma$.

Proof. (i) By Proposation 2.35, let $z = y^*y \in M_m(L_2(M)), y \in M_m(L_4(M))$ and $u_t(y_1)^*u_t(y_2) = T_t(y_1^*y_2)$ be the map obtained from the GNS construction of T_t. Then let $u_{\alpha_q}(y)(t) = t^{(\alpha_q-1)/2}u_t(y)$ satisfying

$$u_{\alpha_q}(y)^*u_{\alpha_q}(y) = \int T_t(y^*y)t^{\alpha_q-1}dt = A^{-\alpha_q}(y^*y)$$

Then we may choose the following decomposation

$$u_{\alpha_q}(y) = 1_{[2^k,\infty)}u_{\alpha_q}(y) + 1_{[0,2^k]}u_{\alpha_q}(y) := u^1_{\alpha_q}(k) + u^2_{\alpha_q}(k), \forall k > 0.$$

Thanks to Theorem 4.23, this implies

$$\|u_{\alpha_q}(y)\|_{M_m(L^c_{2s})} \leq \|u^1_{\alpha_q}\|_{M_m(L^c_{2q}(2^\theta))} + \|u^2_{\alpha_q}\|_{M_m(L^c_{2p}(2^{1-\theta}))}$$

with $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$. Now we choose p = 2. Then

$$\begin{aligned}\|u_{\alpha_q}(y)\|_{M_m(L^c_{2s})} &\leq \|(u^1_{\alpha_q})^*u^1_{\alpha_q}\|^{1/2}_{M_m(L_q(2^\theta))} + \|(u^2_{\alpha_q})^*u^2_{\alpha_q}\|^{1/2}_{M_m(L_2(2^{1-\theta}))} \\ &= \|\Phi^{\mathrm{up}}_{\theta,\alpha_q}(y^*y)\|_{M_m(L_q)} + \|\Phi^{\mathrm{lower}}_{1-\theta,\alpha_q}(y^*y)\|_{M_m(L_2)}\end{aligned} \tag{4.2.9}$$

Recall that $\|A^{-\alpha_q}(z)\|_{M_m(L_s)} = \|(u_{\alpha_q}(y))^* u_{\alpha_q}(y)\|^{1/2}_{M_m(L_s)}$. By Hölder's inequality, we get

$$\|A^{-\alpha_q}(z)\|_{M_m(L_s)} \leq 2(\|\Phi^{\mathrm{up}}_{\theta,\alpha_q}(z)\|_{M_m(L_q)} + \|\Phi^{\mathrm{down}}_{1-\theta,\alpha_q}(z)\|_{M_m(L_2)}).$$

(ii) Let $x > 0$ and $x \in M_m(L_2)$. Then there exists $z \in M_m(L_4)$, such that $x = z^*z$. Then we use the map u_t obtained from GNS construction of T_t, and generate $u_\gamma(z)(t) = t^{(\gamma-1)/2}u_t(z)$. We get $A^{-\gamma}(x) = A^{-\gamma}(z^*z) = u_\gamma(z)^*u_\gamma(z)$. Next we repeat the argument in part (i) with $y = u_\gamma(z)$. Thanks to Lemma 2.30, we get

$$A^{-\alpha}x = A^{-\alpha_q}(A^{-\gamma}x) = A^{-\alpha_q}(u_\gamma(z)^*u_\gamma(z)) = A^{-\alpha_q}(y^*y).$$

And by part(i), we obtain

$$\|A^{-\alpha}x\|_{M_m(L_s)} = \|A^{-\alpha_q}(y^*y)\|_{M_m(L_s)} \leq 2(\|\Phi^{\mathrm{up}}_{1-\theta,\alpha_q}(y^*y)\|_{M_m(L_q)} + \|\Phi^{\mathrm{lower}}_{\theta,\alpha_q}(y^*y)\|_{M_m(L_2)})$$

By Theorem 4.5 with $\alpha + \theta = (1 - \frac{2}{q})\frac{n}{4} + \gamma$, we get

$$\|\Phi^{\mathrm{up}}_{\theta,\alpha_q}(y^*y)\|_{M_m(L_q)} = \|\Phi^{\mathrm{up}}_{\theta,\alpha_q}(A^{-\gamma}(z^*z))\|_{M_m(L_q)} \leq \|z^*z\|_{M_m(L_2)} = \|x\|_{M_m(L_2)}.$$

According the Lemma 4.4, the norm of $\Phi^{\mathrm{lower}}_{1-\theta,\alpha_q}(y^*y)$ is controlled by that of $A^{-(\alpha_q-1+\theta)}$. With cancellation of the exponent, we assume $\gamma = \alpha_q - 1 + \theta$, then we obtain

$$\|\Phi^{\mathrm{lower}}_{1-\theta,\alpha_q}(y^*y)\|_{M_m(L_2)} = \|\Phi^{\mathrm{lower}}_{1-\theta,\alpha_q}(A^{-\gamma}(z^*z))\|_{M_m(L_q)} \leq \|z^*z\|_{M_m(L_2)} = \|x\|_{M_m(L_2)}.$$

Therefore we get

$$\|A^{-\alpha}x\|_{M_m(L_s)} \lesssim \|x\|_{M_m(L_2)}.$$

we deduce the assertion for $0 < \alpha < 1$. For the case $\alpha \geq 1$, repeating the same technique in Theorem 4.8 we obtain the assertion.

Remark 4.27. Equation (4.2.9) leads us to the definition of singular operators $\Phi^{\mathrm{up}}_{\theta,\alpha_q}$ and $\Phi^{\mathrm{lower}}_{1-\theta,\alpha_q}$.

We want to illustrate our result for the strongly continuous semigroup of normal selfadjoint subunital completely positive maps $\{T_t\}$ for completely bounded norm on commutative space $\mathbb{R}$. Then $L^p(\mathcal{M})$ coincides with the usual commutative space $L^p(\mathbb{R})$ and $S^p(H; L^p(\mathbb{R})) = L^p(\mathbb{R}; S^p(H))$ for any H and any $1 \leq p < \infty$. Thus a completely bounded map $A^{-\alpha} : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ is a bounded maps whose tensor extension $T \otimes I_{S^q}$ extends to a bounded operator on the vector valued spaces $A^{-\alpha} : L^p(\mathbb{R}; S^q) \to L^q(\mathbb{R}; S^q)$. However we want to show a more general version $A^{-\alpha}: S_q(L^p(\mathbb{R})) \to S_q(L^q(\mathbb{R}))$ as follows:

Corollary 4.28. *Let $\{T_t\}$ be a strongly continuous semigroup of normal selfadjoint subunital completely positive maps such that (R_n^2) holds for $L_p(\mathbb{R})$. Then*

$$\|A^{-\alpha} : S_q(L_p(\mathbb{R})) \longrightarrow S_q(L_q(\mathbb{R}))\| \leq C$$

with $\alpha = \frac{n}{2}(\frac{1}{q} - \frac{1}{p})$.

Remark 4.29.

$$\begin{array}{ccc} S_q(L_p(\mathbb{R})) & \xrightarrow{A^{-\alpha}} & S_q(L_q(\mathbb{R})) \\ \big\downarrow & & \big\downarrow{\scriptstyle id} \\ L_p(\mathbb{R}, S_q) & \xrightarrow{A^{-\alpha}} & L_q(\mathbb{R}, S_q) = S_q(L_q(\mathbb{R})) \end{array}$$

For the vector valued Sobolev inequality, we already have the base line in the following diagram. And Corollary 4.28 proves the up line. $\forall x \in S_q(L_p)$, $\frac{1}{r} = |\frac{1}{p} - \frac{1}{q}|$.

$$\|x\|_{L_p(S_p)} = (\int \sup_{\|\alpha\|_{2r}, \|\beta\|_{2r}=1} \|\alpha x \beta\|_p^p)^{1/p} \leq \sup_{\|\alpha\|_{2r}, \|\beta\|_{2r}=1} (\int \|\alpha x \beta\|_p^p)^{1/p} = \|x\|_{S_p(L_p)}.$$

Therefore there exists a completely contraction from $S_q(L_p(\mathbb{R}))$ to $L_p(\mathbb{R}, S_q)$. Up line's estimate reduces the base line's, but the opposite direction is not. Therefore the cb-version of Sobolev inequality is much stronger than the vector-valued one.

4.3 Application and Examples

In this section, we want to prove the cb-version of Varopoulos's theorem. The scope of the proof is standard by now.

Theorem 4.30. *Let (T_t) be a semigroup of completely positive selfadjoint contractions on a von Neumann algebra $\mathcal{M}$ with negative generator A and $n > 2$. The following are equivalent*

(i) $\|T_t : L_1(\mathcal{M}) \to L_\infty(\mathcal{M})\|_{cb} \leq C_1 t^{-n/2}$

(ii) $\|A^{-\frac{1}{2}} : L_2(\mathcal{M}) \to L_{\frac{2n}{n-2}}(\mathcal{M})\|_{cb} < \infty$;

(iii) $\|x\|^{2+4/n}_{S_2(L_2(\mathcal{M}))} \leq_{cb} C_2 \langle 1 \otimes Ax, x\rangle \|x\|^{4/n}_{S_2(L_1(\mathcal{M}))}$;

Proof. (i)⇒(ii) follows from Theorem 4.26. Next we show (ii)⇒(iii). From the assertion (ii), let $x \in S_2^m(L_2(\mathcal{M}))$, we have

$$\|x\|_{S_2(L_{2n/(n-2)})} \leq \|A^{1/2} x\|_{S_2(L_2)}.$$

This implies

$$\|x\|^2_{S_2(L_{2n/(n-2)})} \le \langle 1\otimes A^{1/2}x, 1\otimes A^{1/2}x\rangle = \langle 1\otimes Ax, x\rangle.$$

From interpolation theory in Pisier's paper [Pis93] for $\frac{1}{2} = \frac{(1-\theta)(n-2)}{2n} + \frac{\theta}{1}$ with $\theta = \frac{2}{n+2}$, we obtain

$$\|x\|_{S_2(L_2)} \le \|x\|^{1-\theta}_{S_2(L_{2n/(n-2)})}\|x\|^{\theta}_{S_2(L_1)}.$$

Therefore we have

$$\|x\|^{2+4/n}_{S_2(L_2)} \le \langle 1\otimes Ax, x\rangle \|x\|^{4/n}_{S_2(L_1)}.$$

For (iii) to (i), fix $x \in S_2(L_2(\mathcal{M})) \cap S_2(L_1(\mathcal{M}))$ with $\|x\|_{S_2(L_1)} = 1$. Because T_t is a completely contraction, $\|T_t x\|_{S_2(L_1)} \le 1$. Denote $V(t) := \|T_t x\|^2_{S_2(L_2)}$,

$$V'(t) = - < AT_t x, T_t x >,$$

the hypothesis yields

$$V'(t) \le -2C_2^{-1}\|T_t x\|^{2+4/n}_{S_2(L_2)}\|T_t x\|^{-4/n}_{S_2(L_1)} \le -2C_2^{-1}\|T_t x\|^{2+4/n}_{S_2(L_2)} \le -2C_2^{-1}V(t)^{1+2/n}$$

i.e.

$$\frac{d}{dt}(V(t)^{-2/n}) \le 2n^{-1}C_2^{-1}.$$

Integrate both sides, we get

$$V(t) \le (nC_2)^{n/2}t^{-n/2}, \forall t > 0.$$

The first assertion (i) immediately holds by taking square root. □

Corollary 4.31. *Suppose $\{T_t\}$ is a strongly continuous semigroup of normal selfadjoint subunital completely positive maps. The following are equivalent:*

(1) there exist $1 < p < q < \infty$ with $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$ such that

$$\|A^{-\alpha} : L_p \longrightarrow L_q\|_{cb} \le c_1;$$

(2) the semigroup $\{T_t\}$ satisfies (R_n^{cb}), i.e.

$$\|T_t : L_1 \longrightarrow L_\infty\|_{cb} \le c_2 t^{-n/4}.$$

As before, let $T_t = e^{-tA}$ be a symmetric submarkovian semigroup and let $S_t = e^{-tB}$ be a semigroup on $L^p(N), 1 \le p \le \infty$. We then have

Theorem 4.32. *Suppose A and B are two completely positive operators that*

1. *there exist $n > 2, C_1 > 0$ such that*

$$\|f\|^2_{2n/(n-2)} \le_{cb} C_1 \langle Af, f\rangle, \tag{4.3.1}$$

2. *there exist $C_2 > 0, d > 0$ and $\varepsilon > 0$ such that*

$$\langle A^{\varepsilon} x, x\rangle \le_{cb} C_2 \langle Bx, x\rangle + d\|x\|^2, \tag{4.3.2}$$

Then there exists $C > 0$ such that

$$\|S_t f\|_2 \le C t^{-n/(4\varepsilon)} e^t \|f\|_1, \forall t > 0, \forall f \in L_1(N).$$

Proof. From assertion (5.1.3) and Theorem 4.30, we obtain $\|T_t : L_1(\mathcal{M}) \to L_\infty(\mathcal{M})\|_{cb} \le C_1 t^{-n/2}$. Then by Theorem 4.26, we get

$$\|A^{-\varepsilon/2} : L_2 \to L_{2m/(m-1)}\|_{cb} \le C.$$

here $\frac{\varepsilon}{2} = \frac{n}{2}(1 - \frac{m-1}{m})$. Then we get $m = \frac{n}{2\varepsilon}$. From the assertion (4.3.2),

$$\|x\|^2_{2m/(m-1)} \le_{cb} C\|A^{\varepsilon/2} x\|_2^2 \le_{cb} CC_2\langle Bx, x\rangle + Cd\|x\|^2 \le_{cb} \tilde{C}\langle (B+I)x, x\rangle$$

i.e.

$$\|x\|_{2m/(m-1)} \le_{cb} \tilde{C}\|(B+I)^{1/2} x\|_2$$

Denote $\tilde{B} = B + I$. The $\tilde{S}_t = e^{-tB} e^{-t}$ is a symmetric submarkovian semigroup. Applying Theorem 4.32 (ii) to (i) for $\tilde{B}$ and $\tilde{S}_t$, we get

$$\|\tilde{S}_t : L_1(\mathcal{M}) \to L_2(\mathcal{M})\|_{cb} \le \tilde{C}_1 t^{-m/2}.$$

This implies that

$$\|S_t : L_1(\mathcal{M}) \to L_2(\mathcal{M})\|_{cb} \le \tilde{C}_1 t^{-m/2} e^t = \tilde{C}_1 t^{-n/(4\varepsilon)} e^t.$$

Example 4.33. Let $X_1, X_2, \cdots, X_k$ be a system of vector fields satifying Hörmander's condition at step r in some open connected Ω of compact Lie group G. For any multiindex $I = (i_1, i_2, \cdots, i_k)$ of length $|I| = k$

we set

$$X_I = X_{i_1} X_{i_2} \cdots X_{i_k} \text{ and } X_{[I]} = [X_{i_1}, [X_{i_2} \cdots [X_{i_{k-1}}, X_{i_k}] \cdots]].$$

If $I = (i_1)$, then $X_I = X_{[I]} = X_{i_1}$. By Hörmander's condition, the vectors $\{(X_{[I]})\}_{|I| \leq d}$ span G. Let $A = \sum_{|I|<d} X_{[I]}^2$ and $B = \sum_{j=1}^{k} X_j^2$. We are now ready to state the result proved showed in Lemma 2.1 of as follows:

$$\langle A^{\epsilon} x, x \rangle \leq \langle Bx, x \rangle + c\|x\|^2$$

Therefore we apply Theorem 4.32 to operator A and B, then we can get

$$\|S_t = e^{-tB} : L_p(G) \longrightarrow L_q(G)\|_{cb} \leq Ct^{-n/4\epsilon} e^t.$$

Example 4.34. Let the semigroup $T_t(e_{rs}) = e^{-t(r-s)^2} e_{rs}$. For $x = (x_{rs}) \in S_1^m(S_1)$ with $T_t(x) = \sum_{k \in \mathbb{Z}} e^{-tk^2} \Big(\sum_s x_{s+k,s} e_{s+k,s} \Big)$,

$$\begin{aligned}
\|T_t(x)\|_{M_m(B(l_2))} &\leq \sum_{k \in \mathbb{Z}} e^{-tk^2} \| \sum_s x_{s+k,s} e_{s,s} \|_{M_m(l_\infty)} \leq \sum_{k \in \mathbb{Z}} e^{-tk^2} \|x\|_{M_m(S_1)} \\
&\leq 2 \Big(\int_0^\infty e^{-ty^2} dy \Big) \|x\|_{M_m(S_1)} = t^{-1/2} \|x\|_{M_m(S_1)},
\end{aligned}$$

therefore the semigroup satifies the (R_n^2) for n = 1.

Example 4.35. With the matrix M_m where $v_k(e_r) = e_{k+r}$ and $u_j(e_r) = e^{\frac{2\pi i j r}{n}} e_r$, let the semigroup $T_t(u_j v_k) = e^{-t(j^2+k^2)} u_j v_k$. For $x = \sum \alpha_{j,k} u_j v_k$ where $\alpha_{j,k} = \frac{1}{m} tr(v_k^* u_j^* x)$.

$$\|T_t x\|_{M_m} \leq \sum_{k \in \mathbb{Z}} e^{-t(k^2+j^2)} \|(\alpha_{j,k})\|_{M_m} \leq t^{-1} \|x\|_{L_1(M_n, tr)}.$$

Therefore the semigroup satifies the (R_n^2) for n = 2.

Example 4.36. Let $\mathcal{A}_\theta$ be the quantumn euclidean space with $UV = \exp(2\pi i \theta) VU$, and the semigroups T_t have

$$T_t(U_k V_j) = \exp(-t(k-j)^2) U_k V_j.$$

Then $\|T_t : L_1(\mathcal{A}_\theta, \tau_\theta) \longrightarrow L_\infty(\mathcal{A}_\theta, \tau_\theta)\|_{cb} \leq ct^{-1}$. See [JMP13] for more details.

Chapter 5

Stricharz estimates

In applied mathematics, Strichartz estimates are used for linear dipersive partial differential equations. These inequalities describe size and decay of solutions in mixed norm Lebesgue spaces. In [Seg76], the author investigates the linear Klein-Gordon equation. In the pioneering paper [S$^+$77], Strichartz builds the connection between space-time estimate and the restriction theorem of Tomas and Stein. See [LS95],[Kap89], [MSS93], [GV95], [Sog95] for many known Strichartz wave equations. See [GV92] [Yaj87] for Strichartz results for the Schrödinger equation.

Let (A_0, A_1) be an interpolation pair and $(A_0, A_1)_{1/r} = A_{1/r}$. Suppose that for each time $t \in \mathbb{R}$, we have an operator $U(t) : \mathcal{H} \longrightarrow A_{1/2}$ which obeys the followings:

- For all t and all $f \in \mathcal{H}$ we have

$$\|U(t)f\|_{A_{1/2}} \leq C_1 \|f\|_{\mathcal{H}} \quad \text{(energy estimate)} \tag{5.0.1}$$

- For some $\sigma > 0$, all $t \neq s$ and all $g \in L^1$

$$\|U(s)(U(t))^* g\|_{A_1} \leq C_2 |t-s|^{-\sigma} \|g\|_{A_0} \quad \text{(dispersive estimate)} \tag{5.0.2}$$

In [KT98], Keel and Tao gave the concrete form of Strichartz estimates on $\mathbb{R}^n$ as follows:

$$\left(\int \left(\int |U(t)f|^r dx\right)^{q/r} dt\right)^{1/q} \lesssim \|f\|_2.$$

Here (q,r) satisfies $\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}$. Therefore our goal is to determine the space-time norms

$$\|F\|_{L_t^q L^{r'}(\mathcal{M})} \equiv \left(\int \|F(t)\|_{A_{1/r}}^q dt\right)^{\frac{1}{q}},$$

where now $A_{1/r} = L^{r'}(\mathcal{M})$ is a noncommutative L_p space and $F(t) = U(t)f, \forall f \in A_{1/r}(\mathcal{M})$.

As for non-commutative space setting, for which noncompact noncommutative n-dimensional space the unitary operator $U(t)$ still satisfy the Strichartz estimates is a natural question to ask. First candidate to check is deformed Euclidean space $\mathcal{R}^n_\theta$, since it's the noncommutative version of Euclidean space $\mathbb{R}^n$. More precisely, our goal is to prove

$$(\int \|U(t)f\|^q_{L_r(R^n_\theta)}dt)^{1/q} \lesssim \|f\|_2.$$

Proposition 5.1. *If $U(t)$ is an operator from some Hilbert space $\mathcal{H}$ to $A_{1/2}(\mathcal{M})$,*

(1) If $U(t)$ obeys (5.0.1), we have

$$\|U(s)^*U(t) : \mathcal{H} \to \mathcal{H}\| \leq C_1^2 \tag{5.0.3}$$

(2) If $U(t)$ obeys (5.0.1) and (5.0.2), we have

$$\|U(s)^*U(t) : A_{1/r} \to A_{1/r'}\| \leq C_1^{\frac{4}{r}} C_2^{1-\frac{2}{r}} |t-s|^{-1-\beta(r,r)} \tag{5.0.4}$$

where $\beta(r,\tilde{r})$ is given by

$$\beta(r,\tilde{r}) = \sigma - 1 - \frac{\sigma}{r} - \frac{\sigma}{\tilde{r}}.$$

Proof. (1) follows from the (5.0.1) directly. We get (2) by interpolation between (5.0.2) and (5.0.3) with $\theta = \frac{1}{r}$. □

Definition 5.2. We say that the exponent pair (q, r) is sharp σ-admissible if $q, r \geq 2, (q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}.$$

Remark 5.3. (1) Note in particular that when $\sigma > 1$ the endpoint

$$P = (2, \frac{2\sigma}{\sigma - 1})$$

is also a sharp σ-admissible.

(2) By the definition of sharp σ-admissible point, one can check that $\beta(r, r) < 0$ for $q > 2$.

In this chapter we will show that, for certain exponent pair $(q, r), (\tilde{q}, \tilde{r})$, the family $\{U(t) : t \in \mathbb{R}\}$ satisfying (5.0.1) and (5.0.2) has Strichartz estimates of the following form:

(i) the homogeneous Strichartz estimate

$$\|U(t)f\|_{L^q_t A_{1/r}} \leq C_3 \|f\|_H$$

(ii) its dual estimate

$$\| \int (U(s))^* F(s) ds \|_H \le C_3 \|F\|_{L_t^{q'} A_{1/r}} \tag{5.0.5}$$

(iii) the inhomogeneous(retarded) Strichartz estimate

$$\| \int_{t<s} U(t)(U(s))^* F(s) ds \|_{L_t^q A_{1/r'}} \le C_4 \|F\|_{L_t^{\tilde{q}'} A_{1/\tilde{r}}} \tag{5.0.6}$$

In order to prove Strichartz estimate, we define the bilinear form $T : L_t^1 A_0 \times L_t^1 A_0 \to \mathbb{C}$ by

$$T(F,G) := \iint_{s \le t} < U(s)^* F(s), U(t)^* G(t) > dsdt.$$

Remark 5.4. Suppose that $q, \tilde{q} \in [1, \infty]$ and $r, \tilde{r} \in [1, \infty]$. Then

(i) The homogeneous Strichartz estimate (5.0.5) is equivalent to the bilinear estimate

$$|T(F,G)| \lesssim \|F\|_{L_t^q A_{1/r}} \|G\|_{L_t^q A_{1/r}} \tag{5.0.7}$$

(ii) The inhomogeneous Strichartz estimate (5.0.6) is equivalent to the bilinear estimate

$$|T(F,G)| \lesssim \|F\|_{L_t^{\tilde{q}} A_{1/\tilde{r}}} \|G\|_{L_t^{q'} A_{1/r}} \tag{5.0.8}$$

5.1 Homogeneous Strichartz estimate

In order to prove the homogeneous Strichartz estimate (5.0.7), we have to consider $p = 2$(endpoint estimate) and $p \neq 2$ (nonendpoint estimate) separately.

5.1.1 Nonendpoint estimate

Lemma 5.5. *Suppose that $q \in (2, \infty]$ and $\theta \in [0, 1]$. If $U(t)$ obeys* (5.0.1) *and* (5.0.2)*, then the estimate*

$$|T(F,G)| \le C_3^2 \|F\|_{L_t^{q'} A_{1/r}} \|G\|_{L_t^{q'} A_{1/r}}$$

holds with (q, r) sharp σ-admissible.

Proof. By the definition of the sharp σ-admissible point (5.0.1) for the case $\frac{1}{q}+\frac{\sigma}{r}=\frac{\sigma}{2}$, we have

$$\frac{1}{q'}-\frac{1}{q}=-\beta(r,r),$$

and if we integral (5.0.4) and using Hardy-Littlewood-Sobolev inequality when $q>q'$.[Ste70]

$$\begin{aligned}|T(F,G)| &\leq \iint |<U(s)^*F(s),U(t)^*G(t)>|dsdt\\ &\leq \iint C_1^{\frac{4}{r}}C_2^{1-\frac{2}{r}}|t-s|^{-1-\beta(r,r)}\|F(s)\|_{A_{1/r}}\|G(t)\|_{A_{1/r}}dsdt\\ &\leq C_1^{\frac{4}{r}}C_2^{1-\frac{2}{r}}\|F\|_{L_t^{q'}A_{1/r}}\|G\|_{L_t^{q'}A_{1/r}}\end{aligned}$$

□

5.1.2 Endpoint estimate

For the endpoint case, i.e. $q=2$, we have to decompose $T(F,G)$ dyadicdally as $\sum_j T_j(F,G)$, where the summation is over the integers $\mathbb{Z}$ and

$$T_j(F,G)=\int_{t-2^{j+1}<s\leq t-2^j} <(U(s))^*F(s),(U(t))^*G(t)>dsdt, \tag{5.1.1}$$

Lemma 5.6. *If $U(t)$ is an operator from some Hilbert space $\mathcal{H}$ to $A_{1/2}(\mathcal{M})$ satisfying* (5.0.1) *and* (5.0.2)*, the estimate*

$$|T_j(F,G)|\lesssim 2^{-j\beta(a,b)}\|F\|_{L_t^2A_{\frac{1}{a}}}\|G\|_{L_t^2A_{\frac{1}{b}}} \tag{5.1.2}$$

holds for all $j\in\mathbb{Z}$ and all $(\frac{1}{a},\frac{1}{b})$ in a neighbourhood of $(\frac{1}{r},\frac{1}{r})$.

Proof. We first prove the case when $j=0$. We may assume that F,G are supported on a time interval of duration $O(1)$, since we need to decompose F and G into linear combinations of (approximate) L^2-normalized characteristic functions. Without lose of generality, we can assume the supports of functions F, G is a small interval near the original point. We shall prove (5.1.8) for the exponents

(1) $a=b=\infty$

(2) $2\leq a<r, b=2$

(3) $2\leq b<r, a=2$;

the lemma will then follow by interpolation and the fact that $2<r<\infty$.

To prove(1), observe that (5.0.2) gives

$$|T_0(F,G)| \lesssim \iint_{t-2<s<t-1} |t-s|^{-\sigma} \|F(s)\|_{A_0} \|G(t)\|_{A_0} dsdt \lesssim \|F\|_{L^1_t A_0} \|G\|_{L^1_t A_0}. \tag{5.1.3}$$

For (2), we prove a general case. Bring the s-integration inside the inner product in (5.1.1) and apply the Cauchy-Schwarz inequality to obtain

$$|T_j(F,G)| \leq (\sup_t \| \int_{t-2^{j+1}<s\leq t-2^j} (U(s))^* F(s) ds \|_H) \int \|(U(t))^* G(t)\|_H dt.$$

Using the energy estimate $\|(U(t))^* G(t)\|_H \leq \|G(t)\|_{A_{\frac{1}{2}}}$ this becomes

$$|T_j(F,G)| \leq (\sup_t \| \int_{t-2^{j+1}<s\leq t-2^j} (U(s))^* F(s) ds \|_H) \|G\|_{L^1_t A_{1/2}}. \tag{5.1.4}$$

Define the quantity $q(a)$ by requiring $(q(a), a)$ to be sharp σ-admissible. By the results of the previous section of non-endpoint estimates for $(q(a), a)$. Now applying this to (5.1.4) for the case j=0, we obtain

$$|T_0(F,G) : L_t^{q(a)'} A_{1/a} \times L^1_t A_{1/2} \to \mathbb{C}| \leq C_1^{\frac{2}{a}+1} C_2^{\frac{1}{2}-\frac{1}{a}}$$
$$|T_0(F,G) : L_t^2 A_{1/a'} \times L^2_t A_{1/2} \to \mathbb{C}| \leq C_1^{\frac{2}{a}+1} C_2^{\frac{1}{2}-\frac{1}{a}}$$

The last inequality holds if we require $q(a) \geq 2$. Same argument, we get

$$|T_0(F,G) : L^1_t A_{1/2} \times L_t^{q(b)'} A_{1/b} \to \mathbb{C}| \leq C_1^{\frac{2}{b}+1} C_2^{\frac{1}{2}-\frac{1}{b}}$$
$$|T_0(F,G) : L^2_t A_{1/2} \times L^2_t A_{1/b} \to \mathbb{C}| \leq C_1^{\frac{2}{b}+1} C_2^{\frac{1}{2}-\frac{1}{b}}$$

The last inequality holds if we require $q(b) \geq 2$. Let $H(a,b) = C_1^{\frac{2}{a}+\frac{2}{b}} C_2^{1-\frac{1}{a}-\frac{1}{b}}$. Interpolate these two inequations, we get for $2 \leq a_0 < r, 2 \leq b_0 < r$,

$$|T_0(F,G) : L^2 A_{\frac{\theta}{a'}+\frac{1-\theta}{2}} \times L^2 A_{\frac{1-\theta}{b}+\frac{\theta}{2}} \to \mathbb{C}| \leq H(a,2)^{\theta} H(2,b)^{1-\theta}$$

$$|T_0(F,G) : L^2_t A_{1/a_0} \times L^2_t A_{1/b_0'} \to \mathbb{C}| \leq H(a_0, b_0) \tag{5.1.5}$$

By interpolating (5.1.3) and (5.1.5), we get all pair of (a,b) in a neighbourhood of $(\frac{1}{r}, \frac{1}{r})$, which have

$$|T_0(F,G) : L^2_t A_{1/a} \times L^2_t A_{1/b'} \to \mathbb{C}| \leq C_1^{\frac{2}{a}+\frac{2}{b}} C_2^{1-\frac{1}{a}-\frac{1}{b}}.$$

For the general case, it's same as the above proof. We need to consider three exponents:

(i) $a = b = \infty$

(ii) $2 \le a < r, b = 2$

(iii) $2 \le b < r, a = 2$;

For case (i) We know

$$| < U(s)^*F(s), U(t)^*G(t) > | \le C_1^{\frac{4}{r}} C_2^{1-\frac{2}{r}} |t-s|^{-1-\beta(r,r)} \|F(s)\|_{A_{1/r}} \|G(t)\|_{A_{1/r}}$$

Now we want to consider the $T_j(F,G) = \int_{t-2^{j+1}<s\le t-2^j} < U(s)^*F(s), U(t)^*G(t) > dsdt$.

Taking integral of (5.1.8), we get

$$|T_j(F,G)| \le C_2 2^{j-j\sigma} \|F\|_{L_t^2 A_0} \|G\|_{L_t^2 A_0} \tag{5.1.6}$$

To prove (ii). By (5.1.4), we have

$$\begin{aligned}
|T_j(\tilde{F},\tilde{G})| &\le C_1^{\frac{2}{a}} C_2^{\frac{1}{2}-\frac{1}{a}} \|\tilde{F}\|_{L_t^{q(a)'} A_{1/a}} C_1 \|\tilde{G}\|_{L_t^1 A_{1/2}} \\
&= C_1^{\frac{2}{a}+1} C_2^{\frac{1}{2}-\frac{1}{a}} \|\tilde{F}\|_{L_t^{q(a)'} A_{1/a}} \|\tilde{G}\|_{L_t^1 A_{1/2}} \\
&\le C_1^{\frac{2}{a}+1} C_2^{\frac{1}{2}-\frac{1}{a}} 2^{j(\frac{1}{q(a)'}-\frac{1}{2})} \|\tilde{F}\|_{L_t^2 A_{1/a}} \|\tilde{G}\|_{L_t^2 A_{1/2}} \\
&\le C_1^{\frac{2}{a}+1} C_2^{\frac{1}{2}-\frac{1}{a}} 2^{j(\frac{1}{2}-\sigma(\frac{1}{2}-\frac{1}{a}))} \|\tilde{F}\|_{L_t^2 A_{1/a}} \|\tilde{G}\|_{L_t^2 A_{1/2}}
\end{aligned}$$

The last inequality holds if we require $q(a) \ge 2$. Repeating the same argument, we get

$$\begin{aligned}
|T_j(\tilde{F},\tilde{G})| &\le C_1^{\frac{2}{b}} C_2^{\frac{1}{2}-\frac{1}{b}} \|\tilde{F}\|_{L_t^1 A_{1/2}} C_1 \|\tilde{G}\|_{L_t^{q(b)'} A_{1/b}} \\
&= C_1^{\frac{2}{b}+1} C_2^{\frac{1}{2}-\frac{1}{b}} \|\tilde{F}\|_{L_t^1 A_{1/2}} \|\tilde{G}\|_{L_t^{q(b)'} A_{1/b}} \\
&\le C_1^{\frac{2}{b}+1} C_2^{\frac{1}{2}-\frac{1}{b}} 2^{j(\frac{1}{q(b)'}-\frac{1}{2})} \|\tilde{F}\|_{L_t^2 A_{1/2}} \|\tilde{G}\|_{L_t^2 A_{1/b}} \\
&\le C_1^{\frac{2}{b}+1} C_2^{\frac{1}{2}-\frac{1}{b}} 2^{j(\frac{1}{2}-\sigma(\frac{1}{2}-\frac{1}{b}))} \|\tilde{F}\|_{L_t^2 A_{1/2}} \|\tilde{G}\|_{L_t^2 A_{1/b}}
\end{aligned}$$

The last inequality holds if we require $q(b) \ge 2$. Taking the interpolation these two inequations, we get for $2 \le a_0 < r,\ 2 \le b_0 < r$

$$|T_j(\tilde{F},\tilde{G})| \le \tilde{H}(a,2)^{\theta} \tilde{H}(2,b)^{1-\theta} 2^{j(1-\theta)(\frac{1}{2}-\sigma(\frac{1}{2}-\frac{1}{b}))} 2^{\theta j(\frac{1}{2}-\sigma(\frac{1}{2}-\frac{1}{a}))}$$

$$\times\|\tilde{F}\|_{L^2A_{\frac{\theta}{a}+\frac{1-\theta}{2}}}\|\tilde{G}\|_{L^2A_{\frac{1-\theta}{b}+\frac{\theta}{2}}}$$

$$|T_j(\tilde{F},\tilde{G})| \leq \tilde{H}(a_0,b_0)2^{j(1-\sigma(1-\frac{1}{a}-\frac{1}{b}))}\|\tilde{F}\|_{L_t^2A_{1/a_0}}\|\tilde{G}\|_{L_t^2A_{1/b_0}} \tag{5.1.7}$$

Let $\tilde{H}(a,b) = C_1^{\frac{2}{a}+\frac{2}{b}}C_2^{1-\frac{1}{a}-\frac{1}{b}}$. Then we set $\tilde{H}(a,2) = C_1^{\frac{2}{a}+1}C_2^{\frac{1}{2}-\frac{1}{a}}$,$\tilde{H}(2,b) = C_1^{\frac{2}{b}+1}C_2^{\frac{1}{2}-\frac{1}{b}}$. Let $\frac{1}{a} = \frac{1-\lambda}{a_1} + \frac{\lambda}{a_2}, \frac{1}{b} = \frac{1-\lambda}{b_1} + \frac{\lambda}{b_2}.\beta(a,b) = \sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}$. Then we know $\beta(a,b) = \beta(a_1,b_1) + \beta(a_2,b_2)$. Since $\beta(a,b)$ is an affine form, then it can interpolate. Take $C = \max\{C_1^2, C_2\}$, by (5.1.6) and (5.1.7), we know

$$\begin{cases} |T_j(F,G)| \leq & C2^{-j\beta(\infty,\infty)}\|F\|_{L_t^2A_0}\|G\|_{L_t^2A_0} \\ |T_j(F,G)| \leq & C2^{-j\beta(a_0,b_0)}\|F\|_{L_t^2A_{1/a_0}}\|G\|_{L_t^2A_{1/b_0}} \end{cases}$$

$$\Longrightarrow |T_j(F,G)| \leq C2^{-j\beta(a,b)}\|F\|_{L_t^2A_{1/a}}\|G\|_{L_t^2A_{1/b}}$$

here $(\frac{1}{a},\frac{1}{b})$ is a neighbourhood of $(\frac{1}{r},\frac{1}{r})$. □

One would hope that the

$$|T(F,G)| = \sum_{j\in\mathbb{Z}}\|T_j(F,G)\| \lesssim \sum_{j\in\mathbb{Z}}2^{-j\beta(r,r)}\|F\|_{L_t^2A_{\frac{1}{r}}}\|G\|_{L_t^2A_{\frac{1}{r}}} \lesssim \|F\|_{L_t^2A_{\frac{1}{r}}}\|G\|_{L_t^2A_{\frac{1}{r}}}.$$

However, $\beta(1/r,1/r) = 0$ so the summation diverges. Therefore we have to slightly perturb the exponent pair (r,r) by abstract real interpolation argument mentioned in the preliminary.

Lemma 5.7 (Endpoint estimate)**.** *If $U(t)$ obeys* (5.0.1) *and* (5.0.2)*, the estimate*

$$|T(F,G) : L_t^2A_{\frac{1}{r}} \times L_t^2A_{\frac{1}{r}} \to \mathbb{C}| \leq C_5 \tag{5.1.8}$$

holds for $r = \frac{2\sigma}{\sigma-1}, \sigma > 1$.

Proof. Thanks to triangle inequality, It suffices to show that

$$\sum|T_j(F,G)| \lesssim \|F\|_{L_t^2A_{\frac{1}{r}}}\|G\|_{L_t^2A_{\frac{1}{r}}}.$$

Recall the definition of l_s^q given by (2.1.1). Then the above inequality is equivalent with the following:

$$\tilde{T} : L_t^2A_{1/r} \times L_t^2A_{1/r} \to l_1^0 \quad \text{is bounded.} \tag{5.1.9}$$

By Lemma 5.6 there is a positive ε such that the map

$$\tilde{T} : L_t^2 A_{1/a} \times L_t^2 A_{1/b} \to l_\infty^{\beta(a,b)}$$

is bounded for all (a, b) in the neighbourhood $N_\varepsilon = \{(a, b)|(1/a - 1/r)^2 + (1/b - 1/r)^2 \le \varepsilon^2\}$. We carefully choose three points (a, b) as follows: Suppose that $a_0 = b_0 = 1/r + \varepsilon/3$ and $a_1 = b_1 = 1/r - 2\varepsilon/3$, then $\beta(a_0, b_1) = \beta(a_1, b_0) \neq \beta(a_0, b_0)$. And the maps

$$\tilde{T} : L_t^2 A_{1/a_0} \times L_t^2 A_{1/b_0} \to l_\infty^{\beta(a_0,b_0)}$$
$$\tilde{T} : L_t^2 A_{1/a_0} \times L_t^2 A_{1/b_1} \to l_\infty^{\beta(a_0,b_1)}$$
$$\tilde{T} : L_t^2 A_{1/a_1} \times L_t^2 A_{1/b_0} \to l_\infty^{\beta(a_1,b_0)}$$

are bounded. By Theorem 2.7 (ii) we deduce that the map

$$\tilde{T} : (L_t^2 A_{1/a_0}, L_t^2 A_{1/a_1})_{\eta_0,2} \times (L_t^2 A_{1/b_0}, L_t^2 A_{1/b_1})_{\eta_1,2} \to (l_\infty^{\beta(a_0,b_0)}, l_\infty^{\beta(a_0,b_1)})_{\eta,1} \tag{5.1.10}$$

is bounded, where $\eta_0 = \eta_1 = \frac{1}{3}$ and $\eta = \eta_0 + \eta_1$. And we have $(1 - \eta)\beta(a_0, b_0) + \eta\beta(a_0, b_1) = \beta(r, r) = 0$. If we combine this with (5.1.10), then we get (5.1.9). □

5.1.3 Inhomogeneous Strichartz estimates

We now want to estimate inhomogeneous Strichartz inequality(5.0.6). By Lemma 5.4, it suffices to prove (5.0.8), i.e.

$$|T(F, G)| \lesssim \|F\|_{L_t^{q'} A_{1/r'}} \|G\|_{L_t^{\tilde{q}'} A_{1/\tilde{r}'}}$$

into three cases.

Sketch of the proof. Suppose that (q, r) and $(\tilde{q}, \tilde{r})$ are sharp σ-admissible. Observe that $(\infty, 2)$ is sharp σ-admissible. By the definition of the bilinear form, we have

$$|T(F, G)| \lesssim \Big(\sup_t \| \int_{s<t} (U(s))^* F(s) ds \|_H \Big) \|G\|_{L_t^1 A_{1/2}},$$

(i) If $(\tilde{q}, \tilde{r}) = (0, 2)$ then we can use the homogeneous Strichartz estimate (5.0.5), we have the map $T : L_t^{q'} A_{1/r'} \times L_t^1 A_{1/2}$ is bounded;

(ii) If $(q, r) = (0, 2)$, by symmetry we have the map $T : L_t^1 A_{1/2} \times L_t^{\tilde{q}'} A_{1/\tilde{r}'}$ is bounded;

(iii) If $(q,r) = (\tilde{q},\tilde{r})$, it's the homogeneous Strichartz estimate (5.0.7). Therefore we have $T : L_t^{q'} A_{1/r'} \times L_t^{q'} A_{1/r'}$ is bounded.

We interpolate between these three special cases we can obtain the result since $(\frac{1}{q},\frac{1}{r}), (\frac{1}{\tilde{q}},\frac{1}{\tilde{r}})$ and $(0,\frac{1}{2})$ are collinear. □

Combining thess above three subsections, we get the Abstract Strichartz estimate theorem as follows:

Theorem 5.8 (Abstract Strichartz estimates)**.** *If $U(t)$ obeys* (5.0.1) *and* (5.0.2)*, then the estimates*

$$\|U(t)f\|_{L_t^q A_{1/r}} \leq C_3 \|f\|_H$$

$$\| \int (U(s))^* F(s) ds \|_H \leq C_3 \|F\|_{L_t^{q'} A_{1/r}}$$

$$\| \int U(t)(U(s))^* F(s) ds \|_{L_t^q A_{1/r'}} \leq C_4 \|F\|_{L_t^{\tilde{q}'} A_{1/\tilde{r}}}$$

hold for all sharp σ-admissible exponent pairs $(q,r),(\tilde{q},\tilde{r})$.

Remark 5.9. So a natural question comes whether strichartz estimates still hold in completely bounded norm setting. Given the operator $U(t) : H \to L^2(M)$ satisfies

$$\|U(t)f\|_{A_{1/2}(M)} \leq_{cb} C_1 \|f\|_H$$
$$\|U(s)(U(t))^* g\|_{A_1(M)} \leq_{cb} C_2 |t-s|^{-\sigma} \|g\|_{A_0(M)}.$$

Then
$$\|U(t)f\|_{L_t^q A_{1/r}} \overset{?}{\leq_{cb}} C_3 \|f\|_H$$

However, the answer is negative. By the above conditions, we get

$$| < U(s)^* F(s), U(t)^* G(t) > | \leq C_1^2 \|F(s)\|_{S_2^m A_{1/2}} \|G(t)\|_{S_2^m A_{1/2}}$$

$$| < U(s)^* F(s), U(t)^* G(t) > | \leq C_2 |t-s|^{-\sigma} \|F(s)\|_{S_2^m A_0} \|G(t)\|_{S_2^m A_0}$$

By interpolation, we know

$$| < U(s)^* F(s), U(t)^* G(t) > | \leq C |t-s|^{-\theta\sigma} \|F\|_{S_2^m A_r} \|G(t)\|_{S_2^m A_r}$$

Then by the Hardy-Littlewood-Sobolev inequality, we have

$$|T(F,G)| \leq C\|F\|_{L_{q'}(S_2^m A_r)}\|G\|_{L_{q'}(S_2^m A_r)}$$

However, $q'<2<q$, therefore, we fail to swap the Schatten-2 class with the $L_{q'}$ space. Therefore, the technique we used above doesn't work any more.

5.2 Application to PDE

In this section, we define an operator

$$U(t)(f) = e^{it\Delta} f$$

on the Hilbert space $\mathcal{H} = L^2(\mathcal{R}_\theta^n)$ and $A_{1/r} = L^r(\mathcal{R}_\theta^n)$.

Lemma 5.10. *The operator $U(t)$ has the following estimates:*

(i) $\|U(t)f\|_{A_{1/2}} \lesssim \|f\|_{A_{1/2}}$

(ii) $\|U(t)U(s)^* f\|_{A_0} \lesssim |t-s|^{-n/2}\|f\|_{A_1}$

Proof. (i) follows from Plancherel's theorem and (ii) is given in Example 3.17. □

Combining Lemma 5.10 with Theorem 5.8, we get the following result:

Corollary 5.11. *The operator $U(t)$ on quantum euclidean space $\mathcal{R}_\theta^n$ has the Strichartz estimates:*

$$\|U(t)f\|_{L_t^q A_{1/r}} \lesssim \|f\|_{A_{1/2}}$$
$$\|\int_{t>s} U(t)(U(s))^* F(s)ds\|_{L_t^q A_{1/r}} \lesssim \|F\|_{L_t^{\tilde{q}'} A_{1/\tilde{r}'}}$$

hold for all sharp $n/2$-admissible exponent pairs $(q,r), (\tilde{q},\tilde{r})$.

Corollary 5.12. *(q,r) and $(\tilde{q},\tilde{r})$ are two pairs with sharp $n/2$-admissible conditons for $n \geq 1, r, \tilde{r}<\infty$. If u is a (weak) solution to the problem*

$$\begin{cases} (i\frac{\partial}{\partial t} + \Delta)u(t,x) = F(t,x), (t,x) \in [0,T] \times \mathcal{R}_\theta^n \\ u(0,\cdot) = f \end{cases}$$

for some data f, F and time $0<T<\infty$*, then*

$$\|u\|_{L^q([0,T];A_{1/r})} \lesssim \|f\|_{A_{1/2}} + \|F\|_{L^{\tilde{q}'}([0,T];A_{1/\tilde{r}'})}.$$

Proof. Accoring to [KT98], the Schrödinger problem can by solved as follows:

$$u = Sf - i\mathcal{G}F$$

with $S(t)(f) = \chi_{[0,T]}U(t)f$ and $\mathcal{G}F(t) = \int_{t>s}\chi_{[0,T]}(t,s)U(t)U(s)^*F(s)ds$. Then by applying above Corollary 5.11, we obtain

$$\begin{aligned}
\|u\|_{L^q([0,T];A_{1/r})} &\leq \|S(t)f\|_{L^q([0,T];A_{1/r})} + \|\mathcal{G}F\|_{L^q([0,T];A_{1/r})} \\
&= \|\chi_{[0,T]}(t)U(t)f\|_{L^q([0,T];A_{1/r})} + \|\chi_{[0,T]}(t)\int_{t>s}U(t)U(s)^*F(s)ds\|_{L^q([0,T];A_{1/r})} \\
&\lesssim \|f\|_{A_{1/2}} + \|F\|_{L^{\tilde{q}'}([0,T];A_{1/\tilde{r}'})}.
\end{aligned}$$

□

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