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GENERICALLY NONDEGENERATE POISSON STRUCTURES
AND THEIR LIE ALGEBROIDS

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DISSERTATION

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Abstract

In this dissertation, generically nondegenerate Poisson manifolds are studied by lifting them to a Lie algebroid where they can be understood as nondegenerate. This allows standard tools of symplectic geometry to be applied to concretely describe the behavior of the Poisson structure. This study encompasses various Poisson structures and Lie algebroids previously studied in the literature while also developing several new types. The powerful language of Lie algebroids is applied to the computation of Poisson cohomology in a novel way and to the classification of new classes of compact oriented Poisson surfaces.

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List of Symbols

(M, D)	A smooth manifold M and a divisor D .
D	A divisor on a manifold M , that is a set of smooth hypersurfaces $Z \subset M$.
Z defining function	A defining function for a hypersurface $Z \subset M$, usually denoted x . That is, $x \in \mathcal{C}^\infty(M)$ such that $Z = \{p \in M : x(p) = 0\}$ and $dx(p) \neq 0$ for all $p \in Z$.
$\Gamma(E)$	Smooth sections of a vector bundle $E \rightarrow M$.
$(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$	A Lie algebroid over a manifold M , that is, a vector bundle $\mathcal{A} \rightarrow M$, a Lie bracket $[\cdot, \cdot]_{\mathcal{A}}$ on the $\mathcal{C}^\infty(M)$ -module of sections $\Gamma(\mathcal{A})$, and a bundle map $\rho_{\mathcal{A}} : \mathcal{A} \rightarrow TM$ such that $[X, fY] = \mathcal{L}_{\rho_{\mathcal{A}}(X)}f \cdot Y + f[X, Y]$ for $X, Y \in \Gamma(\mathcal{A})$, $f \in \mathcal{C}^\infty(M)$.
${}^{\mathcal{A}}\Omega^k(M)$	The set $\Gamma(\wedge^k \mathcal{A}^*)$ of smooth sections of the k -th exterior power of the dual bundle to $\mathcal{A}$, called the $\mathcal{A}$ -forms on M . This is a differential complex with differential operator defined $(d_{\mathcal{A}}\beta)(\alpha_0, \alpha_1, \dots, \alpha_k) = \sum_{i=0}^k (-1)^i \rho_{\mathcal{A}}(\alpha_i) \cdot \beta(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \beta([\alpha_i, \alpha_j]_{\mathcal{A}}, \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k)$ for $\beta \in {}^{\mathcal{A}}\Omega^k(M)$, and $\alpha_0, \dots, \alpha_k \in \Gamma(\mathcal{A})$.
bTM	b-tangent bundle (often called log tangent bundle) over a pair (M, D) , the vector bundle whose sections are the vector fields on M that are tangent to Z for all $Z \in D$.
${}^0\mathcal{A}$	The vector bundle over (M, Z) whose sections are the sections of $\mathcal{A}$ that are zero at hypersurface Z .
${}^{sc}TM$	Scattering-tangent bundle over a pair (M, Z) , the vector bundle whose sections are sections of ${}^bTM \rightarrow M$ that are zero at Z .
$\mathcal{R}_\pi$	The rigged Lie algebroid of an almost regular Poisson structure (M, π) , which assists in computing Poisson cohomology.

Chapter 1

Introduction

Poisson geometry grew out of a mathematical formulation of classical mechanics and bridges many disparate areas of pure mathematics: for instance integrable systems, representation theory, quantum groups, noncommutative geometry, and singularity theory (p. xii, [12]). Although Poisson geometry emerged two centuries ago in the work of Joseph-Louis Lagrange, as well as Siméon Poisson and Sophus Lie, almost all work on this topic has occurred in the past 30 years and there remains ample room to explore the subject. The primary goal of this thesis is to enlarge the class of Poisson structures that can be studied using standard tools of symplectic geometry. Considering Poisson structures through a symplectic lens has been quite fruitful in a variety of cases for many mathematicians. The method was first used by Ryszard Nest and Boris Tsygan in [41] to study what are called b -symplectic structures; Alan Weinstein pointed out that this is an example of what is called a symplectic Lie algebroid. For further examples following this philosophy, see the works of [6, 7, 17, 18, 19, 21, 22, 30, 47] and [48].

We can unpack the goal statement above by first considering the phrase ‘adapt the tools of symplectic geometry’. A symplectic manifold is an example of a Poisson manifold, one which we call non-degenerate. In this case, the symplectic structure gives a map from the tangent bundle to the cotangent bundle and the Poisson structure gives the inverse map. In this sense, symplectic and non-degenerate Poisson structures are equivalent to one another. These vector bundle maps allow us to carry standard symplectic techniques, such as ‘Moser’s trick’, from the cotangent bundle over to the tangent bundle. To recover this duality with more general Poisson structures, we will replace the tangent bundle by building new vector bundles. These new bundles capture, or encode, how far away the Poisson structure is from

a symplectic one and allow us to view the Poisson structure as non-degenerate. The second key idea we will explore is how to ‘understand Poisson manifolds’. Rigorously, this means to have a concrete understanding of the invariants associated to a Poisson manifold.

1.1 Poisson and symplectic geometry

A *Poisson manifold* (M, π) is a manifold M equipped with a bivector $\pi \in \mathcal{C}^\infty(M; \wedge^2 TM)$ that satisfies the non-linear partial differential equation $[\pi, \pi] = 0$, expressed using a Lie bracket on multivector fields that extends the standard Lie bracket of vector fields. A Poisson bivector defines a skew homomorphism

$$\pi^\sharp : T^*M \rightarrow TM \text{ by the contraction } \xi \mapsto \pi(\xi, \cdot).$$

When this map is an isomorphism, we say that π is *non-degenerate* and the Poisson bivector π can be obtained from a symplectic form: the inverse map

$$\omega^\flat := (\pi^\sharp)^{-1} : TM \rightarrow T^*M$$

is given by contraction with a symplectic differential 2-form $\omega \in \Omega^2(M)$:

$$V \mapsto \omega(V, \cdot).$$

The isomorphisms given by contraction with π and ω are extremely useful, allowing us to explicitly compute standard Poisson invariants.

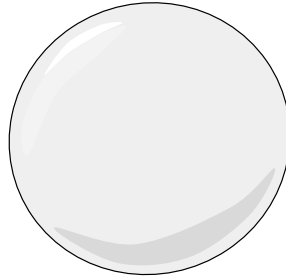
Example 1. *Given a surface S with a bivector π , $[\pi, \pi] = 0$ is trivially satisfied, so every bivector is Poisson. Consider the sphere $\mathbb{S}^2$, depicted in Figure 1, with coordinates*

$0 \leq \theta < 2\pi$, and $0 \leq \rho < \pi$. The bivector

$$\pi = \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \rho}$$

is non-degenerate with corresponding symplectic form $\omega = d\rho \wedge d\theta$.

Figure 1. *Sphere $\mathbb{S}^2$
Degeneracy locus $D = \emptyset$*



We will consider structures (M, π) that are non-degenerate away from a submanifold (or a collection of submanifolds) D of M . In other words, the map $\pi^\sharp : T^*M \rightarrow TM$ - described above - has non-trivial kernel and fails to be an isomorphism at D . We say

π is *degenerate* at D . We colloquially call these structures (M, D, π) *minimally degenerate Poisson manifolds*; a rigorous definition of the structures we consider will be given in Section 2.1. For illustration purposes, consider the following examples.

Example 2. Consider the torus with global coordinates $0 \leq \theta < 2\pi, 0 \leq \rho < 2\pi$. The bivector

$$\pi = \sin^2(\theta) \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \rho}$$

defines a minimally degenerate Poisson structure. The bivector π is non-degenerate except on two disjoint circles that occur at

$$\theta = 0 \text{ and } \theta = \pi,$$

shown in Figure 2.

Example 3. Consider the sphere with coordinates $0 \leq \theta < 2\pi, 0 \leq \rho < \pi$. Consider the bivector

$$\pi = \sin(2\theta) \left(\sin(\theta) + \cos(\theta) \right) \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \rho}.$$

This Poisson bivector is non-degenerate except on the three intersecting great circles that occur at

$$\sin(\theta) = 0, \cos(\theta) = 0, \sin(\theta) + \cos(\theta) = 0,$$

shown in Figure 3.

Figure 2. Torus $\mathbb{T}^2$
Degeneracy locus $D = \{\mathbb{S}_\theta^1, \mathbb{S}_{\theta-\pi}^1\}$

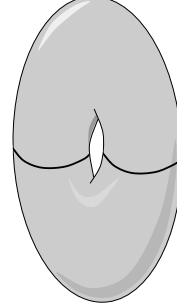
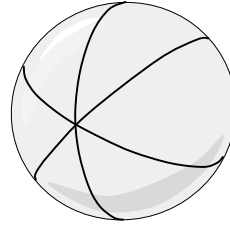


Figure 3. Sphere $\mathbb{S}^2$
Degeneracy locus
 $D = \{\mathbb{S}_{\sin \theta}^1, \mathbb{S}_{\cos \theta}^1, \mathbb{S}_{\sin \theta + \cos \theta}^1\}$



We have restricted our attention here to two-dimensional manifolds purely for sake of illumination. More generally throughout this dissertation, we will consider manifolds of arbitrary dimension with such Poisson structures. These can be quite complicated and even locally do not simply arise as the product of surfaces such as those seen in Examples 1 through 3. For example, we will thoroughly analyze the structure

$$\pi = x_1^3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial y_1} \wedge \left(y_2 \frac{\partial}{\partial y_2} + x_2 \frac{\partial}{\partial x_2} \right) + x_1^2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}$$

on $\mathbb{R}^4$ with coordinates x_1, y_1, x_2, y_2 .

We further note that in each of the above examples, we considered a degeneracy locus D

that had co-dimension 1 inside M . All results and examples in this dissertation will concern this particular case. Our methods do extend to other codimensions, as we will show in future work. For instance, consider the following example of a Poisson torus where D has co-dimension 2.

Example 4. *Again consider the torus with global coordinates $0 \leq \theta < 2\pi, 0 \leq \rho < 2\pi$.*

The bivector

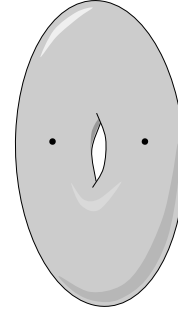
$$\pi = (\sin^2(\theta) + \sin^2(\rho)) \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \rho}$$

defines a Poisson structure. This example demonstrates that the submanifold D need not have co-dimension 1. Here, the bivector on the torus is non-degenerate except for at 4 distinct points (θ, ρ) :

$$(0, 0), (0, \pi), (\pi, 0), \text{ and } (\pi, \pi),$$

two of which are shown in Figure 4. The other two appear on the hidden side of the torus.

Figure 4. *Torus $\mathbb{T}^2$
Degeneracy locus $D = \{p_1, p_2, p_3, p_4\}$*



1.2 Main contributions

1.2.1 New Lie algebroids.

The use of Lie algebroids to understand Poisson structures dates back at least to the work of Nest-Tsygan mentioned above. Our work will involve both Lie algebroids in current use and others previously unstudied in this context.

Melrose rescaling. We construct new Lie algebroids by adapting Melrose rescaling - a method introduced by Richard Melrose for building vector bundles - which we fully explain in Section 2.6.

Example 5. *We view the Poisson structure on the torus in Example 2 as non-degenerate on a Lie algebroid called the “zero-tangent bundle”. In this case, this vector bundle - defined*

in full generality in 2.6.1 - is locally generated by

$$\sin(\theta) \frac{\partial}{\partial \theta} \text{ and } \sin(\theta) \frac{\partial}{\partial \rho}.$$

Fiber bundle construction theorem. When a bundle can not be constructed using Melrose rescaling, we next turn to the fiber bundle construction theorem. If we provide a fiber-wise description of the bundle over a base, and provide a suitable set of transition functions, then we have built a vector bundle. See Section 2.6 for full details.

Example 6. We view the sphere in Example 3 as non-degenerate on the star log tangent bundle. Consider the set of three lines $x = 0$, $y = 0$, and $x + y = 0$ that intersect at the origin in $\mathbb{R}^2$. All vector fields that are tangent to these three lines can be formed using smooth $C^\infty(\mathbb{R}^2)$ -linear combination of the generators

$$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \text{ and } (x + y) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right).$$

This is an example of a star log tangent bundle.

1.2.2 Normal forms & classifications

Once we have a Lie algebroid where we can view our minimally degenerate Poisson structures as non-degenerate, we generalize certain tools of symplectic geometry to the Lie algebroid.

Moser. We use the standard Moser technique to prove a Darboux theorem for certain algebroid symplectic structures.

Example 7. We show that all Poisson structures that can be expressed non-degenerately in the generators from Example 5 are locally expressible as the bi-vector given in Example 2.

Classification of surfaces. As is well-known, given two symplectic forms ω_0 and ω_1 (corresponding to two non-degenerate Poisson bivectors) on a compact surface S , there is a diffeomorphism

$$\phi : S \rightarrow S \text{ such that } \phi^* \omega_0 = \omega_1$$

if and only if ω_0 and ω_1 represent the same degree-2 de Rham cohomology class:

$$[\omega_0] = [\omega_1] \in H^2(S).$$

The classification of Poisson bivectors becomes much more complex when the Poisson structure can degenerate and in general there is no known classification. In 2002, Olga Radko [44, 45] gave a classification of Poisson structures that degenerate linearly at a set of transversely intersecting curves. We generalize this classification scheme to the case of Poisson structures π that degenerate linearly at a set of pairwise transverse curves. The sphere in Example 3 is such a Poisson surface. Instead of classifying using the de Rham cohomology of the tangent bundle, we use the Lie algebroid cohomology of a Lie algebroid called the b -tangent bundle.

In the case when at least three curves intersect at a point, we need further information. Given a vector bundle $E \rightarrow S$, an *orientation form* is a nowhere-vanishing section β of $\wedge^{\text{rank} E} E^*$. Two orientation forms β and β' are said to induce the same *orientation* if $\beta = f\beta'$ for a positive function $f \in \mathcal{C}^\infty(S)$. In our classification scheme, we require that two b -symplectic forms induce the same orientation of the b -tangent bundle.

Theorem 1. (Classification of b -Poisson Surfaces) *Given two b -symplectic forms ω_0 and ω_1 on a compact surface S that degenerate on the same set of pairwise transversely intersecting curves, there is a diffeomorphism*

$$\phi : S \rightarrow S \text{ such that } \phi^* \omega_0 = \omega_1$$

if and only if ω_0 and ω_1 represent the same degree-2 Lie algebroid cohomology class

$$[\omega_0] = [\omega_1] \in {}^b H^2(S)$$

and ω_0 and ω_1 induce the same orientation.

For each of these surfaces, we provide an explicit description of the cohomology ${}^b H^*(S)$. In the sphere example, we have ${}^b H^2(\mathbb{S}^2) \simeq H^2(\mathbb{S}^2) \oplus [H^1(\mathbb{S}^1)]^3 \oplus H^0(\{p\})^6$.

1.2.3 Computing Poisson cohomology

The Poisson bivector determines a differential on multi-vector fields. This complex $(\mathcal{V}^*, d_\pi)$ is the Lichnerowicz complex and its homology groups are Poisson cohomology. Poisson cohomology is an important invariant in the study of Poisson structures. Unfortunately, the computation of Poisson cohomology is quite difficult in general and explicit results are

known in only very select cases ([12], p.43). The simplest case is that of a symplectic manifold, where the Poisson bivector π is non-degenerate and the Poisson cohomology is isomorphic to the de Rham cohomology. The non-degeneracy of π allows us to define an isomorphism between the tangent bundle and the cotangent bundle that intertwines the respective differentials. This induces an isomorphism of complexes,

$$(\mathcal{V}^*, d_\pi) \begin{array}{c} \xrightarrow{\omega^\flat} \\ \xleftarrow{\pi^\sharp} \end{array} (\Omega^*, d_{dR})$$

In order to compute the Poisson cohomology of a minimally degenerate Poisson manifold, we develop and present a new way of computing Poisson cohomology for this type of structure inspired by this isomorphism. We define an isomorphism between the Lichnerowicz complex and the de Rham complex of a Lie algebroid $\mathcal{R}$ called the *rigged*¹ algebroid. Of course, the cohomology of the Lichnerowicz complex can be viewed as the Lie algebroid cohomology of the Poisson Lie algebroid $(T^*M, \pi^\sharp)$. However, viewing the computation in terms of $\mathcal{R}$ -forms turns the intractable differential d_π into a computation using the better understood standard exterior derivative d .

Example 8. *The Poisson torus in Example 2 has Poisson cohomology*

$$H_\pi^0(\mathbb{T}^2) \simeq H^0(\mathbb{T}^2), \quad H_\pi^1(\mathbb{T}^2) \simeq H^1(\mathbb{T}^2) \oplus [H^0(\mathbb{S}^1) \oplus \Omega^0(\mathbb{S}^1)]^2, \text{ and} \\ H_\pi^2(\mathbb{T}^2) \simeq H^2(\mathbb{T}^2) \oplus [H^1(\mathbb{S}^1) \oplus \Omega^1(\mathbb{S}^1)]^2.$$

The Poisson Sphere in Example 3 has Poisson cohomology

$$H_\pi^0(\mathbb{S}^2) \simeq H^0(\mathbb{S}^2), \quad H_\pi^1(\mathbb{S}^2) \simeq H^1(\mathbb{S}^2) \oplus [H^0(\mathbb{S}^1)]^3, \text{ and} \\ H_\pi^2(\mathbb{S}^2) \simeq H^2(\mathbb{S}^2) \oplus [H^1(\mathbb{S}^1)]^3 \oplus H^0(\{p\})^{18}.$$

¹The name is inspired by rigged Hilbert spaces from functional analysis.

Chapter 2

Almost regular Poisson structures & their Lie algebroids

In this chapter, we will discuss relevant background information and lay the ground work for the particular types of Poisson structures discussed in later chapters. Throughout this chapter we will return to the following simple example to illustrate key concepts. This ‘pocket example’ will serve not only as elucidation, but also as a sanity check.

Example 9. *On $\mathbb{R}^2$, fix standard coordinates x, y . We will consider the Poisson bivector*

$$\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

2.1 Almost regular Poisson structures

Almost regular Poisson structures were introduced by Iakovos Androulidakis and Marco Zambon in [2]. These are a large class of structures that includes b -symplectic (log-symplectic), b^k -symplectic, scattering-symplectic, and zero-symplectic structures, structures which will be our primary focus in Chapter 3. Note that Androulidakis and Zambon provide an equivalent definition to the one provided here. However, as personal preference, we will simply define these objects via their characterization theorem.

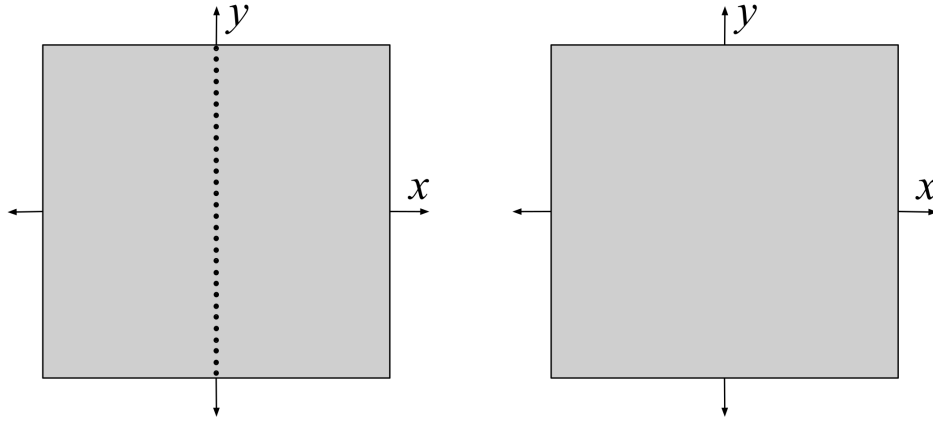
Definition 1. *(Theorem B, [2].) A Poisson manifold (M, π) is **almost regular** if*

1. *the subset M_{reg} where $\pi^\sharp$ has maximal rank is dense in M , and*
2. *there is a unique integrable distribution D of M , such that at every $x \in M_{reg}$ the symplectic leaf of (M, π) at x is an open subset of the leaf of D at x .*

Note that the leaves of the distribution D are Poisson submanifolds of (M, π) .

Example 10. Our pocket Poisson structure $\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ is almost regular. The symplectic foliation of π has two open leaves: the open half planes $\{(x, y) \in \mathbb{R}^2 | x > 0\}$ and $\{(x, y) \in \mathbb{R}^2 | x < 0\}$. The closed leaves are the points in the vertical line $x = 0$. Since the maximal rank of π is 2, the set $M_{reg} = \mathbb{R}^2 \setminus \{x = 0\}$. Because this set is dense in $\mathbb{R}^2$, the bivector π satisfies condition 1 in the definition of almost regular given above.

Figure 5. Symplectic foliation of π and foliation of $D = T\mathbb{R}^2$.



To verify condition 2, we take the distribution $D = T\mathbb{R}^2$. The foliation of D is a single leaf, the entire plane $\mathbb{R}^2$. Since the set M_{reg} is an open subset of $\mathbb{R}^2$, we have verified that π is almost regular.

A symplectic manifold is trivially almost regular. The Poisson surfaces in Examples 2, 3, and 4 in Chapter 1 are all almost regular with distribution D being the entire tangent bundle. We will primarily be concerned with structures like these, called *generically symplectic* Poisson structures, where D is the tangent bundle. In other words, generically symplectic structures are almost regular Poisson structures (M, π) where the maximal rank of $\pi^\#$ is the dimension of M .

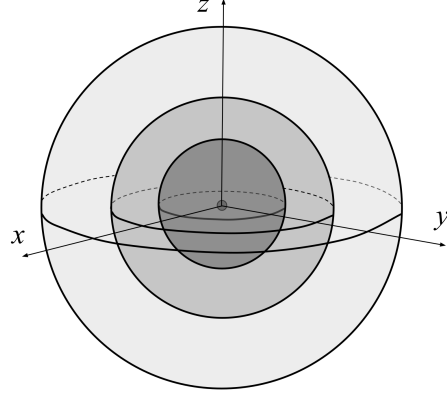
While almost regular structures are a quite large subclass of all Poisson manifolds, there are some very ‘naturally’ occurring structures that fail to be almost regular. Considering such cases illuminates the role of D in the definition.

Example 11. ($\mathfrak{so}(3)$) Consider $\mathbb{R}^3$ with standard coordinates x, y, z and Poisson structure

$$\pi = x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The associated foliation is comprised of leaves that are the spheres $x^2 + y^2 + z^2 = r$ for each real number $r > 0$ and a leaf that is the origin. Note that in this case M_{reg} is $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Thus this structure does satisfy the first condition of definition 1. However, there is no integrable distribution satisfying condition 2: There are no rank 2 foliations with each sphere centered at the origin occurring as a leaf.

Figure 6. Symplectic foliation of $\mathfrak{so}(3)$

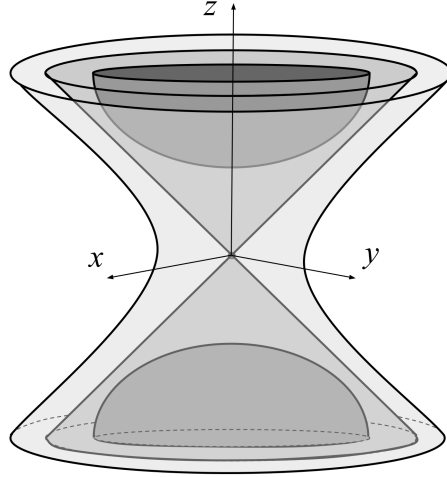


Example 12. ($\mathfrak{sl}(2, \mathbb{R})$) Consider $\mathbb{R}^3$ with standard coordinates and Poisson structure

$$\pi = x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} - z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The associated foliation has a single rank 0 leaf: the origin. Every other leaf has rank 2: a one- or two- sheeted hyperboloid. As in the previous example, M_{reg} is $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and thus this structure satisfies the first condition of definition 1. However, as before there is no integrable distribution satisfying condition 2.

Figure 7. Symplectic foliation of $\mathfrak{sl}(2, \mathbb{R})$



We emphasize that almost regular Poisson structures are not simply Poisson structures where the associated symplectic foliation is regular almost everywhere. There must also be some regular foliation that we can ‘superimpose’ on the symplectic foliation. One can think of this as ‘patching’ the leaves of the singular foliation until we obtain a regular foliation whose rank matches the maximal rank of π .

2.2 Lifting to a Lie algebroid

Lie algebroids give a context for us to view a Poisson structure as non-degenerate. They provide a starting point from which we have hope of applying standard techniques of symplectic geometry.

2.2.1 Lie algebroids

We will begin by recounting some facts about Lie algebroids. See Chapter 8 of [12] for more information. Lie algebroids are to Lie groupoids as Lie algebras are to Lie groups. We prefer to think of a Lie algebroid as a vector bundle that can stand in for the tangent bundle. In particular, Lie algebroids have many of the nice properties that we love about the tangent bundle: a Lie bracket and an associated de Rham theory with exterior derivative.

Definition 2. A **Lie algebroid** $(E, [\cdot, \cdot], \rho)$ is a vector bundle E over a manifold M with a Lie bracket $[\cdot, \cdot]$ on its module of sections $\Gamma(E)$ and a vector bundle morphism $\rho : E \rightarrow TM$ - called the **anchor map** - such that

$$[X, fY] = \mathcal{L}_{\rho(X)}f \cdot Y + f[X, Y]$$

for all $X, Y \in \Gamma(E)$, and $f \in C^\infty(M)$.

An immediate consequence of this definition is the compatibility of the Lie algebroid's bracket with the standard Lie bracket on the tangent bundle:

$$\rho([X, Y]) = [\rho(X), \rho(Y)].$$

In fact, the tangent bundle is trivially an example of a Lie algebroid.

Example 13. Given a manifold M , the tangent bundle TM is a Lie algebroid over M with the standard Lie bracket and anchor map given by the identity map.

Further, any vector bundle can be equipped with a Lie algebroid structure.

Example 14. Any vector bundle $E \rightarrow M$ can be equipped with the structure of a Lie algebroid. Simply take $[\cdot, \cdot] \equiv 0$ and the anchor $\rho \equiv 0$.

Lie algebroids whose anchor maps are an isomorphism almost everywhere will prove most

useful for us. One class of Lie algebroids that are quite similar to the tangent bundle are called rescaled tangent bundles.

Example 15. Consider $\mathbb{R}^2$ with fixed standard coordinates x and y . We will examine a few Lie algebroids whose construction we will explain in greater generality in Section 2.6.

b-tangent bundle

The vector fields $x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ generate a rank-2 vector-bundle called the *b-tangent bundle* (introduced by Melrose [35]), denoted ${}^bT\mathbb{R}^2$. This bundle is called a rescaled tangent bundle because its sections are the vector fields of $T\mathbb{R}^2$ that are tangent to the line $Z = \{x = 0\}$. We would say this is the tangent bundle ‘rescaled’ by TZ at Z . We can equip this bundle with a Lie bracket induced by the standard Lie bracket on $\mathbb{R}^2$ and with anchor map given by evaluation in $T\mathbb{R}^2$. In other words, the expression $x\frac{\partial}{\partial x}$ is treated as a formal object and is non-vanishing as a section of ${}^bT\mathbb{R}^2$. However we evaluate the function x when we view the expression $x\frac{\partial}{\partial x}$ as a section of $T\mathbb{R}^2$.

0-tangent bundle

The vector fields $x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}$ also generate a rank-2 vector-bundle ${}^0T\mathbb{R}^2$ called the *zero-tangent bundle* (introduced by Mazzeo-Melrose [32]). This bundle is called a rescaled tangent bundle because its sections are the vector fields of $T\mathbb{R}^2$ that are zero at the line $Z = \{x = 0\}$. We would say this is the tangent bundle ‘rescaled’ by 0 at Z . This is also a Lie algebroid with the standard Lie bracket and anchor map given by evaluation in $T\mathbb{R}^2$.

b^2 -tangent bundle

The vector fields $x^2\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ generate a rank-2 vector-bundle called the *b^2 -tangent bundle* (introduced by Scott [47]) and denoted ${}^{b^2}T\mathbb{R}^2$. As above, we can equip this bundle with a Lie bracket induced by the standard Lie bracket on $\mathbb{R}^2$ and with anchor map by evaluation.

For each rescaled tangent bundle, we use a Lie bracket induced by the standard tangent bundle bracket. This is another important feature that we need when choosing a Lie algebroid for lifting.

Example 16. Again consider $\mathbb{R}^2$ with fixed standard coordinates x, y . The vector fields

$$\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial y}$$

do not generate a Lie algebroid with the standard bracket because

$$\left[\frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial y} \right] = 2x \frac{\partial}{\partial y}$$

which is not a linear combination of the proposed generators. While this bundle does admit other Lie algebroid structures, we will focus on Lie algebroids with a bracket that is “the same” as the standard tangent bundle bracket so that a bivector on the algebroid is Poisson (i.e. $[\pi, \pi] = 0$) “in the same way” as on the tangent bundle.

2.2.2 $\mathcal{A}$ -multivector fields

Given a Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$, we can build **$\mathcal{A}$ -multivector fields**. These are

$$\mathcal{AV}^k(M) = \mathcal{C}^\infty(M; \wedge^k \mathcal{A})$$

and are analogous to multi vectorfields constructed using $\mathcal{A} = TM$. We can define an $\mathcal{A}$ -Schouten bracket for these sections with a slight modification of the standard formula: The bracket of two $\mathcal{A}$ -multivector fields is given by

$$[a_1 \cdots a_m, b_1 \cdots b_n] = \sum_{i,j} (-1)^{i+j} [a_i, b_j] a_1 \cdots a_{i-1} a_{i+1} \cdots a_m b_1 \cdots b_{j-1} b_{j+1} \cdots b_n$$

for $\mathcal{A}$ -vector fields a_i, b_j and where $[a_i, b_j]$ denotes the $\mathcal{A}$ -bracket. The bracket with a function f is $[f, a_1 \cdots a_m] = -i_{df}(a_1 \cdots a_m)$ for $\mathcal{A}$ -vector fields a_i where i_{df} denotes the interior product.

This bracket provides enough structure for us to define an $\mathcal{A}$ -Poisson structure on $\mathcal{A}$.

Definition 3. Given a Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$, an $\mathcal{A}$ -bivector $\pi \in \mathcal{AV}^2(M)$ is **Poisson** if $[\pi, \pi] = 0$ where $[\cdot, \cdot]$ is the $\mathcal{A}$ -Schouten bracket.

2.2.3 Lifting

Given a Poisson structure (M, π) , we want to find an algebroid $\mathcal{A}$ where we can ‘view’ π as non-degenerate. Rigorously, we do this through lifts.

Definition 4. Given a Poisson manifold (M, π) and algebroid $\mathcal{A} \rightarrow M$, we say that π is **$\mathcal{A}$ -Poisson** if there exists a Poisson bivector $\pi_{\mathcal{A}} \in \mathcal{C}^\infty(M; \wedge^2 \mathcal{A})$ such that $\rho(\pi_{\mathcal{A}}) = \pi$. The bivector $\pi_{\mathcal{A}}$ is called an **$\mathcal{A}$ -lift** of π .

Example 17. The structure $\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ admits lifts to the b -, 0 -, and b^2 -tangent bundles, introduced in Example 15. To make clear that these are different lifts, we suggestively write

$$\pi_b = x \left(x \frac{\partial}{\partial x} \right) \wedge \frac{\partial}{\partial y}$$

for the lift to the b -tangent bundle,

$$\pi_0 = \left(x \frac{\partial}{\partial x} \right) \wedge \left(x \frac{\partial}{\partial y} \right)$$

for the lift to the 0 -tangent bundle, and

$$\pi_{b^2} = \left(x^2 \frac{\partial}{\partial x} \right) \wedge \frac{\partial}{\partial y}$$

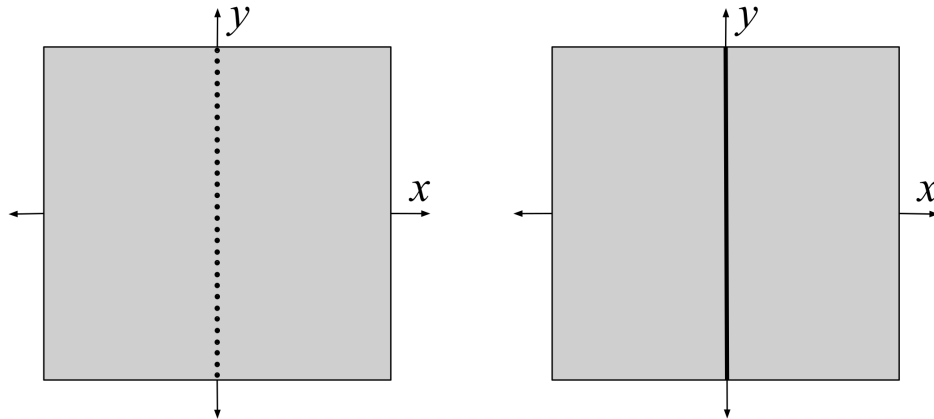
for the lift to the b^2 -tangent bundle. Note that π_{b^2} and π_0 are non-degenerate Poisson structures on their respective algebroids, meaning the sharp maps $\pi_{b^2}^\sharp : {}^{b^2}T^*\mathbb{R}^2 \rightarrow {}^{b^2}T\mathbb{R}^2$ and $\pi_0^\sharp : {}^0T^*\mathbb{R}^2 \rightarrow {}^0T\mathbb{R}^2$ are isomorphisms. On the other hand π_b vanishes at the line $\{x = 0\}$ and is degenerate as a Poisson structure on the b -tangent bundle.

Characteristic foliation of a Lie algebroid.

Given a Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$ over M , we can construct a singular foliation on M , called the characteristic foliation, given by the distribution $\rho(\mathcal{A}) \subseteq TM$. For more information on the properties of this foliation, see section 8.1.4 of [12].

Example 18. We will consider the characteristic foliations of the Lie algebroids introduced in Example 15.

Figure 8. Characteristic foliation of ${}^0T\mathbb{R}^2$ to the left with ${}^bT\mathbb{R}^2$ and ${}^{b^2}T\mathbb{R}^2$ on the right.



Since the 0-tangent bundle is generated by $x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}$, away from $x = 0$, the image of the anchor map is the entire tangent bundle. At $x = 0$, the image is zero. Thus the foliation consists of the 2 open half planes of $\mathbb{R}^2 \setminus \{x = 0\}$ and the points along the line $x = 0$.

Since the b -tangent bundle is generated by $x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, away from $x = 0$, the image of the anchor map is the entire tangent bundle. At $x = 0$, the image is $\frac{\partial}{\partial y}$. Thus the foliation consists of the 2 open half planes of $\mathbb{R}^2 \setminus \{x = 0\}$ and the line $x = 0$. For similar reasons, this is also the characteristic foliation of the b^2 -tangent bundle.

Relating characteristic foliations to symplectic foliations. Given a Poisson manifold (M, π) , we will see that if we can lift π to a Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$, then the characteristic foliation of $\mathcal{A}$ “combines” the leaves of the Poisson foliation. We describe this in an example and refer to [22] for full details about this phenomenon.

Example 19. The Poisson bivector $\pi = x^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ lifts to the b^2 - and 0-tangent bundles. By comparing the symplectic foliation of π - depicted in Figure 5 - to the characteristic foliation of the b^2 -tangent bundle - depicted in Figure 8 - we see that the latter combines the points in $\{x = 0\}$ to make the leaf $x = 0$. On the other hand, sometimes no leaves of the symplectic foliation are combined to create leaves of the characteristic foliation, as seen by comparing the characteristic foliation of the 0-tangent bundle to the symplectic foliation of π . Note that it is not enough to simply compare foliations to determine ‘liftability’. The structure $\pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ has the same foliation as the 0-tangent bundle, but π does not admit a lift to the 0-tangent bundle.

2.2.4 Anchored Lie algebroids

In Example 17, notice that π lifted to π_b . However π_b was degenerate. To obtain non-degenerate lifts for π , we considered π_{b^2} and π_0 . This is an example of a more general phenomenon where we lift a bivector in stages. To do this, we ‘stack’ Lie algebroids.

Definition 5. ([21], Section 2.6). Let $(\mathcal{B}, [\cdot, \cdot], \rho_{\mathcal{B}})$ be a Lie algebroid of a manifold M . Another Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho_{\mathcal{A}})$ is **$\mathcal{B}$ -anchored** if there exists a Lie algebroid morphism

$\phi : \mathcal{A} \rightarrow \mathcal{B}$ over the same base covering the identity:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi} & \mathcal{B} \\ & \searrow \rho_{\mathcal{A}} \quad \swarrow \rho_{\mathcal{B}} & \\ & TM & \end{array}$$

We call ϕ the **factoring map**.

Example 20. Continuing with Example 2.6, the anchor map ρ_{b^2} of the b^2 -tangent bundle factors through the anchor map of the b -tangent bundle by the map ϕ :

$$\left(x^2 \frac{\partial}{\partial x}\right) \xrightarrow{\phi} x \left(x \frac{\partial}{\partial x}\right) \xrightarrow{\rho_b} x^2 \left(\frac{\partial}{\partial x}\right).$$

Since

$$[\partial_y, (x^2 \partial_x)] = [\partial_y, x(x \partial_x)] = [\partial_y, x^2(\partial_x)] = 0$$

and

$$[(x^2 \partial_x), (x^2 \partial_x)] = [x(x \partial_x), x(x \partial_x)] = [x^2(\partial_x), x^2(\partial_x)] = 0,$$

this factoring map preserves the Lie brackets and is in fact a Lie algebroid morphism.

This is an example of a more general phenomenon pointed out by Geoffrey Scott in Section 3.3 of [47] where, given $m > \ell$, a vector field of the form $x^m \frac{\partial}{\partial x}$ can be viewed as $x^{m-\ell} \left(x^\ell \frac{\partial}{\partial x}\right)$.

The fact that we can anchor a Lie algebroid with another will be extremely useful in the computation of an invariant called Lie algebroid cohomology.

2.3 Lie algebroid cohomology.

To every algebroid there is an associated cohomology theory. Recall, for any Lie algebroid $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ over M , with dual $\mathcal{A}^*$, the degree k $\mathcal{A}$ -forms are

$${}^{\mathcal{A}}\Omega^k(M) = \Gamma(\wedge^k \mathcal{A}^*),$$

the sections of the k th exterior power of the dual bundle $\mathcal{A}^*$. The differential operator $d_{\mathcal{A}}$ acting on ${}^{\mathcal{A}}\Omega^*(M)$, $d_{\mathcal{A}} : {}^{\mathcal{A}}\Omega^k(M) \rightarrow {}^{\mathcal{A}}\Omega^{k+1}(M)$ is defined by

$$\begin{aligned} (d_{\mathcal{A}}\beta)(\alpha_0, \alpha_1, \dots, \alpha_k) &= \sum_{i=0}^k (-1)^i \rho_{\mathcal{A}}(\alpha_i) \cdot \beta(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \beta([\alpha_i, \alpha_j]_{\mathcal{A}}, \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k) \end{aligned}$$

for $\beta \in {}^{\mathcal{A}}\Omega^k(M)$, and $\alpha_0, \dots, \alpha_k \in \Gamma(\mathcal{A})$. This is a complex whose cohomology is called the Lie algebroid cohomology of $\mathcal{A}$ or $\mathcal{A}$ -cohomology.

We are now capable of generalizing the notion of a symplectic structure to any Lie algebroid.

Definition 6. *Given a rank $2k$ Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$ over a manifold M , an $\mathcal{A}$ -symplectic structure is a degree 2 $\mathcal{A}$ -form $\omega \in {}^{\mathcal{A}}\Omega^2(M)$ such that $d_{\mathcal{A}}\omega = 0$ and $\wedge^k \omega$ is a nowhere vanishing section of ${}^{\mathcal{A}}\Omega^{2k}(M)$.*

When $\mathcal{A} = TM$ with the standard Lie bracket, we recover the definition of a symplectic form.

Example 21. *Consider the Lie algebroids introduced in Example 15.*

b-forms

The b-one forms are generated by the co-vectors $\frac{dx}{x}$ and dy . The exterior derivative for this complex is given by the standard exterior derivative of the tangent bundle away from $\{x = 0\}$. By continuity, this description extends to all of $\mathbb{R}^2$. The form $\omega = \frac{dx}{x} \wedge dy$ is a b-symplectic structure on $\mathbb{R}^2$.

0-forms

The 0-one forms are generated by $\frac{dx}{x}$ and $\frac{dy}{x}$. As with the b-forms, the 0-exterior derivative is given by the standard d . Interestingly, in this case there exists no choice of local generators that are all closed. Consider $d\left(\frac{dy}{x}\right) = -\frac{dx}{x} \wedge \frac{dy}{x}$. The form $\omega = \frac{dx}{x} \wedge \frac{dy}{x}$ is a 0-symplectic structure on $\mathbb{R}^2$.

b²-forms

The b²-one forms are generated by $\frac{dx}{x^2}$ and dy . The form $\omega = \frac{dx}{x^2} \wedge dy$ is a b²-symplectic structure on $\mathbb{R}^2$.

2.3.1 Techniques for computation

Once we have identified a Lie algebroid where we can lift a Poisson bivector of interest, we need an explicit description of its Lie algebroid cohomology. In this section we will briefly discuss the computational techniques that we have found useful.

Split short exact sequences. Recall that an $\mathcal{A}$ -anchored Lie algebroid $(\mathcal{B}, [\cdot, \cdot]_{\mathcal{B}}, \rho_{\mathcal{B}})$, admits a ‘factoring’ of its anchor map through the anchor map of the Lie algebroid $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$: there exists a Lie algebroid map $\phi : \mathcal{B} \rightarrow \mathcal{A}$ such that $\rho_{\mathcal{B}} = \rho_{\mathcal{A}} \circ \phi$. We will only be considering instances when the factoring map ϕ is an isomorphism away from a collection of smooth hypersurfaces $Z \subset M$. In these cases we compute the Lie algebroid cohomology of $\mathcal{B}$ by forming a short exact sequence of complexes:

$$0 \rightarrow {}^{\mathcal{A}}\Omega^k(M) \rightarrow {}^{\mathcal{B}}\Omega^k(M) \rightarrow \mathcal{C}^k(\mathcal{B}, \mathcal{A}) \rightarrow 0 \quad (2.1)$$

where $\mathcal{C}^k(\mathcal{B}, \mathcal{A})$ is the quotient ${}^{\mathcal{B}}\Omega^k(M)/{}^{\mathcal{A}}\Omega^k(M)$. The quotient complex inherits a differential $d_{\mathcal{C}}$ induced from the differential $d_{\mathcal{B}}$ on ${}^{\mathcal{B}}\Omega^k(M)$. Indeed, if P is the projection $P : {}^{\mathcal{B}}\Omega^k(M) \rightarrow \mathcal{C}^k(\mathcal{B}, \mathcal{A})$, then $d_{\mathcal{C}}(\eta) = P(d_{\mathcal{B}}(\theta))$ where $\theta \in {}^{\mathcal{B}}\Omega^k(M)$ is any form such that $P(\theta) = \eta$.

This short exact sequence allows us to break the computation of cohomology into a series of tractable bite-sized pieces. In particular, the short exact sequence induces a long exact sequence in cohomology:

$$\dots \xrightarrow{\partial} {}^{\mathcal{A}}H^n(M) \rightarrow {}^{\mathcal{B}}H^n(M) \rightarrow H^n(\mathcal{C}) \xrightarrow{\partial} {}^{\mathcal{A}}H^{n+1}(M) \rightarrow \dots$$

When the short exact sequence above is split, the boundary map ∂ is trivial and we have that

$${}^{\mathcal{B}}H^k(M) \simeq {}^{\mathcal{A}}H^k(M) \oplus H^k(\mathcal{C}).$$

Example 22. Consider $\mathbb{R}^2$ with standard coordinates x, y . The vector fields

$$x^2 \frac{\partial}{\partial x}, \text{ and } x \frac{\partial}{\partial y}$$

generate a Lie algebroid, called the scattering tangent bundle, which we will define in full generality in Section 2.6. The anchor map is given by evaluation and the Lie bracket is induced by the standard bracket on $T\mathbb{R}^2$. The corresponding degree 1 scattering-forms are

$$\frac{dx}{x^2}, \text{ and } \frac{dy}{x}.$$

Note that the anchor map for this Lie algebroid factors through the anchor of ${}^{b^2}T\mathbb{R}^2$, the algebroid locally generated by the vector fields

$$x^2 \frac{\partial}{\partial x}, \text{ and } \frac{\partial}{\partial y}.$$

Thus we can form a short exact sequence

$$0 \rightarrow {}^{b^2}\Omega^k(M) \rightarrow {}^{sc}\Omega^k(M) \rightarrow \mathcal{C}^k(sc, b^2) \rightarrow 0.$$

However this quotient is not optimal for computing. Note that $P\left(\frac{dy}{x}\right)$ is a nontrivial element of $\mathcal{C}^k(sc, b^2)$ while $P\left(d\left(\frac{dy}{x}\right)\right) = P\left(-\frac{dx}{x^2} \wedge dy\right) = 0$. It happens to be the case that the boundary map ∂ in the long exact sequence above is trivial. However showing this requires further computation and is not immediately clear. In all of our computations we will use short exact sequences that we can see are split ‘by inspection’, a process we will clarify in Example 24.

Example 23. Any Lie algebroid is trivially TM-anchored. Consider the 0-tangent bundle from Example 15. The anchor map is evaluation in $T\mathbb{R}^2$. The sequence in equation (2.1) becomes

$$0 \rightarrow \Omega^k(\mathbb{R}^2) \rightarrow {}^0\Omega^k(\mathbb{R}^2) \rightarrow \mathcal{C}^k \rightarrow 0.$$

This sequence happens to be split. However, we typically can ‘do better’ in our choice of split short exact sequence. Given the b -tangent bundle also from Example 15, the Mazzeo-Melrose theorem tells us that the b -cohomology ${}^bH^p(\mathbb{R}^2) \simeq H^p(\mathbb{R}^2) \oplus H^{p-1}(\{x=0\})$. Since we already understand b -cohomology, we would be left with an easier computation if we anchored the 0-tangent bundle with the b -tangent bundle instead of the tangent bundle. Since the anchor maps for both of these algebroids are evaluation, the 0-tangent bundle is b -anchored and we have a short exact sequence

$$0 \rightarrow {}^b\Omega^k(\mathbb{R}^2) \rightarrow {}^0\Omega^k(\mathbb{R}^2) \rightarrow \mathcal{C}^k \rightarrow 0.$$

Taylor series. Another powerful tool in computing Lie algebroid cohomology is a Taylor series expansion. The idea is to consider an $\mathcal{A}$ -anchored Lie algebroid $(\mathcal{B}, [\cdot, \cdot]_{\mathcal{B}}, \rho_{\mathcal{B}})$ where the factoring map $\phi : \mathcal{B} \rightarrow \mathcal{A}$ is an isomorphism outside of a hypersurface (or collection of hypersurfaces) $Z \subset M$. In a tubular neighborhood $U \supset Z$, we express a $\mathcal{B}$ -form μ as $\mu = \eta + \theta$ where $\eta \in \mathcal{C}^k$ and $\theta \in {}^{\mathcal{A}}\Omega^k(M)$. We say that η is the singular part of μ and θ is the regular part.

Example 24. We will continue with the short exact sequence from Example 23:

$$0 \rightarrow \Omega^1(\mathbb{R}^2) \rightarrow {}^0\Omega^1(\mathbb{R}^2) \rightarrow \mathcal{C}^1 \rightarrow 0.$$

In a neighborhood of $\{x = 0\}$, we can express a degree 1 0-form μ as

$$\mu = \underbrace{f \frac{dx}{x} + \frac{A}{x}}_{\text{'singular part'}} + B$$

where $f \in \Omega^0(\{x = 0\})$ and $A, B \in \Omega^1(\{x = 0\})$. We will give a brief sketch of how we can use this Taylor series expansion to compute the 0- cohomology. By inspecting $d\mu$ we verify that the sequence splits in that the singular part of $d\mu$ only involves the singular part of μ and similarly the regular part of $d\mu$ only involves the regular part of μ :

$$d\mu = \underbrace{-\frac{dx}{x} \wedge df - \frac{dx}{x^2} \wedge A + \frac{dA}{x}}_{\text{'singular part'}} + dB.$$

Thus ${}^0H^1(\mathbb{R}^2) \simeq H^1(\mathcal{C}^*) \oplus H^1(\mathbb{R}^2)$ Further inspecting this Taylor series expansion allows us to give an explicit description of $H^1(\mathcal{C}^*)$ in that the singular part of $d\mu$ only involves the singular part of μ and the regular part of $d\mu$ only involves the regular part of μ . In order for $d\mu = 0$ to be satisfied, the singular terms in our expansion must satisfy relations

$$df = 0 \text{ and } A = 0.$$

Since ${}^0\Omega^0(\mathbb{R}^2) = C^\infty(\mathbb{R}^2)$, cohomology classes in $H^1(\mathcal{C}^*)$ have representatives

$$\mu - d\tilde{\mu} = \frac{dx}{x} f$$

for closed functions f defined on $\{x = 0\}$ and thus we have identified

$${}^0H^1(\mathbb{R}^2) \simeq H^0(\{x = 0\}) \oplus H^1(\mathbb{R}^2).$$

Briefly let us also consider the other short exact sequence introduced in Example 23: $0 \rightarrow {}^b\Omega^k(\mathbb{R}^2) \rightarrow {}^0\Omega^k(\mathbb{R}^2) \rightarrow \mathcal{C}^k \rightarrow 0$. In a neighborhood of $\{x = 0\}$, we can express a degree 1 0-form μ as

$$\mu = \frac{A}{x} + B$$

where $A \in \Omega^1(\{x = 0\})$ and $B \in {}^b\Omega^1(\mathbb{R}^2)$. Then

$$d\mu = -\frac{dx}{x^2} \wedge A + \frac{dA}{x} + dB.$$

By inspection, we see that the singular part remains singular and that the regular part remains regular. Thus the sequence splits. The kernel relations of the singular part are simply $A = 0$. Thus we conclude ${}^0H^1(\mathbb{R}^2) \simeq {}^bH^1(\mathbb{R}^2)$.

2.4 Nondegenerate lifts & $\mathcal{A}$ -symplectic forms

Definition 7. *Given a Poisson manifold (M, π) and a Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$ over M , we call π an $\mathcal{A}$ -symplectic structure if there exists a non-degenerate Poisson $\mathcal{A}$ -lift $\pi_{\mathcal{A}}$.*

Given a non-degenerate Poisson structure $\pi_{\mathcal{A}}$, the sharp map $\pi_{\mathcal{A}}^{\sharp} : \mathcal{A}^* \rightarrow \mathcal{A}$ is an isomorphism. We denote the inverse map $(\pi_{\mathcal{A}}^{\sharp})^{-1} : \mathcal{A} \rightarrow \mathcal{A}^*$ by $\omega_{\mathcal{A}}^b$. The associated two form $\omega_{\mathcal{A}} \in {}^{\mathcal{A}}\Omega^2(M)$ is an $\mathcal{A}$ -symplectic form.

Example 25. *As noted in Example 17, the Poisson bivector $\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ admits non-degenerate lifts to the b^2 - and 0-tangent bundles. The corresponding symplectic forms are*

$$\omega_{b^2} = dy \wedge \frac{dx}{x^2}$$

and

$$\omega_0 = \frac{dy}{x} \wedge \frac{dx}{x}.$$

Remark 1. *Given an $\mathcal{A}$ -symplectic structure (M, π) , the condition that the $\mathcal{A}$ -lift $\pi_{\mathcal{A}}$ is Poisson on the Lie algebroid $\mathcal{A}$, i.e. $[\pi_{\mathcal{A}}, \pi_{\mathcal{A}}] = 0$, is equivalent to the corresponding $\mathcal{A}$ -symplectic form $\omega_{\mathcal{A}}$ being closed, i.e. $d\omega_{\mathcal{A}} = 0$. This fact follows immediately from the definition of the exterior derivative d for $\mathcal{A}$ -forms, which uses the algebroid Lie bracket and anchor map in the exact same way as the classical exterior derivative on the tangent bundle.*

2.4.1 Moser

When we consider a Lie algebroid that is isomorphic to the tangent bundle away from a hypersurface (or collection of hypersurfaces), we can adapt the standard Moser technique of symplectic geometry to establish normal forms for an $\mathcal{A}$ -symplectic form $\omega_{\mathcal{A}}$.

Lemma 1. ($\mathcal{A}$ -Moser) *Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie algebroid over M with $Z \subset M$ a hypersurface and assume that $\rho_{\mathcal{A}}$ is an isomorphism on $M \setminus Z$. Let $\omega_1, \omega_2 \in {}^{\mathcal{A}}\Omega^2(M)$ be $\mathcal{A}$ -symplectic forms such that*

- $[\omega_1] = [\omega_2] \in {}^{\mathcal{A}}H^2(U)$ for some tubular neighborhood U of Z , and
- $\omega_1|_Z = \omega_2|_Z$

Then there exists a neighbourhood U' of Z and a Lie algebroid isomorphism $\phi : (\mathcal{A}|_{U'}, \omega_1) \rightarrow (\mathcal{A}|_U, \omega_2)$ such that $\phi^*\omega_1 = \omega_2$ and $\phi|_Z$ covers the identity map on Z .

Proof. Let U be a tubular neighborhood of Z such that $[\omega_1] = [\omega_2] \in {}^{\mathcal{A}}H^2(U)$. Consider the convex combination of ω_1 and ω_2 :

$$\omega_t = \omega_1 + t(\omega_2 - \omega_1).$$

The forms ω_1 and ω_2 representing the same class in $\mathcal{A}$ -cohomology is insufficient for ω_t to be $\mathcal{A}$ -symplectic. However, since $\omega_1|_Z = \omega_2|_Z$, there exists some neighborhood $U \supset U_0 \supset Z$ such that ω_t is non-degenerate for all $t \in [0, 1]$. Note that $\omega_2 - \omega_1$ is exact as an $\mathcal{A}$ -form on U_0 because $[\omega_1] = [\omega_2] \in {}^{\mathcal{A}}H^2(U)$. Thus,

$$\frac{d}{dt}\omega_t = \omega_2 - \omega_1 = d\sigma$$

for some degree 1 $\mathcal{A}$ -form σ . Following the standard Moser argument (Sec. 7.3 [11]), we want to solve the equation

$$\mathcal{L}_{v_t}\omega_t + \frac{d}{dt}\omega_t = di_{v_t}\omega_t + d\sigma = 0.$$

Note that the non-degeneracy of ω_t implies the existence of a smooth family of sections $v_t \in \Gamma(\mathcal{A})$ satisfying

$$i_{v_t}\omega_t + \sigma = 0.$$

Since $\rho_{\mathcal{A}}$ is an isomorphism on $M \setminus Z$, the induced map $\rho : \Gamma(\mathcal{A}) \rightarrow \Gamma(TM)$ on sections is injective. Thus $X_t := \rho_{\mathcal{A}}(v_t)$ is a smooth family of vector fields on M that we can integrate, further shrinking our neighborhood if necessary, to obtain an isotopy $\phi : U_0 \times [0, 1] \rightarrow U'$ with $\phi^*\omega_t = \omega_0$ for all $t \in [0, 1]$. Since $v_t|_Z = 0$, we have $\phi_t|_Z = id_Z$. $\square$

Example 26. We can use the $\mathcal{A}$ -Moser Lemma to show that all 0-symplectic structures on

$\mathbb{R}^2$ are locally of the form $\omega = \frac{dx}{x} \wedge \frac{dy}{x}$.

2.5 Poisson cohomology & rigged Lie algebroids

We develop the ideas of this section with examples in Preprints [24, 25] and [26]. A more complete treatment is joint work with Ralph Klaasse and will appear soon [23].

Recall ([12], p. 39) for a general Poisson manifold (M, π) , the Poisson cohomology $H_{\pi}^*(M)$ is defined as the cohomology groups of the Lichnerowicz complex: This complex is

formed using $\mathcal{V}^k(M) := \mathcal{C}^\infty(M; \wedge^k TM)$, smooth multivector fields on M .

$$\dots \rightarrow \mathcal{V}^{k-1}(M) \xrightarrow{d_\pi} \mathcal{V}^k(M) \xrightarrow{d_\pi} \mathcal{V}^{k+1}(M) \rightarrow \dots$$

The differential

$$d_\pi : \mathcal{V}^k(M) \rightarrow \mathcal{V}^{k+1}(M)$$

is defined as

$$d_\pi = [\pi, \cdot],$$

where $[\cdot, \cdot]$ is the Schouten bracket extending the standard Lie bracket on vector fields $\mathcal{V}^1(M)$.

We can easily generalize this cohomology to any Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$ over M . Given an $\mathcal{A}$ -Poisson bivector $\pi_{\mathcal{A}} \in \mathcal{C}^\infty(M; \wedge^2 \mathcal{A})$, we define the $\mathcal{A}$ -Poisson cohomology in the same way we define Poisson cohomology: The operator $d_{\pi_{\mathcal{A}}} = [\pi_{\mathcal{A}}, \cdot]$, where $[\cdot, \cdot]$ is the $\mathcal{A}$ -Schouten bracket, defines a differential on $\mathcal{A}$ -multivector fields ${}^{\mathcal{A}}\mathcal{V}^k(M)$. The **$\mathcal{A}$ -Poisson cohomology** of $(M, \pi_{\mathcal{A}})$, denoted ${}^{\mathcal{A}}H_\pi^*(M)$, is the cohomology of the **$\mathcal{A}$ -Lichnerowicz complex**:

$$\dots \rightarrow {}^{\mathcal{A}}\mathcal{V}^{k-1}(M) \xrightarrow{d_{\pi_{\mathcal{A}}}} {}^{\mathcal{A}}\mathcal{V}^k(M) \xrightarrow{d_{\pi_{\mathcal{A}}}} {}^{\mathcal{A}}\mathcal{V}^{k+1}(M) \rightarrow \dots$$

Example 27. Consider $\mathbb{R}^2$ with bivector $\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. In this case, $\mathcal{V}^1(\mathbb{R}^2)$ is generated by $\mathcal{C}^\infty(\mathbb{R}^2)$ linear combinations of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. As we have previously noted, π admits a lift to the 0-tangent bundle of $\mathbb{R}^2$ constructed in Example 15. Recall that this Lie algebroid is generated by the vector fields $x \frac{\partial}{\partial x}$ and $x \frac{\partial}{\partial y}$. Thus the 0-Lichnerowicz complex has ${}^0\mathcal{V}^1(\mathbb{R}^2)$ generated by $\mathcal{C}^\infty(\mathbb{R}^2)$ linear combinations of $x \frac{\partial}{\partial x}$ and $x \frac{\partial}{\partial y}$. Thus, in this instance, the 0-Poisson cohomology is equivalent to computing standard Poisson cohomology, but on a subset of the multi-vector fields.

2.5.1 Modular vector fields

Once we have local normal forms for our Poisson structure, it is much easier to compute standard Poisson invariants. One of the easier examples is called the modular vector field.

To compute the modular vector field we will use the description provided in section 2.6 of Dufour-Zung [12]. Recall that given a volume form $\Lambda \in \Omega^{2n}(M)$, we can define maps

$$\Lambda^\flat : \mathcal{V}^p(M) \rightarrow \Omega^{2n-p}(M) \text{ and } \Lambda^\sharp : \Omega^{2n-p}(M) \rightarrow \mathcal{V}^p(M)$$

as follows:

$$\Lambda^b(A) = i_A \Lambda \text{ and } \Lambda^\sharp(\eta) = -i_\eta \widehat{\Lambda}$$

where $\widehat{\Lambda} \in \mathcal{V}^{2n}(M)$ is the dual to Λ , i.e. $\langle \Lambda, \widehat{\Lambda} \rangle = 1$. Then the modular vector field $D_\pi \Lambda$ is given by $\Lambda^\sharp \circ d \circ \Lambda^b(\pi)$.

Example 28. *Given the pocket Poisson structure $\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, we will compute the modular vector field using the volume form $\Lambda = dx \wedge dy$. Then*

$$\Lambda^\sharp(d(\Lambda^b \pi)) = \Lambda^\sharp(d(x^2)) = \Lambda^\sharp(2x dx) = 2x \frac{\partial}{\partial y}.$$

Note that the modular vector field depends on the choice of volume form Λ . Consider another volume form $\widetilde{\Lambda} = f\Lambda$ for a positive function $f \in \mathcal{C}^\infty(M)$. Then

$$D_\pi \widetilde{\Lambda} = D_\pi \Lambda - \pi^\sharp(\ln |f|).$$

But note that f is non-zero, so $\ln |f|$ is a smooth function. Further, $-\pi^\sharp(\ln |f|) = d_\pi(\ln |f|)$, an exact element in $\mathcal{V}^1(M)$. Thus the Poisson cohomology class $[D_\pi \Lambda] \in H_\pi^1(M)$ does not depend on Λ .

2.5.2 Computing Poisson cohomology

Because Poisson cohomology is quite challenging to compute, there are only very select cases where the answer is known. In the case of a symplectic manifold where the Poisson bivector is non-degenerate, the Poisson cohomology is isomorphic to the de Rham cohomology of the manifold M . The non-degeneracy of π allows us to define an isomorphism $T^*M \rightarrow TM$ that induces this isomorphism in cohomology: $H^p(M) \simeq H_\pi^p(M)$.

In the case when an $\mathcal{A}$ -Poisson bivector $\pi_{\mathcal{A}}$ is non-degenerate, we recover an isomorphism of complexes analogous to the symplectic case.

Proposition 1. *Let $(M, \omega_{\mathcal{A}})$ be an $\mathcal{A}$ -symplectic manifold, and $\pi_{\mathcal{A}}$ the corresponding non-degenerate $\mathcal{A}$ -Poisson structure. Then, the $\mathcal{A}$ -Poisson cohomology ${}^{\mathcal{A}}H_\pi^*(M)$ is isomorphic to the $\mathcal{A}$ -cohomology ${}^{\mathcal{A}}H^*(M)$.*

This proposition follows from the fact that we have an isomorphism of complexes given by the map $\omega_{\mathcal{A}}^b : \mathcal{A} \rightarrow \mathcal{A}^*$.

Example 29. The 0-Poisson bivector $\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, as a non-degenerate structure, is dual to 0-symplectic form $\omega = \frac{dy \wedge dx}{x^2}$. We can check that π defines a quasi-isomorphism of the 0-Lichnerowicz complex and the 0-de Rham complex. The form ω maps

$$x \frac{\partial}{\partial x} \mapsto -\frac{dy}{x} \quad \text{and} \quad x \frac{\partial}{\partial y} \mapsto \frac{dx}{x}.$$

Finally we check that $d_\pi x = -x^2 \frac{\partial}{\partial y} \mapsto -dx$ and $d_\pi y = x^2 \frac{\partial}{\partial x} \mapsto -dy$. Thus we have a map of complexes that intertwines the exterior derivative d with d_π .

Thus far we have constructed the following diagram of complexes.

$$\begin{array}{ccc} (\mathcal{A}\mathcal{V}^*, d_{\pi_{\mathcal{A}}}) & \xleftarrow[\simeq]{\pi_{\mathcal{A}}^\#} & (\mathcal{A}\Omega^*, d) \\ \downarrow \rho & & \\ (\mathcal{V}^*, d_\pi) & & \end{array}$$

In very special cases, ρ induces an isomorphism between $\mathcal{A}$ -Poisson and Poisson cohomology. One such instance is the b -symplectic setting, a fact first shown by Ioan Mărcuț and Boris Osorno Torres [29]. However, for general $\mathcal{A}$ -Poisson structures, this will not be the case. In order to compute the Poisson cohomology of a generically symplectic Poisson manifold (M, π) , we will construct a complex $(\mathcal{R}\Omega^*, d)$, called the Rigged de Rham complex, that is isomorphic to the Lichnerowicz complex.

$$\begin{array}{ccc} (\mathcal{A}\mathcal{V}^*, d_{\pi_{\mathcal{A}}}) & \xleftarrow[\simeq]{\pi_{\mathcal{A}}^\#} & (\mathcal{A}\Omega^*, d) \\ \downarrow \rho & & \\ (\mathcal{V}^*, d_\pi) & \xleftarrow[\simeq]{\pi^\#} & (\mathcal{R}\Omega^*, d) \end{array}$$

While computing cohomology using the Lichnerowicz complex can be quite intractable, understanding rigged Lie algebroid cohomology is typically easier.

We begin by considering the $\mathcal{C}^\infty(M)$ -submodule of $\mathcal{V}^1(M)$ generated by Hamiltonian vector fields, that is the $\mathcal{C}^\infty(M)$ -module of vector fields

$$\pi^\#(\Gamma(T^*M)). \tag{2.2}$$

Androulidakis and Zambon show that almost regular Poisson structures (Definition 2.5 [2]) are precisely those whose module of Hamiltonian vector fields is projective, i.e. equation

2.2 defines the sections of a vector bundle. We will provide a sketch of their proof for the case when (M, π) is generically symplectic.

Proposition 2. *If (M, π) is generically symplectic, then equation 2.2 defines the sections of a vector bundle $\mathcal{R}_\pi$.*

Proof. By the definition of generically symplectic, M_{reg} - the set where $\pi^\sharp$ has maximal rank - is dense in M . Further, the distribution $D = TM$ satisfies the property that for all $x \in M_{reg}$, $D_x = T_x M$, the symplectic leave through x . Our task is to show that $\pi^\sharp(\Gamma(T^*M))$ is projective. We will follow the steps of Theorem 2.8 [2]. Fix $x \in M$. Let I_x stand for the ideal of functions in $\mathcal{C}^\infty(M)$ that vanish at x . The anchor map $\pi^\sharp$ at the level of sections $\pi^\sharp : \Gamma(T^*M) \rightarrow \Gamma(TM)$ induces a short exact sequence of vector spaces (a special case of equation (1.4) in [2]):

$$0 \rightarrow \mathfrak{h}_x \rightarrow E_x \xrightarrow{\pi^\sharp} A_x \rightarrow 0$$

where $\mathfrak{h}_x$ is the set

$$\left\{ \xi \in T_x^*M \mid \exists \text{ a local extension } \tilde{\xi} \text{ such that } \tilde{\xi}|_{TL} = 0 \text{ for all symplectic leaves } L \text{ of } \pi \right\},$$

$$E_x = \Gamma(T^*M)/I_x\Gamma(T^*M), \text{ and } A_x = \pi^\sharp(\Gamma(T^*M))/I_x\pi^\sharp(\Gamma(T^*M)).$$

Note that showing that $\pi^\sharp(\Gamma(T^*M))$ is projective is equivalent to showing that the rank of A_x is the same for all $x \in M$. Because E_x has constant rank as we vary x , if we can also show that $\mathfrak{h}_x$ has constant rank as we vary x , then we have shown projectivity. So consider any $x \in M$. Because $D_x = T_x M$ and M_{reg} is dense in M , a covector in $\xi \in T_x^*M$ can be extended locally to a 1-form annihilating the symplectic leaves of (M, π) if and only if $\xi = 0$. In other words, $\mathfrak{h}_x = 0$ for all x . $\square$

Definition 8. *Given a generically symplectic Poisson manifold (M, π) , the **rigged Lie algebroid** $(\mathcal{R}_\pi, [\cdot, \cdot], \rho)$ of π is the vector bundle $\mathcal{R}_\pi$ equipped with anchor map ρ being evaluation in TM and Lie bracket $[\cdot, \cdot]$ induced by the standard Lie bracket on TM .*

Example 30. *In the case of $\mathbb{R}^2$ with Poisson bivector $\pi = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, we can check that the image of $\pi^\sharp(\Gamma(T^*\mathbb{R}^2))$ is generated by*

$$x^2 \frac{\partial}{\partial x} \text{ and } x^2 \frac{\partial}{\partial y}.$$

These are the generators of a Lie algebroid called the 0^2 -tangent bundle. Thus $\mathcal{R}_\pi = {}^{0^2}T\mathbb{R}^2$.

We conclude this subsection with a few more remarks to clarify the role of the rigged Lie algebroid.

Lemma 2. *If (M, π) is generically symplectic, then the Lie algebroid cohomology of the rigged Lie algebroid computes the Poisson cohomology of π .*

Proof. To build a map between the two complexes will require several steps.

Begin by considering (1): since π is generically symplectic, the set where $\pi^\sharp$ - viewed as a map between vector bundles - defines an isomorphism is dense. This induces (2), an injective map on sections. We obtain (3), a bijection, by restricting to the codomain. (4) is the inverse map. Recall that the sheaf of sections $\Gamma(\mathcal{R})$ defines a vector bundle. Further, notice that $\pi^\sharp$ in step (2) is a $\mathcal{C}^\infty(M)$ -linear map since it was induced by a bundle morphism. The operation of restriction still gives a $\mathcal{C}^\infty(M)$ -linear map.

$$\begin{aligned}
 (1) \quad & \pi^\sharp : T^*M \rightarrow TM \\
 & \downarrow \\
 (2) \quad & \pi^\sharp : \Gamma(T^*M) \rightarrow \Gamma(TM) \\
 & \downarrow \\
 (3) \quad & \pi_{\text{rest}}^\sharp : \Gamma(T^*M) \rightarrow \pi^\sharp(\Gamma(T^*M)) \simeq \Gamma(R) \\
 & \downarrow \\
 (4) \quad & (\pi_{\text{rest}}^\sharp)^{-1} : \Gamma(\mathcal{R}) \rightarrow \Gamma(T^*M) \\
 & \downarrow \\
 (5) \quad & (\pi_{\text{rest}}^\sharp)^{-1} : \mathcal{R} \rightarrow T^*M \\
 & \downarrow \\
 (6) \quad & \omega^\flat : TM \rightarrow \mathcal{R}^* \\
 & \downarrow \\
 (7) \quad & \omega^\flat : \Gamma(TM) \rightarrow \Gamma(\mathcal{R}^*)
 \end{aligned}$$

Thus (4) is also a $\mathcal{C}^\infty(M)$ -linear map and induces (5), an invertible bundle map. We dualize to obtain (6). This induces (7), a map on sections.

By taking exterior powers of the map $\omega^\flat$, we can extend it to a $\mathcal{C}^\infty(M)$ -linear isomorphism

$$\bar{\omega} : \mathcal{V}^p(M) \rightarrow \mathcal{R}\Omega^p(M).$$

This will be the desired quasi-isomorphism of complexes.

Next we will show that for any smooth multivector field η on a given generically symplectic Poisson manifold (M, π) , we have $\bar{\omega}(d_\pi(\eta)) = -d(\bar{\omega}(\eta))$. We will proceed by induction on the degree of η and by using the Leibniz rule. Let η be a degree 0 form, that is $\eta \in \mathcal{C}^\infty(M)$. Then $\bar{\omega}(\eta) = \eta$ and $-d(\bar{\omega}(\eta)) = -d\eta$. Consider $d_\pi(\eta) = [\pi, \eta] = -X_\eta$, the Hamiltonian vector field of η . On M_{reg} , we have that π defines a symplectic form ω . Further, on M_{reg} , $\bar{\omega}$ is given by contraction with ω . Thus on M_{reg} , we have that

$$\bar{\omega}(-X_\eta) = -i_{X_\eta}\omega = -d\eta,$$

by the correspondence of Hamiltonian vector fields between non-degenerate π and corresponding symplectic form ω . By continuity this identity extends to all of M . If $\eta = d_\pi f$ is an exact 1-vector field, then $\bar{\omega}(d_\pi(d_\pi f)) = \bar{\omega}(0) = 0$ and

$$d(\bar{\omega}(d_\pi f)) = d(\bar{\omega}(X_f)) = d(df) = 0.$$

By the Leibniz rule, the statement is true for all multivector fields. Thus we have shown that, up to a sign, the map $\bar{\omega}$ is an isomorphism that intertwines the differential d of the rigged algebroid $\mathcal{R}$ de Rham complex with the differential operator d_π of the Lichnerowicz complex. Hence $\bar{\omega} : H_\pi^p(M) \rightarrow {}^{\mathcal{R}}H^p(M)$ is an isomorphism. $\square$

The notation for $\bar{\omega}$ suggests it arises from a possibly ‘singular two-form’. Indeed, in the case where we have lifted π non-degenerately to a Lie algebroid $\mathcal{A}$, the map $\bar{\omega}$ is given by an $\mathcal{A}$ -symplectic form $\omega_{\mathcal{A}}$.

Remark 2. Equation 2.2 defines a vector bundle $\mathcal{R}_\pi$ for any (M, π) almost regular Poisson. However, if the maximal rank of π is less than the dimension of M , in general the Lie algebroid cohomology of $\mathcal{R}_\pi$ does not directly compute the Poisson cohomology of π . This is because $\mathcal{R}_\pi$ encodes information about the leaves of the regular foliation arising from distribution D and does not contain ‘transverse’ information.

2.5.3 Modular class

As we discussed above, while the modular vector field depends on our choice of volume form Λ , the class $[D_\pi \Lambda] \in H_\pi^1(M)$ - called the modular class of π - does not. The rigged algebroid provides a clear way to see if $[D_\pi \Lambda]$ is trivial or not. Poisson manifolds (M, π) with trivial modular class are called unimodular. A Poisson manifold is unimodular if and only if M admits a volume form invariant under the Hamiltonian flows of π . Most of the cases we will consider are not unimodular, but $[D_\pi \Lambda]$ still plays an important role.

Example 31. Continuing with the modular vector field from Example 28, we can use the 0-symplectic form $\omega = \frac{dy}{x} \wedge \frac{dx}{x}$ to see that $\bar{\omega}\left(2x \frac{\partial}{\partial y}\right) = 2 \frac{dx}{x}$. Since $\left[\frac{dx}{x}\right] \in {}^{0^2}H^1(\mathbb{R}^2)$ is nonvanishing, this Poisson structure is NOT unimodular.

2.5.4 Commentary on the various cohomologies

Recall that $H_\pi^2(M)$ describes formal infinitesimal deformations of a Poisson structure. On the other hand, there is also a deformation theory for Lie algebroids. In this section, we will discuss the relationship between the two theories.

Let us begin by zooming all the way out to a general Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$. Marius Crainic and Ieke Moerdijk's work [9] introduces a way to study deformations of the bracket $[\cdot, \cdot]$ using a cohomology theory aptly called deformation cohomology. We in particular are concerned with the Poisson Lie algebroid $(T^*M, [\cdot, \cdot], \pi^\sharp)$ where the Lie bracket is defined as

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha, \beta))$$

for all smooth de Rham 1-forms α, β on M . In Corollary 3 of [9], Crainic and Moerdijk explain how a class in second degree Poisson cohomology gives a deformation of the Poisson Lie algebroid. In other words, Poisson cohomology measures deformations of the Lie bracket on T^*M that remain Poisson, i.e. the bracket can always be defined by a Poisson bivector π on M . We compute Poisson cohomology by constructing the rigged Lie algebroid $\mathcal{R}_\pi$, a Lie algebroid isomorphic to $(T^*M, \pi^\sharp)$.

Once we have a non-degenerate lift $\pi_{\mathcal{A}}$ of π , we also consider $\mathcal{A}$ -cohomology, which we use to establish normal forms for π . The $\mathcal{A}$ -forms are a subcomplex of $\mathcal{R}$ -forms. The $\mathcal{A}$ -cohomology zooms in and studies an even more restricted type of deformation of the Poisson Lie algebroid: deformations of the bracket that can always be defined by an $\mathcal{A}$ -symplectic structure.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{C}^\infty(M) & \xrightarrow{\partial_\pi} & \mathcal{C}^\infty(M, TM) & \xrightarrow{\partial_\pi} & \mathcal{C}^\infty(M, \wedge^2 TM) \longrightarrow 0 \\
 & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\
 0 & \longrightarrow & \mathcal{C}^\infty(M) & \xrightarrow{d} & \mathcal{R}\Omega^1(M) & \xrightarrow{d} & \mathcal{R}\Omega^2(M) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \mathcal{C}^\infty(M) & \xrightarrow{d} & \mathcal{A}\Omega^1(M) & \xrightarrow{d} & \mathcal{A}\Omega^2(M) \longrightarrow 0
 \end{array}$$

Lichnerowicz complex:
 defines Poisson cohomology

 $\mathcal{R}$ -forms: computes Poisson
 cohomology

 $\mathcal{A}$ -forms: studies $\mathcal{A}$ -symplectic
 structures

Notice that the Lichnerowicz and $\mathcal{R}$ -de Rham complexes are completely determined by π . On the other hand, there often are many possible choices of Lie algebroid $\mathcal{A}$ with lift $\pi_{\mathcal{A}}$. In certain very special cases the zoomed in picture to $\mathcal{A}$ -forms in fact captures all of the information about Poisson deformations. In [19], Guillemin, Miranda, and Pires showed

that for b -symplectic manifolds the b -de Rham complex computes Poisson cohomology and b -symplectomorphisms give all Poisson isomorphisms. However Geoffrey Scott (see [47], p. 18) points out that in general this is not the case. We will provide many many more examples where the $\mathcal{A}$ -de Rham complex does not compute Poisson cohomology.

2.6 Constructing Lie algebroids

At this point we have explored several features that we desire in a choice of algebroid for an $\mathcal{A}$ -lift $\pi_{\mathcal{A}}$. Finding a suitable algebroid that satisfies these properties is often a labor-intensive task. There are a variety of ways to construct Lie algebroids that we can use to understand geometric structures.

In review, since we are primarily concerned with Poisson bivectors that are symplectic almost everywhere, we will focus on Lie algebroids with a dense isomorphism locus. In other words, given a Lie algebroid $(\mathcal{A}, [\cdot, \cdot], \rho)$ over a manifold M , we want the set $\{p \in M \mid \rho : \mathcal{A}_p \rightarrow T_p M \text{ is an isomorphism}\}$ to be dense in M . Further, we will restrict our attention to the case where the complement of the isomorphism locus is the union of closed codimension one submanifolds - i.e., hypersurfaces. Borrowing from the language of algebraic geometry, such a set is called a *divisor*. For more on this topic we refer to [20]; in our present discussion we will only see the bare minimum of facts needed.

Following the definitions provided by Saito [46], we need divisors D with the property that the sheaf of vector fields tangent to all of the hypersurfaces is locally free. We make this demand because there is a bijection between isomorphism classes of vector bundles of rank n on M and isomorphism classes of locally free sheaves of rank n . And of course Lie algebroids are vector bundles first and foremost. Fittingly, we call such divisors *free*.

The notion of a free divisor is distinct from another commonly studied type of divisor called normal crossing. A divisor D is called *normal crossing* if for each $x \in D$, there is a chart (U, ϕ) of M , $x \in U$, such that $\phi(D)$ is a union of a subset of the coordinate hyperplanes in $\mathbb{R}^n$ intersected with $\phi(D)$. Normal crossing divisors happen to also be free divisors. See Figure 9 to see how these types of divisors are related.

Figure 9. *Divisors*

$$\left\{ \begin{array}{c} \textit{normal} \\ \textit{crossing} \\ \textit{divisors} \end{array} \right\} \subset \left\{ \begin{array}{c} \textit{free} \\ \textit{divisors} \end{array} \right\} \subset \left\{ \begin{array}{c} \textit{divisors} \end{array} \right\}$$

2.6.1 Building algebroids using the vanishing ideal of functions $\mathcal{I}_D$

If the ideal of functions that vanish at D - denoted $\mathcal{I}_D$ - is locally free of rank 1, then we can use it to construct a vector bundle whose sections are the vector fields that are tangent to D . In this section we will assume that our divisor is normal crossing. Normal crossing divisors have this property that $\mathcal{I}_D$ is locally free of rank 1. We have a menagerie of ways to construct vector bundles using $\mathcal{I}_D$. Sections 1.1 and 2.7 of Ralph Klaasse's thesis [21] gives a nice general framework for the relationship between divisors D , their ideals $\mathcal{I}_D$, and the type of constructions we will discuss. Here we will focus only on our viewpoint, a procedure called Melrose rescaling. Note that Melrose rescaling is equivalent to lower elementary modification introduced by Travis Li in his thesis [17, 28].

Rescaling Lie algebroids. Richard Melrose explained how to rescale a vector bundle with a filtration over a hypersurface; see Proposition 8.1 in [36]. We adapt his construction to Lie algebroids. We will describe how to rescale a Lie algebroid with respect to a suitable subbundle over a hypersurface.

Let (M, Z) be a manifold M with hypersurface $Z \subset M$ and let $(\mathcal{A}, \rho, [\cdot, \cdot])$ be a Lie algebroid over M . Given a Lie subalgebroid $F \subseteq \mathcal{A}|_Z \rightarrow Z$, consider the space of sections:

$$\mathcal{D} = \{u \in \mathcal{C}^\infty(M; \mathcal{A}) : u|_Z \in \mathcal{C}^\infty(Z; F)\}. \quad (2.3)$$

$\mathcal{D}$ consists of the set of smooth sections of $\mathcal{A}$ that take values in F at Z . There exists a Lie algebroid $({}^F\mathcal{A}, \rho_F, [\cdot, \cdot]_F)$ over M whose space of sections 'is' $\mathcal{D}$ as defined in (2.3).

Theorem 2. *There exists a vector bundle ${}^F\mathcal{A} \rightarrow M$ with a Lie algebroid morphism*

$$i : {}^F\mathcal{A} \rightarrow \mathcal{A}$$

that is an isomorphism over $M \setminus Z$ such that $i_\mathcal{C}^\infty(M; \mathcal{A}) = \mathcal{D}$.*

Proof. For completeness and to establish notation we will include the steps to show that ${}^F\mathcal{A}$ is a vector bundle. However, as mentioned above, these steps are solely due to Melrose.

We begin by noting that $\mathcal{D}$ is preserved under multiplication by any smooth function on M and thus is a $\mathcal{C}^\infty(M)$ module. For any $p \in M$, we will consider the ideal

$$\mathcal{I}_p := \{f \in \mathcal{C}^\infty(M) : f(p) = 0\}.$$

Let

$${}^F\mathcal{A}_p = \mathcal{D}/(\mathcal{I}_p \cdot \mathcal{D}) \quad \text{and} \quad {}^F\mathcal{A} = \bigsqcup_{p \in M} {}^F\mathcal{A}_p.$$

Then because we can describe $\mathcal{A}_p$ as $\mathcal{C}^\infty(M; \mathcal{A})/(\mathcal{I}_p \cdot \mathcal{C}^\infty(M; \mathcal{A}))$, there exists a natural map $i : {}^F\mathcal{A}_p \rightarrow \mathcal{A}_p$ taking a section in $\mathcal{D}$ and evaluating it at the point p . For all $p \in M \setminus Z$, this map is an isomorphism.

Suppose $p \in Z$ and let $\{v_1, \dots, v_k\}$ be a local basis of smooth sections of F . We can smoothly extend this to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_M\}$ of $\mathcal{A}$. Given a Z defining function x , any element $X \in \mathcal{D}$ is locally of the form

$$X = \sum_{i=1}^k g_i v_i + \sum_{j=k+1}^M x g_j v_j$$

for smooth functions g_i . Thus $\{v_1, \dots, v_k, x v_{k+1}, \dots, x v_M\}$ is a local basis of ${}^F\mathcal{A}$ and the coefficients $g_1, \dots, g_M$ give a local trivialization.

Given any other local basis $\{v'_1, \dots, v'_k\}$ for F , we can extend smoothly to a local basis $\{v'_1, \dots, v'_k, v'_{k+1}, \dots, v'_M\}$ of $\mathcal{A}$. Each of $\{v'_1, \dots, v'_k, x v'_{k+1}, \dots, x v'_M\}$ can be expressed as a smooth linear combination of $\{v_1, \dots, v_k, x v_{k+1}, \dots, x v_M\}$. So this induces smooth transformations among the coefficients g_i and ${}^F\mathcal{A}$ inherits a natural smooth bundle structure from $\mathcal{A}$ with bundle map $i : {}^F\mathcal{A} \rightarrow \mathcal{A}$.

This ends the proof of Melrose's vector bundle construction. At this point we will explain how to equip the bundle with a Lie algebroid structure. Given $(\mathcal{A}, [\cdot, \cdot], \rho)$, we can consider the rescaled vector bundle ${}^F\mathcal{A}$ as a Lie algebroid by taking anchor map ρ_F to be the composition of the bundle map $i : {}^F\mathcal{A} \rightarrow \mathcal{A}$ with bundle map $\rho : \mathcal{A} \rightarrow TM$ and by taking as a Lie bracket $[\cdot, \cdot]_F = i^{-1}([i(\cdot), i(\cdot)])$, the bracket induced from the bracket $[\cdot, \cdot]$ on $\mathcal{A}$.

We must verify that $[\cdot, \cdot]_F$ is a well-defined map $\Gamma({}^F\mathcal{A}) \times \Gamma({}^F\mathcal{A}) \rightarrow \Gamma({}^F\mathcal{A})$, where $\Gamma({}^F\mathcal{A})$ is the $\mathcal{C}^\infty(M)$ -module of smooth sections of ${}^F\mathcal{A} \rightarrow M$. As described above, given a Z

defining function x , any elements $X, Y \in \mathcal{D}$ are locally of the form

$$X = \sum_{i=1}^k g_i v_i + \sum_{j=k+1}^M x g_j v_j \quad \text{and} \quad Y = \sum_{i=1}^k f_i v_i + \sum_{j=k+1}^M x f_j v_j.$$

Then

$$[X, Y]_F = \sum_{i,j=1}^k [g_i v_i, f_j v_j] + \sum_{i=1}^k \sum_{j=k+1}^M [g_i v_i, x f_j v_j] + \sum_{i=1}^k \sum_{j=k+1}^M [x g_i v_i, f_j v_j] + \sum_{i,j=k+1}^M [x g_i v_i, x f_j v_j].$$

It suffices to show that each term of the sum restricts to a section of $\Gamma(F)$ or vanishes at Z . For $1 \leq i, j \leq k$, we have that $[g_i v_i, f_j v_j]|_Z \in \Gamma(F)$ by assumption and consequently $[g_i v_i, f_j v_j] \in \Gamma({}^F \mathcal{A})$. Consider $[g_i v_i, x f_j v_j] = g_i \rho(v_i)(x) \cdot f_j v_j + x [g_i v_i, f_j v_j]$. Since we assumed that $\rho(F)|_Z \subseteq TZ$, $\rho(v_i)(x) = 0$ which implies $[g_i v_i, x f_j v_j] \in \Gamma({}^F \mathcal{A})$. By antisymmetry of the Lie bracket, $[x g_i v_i, f_j v_j] \in \Gamma({}^F \mathcal{A})$. Finally we consider the term $[x g_i v_i, x f_j v_j] = x g_i \rho(v_i)(x) \cdot f_j v_j + x [x g_i v_i, f_j v_j] \in \Gamma({}^F \mathcal{A})$.

Our final task is to check that $[\cdot, \cdot]_F$ satisfies the Leibniz rule:

$$[X, fY]_F = \rho_F(X)f \cdot Y + f[X, Y]_F$$

where $X, Y \in \Gamma({}^F \mathcal{A})$, $f \in C^\infty(M)$ and $\rho_F(X)f$ is the Lie derivative of f with respect to the vector field $\rho_F(X)$. Consider $[X, fY]_F = i^{-1}([i(X), i(fY)]) =$

$$i^{-1}(\rho(i(X))f \cdot i(Y) + f[i(X), i(Y)]) = \rho_F(X)f \cdot Y + f[X, Y]_F$$

because i is an injective bundle map on $M \setminus Z$ and this identity extends by continuity. $\square$

Note that the restriction of ${}^F \mathcal{A}$ to Z is not F , but rather is a vector bundle of the same rank as $\mathcal{A}$. As explained by Melrose in Lemma 8.5 of [36], ${}^F \mathcal{A}|_Z$ is isomorphic to the graded bundle

$$F \oplus (N^*Z \otimes \mathcal{A}|_Z / F). \quad (2.4)$$

The conormal bundle here makes this bundle invariant of choice of Z defining function. Further, ${}^F \mathcal{A}$ is, non-canonically, isomorphic to $\mathcal{A}$: Given a Z defining function x , we can map a local expression of an element of $\Gamma({}^F \mathcal{A})$ to an element in $\Gamma(\mathcal{A})$ by

$$\sum_{i=1}^k g_i v_i + \sum_{j=k+1}^M x g_j v_j \rightarrow \sum_{i=1}^k g_i v_i + \frac{1}{x} \sum_{j=k+1}^M x g_j v_j.$$

Next, we will explore some specific applications of this construction.

Zero tangent bundle. We can consider the tangent bundle of a manifold $TM \rightarrow M$ as a Lie algebroid with Lie bracket the standard bracket on vector fields and with anchor map the identity map. We can apply Theorem 2 to a pair (M, Z) and rescale TM using the subbundle $0 \rightarrow Z$. The rescaled bundle ${}^0TM \rightarrow M$ is called the *zero tangent bundle* and was introduced by Rafe Mazzeo and Richard Melrose in the context of manifolds with boundary [31, 32]. In this case $\mathcal{D}$ is the set of vector fields that vanish at Z . If $x, y_1, \dots, y_n$ are local coordinates near a point in Z , and x is a defining function for Z , then vector fields

$$x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_1}, \dots, x \frac{\partial}{\partial y_n}$$

form a local basis for 0TM . Note that these do not vanish at Z as sections of 0TM . The dual bundle ${}^0T^*M \rightarrow M$ is called the *zero cotangent bundle* and is locally generated by

$$\frac{dx}{x}, \frac{dy_1}{x}, \dots, \frac{dy_n}{x}.$$

The anchor map of the 0TM algebroid is evaluation in the tangent bundle and the bracket is induced by the standard Lie bracket on TM .

b-tangent bundle. We recover the *b-tangent bundle* as formulated in [19], by applying Theorem 2 to the tangent bundle TM over a pair (M, Z) , and taking as subbundle $TZ \rightarrow Z$. The *b-tangent bundle* is the vector bundle whose space of sections is

$$\mathcal{D} = \{u \in \mathcal{C}^\infty(M; TM) : i \circ u|_Z \in \mathcal{C}^\infty(Z; TZ)\},$$

the vector fields that are tangent to Z . If $x, y_1, \dots, y_n$ are local coordinates near a point in Z , and x is a defining function for Z , then the vector fields and co-vectors

$$x \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \quad \frac{dx}{x}, dy_1, \dots, dy_n$$

respectively form local bases for bTM and ${}^bT^*M$. The anchor map of the bTM algebroid is evaluation in the tangent bundle and the bracket is induced by the standard Lie bracket on TM .

Scattering tangent bundle. Next we rescale the *b-tangent bundle* ${}^bTM \rightarrow M$ over a pair (M, Z) using the subbundle $0 \rightarrow Z$. The resulting bundle ${}^{sc}TM \rightarrow M$ is called the *scattering tangent bundle*. If x is a defining function for Z , and $y_1, \dots, y_n$ are local coordinates in Z , then the vector fields

$$x^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_1}, \dots, x \frac{\partial}{\partial y_n}$$

form a local basis for ${}^{sc}TM$. The dual bundle ${}^{sc}T^*M \rightarrow M$ is called the *scattering cotangent bundle* and is locally generated by

$$\frac{dx}{x^2}, \frac{dy_1}{x}, \dots, \frac{dy_n}{x}.$$

The anchor map is evaluation in the b -tangent bundle and then in the tangent bundle TM .

In the same way, the bracket is induced by the standard Lie bracket on TM .

The next example is a generalization of the scattering tangent bundle that is comparable to Geoffrey Scott's b^k generalization of the b -tangent bundle [47].

Scattering- k tangent bundle. Consider the scattering tangent bundle ${}^{sc}TM$ associated to (M, Z) . The scattering-2 tangent bundle ${}^{sc^2}TM$ is the vector bundle whose space of sections is $\mathcal{D} = \{u \in \mathcal{C}^\infty(M; {}^{sc}TM) : u|_Z = 0\}$. In other words, ${}^{sc^2}TM$ is rescaling ${}^{sc}TM$ by the subbundle $0 \rightarrow Z$.

In this fashion, given the scattering- $(k-1)$ tangent bundle ${}^{sc^{(k-1)}}TM$ associated to (M, Z) , the *scattering- k tangent bundle* ${}^{sc^k}TM$ is the vector bundle whose space of sections is $\mathcal{D} = \{u \in \mathcal{C}^\infty(M; {}^{sc^{(k-1)}}TM) : u|_Z = 0\}$. If x is a defining function for Z , and $y_1, \dots, y_n$ are local coordinates in Z , then the vector fields

$$x^{k+1} \frac{\partial}{\partial x}, x^k \frac{\partial}{\partial y_1}, \dots, x^k \frac{\partial}{\partial y_n}$$

form a local basis for ${}^{sc^k}TM$. The dual bundle ${}^{sc^k}T^*M \rightarrow M$ is called the *scattering- k cotangent bundle* and is locally generated by

$$\frac{dx}{x^{k+1}}, \frac{dy_1}{x^k}, \dots, \frac{dy_n}{x^k}.$$

For our final example, we will iterate Theorem 2 on the b^k -tangent bundle. For ease, we fix a Z defining function so as to avoid needing to specifying jet data in our description.

$(0^m; b^k)$ -tangent bundle. Given a pair (M, Z) , the idea of the b^k -tangent bundle is to be the vector bundle whose sections are all tangent to Z and have order k degeneracy in the direction normal to Z . If x is a defining function for Z , and $y_1, \dots, y_n$ are local coordinates in Z , then the vector fields

$$x^k \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$$

form a local basis for ${}^{b^k}TM$.

By iteratively applying Theorem 2 to the b^k -tangent bundle ${}^{b^k}TM \rightarrow M$ and rescaling by the subbundle $0 \rightarrow Z$, the resulting bundle ${}^{0^m}({}^{b^k}TM) \rightarrow M$ is called the $(0^m; b^k)$ -*tangent bundle*. The vector fields

$$x^{k+m} \frac{\partial}{\partial x}, x^m \frac{\partial}{\partial y_1}, \dots, x^m \frac{\partial}{\partial y_n}$$

form a local basis for ${}^{0^m}({}^{b^k}TM)$. The dual bundle ${}^{0^m}({}^{b^k}T^*M) \rightarrow M$ is called the $(0^m; b^k)$ -*cotangent bundle* and is locally generated by

$$\frac{dx}{x^{k+m}}, \frac{dy_1}{x^m}, \dots, \frac{dy_n}{x^m}.$$

Note that this construction is not obtained from the rescaling construction without more data, e.g., a tubular neighborhood decomposition of M near Z . In fact, it is easy to check that a different choice x of Z defining function can produce a different bundle.

Z defining functions and density bundles

The Lie algebroids $\mathcal{A}$ obtained from rescaling the tangent bundle, e.g. ${}^bTM, {}^0TM, {}^{sc}TM$, but not ${}^{b^k}TM$, do not depend on a choice of Z defining function, hence neither do their $\mathcal{A}$ -de Rham cohomologies. However, it is convenient for computations to work in a fixed tubular neighborhood, and so with a fixed Z defining function.

Example 32. Let x be a Z defining function on a manifold (M, Z) . Then $v \in {}^b\Omega^k(M)$ can be expressed as $v = \frac{dx}{x} \wedge \alpha + \beta$ for $\alpha, \beta \in \Omega^*(M)$. Note $dv = \frac{dx}{x} \wedge (x\partial_x\beta - d\alpha) + d\beta$ is a b -form with d the usual differential applied on $M \setminus Z$. Notice that $dv = \frac{dx}{x} \wedge (x\partial_x\beta - d\alpha) + d\beta = 0$ if and only if

$$d\alpha = x\partial_x\beta \text{ and } d\beta = 0.$$

We can express any other Z defining function $\tilde{x}$ as $x = f\tilde{x}$ for some nowhere vanishing function $f \in C^\infty(M)$. Then

$$\frac{dx}{x} = \frac{\tilde{x}df + f d\tilde{x}}{\tilde{x}f} = \frac{df}{f} + \frac{d\tilde{x}}{\tilde{x}} \quad \text{and} \quad \frac{dx}{x} \wedge \alpha + \beta = \frac{df}{f} \wedge \alpha + \frac{d\tilde{x}}{\tilde{x}} \wedge \alpha + \beta.$$

Notice that $\log|f|$ is a smooth function because f is non-vanishing. Then $d(\log|f| \wedge \alpha) = \frac{df}{f} \wedge \alpha$ because α is closed. Thus, in cohomology

$$\left[\frac{dx}{x} \wedge \alpha + \beta \right] = \left[\frac{d\tilde{x}}{\tilde{x}} \wedge \alpha + \beta \right]$$

and the cohomology class is unambiguous despite a representative being expressed using a particular Z defining function.

The change of Z defining function for a scattering cohomology representative, on the other hand, is not canonically trivial in this way.

Example 33. Let x be a Z defining function on a manifold (M, Z) . Consider the scattering-form $v = \frac{dx}{x^{k+1}} \wedge \alpha + \frac{\beta}{x^k}$ for $\alpha, \beta \in \Omega^*(M)$. As above, we express any other Z defining function $\tilde{x}$ as $x = f\tilde{x}$ for some nowhere vanishing positive function $f \in C^\infty(M)$. Then

$$\frac{dx}{x^{k+1}} \wedge \alpha + \frac{\beta}{x^k} = \frac{d\tilde{x}}{\tilde{x}^{k+1}} \wedge \left(\frac{1}{f^k} + \frac{\partial_{\tilde{x}} f}{f^{k+1}} \tilde{x} \right) \alpha + \frac{1}{\tilde{x}^k} \left(\frac{d_Z f}{f^{k+1}} \wedge \alpha + \frac{\beta}{f^k} \right).$$

Thus the α and β decomposition is highly dependent on the choice of Z defining function.

We will show in the course of proving Theorem 8 that the only real ambiguity above is the scaling of α by $\frac{1}{f^k}$. This apparent dependence on x is accounted for by the density bundle in equation (2.4), the restriction of a rescaled bundle to Z . A more complete discussion of densities can be found in section 4.5 of [36].

Definition 9. Let $p \in Z$. The space of s -densities on N_p^*Z is, for $s \in \mathbb{R}$, the space

$$|N_p^*Z|^s = \left\{ \psi : N_p^*Z \setminus \{0\} \rightarrow \mathbb{R} \left| \psi(\lambda V) = |\lambda|^s \psi(V) \ \forall V \in N_p^*Z \setminus \{0\}, \lambda \neq 0 \right. \right\}$$

and the s -densities on N^*Z over Z is the bundle

$$|N^*Z|^s = \bigsqcup_{p \in Z} |N_p^*Z|^s.$$

A Z defining function x induces a trivialization $|dx|^s$ of $|N^*Z|^s \rightarrow Z$. If $\tilde{x} = fx$ is another Z defining function, then $|d\tilde{x}|^s = f^s |dx|^s$ as sections of $|N^*Z|^s$.

Because ${}^{sc}TM|_Z \simeq (TZ \oplus (N^*Z \otimes TM|_Z/TZ)) \otimes N^*Z$, a density bundle appears in the cohomology groups to account for changes of Z defining function and our presentation of the cohomology is independent of Z defining function.

Morphisms of rescaled tangent bundles In 1996 Ryszard Nest and Boris Tsygan introduced the notion of b -symplectic structures [41] on Melrose's b -tangent bundle. To elaborate more on the properties of b -symplectic manifolds, Victor Guillemin, Eva Miranda,

and Ana Rita Pires [19] define the b -category. We will use this category for all rescaled bundles that arise from a sequence of rescalings of TM .

Definition 10. [19] *A b -manifold is a pair (M, Z) consisting of an oriented manifold M and an oriented hypersurface $Z \subset M$. A b -map is a map*

$$f : (M_1, Z_1) \rightarrow (M_2, Z_2),$$

(a map $f : M_1 \rightarrow M_2$ such that $f(Z_1) \subseteq Z_2$) with f transverse to Z_2 such that $f^{-1}(Z_2) = Z_1$.

The b -category is the category whose objects are b -manifolds and morphisms are b -maps.

Quick Summary. We cannot overstate the value of having $\mathcal{I}_D$ at our disposal. It is the fundamental ingredient in Melrose rescaling/lower elementary modification. The charm of this approach is that you can do sequences of rescalings to create more and more complicated Lie algebroids. We also obtain a concrete local description of sections which becomes very useful in computing. With Li's lower elementary modification there is the additional bonus of an 'undo button' called upper elementary modification. In other words, given a rescaled bundle ${}^F\mathcal{A}$, you can work backwards to recover F and $\mathcal{A}$. Further, Ralph Klaasse's language allows you to zoom out and perform operations on the Lie algebroids in a coordinate invariant way.

Unfortunately this approach cannot be used for free divisors whose vanishing ideal of functions is not locally free. In these cases we are left to directly build the desired vector bundles using the fiber bundle construction theorem.

2.6.2 Building Lie algebroids if $\mathcal{I}_D$ is not locally free

The fiber bundle construction theorem is about as down-and-dirty as it gets when it comes to building a geometric object. To show that we have a vector bundle we need the following ingredients: a base manifold M , a description of the fibers in a neighborhood of each point $p \in M$, and a set of transition functions that preserve these fibers.

Star log tangent bundle.

The star log tangent bundle is an example of a situation where we cannot use the ideal $\mathcal{I}_D$ and resort to the fiber bundle construction theorem. We will first introduce this construction in the setting of surfaces before generalizing to a manifold of any dimension.

Definition 11. *A star divisor D on a surface S is a finite collection*

$$D = \{c_1, c_2, \dots, c_n\}$$

of smooth closed curves that are pairwise transverse. For any $p \in S$, the degree of p is the number of curves of D that contain p . In other words, given $p \in S$, $\deg(p) = \#\{c_i \in D : p \in c_i\}$. Note that intersection points of D are precisely points $p \in D$ where $\deg(p) \geq 2$.

Astral atlas. We can equip a surface with a star divisor (S, D) with an atlas $\{(U_\alpha, \phi_\alpha)\}$, which we call an astral atlas, that models D as the intersection of lines at the origin in the xy -plane.

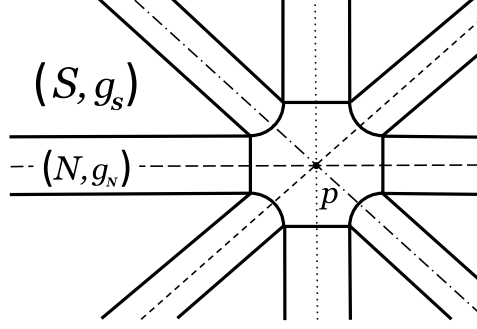
Star metric. Let (S, D) be a surface with a star divisor D . A *star metric* on S is a Riemannian metric g such that for all intersection points $p \in S$, there exists a chart (U, ϕ) around p such that each arc in $D \cap U$ is a geodesic. We will provide a sketch of Max Neumann-Coto's construction of such a metric. A more detailed argument can be found in Lemma 1.2 of his paper [42].

We start with any Riemannian metric g_S on our surface S and a regular neighborhood N of D . By regular neighborhood we mean a union of neighborhoods $c_i \times (-\varepsilon, \varepsilon)$ such that the intersection of any two is either empty or diffeomorphic to $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$.

We put a flat metric g_N on N to make the rectangles and polygons Euclidean and so that each c_i is a geodesic with respect to g_N . Let $\{p_1, \dots, p_m\}$ be the intersection points of (S, D) . Recall that in a closed surface S , an essential curve γ is one that is not homotopic to a point, that is $[\gamma] \in \pi^1(S)$ is nontrivial. In any compact neighborhood of each p_j , there is a lower bound for the lengths of essential curves in $S \setminus N$. We scale g_S by some positive real number k so that the lengths of these essential curves with respect to kg_S are greater

than the lengths of each c_i with respect to g_N .

Figure 10. *Constructing g .*



Using appropriate bump functions, we construct a new metric on S by patching kg_S and g_N together. Intuitively, this new metric makes S look like a mountain range around each intersection point p_j with the curves in D cutting out the ravines between peaks.

Star charts. A star metric allows us to construct charts that model the intersection of curves in D as the intersection of lines in the plane.

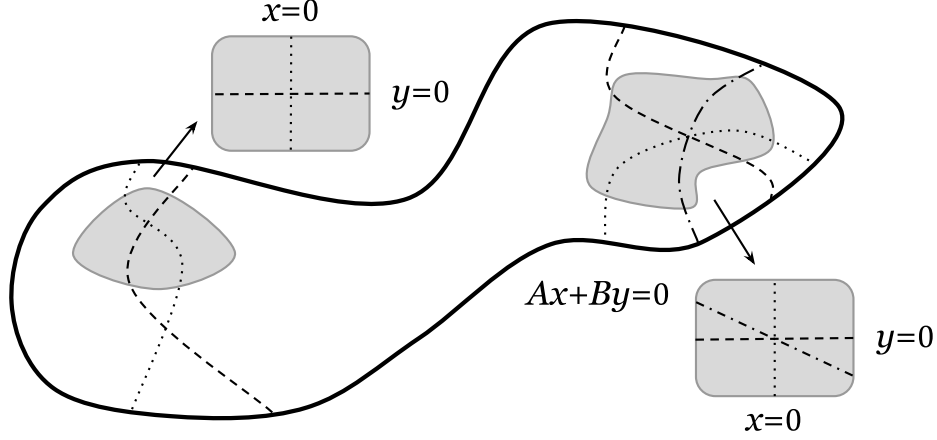
Lemma 3. *Let (S, D) be a surface with a star divisor D . Let g be a star metric on (S, D) . Fix an ordering $\{c_1, \dots, c_n\}$ of the curves in D . At every point $p \in S$ of degree ≥ 2 , there exists a coordinate chart $(U, (x, y))$ centered at p such that the following conditions are satisfied.*

1. *Given $(\alpha, \beta) = \min \{(i, j) \in \mathbb{N} \times \mathbb{N} : i < j \text{ and } p \in c_i, p \in c_j\}$,
 $c_\alpha \cap U = \{x = 0\}$ and $c_\beta \cap U = \{y = 0\}$.*
2. *For each c_ℓ with $p \in c_\ell$ and $\ell \notin \{\alpha, \beta\}$, there exist non-zero real numbers A_ℓ and B_ℓ such that*

$$c_\ell \cap U = \{A_\ell x + B_\ell y = 0\}.$$

If $p \in S$ is an intersection point, then charts centered at p satisfying conditions (1) and (2) from Lemma 3 are called *star charts*. If $p \in S$ is not an intersection point, then any chart around p not containing an intersection point is called a *star chart*. An *astral atlas* is an atlas consisting of star charts.

Figure 11. *Star charts of (S, D) .*



Proof. (of Lemma 3) Let (S, D) be a surface with a star divisor D . Consider an atlas of Riemannian normal coordinate charts. Refine this to another atlas $\{(U, \phi)\}$ by taking charts that are also star charts.

Fix an ordering $\{c_1, \dots, c_n\}$ on the elements of D . Given an intersection point $p \in S$ and a chart (U, ϕ) , centered at p , from our atlas, we will take a linear isomorphism of this chart such that the two curves passing through p with the smallest indices are mapped to $\{x = 0\}$ and $\{y = 0\}$ respectively. An astral atlas consists of charts that have been transformed according to this ordering on D . $\square$

Transition functions between star charts. Let (S, D, g) be a surface, star divisor, and star metric. Given an ordering on D , let $\{(U_\alpha, \phi_\alpha)\}$ be an astral atlas for the triple (S, D, g) . Let $(U, \phi) = (U, (x, y))$ and $(V, \rho) = (V, (\tilde{x}, \tilde{y}))$ be two star charts around an intersection point $p \in S$. Further assume $\deg(p) \geq 3$.

Then $\phi(U \cap D)$ is the intersection of at least 3 lines at a point. Let c_1, c_2, c_3 denote the curves of minimal index passing through p . Then c_1 is locally given by $\{x = 0\}$ and c_2 is locally given by $\{y = 0\}$. Further, c_3 is defined by an equation of the form $ax + by = 0$ for non-zero real numbers a, b .

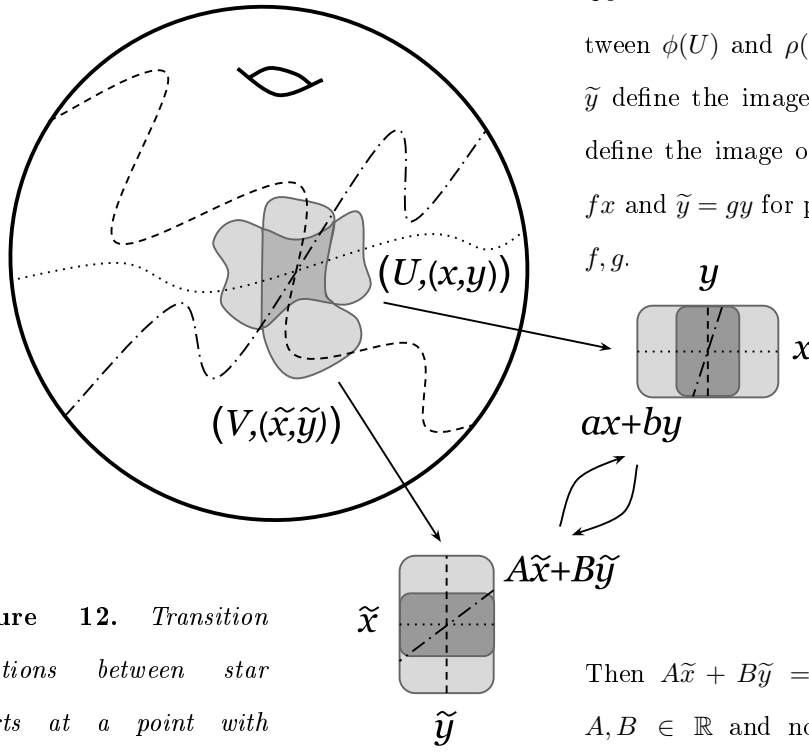


Figure 12. *Transition functions between star charts at a point with degree at least three.*

Consider the transition functions between $\phi(U)$ and $\rho(V)$. In $\rho(V)$, let $\tilde{y}$ define the image of c_1 and let $\tilde{x}$ define the image of c_2 . Then $\tilde{x} = fx$ and $\tilde{y} = gy$ for positive functions f, g .

$$y$$

$$x$$

$$ax+by$$

$$A\tilde{x}+B\tilde{y}$$

$$\tilde{x}$$

$$\tilde{y}$$

Then $A\tilde{x} + B\tilde{y} = h(ax + by)$ for $A, B \in \mathbb{R}$ and nonvanishing $h \in C^\infty(U \cap V)$. Thus we have $(Af - ha)x = (hb - Bg)y$.

Accordingly, there is a function $M \in C^\infty(U \cap V)$ such that

$$(Af - ha) = My \text{ and } (hb - Bg) = Mx.$$

Thus

$$f = (ax + by) \frac{M}{Ab} + \frac{aB}{Ab} g.$$

Notice that when $ax + by = 0$,

$$f = \frac{aB}{Ab} g.$$

Thus at the curve c_3 , transition functions scale the x and y direction by the same function up to a constant. Note that off of the lines, the transition functions are less restricted - due to the function M - and need only give a local diffeomorphism.

In the case when we have at least four curves, we can extract further restrictions for transition functions. Given a chart centered at the intersection of $n \geq 4$ curves, we will show that this constant must be the same at each curve $c_3, \dots, c_n$. Consider the line c_4

given by $cx + ey = 0 = C\tilde{x} + E\tilde{y}$, for some $c, e, C, E \in \mathbb{R}$. Then for some smooth function $N \in \mathcal{C}^\infty(U \cap V)$,

$$f = (cx + ey)\frac{N}{Ce} + \frac{cE}{Ce}g.$$

Thus

$$(ax + by)\frac{M}{Ab} + \frac{aB}{Ab}g = (cx + ey)\frac{N}{Ce} + \frac{cE}{Ce}g \quad (2.5)$$

By considering the value of equation (2.5) at the origin, we conclude that $\frac{cE}{Ce} = \frac{aB}{Ab}$.

In the next subsection we will use our understanding of transition functions in an astral atlas to argue that we have a certain fiber bundle structure. For a technical reason in that computation, in degree 3 charts we must impose the condition that our transition functions fix four lines. Below is a summary of the properties we have observed and this technical assumption.

Properties of transition functions between star charts at intersections of $\deg \geq 3$. *Let (S, D, g) be a surface, star divisor, and star metric. Given an ordering on D , let $\{(U_\alpha, \phi_\alpha)\}$ be an astral atlas for the triple (S, D, g) . Let $(U, \phi) = (U, (x, y))$ and $(V, \rho) = (V, (\tilde{x}, \tilde{y}))$ be two star charts around an intersection point $p \in S$. Further assume $\deg(p) \geq 3$.*

There are non-vanishing functions $f, g \in \mathcal{C}^\infty(U \cap V)$ so that x and y transition to $\tilde{x} = fx$ and $\tilde{y} = gy$. For every line ℓ in $D \cap (U \cap V)$, excluding the lines defined by $\{x = 0\}$ and $\{y = 0\}$, there is a real number c such that $f|_\ell = cg|_\ell$.

A necessary technical assumption at intersections of $\deg = 3$. *In charts centered at points of intersection with $\deg = 3$, there exists a line through the origin $L \notin D \cap (U \cap V)$ such that $f|_L = cg|_L$.*

b-tangent bundle over an astral atlas. Given a surface S with star divisor D , we will use an astral atlas to construct a vector bundle whose smooth sections are vector fields tangent to D . We will begin by defining a vector bundle over each chart. Then, by arguing that all the transition maps are compatible with this structure, we will show that we have in fact constructed a rank 2 vector bundle. Let $p_1, \dots, p_m$ be all the points of S with degree greater than 2. Consider the surface $\tilde{S} = S \setminus \{p_1, \dots, p_m\}$. Then

$$(\tilde{S}, \tilde{D}) = (S \setminus \{p_1, \dots, p_m\}, D \cap \tilde{S})$$

is a surface with normal crossing divisor, that is, a surface with a set of smooth curves that intersect transversely. The b-tangent bundle ${}^bT\tilde{S}$, referred to as the log tangent bundle ${}^{log}T\tilde{S}$ in [18], is the vector bundle whose smooth sections are the vector fields of $\tilde{S}$ that are tangent to $\tilde{D}$. In other words, the b-tangent bundle has smooth sections

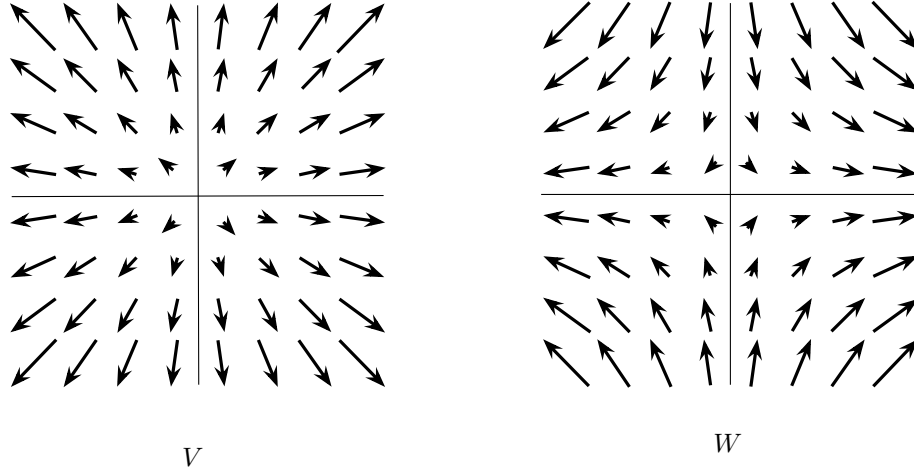
$$\left\{ u \in \mathcal{C}^\infty(\tilde{S}, T\tilde{S}) : u|_c \in \mathcal{C}^\infty(c, Tc) \text{ for all } c \in \tilde{D} \right\}.$$

This will be our vector bundle away from $p_1, \dots, p_m$, the intersection points of degree greater than two. Let $\{(U_\alpha, \phi_\alpha)\}$ be an astral atlas of (S, D) .

Consider a degree k point $p \in S$ where $k \geq 3$. Then p sits in some curves $c_1, \dots, c_k$. Let $(U, (x, y))$ be a star chart around p . Consider the vector fields, depicted in Figure 13,

$$V = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \text{ and } W = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

Figure 13. *Vector fields on TU .*



Away from c_1 and c_2 , these generate TU . Further, away from the origin, V is tangent to all lines passing through the origin.

Let $c_3, \dots, c_k$ be locally defined by $A_3x + B_3y, \dots$, and $A_kx + B_ky$ respectively, i.e. $c_i \cap U = \{A_ix + B_iy = 0\}$. Then we define two vector fields

$$V = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and} \quad \tilde{W} = \prod_{i=3}^k (A_ix + B_iy) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right). \quad (2.6)$$

In the following lemma we will show that V and $\widetilde{W}$ provide a basis for the space of smooth vector fields tangent to $D \cap U$.

Lemma 4. *Let $(U, (x, y))$ be a star chart around a point $p \in S$ of degree at least 3. Any vector field $V \in \mathcal{C}^\infty(U; TU)$ that is tangent to $D \cap U$ can be expressed as a $\mathcal{C}^\infty(U)$ -linear combination of the vector fields given in equation (2.6).*

Proof. Given a star chart $(U, (x, y))$ centered at p , by definition the set $D \cap U$ is given by the lines $x = 0, y = 0, A_3x + B_3y = 0, \dots, A_kx + B_ky = 0$. We want to find all vector fields that are tangent to these lines.

An arbitrary smooth vector field in U has the form

$$V = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}$$

for $a(x, y), b(x, y) \in \mathcal{C}^\infty(U)$.

Claim: The functions a and b satisfy

$$a = x \left(\frac{hy \prod_{i=4}^k (A_i x + B_i y)}{A_3} + c_3 \right) \text{ and } b = y \left(c_3 - \frac{hx \prod_{i=4}^k (A_i x + B_i y)}{B_3} \right)$$

for smooth functions $h, c_3 \in \mathcal{C}^\infty(U)$. Moreover, any choice of h and c_3 determine a vector field tangent to $D \cap U$.

We will prove this claim by induction on the degree of p .

Base case: Assume $\deg p = 3$. Notice that V is tangent to the line $x = 0$ if and only if

$$\left\langle V, \frac{\partial}{\partial x} \right\rangle_{g_{\mathbb{R}^2}} \Big|_{\{x=0\}} = 0.$$

Equivalently, V is tangent to $x = 0$ if and only if $a(x, y)|_{\{x=0\}} = 0$, i.e.

$$a(x, y) = a'(x, y)x \tag{2.7}$$

for $a'(x, y) \in \mathcal{C}^\infty(U)$. Similarly, V is tangent to the line $y = 0$ if and only if $b(x, y)|_{\{y=0\}} = 0$.

That is,

$$b(x, y) = b'(x, y)y \tag{2.8}$$

for $b'(x, y) \in \mathcal{C}^\infty(U)$. Because the gradient of a function is normal to its level sets, V is tangent to $A_3x + B_3y = 0$ precisely when

$$\left\langle V \cdot \left(A_3 \frac{\partial}{\partial x} + B_3 \frac{\partial}{\partial y} \right) \right\rangle_{g_{\mathbb{R}^2}} \Big|_{\{A_3x+B_3y=0\}} = 0.$$

Equivalently, V is tangent to $A_3x + B_3y = 0$ if and only if

$$A_3a(x, y) + B_3b(x, y) = c_3(x, y)(A_3x + B_3y) \quad (2.9)$$

for $c_3(x, y) \in \mathcal{C}^\infty(U)$.

We want to find all possible a, b satisfying equations (2.7), (2.8), and (2.9). Substituting $a = a'x$ and $b = b'y$, we have

$$A_3a'x + B_3b'y = c_3(A_3x + B_3y).$$

Rearranging terms,

$$xA_3(a' - c_3) = yB_3(c_3 - b').$$

Notice that y divides both the right and left hand sides, and thus

$$A_3(a' - c_3) = ey$$

for some $e \in \mathcal{C}^\infty(U)$. Further, $B_3(c_3 - b') = ex$. We can solve for a' and b' respectively:

$$a' = \frac{ey}{A_3} + c_3 \text{ and } b' = c_3 - \frac{ex}{B_3}.$$

Thus, we have shown that if V is tangent to the lines $x = 0, y = 0$, and $A_3x + B_3y = 0$, then

$$a = x \left(\frac{ey}{A_3} + c_3 \right) \text{ and } b = y \left(c_3 - \frac{ex}{B_3} \right).$$

Note that for any smooth functions $c_3, e \in \mathcal{C}^\infty(U)$, we have

$$a = \mathcal{O}(x), b = \mathcal{O}(y), \text{ and } A_3a + B_3b = \mathcal{O}(A_3x + B_3y).$$

Consequently, there are no restrictions on c_3 or e and we have a characterization of all vector fields that are tangent to these three lines.

Induction step: Assume that the claim is true for a point p satisfying $\deg p = n - 1$. We will verify the formula at a point p of degree n .

The set $D \cap U$ is given by the lines

$$x = 0, y = 0, A_3x + B_3y = 0, \dots, A_nx + B_ny = 0.$$

We want to find all vector fields that are tangent to these lines. Since a vector field tangent to these lines is certainly also tangent to the lines

$$x = 0, y = 0, A_3x + B_3y = 0, \dots, A_{n-1}x + B_{n-1}y = 0,$$

then we know that a and b at least satisfy

$$a = x \left(\frac{hy \prod_{i=4}^{n-1} (A_i x + B_i y)}{A_3} + c_3 \right), b = y \left(c_3 - \frac{hx \prod_{i=4}^{n-1} (A_i x + B_i y)}{B_3} \right).$$

Notice that V is tangent to the line $A_n x + B_n y = 0$ precisely when

$$\left\langle V, \left(A_n \frac{\partial}{\partial x} + B_n \frac{\partial}{\partial y} \right) \right\rangle_{g_{\mathbb{R}^2}} \Big|_{\{A_n x + B_n y = 0\}} = 0.$$

Equivalently, V is tangent to $A_n x + B_n y = 0$ if and only if

$$A_n a(x, y) + B_n b(x, y) = c_n(x, y)(A_n x + B_n y) \quad (2.10)$$

for $c_n(x, y) \in \mathcal{C}^\infty(U)$.

By substituting in our expressions for a and b and rearranging terms, we have

$$c_3(A_n x + B_n y) + hxy \prod_{i=4}^{n-1} (A_i x + B_i y) \left(\frac{A_n}{A_3} - \frac{B_n}{B_3} \right) = c_n(A_n x + B_n y).$$

Note that $\frac{A_n}{A_3} - \frac{B_n}{B_3} = 0$ if and only if $A_n x + B_n y = 0$ defines the same line as $A_3 x + B_3 y = 0$.

Notice that $A_n x + B_n y$ divides both the right hand side and the term $c_3(A_n x + B_n y)$, and thus $h = k(A_n x + B_n y)$ for some smooth function $k \in \mathcal{C}^\infty(U)$.

Thus, we have shown if V is tangent to the lines

$$x = 0, y = 0, A_3 x + B_3 y = 0, \dots, A_n x + B_n y = 0,$$

then

$$a = x \left(\frac{ky \prod_{i=4}^n (A_i x + B_i y)}{A_3} + c_3 \right), b = y \left(c_3 - \frac{kx \prod_{i=4}^n (A_i x + B_i y)}{B_3} \right).$$

Note that for any smooth functions $c_3, k \in \mathcal{C}^\infty(U)$, we have

$$a = \mathcal{O}(x), b = \mathcal{O}(y), \dots, \text{ and } A_n a + B_n b = \mathcal{O}(A_n x + B_n y).$$

Consequently, there are no restrictions on c_3 or k and we have a characterization of all vector fields that are tangent to a set of n lines passing through the origin. By choosing $c_3 = 1, k = 0$ and $c_3 = (A_3 x - B_3 y) \prod_{i=4}^n (A_i x + B_i y)$, $k = 2A_3 B_3$, we have a local frame provided by the vector fields defined in equation 2.6.

□

Our final task to construct the b -tangent bundle over a star divisor is to show that the local trivializations V and $\widetilde{W}$ are preserved under the transition functions of an astral atlas.

Lemma 5. *Let (S, D, g) be a surface, star divisor, and star metric with an associated astral atlas $\{(U_\alpha, \phi_\alpha)\}$. The local vector fields defined in equation (2.6) and transition functions given in Section 2.6.2 define a vector bundle over S .*

Proof. Assume we are considering a chart U centered at a point p where at least three curves intersect. Let x, y , and $ax+by$ define the first three of these curves. As discussed in Section 2.6.2, when transitioning from x and y to a chart T with $\tilde{x}$ and $\tilde{y}$, we have the following relations: $\tilde{x} = fx$, $\tilde{y} = gy$, and $A\tilde{x} + B\tilde{y} = h(ax+by)$ for some nowhere vanishing functions $f, g, h \in \mathcal{C}^\infty(U \cap T)$. Further,

$$f = (ax+by)\frac{M}{Ab} + \frac{aB}{Ab}g$$

for some smooth function $M \in \mathcal{C}^\infty(U \cap T)$.

To check that V and $\widetilde{W}$ are preserved under our transition functions, it is enough to show that the corresponding co-vectors V^* and $\widetilde{W}^*$ are preserved using the same open cover and transition functions. That is, we need to check that we still have generators of the form

$$V^* = \frac{dx}{x} + \frac{dy}{y} \quad \text{and} \quad \widetilde{W}^* = \prod_{i=3}^k (A_i x + B_i y)^{-1} \left(\frac{dx}{x} - \frac{dy}{y} \right)$$

when we transition from x, y to $\tilde{x}, \tilde{y}$.

We will first consider V^* . Note that

$$\frac{d\tilde{x}}{\tilde{x}} + \frac{d\tilde{y}}{\tilde{y}} = \frac{dx}{x} + \frac{dy}{y} + \frac{df}{f} + \frac{dg}{g}.$$

Since f and g are nowhere vanishing functions, the fiber generated by V^* expressed in x, y maps to the fiber generated by V^* expressed in $\tilde{x}, \tilde{y}$.

We are left to consider $\widetilde{W}^*$. Note that

$$\prod_{i=3}^k (A_i \tilde{x} + B_i \tilde{y})^{-1} \left(\frac{d\tilde{x}}{\tilde{x}} - \frac{d\tilde{y}}{\tilde{y}} \right) = \prod_{i=3}^k (A_i f x + B_i g y)^{-1} \left(\frac{dx}{x} - \frac{dy}{y} + \frac{df}{f} - \frac{dg}{g} \right).$$

Recall from Section 2.6.2 that at the origin $f = Cg$ for some constant C . Thus

$$\widetilde{W}^* \Big|_{(0,0)} = \left(\frac{1}{Cg} \right)^{k-3+1} \prod_{i=3}^k (A_i x + B_i y)^{-1} \left(\frac{dx}{x} - \frac{dy}{y} + \frac{df}{f} - \frac{dg}{g} \right) \Big|_{(0,0)}.$$

Since g is nowhere vanishing, if we can show that $\frac{df}{f} - \frac{dg}{g}$ vanishes at the origin, we have shown the fiber generated by $\widetilde{W}^*$ expressed in x, y is preserved under transition to the fiber generated by $\widetilde{W}^*$ expressed in $\tilde{x}, \tilde{y}$.

Let ℓ and L be two distinct lines in $U \cap V$, intersecting at the origin, such that $f|_\ell = cg|_\ell$ and $f|_L = cg|_L$ for a real number c . The one-form $\frac{df}{f} - \frac{dg}{g}$ is zero when evaluated on ℓ and when evaluated on L ; since ℓ and L span the dual space at the origin, the one form must

vanish at the origin.

To conclude, because we have local trivializations of our fibers with a suitable set of transition functions, by the fiber bundle construction theorem we have constructed a vector bundle over (S, D) . $\square$

This vector bundle comes equipped with a Lie algebroid structure. The anchor map is evaluation in TS and the Lie bracket is induced by the standard Lie bracket on TS .

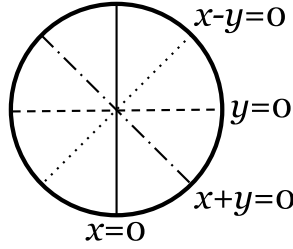
Let us consider an explicit example of a b -tangent bundle over a star divisor.

Example 34. Let $\mathbb{R}^2$ be equipped with the standard Euclidean coordinates and let star divisor D consist of the lines

$$\{x = 0\}, \{y = 0\}, \{x + y = 0\}, \text{ and } \{x - y = 0\},$$

depicted in Figure 14.

Figure 14. star divisor D



The b -tangent bundle is generated by the vector fields

$$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad (x - y)(x + y) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$$

and the b -cotangent bundle is generated by

$$\frac{dx}{x} + \frac{dy}{y}, \quad \frac{1}{(x - y)(x + y)} \left(\frac{dx}{x} - \frac{dy}{y} \right).$$

Star log tangent bundle in higher dimensions. The star log tangent bundle in two dimensions gives us most of the ingredients that we need to extend this construction to higher dimensions. Because the description is a bit notation heavy, we will first consider a simple example in $\mathbb{R}^3$.

Example 35. Consider $\mathbb{R}^3$ with standard coordinates x, y, z . We begin by considering the

normal crossing divisor - shown in grey in Figure 15 - made by the union of the coordinate planes

$$x = 0, y = 0, \text{ and } z = 0.$$

We turn this into a star divisor by taking the union with the planes

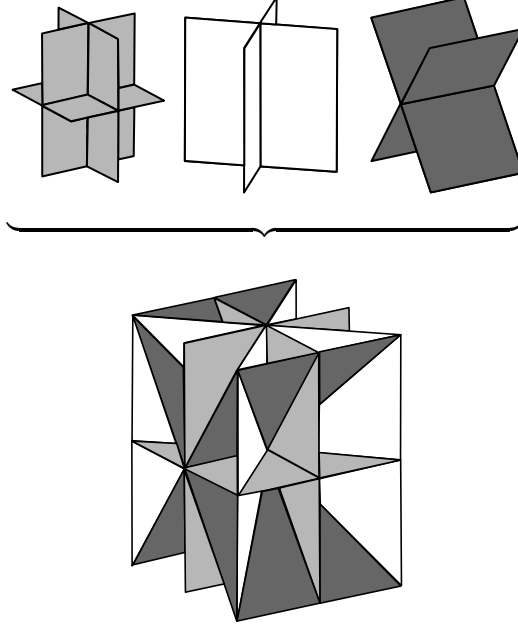
$$x - y = 0, x + y = 0$$

(the white planes) and the planes

$$x - z = 0, x + z = 0$$

(the pewter planes). The resulting star divisor is shown at the bottom of the figure.

Figure 15. *Constructing a star divisor in 3d*



All vector fields tangent to these planes can be expressed as a smooth linear combination of the vector fields

$$V = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z},$$

$$W_y = (x + y)(x - y) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right), \text{ and } W_z = (x + z)(x - z) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \right).$$

These vector fields define a rank 3 vector bundle over $\mathbb{R}^3$. Further, we can equip this bundle with a Lie bracket given by the standard Lie bracket on $\mathbb{R}^3$. To see this, we verify that our generators are closed under the Lie bracket: $[V, W_y] = 2W_y$, $[V, W_z] = 2W_z$, and

$$[W_y, W_z] = (x + y)(x - y)2W_z - (x + z)(x - z)2W_y.$$

The anchor map is given by evaluation in $T\mathbb{R}^3$.

Next we will present the general construction of the star log tangent bundle in $\mathbb{R}^{n+1}$.

Definition 12. Given $\mathbb{R}^{n+1}$ with standard coordinates $(x, z_1, \dots, z_n)$ a star divisor is a union of pairwise transverse planes of the form $x = 0$; $z_i = 0$ for all $1 \leq i \leq n$; and $A_k^i x + B_k^i z_j = 0$ for A_k^i and B_k^i real constants with indices ranging $1 \leq i \leq n$, $1 \leq k \leq m_i$ for m_i a natural number.

Of course there are free divisors other than those given by this definition that could easily be called ‘star’. Those are left for an enterprising mathematician with a penchant for index chasing. We will next show that the following vector fields provide a basis for all vector fields tangent to D :

$$V = x \frac{\partial}{\partial x} + \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}, \quad W_j = \prod_{k=3}^{m_j} (A_k^j x + B_k^j z_j) \left(x \frac{\partial}{\partial x} - 2z_j \frac{\partial}{\partial z_j} + \sum_{i=1}^n z_i \frac{\partial}{\partial z_i} \right) \quad (2.11)$$

Note that we can express W_j as $\prod_{k=3}^{m_j} (A_k^j x + B_k^j z_j)(V - 2z_j \frac{\partial}{\partial z_j})$, a fact that becomes useful in computations.

Lemma 6. *Given $(\mathbb{R}^{n+1}, x, z_1, \dots, z_n)$ with a star divisor D as described in Definition 12, any vector field that is tangent to D can be expressed as a smooth linear combination of the vector fields given in equation (2.11).*

Proof. Let X be an arbitrary vector field tangent to D . We can express X as a smooth linear combination

$$X = a \frac{\partial}{\partial x} + \sum_{i=1}^n b_i \frac{\partial}{\partial z_i}$$

for functions a, b_i . Following the procedure from the proof of Lemma 4, we take the standard dot product of A with a vector field normal to each plane and obtain the following equations (up to a nonzero real constant):

$$a = x \left(\frac{m_i z_i \prod_{k=4}^{m_i} (A_k^i x + B_k^i z_i)}{A_3^i} + n_i \right), \quad b_i = z_i \left(\ell_i - \frac{k_i x \prod_{k=4}^{m_i} (A_k^i x + B_k^i z_i)}{B_3^i} \right)$$

Note that we have n different expressions for the function a . To obtain the desired basis we will give the appropriate choice of function m_i and n_i in each of these, while being careful to maintain equality between all of the expressions.

For the basis given in equation (2.11), we take $m_i = 0$, $n_i = 1$, $\ell_i = 1$, and $k_i = 0$ to obtain V . Each W_j is obtained by taking $k_i = 0$ unless $i = j$ in which case take $k_j = 2A_3 B_3$. We take $\ell_j = (A_3^j x - B_3^j z_j) \prod_{k=4}^{m_j} (A_k^j x + B_k^j z_j)$ which forces the other ℓ_i ’s to be $\prod_{k=3}^{m_i} (A_k^j x + B_k^j z_j)$. □

To complete the proof that these fibers define a vector bundle, we need an appropriate

atlas. We will take an Affine atlas on our manifold, but restrict to those transformations which preserve the origin. In other words, if X and Y are affine spaces, we consider transformations $f: X \rightarrow Y$ of the form $x \mapsto Qx$, where Q is a linear transformation on X .

The Lie bracket will be induced by the standard Lie bracket on TM . To see that the tangent vector fields are closed under this bracket, it suffices to check that

$$[V, W_j] = \left[V, \prod_{k=3}^{m_j} (A_k^j x + B_k^j z_j) (V - 2z_j \frac{\partial}{\partial z_j}) \right] = (m_j - 3 + 1)W_j$$

and

$$[W_i, W_j] = \prod_{k=3}^{m_i} (A_k^i x + B_k^i z_i) (m_j - 3 + 1)W_j - \prod_{k=3}^{m_j} (A_k^j x + B_k^j z_j) (m_i - 3 + 1)W_i.$$

2.6.3 A final word of warning

Throughout the following chapter there will be times when we can prove that a Lie algebroid exists and can even give a description of local sections. However, we do not yet have a method - such as Melrose rescaling - for explicitly constructing the Lie algebroid. As they say, them's the breaks.

Chapter 3

Symplectic geometries

3.1 Zero symplectic geometry

3.1.1 Introduction

We will see in later sections that zero tangent bundles, and other Lie algebroids rescaled by 0, play an extremely vital role in understanding almost regular Poisson structures. However, interestingly, many of these algebroids do not admit symplectic structures on manifolds of dimension greater than two! For example, each of the bundles ${}^{0^m}({}^{b^k}TM)$ defined in Example 2.6.1 fail to admit a symplectic form for M of dimension greater than 2. These examples all satisfy a certain fiber description: let the fibers of an algebroid $\mathcal{A}$ over (M, Z) and its dual $\mathcal{A}^*$ satisfy

$$\mathcal{A}_p \simeq T_p M \text{ if } p \notin Z \text{ and } \mathcal{A}_p \simeq x^m(T_p Z) + \langle x^{k+m} \frac{\partial}{\partial x} \rangle \text{ if } p \in Z \quad (3.1)$$

$$\mathcal{A}_p^* \simeq T_p^* M \text{ if } p \notin Z \text{ and } \mathcal{A}_p^* \simeq \frac{1}{x^m}(T_p^* Z) + \langle \frac{dx}{x^{k+m}} \rangle \text{ if } p \in Z$$

where x is a defining function for Z .

Proposition 3. *Let M be a manifold of dimension greater than 2 and Z any non-empty hypersurface $Z \subset M$. Let $\mathcal{A}$ be an algebroid over (M, Z) satisfying (3.1). If $m > 0$, and $k \neq 1$, then M does not admit an $\mathcal{A}$ -symplectic structure.*

Proof. An $\mathcal{A}$ -symplectic form $\omega_{\mathcal{A}} \in {}^{\mathcal{A}}\Omega^2(M)$ would be expressible in a tubular neighborhood of Z as

$$\omega_{\mathcal{A}} = \frac{dx}{x^{k+m}} \wedge \alpha + \frac{\beta}{x^m}$$

for a smooth 1-form α and smooth 2-form β on M . Consider

$$d\omega_{\mathcal{A}} = -\frac{dx}{x^{k+m}} \wedge d\alpha + \frac{d\beta}{x^m} - m\frac{dx}{x^{m+1}} \wedge \beta.$$

Because $k + m \neq m + 1$, closedness of $\omega_{\mathcal{A}}$ provides the relation $\beta = 0$. However, $\omega_{\mathcal{A}}$ non-degenerate means

$$\wedge^n(\omega_{\mathcal{A}}) = \frac{dx}{x^{(k+mn)}} \wedge \alpha \wedge \beta^{n-1} \neq 0.$$

Thus manifolds of dimension greater than 2 do not admit degree 2, non-degenerate, closed $\mathcal{A}$ -forms for algebroids with fibers given by equation (3.1). $\square$

However, there is still a rich zero-symplectic theory of surfaces. In order to classify these structures, we will need an explicit description of zero-cohomology. Further, the Lie algebroid cohomology of the zero tangent bundle of any even dimensional manifold computes the Poisson cohomology of certain almost regular Poisson manifolds.

3.1.2 Zero cohomology

Theorem 3. *Let (M, Z) be a manifold M with hypersurface $Z \subset M$. Then the Lie algebroid cohomology of the zero tangent bundle 0TM over (M, Z) is*

$${}^0H^p(M) \simeq H^p(M) \oplus H^{p-1}(Z).$$

Proof. We begin by considering the short exact sequence introduced in equation (2.1):

$$0 \rightarrow \Omega^p(M) \xrightarrow{i^*} {}^0\Omega^p(M) \xrightarrow{P} \mathcal{C}^p \rightarrow 0.$$

Our short exact sequence gives us a long exact sequence in cohomology. We will compute the boundary map $\delta : H^p(\mathcal{C}) \rightarrow H^{p+1}(M)$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^p(M) & \longrightarrow & {}^0\Omega^p(M) & \xrightarrow{P} & \mathcal{C}^p \longrightarrow 0 \\ & & \downarrow & & \downarrow d & & \downarrow \\ 0 & \longrightarrow & \Omega^{p+1}(M) & \xrightarrow{i^*} & {}^0\Omega^{p+1}(M) & \longrightarrow & \mathcal{C}^{p+1} \longrightarrow 0 \end{array}$$

Given a choice x of Z defining function, any element in $\mathcal{C}^p$ can be represented by a Taylor series expansion in x of the form

$$c = \frac{\alpha_0}{x^p} + \frac{\alpha_1 x}{x^p} + \dots + \frac{\alpha_{p-1} x^{p-1}}{x^p}$$

where α_i is a degree- p smooth form with support in Z . Given any $y \in {}^0\Omega^p(M)$,

$$y = \nu + \frac{\alpha_0}{x^p} + \frac{\alpha_1 x}{x^p} + \dots + \frac{\alpha_{p-1} x^{p-1}}{x^p}$$

for some smooth form ν . Then we may write y as

$$\frac{x^p}{x^p} \nu + \frac{\alpha_0}{x^p} + \frac{\alpha_1 x}{x^p} + \dots + \frac{\alpha_{p-1} x^{p-1}}{x^p}.$$

Then $P(y) = c$. Note that

$$dy = \frac{x^{p+1}}{x^{p+1}} d\nu - \frac{pdx}{x^{p+1}} \wedge \alpha_0 + \frac{xd\alpha_0}{x^{p+1}} - \dots - \frac{dx}{x^{p+1}} \wedge \alpha_{p-1} x^{p-1} + \frac{x^p d\alpha_{p-1}}{x^{p+1}}$$

and no terms in the expression

$$-\frac{pdx}{x^{p+1}} \wedge \alpha_0 + \frac{xd\alpha_0}{x^{p+1}} - \dots - \frac{dx}{x^{p+1}} \wedge \alpha_{p-1} x^{p-1} + \frac{x^p d\alpha_{p-1}}{x^{p+1}}$$

are smooth. Thus terms in $\mathcal{C}^p$ are closed if and only if their counterparts in ${}^0\Omega^p(M)$ are closed.

Now assume $c \in \ker({}^{\mathcal{C}}d : \mathcal{C}^p \rightarrow \mathcal{C}^{p+1})$. Given any $y \in {}^0\Omega^p(M)$, $y = \nu + c$ for some smooth form ν . Then we may write y as $\frac{x^p}{x^p} \nu + c$ and $P(y) = c$. Note by the observation above that $dy = \frac{x^{p+1}}{x^{p+1}} d\nu$. Thus the unique element b that satisfies $i^*(b) = dy$ is $d\nu$. Since $d\nu$ is exact, we can conclude that $[b] = 0$ in cohomology. Hence our boundary map is zero and

$${}^0H^p(M) \simeq H^p(M) \oplus H^p((\mathcal{C}^*, {}^{\mathcal{C}}d)).$$

Next we will compute $H^p((\mathcal{C}^*, {}^{\mathcal{C}}d))$. Given a choice x of Z defining function, we can explicitly write

$$P\left(\frac{dx}{x^p} \wedge \alpha + \frac{\beta}{x^p}\right) = \frac{dx}{x^p} \wedge \left(\sum_{i=0}^{p-1} \bar{\alpha}_i x^i\right) + \frac{1}{x^p} \left(\sum_{i=0}^{p-1} \bar{\beta}_i x^i\right).$$

Then

$$\begin{aligned} P\left({}^0d\left(\frac{dx}{x^p} \wedge \alpha + \frac{\beta}{x^p}\right)\right) &= P\left(\frac{dx}{x^{p+1}} \wedge (-xd\alpha + x\partial_x\beta - p\beta) + \frac{x}{x^{p+1}} d_Z\beta\right) \\ &= \frac{dx}{x^{p+1}} \wedge \left(-p\bar{\beta}_0 + \sum_{i=1}^{p-1} x^i (-d_z\bar{\alpha}_{i-1} - (p-1)\bar{\beta}_i) + x^p(-d_z\bar{\alpha}_{p-1})\right) + \frac{1}{x^{p+1}} \left(\sum_{i=1}^p x^i d_z\bar{\beta}_{i-1}\right), \end{aligned}$$

where $\bar{\alpha}_i, \bar{\beta}_i$ are smooth forms supported in Z . Thus elements in the kernel of ${}^{\mathcal{C}}d$ satisfy

the relations $\bar{\beta}_0 = 0$, $-d_z\bar{\alpha}_i = (p-i)\bar{\beta}_i$ for all $i = 1, 2, \dots, p-1$, and $d_z\bar{\alpha}_{p-1} = 0$.

Given $\frac{x^i}{x^{p-1}} \bar{\alpha}_i \in \mathcal{C}^{p-1}$ for $i = 0, \dots, p-2$, we have

$${}^{\mathcal{C}}d\left(\frac{x^i}{x^{p-1}} \bar{\alpha}_i\right) = \frac{dx}{x^p} \wedge (-(p-1-i)\bar{\alpha}_i x^i) + \frac{1}{x^p} x^{i+1} d_z\bar{\alpha}_i.$$

Given

$$\frac{dx}{x^{p-1}} \wedge \bar{\gamma}_{p-2} x^{p-2} \in \mathcal{C}^{p-1},$$

we have

$$\mathcal{C}d\left(\frac{dx}{x^{p-1}} \wedge \bar{\gamma}_{p-2}x^{p-2}\right) = \frac{dx}{x^p} \wedge x^{p-1}d\bar{\gamma}_{p-2}.$$

Thus, given Z defining function x , elements of $H^p(\mathcal{C})$ are

$$\left[\frac{dx}{x^p} \wedge \bar{\alpha}_{p-1}x^{p-1}\right]$$

where

$$\left[\frac{dx}{x^p} \wedge d\bar{\gamma}_{p-2}x^{p-1}\right] = 0.$$

Next we will show that this cohomology class is independent of the choice x of Z defining function. Let $\tilde{x} = \phi x$ for a positive function $\phi \in \mathcal{C}^\infty(M)$. Then $\frac{d\tilde{x}}{\tilde{x}} = \frac{dx}{x} + \frac{d\phi}{\phi}$. Because ϕ is positive, $\frac{d\phi}{\phi}$ gives no contribution to the cohomology of $\mathcal{C}^p$. Then $\frac{d\tilde{x}}{\tilde{x}^p} \wedge \bar{\alpha}_{p-1}\tilde{x}^{p-1} = \frac{1}{\phi^{p-1}} \frac{dx}{x^p} \wedge \bar{\alpha}_{p-1}\phi^{p-1}x^{p-1}$.

Thus

$$H^p(\mathcal{C}) \simeq \frac{\{\bar{\alpha}_{p-1} \in \Omega^{p-1}(Z) : d_z \bar{\alpha}_{p-1} = 0\}}{\{d_z \bar{\gamma}_{p-2} : \bar{\gamma}_{p-2} \in \Omega^{p-2}(Z)\}}$$

and

$${}^0H^p(M) \simeq H^p(M) \oplus H^{p-1}(Z).$$

□

3.2 Scattering symplectic geometry

3.2.1 Introduction

Scattering-symplectic geometry includes the study of the standard Euclidean symplectic form at infinity. To be precise, let $(p_1, q_1, \dots, p_n, q_n)$ be the standard coordinates on $\mathbb{R}^{2n}$. Let

$$R = \sqrt{p_1^2 + q_1^2 + \dots + p_n^2 + q_n^2}$$

be the radial coordinate and away from the origin let $x = \frac{1}{R}$. Then in coordinates $t_i = p_i x$, $s_i = q_i x$ the standard symplectic form is expressible as

$$\omega = \sum_i dp_i \wedge dq_i = \sum_i \frac{dx}{x^3} \wedge (s_i dt_i - t_i ds_i) + \frac{1}{x^2} dt_i \wedge ds_i = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2}$$

where $\alpha = s_i dt_i - t_i ds_i$ defines the standard contact structure on $\mathbb{S}^{2n-1}$. We compactify $\mathbb{R}^{2n}$ with a sphere at infinity given by the zero set of x . This compactified space equipped with ω is an example of a scattering-symplectic manifold (with boundary).

In fact, any scattering-symplectic form ω on any manifold M will define a contact structure on the singular locus of ω . Accordingly, scattering-symplectic geometry not only gives us an approach to studying Poisson geometry, but contact geometry as well.

Simple Poisson Structures on Spheres. Because every closed symplectic manifold has non-trivial degree 2 de Rham cohomology group, the only sphere that admits a symplectic structure is the 2 dimensional sphere $\mathbb{S}^2$. An immediate question when expanding the notion of symplectic is to look for these structures on spheres.

Theorem 4. *Every even dimensional sphere $\mathbb{S}^{2n}$ admits a scattering symplectic structure ω such that the equator $\mathbb{S}^{2n-1}$ is the singular hypersurface and ω induces the standard contact structure on $\mathbb{S}^{2n-1}$.*

One very natural context that provides a generalized symplectic structure on spheres is folded symplectic geometry. Richard Melrose [34], by defining folded contact structures, introduced a particular idea of minimally degenerate differential form. Building on [34], Ana Cannas da Silva, Victor Guillemin, and Christopher Woodward [10] define a folded symplectic manifold (M^{2n}, Z, ω) to be a $2n$ -dimensional manifold M equipped with a closed

two-form ω that is non-degenerate except on a hypersurface Z where there exist coordinates such that locally $Z = \{x_1 = 0\}$ and

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i.$$

By allowing this very mild degeneracy, all even dimensional spheres admit a folded symplectic structure. Folded symplectic geometry is similar in spirit to our main object of study (or to $\mathcal{A}$ -symplectic structures). One significant difference is that $\mathcal{A}$ -symplectic structures correspond to classical minimally-degenerate Poisson structures. The inverse of the morphism $\omega^\flat : TM \rightarrow T^*M$ (where it is invertible) for a folded symplectic form does not extend to define a Poisson structure on the entire manifold.

Unfortunately, there are no b -Poisson spheres in dimensions greater than two: Ioan Mărcuț, and Boris Osorno Torres [30] showed that a compact b -symplectic manifold (M, Z) of dimension $2n$ has a class c in $H^2(M)$ such that c^{n-1} is nonzero in $H^{2n-2}(M)$.

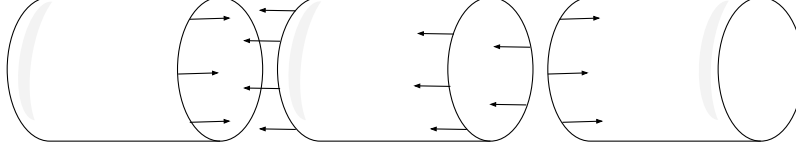
Symplectic Gluing. Recall that a symplectic manifold (M, ω) is a strong symplectic filling of a contact manifold (Z, ξ) if Z is the boundary of M and near Z there is a Liouville vector field V transverse to Z with $i_V \omega$ defining the contact structure ξ such that $\mathcal{L}_V \omega = \omega$. These fillings come in two flavors: *convex* means the Liouville vector field V points outward at the boundary Z and *concave* means V points inward at the boundary.

In order to glue two strong symplectic fillings along a common contact boundary to form a symplectic manifold, one side must be a concave filling and the other must be convex. In this paper, we demonstrate a natural way to expand the symplectic category to allow gluings of convex to convex and concave to concave fillings.

Theorem 5. *Given two strong convex symplectic fillings of a contact manifold, their union over Z admits a scattering symplectic structure that coincides with the existing symplectic structures away from Z .*

Given two strong concave symplectic fillings of a contact manifold, their union over Z admits a folded symplectic structure that coincides with the existing symplectic structures away from Z .

Figure 16. *On the left, convex to convex produces scattering symplectic. On the right, concave to concave gives folded symplectic.*



We can use this theorem to construct many examples of scattering-symplectic manifolds, see section 3.2.4. For instance, $\mathbb{T}^2 \times \mathbb{S}^2$ is scattering-symplectic with singular hypersurface three torus $\mathbb{T}^3$. We also have that $\mathbb{S}^3 \times \mathbb{S}^1$ is scattering-symplectic with singular hypersurface $\mathbb{S}^2 \times \mathbb{S}^1$. While many scattering symplectic manifolds arise in this way, not all such structures can be obtained by gluing two strong convex fillings, see proposition 11.

Poisson Cohomology. For simplicity, we state the result here using a fixed tubular neighborhood of Z . An invariant version can be found in Theorem 11. Given a contact structure ξ on Z , let

$$\Omega_\xi^k(Z) := \left\{ \sigma \in \Omega^k(Z) \mid \forall p \in Z, \text{supp}(\sigma_p) \subseteq \wedge^k \xi_p \right\}$$

and let α be a contact form on Z such that $\ker \alpha = \xi$.

Theorem 6. *If (M, π) is a scattering-Poisson manifold, with induced contact structure ξ on Z , then the Poisson cohomology $H_\pi^p(M)$ is*

$$H^p(M) \oplus H^{p-1}(Z) \oplus \Omega^{p-1}(Z) \oplus \Omega_\xi^{p-1}(Z) \oplus \ker(d\alpha \wedge : \Omega_\xi^{p-2}(Z) \rightarrow \Omega_\xi^p(Z)).$$

3.2.2 Scattering symplectic forms

Given a manifold and hypersurface (M, Z) , a *scattering symplectic structure* on M is a closed, non-degenerate, degree-2 scattering-form $\omega \in \mathcal{C}^\infty(M; \wedge^2({}^{sc}T^*M)) =: {}^{sc}\Omega^2(M)$.

Similar to the role that second degree de Rham cohomology $\Omega^2(M)$ plays in determining the behavior of a symplectic form ω , the second degree scattering-cohomology ${}^{sc}\Omega^2(M)$ will play a key role in determining the behavior of a scattering-symplectic form ω . Next we obtain an explicit description of this cohomology.

Scattering de-Rham cohomology

Our computation of scattering de-Rham cohomology utilizes the following result of Rafe Mazzeo and Richard Melrose ([36], Prop. 2.49).

Theorem 7 (Mazzeo-Melrose). *Let bTM be the b -tangent bundle associated to (M, Z) . The b -cohomology is*

$${}^bH^p(M) \simeq H^p(M) \oplus H^{p-1}(Z).$$

Theorem 8. *Let (M, Z) be a manifold M with hypersurface $Z \subset M$. Then ${}^{sc}H^p(M)$, the Lie algebroid cohomology of the scattering tangent bundle ${}^{sc}TM$ over (M, Z) , is isomorphic to*

$${}^bH^p(M) \oplus \Omega^{p-1}(Z; |N^*Z|^{-p}) \simeq H^p(M) \oplus H^{p-1}(Z) \oplus \Omega^{p-1}(Z; |N^*Z|^{-p}).$$

Proof. The bundle map $i : {}^{sc}TM \rightarrow {}^bTM$ constructed in Theorem 2 is given by evaluation and hence fits into a short exact sequence of complexes

$$0 \rightarrow {}^b\Omega^p(M) \xrightarrow{i^*} {}^{sc}\Omega^p(M) \xrightarrow{\pi} \mathcal{C}^p \rightarrow 0$$

where

$$\mathcal{C}^p = {}^{sc}\Omega^p(M) / {}^b\Omega^p(M).$$

The differential on $\mathcal{C}^p$ is induced by the differential ${}^{sc}d$ on ${}^{sc}\Omega^p(M)$: if π is the projection ${}^{sc}\Omega^p(M) \rightarrow {}^{sc}\Omega^p(M) / {}^b\Omega^p(M)$, then $\mathcal{C}^p d(\eta) = \pi({}^{sc}d(\theta))$ where $\theta \in {}^{sc}\Omega^p(M)$ is any form such that $\pi(\theta) = \eta$. Hence $(\mathcal{C}^p d)^2 = 0$ and $(\mathcal{C}^*, \mathcal{C}^p d)$ is in fact a complex.

Given a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of M near Z , note that x defines a trivialization $t_x : N^*Z \rightarrow \mathbb{R}$ of N^*Z . We can write a degree p scattering form $\nu \in {}^{sc}\Omega^p(M)$ as

$$\nu = \theta + \sum_{i=0}^{p-1} \left(\frac{dx}{x^{p+1}} \wedge \alpha_i x^i + \frac{\beta_i x^i}{x^p} \right)$$

where $\theta \in {}^b\Omega^p(M)$, and $\alpha_i, \beta_i \in \Omega^*(Z) \simeq \Omega^*(Z; |N^*Z|^{-p})$ by $(t_x)_*$.

We write $\mathcal{R}_b(\nu) = \theta$ and $\mathcal{S}_b(\nu) = \nu - \mathcal{R}_b(\nu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\mathcal{R}_b({}^{sc}d\nu) = {}^{sc}d(\mathcal{R}_b(\nu))$ and $\mathcal{S}_b({}^{sc}d\nu) = {}^{sc}d(\mathcal{S}_b(\nu))$. Thus the trivialization τ induces a splitting ${}^{sc}\Omega^*(M) = {}^b\Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence we have ${}^{sc}H^p(M) = {}^bH^p(M) \oplus H^p(\mathcal{C}^*)$ and are left to compute the cohomology of the quotient complex.

Because

$$\pi({}^{sc}d\nu) = \frac{1}{x^{p+1}} \left(\sum_{i=0}^{p-1} \frac{dx}{x} \wedge (x^{i+1}(-d\alpha_i - (p-i)\beta_i)) + x^{i+1}d\beta_i \right)$$

then $\pi({}^{sc}d\nu) = 0$ if and only if $\beta_i = \frac{-d\alpha_i}{(p-i)}$ for all $i = 0, \dots, p-1$. Thus

$$\ker({}^{\mathcal{C}}d : \mathcal{C}^p \rightarrow \mathcal{C}^{p+1}) = \left\{ \sum_{i=0}^{p-1} \left(\frac{dx}{x^{p+1}} \wedge \alpha_i x^i - \frac{d\alpha_i x^i}{(p-i)x^p} \right) \middle| \alpha_i \in \Omega^{p-1}(Z) \right\}.$$

Now we will consider the image of ${}^{\mathcal{C}}d : \mathcal{C}^{p-1} \rightarrow \mathcal{C}^p$. There exists

$$\tilde{\nu} = \sum_{i=1}^{p-1} \frac{\alpha_i x^{i-1}}{-(p-i)x^{p-1}} \in \mathcal{C}^{p-1} \text{ such that } {}^{\mathcal{C}}d\tilde{\nu} = \sum_{i=1}^{p-1} \left(\frac{dx}{x^{p+1}} \wedge \alpha_i x^i - \frac{d\alpha_i x^i}{(p-i)x^p} \right), \text{ and hence}$$

$$\nu - d\tilde{\nu} = \frac{dx}{x^{p+1}} \wedge \alpha_0 - \frac{d\alpha_0}{px^p}.$$

Since

$$\text{Im}({}^{\mathcal{C}}d) \subseteq \{\alpha_0 = 0\},$$

this shows that each such form represents a distinct cohomology class. Thus we have identified $H^p(\mathcal{C}^*)$.

Next, consider the effect of choice of Z defining function. We will show that we can identify

$$H^p((\mathcal{C}^*, {}^{\mathcal{C}}d)) \simeq \Omega^{p-1}(Z; |N^*Z|^{-p}) \text{ by identifying } \frac{dx}{x^{p+1}} \wedge \alpha_0 - \frac{d\alpha_0}{px^p} \text{ with } \alpha_0.$$

Indeed, each choice of x defines a trivialization of N^*Z , $t_x : N^*Z \rightarrow \mathbb{R}$. Then $(t_x)_*$ gives an isomorphism $\Omega^{p-1}(Z) \simeq \Omega^{p-1}(Z; |N^*Z|^{-p})$. To see that this is well-defined, note that changing x to another Z defining function $\tilde{x}$, means that $\tilde{x} = \phi x$ for some non-vanishing function $\phi \in C^\infty(M)$. Then $\frac{d\tilde{x}}{\tilde{x}} = \frac{dx}{x} + \frac{d\phi}{\phi}$. Since ϕ is a positive function,

$$\left[\frac{d\tilde{x}}{\tilde{x}^{p+1}} \wedge \alpha_0 \right] \text{ and } \left[\frac{dx}{x^{p+1}} \wedge \frac{\alpha_0}{\phi^p} \right]$$

are representatives of the same cohomology class in $H^p(\mathcal{C})$ and

$$|d\tilde{x}|^{-p} = \phi^{-p} |dx|^{-p}$$

gives the change of trivialization of the density bundle $|N^*Z|^{-p}$. Hence, the cohomology group at $\mathcal{C}^p$ is $\Omega^{p-1}(Z; |N^*Z|^{-p})$, smooth $p-1$ forms on Z valued in $|N^*Z|^{-p}$.

We have shown that ${}^{sc}H^p(M) \simeq {}^bH^p(M) \oplus \Omega^{p-1}(Z; |N^*Z|^{-p})$. The final isomorphism is a consequence of ${}^bH^p(M) \simeq H^p(M) \oplus H^{p-1}(Z)$. $\square$

Connection to contact geometry. As we noted above, scattering-symplectic geometry includes the study of the standard Euclidean symplectic form at infinity. We also observed

that the standard form on $\mathbb{R}^{2n}$ extends to a scattering-symplectic form that induces a contact structure on the boundary sphere at infinity. Contact structures arising in this way are imposed by all scattering-symplectic structures. The existence of a global Z defining function x means Z is co-orientable. The scattering-symplectic structure imposes the further restriction that Z is also co-orientable as a contact manifold.

Proposition 4. *If (M, Z, ω) is a scattering-symplectic manifold, then ω induces a co-oriented contact structure ξ on Z .*

Proof. Let x be a Z defining function. A scattering 2-form ω can be expressed near Z as

$$\omega = \frac{dx}{x^3} \wedge \alpha + \frac{\beta}{x^2}$$

for some smooth forms $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$. Because ω is closed, $\beta = \frac{-d\alpha}{2}$ at Z .

Further, because ω is a non-degenerate scattering 2-form, we know that

$$\frac{dx}{x^{2n+1}} \wedge \alpha \wedge \left(\frac{-d\alpha}{2} \right)^{n-1} \neq 0$$

at Z . Thus $\alpha \wedge (d\alpha)^{n-1} \neq 0$ as a smooth form on Z . If we express ω using a different Z

defining function $\tilde{x}$ such that $\phi\tilde{x} = x$ for some non-vanishing function $\phi \in \mathcal{C}^\infty(M)$, then

$$\omega|_Z = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2} = \frac{1}{(\phi\tilde{x})^2} \left(\frac{d\phi}{\phi} + \frac{d\tilde{x}}{\tilde{x}} \right) \wedge \alpha - \frac{d\alpha}{2(\phi\tilde{x})^2} = \frac{d\tilde{x}}{\tilde{x}^3} \wedge \left(\frac{\alpha}{\phi^2} \right) - \frac{d(\alpha/\phi^2)}{2\tilde{x}^2}.$$

The contact form $\tilde{\alpha}$ induced by ω expressed in $\tilde{x}$ satisfies $\tilde{\alpha} = \frac{\alpha}{\phi^2}$, and thus is conformally equivalent to α . Thus the scattering symplectic form ω induces a conformal class of contact forms defining the contact structure $\ker \alpha = \xi$ on Z . $\square$

We will explore the relationship between a contact hypersurface and a scattering-symplectic manifold in Section 3.2.3. The existence of an induced contact structure evidences the fact that scattering-symplectic structures are sufficiently rigid to all locally look the same.

Proposition 5. *(a sc-Darboux theorem) Let ω be a sc-symplectic form on (M, Z) and let $p \in Z$. There exists a coordinate chart $(U, x_1, y_1, \dots, x_n, y_n)$ centered at p such that on U , the hypersurface Z is locally defined by $\{x_1 = 0\}$, and*

$$\omega = \frac{dx_1}{x_1^3} \wedge \left(dy_1 + \sum_{i=2}^n y_i dx_i - x_i dy_i \right) + \sum_{i=2}^n \frac{dx_i \wedge dy_i}{x_1^2}.$$

Proof. Let ω be a scattering symplectic form on (M, Z) . By Proposition 4, given a Z defining function x ,

$$\omega|_Z = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2}$$

where α is a contact form on Z . Let $p \in Z$. In a neighborhood $U_p \subset Z$, there exist contact-Darboux coordinates $y_1, x_2, y_2, \dots, x_n, y_n$ such that

$$\alpha = dy_1 + \sum_{i=2}^n (y_i dx_i - x_i dy_i).$$

Then

$$\omega|_Z = \frac{dx}{x^3} \wedge \left(dy_1 + \sum_{i=2}^n y_i dx_i - x_i dy_i \right) + \sum_{i=2}^n \frac{dx_i \wedge dy_i}{x^2}.$$

In the set $U_p \times \{|x| < \varepsilon\}$, choose

$$\omega_0 = \frac{dx}{x^3} \wedge \left(dy_1 + \sum_{i=2}^n y_i dx_i - x_i dy_i \right) + \sum_{i=2}^n \frac{dx_i \wedge dy_i}{x^2}.$$

Note that $(\rho_{sc}^{-1})^*(\omega - \omega_0)$ is a ‘standard’ differential form that is closed on a contractible set, so by the usual Poincaré lemma, $(\rho_{sc}^{-1})^*(\omega - \omega_0) = d\nu$ for some smooth form ν on $U_p \times \{|x| < \varepsilon\}$. By using a homotopy operator, we can assume $\nu|_Z = 0$ and we have a smooth extension of $(\rho_{sc}^{-1})^*(\omega - \omega_0)$ to 0 on Z . By Lemma 1, we have the desired result. $\square$

We can further employ Lemma 1 to establish a tubular neighborhood theorem for ω near Z .

Recall from Theorem 8 that

$${}^{sc}H^2(M) \simeq H^2(M) \oplus H^1(Z) \oplus \Omega^1(Z; |N^*Z|^{-2}).$$

Given a cohomology class $[\sigma] \in {}^{sc}H^2(M)$, we can associate to it a *decomposition* (a, b_1, b_2) where $a \in \Omega^1(Z; |N^*Z|^{-2})$, $b_1 \in H^1(Z)$, and $b_2 \in H^2(M)$. We will consider scattering symplectic forms ω and their cohomology decompositions (a, b_1, b_2) . A given Z defining function x gives us a trivialization $t_x : N^*Z \rightarrow \mathbb{R}$ and defines a smooth contact form $\alpha = (t_x)_*(a) \in \Omega^1(Z)$. We will show that for any $\beta_i \in b_i$, there is a tubular neighborhood of Z such that

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2 \beta_1) - \frac{d\alpha}{2x^2} + \beta_2. \quad (3.2)$$

Proposition 6. *Let (M, Z, ω) be a scattering symplectic manifold. Given a Z defining function x , there exists a tubular neighborhood $U \supset Z$, a contact form α , and closed forms $\beta_1 \in \Omega^1(Z)$, $\beta_2 \in \Omega^2(Z)$ such that on U there exists a scattering-symplectomorphism pulling ω back to (3.2).*

Proof. Let ω be a scattering symplectic form on a manifold (M, Z) with cohomology class

decomposition (a, b_1, b_2) . Let x be a Z defining function and $t_x : N^*Z \rightarrow \mathbb{R}$ the associated trivialization. Let α denote $(t_x)_*(a)$ and let U be a tubular neighborhood of Z . Choose a closed form β_2 that is cohomologous to $b_2|_U$ and choose a representative $\beta_1 \in b_1$. Let

$$\omega_0 = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2.$$

Then $\omega|_U - \omega_0 = d\nu$. By using a homotopy operator, we can assume $\nu|_Z = 0$ and we have a smooth extension of $(\rho_{sc}^{-1})^*(\omega|_U - \omega_0)$ to 0 on Z . By Lemma 1, we have the desired result. $\square$

In the previous two results, we wrote a scattering-symplectic form in a standard way by assuming that ω fixed a contact structure on Z . In general a scattering-symplectomorphism will induce a contactomorphism rather than merely fix the contact structure.

Proposition 7. *If there exists a scattering-symplectomorphism*

$$\Phi : (M_1, Z_1, \omega_1) \rightarrow (M_2, Z_2, \omega_2),$$

then $\Phi|_{Z_1} : Z_1 \rightarrow Z_2$ is a contactomorphism between the contact structures induced by ω_1 and ω_2 respectively.

Proof. Given a Z_2 defining function x_2 , we can write

$$\omega_2|_{Z_2} = \frac{dx_2}{x_2^3} \wedge \alpha_2 - \frac{1}{2} \frac{d\alpha_2}{x_2^2}$$

for some contact form α_2 on Z_2 . Then

$$\Phi^*\omega_2|_{Z_2} = \frac{d(\Phi^*x_2) \wedge \Phi^*\alpha_2}{(\Phi^*x_2)^3} - \frac{1}{2} \frac{\Phi^*d\alpha_2}{(\Phi^*x_2)^2} = \frac{dx_1}{x_1^3} \wedge \alpha_1 - \frac{1}{2} \frac{d\alpha_1}{x_1^2} = \omega_1|_{Z_1}$$

for some Z_1 defining function x_1 and contact form α_1 . We will compare the terms in this equality. Note since Φ preserves the singular locus of ω_1 and ω_2 , then $\Phi^*x_2 = fx_1$ for positive $f \in \mathcal{C}^\infty(M_1)$. So

$$d(\Phi^*x_2) = d(fx_1) = \frac{x_1df + fdx_1}{x_1^3f^3} \quad \text{and} \quad \frac{x_1df + fdx_1}{x_1^3f^3} \Big|_Z = \frac{dx_1}{x_1^3f^2}.$$

Then $\frac{dx_1}{x_1^3f^2} \wedge \Phi^*\alpha_2 = \frac{dx_1}{x_1^3} \wedge \alpha_1$ and thus $\Phi^*\alpha_2 = f^2\alpha_1$. $\square$

In fact, we can use certain contactomorphisms to construct local scattering-symplectomorphisms.

Proposition 8. *Let ω and $\tilde{\omega}$ be scattering-symplectic forms on (M, Z) with cohomology decompositions (a, b_1, b_2) for $[\omega]$ and $(\tilde{a}, \tilde{b}_1, \tilde{b}_2)$ for $[\tilde{\omega}]$. Let x be a Z defining function and*

consider the induced trivialization $t_x : N^*Z \rightarrow \mathbb{R}$. If there is a contactomorphism $\Phi : Z \rightarrow Z$ such that

1. $\Phi^*(t_x)_*a = f \cdot (t_x)_*\tilde{a}$ for positive $f \in C^\infty(Z)$,
2. $\Phi^*b_1 = \tilde{b}_1$, and
3. $\Phi^*b_2|_Z = \tilde{b}_2|_Z$,

then there exists a neighborhood $U \supset Z$ and a scattering-symplectomorphism $\phi : U \rightarrow U$ such that $\phi^*\omega_1 = \omega_2$.

Proof. Fix a Z defining function x and consider the induced trivialization $t_x : N^*Z \rightarrow \mathbb{R}$. Let $\alpha = \Phi^*(t_x)_*a$ and $\tilde{\alpha} = (t_x)_*\tilde{a}$. Let $\beta_1 \in b_1$ and $\beta_2 \in b_2$. By Proposition 6, there exists a tubular neighborhood $Z \times \{x \in (-\delta, \delta)\}$ on which

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2.$$

Let $\Phi : Z \rightarrow Z$ be a contactomorphism such that $\Phi^*(\alpha) = e^g\tilde{\alpha}$, for a smooth function $g \in C^\infty(Z)$. Define a function

$$f : Z \times (-\delta, \delta) \rightarrow \mathbb{R} \quad \text{by} \quad f(z, x) = \sqrt{e^{g(z)}}x.$$

Then consider

$$\Psi : Z \times (-\delta, \delta) \rightarrow Z \times (-\delta, \delta) \quad \text{given by} \quad \Psi(z, x) = (\Phi(z), f(z, x)).$$

Since

$$\frac{d(e^{g/2}x)}{e^{3g/2}x^3} = \frac{dg}{2e^g x^2} + \frac{dx}{e^g x^3},$$

we have that

$$\Psi^*(\omega) = \frac{dx}{x^3} \wedge \tilde{\alpha} + \frac{dx}{x} \wedge \Phi^*\beta_1 - \frac{d\Phi^*\alpha}{2e^g x^2} + \Phi^*\beta_2 + \frac{dg}{2x^2} \wedge \tilde{\alpha} + \frac{dg}{2} \wedge \Phi^*\beta_1.$$

Because

$$-\frac{d\Phi^*\alpha}{2e^g x^2} = -\frac{e^g dg}{2e^g x^2} \wedge \tilde{\alpha} - \frac{e^g d\tilde{\alpha}}{2e^g x^2},$$

we have that

$$\Psi^*(\omega) = \frac{dx}{x^3} \wedge \tilde{\alpha} - \frac{d\tilde{\alpha}}{2x^2} + \frac{dx}{x} \wedge \Phi^*\beta_1 + \Phi^*\beta_2 + \frac{dg}{2} \wedge \Phi^*\beta_1.$$

Note that $\Phi^*\beta_1$ is closed, so

$$d\left(\frac{g}{2}\Phi^*\beta_1\right) = \frac{dg}{2} \wedge \Phi^*\beta_1$$

is an exact form. Since $\Phi^*\tilde{b}_1 = b_1$ and $\Phi^*\tilde{b}_2|_Z = b_2|_Z$, we have $\Phi^*\beta_1 \in \tilde{b}_1$ and

$$\Phi^* \beta_2 + \frac{dg}{2} \wedge \Phi^* \beta_1 \in \tilde{b}_2.$$

Thus by Proposition 6, there exists a scattering-symplectomorphism between $\Psi^* \omega$ and $\tilde{\omega}$ on a tubular neighborhood of Z . $\square$

Scattering-Symplectic Spheres

We will conclude this section by providing an example of scattering-symplectic manifolds.

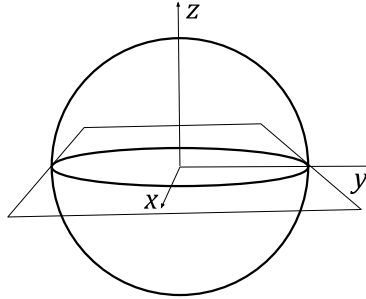
All even dimensional spheres admit scattering-symplectic structures. Let $(x_1, y_1, \dots, x_n, y_n, z)$ be global coordinates in $\mathbb{R}^{2n+1}$. Consider the sphere

$$\mathbb{S}^{2n} = \left\{ \sum_{i=1}^n (x_i^2 + y_i^2) + z^2 = 1 \right\}$$

with equator

$$\mathbb{S}^{2n-1} = \left\{ \sum_{i=1}^n (x_i^2 + y_i^2) = 1, z = 0 \right\}.$$

Figure 17. $\mathbb{S}^{2n}$ embedded in $\mathbb{R}^{n+1}$



We define a one form $\sigma = \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$. Consider the scattering form

$$\beta = -2 \frac{dz}{z^3} \wedge \sigma + \frac{1}{z^2} d\sigma$$

restricted to $\mathbb{S}^{2n}$.

Proposition 9. $(\mathbb{S}^{2n}, \mathbb{S}^{2n-1}, \beta)$ is a scattering-symplectic manifold.

Proof. First notice that $\beta = d\left(\frac{\sigma}{z^2}\right)$. Thus β is closed. We point out that this does not make β exact as a scattering form. We are left to show that β is non-degenerate on the $2n$ -sphere.

In the set $U_{x_1} := \mathbb{S}^{2n} \setminus \{x_1 = 0\}$, we have smooth coordinates $(y_1, x_2, y_2, \dots, x_n, y_n, z)$. By rewriting

$$x_1 = \sqrt{1 - y_1^2 - \sum_{i=2}^n (x_i^2 + y_i^2) - z^2}$$

and

$$dx_1 = \frac{-\left(y_1 dy_1 + \sum_{i=2}^n (x_i dx_i + y_i dy_i) + z dz\right)}{x_1},$$

we see that in U_{x_1} ,

$$\begin{aligned} \beta = & -\frac{dz}{z^3} \wedge \left(x_1 dy_1 + \frac{y_1^2 dy_1}{x_1} + \frac{y_1}{x_1} \left(\sum_{i=2}^n (x_i dx_i + y_i dy_i) \right) + \sum_{i=2}^n (x_i dy_i - y_i dx_i) \right) \\ & + \frac{1}{z^2} \frac{-\left(\sum_{i=2}^n (x_i dx_i + y_i dy_i) + z dz \right)}{x_1} \wedge dy_1 + \frac{1}{z^2} \sum_{i=2}^n (dx_i \wedge dy_i). \end{aligned}$$

The coefficient is 1 for terms of the form

$$\frac{1}{z^2} dx_i \wedge dy_i$$

for $i = 2, \dots, n$. Thus to show non-degeneracy, it suffices to show that the coefficient of the term $\frac{dz}{z^3} \wedge dy_1$ is always nonzero. This coefficient is

$$-(x_1 + \frac{y_1^2}{x_1} + \frac{z^2}{x_1}).$$

Since, $x_1 \neq 0$, this function is always nonzero in U_{x_1} . By symmetry, this argument shows that β is non-degenerate in the sets $U_{x_i} = \mathbb{S}^{2n} \setminus \{x_i = 0\}$, and $U_{y_i} = \mathbb{S}^{2n} \setminus \{y_i = 0\}$ for $i = 1, \dots, n$. We are left to consider β at the poles where z is ± 1 , that is at the points $(0, 0, 0, \dots, 0, 0, \pm 1)$. Here, $\beta = \sum_{i=1}^n dx_i \wedge dy_i$. Thus β is non-degenerate on the $2n$ -sphere. $\square$

Remark 3. For cohomological reasons, there are no symplectic spheres in dimensions greater than 2. Similarly, Mărcuț and Orsoni-Torres [30] proved that a compact b -symplectic manifold (M, Z) of dimension $2n$ has a class c in $H^2(M)$ such that c^{n-1} is nonzero in $H^{2n-2}(M)$. Thus there are also no b -symplectic spheres in dimensions greater than 2.

Remark 4. This shows for a scattering symplectic structure to exist on a compact manifold (M, Z) , Z must admit a co-orientable contact structure. Strikingly, it also shows that sometimes this is all you need!

3.2.3 Contact hypersurfaces

We have seen that every scattering-symplectic manifold has a co-oriented contact hypersurface. Now we will explore the opposite question: given a contact hypersurface, does it appear as the singular hypersurface of a scattering symplectic manifold? Given a co-oriented contact manifold (Z, α) , there always exists a non-compact scattering symplectic manifold (M, ω) with Z as singular hypersurface such that ω induces α on Z .

Proposition 10. *Let Z be a $2n - 1$ dimensional contact manifold with globally defined contact form α . Let $\tilde{Z} = Z \times \mathbb{R}$ and let $\pi : \tilde{Z} \rightarrow Z$ be the obvious projection. Let x be a coordinate on $\mathbb{R}$. Then*

$$\omega = d\left(\frac{\pi^*\alpha}{x^2}\right)$$

*is a scattering symplectic form on $(\tilde{Z}, Z)$. We call $(\tilde{Z}, Z, \omega)$ the **scattering symplectization** of (Z, α) .*

Proof. It is clear by construction that ω is closed. This does not make ω exact as a scattering form. For ease of notation we will write α rather than $\pi^*\alpha$. Then

$$\omega = -\frac{2}{x^3}dx \wedge \alpha + \frac{d_Z\alpha}{x^2}.$$

To check non-degeneracy of ω , notice that

$$\omega^n = -\frac{2}{x^{2n+1}}dx \wedge \alpha \wedge (d_Z\alpha)^{n-1}.$$

A consequence of Z being contact is that $\alpha \wedge (d_Z\alpha)^{n-1} \neq 0$. Thus ω is a closed non-degenerate scattering form on $(\tilde{Z}, Z)$. $\square$

In this construction, away from $\{x = 0\}$, the one form

$$\frac{\alpha}{x^2}$$

is a smooth primitive for ω . In particular, the vector field

$$V = -\frac{x}{2} \frac{\partial}{\partial x}$$

is non-zero when $x \neq 0$, is transverse to each level set $Z = \{x = c\}$ for nonzero constants $c \in \mathbb{R}$, and satisfies

$$i_V\omega = \alpha.$$

This precisely means that $\tilde{Z} \setminus \{|x| < \varepsilon\}$, for any $\varepsilon > 0$, is a disjoint union of strong symplectic fillings of the contact manifold (Z, α) .

This additional structure - a Liouville vector field V giving this relation between ω and α - is not a feature of all scattering-symplectic manifolds. Indeed, we will now show that whether or not a scattering-symplectic manifold is a strong symplectic filling in this sense can be read off of the sc- cohomology class of the scattering symplectic form.

Recall from Section 3.2.2, that we can associate to $[\omega] \in {}^{sc}H^2(M)$ a decomposition (a, b_1, b_2) where $a \in \Omega^1(Z; |N^*Z|^{-2})$, $b_1 \in H^1(Z)$, and $b_2 \in H^2(M)$. There is a tubular neighborhood of Z with a given Z defining function x giving us a trivialization $t_x : N^*Z \rightarrow \mathbb{R}$. By Proposition 6, this defines a smooth contact form $\alpha = (t_x)_*(a) \in \Omega^1(Z)$, $\beta_1 \in b_1$, and $\beta_2 \in b_2$ such that

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2.$$

In the following propositions we will always be working with ω in such a tubular neighborhood.

Proposition 11. *Let (M, Z, ω) be a scattering-symplectic manifold with singular contact hypersurface (Z, α) . If $[\omega]$ has cohomology decomposition (a, b_1, b_2) with b_1 or $b_2 \neq 0$, then for $\varepsilon > 0$ small, $(M \setminus \{|x| < \varepsilon\}, \omega)$ is not a strong symplectic filling of (Z, α) .*

The following lemma gives us a normal form for strong convex and strong concave fillings in a neighborhood of the boundary.

Lemma 7. *If (M, ω) is a strong convex symplectic filling of (Z, ξ) , then for some $c > 0$, there exists a collar neighborhood $Z \times [0, c)_r$ of Z on which*

$$\omega = d(e^{-r}\alpha)$$

and, given the projection $p : Z \times [0, c) \rightarrow Z$, $\alpha = p^(\tilde{\alpha})$ for an $\tilde{\alpha}$ satisfying $\ker \tilde{\alpha} = \xi$.*

If (M, ω) is a strong concave symplectic filling of (Z, ξ) , then for some $c > 0$, there exists a collar neighborhood $Z \times [0, c)_r$ of Z on which

$$\omega = d(e^r\alpha)$$

and, given the projection $p : Z \times [0, c) \rightarrow Z$, $\alpha = p^(\tilde{\alpha})$ for an $\tilde{\alpha}$ satisfying $\ker \tilde{\alpha} = \xi$.*

Proof. By definition of convex strong symplectic filling, near Z there exists a nowhere vanishing vector field v transverse to Z such that $\mathcal{L}_{-v}\omega = \omega$. By exponentiating v with respect

to any Riemannian metric, we can choose a collar neighborhood $Z \times [0, c)_r$ of Z such that r is the coordinate for $[0, c)$, $c > 0$, and

$$v|_{Z \times [0, c)} = \frac{\partial}{\partial r}.$$

It follows from the definition of strong symplectic filling that

$$(i_{-\frac{\partial}{\partial r}} \omega)|_Z = \alpha$$

for some contact form α defining ξ on Z . Given the projection

$$p : Z \times [0, c) \rightarrow Z,$$

let $\tilde{\alpha} = p^* \alpha$. Let

$$\gamma = i_{-\frac{\partial}{\partial r}} \omega.$$

Then because $\mathcal{L}_{-v} \omega = \omega$, we have $d\gamma = \omega$.

Let $x_1, x_2, \dots, x_n$ be any set of coordinates in Z . Then in these coordinates,

$$\gamma = gdr + \sum_i f_i dx_i$$

for some smooth functions $g, f_i \in \mathcal{C}^\infty(Z \times [0, c))$. We can compute

$$d\gamma = \sum_j \left(\frac{\partial g}{\partial x_j} dx_j \wedge dr + \sum_i \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i \right) + \sum_i \frac{\partial f_i}{\partial r} dr \wedge dx_i.$$

Then $i_{-\frac{\partial}{\partial r}} d\gamma = \gamma$ gives the relations

$$\sum_i \left(\frac{\partial g}{\partial x_i} - \frac{\partial f_i}{\partial r} \right) dx_i = gdr + \sum_i f_i dx_i.$$

In other words,

$$g = 0, \text{ and } -\frac{\partial f_i}{\partial r} = f_i.$$

Thus $f_i = c_i \cdot e^{-r}$ for some constants c_i . Since $\gamma|_{\{r=0\}} = \alpha$, we have that

$$\gamma = e^{-r} \tilde{\alpha} \text{ and } \omega = d(e^{-r} \tilde{\alpha}).$$

A similar computation shows the statement for a concave filling. □

We are now prepared to prove Proposition 11.

Proof. By Proposition 6, there is a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of Z such that

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2 \beta_1) - \frac{d\alpha}{2x^2} + \beta_2$$

for $\alpha, \beta_1, \beta_2 \in \Omega^*(Z)$. Assume, for a contradiction, that $M \setminus \{x < \varepsilon\}$ for $\varepsilon > 0$ is a strong symplectic filling of (Z, α) .

By Lemma 7, there is a smooth function $f \in \mathcal{C}^\infty(\tau)$ such that

$$\omega = d(f\alpha) = \partial_x f dx \wedge \alpha + d_Z f \wedge \alpha + f d\alpha.$$

Thus

$$\frac{dx}{x^3} \wedge (\alpha + \beta_1 x^2) = \partial_x f dx \wedge \alpha \text{ and } \frac{\alpha}{x^3} + \frac{\beta_1}{x} = \partial_x f \alpha.$$

We can solve for β_1 ,

$$\beta_1 = (x \partial_x f - \frac{1}{x^2}) \alpha.$$

Since β_1 is closed,

$$0 = d\beta_1 = (x \partial_x f - \frac{1}{x^2}) d\alpha + (\partial_x f + x \partial_{xx} f + \frac{2}{x^3}) dx \wedge \alpha + x d_Z(\partial_x f) \wedge \alpha.$$

Thus

$$\partial_x f + x \partial_{xx} f = -\frac{2}{x^3}$$

with solution $f = -\frac{1}{2x^2}$.

Then

$$\omega = d(-\frac{1}{2x^2} \alpha) = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2} \text{ and } \beta_1 = \beta_2 = 0.$$

Thus we have reached a contradiction. $\square$

On the other hand, if $b_1, b_2 = 0$ in a cohomological decomposition of a scattering symplectic form $[\omega]$, then we always have the additional structure of a strong symplectic filling. And in fact, that filling is always convex, meaning the Liouville vector field points outward at Z .

Proposition 12. *Let (M, Z, ω) be a scattering-symplectic manifold with singular contact hypersurface (Z, α) . If $[\omega]$ has cohomology decomposition $(a, 0, 0)$, then for $\varepsilon > 0$ small, $(M \setminus \{|x| < \varepsilon\}, \omega) = (M_{x \geq \varepsilon}, \omega) \cup (M_{x \leq -\varepsilon}, \omega)$ is a collection of symplectic manifolds each with contact boundary (Z, α) such that ω is a convex strong symplectic filling of α .*

Proof. Choose a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of Z as in Proposition 6. Then

$$\omega|_\tau = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2}.$$

Define $M_{x \geq \varepsilon}$ to be the connected component of $M \setminus (Z \times (-\varepsilon, \varepsilon))$ containing $Z \times \{\varepsilon\}$ and let its symplectic form be the scattering symplectic form on M restricted to $M_{x \geq \varepsilon}$. Similarly, define $M_{x \leq -\varepsilon}$ to be the connected component of $M \setminus (Z \times (-\varepsilon, \varepsilon))$ containing $Z \times \{-\varepsilon\}$ and let its symplectic form be the scattering symplectic form on M restricted to $M_{x \leq -\varepsilon}$.

Let $V = -\frac{x}{2} \partial_x$. Notice for all points in $Z \times \{\varepsilon\}$ and $Z \times \{-\varepsilon\}$ that V is tranverse to Z . Next, observe that $i_V \omega = \frac{\alpha}{x^2}$. Thus $\mathcal{L}_V \omega = di_V \omega = \omega$. Notice that ω on $M_{x \geq \varepsilon}$ and ω on

$M_{x \leq \varepsilon}$ induce the same contact form. In particular,

$$-\frac{\alpha}{2x^2} \Big|_{Z \times \{\varepsilon\}} = -\frac{\alpha}{2(\varepsilon)^2} = -\frac{\alpha}{2(-\varepsilon)^2} = -\frac{\alpha}{2x^2} \Big|_{Z \times \{-\varepsilon\}}.$$

Further, notice that $V|_{Z \times \{\varepsilon\}} = -\frac{\varepsilon}{2} \partial_x$ is an outward pointing vector. Similarly,

$$V|_{Z \times \{-\varepsilon\}} = -\frac{-\varepsilon}{2} \partial_x$$

is outward pointing. Thus both fillings are convex. $\square$

3.2.4 Symplectic Gluing

We will demonstrate how to glue strong symplectic fillings along a common boundary. In particular, by allowing scattering-symplectic structures, we can glue convex fillings to convex fillings and by allowing folded-symplectic structures, we can glue concave fillings to concave fillings.

We will begin by recalling how a strong concave and strong convex symplectic filling are glued to form a symplectic manifold; see for example Theorem 5.4, [40].

Proposition 13. *Let (M_1, ω_1) and (M_2, ω_2) be a strong convex and strong concave symplectic filling respectively of (Z, ξ) . Then $M_1 \cup_Z M_2$, the union of M_1 to M_2 at Z , has a symplectic structure ω such that $\omega|_{M_1 \setminus Z} \simeq \omega_1$, $\omega|_{M_2 \setminus Z} \simeq \omega_2$.*

Proof. By Lemma 7, since ω_1 is a strong convex symplectic filling of (Z, ξ) , there exists $Z \times [0, c_1)$ a tubular neighborhood of Z in M_1 on which $\omega_1 = d(e^{-r_1} \alpha)$ for r_1 the coordinate for the interval $[0, c_1)$ and where $\ker \alpha = \xi$. Similarly, because (M_2, ω_2) is a strong concave symplectic filling, we have $Z \times [0, c_2)$ a tubular neighborhood of Z in M_2 on which $\omega_2 = d(e^{r_2} \alpha)$ for r_2 the coordinate of the interval $[0, c_2)$. Without loss of generality, assume $c_1 = c_2 = c$.

We attach a collar neighborhood $Z \times (-c, 0)_{r_1}$ to M_1 and a collar neighborhood $Z \times (-c, 0)_{r_2}$ to M_2 . The union $M_1 \cup_Z M_2$ is formed from identifying $Z \times (-c, c)_{r_1}$ with $Z \times (-c, c)_{r_2}$ by mapping Z to itself and setting $r_1 = -r_2 = r$. The smooth structure on $M_1 \cup_Z M_2$ is obtained from the charts on M_1 and M_2 respectively.

We define a symplectic form on $M_1 \cup_Z M_2$ by $\omega = d(e^r \alpha)$. We interpret ω to extend as ω_2 into $M_2 \setminus (Z \times [0, c)_{r_2})$. Similarly, we interpret ω to extend as ω_1 into $M_1 \setminus (Z \times [0, c)_{r_1})$. $\square$

Next, we introduce a method for gluing two strong convex symplectic fillings by using a scattering-symplectic structure.

Theorem 9. *Let (M_1, ω_1) and (M_2, ω_2) be strong convex symplectic fillings of (Z, ξ) . Then $M_1 \cup_Z M_2$, the union of M_1 to M_2 at Z , has a scattering symplectic structure ω such that $\omega|_{M_1 \setminus Z} \simeq \omega_1$, $\omega|_{M_2 \setminus Z} \simeq \omega_2$, and the singular hypersurface of ω is Z .*

Proof. By Lemma 7, since ω_1 is a strong convex symplectic filling of (Z, ξ) , there exists $Z \times [0, c_1)$ a tubular neighborhood of Z in M_1 on which

$$\omega_1 = d(e^{-r_1} \alpha)$$

for r_1 the coordinate for the interval $[0, c_1)$ and where $\ker \alpha = \xi$. Similarly, let $Z \times [0, c_2)$ be a tubular neighborhood of Z in M_2 on which

$$\omega_2 = d(e^{-r_2} \alpha)$$

for r_2 the coordinate of the interval $[0, c_2)$. Without loss of generality, assume $c_1, c_2 > 2$.

The union of M_1 to M_2 at Z is formed from the disjoint union

$$(M_1 \setminus \{Z \times [0, 1/2]_{r_1}\}) \sqcup (M_2 \setminus \{Z \times [0, 1/2]_{r_2}\})$$

by identifying $Z \times (1/2, 2)_{r_1}$ with $Z \times (1/2, 2)_{r_2}$ by mapping Z to itself and setting $r_1 = \frac{1}{r_2}$.

In other words, we have identified the annulus $Z \times (1/2, 2)_{r_1}$ with the annulus $Z \times (1/2, 2)_{r_2}$ by inverting the first about $r_1 = 1$ and gluing it to the latter. The smooth structure on $M_1 \cup_Z M_2$ is obtained from the charts on $M_1 \setminus \{Z \times [0, 1/2]_{r_1}\}$ and $M_2 \setminus \{Z \times [0, 1/2]_{r_2}\}$ respectively.

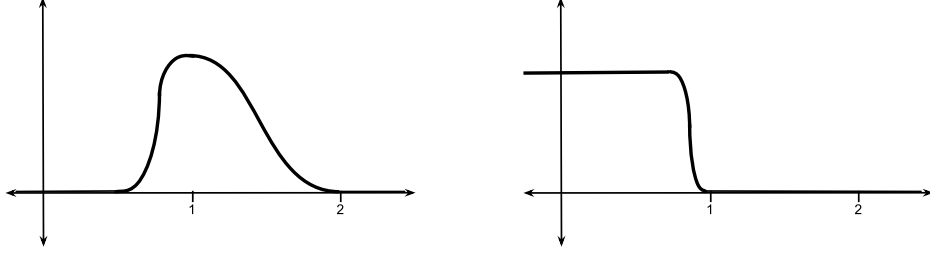
Next, we will define a scattering-symplectic form ω on $M_1 \cup_Z M_2$ with singular locus $Z \times \{r_1 = r_2 = 1\}$. Let $\gamma = e^{-r_1} \alpha$ denote a primitive for ω_1 and let $\tilde{\gamma} = e^{-r_2} \alpha$ denote a primitive for ω_2 . We define

$$\omega = d\left(\left(\frac{\phi(r_1)}{(r_1 - 1)^2} + \psi(r_1)\right)\tilde{\gamma}\right) + d\left(\left(\frac{\phi(r_2)}{(r_2 - 1)^2} + \psi(r_2)\right)\gamma\right) \quad (3.3)$$

where

- $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth bump function supported in $(1/2, 2)$ and
- $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function supported in $(-\infty, 1)$ such that $\psi|_{(-\infty, 7/8)} \equiv 1$.

Figure 18. The graphs of functions ϕ and ψ defined in equations (3.4) and (3.5).



By definition, ω is closed. Since $\psi(r_1) = 1$ for $r_1 \leq 7/8$, we interpret ω to extend as ω_2 into $M_2 \setminus (Z \times [0, 2)_{r_2})$. Similarly, since $\psi(r_2) = 1$ for $r_2 \leq 7/8$, we interpret ω to extend as ω_1 into $M_1 \setminus (Z \times [0, 2)_{r_1})$. By Lemma 8, whose statement and proof immediately follows this proof, functions ϕ and ψ exist that make ω into a non-degenerate scattering form. $\square$

Lemma 8. *There exists a smooth function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, ϕ supported in $(1/2, 2)$, and a smooth function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, ψ supported in $(-\infty, 1)$ with $\psi|_{(-\infty, 7/8)} \equiv 1$ such that ω as in equation (3.3) is non-degenerate.*

Proof. Let

$$\phi(r) = \begin{cases} e^{\frac{r}{(r-1/2)(r-2)}} & \text{if } r \in (1/2, 2); \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

and

$$\psi(r) = \begin{cases} 1 & \text{if } r \leq 7/8; \\ 1 - \frac{e^{\frac{-1}{(r-1)}}}{e^{\frac{-1}{(r-1)}} + e^{\frac{-1}{(7/8-r)}}} & \text{if } r \in (7/8, 1); \\ 0 & \text{if } r \geq 1. \end{cases} \quad (3.5)$$

Let

$$\omega = d \left(\left(\frac{\phi(r_1)}{(r_1 - 1)^2} + \psi(r_1) \right) \tilde{\gamma} \right) + d \left(\left(\frac{\phi(r_2)}{(r_2 - 1)^2} + \psi(r_2) \right) \gamma \right)$$

where $r_1 r_2 = 1$,

$$\tilde{\gamma} = \frac{e^{-r_2}}{e^{-r_1}} \gamma,$$

and $\gamma = e^{-r_1} \alpha$.

By symmetry of expression, the following are equivalent:

- verifying non-degeneracy on $Z \times (1/2, 2)_{r_1}$ when $r_1 - 1 < 0$
- verifying non-degeneracy on $Z \times (1/2, 2)_{r_2}$ when $r_2 - 1 < 0$.

Further, because $\phi(r)$ is multiplicatively symmetric on the interval $(1/2, 2)$ about 1, the following are equivalent:

- verifying that ω is non-degenerate on $Z \times (1/2, 2)_{r_1}$ when $r_1 - 1 < 0$
- verifying that ω is non-degenerate on $Z \times (1/2, 2)_{r_2}$ when $r_2 - 1 > 0$.

Thus to show ω is non-degenerate on all of $M_1 \cup_Z M_2$, it suffices to check that ω is non-degenerate in coordinate r_1 when $r_1 - 1 < 0$.

$$\begin{aligned} \omega = & \left[\frac{\phi'(r_1)}{(r_1 - 1)^2} - 2 \frac{\phi(r_1)}{(r_1 - 1)^3} + \psi'(r_1) \right] dr_1 \wedge \tilde{\gamma} + \left[\frac{\phi(r_1)}{(r_1 - 1)^2} + \psi(r_1) \right] d\tilde{\gamma} \\ & + \left[\frac{\phi'(r_2)}{(r_2 - 1)^2} - 2 \frac{\phi(r_2)}{(r_2 - 1)^3} + \psi'(r_2) \right] dr_2 \wedge \gamma + \left[\frac{\phi(r_2)}{(r_2 - 1)^2} + \psi(r_2) \right] d\gamma. \end{aligned}$$

Since

$$r_2 = \frac{1}{r_1}, \text{ we can compute } dr_2 = \frac{-1}{r_1^2} dr_1.$$

Note that

$$\tilde{\gamma} = \frac{e^{-r_2}}{e^{-r_1}} \gamma = e^{-1/r_1 + r_1} \gamma.$$

Thus

$$dr \wedge \tilde{\gamma} = e^{-1/r_1 + r_1} dr \wedge \gamma$$

and

$$d\tilde{\gamma} = e^{-1/r_1 + r_1} \left(\frac{1}{r_1^2} + 1 \right) dr \wedge \gamma + e^{-1/r_1 + r_1} d\gamma.$$

Then

$$\begin{aligned} & \left[\frac{\phi'(\frac{1}{r_1})}{(\frac{1}{r_1} - 1)^2} - 2 \frac{\phi(\frac{1}{r_1})}{(\frac{1}{r_1} - 1)^3} + \psi' \left(\frac{1}{r_1} \right) \right] d \left(\frac{1}{r_1} \right) \wedge \gamma + \left[\frac{\phi(\frac{1}{r_1})}{(\frac{1}{r_1} - 1)^2} + \psi \left(\frac{1}{r_1} \right) \right] d\gamma \\ & = \left[-\frac{\phi' \left(\frac{1}{r_1} \right)}{(r_1 - 1)^2} - \frac{2\phi(\frac{1}{r_1})r_1}{(r_1 - 1)^3} - \frac{\psi' \left(\frac{1}{r_1} \right)}{r_1^2} \right] dr_1 \wedge \gamma + \left[\frac{\phi(\frac{1}{r_1})r_1^2}{(r_1 - 1)^2} + \psi \left(\frac{1}{r_1} \right) \right] d\gamma. \end{aligned}$$

Thus

$$\omega = A dr \wedge \gamma + B d\gamma,$$

where

$$A = e^{-1/r_1 + r_1} \frac{\phi'(r_1)}{(r_1 - 1)^2} - 2e^{-1/r_1 + r_1} \frac{\phi(r_1)}{(r_1 - 1)^3} + e^{-1/r_1 + r_1} \psi'(r_1) - \frac{\psi'(\frac{1}{r_1})}{r_1^2} +$$

$$\left(\frac{1}{r_1^2} + 1\right) e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1-1)^2} + \left(\frac{1}{r_1^2} + 1\right) e^{-1/r_1+r_1} \psi(r_1) - \frac{\phi'(\frac{1}{r_1})}{(r_1-1)^2} - \frac{2\phi(\frac{1}{r_1})r_1}{(r-1)^3},$$

and

$$B = e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1-1)^2} + e^{-1/r_1+r_1} \psi(r_1) + \frac{\phi(\frac{1}{r_1})r_1^2}{(r_1-1)^2} + \psi\left(\frac{1}{r_1}\right).$$

Then

$$\omega^n = AB^{n-1}dr_1 \wedge \gamma \wedge (d\gamma)^{n-1} + B^n(d\gamma)^n.$$

Notice

$$(d\gamma)^n = -e^{-r_1n}dr_1 \wedge \alpha \wedge (d\alpha)^{n-1}$$

and

$$dr_1 \wedge \gamma \wedge (d\gamma)^{n-1} = e^{-r_1n}dr_1 \wedge \alpha \wedge (d\alpha)^{n-1}.$$

Thus we are left to show that $AB^{n-1} - B^n > 0$. Notice that $B > 0$. So we will show that

$A - B > 0$.

$$\begin{aligned} A - B &= e^{-1/r_1+r_1} \left(\frac{\phi'(r_1)}{(r_1-1)^2} - 2\frac{\phi(r_1)}{(r_1-1)^3} + \psi'(r_1) \right) + \frac{1}{r_1^2} e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1-1)^2} \\ &\quad + \frac{1}{r_1^2} e^{-1/r_1+r_1} \psi(r_1) - \frac{\phi'(\frac{1}{r_1})}{(r_1-1)^2} - \frac{2\phi(\frac{1}{r_1})r_1}{(r-1)^3} - \frac{\psi'(\frac{1}{r_1})}{r_1^2} - \frac{\phi(\frac{1}{r_1})r_1^2}{(r_1-1)^2} - \psi\left(\frac{1}{r_1}\right). \end{aligned}$$

Notice

$$e^{-1/r_1+r_1} \left(\frac{\phi'(r_1)}{(r_1-1)^2} - 2\frac{\phi(r_1)}{(r_1-1)^3} + \psi'(r_1) \right) > 0$$

because on the interval $(7/8, 1)$ we have $\psi' \geq -128$ and

$$\frac{\phi'(r_1)}{(r_1-1)^2} - 2\frac{\phi(r_1)}{(r_1-1)^3} > 139.$$

Next, observe that

$$-\frac{2\phi(\frac{1}{r_1})r_1}{(r-1)^3} - \frac{\phi(\frac{1}{r_1})r_1^2}{(r_1-1)^2} > 0$$

because

$$-\frac{2r_1}{(r_1-1)^3} - \frac{r_1^2}{(r_1-1)^2} > 0$$

for $r_1 \in (1/2, 1)$.

Finally, notice

$$-\frac{\phi'(\frac{1}{r_1})}{(r_1-1)^2}, \quad \frac{1}{r_1^2} e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1-1)^2}, \quad \text{and} \quad \frac{1}{r_1^2} e^{-1/r_1+r_1} \psi(r_1)$$

are positive while

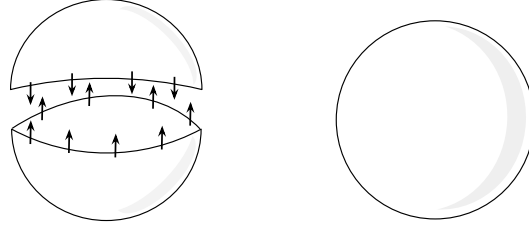
$$-\frac{\psi'(\frac{1}{r_1})}{r_1^2} \quad \text{and} \quad \psi\left(\frac{1}{r_1}\right)$$

are zero for $r_1 - 1 < 0$. □

Remark 5. Because $\mathbb{D}^{2n}$ with the standard symplectic form ω_{st} is a strong convex symplectic

filling of the unit sphere $\mathbb{S}^{2n-1}$ with standard contact structure ξ_{st} , Theorem 9 provides an alternate way of constructing the scattering symplectic spheres described in Proposition 9.

Figure 19. $\mathbb{D}^{2n}$ glues with $\mathbb{D}^{2n}$ to yield scattering symplectic $(\mathbb{S}^{2n}, \mathbb{S}^{2n-1})$.

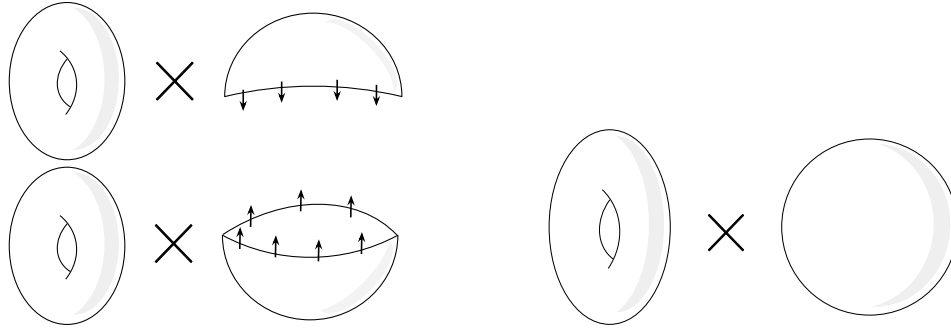


Theorem 9 provides us with a treasure trove of additional examples, particularly in dimension 4 where constructing strong convex symplectic fillings has been an industry in its own right. For a certainly incomplete list, see [13, 14, 16], or [33]. For the sake of brevity, we will limit our attention to a couple of examples.

Example 36. The pair $(\mathbb{T}^2 \times \mathbb{S}^2, \mathbb{T}^3)$ is scattering symplectic.

Let (q_1, q_2) be coordinates for the torus $\mathbb{T}^2$ and let (p_1, p_2) be coordinates for the disk $\mathbb{D}^2$. Then $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ is a symplectic form on $\mathbb{T}^2 \times \mathbb{D}^2$. We can rewrite this expression using polar coordinates by setting $p_1 = r \cos \theta$ and $p_2 = r \sin \theta$. Then $\gamma = r \cos \theta dq_1 + r \sin \theta dq_2$ is a primitive for ω near the boundary $\partial(\mathbb{T}^2 \times \mathbb{D}^2)$. At the boundary, $\gamma|_{r=1} = \cos \theta dq_1 + \sin \theta dq_2$ is a contact form on $\mathbb{T}^2 \times \mathbb{S}^1 = \mathbb{T}^3$. Notice that $i_{r\partial_r} d\gamma = \gamma$ for outward pointing normal vector $r\partial_r$.

Figure 20. $\mathbb{T}^2 \times \mathbb{D}^2$ glues with $\mathbb{T}^2 \times \mathbb{D}^2$ to yield scattering symplectic $(\mathbb{T}^2 \times \mathbb{S}^2, \mathbb{T}^3)$.

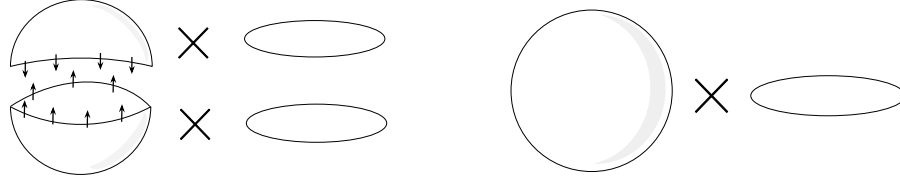


Thus $(\mathbb{T}^2 \times \mathbb{D}^2, \omega)$ is a strong convex symplectic filling of the torus $\mathbb{T}^3$ with contact structure $\gamma|_{r=1} = 0$. As described in Theorem 9, we construct the union of $\mathbb{T}^2 \times \mathbb{D}^2$ with itself at $\partial(\mathbb{T}^2 \times \mathbb{D}^2)$. Thus the pair $(\mathbb{T}^2 \times \mathbb{S}^2, \mathbb{T}^3)$, where $\mathbb{T}^3$ is identified as $\mathbb{T}^2 \times \mathbb{S}^1$ and the factor $\mathbb{S}^1$ is the equator of $\mathbb{S}^2$, admits a scattering symplectic structure.

Example 37. The pair $(\mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1)$ is scattering symplectic.

Let (x, y, z) be the standard Euclidean coordinates for $\mathbb{D}^3$ and let θ be a coordinate for $\mathbb{S}^1$. The manifold $\mathbb{D}^3 \times \mathbb{S}^1$ admits the symplectic form $\omega = 2dx \wedge dy + dz \wedge d\theta$. Then $\gamma = xdy - ydx + zd\theta$ is a primitive for ω . The radial vector field in $\mathbb{D}^3$, $R = x\partial_x + y\partial_y + z\partial_z$, is an outward pointing normal vector at the boundary $\partial(\mathbb{D}^3 \times \mathbb{S}^1)$. Further $i_R d\gamma = \gamma$ and $\gamma|_{\partial(\mathbb{D}^3 \times \mathbb{S}^1)} = 0$ defines a contact structure on $\mathbb{S}^2 \times \mathbb{S}^1$. Thus $(\mathbb{D}^3 \times \mathbb{S}^1, \omega)$ is a strong convex symplectic filling of $\mathbb{S}^2 \times \mathbb{S}^1$ with contact structure $\gamma = 0$.

Figure 21. $\mathbb{D}^3 \times \mathbb{S}^1$ glues with $\mathbb{D}^3 \times \mathbb{S}^1$ to yield scattering symplectic $(\mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1)$.



By Theorem 9, we construct the union of $\mathbb{D}^3 \times \mathbb{S}^1$ with itself at $\partial(\mathbb{D}^3 \times \mathbb{S}^1)$. Then the pair $(\mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1)$, where the factor $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{S}^1$ is identified as the equator of $\mathbb{S}^3$, admits a scattering symplectic structure.

By expanding the symplectic category to allow scattering symplectic structures, we can overcome the obstacle preventing convex fillings from being glued to other convex fillings. In the next theorem, we show that folded symplectic structures can similarly overcome this obstacle for concave fillings, allowing a concave filling to be glued to another concave filling.

Recall that a folded symplectic manifold (M^{2n}, Z, ω) is a $2n$ -dimensional manifold M equipped with a closed two-form ω that is non-degenerate except on a hypersurface Z , called the folding hypersurface, where there exist coordinates such that locally $Z = \{x_1 = 0\}$ and

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i.$$

In section 6 of [10], Ana Cannas da Silva, Victor Guillemin, and Christopher Woodward prove that given any two compact oriented 2d-dimensional symplectic manifolds W_1 , W_2 with common boundary Z , their union over their boundary $W_1 \cup_Z W_2$ admits a folded-symplectic structure. We will consider the special case when the two manifolds W_1 and W_2 are strong concave symplectic fillings of a contact boundary (Z, α) . We will prove that $W_1 \cup_Z W_2$ can be endowed with a folded-symplectic structure that preserves this strong concavity on either side of the hypersurface Z .

Theorem 10. *Let (M_1, ω_1) and (M_2, ω_2) be strong concave symplectic fillings of (Z, ξ) . Then $M_1 \cup_Z M_2$, the union of M_1 to M_2 at Z , has a folded symplectic structure ω such that $\omega|_{M_1 \setminus Z} \simeq \omega_1$, $\omega|_{M_2 \setminus Z} \simeq \omega_2$, and the folding hypersurface of ω is Z .*

Proof. Since ω_1 is a strong concave symplectic filling, by Lemma 7, there exists $Z \times [0, c_1)$ a tubular neighborhood of Z in M_1 on which $\omega_1 = d(e^{r_1} \alpha)$ for r_1 the coordinate for the interval $[0, c_1)$ and where $\ker \alpha = \xi$. Similarly, let $Z \times [0, c_2)$ be a tubular neighborhood of Z in M_2 on which $\omega_2 = d(e^{r_2} \alpha)$ for r_2 the coordinate of the interval $[0, c_2)$. Without loss of generality, assume $c_1, c_2 > 2$.

We attach a collar neighborhood $Z \times (-2, 0)_{r_1}$ to M_1 and a collar neighborhood $Z \times (-2, 0)_{r_2}$ to M_2 .

The union $M_1 \cup_Z M_2$ is formed from identifying $Z \times (-2, 2)_{r_1}$ with $Z \times (-2, 2)_{r_2}$ by mapping Z to itself and setting $r_1 = -r_2$. The smooth structure on $M_1 \cup_Z M_2$ is obtained from the charts on M_1 and M_2 respectively.

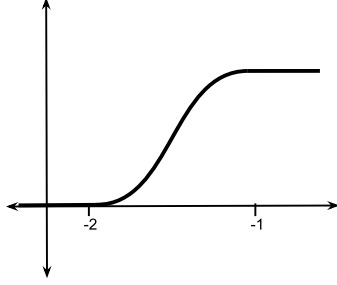
Next, we will define a folded symplectic form ω on $M_1 \cup_Z M_2$ with folding hypersurface $Z \times \{r_1 = r_2 = 0\}$. We define

$$\omega = d(\psi(r_1)e^{r_1}\alpha) + d(\psi(r_2)e^{r_2}\alpha) \quad (3.6)$$

where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function supported in $(-2, \infty)$ with $\psi|_{(-1, \infty)} \equiv 1$.

By definition, ω is closed. Since $\psi(r_1) = 1$ for $r_1 > -1$, we interpret ω to extend as ω_1 into $M_1 \setminus (Z \times [0, 2)_{r_1})$. Similarly, since $\psi(r_2) = 1$ for $r_2 > -1$, we interpret ω to extend as ω_2 into $M_2 \setminus (Z \times [0, 2)_{r_2})$.

Figure 22. The graph of the function ψ defined in equation (3.7).



By Lemma 9, which can be found immediately following this proof, such a function ψ exists and ω is a folded symplectic form. $\square$

Lemma 9. *There exists a smooth function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, ψ supported in $(-2, \infty)$ with $\psi|_{(-1, \infty)} \equiv 1$ such that ω as in equation (3.6) is non-degenerate as a folded symplectic form.*

Proof. We define $\psi(r)$ as

$$\psi(r) = \begin{cases} 0 & \text{if } r \leq -2; \\ \frac{e^{\frac{-1}{(r+2)}}}{e^{\frac{-1}{(r+2)}} + e^{\frac{-1}{(-1-r)}}} & \text{if } r \in (-2, -1); \\ 1 & \text{if } r \geq -1. \end{cases} \quad (3.7)$$

Let

$$\omega = d(\psi(r_1)e^{r_1}\alpha + \psi(r_2)e^{r_2}\alpha)$$

where $r_1 + r_2 = 0$. By the symmetry of this expression, to show ω is non-degenerate on all of $(M_1 \cup_Z M_2) \setminus Z$, it suffices to check that ω is non-degenerate when $r_1 > 0$. We have,

$$\begin{aligned} \omega &= (e^{r_1}\psi'(r_1) + e^{r_1}\psi(r_1) - e^{-r_1}\psi'(-r_1) - e^{-r_1}\psi(-r_1)) dr_1 \wedge \alpha \\ &\quad + (\psi(r_1)e^{r_1} + \psi(-r_1)e^{-r_1}) d\alpha. \end{aligned}$$

The non-degeneracy of ω will follow from checking that the coefficients of $d\alpha$ and $dr_1 \wedge \alpha$ are strictly positive. Note

$$\psi(r_1)e^{r_1} + \psi(-r_1)e^{-r_1}$$

is always positive. Since $\psi'(r_1) = 0$, we are left to check that

$$e^{r_1}\psi(r_1) - e^{-r_1}\psi'(-r_1) - e^{-r_1}\psi(-r_1) > 0.$$

Notice for $r_1 \in (1, 2)$, we have

$$\psi'(-r_1) < 3, \quad \psi(r_1) = 1, \text{ and } \psi(-r_1) \leq 1.$$

Thus for $r_1 \in (1, 2)$, we must show

$$e^{r_1} - 4e^{-r_1} > 0.$$

In other words, $e^2 > 4$, which is true. For $r_1 \in (0, 1]$, note that

$$\psi'(-r_1) = 0, \text{ and } \psi(r_1) = \psi(-r_1) = 1.$$

Thus the inequality is reduced to showing $e^{r_1} - e^{-r_1} > 0$. In other words for $r_1 \in (1, 2)$, we need $e^{2r_1} > 1$. But this inequality is true.

We will verify that ω is folded-symplectic using an equivalent definition of a folded symplectic form [10]: Let M be a $2d$ -dimensional manifold and $\omega \in \Omega^2(M)$ closed. Let Z be the set of points where $\omega^d = 0$. Let ω^d intersect the zero section of $\wedge^{2d} T^*M$ transversally. Let i be the inclusion map of Z into M . If the form $i^*\omega^{d-1} \in \Omega^{2d-2}(Z)$ is non-vanishing, ω is said to be a folded symplectic form.

Note $\omega|_Z = 2d\alpha$ and $\alpha \wedge d\alpha^{d-1}|_Z \neq 0$. Thus $d\alpha^{d-1}$ is non-vanishing and we have shown that ω is a folded symplectic structure. $\square$

Because $\mathbb{R}^{2n} \setminus \mathbb{D}^{2n}$ with the standard symplectic form is a concave symplectic filling of $\mathbb{S}^{2n-1}$ with the standard contact form, this construction immediately gives us a folded-symplectic connect sum over the sphere with its standard contact structure. This construction is due to Cannas da Silva, Guillemin, and Woodward; see Example (2), Section 3 in [10].

Corollary 1 (Cannas da Silva-Guillemin-Woodward). *Given $2n$ dimensional symplectic manifolds (M_1, ω_1) and (M_2, ω_2) , their connect sum $M_1 \# M_2$ is folded-symplectic with folding hypersurface $\mathbb{S}^{2n-1}$ with its standard contact structure.*

By combining Theorems 9 and 10, we can construct examples of scattering-folded symplectic manifolds:

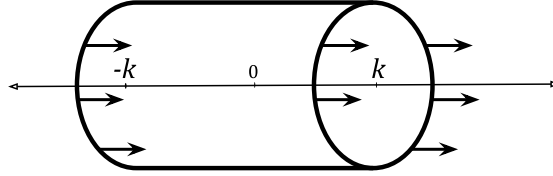
Definition 13. *A scattering-folded symplectic manifold (M, Z_{sc}, Z_f, ω) is a manifold M , with two distinct hypersurfaces Z_{sc} and Z_f , equipped with a two form ω that is symplectic*

everywhere except for on a singular hypersurface Z_{sc} where it is a scattering symplectic form and on a folding hypersurface Z_f where it is a folded symplectic form.

Example 38. Recall [4] that a symplectic cone is a triple (M, ω, X) of a manifold M with a symplectic form ω and a vector field X such that X generates a proper action of the real numbers on M and $\mathcal{L}_X \omega = \omega$. Any symplectic cone is of the form $B \times \mathbb{R}$ where B is a co-oriented contact manifold with contact form α . In fact, all symplectic cones can be written as $(B \times \mathbb{R}, d(e^t \alpha), \frac{\partial}{\partial t})$ where t is the $\mathbb{R}$ coordinate.

Let $(B \times \mathbb{R}, d(e^t \alpha), \frac{\partial}{\partial t})$ be any symplectic cone of a co-oriented contact manifold (B, α) . We will truncate $B \times \mathbb{R}$ by considering $B \times [-k, k]$ for any real number $k > 0$.

Figure 23. Truncated symplectic cone



At the level set $t = k$, the vector field $\frac{\partial}{\partial t}$ is outward pointing and satisfies $i_{\frac{\partial}{\partial t}} \omega = \alpha$. Thus at $t = k$, we have a strong convex symplectic filling of (B, α) . At the level set $t = -k$, the vector field $\frac{\partial}{\partial t}$ is inward pointing and satisfies $i_{\frac{\partial}{\partial t}} \omega = \alpha$. Thus at $t = -k$, we have a strong concave symplectic filling of (B, α) .

By Theorems 9 and 10, we can glue $B \times [-k, k]$ to a copy of itself to form the scattering-folded symplectic manifold $(B \times \mathbb{S}^1, B \times \{k\}, B \times \{-k\}, \omega)$.

Our next example provides an explicit description of a scattering-folded symplectic manifold.

Example 39. The triple $(\mathbb{T}^{2n}, \cup_{2m} \mathbb{T}^{2n-1}, \cup_{2m} \mathbb{T}^{2n-1})$ is scattering-folded symplectic for all $n, m \in \mathbb{N}$.

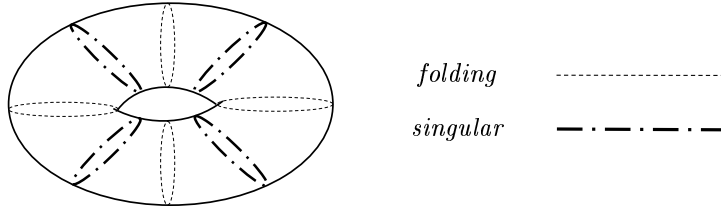
In [5], Frédéric Bourgeois defines a contact form β on all odd dimensional tori $\mathbb{T}^{2n-1}$. We identify $\mathbb{T}^{2n}$ with $\mathbb{T}^{2n-1} \times \mathbb{S}^1$ and denote the angular coordinate on $\mathbb{S}^1$ by θ . We define a scattering-folded form on $\mathbb{T}^{2n}$ by

$$\omega_m = d \left(\frac{\beta}{\sin^2(m\theta)} \right) = \frac{-2m \cos(m\theta) d\theta \wedge \beta}{\sin^3(m\theta)} + \frac{d\beta}{\sin^2(m\theta)}$$

for any $m \in \mathbb{N}$.

Then for each zero z of $\sin(m\theta)$ in $[0, 2\pi)$, we have a singular hypersurface $\mathbb{T}^{2n-1} \times \{z\}$. Since there are $2m$ zeroes of $\sin(m\theta)$ in $[0, 2\pi)$, $Z_{sc} = \cup_{2m} \mathbb{T}^{2n-1}$. Similarly, for each zero z of $\cos(m\theta)$ in $[0, 2\pi)$, we have a folding hypersurface $\mathbb{T}^{2n-1} \times \{z\}$. Since there are $2m$ zeroes of $\cos(m\theta)$ in $[0, 2\pi)$, $Z_f = \cup_{2m} \mathbb{T}^{2n-1}$.

Figure 24. $(\mathbb{T}^2, \cup_4 \mathbb{S}^1, \cup_4 \mathbb{S}^1)$.



We equip $\mathbb{T}^2$ with the form

$$d \left(\frac{d\theta_1}{\sin^2(2\theta_2)} \right) = \frac{-4 \cos(2\theta_2) d\theta_2 \wedge d\theta_1}{\sin^3(2\theta_1)}.$$

The folding hypersurfaces occur at $\theta_2 = \pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$. The singular hypersurfaces occur at $\theta_2 = 0, \pi/2, \pi$, and $3\pi/2$.

3.2.5 Scattering Poisson geometry

A scattering-Poisson structure is dual to a scattering-symplectic structure. In this section we will explore these structures utilizing the language of Poisson geometry.

Definition 14. A Poisson manifold (M, π) is scattering-Poisson if there exists an oriented hypersurface $Z \subset M$ such that there is a bivector $\pi_{sc} \in \bigwedge^2({}^{sc}TM)$ with $\rho(\pi_{sc}) = \pi$.

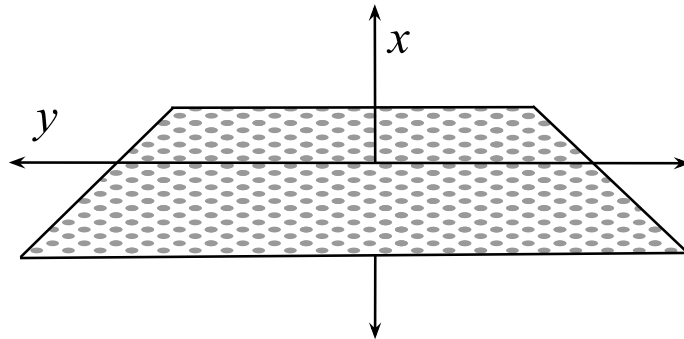
It is assumed that π_{sc} is non-degenerate unless otherwise stated.

By dualizing the scattering-symplectic form as in Proposition 5, for any non-degenerate scattering-Poisson manifold (M, π) , for all $p \in Z$ there exists a coordinate chart $(U, x_1, y_1, \dots, x_n, y_n)$ centered at p such that on U , the hypersurface Z is locally defined by $\{x_1 = 0\}$, and

$$\pi = x_1^3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial y_1} \wedge \left(\sum_{i=2}^n y_i \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial x_i} \right) + x_1^2 \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}. \quad (3.8)$$

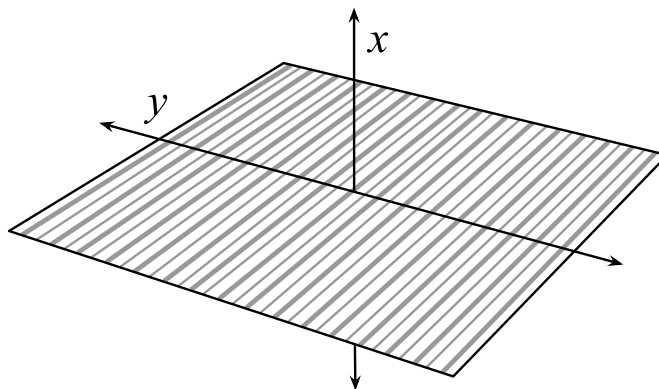
Symplectic foliations. Every Poisson structure on a manifold induces a foliation by symplectic manifolds. The symplectic foliation for a non-degenerate scattering Poisson structure contains the open leaves $M \setminus Z$, locally given by $\{x < 0\}$ and $\{x > 0\}$ as depicted in Figure 25. The individual points of the hyperplane Z are zero dimensional symplectic leaves.

Figure 25. *sc-Poisson foliation*



Recall from [19] and [47], in the case of b^- - and b^k -Poisson structures, the symplectic foliation contains the open leaves $\{x > 0\}$ and $\{x < 0\}$, and a regular codimension 1 foliation of the hypersurface Z - depicted in Figure 26 as the level sets $y = c$ for each $c \in \mathbb{R}$.

Figure 26. *b^k -Poisson foliation*



Note that the Hamiltonian vector fields associated to a scattering-Poisson manifold are all zero at Z . However, as we will show in the following discussion, π contains information about

the induced contact structure on the degeneracy hypersurface. Since contact structures are maximally non-integrable, it is fitting that the associated symplectic foliation is maximally trivial.

Modular vector field. We can use the local normal form for a non-degenerate scattering Poisson bivector, given in equation (3.8), to compute the modular vector field with respect to the standard volume form $\Lambda = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$. We have that $\Lambda^\flat \pi = -x_1^3 dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n - \sum_{i=2}^n x_1^2 y_i dx_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \widehat{dy_i} \cdots \wedge dx_n \wedge dy_n + x_1^2 \sum_{i=2}^n (x_i dx_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \widehat{dx_i} \cdots \wedge dy_n + dx_1 \wedge dy_1 \wedge \cdots \widehat{dx_i} \wedge \widehat{dy_i} \cdots \wedge dx_n \wedge dy_n)$.

Next we compute that $d(\Lambda^\flat \pi) = -(2n-1)x_1^2 dx_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n$. Thus

$$D_\pi \Lambda = (2n-1)x_1^2 \frac{\partial}{\partial y_1}.$$

Poisson cohomology. Consider a non-degenerate scattering Poisson manifold (M, Z, π) and let ω be the corresponding scattering-symplectic form. We will now give an explicit description of the scattering-rigged algebroid. Given a choice of Z defining function x , ω induces a contact form α on Z .

For local coordinates $(r, s_1, t_1, \dots, s_m, t_m)$, in Z such that $\alpha = dr + \sum_{i=1}^m s_i dt_i$, the local sections of $\mathcal{R}$ are smooth linear combinations of

$$x^3 \frac{\partial}{\partial x}, x^3 \frac{\partial}{\partial r}, x^2 \frac{\partial}{\partial s_1}, \dots, x^2 \frac{\partial}{\partial s_m}, x^2 \left(s_1 \frac{\partial}{\partial r} - \frac{\partial}{\partial t_1} \right), \dots, x^2 \left(s_m \frac{\partial}{\partial r} - \frac{\partial}{\partial t_m} \right). \quad (3.9)$$

Notice that we can locally identify $\mathcal{R}^*$ as the span of

$$\frac{dx}{x^3}, \frac{\alpha}{x^3}, \frac{ds_1}{x^2}, \dots, \frac{ds_m}{x^2}, \frac{dt_1}{x^2}, \dots, \frac{dt_m}{x^2}.$$

Lemma 10. *The Poisson cohomology of a non-degenerate scattering-Poisson manifold (M, Z, π) is isomorphic to the Lie algebroid cohomology ${}^{\mathcal{R}}H^*(M)$ of the scattering rigged algebroid $\mathcal{R}$ as locally identified in equation (3.9).*

Proof. We know that the rigged Lie algebroid computes Poisson cohomology. We are left to prove that we have correctly identified $\mathcal{R}$. Let ω be the scattering symplectic form dual to π . We define a map $\bar{\omega} : TM \rightarrow \mathcal{R}^*$ by $v \mapsto i_v \omega$. For all $p \in M \setminus Z$, $\mathcal{R}_p^* \simeq T_p^* M$ and $\bar{\omega}_p$ is

a symplectic form. Thus for all $p \in M \setminus Z$, we have that $\bar{\omega}_p$ is an isomorphism. For $p \in Z$, it follows from ω being non-degenerate as a scattering 2-form that $\bar{\omega}$ is injective. One can verify using Proposition 6 and local coordinates that $\bar{\omega}$ is surjective. Thus the map $\bar{\omega}$ is a bundle isomorphism. $\square$

Given a $2n$ -dimensional non-degenerate scattering Poisson manifold (M, Z, π) , let ξ denote the contact distribution on Z induced by π and let $\Omega_\xi^k(Z)$ denote degree- k forms σ on Z such that for all $p \in Z$, σ_p is supported in $\wedge^k \xi_p$. That is,

$$\Omega_\xi^k(Z) := \left\{ \sigma \in \Omega^k(Z) \mid \forall p \in Z, \text{supp}(\sigma_p) \subseteq \wedge^k \xi_p \right\}.$$

Let α be a contact form on Z such that $\ker \alpha = \xi$. Note that

$$\mathcal{K}^k := \ker(d\alpha \wedge : \Omega_\xi^k(Z) \rightarrow \Omega_\xi^{k+2}(Z))$$

is independent of the choice of α because any other choice of contact form will give a symplectic structure conformal to $d\alpha$ on ξ . We adopt the convention that $\mathcal{K}^k = 0$ for $k \leq 0$.

In the following theorem, we will show that the cohomology class of $\mu \in \mathcal{R}H^k(M)$ is uniquely determined by a smooth b -form and a 1-jet at Z of a closed form in $\mathcal{R}\Omega^k(M)$. Let $\mathcal{J}_Z^1(\mathcal{R}\Omega_{cl}^k(M))$ denote the 1-jets at Z of closed forms in $\mathcal{R}\Omega^k(M)$.

Theorem 11. *Given (M, Z, π) a $2n$ -dimensional non-degenerate scattering Poisson manifold, let ξ denote the contact distribution on Z induced by π . The Poisson cohomology $H_\pi^p(M)$ of (M, Z, π) is*

$$H^p(M) \oplus H^{p-1}(Z) \oplus \mathcal{J}_Z^1(\mathcal{R}\Omega_{cl}^p(M)).$$

Given a fixed Z defining function x ,

$$H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z) \oplus \Omega^{p-1}(Z) \oplus \Omega_\xi^{p-1}(Z) \oplus \mathcal{K}^{p-2}.$$

Remark 6. *Given a contact hypersurface (Z, α) of dimension $2n + 1$, the map $d\alpha \wedge : \Omega_\xi^k(Z) \rightarrow \Omega_\xi^{k+2}(Z)$, by a local computation at a point, is injective for $k = 0, \dots, n - 1$ and is identically equal to zero for $k = 2n, 2n + 1$. Thus $\mathcal{K}^k = 0$ for $k = 0, \dots, n - 1$ and $\mathcal{K}^k = \Omega_\xi^k(Z)$ for $k = 2n, 2n + 1$.*

Proof. We are left to compute $\mathcal{R}H^p(M)$. We have a short exact sequence

$$0 \rightarrow {}^b\Omega^k(M) \xrightarrow{i^*} \mathcal{R}\Omega^k(M) \xrightarrow{P} \mathcal{C}^k \rightarrow 0$$

where

$$\mathcal{C}^p = \mathcal{R}\Omega^p(M)/{}^b\Omega^p(M)$$

is the quotient under the evaluation bundle map $i : \mathcal{R}TM \rightarrow {}^bTM$. We have a differential $\mathcal{C}d$ induced by the differential $\mathcal{R}d$ on $\mathcal{R}\Omega^p(M)$. In particular, if P is the projection $\mathcal{R}\Omega^p(M) \rightarrow \mathcal{R}\Omega^p(M)/{}^b\Omega^p(M)$, then $\mathcal{C}d(\eta) = P(\mathcal{R}d(\theta))$ where $\theta \in \mathcal{R}\Omega^p(M)$ is any form such that $P(\theta) = \eta$. Hence $(\mathcal{C}d)^2 = 0$ and $(\mathcal{C}^*, \mathcal{C}d)$ is in fact a complex.

Given a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of M near Z , we can write a degree- k form ν in $\mathcal{R}\Omega^k(M)$ as an expansion in x of the form

$$\nu = \mu_b + \frac{dx}{x^{2k+1}} \wedge \left(\sum_{i=0}^{2k-1} \eta_i x^i \right) + \frac{1}{x^{2k}} \sum_{i=0}^{2k-1} \beta_i x^i + \frac{dx \wedge \alpha \wedge \theta}{x^{2k+2}} + \frac{\alpha \wedge \gamma}{x^{2k+1}},$$

μ_b is a b-form, α is the contact form on Z induced by ω and x , $\eta_i \in \Omega^{k-1}(Z)$, $\beta_i \in \Omega^k(Z)$, $\theta \in \Omega^{k-2}(Z)$ such that

$$\text{supp}(\theta) \subseteq \wedge^{k-2}\xi,$$

and $\gamma \in \Omega^{k-1}(Z)$ such that

$$\text{supp}(\gamma) \subseteq \wedge^{k-1}\xi.$$

We write $R_b(\nu) = \mu_b$ and $S_b(\nu) = \nu - R_b(\nu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $R_b(\mathcal{R}d\nu) = \mathcal{R}d(R_b(\nu))$ and $S_b(\mathcal{R}d\nu) = \mathcal{R}d(S_b(\nu))$. Thus the trivialization τ induces a splitting $\mathcal{R}\Omega^*(M) = {}^b\Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence $\mathcal{R}H^k(M) = {}^bH^k(M) \oplus H^k(\mathcal{C}^*)$ and we are left to compute the cohomology of the quotient complex.

We have that

$$\begin{aligned} \mathcal{R}d(S_b(\nu)) = & - \sum_{i=0}^{2k-1} \frac{dx}{x^{2k+1}} \wedge d\eta_i x^i - \sum_{i=0}^{2k-1} \frac{(2k-i)dx}{x^{2k+1}} \wedge \beta_i x^i + \sum_{i=0}^{2k-1} \frac{d\beta_i}{x^{2k}} x^i \\ & - \frac{dx \wedge d\alpha \wedge \theta}{x^{2k+2}} + \frac{dx \wedge \alpha \wedge d\theta}{x^{2k+2}} - \frac{2k+1}{x^{2k+2}} dx \wedge \alpha \wedge \gamma + \frac{d\alpha \wedge \gamma}{x^{2k+1}} - \frac{\alpha \wedge d\gamma}{x^{2k+1}}. \end{aligned}$$

Thus the kernel $\mathcal{C}d : \mathcal{C}^k \rightarrow \mathcal{C}^{k+1}$ is defined by the relation

$$-d\eta_i x^{i+1} - (2k-i)\beta_i x^{i+1} - d\alpha \wedge \theta + \alpha \wedge d\theta - (2k+1)\alpha \wedge \gamma = 0.$$

In order for the expression to be zero, the coefficients of the polynomial must be zero and thus

$$\beta_i = \frac{-d\eta_i}{(2k-i)}$$

for $i = 0, \dots, 2k-1$.

Now we consider the kernel relation given by the coefficient

$$-d\alpha \wedge \theta + \alpha \wedge d\theta - (2k+1)\alpha \wedge \gamma = 0.$$

By contracting with R , the Reeb vector field associated to α , we recover

$$d\theta - \alpha \wedge i_R d\theta - (2k+1)\gamma = 0.$$

Thus

$$\gamma = \frac{d\theta - \alpha \wedge i_R d\theta}{(2k+1)}.$$

Substituting this into the original expression, we have that

$$-d\alpha \wedge \theta + \alpha \wedge d\theta - (2k+1)\alpha \wedge \frac{d\theta}{(2k+1)} = 0$$

since $\alpha^2 = 0$. Thus $d\alpha \wedge \theta = 0$.

Thus all closed forms in $\mathcal{C}^k$ are of the form

$$\frac{dx}{x^{2k+1}} \wedge \left(\sum_{i=0}^{2k-1} \eta_i x^i \right) + \frac{1}{x^{2k}} \sum_{i=0}^{2k-1} \frac{-d\eta_i}{(2k-i)} x^i + \frac{dx \wedge \alpha \wedge \theta}{x^{2k+2}} + \frac{\alpha}{x^{2k+1}} \wedge \frac{(d\theta - \alpha \wedge i_R d\theta)}{(2k+1)},$$

where $\theta \in \ker(d\alpha \wedge : \Omega_\xi^{k-2}(Z) \rightarrow \Omega_\xi^k(Z))$.

Elements in $\mathcal{C}^k d(\mathcal{C}^{k-1})$ are of the form

$$\begin{aligned} & - \sum_{i=0}^{2k-3} \frac{dx}{x^{2k-1}} \wedge d\eta_i x^i - \sum_{i=0}^{2k-3} \frac{(2k-2-i)dx}{x^{2k-1}} \wedge \beta_i x^i + \sum_{i=0}^{2k-3} \frac{d\beta_i}{x^{2k-2}} x^i \\ & - \frac{dx \wedge d\alpha \wedge \theta}{x^{2k}} + \frac{dx \wedge \alpha \wedge d\theta}{x^{2k}} - \frac{2k-1}{x^{2k}} dx \wedge \alpha \wedge \gamma + \frac{d\alpha \wedge \gamma}{x^{2k-1}} - \frac{\alpha \wedge d\gamma}{x^{2k-1}}. \end{aligned} \quad (3.10)$$

Thus there is the element

$$\sum_{j=2}^{2k-1} \frac{-x^{j-2}\eta_j}{(2k-j)x^{2k-2}}$$

in $\mathcal{C}^{k-1}$ such that

$$d \left(\sum_{j=2}^{2k-1} \frac{-x^{j-2}\eta_j}{(2k-j)x^{2k-2}} \right) = \sum_{j=2}^{2k-1} \left(\frac{dx}{x^{2k+1}} \wedge \eta_j x^j - \frac{d\eta_j x^j}{(2k-j)x^{2k}} \right).$$

If we express $\eta_i = \delta_i + \alpha \wedge \gamma_i$ for $\delta_i, \gamma_i \in \Omega_\xi^k(Z)$, then there is the element

$$\frac{-\alpha \wedge \gamma_1}{(2k-1)x^{2k-1}}$$

in $\mathcal{C}^{k-1}$ such that

$$d \left(\frac{-\alpha \wedge \gamma_1}{(2k-1)x^{2k-1}} \right) = \frac{dx \wedge \alpha \wedge \gamma_1 x}{x^{2k+1}} - \frac{d(\alpha \wedge \gamma_1)x}{(2k-1)x^{2k}}.$$

By (3.10), the remaining terms in a closed form in $\mathcal{C}^k$ are too singular to appear in the image $d(\mathcal{C}^{k-1})$. Thus an element of $H^k(\mathcal{C})$ has a representative of the form

$$\begin{aligned} \nu &= \frac{dx}{x^{2k+1}} \wedge (\delta_0 + \alpha \wedge \gamma_0) + \frac{dx}{x^{2k+1}} \wedge x\delta_1 + \frac{dx}{x^{2k+2}} \wedge \alpha \wedge \theta \\ & - \frac{d(\delta_0 + \alpha \wedge \gamma_0)}{(2k)x^{2k}} - \frac{d\delta_1}{(2k-1)x^{2k-1}} - \frac{d(\alpha \wedge \theta)}{(2k+1)x^{2k+1}} \end{aligned}$$

where $\delta_0, \gamma_0, \delta_1 \in \Omega_\xi^*(Z)$ and $\theta \in \mathcal{K}^{k-2} = \ker(d\alpha \wedge : \Omega_\xi^{k-2}(Z) \rightarrow \Omega_\xi^k(Z))$ and each such form

represents a separate cohomology class. Thus for a fixed Z defining function x , by the map

$$\nu \mapsto (\delta_0 + \alpha \wedge \gamma_0, \delta_1, \theta),$$

$$H^k(\mathcal{C}) \simeq \Omega^{k-1}(Z) \oplus \Omega_\xi^{k-1}(Z) \oplus \mathcal{K}^{k-2}.$$

To conclude, we consider what happens under change of Z defining function x . Note that ν is completely determined by $i_{\partial x}\nu$. Further, note that representatives ν are equivalent to closed forms because the image $d(\mathcal{C}^{k-1})$ has empty intersection with the collection of forms of type ν .

Next, we can rearrange the dx coefficient in ν as

$$\underbrace{\frac{dx}{x^{2k+1}} \wedge \delta_0 + \frac{dx}{x^{2k+2}} \wedge \alpha \wedge \theta}_{0\text{-jet}} + \underbrace{\frac{dx}{x^{2k+1}} \wedge \alpha \wedge \gamma_0 x + \frac{dx}{x^{2k+1}} \wedge \delta_1 x}_{1\text{-jet}}.$$

Thus, $H^k(\mathcal{C})$ is in bijective correspondence with all 1-jets at Z of closed forms in $\mathcal{R}\Omega^k(M)$, denoted $\mathcal{J}_Z^1(\mathcal{R}\Omega_{cl}^k(M))$.

We have shown that the Poisson cohomology $H_\pi^k(M)$ of (M, Z, π) is

$${}^b H^k(M) \oplus \mathcal{J}_Z^1(\mathcal{R}\Omega_{cl}^k(M))$$

and, given a fixed Z defining function x , is

$${}^b H^k(M) \oplus \Omega^{k-1}(Z) \oplus \Omega_\xi^{k-1}(Z) \oplus \mathcal{K}^{k-2}.$$

The final isomorphism is due to Mazzeo-Melrose that ${}^b H^k(M) \simeq H^k(M) \oplus H^{k-1}(Z)$. $\square$

Modular class. Recall that locally the modular vector field of a non-degenerate scattering Poisson bivector associated to the volume form $\Lambda = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is

$$D_\pi \Lambda = (2n-1)x_1^2 \frac{\partial}{\partial y_1}.$$

We can use the computation of Poisson cohomology above to analyze the modular class. Consider $\bar{\omega} \left((2n-1)x_1^2 \frac{\partial}{\partial y_1} \right) = -(2n-1) \frac{dx_1}{x_1}$, which represents a non-vanishing cohomology class in $\mathcal{R}H^1(M)$. Thus non-degenerate scattering Poisson manifolds are NOT unimodular.

3.3 b^k symplectic geometry

3.3.1 Introduction

The notion of a b^k -Poisson manifold, a type of generically symplectic Poisson structure, was introduced by Geoffrey Scott [47]. A b^k -Poisson manifold is an $2n$ -dimensional manifold M equipped with a Poisson bivector π that is non-degenerate except on a hypersurface Z where there exist coordinates such that locally $Z = \{x_1 = 0\}$ and

$$\pi = x_1^k \partial x_1 \wedge \partial y_1 + \sum_{i=2}^n \partial x_i \wedge \partial y_i.$$

Each b^k -symplectic structure induces a cosymplectic structure (θ, η) on Z for each choice of Z defining function. The flow of the Reeb vector field associated to the cosymplectic structure defines a foliation on Z , which we will denote $\mathcal{F}_R$. Consider the horizontal forms on this foliation:

$$\Omega_h^p(Z) = \{\sigma \in \Omega^p(Z) \mid i_R \sigma = 0\}.$$

We define an exterior derivative

$$d_h = d - \theta \wedge \mathcal{L}_R.$$

This forms a complex and we call its homology groups $H_h^*(\mathcal{F}_R)$ the *horizontal foliation cohomology of $\mathcal{F}_R$* .

Theorem 12. *If (M, π) is a b^k -Poisson manifold, then the Poisson cohomology $H_\pi^p(M)$ is*

$$H^p(M) \oplus H^{p-1}(Z) \oplus (H_h^{p-2}(\mathcal{F}_R))^{k-1} \oplus (H_h^{p-1}(\mathcal{F}_R))^{k-1}$$

for $k \geq 2$.

For $k = 1$, our method recovers the result of Ioan Mărcuț and Boris Osorno Torres [29] that

$$H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z).$$

3.3.2 Poisson cohomology of a b^k -Poisson manifold

Let (M, Z, π) be a non-degenerate b^k -Poisson manifold and let ω be the corresponding b^k -symplectic form. Guillemin, Miranda, and Pires ([19], $k = 1$) and Scott ([47], $k \geq 2$) showed that given a fixed Z defining function x , ω induces a cosymplectic structure

$$(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$$

on Z . That is, there exists a pair of closed forms such that

$$\theta \wedge \eta^{n-1} \neq 0$$

where the dimension of Z is $2n - 1$ and

$$\omega = \frac{dx}{x^k} \wedge \theta + \eta. \quad (3.11)$$

Let (M, Z, ω) be a b^k -symplectic manifold for $k \geq 2$. Following Scott, we will fix and work in a tubular neighborhood $Z \times (-\varepsilon, \varepsilon)_x$ where the form ω is expressible near Z in equation (3.11). However it suffices to only fix a finite jet of Z defining function, and for $k = 1$ it is unnecessary to fix anything.

Let R be the Reeb vector field associated to (θ, η) . That is the non-vanishing vector field R on Z such that

$$\theta(R) = 1 \text{ and } i_R \eta = 0.$$

Then TZ splits as

$$\mathbb{R}\langle R \rangle \oplus \mathbb{R}\langle \ker \theta \rangle.$$

We identify the *b-rigged algebroid* $\mathcal{R}$ as the algebroid whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; TM) : u|_Z \in \mathcal{C}^\infty(Z; \ker \theta)\}.$$

For local coordinates $(r, s_1, t_1, \dots, s_m, t_m)$ in Z such that

$$\theta = dr \text{ and } \eta = \sum_{i=1}^m ds_i \wedge dt_i,$$

the local sections of $\mathcal{R}$ are smooth linear combinations of

$$x\partial x, \quad x\partial r, \quad \partial s_1, \partial t_1, \dots, \partial s_m, \partial t_m.$$

Notice that we can locally identify the dual elements of $\mathcal{R}^*$ as

$$\frac{dx}{x}, \quad \frac{\theta}{x}, \quad ds_1, dt_1, \dots, ds_m, dt_m.$$

The b^k *rigged algebroid* $\mathcal{R}$ is defined iteratively using the *b-rigged algebroid* ${}^b\mathcal{R}$. The b^2 -rigged algebroid is the vector bundle whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; {}^b\mathcal{R}) : u|_Z \in \mathcal{C}^\infty(Z; \ker \theta)\}.$$

Given the b^{k-1} -rigged algebroid, the b^k -rigged algebroid is the the vector bundle whose space of sections is

$$\left\{ u \in \mathcal{C}^\infty(M; b^{k-1}\mathcal{R}) : u|_Z \in \mathcal{C}^\infty(Z; \ker \theta) \right\}.$$

For local coordinates $(r, s_1, t_1, \dots, s_m, t_m)$ in Z such that

$$\theta = dr \text{ and } \beta = \sum_{i=1}^m ds_i \wedge dt_i,$$

the local sections of $\mathcal{R}$ are smooth linear combinations of

$$x^k \partial x, \quad x^k \partial r, \quad \partial s_1, \partial t_1, \dots, \partial s_m, \partial t_m.$$

Notice that we can locally identify the dual elements of $\mathcal{R}^*$ as

$$\frac{dx}{x^k}, \quad \frac{\theta}{x^k}, \quad ds_1, dt_1, \dots, ds_m, dt_m.$$

For $k \geq 2$, the Lie algebroid cohomology of the b^k rigged algebroid will contain information about the symplectic foliation of the Poisson structure. The flow of the Reeb vector field associated to the cosymplectic structure defines a foliation on Z , which we will denote $\mathcal{F}_R$. Consider the horizontal forms on this foliation:

$$\Omega_h^p(Z) = \{\sigma \in \Omega^p(Z) \mid i_R \sigma = 0\}.$$

We define an exterior derivative

$$d_h = d - \theta \wedge \mathcal{L}_R.$$

First note that this is well defined on the complex: Indeed showing $d_h \sigma \in \Omega_h^*(Z)$ is equivalent to showing that $i_R d_h \sigma = 0$. Further $\sigma \in \Omega_h^*(Z)$ means that $i_R \sigma = 0$ and thus $\mathcal{L}_R \sigma = i_R d \sigma$. Thus

$$i_R(d_h \sigma) = i_R(d \sigma - \theta \wedge i_R d \sigma) = i_R d \sigma - i_R d \sigma = 0.$$

Next we will show that d_h squares to zero. Given $\sigma \in \Omega_h^p(Z)$,

$$\begin{aligned} d_h^2 \sigma &= d^2 \sigma - d(\theta \wedge \mathcal{L}_R \sigma) - \theta \wedge \mathcal{L}_R d \sigma + \theta \wedge \mathcal{L}_R(\theta \wedge \mathcal{L}_R \sigma) \\ &= \theta \wedge d \mathcal{L}_R \sigma - \theta \wedge \mathcal{L}_R d \sigma + \theta \wedge \mathcal{L}_R \theta \wedge \mathcal{L}_R \sigma = 0. \end{aligned}$$

Thus $(\Omega_h^*(Z), d_h)$ is a complex. We call its cohomology groups $H_h^*(\mathcal{F}_R)$ the *horizontal foliation cohomology of $\mathcal{F}_R$* .

While computing the Poisson cohomology of a non-degenerate b^k -Poisson manifold for all k , we recover the result of result of Ioan Mărcuț and Boris Osorno Torres [29] when $k = 1$.

Theorem 13. *Given a $2n$ -dimensional non-degenerate b^k -Poisson manifold (M, Z, π) , the Poisson cohomology $H_\pi^p(M)$ is*

$$H^p(M) \oplus H^{p-1}(Z) \text{ for } k = 1$$

$$H^p(M) \oplus H^{p-1}(Z) \oplus (H_h^{p-2}(\mathcal{F}_R))^{k-1} \oplus (H_h^{p-1}(\mathcal{F}_R))^{k-1} \text{ for } k \geq 2.$$

Proof. All that remains is to compute ${}^{\mathcal{R}}H^p(M)$. The evaluation bundle map $i : {}^{\mathcal{R}}TM \rightarrow {}^bTM$ induces an inclusion of complexes

$$0 \rightarrow {}^b\Omega^p(M) \xrightarrow{i^*} {}^{\mathcal{R}}\Omega^p(M) \xrightarrow{P} \mathcal{C}^p \rightarrow 0$$

where

$$\mathcal{C}^p = {}^{\mathcal{R}}\Omega^p(M) / {}^b\Omega^p(M)$$

is the quotient. We have a differential ${}^{\mathcal{C}}d$ induced by the differential ${}^{\mathcal{R}}d$ on ${}^{\mathcal{R}}\Omega^p(M)$. In particular, if P is the projection ${}^{\mathcal{R}}\Omega^p(M) \rightarrow {}^{\mathcal{R}}\Omega^p(M) / {}^b\Omega^p(M)$, then ${}^{\mathcal{C}}d(\eta) = P({}^{\mathcal{R}}d(\theta))$ where $\theta \in {}^{\mathcal{R}}\Omega^p(M)$ is any form such that $P(\theta) = \eta$. Hence $({}^{\mathcal{C}}d)^2 = 0$ and $(\mathcal{C}^*, {}^{\mathcal{C}}d)$ is a complex.

Given a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of M near Z , we can write a degree k form μ in ${}^{\mathcal{R}}\Omega^k(M)$ as

$$\mu = \nu_b + \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge L_i x^i + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge (\theta \wedge M_i + N_i) x^i + \sum_{i=0}^{k-1} \frac{\theta}{x^k} \wedge P_i x^i$$

for ν_b a smooth b-form, and $L_i, M_i, N_i, P_i \in \Omega_h^*(Z)$.

We write $R_b(\nu) = \mu_b$ and $S_b(\nu) = \nu - R_b(\nu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $R_b({}^{\mathcal{R}}d\nu) = {}^{\mathcal{R}}d(R_b(\nu))$ and $S_b({}^{\mathcal{R}}d\nu) = {}^{\mathcal{R}}d(S_b(\nu))$. Thus the trivialization τ induces a splitting ${}^{\mathcal{R}}\Omega^*(M) = {}^b\Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence ${}^{\mathcal{R}}H^p(M) = {}^bH^p(M) \oplus H^p(\mathcal{C}^*)$ and we are left to compute the cohomology of the quotient complex.

After identifying $\mathcal{C}^p = \{\mu \in {}^{\mathcal{R}}\Omega^k(M) : \nu_b = 0\}$, the differential is given by

$$\begin{aligned} d\mu = & \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge dL_i x^i + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge (\theta \wedge dM_i - dN_i) x^i \\ & - \sum_{i=0}^{k-1} \frac{(k-i)dx}{x^{k+1}} \wedge \theta \wedge P_i x^i - \sum_{i=0}^{k-1} \frac{\theta}{x^k} \wedge dP_i x^i. \end{aligned}$$

Then $d\nu = 0$ if and only if

- $\theta \wedge dL_i = 0$ for $i = 0, \dots, k-2$
- $\theta \wedge dL_{k-1} - \theta \wedge kP_0 = 0$
- $-(k-i)\theta \wedge P_i + \theta \wedge dM_{i-1} - dN_{i-1} = 0$ for $i = 1, \dots, k-1$.

Note that there is $\tilde{\mu} \in \mathcal{C}^{p-1}$ of the form

$$\tilde{\mu} = \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge l_i x^i + \sum_{i=0}^{k-2} \frac{\theta}{x^k} \wedge \frac{M_i x^{i+1}}{(-k+i+1)} + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge n_i x^i - \frac{\theta}{kx^k} \wedge L_{k-1}$$

satisfying

$$\begin{aligned} d\tilde{\mu} &= \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge dl_i x^i + \sum_{i=0}^{k-2} \left(\frac{dx}{x^k} \wedge \theta \wedge M_i x^i - \frac{\theta}{x^k} \wedge \frac{dM_i x^{i+1}}{(-k+i+1)} \right) \\ &\quad + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge dn_i x^i - \frac{dx}{x^{k+1}} \wedge \theta \wedge L_{k-1} + \frac{\theta}{kx^k} \wedge dL_{k-1}. \end{aligned}$$

Thus $[\mu - d\tilde{\mu}] \in H^p(\mathcal{C})$ has a representative

$$\sum_{i=0}^{k-2} \frac{dx}{x^{2k}} \wedge \theta \wedge (L_i - dl_i) x^i + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge (N_i - dn_i) x^i + \sum_{i=1}^{k-1} \frac{1}{x^k} \frac{dN_{i-1}}{(k-i)} x^i$$

where $L_i, l_i, N_i, n_i \in \Omega_h^*(Z)$. Note that $d_h N_i \equiv 0$ since $dN_{i-1} = (-k+i)\theta \wedge P_i$.

Notice if two forms ν_1, ν_2 are representatives of the same cohomology class in $H^p(\mathcal{C})$,

then the coefficients of the expression $\nu_1 - \nu_2$ must be exact. Thus, we have shown

$$H^p(\mathcal{C}) = \frac{\left\{ L_i \in \Omega_h^{p-2}(Z) : d_h L_i = 0 \right\}}{\left\{ L_i : L_i = d_h l_i, l_i \in \Omega_h^{p-3}(Z) \right\}} \oplus \frac{\left\{ N_i \in \Omega_h^{p-1}(Z) : d_h N_i = 0 \right\}}{\left\{ N_i : N_i = d_h n_i, n_i \in \Omega_h^{p-2}(Z) \right\}}$$

and

$$H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z) \oplus (H_h^{p-2}(\mathcal{F}_R))^{k-1} \oplus (H_h^{p-1}(\mathcal{F}_R))^{k-1}. \quad \square$$

Remark 7. *It may seem surprising to some readers that the Poisson cohomology involves the horizontal forms of the foliation, rather than the basic forms.*

Given the foliation $\mathcal{F}_R$, the basic forms are

$$\Omega_{b_{sc}}^p(Z) = \{ \sigma \in \Omega_h^p(Z) : \mathcal{L}_R \sigma = 0 \}.$$

We will provide an example of a b^2 Poisson manifold to help convince readers of this fact.

Example 40. *Let $M = \mathbb{R} \times \mathbb{T}^3$. let x be the coordinate on $\mathbb{R}$ and let θ_1, θ_2 , and θ_3 be the respective angular coordinates on the three copies of $\mathbb{S}^1$ in $\mathbb{T}^3$. We define a b^2 -symplectic form on $\mathbb{R} \times \mathbb{T}^3$ as*

$$\omega = \frac{dx}{x^2} \wedge d\theta_1 + d\theta_2 \wedge d\theta_3.$$

The singular hypersurface is $Z = \{0\} \times \mathbb{T}^3$. The symplectic form induces the cosymplectic structure $(d\theta_1, d\theta_2 \wedge d\theta_3)$ with Reeb vector field $\frac{\partial}{\partial\theta_1}$. The associated rigged algebroid $\mathcal{R}$ is generated by

$$x^2 \frac{\partial}{\partial x}, \quad x^2 \frac{\partial}{\partial\theta_1}, \quad \frac{\partial}{\partial\theta_2}, \quad \text{and} \quad \frac{\partial}{\partial\theta_3}.$$

Consider the rigged form

$$\nu = \frac{dx}{x^2} \wedge \frac{d\theta_1}{x^2} \wedge \cos(\theta_1)d\theta_3.$$

The one form $\cos(\theta_1)d\theta_3$ is a horizontal form because

$$i_{\frac{\partial}{\partial \theta_1}} \cos(\theta_1)d\theta_3 = 0.$$

Further, this form is closed under the horizontal differential because

$$\begin{aligned} d_h \cos(\theta_1)d\theta_3 &= (d - d\theta_1 \wedge \mathcal{L}_{\frac{\partial}{\partial \theta_1}}) \cos(\theta_1)d\theta_3 \\ &= -\sin(\theta_1)d\theta_1 \wedge d\theta_3 - d\theta_1 \wedge (i_{\frac{\partial}{\partial \theta_1}}(-\sin(\theta_1)d\theta_1 \wedge d\theta_3)) = 0. \end{aligned}$$

However, $\cos(\theta_1)d\theta_3$ is not a basic form because

$$\mathcal{L}_{\frac{\partial}{\partial \theta_1}} \cos(\theta_1)d\theta_3 = i_{\frac{\partial}{\partial \theta_1}}(-\sin(\theta_1)d\theta_1 \wedge d\theta_3) = -\sin(\theta_1)d\theta_3 \neq 0.$$

Consider

$$d_{\mathcal{R}}\nu = d \left(\frac{dx}{x^2} \wedge \frac{d\theta_1}{x^2} \wedge \cos(\theta_1)d\theta_3 \right) = \frac{dx}{x^2} \wedge \frac{d\theta_1}{x^2} \wedge [-\sin(\theta_1)d\theta_1 \wedge d\theta_3] = 0$$

and thus $[\nu] \in H_h^3(\mathbb{T}^3)$. Further, we can identify the class $[\nu]$ with $[\cos(\theta_1)d\theta_3] \in H_h^3(\mathbb{T}^3)$.

We will conclude with an example of a b^k Poisson manifold and give an explicit expression of its cohomology.

Example 41. Poisson cohomology of the b^k manifold $(\mathbb{T}^{2n}, \cup_2 \mathbb{T}^{2n-1})$

We will identify $\mathbb{T}^{2n}$ with $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{T}^{2n-2}$. Let $\theta \in [0, 2\pi)$ be the angular coordinate on the first copy of $\mathbb{S}^1$, let $d\phi$ be the volume form on the second $\mathbb{S}^1$, and let β be the standard symplectic form on $\mathbb{T}^{2n-2}$.

We define a b^k -symplectic form by

$$\omega = \frac{d\theta}{\sin^k \theta} \wedge d\phi + \beta.$$

The singular hypersurface of this structure is the disjoint union of $\{0\} \times \mathbb{T}^{2n-1}$ and $\{\pi\} \times \mathbb{T}^{2n-1}$. The symplectic foliation of the dual Poisson structure contains two open symplectic leaves: $\mathbb{T}^{2n} \setminus (\{0\} \times \mathbb{T}^{2n-1} \cup \{\pi\} \times \mathbb{T}^{2n-1})$. The hypersurfaces $\{0\} \times \mathbb{T}^{2n-1}$ and $\{\pi\} \times \mathbb{T}^{2n-1}$ are foliated by leaves of the form $\{cst.\} \times \mathbb{T}^{2n-2}$.

The induced cosymplectic structure is $(d\phi, \beta)$ and the associated Reeb vector field is $\frac{\partial}{\partial \phi}$.

Consider

$$\Omega_h^p(\mathbb{T}^{2n-1}) = \{\sigma \in \Omega^p(\mathbb{T}^{2n-1}) \mid i_{\partial \phi} \sigma = 0\} = \mathcal{C}^\infty(\mathbb{S}^1; \Omega^p(\mathbb{T}^{2n-2}))$$

and $d_h = d_{\mathbb{T}^{2n-2}}$. Then $H_h^p(\mathcal{F}_{\partial \phi}) = \mathcal{C}^\infty(\mathbb{S}^1; H^p(\mathbb{T}^{2n-2}))$. By Theorem 13, $H_\pi^p(\mathbb{T}^{2n})$ is computable as

$$H^p(\mathbb{T}^{2n}) \oplus [H^{p-1}(\mathbb{T}^{2n-1}) \oplus (C^\infty(\mathbb{S}^1; H^{p-2}(\mathbb{T}^{2n-2})) \oplus C^\infty(\mathbb{S}^1; H^{p-1}(\mathbb{T}^{2n-2})))^{k-1}]^2$$

and we note that the factors coming from the singular hypersurface are squared because we have a disjoint union of two tori.

3.4 log symplectic geometry

3.4.1 Intruduction

One class of manifolds where the Poisson cohomology is known is the case of b -symplectic manifolds. A *b-symplectic manifold*, defined by Ryszard Nest and Boris Tsygan in [41], is a $2n$ -dimensional manifold M equipped with a Poisson bi-vector π that is non-degenerate except on a hypersurface Z where there exist coordinates such that locally $Z = \{z = 0\}$ and

$$\pi = z \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial w} + \sum_{k=1}^{n-1} \frac{\partial}{\partial u_k} \wedge \frac{\partial}{\partial v_k}.$$

Recently, much work has been done studying the various facets of b -symplectic structures.

In particular, Ioan Mărcuț and Boris Osorno Torres ([29], Prop. 1) and Guillemin, Miranda, and Pires ([19], Thm. 30) showed that the Poisson cohomology $H_\pi^p(M)$ is isomorphic to the Lie algebroid cohomology of a specific Lie algebroid, the b -tangent bundle, and hence,

$$H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z).$$

We will employ the rigged algebroid method to study a class, which we call *partitionable*, of log symplectic manifolds, a generalization of the b -symplectic case formulated by Marco Gualtieri, Songhao Li, Alvaro Pelayo, and Tudor Ratiu [18]. We will consider Poisson manifolds (M, π) where π is non-degenerate except on a collection of hypersurfaces and at the degeneracy locus π locally takes the form

$$\pi = \sum_{i=1}^n x_i y_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{j=1}^m z_j \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial w_j} + \sum_{k=1}^\ell \frac{\partial}{\partial u_k} \wedge \frac{\partial}{\partial v_k}.$$

Note that n , m , and ℓ will vary depending on the number of hypersurfaces intersecting at the point and the hypersurfaces are locally defined by the coordinates: $Z_{x_i} = \{x_i = 0\}$, $Z_{y_i} = \{y_i = 0\}$, and $Z_{z_j} = \{z_j = 0\}$. We call these log symplectic structures *partitionable* because they partition the hypersurfaces in the degeneracy locus into two types: those hypersurfaces Z_{x_i} that are paired with a hypersurface Z_{y_i} and then hypersurfaces Z_{z_j} that are singletons.

The main result of this section is an explicit description of the Poisson cohomology for this class of Poisson manifold. We state our result here using a fixed tubular neighborhood of D , the degeneracy locus of π . A more detailed discussion is given in subsection 3.4.3. We will elaborate on the meaning of certain terms in the exposition following the statement.

Theorem 14. *Let (M, π) be a partitionable log symplectic structure for a partitioned set of compact, co-orientable, and transversely intersecting hypersurfaces:*

$$D = \{Z_{x_1}, Z_{y_1}, \dots, Z_{x_n}, Z_{y_n}, Z_{z_1}, \dots, Z_{z_m}\}.$$

The Poisson cohomology $H_\pi^p(M)$ of (M, π) is a direct sum of the following vector spaces:

1. *A single copy of $H^p(M)$.*
2. *For every non-empty subset $E \subseteq D$ with $|E| \leq p$, a copy of*

$$H^{p-|E|} \left(\bigcap_{Z \in E} Z \right).$$
3. *For every subset $F \subseteq D$ containing at least one pair of elements Z_{x_i}, Z_{y_i} and with $|F| \leq p$, a copy of*

$$H^{p-|F|} \left(\bigcap_{Z \in F} Z \right).$$

To understand this theorem, let us begin by considering a smooth manifold M together with a finite set D of smooth hypersurfaces $Z_i \subset M$ that intersect transversely. In other words, D is a smooth normal crossing divisor on M . Throughout this work we will further assume that every hypersurface Z_i is compact and co-orientable. We view log symplectic Poisson bi-vectors as non-degenerate on Richard Melrose's b -tangent bundle bTM [36], called the log tangent bundle in [18].

Definition 15. [18] *A log symplectic structure on (M, D) is a closed non-degenerate 2-form ω in the de Rham complex of the b -tangent bundle. In other words,*

$$\omega \in {}^b\Omega^2(M) = \mathcal{C}^\infty(M; \wedge^2({}^bT^*M))$$

satisfying $d\omega = 0$ and $\omega^n \neq 0$. The form ω induces a map $\omega^\flat$ between the b -tangent and b -cotangent bundles.

$${}^bTM \begin{array}{c} \xrightarrow{\omega^\flat} \\ \xleftarrow{\pi^\sharp = (\omega^\flat)^{-1}} \end{array} {}^bT^*M$$

The inverse map is induced by a bi-vector $\pi \in C^\infty(M; \wedge^2(^bTM))$. This bi-vector is called a log Poisson structure on (M, D) .

As we mentioned above, log symplectic manifolds are a broad generalization of b -symplectic manifolds as developed in [41]. We restrict our attention to a class of structures that share some nice geometric features of b -symplectic structures that are lost in the whole of the log symplectic category.

Definition 16. A log symplectic structure ω is partitionable if it determines - in a procedure explicitly described below - a partition Λ_D of the set D into pairs Z_{x_i}, Z_{y_i} and singletons Z_{z_j} . We call the hypersurface pairs type x and type y . We call the singleton hypersurfaces type z . Thus the set D admits a relabeling

$$Z_{x_1}, Z_{y_1}, \dots, Z_{x_k}, Z_{y_k}, \quad Z_{z_1}, \dots, Z_{z_\ell}.$$

Up to switching the labels x_i and y_i , and permuting the set $\{1, \dots, k\}$ and $\{1, \dots, \ell\}$, this partition is unique.

Determining the partition Λ_D :

Victor Guillemin, Eva Miranda, and Ana Rita Pires[19] showed that every b -symplectic form induces a cosymplectic structure $(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$ on its singular hypersurface Z for every choice of Z defining function. That is, there exists a pair of closed forms θ, η on Z such that $\theta \wedge \eta^{n-1} \neq 0$ where the dimension of Z is $2n - 1$. Further, they showed that on Z , $\ker \theta$ defines the induced symplectic foliation of the associated Poisson structure, depicted in Figure 27.

A log symplectic structure is partitionable when it induces a generalization of cosymplectic structures on the intersection of any subset of hypersurfaces in D . In particular we would like the induced cosymplectic structures on each $Z \in D$ to intersect ‘nicely’. Let us

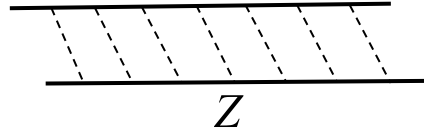
Figure 27. single hypersurface

local expression

$$\omega = \frac{dz}{z} \wedge \theta + \eta$$

for $\theta, \eta \in \Omega^*(Z)$

induced foliation



consider the symplectic foliation induced by a log Poisson structure. Our class of log symplectic structures induces regular foliations on the intersections of hypersurfaces in D . We desire one of two types of behavior for a log symplectic form at the nonempty intersection of two hypersurfaces Z_1, Z_2 in divisor D and away from $D \setminus \{Z_1, Z_2\}$.

Type x and y: In the first instance, we have the intersection of hypersurfaces $Z_{x_1} = \{x_1 = 0\}$, $Z_{y_1} = \{y_1 = 0\}$, depicted in Figure 28. The intersection $Z_{x_1} \cap Z_{y_1}$ is a symplectic leaf in the foliation induced by $\pi = \omega^{-1}$. This leaf extends the foliation on Z_{x_1} away from $Z_{x_1} \cap Z_{y_1}$ and extends the foliation on Z_{y_1} away from $Z_{y_1} \cap Z_{x_1}$.

Figure 28. *Two hypersurfaces*

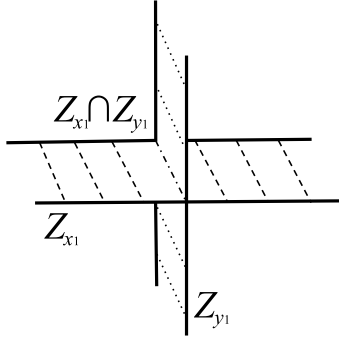
Type x/y

local expression:

$$\omega = \frac{dx_1}{x_1} \wedge \frac{dy_1}{y_1} + \delta$$

for $\delta \in \Omega^2(M)$.

induced foliation:



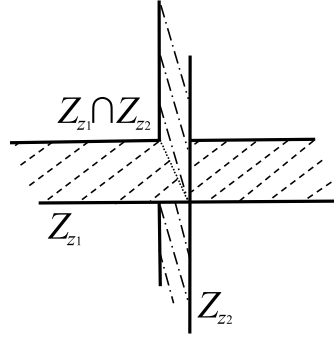
Type z

local expression:

$$\omega = \frac{dz_1}{z_1} \wedge \alpha + \frac{dz_2}{z_2} \wedge \beta + \delta$$

for $\alpha, \beta \in \Omega^1(M)$, and $\delta \in \Omega^2(M)$.

induced foliation:



Type z: In the second instance, we have the intersection of hypersurfaces $Z_{z_1} = \{z_1 = 0\}$, $Z_{z_2} = \{z_2 = 0\}$, shown in Figure 2. Then $\pi = \omega^{-1}$ induces a codimension 2 symplectic foliation on $Z_{z_1} \cap Z_{z_2}$.

For more general hypersurface intersections, $\pi = \omega^{-1}$ induces a regular symplectic foliation with co-dimension dependent on the number and type of hypersurfaces meeting at the intersection. Partitionable log symplectic structures are precisely those log symplectic structures whose induced foliation is given by what is called a k -cosymplectic structure.

Definition 17. [43] A *k*-cosymplectic structure¹ on a $k + 2\ell$ dimensional manifold M is a family (α_i, β) of k closed one forms $\alpha_i \in \Omega^1(M)$ and a closed two form $\beta \in \Omega^2(M)$ such that $(\wedge_i \alpha_i) \wedge \beta^\ell \neq 0$.

3.4.2 Partitionable log symplectic structures

In this section we will discuss various features of partitionable log Poisson structures. In particular, we will discuss how to easily identify partitionable structures from among all log symplectic forms.

Let M be a manifold and D a set of compact, co-orientable, and transversely intersecting hypersurfaces. Recall that the b -tangent bundle is the vector bundle whose smooth sections are the vector fields tangent to D , that is

$$\{u \in \mathcal{C}^\infty(M; TM) : u|_Z \in \mathcal{C}^\infty(Z, TZ) \text{ for all } Z \in D\}.$$

This vector bundle is a Lie algebroid with anchor map the evaluation in TM and with bracket induced by the standard Lie bracket on TM .

Definition 18. A map $\phi : (M_1, D_1 = \{Z_1, \dots, Z_m\}) \rightarrow (M_2, D_2 = \{W_1, \dots, W_m\})$ is a *b-map* if $f(Z_i) \subseteq W_i$, f is transverse to each W_i , and $f^{-1}(W_i) = Z_i$. Given two log symplectic forms ω_1, ω_2 on (M, D) , a log-symplectomorphism is a *b-map* $\phi : M \rightarrow M$ that is a diffeomorphism such that $\phi^* \omega_2 = \omega_1$.

Let τ denote a nonempty subset of $\{1, \dots, |D|\}$. In [18], they point out that the Lie algebroid cohomology of the b -tangent bundle over (M, D) is

$${}^b H^p(M) \simeq H^p(M) \oplus \bigoplus_{|\tau| \leq p} H^{p-|\tau|} \left(\bigcap_{t \in \tau} Z_t \right). \quad (3.12)$$

We can identify partitionable log symplectic structures on (M, D) by their b -cohomology classes.

To any log symplectic form ω we can associate a *decomposition* of its cohomology class $[\omega] \in {}^b H^2(M)$ in terms of the isomorphism (3.12):

¹The term *k*-cosymplectic has also been used in classical field theories, with a different meaning.

$$(a, \underbrace{b_1, \dots, b_k}_{b_i \in H^1(Z_i)}, \underbrace{c_{1,2}, \dots, c_{k-1,k}}_{c_{i,j} \in H^0(Z_i \cap Z_j)}) \in H^2(M) \oplus \bigoplus_i H^1(Z_i) \oplus \bigoplus_{i \neq j} H^0(Z_i \cap Z_j).$$

Given ω any log symplectic form on a manifold (M, D) , the structure ω is *partitionable* if its b -cohomology class decomposition

$$(a, b_1, \dots, b_k, c_{\{1,2\}}, \dots, c_{\{k-1,k\}})$$

satisfies the following conditions.

1. For all s , either $b_s \neq 0$ or $c_{\{s,i\}} \neq 0$ for some i .
2. If $b_s \neq 0$, then $c_{\{i,s\}} = 0$ for all i .
3. If $c_{\{s,t\}} \neq 0$, then $c_{\{s,j\}} = c_{\{i,t\}} = 0$ for all $j \neq t, i \neq s$.

Note that this cohomology decomposition is sufficient to guarantee a log symplectic is partitionable, but it is not a necessary condition. We will prove this characterization in Proposition 3.13.

Recall by definition, given a partitionable log symplectic form ω on a manifold (M, D) , ω gives us a partition Λ_D of the set D as

$$Z_{x_1}, Z_{y_1}, \dots, Z_{x_k}, Z_{y_k}, Z_{z_1}, \dots, Z_{z_\ell}.$$

Given a subset S of a divisor D , we call

$$X_S = \bigcap_{Z \in S} Z$$

a *maximal intersection* if X_S is non-empty and if $Z \cap X_S = \emptyset$ for all $Z \in D \setminus S$.

We can assign a subpartition Λ_S to the subset S according to the decomposition $(a, b_1, \dots, b_k, c_{\{1,2\}}, \dots, c_{\{k-1,k\}})$ of $[\omega] \in {}^b H^2(U_S)$ where U_S is any neighborhood of X_S . In particular, hypersurfaces of type z in Λ_D will remain type z in the subpartition Λ_S . However if there is a hypersurface Z_{x_i} of type x in S and its type y counterpart Z_{y_i} is not in S , then Z_{x_i} will have a type z designation in the subpartition Λ_S . We will show that there are hypersurface defining functions, i.e. $x \in \mathcal{C}^\infty(M)$ such that $Z_x = \{x = 0\}$ and $dx(p) \neq 0$ for all $p \in Z$, so we can express ω near X_S as

$$\omega_0 = \sum_{i=1}^k \lambda_i \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i} + \sum_{j=1}^m \frac{dz_j}{z_j} \wedge \alpha_j + \delta. \quad (3.13)$$

where $\lambda_i \in \Omega^0(X_S)$ is a closed form representing $c_{\{i,j\}}$, $\alpha_j \in \Omega^1(X_S)$ is a closed form representing b_{z_j} , and $\delta \in \Omega^2(X_S)$ is a closed form representing a .

Proposition 14. *Let ω be a partitionable log symplectic form on a manifold (M, D) . Let S be any subset of D such that $X_S = \bigcap_{Z \in S} Z$ is a maximal intersection and let Λ_S be a subpartition of S . If $(a, b_1, \dots, b_k, c_{\{1,2\}}, \dots, c_{\{k-1,k\}})$ is a decomposition of $[\omega] \in {}^b H^2(U_S)$ where U_S is a tubular neighborhood of X_S , then there exist*

- *hypersurface defining functions x_i, y_i, z_i partitioned according to Λ_S ,*
- *a tubular neighborhood $U \supset X_S$, and*
- *$\alpha_j \in \Omega^1(X_S)$ a closed form representing b_{z_j} and $\delta \in \Omega^2(X_S)$ a closed form representing a ,*

such that on U there is a log-symplectomorphism pulling ω back to (3.13).

Proof. Let ω be a partitionable log symplectic form on (M, D) . Given a maximal intersection X_S given by $S \subseteq D$, let $(a, b_1, \dots, b_k, c_{\{1,2\}}, \dots, c_{\{k-1,k\}})$ be a decomposition of $[\omega] \in {}^b H^2(U_S)$. Let x_i, y_i, z_i be hypersurface defining functions partitioned according to Λ_S . By the isomorphism from equation (3.12), in these coordinates $\omega|_{U_S} =$

$$\begin{aligned} & \sum_{i,j} \left(\frac{dx_i}{x_i} \wedge \left(a_{ij} \frac{dx_j}{x_j} + b_{ij} \frac{dy_j}{y_j} + c_{ij} \frac{dz_j}{z_j} \right) + \frac{dy_i}{y_i} \wedge \left(d_{ij} \frac{dy_j}{y_j} + e_{ij} \frac{dz_j}{z_j} \right) \right) \\ & + \sum_{i,j} f_{ij} \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j} + \sum_{\ell} \left(\frac{dx_{\ell}}{x_{\ell}} \wedge A_{\ell} + \frac{dy_{\ell}}{y_{\ell}} \wedge B_{\ell} + \frac{dz_{\ell}}{z_{\ell}} \wedge C_{\ell} \right) + \delta. \end{aligned}$$

From $d\omega = 0$, it follows that $a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}$ are all closed 0-forms and thus are real numbers. By the conditions provided above, the only non-zero numbers among these are b_{ii} . Further, by the cohomological characterization of partitionable, the one-forms $A_{\ell}|_{Z_{x_{\ell}} \cap Z_{y_{\ell}}}$ is exact and $B_{\ell}|_{Z_{x_{\ell}} \cap Z_{y_{\ell}}}$ is exact. Thus $A_{\ell}|_{X_S}$ and $B_{\ell}|_{X_S}$ are exact.

Thus under an appropriate relabeling and change of X_S defining functions

$$\omega|_{U_S} = \sum_{i=1}^k \lambda_i \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i} + \sum_{j=1}^m \frac{dz_j}{z_j} \wedge \alpha_j + \delta.$$

Now we will proceed by the standard relative Moser argument (See [11] Sec. 7.3 for the smooth setting, and [47] Thm 6.4 for the b -symplectic version). Pick a tubular neighborhood $U_0 \subseteq U_S$ of X_S . Define ω_0 to be

$$\sum_{i=1}^k \tilde{\lambda}_i \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i} + \sum_{j=1}^m \frac{dz_j}{z_j} \wedge \tilde{\alpha}_j + \tilde{\delta}$$

where $\tilde{\lambda}_i$, $\tilde{\alpha}_j$, and $\tilde{\delta}$ are the pullbacks of λ_i , α_j , and δ respectively. Then in U_0 ,

$$\omega - \omega_0 = \sum_{j=1}^m \frac{dz_j}{z_j} \wedge (\alpha_j - \tilde{\alpha}_j) + \delta - \tilde{\delta}.$$

Since the form $\omega - \omega_0$ is closed on U_0 , and $(\omega - \omega_0)|_{X_S} = 0$ and $\delta - \tilde{\delta} = 0$, by the relative Poincaré Lemma, there exist primitives μ_j of $\alpha_j - \tilde{\alpha}_j$ and a primitive σ of $\delta - \tilde{\delta}$ such that $\mu_j|_{X_S} = \sigma|_{X_S} = 0$. Define

$$\mu = -\sum_{j=1}^m \frac{dz_j}{z_j} \mu_j + \sigma.$$

Then $\omega - \omega_0 = d\mu$. Let $\omega_t = (1-t)\omega_0 + t\omega$. Then $\frac{d\omega_t}{dt} = \omega - \omega_0 = d\mu$. Because μ is a log one form, the vector field defined by $i_{v_t}\omega_t = -\mu$ is a log vector field and its flow fixes the divisor D . Further, $v_t = 0$ on X_S . Thus we can integrate v_t to an isotopy that is the identity on X_S and fixes D . This isotopy is the desired log-symplectomorphism. $\square$

This proposition gives us k -cosymplectic structures on every intersection of subsets of D :

Proposition 15. *Let ω be a partitionable log symplectic form ω on a manifold (M, D) . For any set $S \subseteq D$ such that $\bigcap_{z \in S} Z$ is nonempty, let*

$$X = \left(\bigcap_{z \in S} Z \right) \setminus \left(\bigcap_{z \in S} Z \cap (D \setminus S) \right).$$

In other words, the set X is the intersection of elements in S away from hypersurfaces in $D \setminus S$. For each choice of a set of defining functions for the hypersurfaces in S , the form ω induces an ℓ -cosymplectic structure on X where ℓ is the number of type z forms in Λ_S .

Proof. For any $S \subseteq D$ such that $\bigcap_{z \in S} Z$ is nonempty, let $X = \bigcap_{z \in S} Z$ away from higher order intersections. By equation (3.13),

$$\omega|_X = \sum_{i=1}^k \lambda_i \frac{dr_i}{r_i} \wedge \frac{ds_i}{s_i} + \sum_{j=1}^m \frac{dt_j}{t_j} \wedge \theta_j + \beta$$

for closed $\lambda_i \in \Omega^0(X)$, $\theta_j \in \Omega^1(X)$, $\beta \in \Omega^2(X)$. By the non-degeneracy of ω ,

$$0 \neq \wedge^n \omega|_X = \left(\bigwedge_i \lambda_i \frac{dr_i}{r_i} \wedge \frac{ds_i}{s_i} \right) \wedge \left(\bigwedge_j \frac{dt_j}{t_j} \wedge \theta_j \right) \wedge \beta^{n-k-m}.$$

Thus

$$\left(\bigwedge_j \theta_j \right) \wedge \beta^{n-k-m} \neq 0$$

and (θ_j, β) is an ℓ -cosymplectic structure on X . $\square$

By the standard symplectic linear algebra argument (for instance see [11] Sec. 1.1) we

can express ω at point $p \in D$ as

$$\omega_p = \sum_{i=1}^k \lambda_i \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i} + \sum_{j=1}^m \frac{dz_j}{z_j} \wedge ds_j + \sum_{k=1}^n dp_k \wedge dq_k.$$

By the proof of Proposition 14, a relative Moser's argument gives an analogue of Darboux's theorem for partitionable log symplectic structures.

Corollary 2. *Let ω be a partitionable log symplectic form ω on a manifold (M, D) . Let $p \in D$ and let S be the subset of D of hypersurfaces containing p . Let Λ_S be a subpartition of S . Then there exist local coordinates centered at p with local hypersurface defining functions x_i, y_i, z_i partitioned according to Λ_S such that*

$$\omega = \sum_{i=1}^k \lambda_i \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i} + \sum_{j=1}^m \frac{dz_j}{z_j} \wedge ds_j + \sum_{k=1}^n dp_k \wedge dq_k$$

and the Poisson bi-vector associated to ω has the form

$$\pi = \sum_{i=1}^k \frac{1}{\lambda_i} x_i y_i \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_i} + \sum_{j=1}^m z_j \frac{\partial}{\partial s_j} \wedge \frac{\partial}{\partial z_j} + \sum_{k=1}^n \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial p_k}.$$

We will next show that every k -cosymplectic structure (α_j, β) can sit inside a larger manifold (M, D) such that a log symplectic form on M induces (α_j, β) on W .

Examples from products

In Example 18 of [19], Guillemin, Miranda, and Pires explained how given any compact b -symplectic surface (M_b, π_b) and any compact symplectic surface (M_s, π_s) , the product $(M_b \times M_s, \pi_b + \pi_s)$ is a b -Poisson manifold. Similarly, the product of any partitionable log symplectic surfaces will produce a partitionable log symplectic manifold.

For instance, the torus $\mathbb{T}^2 \simeq \mathbb{S}_\theta^1 \times \mathbb{S}_\rho^1$ is a log symplectic surface with partitionable form

$$\omega = \frac{d\theta}{\sin(\theta)} \wedge \frac{d\rho}{\sin(\rho)}.$$

In general, we have a product structure for partitionable log symplectic manifolds: Given two partitionable log symplectic manifolds (M_1, D_1, ω_1) and (M_2, D_2, ω_2) , one can show that the product

$$(M_1 \times M_2, (D_1 \times M_2) \cup (D_2 \times M_1), \omega_1 + \omega_2)$$

is also a partitionable log symplectic manifold. Further, this product respects the existing partitions of D_1 and D_2 : For instance, if $Z \in D_1$ was type x , then $Z \times M_2$ is type x in $(D_1 \times M_2) \cup (D_2 \times M_1)$.

We can also explicitly construct partitionable log symplectic manifolds from a given k -cosymplectic structure by taking a product with a torus.

Example 42. *Let (M, β) be any symplectic manifold. Then $\mathbb{T}^k \times \mathbb{T}^k \times M$ with angular coordinates $\theta_1, \dots, \theta_k, \rho_1, \dots, \rho_k$ on $\mathbb{T}^k \times \mathbb{T}^k$ is a partitionable log symplectic structure with the form*

$$\omega = \sum_{i=1}^k \frac{d\theta_i}{\sin(\theta_i)} \wedge \frac{d\rho_i}{\sin(\rho_i)} + \beta.$$

Let (M, α_j, β) be any k -cosymplectic manifold. Then $\mathbb{T}^k \times M$ with θ_i the angle coordinates on $\mathbb{S}^1$ is a partitionable log symplectic structure with the form

$$\omega = \sum_{i=1}^k \frac{d\theta_i}{\sin(\theta_i)} \wedge \alpha_i + \beta.$$

Symplectic foliations

Next we continue our discussion from the introduction, where we mentioned that the class of partitionable log symplectic structures induce regular foliations on the intersection of hypersurfaces in D . For general intersections, let $I, J, K \subseteq \{1, \dots, k\}$ and $L \subseteq \{1, \dots, m\}$ such that $I \cap J = J \cap K = I \cap K = \emptyset$. On

$$W = \bigcap_{i \in I} (Z_{x_i} \cap Z_{y_i}) \bigcap_{j \in J} Z_{x_j} \bigcap_{k \in K} Z_{y_k} \bigcap_{\ell \in L} Z_{z_\ell}$$

away from $W \cap (D \setminus \{Z_{x_i}, Z_{y_i}, Z_{x_j}, Z_{y_k}, Z_{z_\ell}\})$, ω induces a regular codimension $|J| + |K| + |L|$ symplectic foliation. Further, the foliation is given by the k -cosymplectic structure provided in Proposition 15.

3.4.3 Proof of Theorem 14

In this section we will prove a coordinate independent version of Theorem 14. To state the theorem coordinate independently, we first give a brief recall of densities bundles, which are more thoroughly discussed in section 4.5 of [36].

Definition 19. *Let $p \in Z$. The space of s -densities on N_p^*Z is, for $s \in \mathbb{R}$, the space*

$$|N_p^*Z|^s = \left\{ \psi : N_p^*Z \setminus \{0\} \rightarrow \mathbb{R} \mid \psi(\lambda V) = |\lambda|^s \psi(V) \ \forall V \in N_p^*Z \setminus \{0\}, \lambda \neq 0 \right\}$$

*and the s -densities on N^*Z over Z is the bundle*

$$|N^*Z|^s = \bigsqcup_{p \in Z} |N_p^*Z|^s.$$

A Z defining function x induces a trivialization $|dx|^s$ of $|N^*Z|^s \rightarrow Z$. If $\tilde{x} = fx$ is another Z defining function, then $|d\tilde{x}|^s = f^s|dx|^s$ as sections of $|N^*Z|^s$.

In the following theorem, a density bundle appears in the cohomology groups to account for changes of Z defining function, making our presentation of the cohomology independent of Z defining function.

Theorem 15. *Let (M, π) be a partitionable log symplectic structure for*

$$D = \{Z_{x_1}, Z_{y_1}, \dots, Z_{x_k}, Z_{y_k}, Z_{z_1}, \dots, Z_{z_\ell}\}.$$

Let $\mathcal{M}$ denote all collections of sets I, J, K, L satisfying

$$I, J, K \subseteq \{1, \dots, k\}, L \subseteq \{1, \dots, \ell\}$$

$$\text{such that } I \neq \emptyset \text{ and } I \cap J = I \cap K = \emptyset.$$

Set

$$m = 2|I| + |J| + |K| + |L|$$

and let v_i denote $Z_{x_i} \cap Z_{y_i}$. Then the Poisson cohomology $H_\pi^p(M)$ of (M, π) is

$${}^bH^p(M) \oplus \bigoplus_{\mathcal{M}} H^{p-m}(\underbrace{\bigcap_{i \in I, j \in J, k \in K, \ell \in L} Z_{x_i} \cap Z_{y_i} \cap Z_{x_j} \cap Z_{y_k} \cap Z_{z_\ell}}; \bigotimes_{i \in I} |N_{v_i}^* Z_{x_i}|^{-1} \otimes |N_{v_i}^* Z_{y_i}|^{-1})$$

for $m \leq p$.

Recall, given a Poisson manifold (M, π) , the Poisson cohomology $H_\pi^*(M)$ is defined as the cohomology groups of the Lichnerowicz complex (see for instance [12], p. 39). The k -th element in the sequence is made up of smooth k -multivector fields on M , $\mathcal{V}^k(M) := \mathcal{C}^\infty(M; \wedge^k TM)$.

$$\dots \rightarrow \mathcal{V}^{k-1}(M) \xrightarrow{d_\pi} \mathcal{V}^k(M) \xrightarrow{d_\pi} \mathcal{V}^{k+1}(M) \rightarrow \dots$$

The differential

$$d_\pi : \mathcal{V}^k(M) \rightarrow \mathcal{V}^{k+1}(M)$$

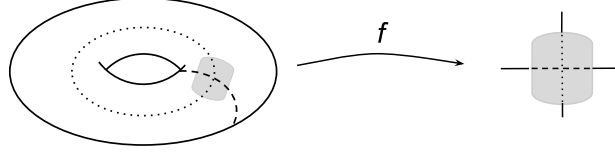
is defined as

$$d_\pi = [\pi, \cdot],$$

where $[\cdot, \cdot]$ is the Schouten bracket extending the standard Lie bracket on vector fields $\mathcal{V}^1(M)$.

Good tubular neighborhoods

Consider a manifold M with a set D of smooth transversely intersecting hypersurfaces. For each point $p \in D$, there is a chart (U, f) of M centered at p such that for each hypersurface $Z \in D$ with $Z \cap U \neq \emptyset$, $f(U \cap Z)$ is a coordinate hyperplane in $\mathbb{R}^n$ intersected with $f(U)$.



A good tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)$ of $Z \in D$ is a neighborhood that extends charts of the type U above at the intersection of $Z \cap (D \setminus Z)$. In our computations below we will always use good tubular neighborhoods. For existence of such neighborhoods, see for instance section 5 of [1].

Constructing the rigged algebroid

Next, we will identify the rigged Lie algebroid of a non-degenerate partitionable log Poisson structure, denoted ${}^{\log}\mathcal{R}$. For the purposes of computing Poisson cohomology, it will be useful to construct ${}^{\log}\mathcal{R}$ through a sequence of Melrose rescalings, explained in Section 2.6.

The Lie algebroid $\mathcal{A}_i$. Consider an expression of ω as in equation (3.13):

$$\omega = \sum_{i=1}^k \lambda_i \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i} + \sum_{j=1}^m \frac{dz_j}{z_j} \wedge \alpha_j + \delta.$$

We will begin by rescaling TM at Z_{z_1} by the vector bundle $\ker \alpha_1 \rightarrow Z_{z_1}$. In order to employ Theorem 2, we must verify that

$$[\ker \alpha_1, \ker \alpha_1] \subseteq \ker \alpha_1.$$

Let $X, Y \in \ker \alpha_1$. Consider

$$d\alpha_1(X, Y) = X\alpha_1(Y) - Y\alpha_1(X) - \alpha_1([X, Y]). \quad (3.14)$$

Because $d\alpha_1 = 0$ and $X, Y \in \ker \alpha_1$, this reduces to $\alpha_1([X, Y]) = 0$. Thus $[X, Y] \in \ker \alpha_1$.

Thus by Theorem 2, there is a Lie algebroid $\mathcal{A}_1$ whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; TM) : u|_{Z_{z_1}} \in \mathcal{C}^\infty(Z_{z_1}; \ker \alpha_1)\}.$$

Note that the cotangent bundle T^*M includes into $\mathcal{A}_1^*$. Thus the one form $\alpha_2 \in \Omega^1(M)|_{Z_{z_2}}$ can be lifted to a one form $\tilde{\alpha}_2 = i(\alpha_2) \in \mathcal{A}_1^* \Omega^1(M)|_{Z_{z_2}}$.

$$\begin{array}{ccc} & \mathcal{A}_1^*|_{Z_{z_2}} & \\ & \uparrow i & \nwarrow \tilde{\alpha}_2 \\ T^*M|_{Z_{z_2}} & \xleftarrow{\alpha_2} & M \end{array}$$

Note that we can always lift forms in this way, however the lifted form may vanish at $Z_{z_1} \cap Z_{z_2}$ while the original does not. In order to employ Theorem 2, we must verify that $\ker \tilde{\alpha}_2$ is a subbundle of $\mathcal{A}_1|_{Z_{z_2}}$. Because α_2 is a closed one-form in a k -cosymplectic structure at Z_{z_2} , $\ker \alpha_2$ is a subbundle of $TM|_{Z_{z_2}}$. Away from $Z_{z_1} \cap Z_{z_2}$, the inclusion i gives us an isomorphism $\mathcal{A}_1^*|_{Z_{z_2}} \simeq T^*M|_{Z_{z_2}}$ and it is clear that $\ker \tilde{\alpha}_2$ is a subbundle of $\mathcal{A}_1^*|_{Z_{z_2}}$.

Let $p \in Z_{z_1} \cap Z_{z_2}$. There exist local coordinates $z_1, s_1, z_2, s_2, p_1, \dots, p_n$ of M at $Z_{z_1} \cap Z_{z_2}$ such that $\alpha_1 = ds_1$ and $\alpha_2 = ds_2$. Note that locally near p sections of

$$T^*M \text{ are generated by } dz_1, ds_1, dz_2, ds_2, dp_1, \dots, dp_n$$

and

$$\mathcal{A}_1^* \text{ are generated by } \frac{dz_1}{z_1}, \frac{ds_1}{z_1}, dz_2, ds_2, dp_1, \dots, dp_n.$$

Note that the support of α_2 is the image of the support of $\tilde{\alpha}_2$ under the anchor map of $\mathcal{A}_1$. Thus $\text{rank}(\ker \alpha_2) = \text{rank}(\ker \tilde{\alpha}_2)$ and $\ker \tilde{\alpha}_2$ is a subbundle of $\mathcal{A}_1|_{Z_{z_2}}$.

We will form $\mathcal{A}_2$ by rescaling $\mathcal{A}_1$ by $\ker \tilde{\alpha}_2$ at Z_2 . By the computation analogous to equation (3.14) with respect to $\tilde{\alpha}_2$, there exists a Lie algebroid $\mathcal{A}_2$ whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; \mathcal{A}_1) : u|_{Z_{z_2}} \in \mathcal{C}^\infty(Z_{z_2}; \ker \tilde{\alpha}_2)\}.$$

To form $\mathcal{A}_j$ we will rescale $\mathcal{A}_{j-1}$ at Z_{z_j} . As above, we lift the one form $\alpha_j \in \Omega^1(Z_{z_j})$ to the one form $\tilde{\alpha}_j = i(\alpha_j) \in \mathcal{A}_{j-1}^1 \Omega^1(M)|_{Z_{z_j}}$. Analogous to the argument above, one can verify in local coordinates that $\ker \tilde{\alpha}_j$ is a subbundle of $\mathcal{A}_{j-1}|_{Z_{z_j}}$. By computation (3.14), there exists a Lie algebroid $\mathcal{A}_j$ whose space of sections is

$$\left\{ u \in \mathcal{C}^\infty(M; \mathcal{A}_{j-1}) : u|_{Z_{z_j}} \in \mathcal{C}^\infty(Z_{z_j}; \ker \tilde{\alpha}_j) \right\}.$$

The Lie algebroid $\mathcal{B}_i$. Next, we rescale $\mathcal{A}_m$ at Z_{x_i} and Z_{y_i} .

As previously described, we can lift the one form $dx_1 \in \Omega^1(M)|_{Z_{x_1}}$ to the one form $\tilde{dx}_1 = i(dx_1) \in \mathcal{A}_m \Omega^1(M)|_{Z_{x_1}}$ and we can lift $dy_1 \in \Omega^1(M)|_{Z_{y_1}}$ to the one form $\tilde{dy}_1 = i(dy_1) \in \mathcal{A}_m \Omega^1(M)|_{Z_{y_1}}$. The $\mathcal{A}_m$ -one form $\tilde{dx}_1$ is non-zero at Z_{x_1} and the $\mathcal{A}_m$ -one form $\tilde{dy}_1$ is non-zero at Z_{y_1} . Further, by (3.14) above, $[\ker \tilde{dx}_1, \ker \tilde{dx}_1] \subseteq \ker \tilde{dx}_1$ and $[\ker \tilde{dy}_1, \ker \tilde{dy}_1] \subseteq \ker \tilde{dy}_1$. Additionally, since we are working in a good tubular neighborhood,

$$[\partial_{x_1}, \partial_{y_1}] = 0.$$

Thus $[\ker \tilde{dx}_1 \cap \ker \tilde{dy}_1, \ker \tilde{dx}_1 \cap \ker \tilde{dy}_1] \subseteq \ker \tilde{dx}_1 \cap \ker \tilde{dy}_1$

and by Theorem 2, there exists a Lie algebroid $\mathcal{B}_1$ whose space of sections is

$$\left\{ u \in \mathcal{C}^\infty(M; \mathcal{A}_m) \left| \begin{array}{l} u|_{Z_{x_1}} \in \mathcal{C}^\infty(Z_{x_1}, \ker \tilde{dx}_1) \\ u|_{Z_{y_1}} \in \mathcal{C}^\infty(Z_{y_1}, \ker \tilde{dy}_1) \end{array} \right. \right\}.$$

We iteratively form $\mathcal{B}_j$ by rescaling $\mathcal{B}_{j-1}$. First, we lift the one form $dx_j \in \Omega^1(M)|_{Z_{x_j}}$ to the one form $\tilde{dx}_j = i(dx_j) \in \mathcal{B}_{j-1}^1 \Omega^1(M)|_{Z_{x_j}}$ and we lift $dy_j \in \Omega^1(M)|_{Z_{y_j}}$ to the one form $\tilde{dy}_j = i(dy_j) \in \mathcal{B}_{j-1}^1 \Omega^1(M)|_{Z_{y_j}}$. Similar to above, the algebroid $\mathcal{B}_n$ is the vector bundle whose space of sections is

$$\left\{ u \in \mathcal{C}^\infty(M; \mathcal{B}_{j-1}) \left| \begin{array}{l} u|_{Z_{x_j}} \in \mathcal{C}^\infty(Z_{x_j}, \ker \tilde{dx}_j) \\ u|_{Z_{y_j}} \in \mathcal{C}^\infty(Z_{y_j}, \ker \tilde{dy}_j) \end{array} \right. \right\}.$$

By using our local expression for a partitionable log symplectic form ω , one can check that $\mathcal{B}_k = {}^{\log}\mathcal{R}$ as a vector bundle and, by the continuity of the standard Lie bracket off of D , these are in fact isomorphic as Lie algebroids. If $W = \bigcap_{Z \in D} Z$ is non-empty, then for all $p \in W$ there exist local coordinates

$$(x_1, y_1, \dots, x_k, y_k, z_1, v_1, \dots, z_\ell, v_\ell, p_1, q_1, \dots, p_m, q_m)$$

such that local sections of ${}^{\log}\mathcal{R}$ are smooth linear combinations of

$$x_1 y_1 \frac{\partial}{\partial x_1}, x_1 y_1 \frac{\partial}{\partial y_1}, \dots, x_k y_k \frac{\partial}{\partial x_k}, x_k y_k \frac{\partial}{\partial y_k},$$

$$z_1 \frac{\partial}{\partial z_1}, z_1 \frac{\partial}{\partial v_1}, \dots, z_\ell \frac{\partial}{\partial z_\ell}, z_\ell \frac{\partial}{\partial v_\ell}, \text{ and } \frac{\partial}{\partial p_1}, \frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial p_m}, \frac{\partial}{\partial q_m}.$$

We can locally identify sections of ${}^{\log}\mathcal{R}^*$ as the space generated by

$$\frac{dx_i}{x_i y_i}, \frac{dy_i}{x_i y_i}, \frac{dz_j}{z_j}, \frac{\alpha_j}{z_j}, dp_n, dq_n.$$

The log *rigged de-Rham forms* are

$${}^{\log}\mathcal{R}\Omega^p(M) = \mathcal{C}^\infty(M; \wedge^p({}^{\log}\mathcal{R}^*)),$$

smooth sections of the p -th exterior power of ${}^{\log}\mathcal{R}^*$. This complex has exterior derivative d given by extending the standard smooth differential on $M \setminus D$ to M .

Computing the Lie algebroid cohomology of ${}^{\log}\mathcal{R}$

Note that we have inclusions of complexes:

$$\mathcal{A}_1\Omega^*(M) \rightarrow \dots \rightarrow \mathcal{A}_m\Omega^*(M) \rightarrow \mathcal{B}_1\Omega^*(M) \rightarrow \dots \rightarrow \mathcal{B}_k\Omega^*(M) = {}^{\log}\mathcal{R}\Omega^*(M).$$

Lemma 11. *Let $\tilde{D}$ be the subset of D consisting of hypersurfaces labeled Z_{z_i} . The Lie algebroid cohomology of $\mathcal{A}_m$ is isomorphic to the Lie algebroid cohomology of the log $\tilde{D}$ tangent bundle. That is,*

$$\mathcal{A}_m H^p(M) \simeq H^p(M) \bigoplus_{\tau \in \mathcal{T}} H^{p-|\tau|} \left(\bigcap_{j \in \tau} Z_{z_j} \right)$$

where $\mathcal{T}$ denotes all of the nonempty subsets of $\{1, \dots, |\tilde{D}|\}$.

Proof. The bundle map $i : \mathcal{A}_i \rightarrow \mathcal{A}_{i-1}$ constructed in Theorem 2 fits into a short exact sequence of complexes

$$0 \rightarrow \mathcal{A}_{i-1}\Omega^p(M) \rightarrow \mathcal{A}_i\Omega^p(M) \rightarrow \mathcal{C}^p \rightarrow 0$$

where

$$\mathcal{C}^p = \mathcal{A}_i\Omega^p(M) / \mathcal{A}_{i-1}\Omega^p(M).$$

The differential on $\mathcal{C}^p$ is induced by the differential $\mathcal{A}_i d$ on $\mathcal{A}_i\Omega^p(M)$: if P is the projection $\mathcal{A}_i\Omega^p(M) \rightarrow \mathcal{A}_i\Omega^p(M) / \mathcal{A}_{i-1}\Omega^p(M)$, then $\mathcal{C}d(\eta) = P(\mathcal{A}_i d(\theta))$ where $\theta \in \mathcal{A}_i\Omega^p(M)$ is any form such that $P(\theta) = \eta$. Hence $(\mathcal{C}d)^2 = 0$ and $(\mathcal{C}^*, \mathcal{C}d)$ is in fact a complex.

Given a good tubular neighborhood $\tau = Z_{z_i} \times (-\varepsilon, \varepsilon)_{z_i}$ of M near Z_{z_i} , note that z_i defines a trivialization $t_{z_i} : N^*Z_{z_i} \rightarrow \mathbb{R}$ of $N^*Z_{z_i}$. We will use the notation $\mathcal{A}_{i-1}\Omega^*(Z_{z_i})$ to denote the de-Rham complex of the Lie algebroid $\mathcal{A}_{i-1}|_{Z_{z_i}}$. let $\chi_i : M \rightarrow \mathbb{R}$ be a smooth bump function supported in a neighborhood of Z_{z_i} and with constant value one at Z_{z_i} .

We can write a degree- p $\mathcal{A}_i$ form $\mu \in \mathcal{A}_i \Omega^p(M)$ as

$$\mu = \theta + \chi_i \left(\frac{dz_i \wedge \alpha_i}{z_i^2} \wedge A + \frac{dz_i}{z_i} \wedge B + \frac{\alpha_i}{z_i} \wedge C \right)$$

and $\theta \in \mathcal{A}_{i-1} \Omega^p(M)$, $A, B, C \in \mathcal{A}_{i-1} \Omega^*(Z_{z_i}) \simeq \mathcal{A}_{i-1} \Omega^*(Z_{z_i}; |N^* Z_{z_i}|^{-1})$ by $t_{z_i^*}$.

We write $\mathcal{R}(\mu) = \theta$ and $\mathcal{S}(\mu) = \mu - \mathcal{R}(\mu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\mathcal{R}(d\mu) = d(\mathcal{R}(\mu))$ and $\mathcal{S}(d\mu) = d(\mathcal{S}(\mu))$. Thus the trivialization τ induces a splitting $\mathcal{A}_i \Omega^*(M) = \mathcal{A}_{i-1} \Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence $\mathcal{A}_i H^p(M) = \mathcal{A}_{i-1} H^p(M) \oplus H^p(\mathcal{C}^*)$ and we are left to compute the cohomology of the quotient complex.

After identifying $\mathcal{C}^p = \{\mu \in \mathcal{A}_i \Omega^p(M) : \theta = 0\}$, the differential is given by

$$d\mu = \frac{dz_i \wedge \alpha_i}{z_i^2} \wedge (dA - C) - \frac{dz_i}{z_i} \wedge dB - \frac{\alpha_i}{z_i} \wedge dC.$$

Thus $\ker(d : \mathcal{C}^p \rightarrow \mathcal{C}^{p+1}) = \{C = dA, \quad dB = 0\}$. If $d\mu = 0$, then there is

$$\tilde{\mu} = \frac{\alpha_i}{z_i} \wedge A - \frac{dz_i}{z_i} \wedge b \in \mathcal{C}^{p-1}$$

such that $d \left(-\frac{\alpha_i}{z_i} \wedge A - \frac{dz_i}{z_i} \wedge b \right) = \frac{dz_i \wedge \alpha_i}{z_i^2} \wedge A + \frac{\alpha_i}{z_i} \wedge dA - \frac{dz_i}{z_i} \wedge db$. Then $[\mu - d\tilde{\mu}] \in H^p(\mathcal{C})$ has representative

$$\frac{dz_i}{z_i} \wedge (B - db).$$

This computation also shows that if $\mu = \frac{dz_i}{z_i} \wedge B \in d(\mathcal{C}^{p-1})$, then $\mu = d\nu$ for some $\nu \in \mathcal{C}^{p-1}$ where

$$\nu = \frac{dz_i}{z_i} \wedge \frac{\alpha_i}{z_i} \wedge a + \frac{dz_i}{z_i} \wedge b + \frac{\alpha_i}{z_i} \wedge c$$

and $B = db$. Thus B is exact. Further, if two forms ν_1, ν_2 are representatives of the same cohomology class in $H^p(\mathcal{C})$, this shows that the coefficients of the expression $\nu_1 - \nu_2$ must be exact.

Let us consider the effect of changing the Z_{z_i} -defining function. By the change of Z_{z_i} defining function computation found in Section 2.6 the cohomology class $[B - db]$ is unambiguous despite a representative being expressed using a particular Z_{z_i} defining function.

Thus

$$H^p(\mathcal{C}) = \frac{\{B \in \mathcal{A}_{i-1} \Omega^{p-1}(M) : dD = 0\}}{\{B = db, b \in \mathcal{A}_{i-1} \Omega^{p-2}(M)\}}$$

and

$$\mathcal{A}_i H^p(M) \simeq \mathcal{A}_{i-1} H^p(M) \oplus \mathcal{A}_{i-1} H^{p-1}(Z_{z_i}).$$

Since $\mathcal{A}_1 H^p(M) \simeq H^p(M) \oplus H^{p-1}(Z_{z_1})$, we have that

$$\mathcal{A}_m H^p(M) \simeq H^p(M) \bigoplus_{\tau \in \mathcal{T}} H^{p-|\tau|} \left(\bigcap_{j \in \tau} Z_{z_j} \right)$$

where $\mathcal{T}$ denotes all of the nonempty subsets of $\{1, \dots, |\tilde{D}|\}$. □

Lemma 12. *The Lie algebroid cohomology of $\mathcal{B}_m$ is isomorphic to*

$${}^b H^p(M) \oplus \bigoplus_{\mathcal{M}} H^{p-m}(\underbrace{\bigcap_{i \in I, j \in J, k \in K, \ell \in L} Z_{x_i} \cap Z_{y_i} \cap Z_{x_j} \cap Z_{y_k} \cap Z_{z_\ell}}_{i \in I, j \in J, k \in K, \ell \in L}; \otimes \underbrace{|N_{v_i}^* Z_{x_i}|^{-1} \otimes |N_{v_i}^* Z_{y_i}|^{-1}}_{i \in I})$$

where $\mathcal{M}$ denotes all collections of sets I, J, K, L satisfying

$$I, J, K \subseteq \{1, \dots, k\}, L \subseteq \{1, \dots, n\} \text{ such that } I \neq \emptyset, \text{ and } I \cap J = I \cap K = \emptyset$$

with $m := 2|I| + |J| + |L| + |K|$ and $v_i = Z_{x_i} \cap Z_{y_i}$.

Proof. We set $\mathcal{B}_0 = \mathcal{A}_m$. The bundle map $i : \mathcal{B}_i \rightarrow \mathcal{B}_{i-1}$ constructed in Theorem 2 fits into a short exact sequence of complexes

$$0 \rightarrow {}^{\mathcal{B}_{i-1}}\Omega^p(M) \rightarrow {}^{\mathcal{B}_i}\Omega^p(M) \rightarrow \mathcal{C}^p \rightarrow 0$$

where

$$\mathcal{C}^p = {}^{\mathcal{B}_i}\Omega^p(M) / {}^{\mathcal{B}_{i-1}}\Omega^p(M).$$

The differential on $\mathcal{C}^p$ is induced by the differential ${}^{\mathcal{B}_i}d$ on ${}^{\mathcal{B}_i}\Omega^p(M)$: if P is the projection ${}^{\mathcal{B}_i}\Omega^p(M) \rightarrow {}^{\mathcal{B}_i}\Omega^p(M) / {}^{\mathcal{B}_{i-1}}\Omega^p(M)$, then $\mathcal{C}^p d(\eta) = P({}^{\mathcal{B}_i}d(\theta))$ where $\theta \in {}^{\mathcal{B}_i}\Omega^p(M)$ is any form such that $P(\theta) = \eta$. Hence $(\mathcal{C}^p d)^2 = 0$ and $(\mathcal{C}^*, \mathcal{C}^p d)$ is in fact a complex.

Given good tubular neighborhoods $\tau_x = Z_{x_i} \times (-\varepsilon, \varepsilon)_{x_i}$ of M near Z_{x_i} and $\tau_y = Z_{y_i} \times (-\varepsilon, \varepsilon)_{y_i}$ of M near Z_{y_i} , note that z_{x_i} defines a trivialization $t_{x_i} : N^*Z_{x_i} \rightarrow \mathbb{R}$ of $N^*Z_{x_i}$ and z_{y_i} defines a trivialization $t_{y_i} : N^*Z_{y_i} \rightarrow \mathbb{R}$ of $N^*Z_{y_i}$. let $\chi_i : M \rightarrow \mathbb{R}$ be a smooth bump function supported in a neighborhood of $Z_{x_i} \cup Z_{y_i}$ and with constant value one at $Z_{x_i} \cup Z_{y_i}$.

We can write a degree- p $\mathcal{B}_i$ form $\mu \in {}^{\mathcal{B}_i}\Omega^p(M)$ as

$$\begin{aligned} \mu = & \theta + \chi_i \frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge (A_{00} + A_{10}x_i + A_{01}y_i + A_{20}x_i^2 + A_{02}y_i^2 + A_{11}x_i y_i) \\ & + \chi_i \frac{dx_i}{x_i y_i} \wedge (B_0 + B_1 x_i) + \chi_i \frac{dy_i}{x_i y_i} \wedge (C_0 + C_1 y_i) + \chi_i \frac{dx_i}{x_i} \wedge D + \chi_i \frac{dy_i}{y_i} \wedge E \end{aligned}$$

where $\theta \in {}^{\mathcal{B}_{i-1}}\Omega^p(M)$,

$$A_{k,l} \in {}^{\mathcal{B}_{i-1}}\Omega^*(Z_{x_i} \cap Z_{y_i}) \simeq {}^{\mathcal{B}_{i-1}}\Omega^*(Z_{x_i} \cap Z_{y_i}; |N_{v_i}^* Z_{x_i}|^{-1} \otimes |N_{v_i}^* Z_{y_i}|^{-1})$$

by $t_{x_i}^*, t_{y_i}^*$ where $v_i = Z_{x_i} \cap Z_{y_i}$,

$$C_i, D \in {}^{\mathcal{B}_{i-1}}\Omega^*(Z_{x_i}) \simeq {}^{\mathcal{B}_{i-1}}\Omega^*(Z_{x_i}; |N^* Z_{x_i}|^{-1}) \text{ by } t_{y_i}^*, \text{ and}$$

$$B_i, E \in {}^{\mathcal{B}_{i-1}}\Omega^*(Z_{y_i}) \simeq {}^{\mathcal{B}_{i-1}}\Omega^*(Z_{y_i}; |N^* Z_{y_i}|^{-1}) \text{ by } t_{x_i}^*.$$

Further, B_0 is independent of x and C_0 is independent of y .

We write $\mathcal{R}(\mu) = \theta$ and $\mathcal{S}(\mu) = \mu - \mathcal{R}(\mu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\mathcal{R}(d\mu) = d(\mathcal{R}(\mu))$ and $\mathcal{S}(d\mu) = d(\mathcal{S}(\mu))$. Thus the trivializations τ_{x_i}, τ_{y_i}

induce a splitting ${}^{\mathcal{B}_i}\Omega^*(M) = {}^{\mathcal{B}_{i-1}}\Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence ${}^{\mathcal{B}_i}H^p(M) = {}^{\mathcal{B}_{i-1}}H^p(M) \oplus H^p(\mathcal{C}^*)$ and we are left to compute the cohomology of the quotient complex.

After identifying $\mathcal{C}^p = \{\mu \in {}^{\mathcal{B}_i}\Omega^p(M) : \theta = 0\}$, the differential of $d\mu$ is given by

$$\begin{aligned} & \frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge (dA_{00} + (dA_{10} + B_0)x_i + (dA_{01} - C_0)y_i + (dA_{20} + B_1)x_i^2 + (dA_{02} - C_1)y_i^2) \\ & + \frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge dA_{11}x_i y_i - \frac{dx_i}{x_i y_i} \wedge dB - \frac{dy_i}{x_i y_i} \wedge dC - \frac{dx_i}{x_i} \wedge dD - \frac{dy_i}{y_i} \wedge dE. \end{aligned}$$

Thus $\ker(d : \mathcal{C}^p \rightarrow \mathcal{C}^{p+1})$ can be identified with the relations

$$\begin{aligned} dA_{00} &= 0, \quad B_0 = -dA_{10}, \quad B_1 = -dA_{20}, \quad C_0 = dA_{01}, \\ C_1 &= dA_{20}, \quad dA_{11} = 0, \quad dD = 0, \quad dE = 0 \end{aligned}$$

even though B_1 could depend on x_i and C_1 could depend on y_i above.

If $d\mu = 0$, then there is an element $\tilde{\mu}$ in $\mathcal{C}^{p-1}$,

$$\begin{aligned} \tilde{\mu} &= -\frac{dx_i}{x_i y_i} \wedge (A_{10} + A_{20}x_i) + \frac{dy_i}{x_i y_i} \wedge (A_{01} + A_{02}y_i) + \frac{dx_i \wedge dy_i}{x_i^2 y_i^2} (a_{00} + a_{11}x_i y_i) \\ &+ \frac{dx_i}{x_i} \wedge \delta + \frac{dy_i}{y_i} \wedge e \end{aligned}$$

such that $\mu - d\tilde{\mu}$ equals

$$\frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge \left((A_{00} - da_{00}) + (A_{11} - da_{11})x_i y_i \right) + \frac{dx_i}{x_i} \wedge (D + d\delta) + \frac{dy_i}{y_i} \wedge (E + de).$$

This computation also shows that if

$$\mu = \frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge (A_{00} + A_{11}x_i y_i) + \frac{dx_i}{x_i} \wedge D + \frac{dy_i}{y_i} \wedge E$$

is in $d(\mathcal{C}^{p-1})$, then $\mu = d\nu$ for some $\nu \in \mathcal{C}^{p-1}$ where

$$\begin{aligned} \nu &= \frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge (a_{00} + a_{10}x_i + a_{01}y_i + a_{20}x_i^2 + a_{02}y_i^2 + a_{11}x_i y_i) \\ &+ \frac{dx_i}{x_i y_i} \wedge (b_0 + b_1x_i) + \frac{dy_i}{x_i y_i} \wedge (c_0 + c_1y_i) + \frac{dx_i}{x_i} \wedge \delta + \frac{dy_i}{y_i} \wedge e \end{aligned}$$

and $A_{00} = da_{00}$, $A_{11} = da_{11}$, $D = d\delta$, and $E = de$. Thus if two forms ν_1, ν_2 are representatives of the same cohomology class in $H^p(\mathcal{C}^p)$, then the coefficients of the expression $\nu_1 - \nu_2$ must be exact.

Note that $\frac{dx_i \wedge dy_i}{x_i y_i} \wedge (A_{11} - da_{11}) + \frac{dx_i}{x_i} \wedge (D + d\delta) + \frac{dy_i}{y_i} \wedge (E + de)$ is a $\log \tilde{D}_i$ symplectic form for $\tilde{D}_i = \{Z_{z_1}, \dots, Z_{z_\ell}, Z_{x_1}, Z_{y_1}, \dots, Z_{x_j}, Z_{y_j}\}$. Further, ${}^{\mathcal{B}_i}\Omega^p(M)$ splits as ${}^{\log \tilde{D}_i}\Omega^p(M) \oplus \mathcal{D}^p$ for the appropriate quotient complex $\mathcal{D}^p$. Thus we know that the representatives $(A_{11} - da_{11})$, $(D + d\delta)$, and $(E + de)$ are invariant under change of Z_{x_i} and Z_{y_i} defining function.

Thus we are left to compute what happens to

$$\frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge (A_{00} - da_{00})$$

under change of Z_{x_i} and Z_{y_i} defining function. By this computation, which occurs at the conclusion of the proof of Theorem 8

$$\frac{dx_i \wedge dy_i}{x_i^2 y_i^2} \wedge (A_{00} - da_{00})$$

can be identified as an element of

$$\mathcal{B}_{i-1} H^{p-2}(Z_{x_i} \cap Z_{y_i}) \simeq \mathcal{B}_{i-1} H^{p-2}(Z_{x_i} \cap Z_{y_i}; |N_{v_i}^* Z_{x_i}|^{-1} \otimes |N_{v_i}^* Z_{y_i}|^{-1})$$

trivialized by $t_{x_i}^*$ and $t_{y_i}^*$, and where $v_i = Z_{x_i} \cap Z_{y_i}$. Thus

$$\begin{aligned} \mathcal{B}_i H^p(M) &\simeq \mathcal{B}_{i-1} H^p(M) \oplus \mathcal{B}_{i-1} H^{p-1}(Z_{x_i}) \oplus \mathcal{B}_{i-1} H^{p-1}(Z_{y_i}) \oplus \mathcal{B}_{i-1} H^{p-2}(Z_{x_i} \cap Z_{y_i}) \\ &\quad \oplus \mathcal{B}_{i-1} H^{p-2}(Z_{x_i} \cap Z_{y_i}; |N^* Z_{x_i}|^{-1} \otimes |N^* Z_{y_i}|^{-1}). \end{aligned}$$

Since $\mathcal{A}_m = \mathcal{B}_0$, by using Lemma 11 we can conclude that the cohomology $\mathcal{B}_m H^p(M)$ is

$${}^b H^p(M) \oplus \bigoplus_{\mathcal{M}} H^{p-m}(\underbrace{\bigcap_{i \in I, j \in J, k \in K, \ell \in L} Z_{x_i} \cap Z_{y_i} \cap Z_{x_j} \cap Z_{y_k} \cap Z_{z_\ell}}_{i \in I, j \in J, k \in K, \ell \in L}; \underbrace{|N_{v_i}^* Z_{x_i}|^{-1} \otimes |N_{v_i}^* Z_{y_i}|^{-1}}_{i \in I})$$

where $\mathcal{M}$ denotes all collections of sets I, J, K, L satisfying

$$I, J, K \subseteq \{1, \dots, k\}, L \subseteq \{1, \dots, n\} \text{ such that } I \neq \emptyset, \text{ and } I \cap J = I \cap K = \emptyset$$

with $m := 2|I| + |J| + |L| + |K|$ and v_i denotes $Z_{x_i} \cap Z_{y_i}$. □

Since $\mathcal{B}_m = {}^{\log} \mathcal{R}$, we have reached the conclusion of Theorem 14.

3.5 Star-log symplectic surfaces

3.5.1 Introduction

As is well-known, two non-degenerate Poisson - i.e. symplectic - structures on a compact connected surface S are the same precisely when they have the same de Rham cohomology class. In this work we give an analogous classification of a more general class of Poisson bivector on a surface which we call star log symplectic structures. Since every bivector on a surface S is Poisson, we organize Poisson structures on S by their degeneracy loci and, in fact, Vladimir Arnol'd introduced a hierarchy for these degeneracies (Appendix 14 C [3]). Olga Radko provided a classification when the Poisson bivector degenerated linearly along a curve [45]. *We will extend her classification to Poisson bivectors whose degeneracy loci are locally modeled by a finite set of lines in the plane intersecting at a point. Additionally, we will compute the Poisson cohomology of such structures π and discuss how our classification relates to deformations of π .*

Classification. Let (S, π) be a Poisson structure on a surface S . The key idea to our approach is to not work with π , but instead to work with the corresponding symplectic form, even when π is degenerate. Recall that π induces a map $\pi^\sharp$ between T^*S and TS . When π is non-degenerate, it admits an inverse.

$$T^*S \begin{array}{c} \xrightarrow{\pi^\sharp} \\ \xleftarrow{\omega^\flat = (\pi^\sharp)^{-1}} \end{array} TS$$

This inverse map defines a symplectic form ω for S . Moser's argument (see Theorem 2.70 in [11]), a fundamental technique in symplectic geometry, provides a classification: two symplectic forms ω_0 and ω_1 on a compact connected surface S are symplectomorphic if and only if they are cohomologous in degree 2 de Rham cohomology. The first main result of this paper is a classification similar in spirit to this symplectic case. We will elaborate on the meaning of certain terms in the exposition following the statement.

Theorem 16. *(Classification of star log symplectic surfaces). Let (S, D) be a compact connected surface with star divisor D . Two log-symplectic forms ω_0 and ω_1 on (S, D) are symplectomorphic if and only if they are cohomologous in the degree 2 Lie algebroid coho-*

mology of the b -tangent bundle and ω_0 and ω_1 induce the same orientation of the b -tangent bundle.

To understand this theorem, let us begin by considering a class of Poisson structures that have some degeneracy. We are not able to immediately work with a corresponding form ω because the map $\pi^\sharp$ will not have an inverse. The trick is to find a class of Poisson structure where we can replace the tangent bundle with a Lie algebroid $\mathcal{A}$. We can view bivectors π in this class as non-degenerate on $\mathcal{A}$ and define an associated symplectic form on $\mathcal{A}^*$.

$$\mathcal{A}^* \begin{array}{c} \xrightarrow{\pi^\sharp} \\ \xleftarrow{\omega^\flat = (\pi^\sharp)^{-1}} \end{array} \mathcal{A}$$

This isomorphism allows us to use Moser-type arguments and we can hope for a classification similar to the symplectic case.

In Section 3.5.3 we will provide a more detailed description of this method, but for now we offer a sampling of relevant recent literature: Victor Guillemin, Eva Miranda, and Ana Rita Pires [19] carried out this procedure for a class of structure called b -Poisson². Geoffrey Scott proved versions of Moser's theorem in what he named the b^k -setting and used it to establish this characterization for b^k -Poisson surfaces (See section 6.1 of [47] and in particular Theorem 6.7). In [24], we use Moser techniques to establish local normal forms for scattering-symplectic structures on manifolds of any even dimension. Eva Miranda and Arnau Planas [39] further use these Moser techniques to establish a version of Scott's classification that is equivariant under the action of a group.

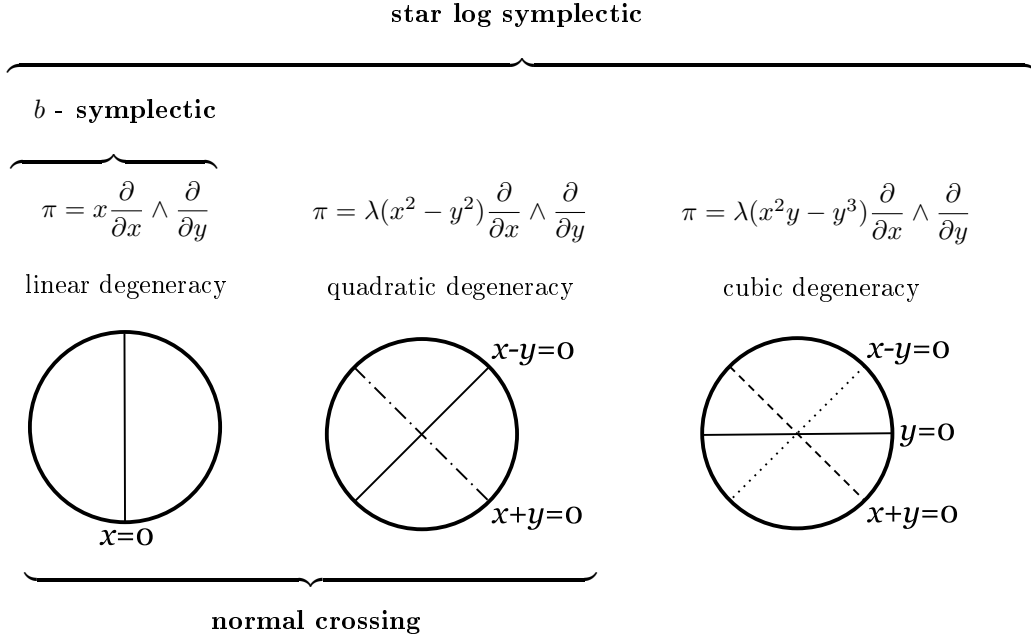
We will add a new class of Poisson structures on surfaces - which we call star log symplectic - to this collection of characterizations of Poisson structures. Work has been done in the setting where Poisson bivectors degenerate linearly on a normal crossing divisor D , i.e. a set of smooth hypersurfaces $Z \subset M$ that intersect transversely, see [18, 19, 25] or [44]. Given a surface S , we will examine Poisson bivectors that degenerate linearly on a set D of smooth hypersurfaces $Z \subset S$ whose intersections are modeled by a finite set of lines in the plane intersecting at a point. Such a set D is called a *star divisor*.

To define the b -tangent bundle, the Lie algebroid we will use, is subtle and requires

²This is a different, but equivalent, approach to Olga Radko's work [45].

more work than in the case of single, or two intersecting, lines. We address the challenge of defining the b -tangent bundle in this more general setting by introducing an adapted atlas, which we call an *astral atlas*. The complete details of this construction are provided in Section 2.6.2.

Figure 29. *Some local models*



In Section 3.5.3 we classify star log symplectic surfaces up to b -symplectomorphism. For simplicity, we state our results here using a fixed tubular neighborhood of D . A more detailed and general discussion can be found in Section 3.5.3.

We realize star log symplectic Poisson bivectors as non-degenerate on Richard Melrose's b -tangent bundle [36]. We will show, given a surface with star divisor $(S, D = \{Z_1, \dots, Z_m\})$, the second Lie algebroid cohomology of the b -tangent bundle is

$${}^b H^2(S) \simeq H^2(S) \oplus \bigoplus_i H^1(Z_i) \oplus \bigoplus_{i < j} H^0(Z_i \cap Z_j).$$

Note that this result is quite surprising since the Lie algebroid cohomology is only noticing pairwise intersection of curves. This is the cohomology featured in the classification Theorem 16. Because the Lie algebroid cohomology only ‘sees’ pair-wise intersection, we also require

that two b -symplectic forms ω_0, ω_1 induce the same orientation of the b -tangent bundle, i.e. $\omega_0 = f\omega_1$ for a positive function $f \in C^\infty(S)$.

Remark 8. (On the Arnol'd hierarchy of planar singularities). *As we briefly mentioned above, Arnol'd introduced a heirarchy of degeneracies for Poisson structures on a surface near a singular point (introduced in Appendix 14 C [3] without proofs). In dimension 2, because the Jacobi identity is trivial, all bivectors are Poisson. Consequently, any Poisson bivector π is locally of the form $\pi = f \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ for some smooth function f and studying the local behavior of π is closely related to understanding the degeneracy of the function f . Radko [45] classified all Poisson structures with degeneracy loci modeled on singularities of type A_0 . We classify Poisson structures that are locally modeled by the cubic degeneracy in Figure 29, a singularity of type $D_4^{0,0}$. At this point it is not clear the relationship between star log symplectic structures with higher order degeneracy and Arnol'd's classification.*

Poisson Cohomology. Next, we turn to Poisson cohomology. We state our result for star log symplectic surfaces (S, D, π) here using a fixed tubular neighborhood of D . A more detailed discussion is given in Section 3.5.4.

Theorem 17. (Poisson cohomology of a log Poisson structure). *Given a log-symplectic surface $(S, D = \{Z_1, \dots, Z_k\}, \pi)$, i.e. a surface with a set D of smooth hypersurfaces $Z_i \subset S$ whose intersections are modeled by a finite set of lines in the plane intersecting at a point, the Poisson cohomology in degrees 0 and 1 is*

$$H_\pi^0(S) \simeq H^0(S) \quad \text{and} \quad H_\pi^1(S) \simeq H^1(S) \oplus \bigoplus_i H^0(Z_i).$$

The cohomology in degree 2 is a direct sum of the following vector spaces:

1. *A single copy of $H^2(S)$.*
2. *For each hypersurfaces Z_i , a copy of $H^1(Z_i)$.*
3. *For each pair wise intersection of two hypersurfaces Z_i, Z_j with $i < j$,*

$$[H^0(Z_i \cap Z_j)]^2$$
4. *For each intersection of three hypersurfaces Z_i, Z_j, Z_k with $i < j < k$,*

$$[H^0(Z_i \cap Z_j \cap Z_k)]^3$$

5. For each intersection of four or more hypersurfaces $Z_{i_1}, Z_{i_2}, Z_{i_3}, \dots, Z_{i_\ell}$ with $i_1 < i_2 < i_3 < \dots < i_\ell$, a copy of

$$[H^0(Z_{i_1} \cap Z_{i_2} \cap \dots \cap Z_{i_\ell})]^4.$$

3.5.2 Star log symplectic surfaces

In this section we consider a setting where our Poisson structures of interest can be viewed as symplectic structures. This will allow us to establish star log symplectic analogues of familiar results, such as Darboux and Moser's theorem.

Definition 20. A star log symplectic structure on a surface S with star divisor D is a 2-form $\omega \in {}^b\Omega^2(S) = \mathcal{C}^\infty(S; \wedge^2({}^bT^*S))$ satisfying

$$d\omega = 0 \text{ and } \omega^n \neq 0.$$

The form ω induces a map ω^b between the b -tangent and b -cotangent bundles.

$${}^bTS \xrightleftharpoons[\pi^\# = (\omega^b)^{-1}]{\omega^b} {}^bT^*S$$

The inverse map is induced by a bivector $\pi \in \mathcal{C}^\infty(S; \wedge^2({}^bTS))$. This bivector is called a log Poisson structure on (S, D) .

Next, we provide an explicit example of a star log symplectic structure on a compact surface.

Example 43. A cubic star log symplectic surface.

Consider the sphere $\mathbb{S}^2$ with spherical coordinates $0 \leq \theta < 2\pi$, $0 \leq h \leq \pi$.

We have a star divisor D consisting of

$$\{\sin(\theta - \pi/6) = 0\}, \{\sin(\theta - \pi/2) = 0\}, \text{ and } \{\sin(\theta - 5\pi/6) = 0\},$$

depicted in Figure 30. The form

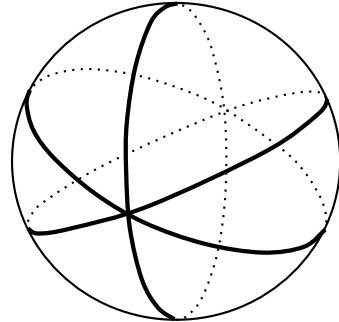
$$\omega = \frac{d\theta \wedge dh}{\sin(\theta - \frac{\pi}{6}) \sin(\theta - \frac{\pi}{2}) \sin(\theta - \frac{5\pi}{6})}$$

is a star log symplectic structure on $\mathbb{S}^2$.

The associated Poisson bivector is given by

$$\pi = \sin(\theta - \frac{\pi}{6}) \sin(\theta - \frac{\pi}{2}) \sin(\theta - \frac{5\pi}{6}) \frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}.$$

Figure 30. $(\mathbb{S}^2, \cup_3 \mathbb{S}^1)$



3.5.3 Classification

The first step in classifying star log symplectic surfaces is computing the Lie algebroid cohomology of the b -tangent bundle.

b -cohomology. Given a surface S with a collection of transverse curves $\{Z_1, \dots, Z_k\}$, the Lie algebroid cohomology of the associated b -tangent bundle is

$${}^b H^p(S) \simeq H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i) \oplus \bigoplus_{i < j} H^{p-2}(Z_i \cap Z_j), \quad (3.15)$$

a fact originally pointed out in Appendix A.23 of [18]. Remarkably, this description still holds for the more general case of a surface with star divisor!

Theorem 18. *Let (S, D) be a surface S with star divisor $D = \{Z_1, \dots, Z_k\}$. The Lie algebroid cohomology of the b -tangent bundle over (S, D) is given by (3.15).*

Proof. Let (S, D) be a surface S with star divisor D . We will compute the Lie algebroid cohomology of the b -tangent bundle over (S, D) using a filtration $\mathcal{F}$ of cochain complexes.

Constructing filtration $\mathcal{F}$.

Given a subset $I \subseteq D$, let ${}^b \Omega^*(S, I)$ denote the complex of b -forms of the b -tangent bundle over (S, I) . In this notation, we want to compute the cohomology of the complex ${}^b \Omega^*(S, D)$.

Consider the sets

$$\mathcal{F}_k^* := {}^{v.s.} \text{Span} \left\{ \bigcup_{I \subseteq D, |I| = k} {}^b \Omega^*(S, I) \right\}.$$

Notice that $\mathcal{F}_k^* \subseteq \mathcal{F}_{k+1}^*$ for all k . Each set $\mathcal{F}_k^*$ is a group under addition and has the structure of a module with scalars from the ring

$$\mathcal{F}_k^0 = \mathcal{C}^\infty(S).$$

Note that ${}^b \Omega^*(S, I) \subseteq {}^b \Omega^*(S, D)$ for all $I \subseteq D$. Consequently, the modules $\mathcal{F}_k^* \subseteq {}^b \Omega^*(S, D)$ for all k . It is easy to check that $\mathcal{F}_k^*$ is closed under the differential of ${}^b \Omega^*(S, D)$. Thus we can define a differential $\mathcal{F}_k^p \xrightarrow{d} \mathcal{F}_k^{p+1}$ inherited from the differential on ${}^b \Omega^*(S, D)$.

For each k , we have a chain complex:

$$0 \rightarrow \mathcal{F}_k^0 \xrightarrow{d} \mathcal{F}_k^1 \xrightarrow{d} \mathcal{F}_k^2 \rightarrow 0.$$

We will employ the following filtration of cochain complexes:

$$\mathcal{F} = \left\{ \Omega^*(S) = \mathcal{F}_0^* \subseteq \mathcal{F}_1^* \subseteq \mathcal{F}_2^* \subseteq \cdots \subseteq \mathcal{F}_{|D|-1}^* \subseteq \mathcal{F}_{|D|}^* = {}^b\Omega^*(S, D) \right\}.$$

In the next step, we consider various short exact sequences of complexes involving the terms in $\mathcal{F}$.

A short exact sequence of complexes.

The inclusions $\mathcal{F}_{k-1}^* \rightarrow \mathcal{F}_k^*$ of cochain complexes from filtration $\mathcal{F}$ fit into a short exact sequence of complexes

$$0 \rightarrow \mathcal{F}_{k-1}^* \rightarrow \mathcal{F}_k^* \rightarrow \mathcal{C}_k^* \rightarrow 0$$

where

$$\mathcal{C}_k^p = \mathcal{F}_k^p / \mathcal{F}_{k-1}^p.$$

The differential ${}^{\mathcal{C}}d$ on $\mathcal{C}_k^p$ is induced by d on $\mathcal{F}_k^p$: if P is the projection $\mathcal{F}_k^p \rightarrow \mathcal{F}_k^p / \mathcal{F}_{k-1}^p$, then ${}^{\mathcal{C}}d(\eta) = P(d(\theta))$ where $\theta \in \mathcal{F}_k^p$ is any form such that $P(\theta) = \eta$. Hence $({}^{\mathcal{C}}d)^2 = 0$ and $(\mathcal{C}_k^*, {}^{\mathcal{C}}d)$ is a complex.

Given one of these short exact sequences, we will show that the boundary maps in the induced long exact sequences in cohomology are zero. This feature will facilitate our arrival at a nice description of ${}^bH^p(S)$.

Computing $H^p(\mathcal{F}_1^*)$.

Let $k = 1$ and consider the short exact sequence

$$0 \rightarrow \mathcal{F}_0^* \rightarrow \mathcal{F}_1^* \rightarrow \mathcal{C}_1^* \rightarrow 0$$

defined above. Note that $\mathcal{F}_0^* = \Omega^*(S)$.

Let $D = \{Z_1, Z_2, \dots, Z_\ell\}$ be our star divisor and consider defining functions $x_1, x_2, \dots, x_\ell$ such that $Z_i = \{x_i = 0\}$. For each i , let $\chi_i : S \rightarrow \mathbb{R}$ be a smooth bump function supported in a neighborhood of Z_i and with constant value one at Z_i .

An element $\mu \in \mathcal{F}_1^p$ can be expressed as

$$\mu = \sum_{i=1}^{\ell} \chi_i \cdot \frac{dx_i}{x_i} \wedge \alpha_i + \beta$$

where $\alpha_i \in \Omega^{p-1}(Z_i)$ and $\beta \in \Omega^p(S)$.

Given $\mu \in \mathcal{F}_1^p$, we write $\Re(\mu) = \beta$ and $\Im(\mu) = \mu - \Re(\mu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\Re(d\mu) = d(\Re(\mu))$ and $\Im(d\mu) = d(\Im(\mu))$. Thus we have a splitting

$$\mathcal{F}_1^* = \Omega^*(S) \oplus \mathcal{C}_1^*$$

as complexes and $H^p(\mathcal{F}_1^*) \simeq H^p(S) \oplus H^p(\mathcal{C}_1^*)$. We are left to compute the cohomology group of the quotient complex $\mathcal{C}_1^*$.

As mentioned above, $P(\mu) \in \mathcal{C}_1^*$ is of the form

$$P(\mu) = \sum_{i=1}^{\ell} \frac{dx_i}{x_i} \wedge \alpha_i$$

for $\alpha_i \in \Omega^{p-1}(Z_i)$. Then

$$dP(\mu) = - \sum_{i=1}^{\ell} \frac{dx_i}{x_i} \wedge d\alpha_i.$$

Note that as an element of $\mathcal{F}_1^*$,

$$\mu|_{Z_i} = \frac{dx_i}{x_i} \wedge \alpha_i.$$

Thus, the condition that $\mu|_{Z_i} = 0$ implies that $\alpha_i = 0$. Consequently, given these choices of Z_i defining functions $x_1, \dots, x_{\ell}$, the set $\ker(d : \mathcal{C}^p \rightarrow \mathcal{C}^{p+1})$ can be identified with

$$\bigoplus_{i=1}^{\ell} \{ \alpha_i \in \Omega^{p-1}(Z_i) : d\alpha_i = 0 \}.$$

Further, the set $\text{im}(d : \mathcal{C}^{p-1} \rightarrow \mathcal{C}^p)$ can be identified with

$$\bigoplus_{i=1}^{\ell} \{ \alpha_i \in \Omega^{p-1}(Z_i) : \alpha_i = d\gamma_i, \gamma_i \in \Omega^{p-2}(Z_i) \}.$$

For completeness, we next consider change of Z_i defining function. However, by a computation which can be found in example 2.11 of [24], these sets are invariant under change of Z_i defining functions. Thus

$$H^p(\mathcal{C}_1^*) \simeq \bigoplus_i H^{p-1}(Z_i) \text{ and } H^p(\mathcal{F}_1^*) \simeq H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i).$$

Computing $H^p(\mathcal{F}_2^*)$.

Let $k = 2$ and consider the short exact sequence

$$0 \rightarrow \mathcal{F}_1^* \rightarrow \mathcal{F}_2^* \rightarrow \mathcal{C}_2^* \rightarrow 0.$$

Let $D = \{Z_1, Z_2, \dots, Z_{\ell}\}$ be our star divisor. For each pair of curves Z_i, Z_j with $i < j$, there is a pair of tubular neighborhoods $\tau_i = Z_i \times (-\varepsilon, \varepsilon)_{x_{ij}}$ of Z_i and $\tau_j = Z_j \times (-\varepsilon, \varepsilon)_{y_{ij}}$ of Z_j such that

$$\left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{ij}} \right] = 0.$$

For existence of such neighborhoods see, for instance, section 5 of [1].

We begin by noting that $\mathcal{F}_2^1 = \mathcal{F}_1^1$.

Let $\chi_{i,j}$ be a smooth bump function supported near $Z_i \cup Z_j$ and with constant value one near $Z_i \cup Z_j$. An element $\mu \in \mathcal{F}_2^2$ can be expressed as

$$\mu = \sum_{i < j} \chi_{ij} \cdot \frac{dx_{ij}}{x_{ij}} \wedge \frac{dy_{ij}}{y_{ij}} \alpha_{ij} + \beta$$

where $\alpha_{ij} \in \Omega^0(Z_i \cap Z_j)$ and $\beta \in \mathcal{F}_1^2$.

Because $\mathcal{F}_1^*$ is closed under d and $d\mu = 0$ for all $\mu \in \mathcal{F}_2^2$, we have a splitting

$$\mathcal{F}_2^* = \mathcal{F}_1^* \oplus \mathcal{C}_2^*$$

as complexes and $H^p(\mathcal{F}_2^*) \simeq H^p(\mathcal{F}_1^*) \oplus H^p(\mathcal{C}_2^*)$. We are left to compute the cohomology group of the quotient complex $\mathcal{C}_2^*$. In this instance, $\mathcal{C}_2^1 = 0$ and we are computing the cohomology of the sequence $0 \rightarrow \mathcal{C}_2^2 \rightarrow 0$. Equivalently, we are left to identify the set $\mathcal{C}_2^2$.

Any $P(\mu) \in \mathcal{C}_2^2$ is of the form

$$P(\mu) = \sum_{i < j} \frac{dx_{ij}}{x_{ij}} \wedge \frac{dy_{ij}}{y_{ij}} \alpha_{ij}$$

for $\alpha_{ij} \in \Omega^0(Z_i \cap Z_j)$. In other words, each α_{ij} is a function defined on a set of discrete points.

Then the set $\mathcal{C}_2^2$ can be identified with

$$\bigoplus_{i < j} \{ \alpha_{ij} \in \Omega^{p-2}(Z_i \cap Z_j) \}.$$

By the computation which can be found in example 2.11 of [24], these sets are invariant under change of Z_i and Z_j defining functions. Thus

$$H^p(\mathcal{C}_2^*) \simeq \bigoplus_{i < j} H^{p-2}(Z_i \cap Z_j)$$

and

$$H^p(\mathcal{F}_2^*) \simeq H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i) \oplus \bigoplus_{i < j} H^{p-2}(Z_i \cap Z_j).$$

In our final step, we will show that the cohomology of $\mathcal{C}_k^*$ vanishes for $k \geq 3$, and thus $H^p(\mathcal{F}_2^*)$ actually computes ${}^b H^p(S)$.

Computing $H^p(\mathcal{F}_k^*)$ for $k \geq 3$.

Fix $k \geq 3$ and consider the short exact sequence

$$0 \rightarrow \mathcal{F}_{k-1}^* \rightarrow \mathcal{F}_k^* \rightarrow \mathcal{C}_k^* \rightarrow 0.$$

Given any $p \in S$ such that at least the k curves in the set $I = \{Z_1, \dots, Z_k\}$ intersect at

p , there exists a star chart $(U_{p,I}, x, y)$ centered at p such that

$$Z_1 = \{x = 0\}, Z_2 = \{y = 0\}, \dots, Z_k = \{A_k x + B_k y = 0\}.$$

Let $\chi_{p,I}$ be a smooth bump function supported in $U_{p,I}$ and with constant value one in a neighborhood of p .

Any element $\mu \in \mathcal{F}_k^1$ can be expressed as

$$\mu = \sum_{\substack{p, I \\ |\mathbf{I}|=k}} \chi_{p,I} \cdot \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} \left(\frac{dx}{x} - \frac{dy}{y} \right) \wedge \alpha_{p,I} + \beta$$

for $\alpha_{p,I} \in \Omega^0(\{p\})$ and $\beta \in \mathcal{F}_{k-1}^1$. Any element $\mu \in \mathcal{F}_k^2$ can be expressed as

$$\mu = \sum_{\substack{p, I \\ |\mathbf{I}|=k}} \chi_{p,I} \cdot \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} \frac{dx \wedge dy}{xy} \wedge \alpha_{p,I} + \beta$$

for $\alpha_{p,I} \in \Omega^0(\{p\})$ and $\beta \in \mathcal{F}_{k-1}^2$. Note that each $\alpha_{p,I}$ is simply a real number.

Given $\mu \in \mathcal{F}_k^\ell$, we write $\Re(\mu) = \beta$ and $\Im(\mu) = \mu - \Re(\mu)$ for ‘regular’ and ‘singular’ parts.

One can check that $\Re(d\mu) = d(\Re(\mu))$ and $\Im(d\mu) = d(\Im(\mu))$. Thus we have a splitting

$$\mathcal{F}_k^* = \mathcal{F}_{k-1}^* \oplus \mathcal{C}_k^*$$

as complexes and $H^p(\mathcal{F}_k^*) \simeq H^p(\mathcal{F}_{k-1}^*) \oplus H^p(\mathcal{C}_k^*)$. We are left to compute the cohomology group of the quotient complex $\mathcal{C}_k^*$.

We will first compute the cohomology at $\mathcal{C}_k^1$. Any $P(\mu) \in \mathcal{C}_k^1$ is of the form

$$P(\mu) = \sum_{\substack{p, I \\ |\mathbf{I}|=k}} \chi_{p,I} \cdot \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} \left(\frac{dx}{x} - \frac{dy}{y} \right) \wedge \alpha_{p,I}$$

for $\alpha_{p,I} \in \Omega^0(\{p\})$. Then, as an element of $\mathcal{C}$,

$$dP(\mu) = \sum_{\substack{p, I \\ |\mathbf{I}|=k}} \chi_{p,I} \cdot \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} \frac{dx \wedge dy}{xy} \wedge (k-2)\alpha_{p,I}. \quad (3.16)$$

Thus $\ker(d : \mathcal{C}_k^1 \rightarrow \mathcal{C}_k^2) = \{\alpha_{p,I} = 0\}$ and $H^1(\mathcal{C}_k^*) = 0$ for all $k \geq 3$.

Next, we compute the cohomology at $\mathcal{C}_k^2$. Any element $P(\mu) \in \mathcal{C}_k^2$ is of the form

$$P(\mu) = \sum_{\substack{p, I \\ |I|=k}} \chi_{p, I} \cdot \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} \frac{dx \wedge dy}{xy} \wedge \alpha_{p, I}$$

for $\alpha_{p, I} \in \Omega^0(\{p\})$. We have $dP(\mu) = 0$ so $\ker(d : \mathcal{C}_k^2 \rightarrow 0) = \mathcal{C}_k^2$. By equation (3.16), $\text{im}(d : \mathcal{C}_k^1 \rightarrow \mathcal{C}_k^2) = \mathcal{C}_k^2$. As desired, we have shown that $H^2(\mathcal{C}_k^*) = 0$ for $k \geq 3$ and have completed the proof of the theorem. $\square$

Remark 9. Spectral Sequences. *We can recontextualize the proof of Theorem 18 in the language of spectral sequences. We will employ the notation used in [8]. The differential complex $({}^b\Omega^*(S), d)$ is a filtered complex with filtration*

$$0 \subseteq \Omega^*(S) = \mathcal{F}_0^* \subseteq \mathcal{F}_1^* \subseteq \mathcal{F}_2^* \subseteq \cdots \subseteq \mathcal{F}_m^* = {}^b\Omega^*(S).$$

Then $E_{d, k}^0 = \mathcal{C}_k^d$ and the associated graded complex of $({}^b\Omega^*(S), d)$ is

$$\bigoplus_{k=1}^m \mathcal{C}_k^*.$$

The computation above shows that the spectral sequence collapses at the first page and

$$\bigoplus_{k=1}^m E_{d, k}^1 = \bigoplus_{k=1}^m H^d(\mathcal{C}_k^*) \simeq {}^bH^d(S),$$

or that the cohomology of the associated graded complex computes the cohomology of $({}^b\Omega^*(S), d)$.

A log Darboux theorem

Darboux's theorem gives us a local description of Log Poisson structures which enables us to identify the rigged algebroid and compute Poisson cohomology. We establish this type of normal form theorem using a Moser-type argument in a neighborhood of $p \in D$. Let (S, D) be a surface S with star divisor D and assume $p \in D$ has degree $k \geq 3$. In a star chart centered at p , consider the star log symplectic form

$$\omega_0 = \left(\frac{h}{xy} \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} + P \right) dx \wedge dy \quad (3.17)$$

where

$$P = \sum_i \frac{\lambda_i}{x(A_i x + B_i y)} + \frac{\delta_i}{y(A_i x + B_i y)} + \sum_{i < j} \frac{\nu_{ij}}{(A_i x + B_i y)(A_j x + B_j y)}$$

for real numbers $\lambda_i, \delta_i, \nu_{ij}$ and $h \in \{-1, 1\}$.

Proposition 16. *Let ω be a star log symplectic form on (S, D) . Given a point $p \in D$ of degree $k \geq 3$, there exists a neighborhood U of p such that on U there is a b -symplectomorphism pulling ω back to a form expressible as in equation (3.17).*

Proof. Let (S, D) be a surface S with star divisor D and assume $p \in D$ has degree k . Let ω be a log-symplectic structure on S . The proof of Theorem 18 tells us that at any intersection point $p \in D$, there exists a star chart U such that ω is expressible as $\omega_0 + \nu$ where $\nu \in {}^b\Omega^2(U)$ is an exact log 2-form and ω_0 is the log 2-form given by equation (3.17). Further,

$$\omega|_p = \frac{c}{xy} \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} dx \wedge dy$$

for $c \in \mathbb{R}$ and $\nu|_p = 0$. We have that

$$\omega - \omega_0 = \nu = d\gamma$$

for some $\gamma \in {}^b\Omega^1(U)$. Note that γ is not a primitive for any terms of the form

$$\frac{c}{xy} \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} dx \wedge dy$$

for $c \in \mathbb{R}$. Consequently, when considered as a b -form, γ vanishes at the point p because it is not ‘maximally singular’.

Now we will proceed by the standard relative Moser argument (See [11] Sec. 7.3 for the smooth setting, and [47] Thm 6.4 for the b -symplectic version). Choose real numbers $\lambda_i, \delta_i, \nu_{ij}$ so that ω and ω_0 are cohomologous in the Lie algebroid cohomology of the b -tangent bundle. Further, if $c > 0$, choose $h = 1$. If $c < 0$, choose $h = -1$. This second condition means that ω and ω_0 induce the same orientation of the b -tangent bundle. It is necessary to ensure that the convex combination $\omega_t = (1 - t)\omega_0 + t\omega$ is non-degenerate for all time t . Then $\frac{d\omega_t}{dt} = \omega - \omega_0 = d\gamma$. Because γ is a log one form, the vector field defined by $i_{v_t}\omega_t = -\gamma$ is a log vector field and its flow fixes the divisor $D \cap U$. Thus we can integrate v_t to an isotopy that fixes $D \cap U$. This isotopy is the desired log-symplectomorphism. $\square$

Global Moser for star log symplectic manifolds

Proposition 17. *Let S be a compact surface with a star divisor D and let ω_0 and ω_1 be two star log symplectic forms on (S, D) . Suppose that there is a family of star log symplectic forms ω_t from ω_0 to ω_1 defined for $0 \leq t \leq 1$. If the b -cohomology class $[\omega_t]$ is independent of t , then there exists a family of diffeomorphisms*

$$\gamma_t : S \rightarrow S \text{ for } 0 \leq t \leq 1$$

such that

$$\gamma_t|_D \text{ is the identity map on } D \text{ and } \gamma_t^* \omega_t = \omega_0.$$

Because the proof of Proposition 17 is very similar to the proof of Proposition 16 and follows closely the proofs of Theorem 38 in [19] or Theorem 6.5 in [47], we will omit the details. This log version of the global Moser theorem completes the proof of Theorem 16 and gives us a classification of star log symplectic surfaces by b -cohomology classes.

3.5.4 Poisson Cohomology

To compute the Poisson cohomology of star log symplectic surfaces, we will use the method of rigged Lie algebroids that we introduced in [24]. In particular, we construct a Lie algebroid that is isomorphic to the Poisson Lie algebroid of π .

Definition 21. *Let (S, D, ω) be a star log symplectic surface of the b -tangent bundle $({}^bTS, \rho : {}^bTS \rightarrow TS)$ over (S, D) . The rigged Lie algebroid of ω is the vector bundle whose space of sections is*

$$\{u \in C^\infty(S; {}^bTS) \mid i_u \omega \in \rho^*(T^*S)\},$$

i.e. the b -vector fields that when contracted into ω “smoothen” it into a smooth one form. Alternatively, the sections of the dual rigged bundle $\Gamma(\mathcal{R}^)$ are an extension to S of the image $\omega^b(\Gamma(TS))$ away from D .*

As discussed in [25], the rigged Lie algebroid is isomorphic to the Poisson Lie algebroid T^*S with anchor map $\pi^\sharp = (\omega^b)^{-1}$. However, completing the computation of cohomology using the Lie algebroid cohomology of the rigged Lie algebroid is much more tractable.

Constructing the rigged Lie algebroid

Using Darboux coordinates for a star log symplectic form ω on (S, D) , we will identify the rigged Lie algebroid. Recall that in a neighborhood of an intersection point p of degree $k \geq 3$, ω can be expressed as

$$\omega_0 = \left(\frac{1}{xy} \prod_{i=3}^k \frac{1}{(A_i x + B_i y)} + P \right) dx \wedge dy$$

where

$$P = \sum_i \frac{\lambda_i}{x(A_i x + B_i y)} + \frac{\delta_i}{y(A_i x + B_i y)} + \sum_{i < j} \frac{\nu_{ij}}{(A_i x + B_i y)(A_j x + B_j y)}$$

for real numbers $\lambda_i, \delta_i, \nu_{ij}$. Then the space

$$\{u \in \mathcal{C}^\infty(S; {}^b TS) | i_u \omega \in \rho^*(T^* S)\}$$

is locally generated by

$$xy \prod_{i=3}^k (A_i x + B_i y) \frac{\partial}{\partial x}, \quad xy \prod_{i=3}^k (A_i x + B_i y) \frac{\partial}{\partial y}.$$

These are local generators of the zero tangent bundle, that is the vector bundle whose space of sections are

$$\{u \in \mathcal{C}^\infty(S; TS) | u|_Z = 0 \text{ for all } Z \in D\}.$$

This vector bundle, first introduced by Rafe Mazzeo and Richard Melrose in the context of manifolds with boundary [31, 32], is a Lie algebroid. We will use anchor map the given by evaluation in the tangent bundle and Lie bracket induced by the standard Lie bracket on TS .

Lemma 13. *The Poisson cohomology of a star log Poisson surface (S, D, π) is isomorphic to the Lie algebroid cohomology ${}^0 H^*(S)$ of the zero tangent bundle of (S, D) .*

The details of the proof of this lemma can be found in Section 5 of [24]. Thus to complete the proof of Theorem 17, it suffices to compute the Lie algebroid cohomology of the 0 tangent bundle of (S, D) .

0-cohomology

Given a surface S with a collection D of transverse curves $\{Z_1, \dots, Z_k\}$, the Lie algebroid cohomology of the associated 0-tangent bundle is

$$H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i) \oplus \bigoplus_{i < j} (H^{p-2}(Z_i \cap Z_j) \oplus H^{p-2}(Z_i \cap Z_j; |N^* Z_i|^{-1} \otimes |N^* Z_j|^{-1})) \quad (3.18)$$

which we originally computed in [25]. Unlike in the case of the b -tangent bundle, this description does not hold in the more general case when D is any star divisor. In other words, the 0-cohomology perceives higher order intersection of curves in D while b -cohomology does not.

Theorem 19. *Let (S, D) be a surface S with ordered star divisor*

$$D = \{Z_1, \dots, Z_k\}.$$

Given a star atlas with respect to this ordering, the Lie algebroid cohomology of the 0-tangent bundle over (S, D) is isomorphic to the expression (3.18) in degree 0 and 1. In degree 2, the cohomology is a direct sum of the following vector spaces:

- *A single copy of $H^2(S)$.*
- *Each hypersurfaces Z_i contributes $H^1(Z_i)$.*
- *Each pair wise intersection of two hypersurfaces Z_i, Z_j with $i < j$ contributes*

$$H^0(Z_i \cap Z_j) \oplus H^0(Z_i \cap Z_j; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}).$$

- *Each intersection of three hypersurfaces Z_i, Z_j, Z_k with $i < j < k$ contributes*

$$H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1} \otimes |N^*Z_k|^{-1}) \oplus$$

$$H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_j|^{-1} \otimes |N^*Z_k|^{-1}) \oplus H^0(Z_i \cap Z_j \cap Z_k; |N^*Z_i|^{-1} \otimes |N^*Z_k|^{-1}).$$

- *Each intersection of four or more hypersurfaces $Z_{i_1}, Z_{i_2}, Z_{i_3}, \dots, Z_{i_\ell}$ with $i_1 < i_2 < i_3 < \dots < i_\ell$ contributes*

$$H^0(\bigcap_i Z_i; \bigotimes_i |N^*Z_i|^{-1}) \oplus H^0(\bigcap_i Z_i; \bigotimes_{i \neq i_1} |N^*Z_i|^{-1}) \oplus$$

$$H^0(\bigcap_i Z_i; \bigotimes_{i \neq i_2} |N^*Z_i|^{-1}) \oplus H^0(\bigcap_i Z_i; \bigotimes_{i \neq i_1, i_2} |N^*Z_i|^{-1}).$$

Example 44. *Consider the star log symplectic structure on the sphere introduced in Example 43. The theorem tells us that in fixed coordinates, we can identify*

$$H_\pi^2(\mathbb{S}^2) \simeq \mathbb{R}^{22}.$$

Further, in the following example we demonstrate how many of the classes in $H_\pi^2(S)$ do not arise as bivectors on the b -tangent bundle.

Example 45. *Consider $\mathbb{R}^2$ with standard coordinates equipped with Poisson bivector*

$$\pi = (x + y)(x - y)x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The degeneracy locus of π is depicted in Figure 29. As described in Theorem 17, the second Poisson cohomology $H_\pi^2(\mathbb{R}^2)$ contains three copies of

$$H^0(\{x=0\} \cap \{x-y=0\} \cap \{x+y=0\}).$$

One of these vector spaces is generated by the bivector

$$\nu = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Since ν does not vanish at the lines $x+y=0$ or $x-y=0$, it does not correspond to an element of the b-de Rham sub-complex. Further, its corresponding element in the rigged algebroid $\mathcal{R}$ is not cohomologous to any b-form. Thus not all non-trivial deformations of π near π can be done through a path of log-symplectic structures.

Proof. Let (S, D) be a surface S with star divisor D . We will compute the Lie algebroid cohomology of the 0-tangent bundle over (S, D) using a filtration ${}^0\mathcal{F}$ of cochain complexes.

Constructing filtration ${}^0\mathcal{F}$.

Given a subset $I \subseteq D$, let ${}^0\Omega^*(S, I)$ denote the complex of 0-forms of the 0-tangent bundle over (S, I) . In this notation, we want to compute the cohomology of the complex ${}^0\Omega^*(S, D)$.

Consider the sets

$${}^0\mathcal{F}_k^* := v.s. \text{Span} \left\{ \bigcup_{I \subseteq D, |I|=k} {}^0\Omega^*(S, I) \right\}.$$

Notice that ${}^0\mathcal{F}_k^* \subseteq {}^0\mathcal{F}_{k+1}^*$ for all k . Each set ${}^0\mathcal{F}_k^*$ is a group under addition and has the structure of a module with scalars from the ring

$${}^0\mathcal{F}_k^0 = \mathcal{C}^\infty(S).$$

Note that ${}^0\Omega^*(S, I) \subseteq {}^0\Omega^*(S, D)$ for all $I \subseteq D$. Consequently, the modules ${}^0\mathcal{F}_k^* \subseteq {}^b\Omega^*(S, D)$ for all k . It is easy to check that ${}^0\mathcal{F}_k^*$ is closed under the differential of ${}^0\Omega^*(S, D)$. Thus we can define a differential ${}^0\mathcal{F}_k^p \xrightarrow{d} {}^0\mathcal{F}_k^{p+1}$ inherited from the differential on ${}^0\Omega^*(S, D)$.

For each k , we have a chain complex:

$$0 \rightarrow {}^0\mathcal{F}_k^0 \xrightarrow{d} {}^0\mathcal{F}_k^1 \xrightarrow{d} {}^0\mathcal{F}_k^2 \rightarrow 0.$$

We will employ the following filtration of cochain complexes:

$${}^0\mathcal{F} = \left\{ \Omega^*(S) = {}^0\mathcal{F}_0^* \subseteq {}^0\mathcal{F}_1^* \subseteq {}^0\mathcal{F}_2^* \subseteq \dots \subseteq {}^0\mathcal{F}_{|D|-1}^* \subseteq {}^0\mathcal{F}_{|D|}^* = {}^0\Omega^*(S, D) \right\}.$$

In the next step, we consider various short exact sequences of complexes involving the terms in ${}^0\mathcal{F}$.

A short exact sequence of complexes.

The inclusions ${}^0\mathcal{F}_{k-1}^* \rightarrow {}^0\mathcal{F}_k^*$ of cochain complexes from filtration ${}^0\mathcal{F}$ fit into a short exact sequence of complexes

$$0 \rightarrow {}^0\mathcal{F}_{k-1}^* \rightarrow {}^0\mathcal{F}_k^* \rightarrow \mathcal{C}_k^* \rightarrow 0$$

where

$$\mathcal{C}_k^p = {}^0\mathcal{F}_k^p / {}^0\mathcal{F}_{k-1}^p.$$

The differential ${}^{\mathcal{C}}d$ on $\mathcal{C}_k^p$ is induced by d on ${}^0\mathcal{F}_k^p$: if P is the projection ${}^0\mathcal{F}_k^p \rightarrow {}^0\mathcal{F}_k^p / {}^0\mathcal{F}_{k-1}^p$, then ${}^{\mathcal{C}}d(\eta) = P(d(\theta))$ where $\theta \in {}^0\mathcal{F}_k^p$ is any form such that $P(\theta) = \eta$. Hence $({}^{\mathcal{C}}d)^2 = 0$ and $(\mathcal{C}_k^*, {}^{\mathcal{C}}d)$ is a complex.

Given one of these short exact sequences, we will show that the boundary maps in the induced long exact sequences in cohomology are zero. This feature will facilitate our arrival at a nice description of ${}^0H^p(S)$.

Computing $H^p({}^0\mathcal{F}_1^*)$.

Let $k = 1$ and consider the short exact sequence

$$0 \rightarrow {}^0\mathcal{F}_0^* \rightarrow {}^0\mathcal{F}_1^* \rightarrow {}^0\mathcal{C}_1^* \rightarrow 0$$

defined above. Note that $\mathcal{F}_0^* = \Omega^*(S)$.

Let $D = \{Z_1, Z_2, \dots, Z_\ell\}$ be our star divisor and consider defining functions $x_1, x_2, \dots, x_\ell$ such that $Z_i = \{x_i = 0\}$. For each i , let $\chi_i : S \rightarrow \mathbb{R}$ be a smooth bump function supported in a neighborhood of Z_i and with constant value one near Z_i . An element $\mu \in \mathcal{F}_1^1$ can be expressed as

$$\mu = \sum_{i=1}^{\ell} \chi_i \cdot \left(\frac{dx_i}{x_i} \alpha_i + \frac{\beta_i}{x_i} \right) + \theta$$

where $\alpha_i \in \mathcal{C}^\infty(Z_i)$, $\beta_i \in \Omega^1(Z_i)$, and $\theta \in \Omega^1(S)$.

Note that

$$d\mu = - \sum_{i=1}^{\ell} \chi_i \left(\frac{dx_i}{x_i} \wedge d\alpha_i - \frac{dx_i \wedge \beta_i}{x_i^2} \right) + d\chi_i \wedge \left(\frac{dx_i}{x_i} \alpha_i + \frac{\beta_i}{x_i} \right) + \frac{d\beta_i}{x_i} + d\theta.$$

An element $\mu \in \mathcal{F}_1^2$ can be expressed as

$$\mu = \sum_{i=1}^{\ell} \chi_i \cdot \frac{dx_i}{x_i^2} \wedge (\alpha_i + \beta_i x_i) + \theta$$

where $\alpha_i, \beta_i \in \Omega^1(Z_i)$ and $\theta \in \Omega^2(S)$. Note that $d\mu = 0$.

Given $\mu \in \mathcal{F}_1^p$, we write $\Re(\mu) = \theta$ and $\Im(\mu) = \mu - \Re(\mu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\Re(d\mu) = d(\Re(\mu))$ and $\Im(d\mu) = d(\Im(\mu))$. Thus we have a splitting

$${}^0\mathcal{F}_1^* = \Omega^*(S) \oplus {}^0\mathcal{C}_1^*$$

as complexes and $H^p({}^0\mathcal{F}_1^*) \simeq H^p(S) \oplus H^p({}^0\mathcal{C}_1^*)$. We are left to compute the cohomology group of the quotient complex ${}^0\mathcal{C}_1^*$.

Any $P(\mu) \in {}^0\mathcal{C}_1^1$ is of the form

$$P(\mu) = \sum_{i=1}^{\ell} \frac{dx_i}{x_i} \alpha_i + \frac{\beta_i}{x_i}$$

for $\alpha_i \in \mathcal{C}^\infty(Z_i)$, $\beta_i \in \Omega^1(Z_i)$. Then

$$dP(\mu) = - \sum_{i=1}^{\ell} \frac{dx_i}{x_i} \wedge d\alpha_i - \frac{dx_i \wedge \beta_i}{x_i^2} + \frac{d\beta_i}{x_i}.$$

By inspecting the expression at each Z_i , this gives us kernel relations $d\alpha_i = 0$, $\beta_i = 0$. Given these choices of Z_i defining functions $x_1, \dots, x_\ell$, the $\ker(d : \mathcal{C}^p \rightarrow \mathcal{C}^{p+1})$ can be identified with

$$\bigoplus_{i=1}^{\ell} \{\alpha_i \in \mathcal{C}^\infty(Z_i) : d\alpha_i = 0\}.$$

Thus

$$H^1({}^0\mathcal{C}_1^*) \simeq \bigoplus_i H^0(Z_i).$$

For $H^2({}^0\mathcal{C}_1^*)$, note that given $\mu \in {}^0\mathcal{F}_1^2$ there exists $\tilde{\mu} \in {}^0\mathcal{F}_1^1$ such that

$$P(\mu - d\tilde{\mu}) = \sum_i \frac{dx_i}{x_i} \wedge (\beta_i - d\alpha_i)$$

for $\beta_i \in \Omega^1(Z_i)$, $\alpha_i \in \Omega^0(Z_i)$, and that elements of this form are not in the image of d . Thus we can identify

$$H^2({}^0\mathcal{C}_1^*) \simeq \bigoplus_i H^1(Z_i).$$

By a computation, which can be found in example 2.11 of [24], these sets are invariant under change of Z_i defining function. Thus

$$H^p({}^0\mathcal{C}_1^*) \simeq \bigoplus_i H^{p-1}(Z_i) \text{ and } H^p({}^0\mathcal{F}_1^*) \simeq H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i).$$

Computing $H^p({}^0\mathcal{F}_2^*)$.

Let $k = 2$ and consider the short exact sequence

$$0 \rightarrow {}^0\mathcal{F}_1^* \rightarrow {}^0\mathcal{F}_2^* \rightarrow {}^0\mathcal{C}_2^* \rightarrow 0.$$

Let $D = \{Z_1, Z_2, \dots, Z_\ell\}$ be our star divisor. For each pair of curves Z_i, Z_j with $i < j$,

there is a pair of tubular neighborhoods $\tau_i = Z_i \times (-\varepsilon, \varepsilon)_{x_{ij}}$ of Z_i and $\tau_j = Z_j \times (-\varepsilon, \varepsilon)_{y_{ij}}$ of Z_j such that

$$\left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{ij}} \right] = 0.$$

For existence of such neighborhoods see, for instance, section 5 of [1].

Let χ_{ij} be a smooth bump function supported near $Z_i \cup Z_j$ and with constant value one near $Z_i \cup Z_j$. An element $\mu \in {}^0\mathcal{F}_2^1$ can be expressed as

$$\mu = \sum_{i < j} \chi_{ij} \cdot \left(\frac{dx_{ij}}{x_{ij}y_{ij}} \alpha_{ij} + \frac{dy_{ij}}{x_{ij}y_{ij}} \beta_{ij} \right) + \theta$$

for $\alpha_{ij}, \beta_{ij} \in \mathcal{C}^\infty(Z_i \cap Z_j)$, and $\theta \in {}^0\mathcal{F}_1^1$. Note that α and β are just constants. Thus

$$d\mu = \sum_{i < j} \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}y_{ij}^2} \alpha_{ij} - \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2 y_{ij}} \beta_{ij} + d\theta.$$

An element $\mu \in {}^0\mathcal{F}_2^2$ can be expressed as

$$\mu = \sum_{i < j} \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2 y_{ij}^2} (\alpha_{ij} + \beta_{ij} x_{ij} + \gamma_{ij} y_{ij} + \delta_{ij} x_{ij} y_{ij}) + \theta$$

for $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij} \in \mathcal{C}^\infty(Z_i \cap Z_j)$, and $\theta \in {}^0\mathcal{F}_1^2$. Note $d\mu = 0$.

Given $\mu \in \mathcal{F}_2^p$, we write $\mathfrak{R}(\mu) = \theta$ and $\mathfrak{S}(\mu) = \mu - \mathfrak{R}(\mu)$ for ‘regular’ and ‘singular’ parts.

It is easy to see that $\mathfrak{R}(d\mu) = d(\mathfrak{R}(\mu))$ and $\mathfrak{S}(d\mu) = d(\mathfrak{S}(\mu))$. Thus we have a splitting

$${}^0\mathcal{F}_2^* = {}^0\mathcal{F}_1^* \oplus {}^0\mathcal{C}_2^*$$

as complexes and $H^p({}^0\mathcal{F}_2^*) \simeq H^p({}^0\mathcal{F}_1^*) \oplus H^p({}^0\mathcal{C}_2^*)$. We are left to compute the cohomology group of the quotient complex ${}^0\mathcal{C}_2^*$.

Any $P(\mu) \in {}^0\mathcal{C}_2^1$ is of the form

$$P(\mu) = \sum_{i < j} \frac{dx_{ij}}{x_{ij}y_{ij}} \alpha_{ij} + \frac{dy_{ij}}{x_{ij}y_{ij}} \beta_{ij}$$

for $\alpha_{ij}, \beta_{ij} \in \mathcal{C}^\infty(Z_i \cap Z_j)$. Then

$$dP(\mu) = \sum_{i < j} \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}y_{ij}^2} \alpha_{ij} - \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2 y_{ij}} \beta_{ij}.$$

Thus our kernel relations are $\alpha_{ij} = 0$ and $\beta_{ij} = 0$.

For $H^2({}^0\mathcal{C}_1^*)$, note that for $\mu \in {}^0\mathcal{F}_2^2$ there exists $\tilde{\mu} \in {}^0\mathcal{F}_2^1$ such that

$$P(\mu - d\tilde{\mu}) = \frac{dx_{ij} \wedge dy_{ij}}{x_{ij}^2 y_{ij}^2} \wedge (\alpha_{ij} + \delta_{ij} x_{ij} y_{ij})$$

for $\alpha_{ij}, \delta_{ij} \in \mathcal{C}^\infty(Z_i \cap Z_j)$. Further, there are no elements of this form in the image of d .

Thus, given these choices x_{ij}, y_{ij} of defining functions, we can identify

$$H^2({}^0\mathcal{C}_2^*) \simeq \bigoplus_{i < j} H^0(Z_i \cap Z_j) \oplus H^0(Z_i \cap Z_j).$$

By a computation which can be found at the conclusion of the proof of Theorem 8, this

cohomology can be identified independently of defining functions as

$$H^2({}^0\mathcal{C}_2^*) \simeq \bigoplus_{i < j} H^0(Z_i \cap Z_j) \oplus H^0(Z_i \cap Z_j; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}).$$

Thus $H^p({}^0\mathcal{F}_2^*)$ is

$$H^p(S) \oplus \bigoplus_i H^{p-1}(Z_i) \oplus \bigoplus_{i < j} H^{p-2}(Z_i \cap Z_j) \oplus H^{p-2}(Z_i \cap Z_j; |N^*Z_i|^{-1} \otimes |N^*Z_j|^{-1}).$$

Computing $H^p({}^0\mathcal{F}_k^*)$ for $k \geq 3$.

Fix $k \geq 3$ and consider the short exact sequence

$$0 \rightarrow {}^0\mathcal{F}_{k-1}^* \rightarrow {}^0\mathcal{F}_k^* \rightarrow {}^0\mathcal{C}_k^* \rightarrow 0.$$

Given any $p \in S$ such that at least the k curves in the set $I = \{Z_1, \dots, Z_k\}$ intersect at p , there exists a star chart with respect to I centered at p , $(U_{p,I}, x, y)$, such that

$$Z_1 = \{x = 0\}, Z_2 = \{y = 0\}, \dots, Z_k = \{A_k x + B_k y = 0\}.$$

Let $\chi_{p,I}$ be a smooth bump function supported in $U_{p,I}$ and with constant value one in a neighborhood of p . Let

$$R := \prod_{i=3}^k (A_i x + B_i y).$$

An element $\mu \in {}^0\mathcal{F}_k^1$ can be expressed as

$$\mu = \sum_{\substack{p, I \\ |I|=k}} \chi_{p,I} \cdot \left(\frac{dx}{xyR} \alpha_{p,I} + \frac{dy}{xyR} \beta_{p,I} \right) + \theta$$

for $\alpha_{p,I}, \beta_{p,I} \in \Omega^0(\{p\})$ (i.e. constants) and $\theta \in {}^0\mathcal{F}_{k-1}^1$. Note

$$\begin{aligned} d\mu &= \sum_{\substack{p, I \\ |I|=k}} \chi_{p,I} \cdot \left(\frac{dx \wedge dy}{xy^2 R} \alpha_{p,I} + \frac{dx \wedge dy}{xy R^2} (\partial_y R) \alpha_{p,I} - \frac{dx \wedge dy}{x^2 y R} \beta_{p,I} - \frac{dx \wedge dy}{xy R^2} (\partial_x R) \beta_{p,I} \right) \\ &\quad + \sum_{\substack{p, I \\ |I|=k}} d\chi_{p,I} \wedge \left(\frac{dx}{xyR} \alpha_{p,I} + \frac{dy}{xyR} \beta_{p,I} \right) + d\theta \end{aligned}$$

Any element $\mu \in {}^0\mathcal{F}_k^2$ can be expressed as

$$\mu = \sum_{\substack{p, I \\ |I|=k}} \chi_{p,I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge (\alpha_{p,I} + \beta_{p,I} x + \gamma_{p,I} y + \delta_{p,I} xy) + \theta$$

for $\alpha_{p,I}, \beta_{p,I}, \gamma_{p,I}, \delta_{p,I} \in \Omega^0(\{p\})$ and $\theta \in {}^0\mathcal{F}_{k-1}^2$. Note $d\mu = 0$.

Given $\mu \in {}^0\mathcal{F}_k^\ell$, we write $\mathfrak{R}(\mu) = \theta$ and $\mathfrak{S}(\mu) = \mu - \mathfrak{R}(\mu)$ for ‘regular’ and ‘singular’ parts. One can check that $\mathfrak{R}(d\mu) = d(\mathfrak{R}(\mu))$ and $\mathfrak{S}(d\mu) = d(\mathfrak{S}(\mu))$. Thus we have a splitting

$${}^0\mathcal{F}_k^* = {}^0\mathcal{F}_{k-1}^* \oplus {}^0\mathcal{C}_k^*$$

as complexes and $H^p({}^0\mathcal{F}_k^*) \simeq H^p({}^0\mathcal{F}_{k-1}^*) \oplus H^p({}^0\mathcal{C}_k^*)$. We are left to compute the cohomology group of the quotient complex ${}^0\mathcal{C}_k^*$.

We will first compute the cohomology at ${}^0\mathcal{C}_k^1$. Any $P(\mu) \in {}^0\mathcal{C}_k^1$ is of the form

$$P(\mu) = \sum_{\substack{p, I \\ |I|=k}} \chi_{p, I} \cdot \left(\frac{dx}{xyR} \alpha_{p, I} + \frac{dy}{xyR} \beta_{p, I} \right)$$

for $\alpha_{p, I}, \beta_{p, I} \in \Omega^0(\{p\})$. Then $dP(\mu)$ follows from our expression of $d\mu$ above. Thus we can identify the set $\ker(d : \mathcal{C}_k^1 \rightarrow \mathcal{C}_k^2) = \{\alpha_{p, I} = 0, \beta_{p, I} = 0\}$ and $H^1(\mathcal{C}_k^*) = 0$ for all $k \geq 3$.

Next, we compute the cohomology at $\mathcal{C}_k^2$. Any element $P(\mu) \in \mathcal{C}_k^2$ is of the form

$$P(\mu) = \sum_{\substack{p, I \\ |I|=k}} \chi_{p, I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge (\alpha_{p, I} + \beta_{p, I} x + \gamma_{p, I} y + \delta_{p, I} xy)$$

for $\alpha_{p, I}, \beta_{p, I}, \gamma_{p, I}, \delta_{p, I} \in \Omega^0(\{p\})$. We have $dP(\mu) = 0$. Note that $d({}^0\mathcal{C}_k^1)$ has empty intersection with elements of the form

$$\frac{dx \wedge dy}{x^2 y^2 R^2} \wedge (\alpha_{p, I} + \beta_{p, I} x + \gamma_{p, I} y).$$

Using the expression of $d\mu$ for $\mu \in \mathcal{C}_k^1$ given above, we are left to consider when does the following equality hold at the point p :

$$\delta_{p, I} = (\partial_y P) \alpha_{p, I} - (\partial_x P) \beta_{p, I}$$

for $\alpha_{p, I}, \beta_{p, I}, \delta_{p, I} \in \Omega^0(\{p\})$. For this equation to hold true, P would need to be a single line $P = Ax + By$. Thus when $k \geq 4$, the term

$$\sum_{\substack{p, I \\ |I|=k}} \chi_{p, I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge \delta_{p, I} xy$$

is not in the image of $d(\mathcal{C}_k^1)$.

Let us consider the case where $k = 3$. In this instance, the numbers $\beta_{p, I} = 0$ and $\alpha_{p, I} = \frac{\delta_{p, I}}{B}$ provide a solution to the equality and

$$\sum_{\substack{p, I \\ |I|=k}} \chi_{p, I} \cdot \frac{dx \wedge dy}{x^2 y^2 R^2} \wedge \delta_{p, I} xy$$

is in the image of $d(\mathcal{C}_3^1)$.

Thus, for these fixed Z_i defining functions, we have identified

$$H^n(\mathcal{C}_3^*) \simeq \bigoplus_{p, |I|=3} (H^{n-2}(\{p\}))^3$$

and, for $k \geq 4$,

$$H^n(\mathcal{C}_k^*) \simeq \bigoplus_{p, |I|=k} (H^{n-2}(\{p\}))^4.$$

For completeness, we will consider what happens under change of defining functions.

By a computation which can be found at the conclusion of the proof of Theorem 8, the cohomology $H^p(\mathcal{C}_k^*)$ can be identified independently of defining functions as

$$\bigoplus_{\substack{i \in J \\ |J|=3}} (H^{p-2}(\bigcap_i Z_j; \bigotimes_i |N^* Z_i|^{-1}) \oplus H^{p-2}(\bigcap_i Z_i; \bigotimes_{i \in J \setminus \{i_1\}} |N^* Z_i|^{-1}) \oplus H^{p-2}(\bigcap_i Z_i; \bigotimes_{i \in J \setminus \{i_2\}} |N^* Z_i|^{-1}))$$

when $k = 3$ and as

$$\begin{aligned} & \bigoplus_{\substack{i \in J \\ |J|=k}} (H^{p-2}(\bigcap_i Z_j; \bigotimes_i |N^* Z_i|^{-1}) \oplus H^{p-2}(\bigcap_i Z_i; \bigotimes_{i \in J \setminus \{i_1\}} |N^* Z_i|^{-1})) \\ & \oplus \bigoplus_{\substack{i \in J \\ |J|=k}} (H^{p-2}(\bigcap_i Z_i; \bigotimes_{i \in J \setminus \{i_1\}} |N^* Z_i|^{-1}) \oplus H^{p-2}(\bigcap_i Z_i; \bigotimes_{i \in J \setminus \{i_1, i_2\}} |N^* Z_i|^{-1})) \end{aligned}$$

when $k \geq 4$, where J is a subset of integers $\{1, \dots, k\}$ and i_1, i_2 are the smallest elements in J .

□

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Appendix A

Differential Geometry

In this appendix, we provide sufficient background information so that a mathematically mature reader may appreciate and understand the content of the dissertation. The topics discussed are hardly complete and we have provided references for suggested further reading. In differential geometry, we use techniques of calculus and linear algebra to study smooth manifolds and a wide variety of structures placed on them.

A.1 Riemannian Geometry

The basic object of study in Riemannian geometry is a Riemannian metric.

Definition 22. *Given a manifold M , a **Riemannian metric** g on M is a smooth map*

$$g : \mathcal{C}^\infty(M; TM) \times \mathcal{C}^\infty(M; TM) \rightarrow \mathcal{C}^\infty(M)$$

that is linear in each entry, symmetric, and satisfies $g(s, s) \geq 0$ and $g(s, s) = 0$ if and only if $s = 0$.

Geometers of different flavors have strong and varying opinions on the place of a Riemannian metric in studying other types of structures in differential geometry.

“As a student at Cambridge, I was taught that in Linear Algebra ‘a gentleman should never use a basis unless he really has to’. By the same token, one should not use auxiliary Riemannian metrics if one can do without them.”

- Hansjörg Geiges

That being said, Riemannian metrics do a lot of heavy lifting in differential geometry. A Riemannian metric is a gadget that lets us determine distances along curves and makes it

possible to define angles, lengths of curves, volumes, curvature, and gradients of functions on manifolds. Given a Riemannian manifold (M, g) , the *gradient of f* is the vector field

$$df = g(\cdot, \nabla f).$$

At each point, the gradient of f will show the direction in which the function changes most quickly.

Example 46. Consider the torus $\mathbb{T}^2$ embedded into $\mathbb{R}^3$, depicted in Figure 31. We define a height function $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ by $f(p) = z$. Since the gradient of f shows the direction in which the function changes most quickly, the vector field flows vertically along the surface, shown in Figure 32.

Figure 31. *embedding*

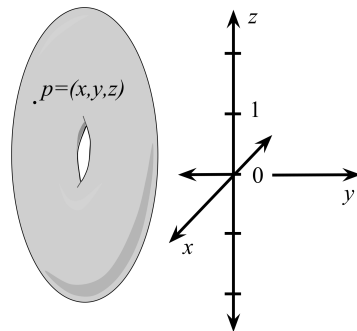
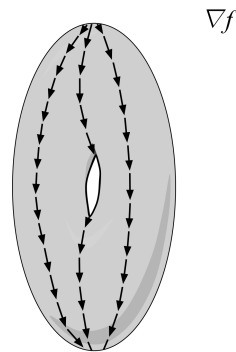


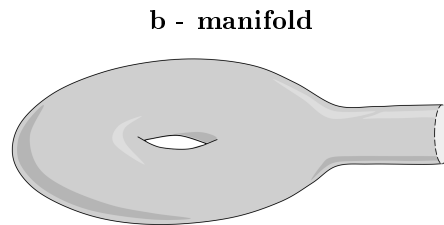
Figure 32. *gradient*



A.1.1 Riemannian metrics and Melrose rescaling.

Earlier, we discussed how to use a rescaled tangent bundle to study a Poisson bivector. These bundles originate from the study of Riemannian metrics on manifolds with ends.

Figure 33. *A torus with end*



A torus with a cylindrical end

The b-setting has its genesis in a compactification of a manifold with a cylindrical end in that the b-tangent bundle is a Lie algebroid that extends the standard tangent bundle to this compactification.

To see the relation with cylindrical geometry, given any compact Riemannian manifold (M, g_M) , let us consider the cylinder and metric:

$$C = \mathbb{R}_t \times M \quad g_C = dt^2 + g_M.$$

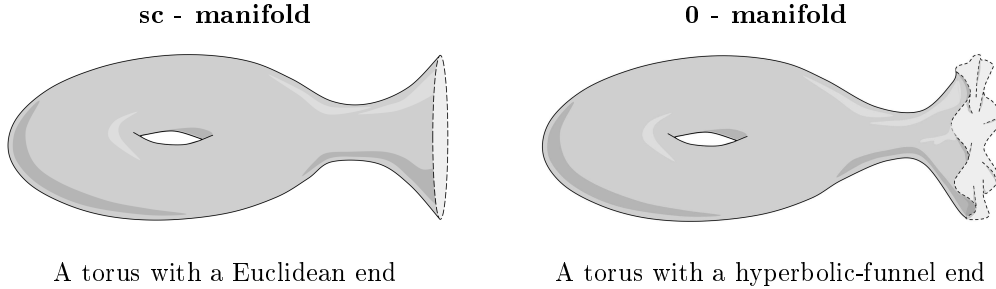
Consider the new coordinate $x = e^{-t}$. Then

$$g_C = \frac{dx}{x^2} + g_M.$$

We can create a new space, denoted $\overline{C}$, by adding the point $t = \{\infty\}$ (i.e. $x = \{0\}$) to the real line. Then the vector fields of bounded pointwise length with respect to g_C are precisely the sections of ${}^bT\overline{C}$, the rescaling of $T\overline{C}$ by the bundle TM at $\{x = 0\} \times M$. So the cylindrical metric is naturally interpreted as a metric on ${}^bT\overline{C}$. For a more complete discussion of the b-tangent bundle and its role in cylindrical geometry, see Chapter 7 of [38].

There are several other naturally occurring geometries on spaces: the scattering setting [37] is a compactification of a manifold with a Euclidean end, and the 0-setting [31, 32] is a compactification of a manifold with a hyperbolic-funnel end.

Figure 34. *Tori with ends*



To see the relation with Euclidean geometry, we consider Euclidean space with its standard metric. By performing the spherical compactification described in the Introduction, near the boundary this metric is

$$g = \frac{dx}{x^4} + \frac{g_{\mathbb{S}}}{x^2}$$

where g_S is the standard metric on the sphere. The vector fields of bounded pointwise length with respect to g are the sections of the scattering-tangent bundle ${}^{sc}T\bar{\mathbb{R}}^n$ and g is naturally interpreted as a metric on ${}^{sc}T\bar{\mathbb{R}}^n$. For more details, see Section 1.8 of [38].

To see the relation with hyperbolic geometry, let us consider the half-plane model of hyperbolic space and its associated metric:

$$\mathbb{H}^n = \{(x, y_1, \dots, y_n) \in \mathbb{R}^n \mid x > 0\} \quad g = \frac{dx^2}{x^2} + \frac{\sum_i dy_i^2}{x^2}.$$

We can create a new space, denoted $\bar{\mathbb{H}}^n$, by adding

$$Z = \{(x, y_1, \dots, y_n) \in \mathbb{R}^n \mid x = 0\}.$$

The vector fields of bounded pointwise length with respect to g are precisely the sections of ${}^0T\bar{\mathbb{H}}^n$, the rescaling of $T\bar{\mathbb{H}}^n$ by the 0 bundle over $Z = \{x = 0\}$. So the hyperbolic metric is naturally interpreted as a metric on ${}^0T\bar{\mathbb{H}}^n$. For a more complete discussion of the 0-tangent bundle and its role in hyperbolic geometry, see Section 8.3 of [38].

A.2 Symplectic geometry

Definition 23. A **symplectic manifold** is a manifold M with a closed, non-degenerate 2-form ω .

Example 47. Consider the torus $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates θ and ρ . The 2-form

$$\omega = d\theta \wedge d\rho$$

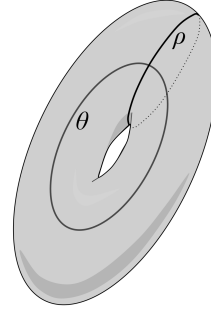
is a symplectic structure. For any even dimensional torus $\mathbb{T}^{2n}$ with global coordinates

$\theta_1, \dots, \theta_n, \rho_1, \dots, \rho_n$, the differential form

$$\omega = \sum_{i=1}^n d\theta_i \wedge d\rho_i$$

is symplectic.

Figure 35. Torus $\mathbb{T}^2$



In fact, every symplectic manifold locally looks like ω as in this example.

Theorem 20. (Darboux) Given a symplectic manifold (M^{2n}, ω) , for each $p \in M$ there exists a chart $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ centered at p such that

$$\omega|_U = \sum_{i=1}^n dx_i \wedge dy_i.$$

Because of its application in physics, a very important example of a symplectic manifold is the cotangent bundle T^*M of any manifold M .

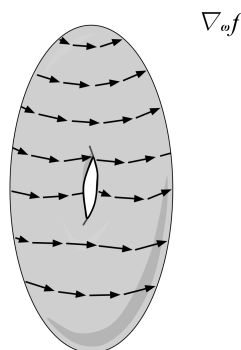
Symplectic geometry has its origins in the Hamiltonian formulation of classical mechanics. In particular, the phase space of certain systems is a symplectic manifold. The symplectic gradient is the analog in symplectic geometry of the gradient in Riemannian geometry. Given a symplectic manifold (M, ω) , the *symplectic gradient* $\nabla_\omega f$ is the vector field satisfying

$$df = \omega(\cdot, \nabla_\omega f).$$

At each point, the symplectic gradient of f will show the direction in which the function changes least quickly. If you are interested in conserving quantities, symplectic geometry can be a powerful setting for studying a physical system.

Example 48. Recall from Example 46 the torus embedded in $\mathbb{R}^3$ with height function f . Since the symplectic gradient of f indicates the direction in which the function changes the least, this vector field flows in horizontal ‘rings’ on the surface.

Figure 36. *symplectic gradient*



Unfortunately, many manifolds do not admit symplectic structures.

Cohomological obstruction to existence. If (M^{2n}, ω) is a compact symplectic manifold, then the de Rham cohomology class $[\omega] \in H^2(M)$ is non-zero. Since $H^2(\mathbb{S}^{2n}) = 0$ for all $n > 1$, there are no symplectic structures on spheres of dimension greater than 2.

Compatible triples. Symplectic geometry and Riemannian geometry are nicely related through something called an almost complex structure.

Definition 24. A complex structure on a vector space V is an endomorphism $J : V \rightarrow V$ such that $J^2 = -Id$. An **almost complex structure** on a manifold M is an endomorphism

of the tangent bundle $J : TM \rightarrow TM$ such that for all $p \in M$, $J_p : T_p M \rightarrow T_p M$ is a complex structure. A symplectic form ω is **compatible** with J if for all $p \in M$

1. $\omega(Ju, Jv) = \omega(u, v)$ for all $u, v \in T_p M$, and
2. $\omega(v, Jv) > 0$ for all non-zero $v \in T_p M$.

Note that J is ω -compatible if and only if $\omega(\cdot, J\cdot)$ defines a Riemannian metric. A *compatible triple* (ω, J, g) on a manifold M is a symplectic form, almost complex structure, and metric such that $g(u, v) = \omega(u, Jv)$ for all $u, v \in C^\infty(M; TM)$.

Lagrangian submanifolds. Lagrangian submanifolds are a very important in symplectic geometry, particularly in the statement of the Arnold-Givental conjecture and the construction of Floer homology. Given a symplectic manifold (M, ω) a Lagrangian submanifold is one where the symplectic form vanishes. Further, the dimension of a Lagrangian submanifold is the maximal dimension where a space can have this vanishing property.

Definition 25. Given a symplectic manifold (M^{2n}, ω) , a submanifold Y is **lagrangian** if, for $i : Y \rightarrow M$ is the inclusion map, $i^* \omega = 0$ and $\dim Y = n$.

Further reading: Ana Cannas da Silva's [11] text is a good source of information on symplectic geometry.

A.3 Poisson Geometry

Definition 26. A **Poisson structure** on a manifold M is a skew-symmetric bracket on functions

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$$

such that

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \text{ and } \{f, gh\} = \{f, g\}h + g\{f, h\}$$

for all $f, g, h \in C^\infty(M)$.

Equivalently, a Poisson structure is a bivector $\pi \in C^\infty(M; \wedge^2 TM)$ such that

$$[\pi, \pi] = 0$$

where this bracket is the Schouten bracket, a type of graded Lie bracket defined on multivector fields that extends the standard Lie bracket of vector fields. To go from one perspective to the other, use the formula

$$\{f, g\} = \pi(df, dg)$$

for all $f, g \in \mathcal{C}^\infty(M)$.

Example 49. *The zero bracket $\{f, g\} = 0$ is a Poisson structure for any manifold M .*

Example 50. *Symplectic manifolds are Poisson structures. Given (M, ω) symplectic, we can define a Poisson structure π by $(\omega^\flat)^{-1} = \pi^\sharp$. Then $d\omega = 0$ is equivalent to $[\pi, \pi] = 0$. Darboux's theorem then gives us a local expression for π :*

$$\pi = \sum_{i=1}^n \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_i}.$$

Weinstein's splitting theorem gives us a local expression for a general Poisson bivector.

Theorem 21. (Weinstein Splitting) *Let (M, π) be a Poisson manifold. For each $p \in M$, there exists a coordinate system $x_1, \dots, x_r, y_1, \dots, y_r$, and $z_1, \dots, z_s$ centered at p such that*

$$\pi = \sum_{i=1}^r \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_i} + \sum_{1 \leq j < k \leq s} \phi_{j,k}(z_1, \dots, z_s) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k}$$

where the functions $\phi_{j,k}(z_1, \dots, z_s)$ only depend on $z_1, \dots, z_s$ and $\phi_{j,k}(0, \dots, 0) = 0$.

Each Poisson bivector π defines a homomorphism $\pi^\sharp : TM \rightarrow TM$ by the contraction $\xi \rightarrow \pi(\xi, \cdot)$. For each $p \in M$, the *rank of π at p* is the rank of $\pi_p^\sharp$. In Weinstein's splitting theorem above, the rank of π at p is $2r$.

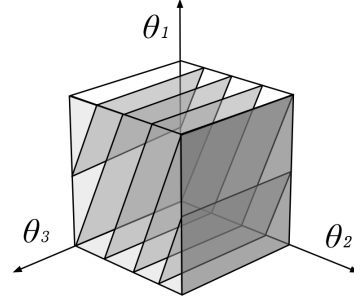
Singular symplectic foliation The image $Im(\pi^\sharp)$ is an integrable distribution. Thus there is a singular foliation $\mathcal{F}$ such that $T\mathcal{F} = Im(\pi^\sharp)$. Further, each leaf S in the foliation is a symplectic manifold. The symplectic structure is obtained by taking $\omega = \pi_S^{-1}$ where $\pi_S(\alpha|_{TS}, \beta|_{TS}) = \pi(\alpha, \beta)$ for all $\alpha, \beta \in \mathcal{C}^\infty(S; T^*M|_S)$. Given $p \in M$, the leaf containing p is locally defined by the $z_1 = c_1, \dots, z_s = c_s$ for real constants c_i and for coordinates z_i provided in Weinstein's splitting theorem above.

Example 51. Let $M = \mathbb{T}^3$ be the torus with global coordinates $(\theta_1, \theta_2, \theta_3)$. The bivector

$$\pi = \left(\frac{\partial}{\partial \theta_1} + \lambda \frac{\partial}{\partial \theta_2} \right) \wedge \frac{\partial}{\partial \theta_3}$$

is Poisson for any $\lambda \in \mathbb{R}$. The leaves are defined by $\theta_1 - \lambda \theta_2 = \text{cnst.}$ If $\lambda \in \mathbb{Q}$, then the symplectic leaves are embedded submanifolds diffeomorphic to $\mathbb{T}^2$. If $\lambda \notin \mathbb{Q}$, then the symplectic leaves are not embedded and are diffeomorphic to $\mathbb{R} \times \mathbb{S}^1$.

Figure 37. $\mathbb{T}^3$ identification space with toroidal leaves.



Further reading: More information on the facts we recount here can be found in Jean-Paul Dufour and Nguyen Zung's book [12].

A.4 Contact Geometry

Given a manifold M , let $\xi \subset TM$ denote a field of hyperplanes, that is a smooth codimension 1 subbundle.

Definition 27. Given an odd dimensional manifold M^{2n+1} , a **contact structure** is a hyperplane field $\xi \subset TM$ such that around each point there is a locally defined 1-form α such that $\ker \alpha = \xi$ and $\alpha \wedge (d\alpha)^n \neq 0$.

Example 52. Consider $\mathbb{R}^3$ with standard coordinates x, y, z . The one form

$$\alpha = dz - ydx$$

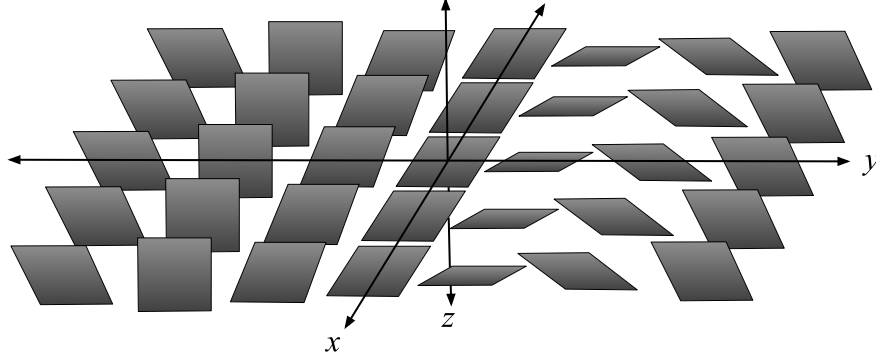
defines a contact structure. In other words $\xi = \ker \alpha$ is a contact distribution. We can verify that this is in fact a contact distribution by checking that $\alpha \wedge d\alpha \neq 0$. Since $d\alpha = -dy \wedge dx$, it is simple to see that $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$ is nowhere vanishing.

To see how α defines the hyperplanes depicted in Figure 38, consider when $y = 0$. Then $\alpha = dz$ and $\ker \alpha$ is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. This corresponds to the 5 'flat' xy -planes depicted along the x -axis.

When $y = 1$, $\alpha = dz - dx$. Then $\ker \alpha$ is spanned by $\frac{\partial}{\partial z} + \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. Accordingly, as we move along the y -axis, the hyperplanes start to tilt upwards. As $y \rightarrow \infty$, the hyperplanes

approach ‘upright’ zy -planes. As $y \rightarrow -\infty$, the hyperplanes twist in the opposite direction, but also approach ‘upright’ zy -planes.

Figure 38. A contact distribution on $\mathbb{R}^3$.



We call a contact distribution “maximally non-integrable”, in other words there is no $2n$ -dimensional submanifold N so that $\xi|_N = TN$, even locally. Intuitively, the hyperplanes twist too much to be fit to a dimension $2n$ submanifold. In fact, n is the maximal dimension of a submanifold whose tangent bundle is contained in ξ . Such submanifolds have a special place in contact geometry.

Definition 28. Given a contact manifold (M, ξ) , a submanifold $L \subset M^{2n+1}$ is **Legendrian** if $\dim L = n$ and $T_p L \subset \xi_p$ for all $p \in L$.

We are primarily concerned about cases where ξ can be defined by a global 1-form α . Such contact manifolds $(M, \xi = \ker \alpha)$ are called *coorientable* with the 1-form α called a coorientation. Note that for any nowhere vanishing function $f \in C^\infty(M)$, $\ker \alpha = \ker f\alpha$ and $(f\alpha) \wedge (d(f\alpha))^n = f^{n+1} \alpha \wedge (d\alpha)^n$. Thus we can make conformal changes to α and still obtain the same coorientable contact structure. The fact that ξ is defined by a conformal class of 1-forms, rather than just α , is key to defining the correct notion of morphism in contact geometry.

Definition 29. If ξ_1 and ξ_2 are two contact structures on M , then a diffeomorphism $\phi : M \rightarrow M$ is a **contactomorphism** if $d\phi : TM \rightarrow TM$ satisfies $\phi_*(\xi_1) = \xi_2$. If $\ker \alpha_1 = \xi_1$ and $\ker \alpha_2 = \xi_2$, this is equivalent to $\phi^* \alpha_2 = g\alpha_1$ for a nonvanishing function $g \in C^\infty(M)$.

Returning to Example 52, this gives the local model for all three dimensional contact manifolds. In fact there is an analogous local model for all contact manifolds:

Theorem 22. (Darboux) *Let $(M^{2n+1}, \xi = \ker \alpha)$ be a cooriented contact manifold. At every $p \in M$, there exists local coordinates $(U, x_1, \dots, x_n, y_1, \dots, y_n, z)$ such that*

$$\alpha|_U = dz - \sum_{i=1}^n y_i dx_i.$$

Accordingly, as in symplectic geometry, there are no local invariants of a contact manifold (other than dimension).

Reeb vector field Given a cooriented contact manifold $(M, \xi = \ker \alpha)$, we can define a useful vector field called the Reeb vector field.

Definition 30. *Given a cooriented contact manifold $(M, \xi = \ker \alpha)$ the **Reeb vector field** R_α of α is the unique vector field such that $d\alpha(R_\alpha, \cdot) \equiv 0$ and $\alpha(R_\alpha) \equiv 1$.*

Example 53. *Continuing with the contact 1-form $\alpha = dz - ydx$ from Example 52, the Reeb vector field is*

$$R_\alpha = \frac{\partial}{\partial z}.$$

Note that the Reeb vector field is determined by α . If we consider $f\alpha$, then $d(f\alpha) = df \wedge \alpha + f d\alpha$. Thus $d(f\alpha)(R_\alpha, \cdot) = df(\cdot)$, which in general is nonzero.

Also note that the Reeb vector field gives us a specific splitting of TM as $\mathbb{R} \oplus \xi$.

Symplectic geometry of contact manifolds Symplectic geometry is closely related to contact geometry. Given a cooriented contact manifold $(M, \xi = \ker \alpha)$, the condition $\alpha \wedge (d\alpha)^n \neq 0$ implies that $d\alpha|_\xi$ is non-degenerate. In other words, $(\xi, d\alpha)$ is a symplectic vector bundle over M . Further, if we replace α by a conformally equivalent 1-form $f\alpha$, then $d(f\alpha)|_\xi = f d\alpha|_\xi$. Thus a cooriented contact manifold $(M, \xi = \ker \alpha)$ gives a conformal class of symplectic vector bundles over M . Much like almost complex structures in symplectic geometry, there is a notion of a complex structure on ξ that is compatible with the symplectic structure $d\alpha$.

Definition 31. A complex bundle structure J on ξ is a family J_p of complex structures on each hyperplane ξ_p that varies smoothly. We say J is ξ -**compatible** if for all 1-forms α defining ξ and for all $p \in M$, the map $J_p : \xi_p \rightarrow \xi_p$ satisfies

1. $d\alpha(Ju, Jv) = d\alpha(u, v)$ for all $u, v \in \xi_p$, and

2. $d\alpha(v, Jv) > 0$ for all non-zero $v \in \xi_p$.

Observe that this is analogous to an almost complex structure on TM being compatible with a symplectic form ω .

Contact manifolds also arise naturally as the boundaries of symplectic manifolds. Given any cooriented contact manifold $(M, \xi = \ker \alpha)$, we can cook up a symplectic manifold that has M as its boundary.

Definition 32. Given a cooriented contact manifold $(M, \xi = \ker \alpha)$, consider the projection $\pi : \mathbb{R} \times M \rightarrow M$. We can define a symplectic manifold $(\mathbb{R} \times M, \omega = d(e^t \pi^* \alpha))$ where t is the $\mathbb{R}$ -coordinate. The manifold $(\mathbb{R} \times M, \omega)$ is called the **symplectization** of (M, ξ) .

For convenience, we will write $\pi^* \alpha$ as α . Then expanded

$$\omega = e^t dt \wedge \alpha + e^t d\alpha.$$

We decompose $\mathbb{R} \times M$ into two manifolds with boundary: $(-\infty, 0] \times M$ and $[0, \infty) \times M$.

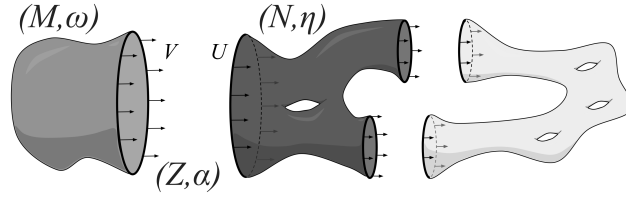
Consider what happens if we contract $V = \frac{\partial}{\partial t}$ into ω :

$$i_V \omega = e^t \alpha.$$

We have recovered a primitive for ω . Note that in $(-\infty, 0] \times M$, V is outward pointing at the boundary and in $[0, \infty) \times M$, V is inward pointing. In general, if you have this type of relationship between symplectic manifolds with contact boundary, you can glue them together.

Definition 33. A symplectic manifold (M, ω) with boundary is a **strong symplectic filling** of a contact manifold $(Z, \xi = \ker \alpha)$ if Z is the boundary of M and near Z there is a Liouville vector field V transverse to Z with $\xi = \ker i_V \omega$ and $\mathcal{L}_V \omega = \omega$. This filling is called **convex** if V points outward at the boundary Z and is called **concave** if V points inward at the boundary.

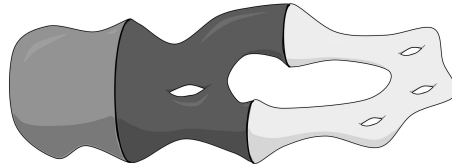
Figure 39. (M, ω) is a convex filling of (Z, α) . (N, η) is a concave filling of (Z, α) .



As described above, we can glue convex fillings to concave ones.

Theorem 23. *Given (M, ω) a convex strong symplectic filling of (Z, α) and (N, η) a concave strong symplectic filling of (Z, α) , there is a symplectic manifold $M \cup_Z N$, the union of M to N at Z , with symplectic structure β such that $\beta|_{M \setminus Z} \simeq \omega$ and $\beta|_{N \setminus Z} \simeq \eta$.*

Figure 40. Symplectic glueing of a convex to a concave strong symplectic filling.



Further reading: Hansjörg Geiges' book gives a good explanation of the basics of contact topology [15].

A.5 Cosymplectic geometry

In the same way contact structures arise as the boundaries of symplectic manifolds, cosymplectic manifolds are an odd-dimensional counterpart of symplectic manifolds. However, while contact manifolds are quite well studied, cosymplectic structures are relatively obscure.

Definition 34. *Given an odd dimensional manifold Z , a cosymplectic structure on Z is a pair (θ, β) where $\theta \in \Omega^1(Z)$ and $\beta \in \Omega^2(Z)$ are both closed and $\theta \wedge \beta^n \neq 0$.*

Unlike contact manifolds, which are maximally non-integrable, cosymplectic structures are as close as an odd-dimensional manifold can possibly be to being symplectic: In particular, the distribution $\ker \theta$ defines a codimension one foliation of Z and β defines a symplectic structure on each leaf.

Darboux's theorem tells us that cosymplectic structures all locally look the same - just as for contact and symplectic forms.

Theorem 24. (Darboux) *Let (Z, θ, β) be a dimension $2n + 1$ cosymplectic manifold. At every $p \in Z$, there exists local coordinates $(U, x_1, \dots, x_n, y_1, \dots, y_n, z)$ such that*

$$\theta|_U = dt \text{ and } \beta|_U = \sum_{i=1}^n dx_i \wedge dy_i.$$

Note that in the above coordinates the vector field $R = \frac{\partial}{\partial t}$ satisfies $\theta(R) = 1$ and $\beta(R, \cdot) = 0$. This is a local description of the Reeb vector field associated to a cosymplectic structure.

Definition 35. *Given a cosymplectic manifold (Z, θ, β) the **Reeb vector field** R is the unique vector field such that $\beta(R, \cdot) \equiv 0$ and $\theta(R) \equiv 1$.*