W-OPERATORS AND GENERATING FUNCTIONS OF HURWITZ NUMBERS

BY

HAO SUN

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2018

Urbana, Illinois

Doctoral Committee:

Professor Rinat Kedem, Chair
Professor Maarten Bergvelt, Director of Research
Professor Philippe Di Francesco
Professor Thomas Nevins

Abstract

This thesis is motivated by the W-operators introduced by Mironov et al. [18]. We prove that the W-operators are generalizations of the cut-and-join operator studied by Goulden and Jackson [11]. We give a new description of the structure of W-operators, using the combinations of symmetric groups. As an application, we prove new formulas about generating functions of connected Hurwitz numbers and give topological recursion formulas for d-Hurwitz numbers.

To my family

Acknowledgments

I owe an incalculable gratitude to my advisor Maarten Bergvelt. This thesis would not have been possible without his infinite support, guidance and encouragement. I am grateful to him for giving me research problems, for meeting with me regularly and teaching me most of the mathematics I learned during graduate school. Thank you Maarten, for reading each one of my papers carefully, making invaluable suggestions and making me go through ten more iterations. Thank you for encouraging me to talk to other mathematicians which helped me befriend many amazing people, and build collaborations.

I feel very fortunate to have amazing friends, near and far. Yiwang Chen, Weikang Fan, Xinghua Gao, Georgios Kydonakis, Yun Shi, Lutian Zhao, Mingyu Zhao, to name a few. Thank you for your friendship and making my time during graduate school enjoyable.

I am grateful to my professors, mentors and colleagues over the years, including Philippe Di Francesco, Sheldon Katz, Rinat Kedem and Thomas Nevins for teaching me various topics, introducing me to research mathematics, and discussing mathematics with me.

My parents Xiaohong Tong and Yunyuan Sun have been a constant source of support throughout my life, and none of this would be possible without their confidence in me.

Finally, I want to thank Beibei, a cute and greedy pug. He joined my family in 2002. Thanks for accompanying my parents and me for sixteen years.

Table of Contents

Chapter 1

Introduction

1.1 Background

1.1.1 W-Operator

The cut-and-join operator Δ was introduced by Goulden [11]. It is an infinite sum of differential operators in variables p_i, $i \geq 1$. It plays an important role in calculating the simple Hurwitz numbers [18].

In 2008, Mironov et al. [29][30] constructed W-operators $W([d])$, where d is a positive integer. They are differential operators acting on the space $\mathbb{C}[[X_{ij}]]_{i,j\geq 1}$ of formal series in variables X_{ij} $(i, j \geq 0)$, where X_{ij} are coordinate functions on the infinite matrix. A subring of $\mathbb{C}[[X_{ij}]]_{i,j\geq 1}$ is $\mathbb{C}[p_1, p_2, ...]$, where $p_k = Tr(X^k)$ and $X = (X_{ij})_{i,j\geq 1}$. A direct calculation shows that $W([2])$ is the cut-and-join operator Δ on the ring $\mathbb{C}[p_1, p_2, ...]$. Mironov et al. proved an equation for the generating function of disconnected Hurwitz numbers as an application of W-operators [29], [30]. We will briefly discuss their results in the next section about Hurwitz number.

1.1.2 Hurwitz Number

The Hurwitz enumeration problem aims at classifying all n-fold coverings $X \to S^2$ (or $X \to \mathbb{C}P^1$), i.e. with k branch points $\{z_1, ..., z_k\}$. One obtains many different problems by imposing conditions on the coverings. The number of solutions of the given type is called the Hurwitz number of that type. For instance, if we want X to be a connected space, we deal with the connected Hurwitz problem and connected Hurwitz numbers. Otherwise, if X can be disconnected, we deal with the disconnected Hurwitz problem and disconnected Hurwitz number. Generally, Hurwitz numbers are collected in generating functions and the relation between disconnected and connected generating functions of the Hurwitz numbers with a specific type is given by

$$e^{H^{con}} = H^{dis},$$

where H^{con} is the generating function for connected Hurwitz numbers and H^{dis} for the disconnected Hurwitz numbers for the specific type. In Chapter 3, Section 2 discusses the generating function of disconnected d-Hurwitz numbers (or d-Frobenius numbers) and the other sections deal with the generating functions of connected Hurwitz numbers.

Given such a covering $X \to S^2$, each branch point z_i corresponds to a permutation σ_i in S_n. Denote by λ_i the partition corresponding to σ_i. The number of all connected n-coverings with k ordered branch points $z_i, 1 \leq i \leq k$, each of which corresponds to a permutation of type $\lambda_i, 1 \leq i \leq k$, is finite. This number is denoted by $\mathrm{Cov}_n(\lambda_1, ..., \lambda_k)$. Equivalently, $\mathrm{Cov}_n(\lambda_1, ..., \lambda_k)$ is the number of k-tuples $(\sigma_1, ..., \sigma_k) \in S_n^k$ satisfying the following conditions [1], [26],

(1) σ_i is of type λ_i,

(2) $\sigma_1 ... \sigma_k = 1$,

(3) The group generated by the elements $\{\sigma_1, ..., \sigma_k\}$ is transitive on the set $\{1, ..., n\}$.

There are many different types of Hurwitz numbers well-studied by different mathematicians. In 1891, Hurwitz first studied the branched covers of the sphere by an n-sheeted Riemann surface [22]. Hurwitz proposed a formula for the *minimal simple Hurwitz number* (see Definition 3.3.1) without a proof. *Simple* means that all but one λ_i are transpositions and *minimal* means that the genus of the covering space X is zero. In 1997, Goulden and Jackson confirmed Hurwitz's formula by calculating this number in a combinatorics way [12]. In 2000, four mathematicians, Ekedahl, Lando, Shapiro and Vainshtein, proved the ELSV formula for the *simple Hurwitz number* using the Hodge integral [5],[6], which is a more general formula working for covering spaces with genus $g \geq 0$. This formula gives an interpretation of the Hurwitz number in algebraic geometry. When $g = 0$, ELSV formula gives the Hurwitz's formula for minimal simple Hurwitz number. At the same time, Okounkov and R. Pandharipande used the branch morphism for the stable maps [7] and localization formula [10] to give another proof of the ELSV formula [33]. All of the above results are about the connected Hurwitz number, which means that the covering space X is connected. Later on, Okounkov proved that the generating function of double Hurwitz number satisfies the Toda equation [31]. In 2005, Goulden, Jackson and Vakil studied the geometry of the double Hurwitz number [15]. The double Hurwitz numbers counts the coverings where all but two of the permutations σ_i are transpositions. In 2006, Okounkov and Pandharipande constructed the Gromov-Witten/Hurwitz correspondence [32], which is a bridge connecting the representation theory (shifted Boson-Fermion correspondence) with algebraic geometry (Gromov-Witten invariants). In their work, they focus on the disconnected Hurwitz number, because there is a natural correspondence between the disconnected Hurwitz number and the representation theory of permutation groups. Later on, Cavalieri, Johnson, Markwig studied the wall-crossings for double Hurwitz number in 2011 [2]. Shadrin, Spitz and Zvonkine studied the r-spin Hurwitz number. Roughly speaking, they wanted to use the r-spin Gromov-Witten invariants to calculate the r-spin Hurwitz number, which is still a conjecture [37]. In 2016, Harnad wrote an overview about the weighted Hurwitz number and used mKP and 2D Toda lattice τ-functions to study the generating functions of the weighted Hurwitz number [21].

Now we will give a brief review about the generating functions of the simple Hurwitz number.

Given α a partition of n, in this paper, the simple Hurwitz number is defined as

$$h_k(\alpha) = \mathrm{Cov}_n(1^{n-2}2, ..., 1^{n-2}2, \alpha).$$

It is the number of $(k+1)$-tuples $(\sigma_1, ..., \sigma_k, \sigma^{-1}) \in S_n^{k+1}$ satisfying the following conditions

(1) σ_i are transpositions (or of type $1^{n-2}2$), where $1 \leq i \leq k$, and σ^{-1} is of type α,

(2) $\sigma_1 ... \sigma_k = \sigma$,

(3) the group generated by $\{\sigma_1, ..., \sigma_k\}$ is transitive on the set $\{1, ..., n\}$.

Simple means that all but one permutation are transpositions. Compared with the classical simple Hurwitz number [11], [18], $\mathrm{Cov}_n(1^{n-2}2, ..., 1^{n-2}2, \alpha)$ counts all possible k-tuples $(\sigma_1, ..., \sigma_k, \sigma^{-1})$ where σ is of type α while the classical Hurwitz number counts all k-tuples $(\sigma_1, ..., \sigma_k, \sigma^{-1})$ with an arbitrary but fixed permutation σ of type α. In this paper, we call $h_k(\alpha) = \mathrm{Cov}_n(1^{n-2}2, ..., 1^{n-2}2, \alpha)$ the simple Hurwitz number.

The generating function H for simple Hurwitz numbers is

$$H(u, p) = H(u, p_1, p_2, ...) = \sum_{n \geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha \vdash n} h_k(\alpha) \frac{u^k}{k!} p_\alpha \ ,$$

where $p_\alpha = p_{\alpha_1}...p_{\alpha_l}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$. This generating function satisfies the following equation [14], [16]

$$\frac{\partial H}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}\left((i+j)p_ip_j\frac{\partial H}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 H}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\right). \tag{1.1.1}$$

We will discuss this equation in detail later and we prove a more general theorem (Theorem 3.4.3) from which Equation (1.1.1) follows (see Section 3.4).

Now we introduce another parameter y to the generating function $H(u,p)$ for the genus g. By Riemann-Hurwitz formula, the genus g is uniquely determined by the degree n, the number of transpositions k and the length $l(\alpha)$ of partition α,

$$g = \frac{k-n-l(\alpha)}{2} + 1.$$

Define $H(u,p)(y)$ as

$$H(u,p)(y) = \sum_{n\geq 1}\frac{1}{n!}\sum_{k=1}^{\infty}\sum_{\alpha\vdash n} h_k(\alpha)\frac{u^k}{k!}p_\alpha y^g \ .$$

We rewrite $H(u,p)(y)$ as

$$H(u,p)(y) = \sum_g H^g(u,p)y^g,$$

where $H^g(u,p) = \sum_{n\geq 1}\frac{1}{n!}\sum_{k=1}^{\infty}\sum_{\alpha\vdash n} h_k^g(\alpha)\frac{u^k}{k!}p_\alpha$. Here $h_k^g(\alpha)$ is the Hurwitz number of coverings with genus g, which is a finite number.

Clearly, $H(u,p)(y)$ also satisfies (1.1.1) and we take the coefficient of y^g on both sides of (1.1.1), we have

$$\frac{\partial H^g}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}(i+j)p_ip_j\frac{\partial H^g}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 H^{g-1}}{\partial p_i \partial p_j} + \sum_{g_1+g_2=g} ijp_{i+j}\frac{\partial H^{g_1}}{\partial p_i}\frac{\partial H^{g_2}}{\partial p_j}. \tag{1.1.2}$$

Goulden and Jackson use formula (1.1.1) and (1.1.2) to calculate the Hurwitz number $h_k^g(\alpha)$ for lower genus $g = 0, 1, 2$ [18], [13], [14], .

Equation (1.1.1) comes from the idea of cut-and-join operator Δ [11]

$$\Delta = \frac{1}{2}\sum_{i\geq 1}\sum_{j\geq 1}(ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} + (i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}),$$

which satisfies the following formula

$$\Phi(K_{(1^{n-2}2)}g) = \Delta\Phi(g). \tag{1.1.3}$$

where g is any element in the permutation group S_n, $K_{(1^{n-2}2)}$ is the central element of $\mathbb{C}S_n$ corresponding to the partition $(1^{n-2}2)$ and Φ is a linear map from the group ring $\mathbb{C}S_n$ to the polynomial ring $\mathbb{C}[p_1, p_2, ...]$. Precise definitions can be found in Section 2.9.

1.2 Outline of the Paper and Statement of results

1.2.1 W-Operator

In Section 2.1, we give the definition of the W-operator $W([n])$. Section 2.2 aims at introducing the main tools, permutations and quivers, to prove Theorem 2.3.1. In Section 2.3, we give the proof of the structure Theorem 2.3.1 of $W([n])$.

Theorem. ***2.3.1*** *$W([n])$ is a well-defined operator on $\mathbb{C}[p_1, p_2, ...]$ and it can be written as the sum of $n!$ summations, each of which corresponds to a unique quiver $\hat{Q}_\beta$ or equivalently a unique permutation $\beta \in S_n$.*

The summation corresponding to permutation β is denoted by FS_β and we call it the free summation. (See Definition 2.3.2.) The first part of the theorem, that $W([n])$ is a well-defined operator on $\mathbb{C}[p_1, p_2, ...]$, is a basic property and the second part shows that there is a relation between the permutation group and the W-operators. This theorem gives a new way of studying the W-operator $W([d])$. This theorem was first proved in paper [38].

In Section 2.4, we study some combinatorics properties of the summations. We define the degree of summations in $W([n])$. The degree of the summation FS_β is the sum of its polynomial part's degree and the order of its differential part. For example, consider the following summation

$$FS_{(1)} = \sum_{i=1} ip_i\frac{\partial}{\partial p_i},$$

where (1) is the unique permutation in S_1. The degree of this summation is 2. The degree of different summations in $W([n])$ can be different. Given a positive integer n, we find that the degree for free summations FS_α can only be $n+1, n-1, ...$, where $\alpha \in S_n$. (See Remark 2.4.3.) An *ordinary summation* (OS) is a summation with maximal degree $n+1$ in $W([n])$ and an (r, s)-type OS is an OS summation such that its polynomial part degree is r and its order of differential part is s, which means $r + s = n + 1$. At the end of Section 2.4, we ask the following question.

Question. ***2.4.4*** *Given a positive integer n, how many summations in $W([n])$ are of degree $n+1$? Equivalently, what is the number of permutations in S_n such that the degree of the corresponding summation is $n+1$?*

In Section 2.5 and 2.6., we prove that this problem is equivalent to a special perfect paring problem in combinatorics (Theorem 2.5.10 and 2.6.4). We figure out this number in Section 2.8 (Theorem 2.8.2).

Theorem. ***2.8.2*** *The number of $(r, n-r+1)$-type OS in $W([n])$ is the Narayana number:*

$$|OS(n,r)| = \frac{1}{n+1}\binom{n+1}{r}\binom{n-1}{r-1}.$$

The number of all summations with degree $n+1$ in $W([n])$ is the Catalan number

$$\sum_{r\geq 1}^{n} \frac{1}{n+1}\binom{n+1}{r}\binom{n-1}{r-1} = \frac{1}{n+1}\binom{2n}{n}.$$

In Section 2.7, we construct the dual non-crossing sequence. (See Construction 2.7.1.) We use this construction to prove the following corollary the number of (r, s)-type OS equals to the number of (s, r)-type OS.

Corollary. ***2.7.3*** *Given two positive integers n, r, we have*

$$|OS(n,r)| = |OS(n, n-r+1)|.$$

In Section 2.9, we prove the following formula about the W-operator $W([d])$, which plays an important rule in studying the generating function of Hurwitz number. It is a generalization of the cut-and-join formula (1.1.3).

Theorem. ***2.9.1*** *For any $g \in \mathbb{C}S_n$,*

$$\Phi(K_{(1^{n-d}d)}g) = W([d])\Phi(g), \tag{1.2.1}$$

where $K_{(1^{n-d}d)}$ is the central element in $\mathbb{C}S_n$ corresponding to the partition $(1^{n-d}d)$.

In the theorem, the map $\Phi : \mathbb{C}S_n \to \mathbb{C}[p_1, p_2, ...]$ is a linear map defined as follows

$$\Phi(g) = p_\lambda,$$

where g is a permutation in S_n of type $\lambda = (\lambda_1, ..., \lambda_m)$ and $p_\lambda = p_{\lambda_1}...p_{\lambda_m}$. This theorem was first proved in the paper [39].

In Section 2.10, we give another, equivalent, construction of $W([d])$ based on the formula we studied in Theorem 2.9.1.

1.2.2 Hurwitz Number

In section 3.1, we review some well-known theorems and properties about simple Hurwitz number based on the cut-and-join operator. In section 3.2, we define the d-Frobenius number $f_k^{[d]}(\alpha)$ and d-Hurwitz number $h_k^{[d]}(\alpha)$ (see Definition 3.2.6). The idea is to replace the transpositions used to define the simple Hurwitz numbers by d-cycles, $d \geq 2$. The d-Frobenius number is the disconnected d-Hurwitz number and the d-Hurwitz number is the connected Hurwitz number. We also define the generating function $F^{[d]}$ for the d-Frobenius number and give another proof of the following theorem, which is first proved in [29].

Theorem. ***3.2.7*** *$F^{[d]}$ is the unique formal series solution in u to the differential equation*

$$\frac{\partial F^{[d]}}{\partial u} = W([d])F^{[d]}$$

with initial condition

$$F^{[d]}(0, p) = e^{p_1}.$$

In the following sections, all Hurwitz numbers we consider are connected.

In section 3.3, we consider the minimal d-Hurwitz number $h^d(\alpha)$, which is the number of coverings X with genus zero, and its generating function $\widetilde{H}_d^{min}(z, u, p_1, p_2, ...)$. We give another proof of the following formula, as an application of the W-operator. The following theorem is first proved by Goulden and Jackson [16].

Theorem. ***3.3.6***

$$\frac{\partial \widetilde{H}_d^{min}}{\partial u} = \widetilde{\widetilde{HW}}([d])(\widetilde{H}_d^{min}),$$

where $\widetilde{\widetilde{HW}}([d])$ is defined in Construction 3.3.3.

In section 3.4, we go back to the d-Hurwitz number $h_k^{[d]}(\alpha)$ (see Definition 3.2.6), which is number of all possible coverings $f : X \to S^2$ such that the genus of X is greater or equal to zero and all but one permutations of the branch points are d-cycles. Its generating function is denoted by $H^{[d]}(u,p)$. We prove the following theorem about $H^{[d]}(u,p)$ [40].

Theorem. ***3.4.3***

$$\frac{\partial H^{[d]}}{\partial u} = \widehat{W}([d])H^{[d]}.$$

In the above theorem, $\widehat{W}([d])$ is a differential operator related to $W([d])$. In fact, we use the entire section (section 3.4) to construct this operator $\widehat{W}([d])$ from $W([d])$. When $d = 2$, Theorem 3.4.3 gives Equation (1.1.1) with $H^{[2]} = H$.

In section 3.5, we consider the topological recursion of the d-Hurwitz number. Similar to Formula (1.1.2) for the generating function of the simple Hurwitz number, we introduce another parameter y to $H^{[d]}(u,p)$ for the genus g and define the generating function $H^{[d],g}(u,p)$ for genus g coverings. We have the following topological recursion formula for d-Hurwitz number [40].

Corollary. ***3.5.2***

$$\frac{\partial H^{[d],g}}{\partial u} = \sum_{\beta \in S_d} \sum_{i=1}^{dD(FS_\beta)} \sum_{\substack{g_1+\ldots+g_i= \\ g-dD(FS_\beta)+i}} \widehat{FS}_{\beta,i}(H^{[d],g_1}, ..., H^{[d],g_i}).$$

where $\widehat{FS}_{\beta,i}$ is a "differential operator" in variables p_i defined in Section 3.4 and $dD(FS_\beta)$ is the differential degree of FS_β, $dP(FS_\beta)$ is the polynomial degree of FS_β introduced in Section 2.4.

Some examples of these operators $\widehat{FS}_{\beta,i}$ are as follows.

$$\widehat{FS}_{(12),1}(H^{[2],g}) = \frac{1}{2}\sum_{i,j\geq 1}(i+j)p_ip_j\frac{\partial H^{[2],g}}{\partial p_{i+j}},$$
$$\widehat{FS}_{(1)(2),1}(H^{[2],g-1}) = \frac{1}{2}\sum_{i,j\geq 1}ijp_{i+j}\frac{\partial^2 H^{[2],g-1}}{\partial p_i\partial p_j},$$
$$\widehat{FS}_{(1)(2),2}(H^{[2],g_1}, H^{[2],g_2}) = \frac{1}{2}\sum_{i,j\geq 1}ijp_{i+j}\frac{\partial H^{[2],g_1}}{\partial p_i}\frac{\partial H^{[2],g_2}}{\partial p_j}.$$

When $d = 2$, Corollary 3.5.2 gives Equation (1.1.2) ($H^{[2],g} = H^g$).

Chapter 2

W-Operator

W-operators were originally defined in Mironov, Morozov and Natanzon's paper [30]. The W-operator was used for studying the Hurwitz number (possibly disconnected) [30] [29]. This chapter is based on our papers about the W-operators $W([n])$ [38] [39]. We use permutation groups and quivers to study the structure of the W-operators $W([n])$.

2.1 Definition of W-Operator

The goal of this section is to give the definition of the W-operator $W([n])$.

Definition 2.1.1. *A variable matrix X is an infinite matrix with variable X_{ab} in the (a,b)-entry, i.e. $X := (X_{ab})_{a\geq 1, b\geq 1}$.*

Definition 2.1.2. *Given $k \geq 1$, p_k is the trace of X^k, i.e. $p_k = tr(X^k)$. p_k is a power series in $\mathbb{C}[[X_{ab}]]_{a,b\geq 1}$. $\mathbb{C}[p_1, p_2, ...]$ is a polynomial ring with infinitely many variables p_k.*

Remark 2.1.3. *If X is a special variable matrix with $X_{ab} = 0$, when $a \neq b$, then p_k is exactly the power sum symmetric function $\sum_{i=1}^{\infty} X_{ii}^k$.*

Definition 2.1.4. *The operator matrix D is the infinite matrix with D_{ab} in the (a,b)-entry, where $D_{ab} = \sum\limits_{c=1}^{\infty} X_{ac}\frac{\partial}{\partial X_{bc}}$.*

In the rest of the paper, we prefer to write $D_{ab} = X_{ac}\frac{\partial}{\partial X_{bc}}$ with the summation over c implied.

Lemma 2.1.5. *Let $F(p)$ be any polynomial (or formal power series) in $\mathbb{C}[p_1, p_2, ...]$ (or $\mathbb{C}[[p_1, p_2, ...]]$. We have*

$$D_{ab}F(p) = \sum_{k=1}^{\infty} k(X^k)_{ab}\frac{\partial F(p)}{\partial p_k}. \tag{2.1.1}$$

For $k \geq 0$, we have

$$D_{cd}(X^k)_{ab} = \sum_{j=0}^{k-1}(X^j)_{ad}(X^{k-j})_{cb}. \tag{2.1.2}$$

In particular, we have

$$\begin{aligned}\sum_{k_j=1}^{\infty} D_{a_{n+1}a_n}(X^{k_j})_{a_ia_j} &= \sum_{k_j=1}^{\infty}\sum_{k_n=0}^{k_j-1}(X^{k_n})_{a_ia_n}(X^{k_j-k_n})_{a_{n+1}a_j} \\ &= \sum_{k_j=1}^{\infty}\sum_{k_n=1}^{\infty}(X^{k_n})_{a_ia_n}(X^{k_j})_{a_{n+1}a_j}.\end{aligned}$$

Proof. We only give the proof for Equation (2.1.1). The proof of the other formulas are similar. Details can be found in [29], [38].

We want to calculate $D_{ab}p_k$, $k \geq 1$. Note that $\frac{\partial}{\partial X_{bc}}(X_{a_1a_2}X_{a_2a_3}...X_{a_na_1})$ is nontrivial if and only if there is some i such that $X_{a_ia_{i-1}} = X_{bc}$. In this case, $b = a_i, c = a_{i-1}$. We have

$$\begin{aligned}D_{ab}p_k &= X_{ac}\frac{\partial}{\partial X_{bc}}X^k \\ &= X_{ac}\frac{\partial}{\partial X_{bc}}\sum_{a_1,\ldots,a_k}\prod\left(X_{a_1a_k}X_{a_ka_{k-1}}...X_{a_2a_1}\right) \\ &= \sum_{a_1,\ldots,a_k}\sum_{i=0}^{k-1}X_{a_1a_k}...X_{a_{i+1}b}X_{ac}...X_{a_ka_1} \\ &= k(X^k)_{ab}.\end{aligned}$$

Equation 2.1.1 holds for all monomials p_k, $k \geq 1$. So, the equation also holds for all polynomial (or formal power series) $F(p)$. $\square$

Definition 2.1.6. *The normal ordered product of D_{ab} and D_{cd} is*

$$: D_{ab}D_{cd} := X_{ae_1}X_{ce_2}\frac{\partial}{\partial X_{be_1}}\frac{\partial}{\partial X_{de_2}}$$

(again with the summation over e_1, e_2 implied).

Lemma 2.1.7. *We consider D_{ab} acting on $p_i = tr(X^i)$. Then, we have*

$$
\begin{aligned}
: D_{a_{n+2}a_{n+1}} D_{a_{n+1}a_n} : = & \sum_{k,j\geq 1} ((k+j)(X^j)_{a_{n+1}a_{n+1}} (X^k)_{a_{n+2}a_n} \frac{\partial}{\partial p_{k+j}}) \\
& + \sum_{k,j\geq 1} (kj(X^k)_{a_{n+1}a_n} (X^j)_{a_{n+2}a_{n+1}} \frac{\partial^2}{\partial p_k \partial p_j}).
\end{aligned}
$$

Proof. See [29], [38]. □

Remark 2.1.8. *The formula of normal ordered product $: D_{a_{n+2}a_{n+1}} D_{a_{n+1}a_n} :$ in Lemma 2.1.7 comes from the calculation of $D_{a_{n+2}a_{n+1}} D_{a_{n+1}a_n}$. By calculation, we have*

$$
\begin{aligned}
D_{a_{n+2}a_{n+1}} D_{a_{n+1}a_n} = & \sum_{k\geq 1, j\geq 0} ((k+j)(X^j)_{a_{n+1}a_{n+1}} (X^k)_{a_{n+2}a_n} \frac{\partial}{\partial p_{k+j}}) \\
& + \sum_{k,j\geq 1} (kj(X^k)_{a_{n+1}a_n} (X^j)_{a_{n+2}a_{n+1}} \frac{\partial^2}{\partial p_k \partial p_j}).
\end{aligned}
$$

The subscript j in the first summation $\sum\limits_{k\geq 1, j\geq 0} (k+j)(X^j)_{a_{n+1}a_{n+1}} (X^k)_{a_{n+2}a_n} \frac{\partial}{\partial p_{k+j}})$ goes from 0 to infinity. If we calculate the normal ordered product $: D_{a_{n+2}a_{n+1}} D_{a_{n+1}a_n} :$, the "zero" term does not appear, which gives the formula in Lemma 2.1.7. In fact, the zero term comes from $[\frac{\partial}{\partial X_{a_{n+1}e_1}}, X_{a_{n+1}e_2}]$, since

$$
\begin{aligned}
& D_{a_{n+2}a_{n+1}} D_{a_{n+1}a_{n-1}} = \\
& : D_{a_{n+2}a_{n+1}} D_{a_{n+1}a_{n-1}} : + X_{a_{n+2}e_1} [\frac{\partial}{\partial X_{a_{n+1}e_1}}, X_{a_{n+1}e_2}] \frac{\partial}{\partial X_{a_n e_2}}.
\end{aligned}
$$

The reader can use the same method to calculate the normal product $: D_{a_{n+2}a_{n+1}} ... D_{a_2a_1} :$ from the product $D_{a_{n+2}a_{n+1}} ... D_{a_2a_1}$. Compared with the product $D_{a_{n+2}a_{n+1}} ... D_{a_2a_1}$, the normal product $: D_{a_{n+2}a_{n+1}} ... D_{a_2a_1} :$ has no "zero term". More precisely, all subscripts go from one to infinity.

Definition 2.1.9. *For any positive integer n, we define the W-operator $W([n])$ as*

$$
W([n]) := \frac{1}{n} : tr(D^n) := \frac{1}{n} \sum_{a_1,...,a_n \geq 1} : D_{a_1a_n} D_{a_na_{n-1}} ... D_{a_2a_1} : .
$$

Notation 2.1.10. *We prefer to us the following notation for the normal ordered product*

$$D_{(a_1,...,a_d)} :=: D_{a_1a_n}D_{a_na_{n-1}}...D_{a_2a_1} : .$$

The W-operators $W([n])$ have many interesting properties in combinatorics and representations of permutation groups. In this chapter, we will show how $W([n])$ relates to the permutation groups. In the next chapter, we will give some applications of $W([n])$ to the Hurwitz number.

2.2 Quiver and Permutation Group

In this section, we give some constructions on quivers and permutations (Constructions 2.2.5 and 2.2.8). These constructions are our main tools to prove the structure theorem (Theorem 2.3.1) in Section 2.3.

We begin with the quiver. A quiver is a directed graph. So, as usual, a **quiver** $Q = (V, A, s, t)$ is a quadruple, where V is the set of vertices, A is the set of arrows, s and t are two maps $A \to V$. If $a \in A$, $s(a)$ is the source of this arrow and $t(a)$ is the target. We assume V and A to be finite sets. If B is a subset of A, $V_B = \{s(a), t(a), a \in B\}$, then we call (V_B, B, s', t') the **subquiver** of Q, where $s' = s|_B$, $t' = t|_B$. A quiver $Q = (V, A, s, t)$ is **connected** if the underlying undirected graph of Q is connected. A connected quiver $Q = (V, A, s, t)$ is a **loop**, if for any vertex $v \in V$, there is a unique arrow $a \in A$ such that $s(a) = v$ and a unique arrow $b \in A$ such that $t(b) = v$. A **chain** is obtained by omitting a single arrow in a loop. $\mathbb{FQ}$ is the set of all quivers with finitely many vertices and finitely many arrows.

Definition 2.2.1. *Let* $\Phi_n : S_n \to \mathbb{FQ}$ *be the map such that* $\Phi_n(\alpha) = Q_\alpha$*, where*

$$Q_\alpha = \{V_\alpha = \{1, ..., n\}, A_\alpha = \{i \to \alpha(i), 1 \leq i \leq n\}, s_\alpha, t_\alpha\}.$$

Q_α *consists of disjoint loops which represent disjoint cycles of* α*.*

Since the source map and target map is well-defined for any arrow in any quiver, we will use the same symbols s, t for the source and targets maps in any quiver from now on.

Remark 2.2.2. *Every permutation can be written as the product of disjoint cycles. For example,* $(123)(45) \in S_6$*. But, in this paper, we prefer to write it as* $(123)(45)(6)$*, which includes the fixed integer* 6 *as "1-cycle".*

Given $\alpha \in S_n$, Q_α is the corresponding quiver. We define a new vertex set $\hat{V}_\alpha = \{1, ..., n, n+1\}$. There is a unique arrow a in Q_α such that $s(a) = 1$. We substitute this arrow by a new one $\hat{a}$, where $s(\hat{a}) = n+1$ and $t(\hat{a}) = t(a)$. Denote by $\hat{A}_\alpha$ the new set of arrows.

Definition 2.2.3. *Denote by $\hat{Q}_\alpha$ the new quiver,*

$$\hat{Q}_\alpha = (\hat{V}_\alpha, \hat{A}_\alpha, s, t).$$

Example 2.2.4. *Take $\alpha = (123) \in S_3$, then Q_α is*

$$Q_\alpha : \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd} .$$

$\hat{Q}_\alpha$ is

$$\hat{Q}_\alpha : \quad 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1 .$$

Clearly, Q_α is a loop and $\hat{Q}_\alpha$ is a chain.

In general, $\hat{Q}_\alpha$ consists of a chain and possibly a number of loops. Clearly, we can construct Q_α uniquely from $\hat{Q}_\alpha$.

We will consider how to construct $\hat{Q}_\alpha$ from $\hat{Q}_\beta$, where $\alpha \in S_n$ and $\beta \in S_{n+1}$. Given any permutation $\alpha \in S_n$ and $\beta \in S_{n+1}$, compared with $\hat{Q}_\beta$, $\hat{Q}_\alpha$ has two properties

- For any $\alpha \in S_n$, there is no arrow $a \in \hat{A}_\alpha$ such that $t(a) = n+1$.
- $n+2 \notin \hat{V}_\alpha$.

Hence, if we want to construct from $\hat{Q}_\beta$, $\beta \in S_{n+1}$, a quiver $\hat{Q}_\alpha$ for some $\alpha \in S_n$, we have to delete the vertex $n+2$ from $\hat{V}_\beta$ and delete one arrow from $\hat{A}_\beta$. Here is the construction.

Construction 2.2.5. *Given $\beta \in S_{n+1}$, we take the arrows $a, b \in \hat{A}_\beta$ such that*

$$s(a) = n+2, \quad t(b) = n+1.$$

We also assume that

$$s(b) = j, \quad t(a) = i.$$

- *If a and b are the same arrow which means $j = n+2, i = n+1$, we delete this arrow from $\hat{A}_\beta$ and delete $n+2$ from $\hat{V}_\beta$.*
- *If $a \neq b$, we delete these two arrows a, b from $\hat{A}_\beta$ and add a new arrow c such that $s(c) = j$, $t(c) = i$. Also, we delete the vertex $n+2$ from $\hat{V}_\beta$.*

Denote by $\hat{Q}'_\beta$ the new quiver we construct from $\hat{Q}_\beta$ in this way.

Example 2.2.6. *In the first example, we consider $\beta = (321)$. The quiver $\hat{Q}_\beta$ is*

$$\hat{Q}_\beta : \quad 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1 \ .$$

In this case, the arrow a, b are the same $4 \to 3$. Then, we delete this arrow and the vertex 4. We get the following quiver $\hat{Q}'_{(321)}$

$$\hat{Q}'_{(321)} : \quad 3 \longrightarrow 2 \longrightarrow 1 \ ,$$

which corresponds to the quiver $\hat{Q}_{(21)}$.

The second example is $\beta = (3)(21)$ with $\hat{Q}_{(3)(21)}$

$$\begin{tikzcd} \hat{Q}_{(3)(21)} : \quad 4 \arrow[r] & 2 \arrow[r] & 1 & & 3 \arrow[loop above] \ . \end{tikzcd}$$

Now a is $4 \to 2$ and b is $3 \to 3$. By Construction 2.2.5, we get the following quiver $\hat{Q}'_{(3)(21)}$

$$\hat{Q}'_{(3)(21)} : \quad 3 \longrightarrow 2 \longrightarrow 1 \ ,$$

which corresponds to the same quiver $\hat{Q}_{(21)}$.

The third example is $\beta = (3)(2)(1)$, the identity permutation in S_3. By the same argument, we find $\hat{Q}'_{(3)(2)(1)} = \hat{Q}_{(2)(1)}$.

The fact that for all $\beta \in S_3$, $\hat{Q}'_\beta$ in this example is of the form $\hat{Q}_\alpha$ for α some permutation in S_n is no accident. In fact, we have the following more general statement.

Lemma 2.2.7. *Given any permutation $\beta \in S_{n+1}$, there is a permutation $\alpha \in S_n$ such that $\hat{Q}_\alpha = \hat{Q}'_\beta$.*

Proof. Let $\beta = \beta_1\beta_2...\beta_k$ be the product of disjoint cycles of β. We prove this lemma in the following three cases.

Case 0 $n+1$ and 1 are in different cycles of β.

We assume that $\beta_1 = (i \quad ... \quad 1)$, the cycle contains 1 and $\beta_1(1) = i$, and $\beta_2 = (j \quad n+1 \quad ...)$, the cycle contains $n+1$ and $\beta_2(j) = n+1$. So, the loops in Q_β correspond to β_1 and β_2 are

$$\begin{tikzcd} Q_{\beta_1\beta_2}: & 1 \arrow[r] & i \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] & , & j \arrow[r] & n+1 \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] & . \end{tikzcd}$$

So, in $\hat{Q}_\beta$, they are

$$\begin{tikzcd} \hat{Q}_{\beta_1\beta_2}: & n+2 \arrow[r] & i \arrow[r] & \cdots \arrow[r] & 1 & , & j \arrow[r] & n+1 \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] & . \end{tikzcd}$$

By the Construction 2.2.5, we get

$$\hat{Q}'_{\beta_1\beta_2} = \hat{Q}_{\alpha_{12}}: \quad n+1 \longrightarrow \cdots \longrightarrow j \longrightarrow i \longrightarrow \cdots \longrightarrow 1\ .$$

Clearly, this corresponds to a cycle α_{12} in S_n by replacing $n+1$ by 1. Hence, $\alpha = \alpha_{12}\beta_3...\beta_k$ is the element in S_n satisfying $\hat{Q}_\alpha = \hat{Q}'_\beta$.

Case 1 $n+1$ and 1 are in the same cycle and $\beta(1) = n+1$.

Say $\beta = \beta_1\beta_2...\beta_k$, where $\beta_1 = (n+1 \ ... \ 1)$. The quiver in $\hat{Q}_\beta$ corresponds to β_1 is

$$\hat{Q}_{\beta_1}: \quad n+2 \longrightarrow n+1 \longrightarrow \cdots \longrightarrow 1\ .$$

By the Construction 2.2.5, we get

$$\hat{Q}_{\alpha_1}: \quad n+1 \longrightarrow \cdots \longrightarrow 1\ .$$

Clearly, this corresponds to a cycle α_1 in S_n by replacing $n+1$ by 1. So, $\alpha = \alpha_1\beta_2...\beta_k$ is the element in S_n satisfying $\hat{Q}_\alpha = \hat{Q}'_\beta$.

Case 2 $n+1$ and 1 are in the same cycle and $\beta(1) = i$, $i \neq n+1$.

Assume $\beta = \beta_1\beta_2 ... \beta_k$, where $\beta_1 = (i \quad ... \quad j \quad (n+1) \quad ... \quad 1)$ (j and i can be the same number). The quiver in $\hat{Q}_\beta$ corresponding to β_1 is

$$\hat{Q}_{\beta_1}: \quad n+2 \longrightarrow i \longrightarrow \cdots \longrightarrow j \longrightarrow n+1 \longrightarrow \cdots \longrightarrow 1 \ .$$

Hence, by the construction above, we get two cycles

$$\hat{Q}'_{\beta_1} = \hat{Q}_{\alpha_1\alpha_2}: \quad \begin{tikzcd} j \arrow[r] & i \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] \end{tikzcd} , \qquad n+1 \longrightarrow \cdots \longrightarrow 1 \ .$$

Clearly, replacing $n+1$ by 1, they correspond to two disjoint cycles α_1, α_2 in S_n. So, $\alpha = \alpha_1\alpha_2\beta_2 ... \beta_k$ is the element in S_n satisfying $\hat{Q}_\alpha = \hat{Q}'_\beta$.

In conclusion, for any $\beta \in S_{n+1}$, there is an element $\alpha \in S_n$ such that $\hat{Q}_\alpha = \hat{Q}'_\beta$. $\square$

Next we want to go in the opposite direction. For each $\alpha \in S_n$, we want to find all $\beta \in S_{n+1}$ such that $\hat{Q}'_\beta = \hat{Q}_\alpha$. Given a fixed permutation $\alpha \in S_n$, there turns out to be $n+1$ choices of β in S_{n+1}.

Given any quiver $\hat{Q}_\alpha$, $\alpha \in S_n$, if we want to construct a new quiver $\hat{Q}_\beta$ representing an element $\beta \in S_{n+1}$, we should add the vertex $n+2$ into $\hat{V}_\alpha$ and add arrows a_1, a_2 in $\hat{A}_\alpha$ such that

$$s(a_1) = n+2, \quad t(a_2) = n+1,$$

where a_1, a_2 can be the same arrow. Here is the construction.

Construction 2.2.8. *Given any $\alpha \in S_n$, we write α as the product of disjoint cycles $\alpha = \alpha_1\alpha_2 ... \alpha_k$. We assume $1 \in \alpha_1$. So, the corresponding subquiver for α_1 in $\hat{Q}_\alpha$ is the chain as following*

$$\hat{Q}_{\alpha_1}: \quad n+1 \longrightarrow \cdots \longrightarrow 1 \ .$$

- ***Case 0***

  We extend the quiver for α_1 directly

$$\hat{Q}_{\beta_1}: \quad n+2 \longrightarrow n+1 \longrightarrow \cdots \longrightarrow 1 \ .$$

Clearly, this subquiver represents a well-defined cycle β_1. In this way, we construct a permutation $\beta \in S_{n+1}$, where $\beta = \beta_1\alpha_2...\alpha_k$. In this case, a_1, a_2 are the same arrow

$$a_1 = a_2 : \qquad n+2 \longrightarrow n+1 \ .$$

Next we consider the general case. Roughly speaking, the idea is cutting an arrow in $\hat{Q}_\alpha$ and reconnect the chain and loops in $\hat{Q}_\alpha$. There are n choices of arrows in $\hat{Q}_\alpha$. We first choose an arbitrary arrow $a : i \to j$ in $\hat{Q}_\alpha$.

- ***Case 1,*** $a \in \hat{Q}_{\alpha_1}$

  In this case, $\hat{Q}_{\alpha_1}$ is

$$\hat{Q}_{\alpha_1} : \quad n+1 \longrightarrow \cdots \longrightarrow i \longrightarrow j \longrightarrow \cdots \longrightarrow 1 \ .$$

  First, cut the arrow $i \to j$, we get

$$n+1 \longrightarrow \cdots \longrightarrow i \ , \qquad j \longrightarrow \cdots \longrightarrow 1 \ .$$

  Then, we add the following two arrows

$$a_1 : n+2 \longrightarrow j \qquad a_2 : i \longrightarrow n+1 \ .$$

  Finally, we get the following quiver,

$$\hat{Q}_{\beta_1\beta_2} : \quad n+2 \longrightarrow j \longrightarrow \cdots \longrightarrow 1 \ , \qquad \begin{tikzcd} i \arrow[r] & n+1 \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] \end{tikzcd} .$$

  They represent two disjoint cycles in S_{n+1} by replacing $n+2$ by 1. Call them β_1 and β_2. So, $\beta = \beta_1\beta_2\alpha_2...\alpha_k$ is the permutation in S_{n+1} constructed by cutting the arrow a.

- ***Case 2,*** $a \notin \hat{Q}_{\alpha_1}$

  Without loss of generality, we can assume $a \in \hat{Q}_{\alpha_2}$. The corresponding quiver for α_1 and α_2 are

$$\hat{Q}_{\alpha_1\alpha_2} : \quad n+1 \longrightarrow \cdots \longrightarrow 1 \ , \qquad \begin{tikzcd} i \arrow[r] & j \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] \end{tikzcd} .$$

*Similar to **Case 1**, we cut the arrow $i \to j$ and we get*

$$n+1 \longrightarrow \cdots \longrightarrow 1 \ , \qquad j \longrightarrow \cdots \longrightarrow i \ .$$

Then, we add the following two arrows

$$a_1 : n+2 \longrightarrow j \qquad a_2 : i \longrightarrow n+1 \ .$$

Finally, we get the chain

$$\hat{Q}_{\beta_1} : \quad n+2 \longrightarrow j \longrightarrow \cdots \longrightarrow i \longrightarrow n+1 \longrightarrow \cdots \longrightarrow 1 \ .$$

It represents a cycle in S_{n+1} by replacing $n+2$ by 1 and denote by β_1. So, $\beta = \beta_1\alpha_3...\alpha_k$ is a permutation in S_{n+1}.

*In all cases, we have $\hat{Q}'_\beta = \hat{Q}_\alpha$. In the quiver $\hat{Q}_\alpha$, there are n arrows. Hence, we can construct n quivers or permutations from **Case 1,2**. In conclusion, there are $n+1$ choices of $\beta \in S_{n+1}$ such that $\hat{Q}'_\beta = \hat{Q}_\alpha$. It is easy to see that β constructed in this way are distinct.*

The three cases in Construction 2.2.8 corresponds to the cases in Lemma 2.2.7.

Construction 2.2.5, Lemma 2.2.7 and Construction 2.2.8 imply the following theorem.

Theorem 2.2.9. *For any $\alpha \in S_n$, we can construct $n+1$ distinct permutations β in S_{n+1} such that $\hat{Q}'_\beta = \hat{Q}_\alpha$. In fact, if we do it for all $\alpha \in S_n$, we will get $(n+1)!$ elements, which are exactly all permutations in the group S_{n+1}.*

Remark 2.2.10. *We can summarize the above construction as following. Given any positive integer n, there is a map*

$$\Psi_n : S_{n+1} \to S_n$$

such that $\Psi_n(\beta) = \alpha$, when $\hat{Q}'_\beta = \hat{Q}_\alpha$. Lemma 2.2.7 says that Ψ_n is well defined and the theorem, Theorem 2.2.9, says that the preimage $\Psi_n^{-1}(\alpha)$ consists of $n+1$ distinct elements β. So, Ψ_n is a $n+1$ to 1 map.

We define the following notation $[\alpha, j]$, which will be used in the next section.

Definition 2.2.11. *Let α be a permutation in S_n. Denote by $[\alpha, j]$ the permutation constructed from α, where j is an integer, $0 \leq j \leq n$. $[\alpha, 0]$ corresponds to the **Case 0** in*

*Construction 2.2.8 and, if $j \geq 1$, $[\alpha, j]$ corresponds to **Case 1,2** by cutting the arrow a such that $t(a) = j$.*

2.3 Structure of $W([n])$

In this section, we discuss the structure theorem 2.3.1 of $W([n])$. It is based on Construction 2.2.8, which produces from a quiver $\hat{Q}_\alpha(\alpha \in S_n)$ $n+1$ quivers $\hat{Q}_\beta(\beta \in S_{n+1})$. In fact, Construction 2.2.8 comes from the calculation of $W([n])$ (see Definition 2.1.9 and Lemma 2.1.7), which gives the basic idea about the structure theorem.

Theorem 2.3.1 (Structure Theorem)**.** *$W([n])$ is a well-defined operator on $\mathbb{C}[p_1, p_2, ...]$ and it can be written as the sum of $n!$ summations, each of which corresponds to a unique quiver $\hat{Q}_\beta$ or equivalently a unique permutation $\beta \in S_n$.*

Proof. We give some examples and ideas about the proof.

To calculate $W([n])$, we have to figure out the operator

$$: D_{a_1 a_n} D_{a_n a_{n-1}} ... D_{a_2 a_1} :$$

for any $a_i \geq 1$, $1 \leq i \leq n$. By Remark 2.1.8, it is equivalent for us to calculate the product $D_{a_1 a_n} D_{a_n a_{n-1}} ... D_{a_2 a_1}$. Since we want to use induction to calculate this product, we replace $D_{a_1 a_n}$ by $D_{a_{n+1} a_n}$. Now let's calculate the base step and we will explain how we construct the summation FS_β corresponding to $\hat{Q}_\beta$, where $\beta \in S_2$. Let $n = 1$, by Lemma 2.1.5, we have

$$D_{a_2 a_1} = \sum_{k_1=1}^{\infty} k_1 (X^{k_1})_{a_2 a_1} \frac{\partial}{\partial p_{k_1}}.$$

We associate this summation to the quiver

$$\hat{Q}_{(1)} : \quad 2 \longrightarrow 1 \ ,$$

which corresponds to the subscript of $(X^{k_1})_{a_2 a_1}$.

Now we calculate $D_{a_3 a_2} D_{a_2 a_1}$,

$$D_{a_3 a_2} D_{a_2 a_1} = \sum_{k_1=1}^{\infty} \left(D_{a_3 a_2} (k_1 (X^{k_1})_{a_2 a_1}) \right) \frac{\partial}{\partial p_{k_1}} + \sum_{k_1=1}^{\infty} k_1 (X^{k_1})_{a_2 a_1} \left(D_{a_3 a_2} \circ \frac{\partial}{\partial p_{k_1}} \right), \quad (2.3.1)$$

By Lemma 2.1.5, we have

$$
\begin{aligned}
D_{a_3a_2}D_{a_2a_1} = & \sum_{k_1\geq 1, k_2\geq 0}((k_1+k_2)(X^{k_2})_{a_2a_2}(X^{k_1})_{a_3a_1}\frac{\partial}{\partial p_{k_1+k_2}}) \\
& + \sum_{k_1,k_2\geq 1}(k_1k_2(X^{k_2})_{a_3a_2}(X^{k_1})_{a_2a_1})\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}).
\end{aligned}
$$

We associate the first summation to the quiver $\hat{Q}_{(1)(2)}$

$$
\begin{tikzcd}
\hat{Q}_{(1)(2)}: & 2 \arrow[loop above] & , & 3 \arrow[r] & 1 \ ,
\end{tikzcd}
$$

which comes from the subscripts of the polynomial part $(X^{k_2})_{a_2a_2}(X^{k_1})_{a_3a_1}$. Similarly, the second summation corresponds to the quiver $\hat{Q}_{(12)}$

$$
\hat{Q}_{(12)}: \quad 3 \longrightarrow 2 \longrightarrow 1 \ .
$$

We know that $D_{a_3a_2}$ acting on $(X^{k_1})_{a_2a_1}$ gives the first summation, which corresponds to **Case 1** of cutting the arrow $2 \to 1$ in $\hat{Q}_{(1)}$ in Construction 2.2.8. The same argument holds for the second summation, where $D_{a_3a_2}$ acts on $\frac{\partial}{\partial p_{k_1}}$ and it corresponds to the **Case 0** in Construction 2.2.8. By Lemma 2.1.7 and Remark 2.1.8, we know that $: D_{a_3a_2}D_{a_2a_1} :$ and $D_{a_3a_2}D_{a_2a_1}$ are almost the same and the only difference comes from the term with subscript $j = 0$ in the first summation. Hence, we can use quivers to describe the summations of $: D_{a_3a_2}D_{a_2a_1} :$ in the same way as $D_{a_3a_2}D_{a_2a_1}$. In conclusion, we find that $: D_{a_3a_2}D_{a_2a_1} :$ can be written as the sum of two summations, which correspond to quivers $\hat{Q}_\alpha$, $\alpha \in S_2$,

$$
\begin{aligned}
: D_{a_3a_2}D_{a_2a_1} : = & \sum_{k_1,k_2\geq 1}((k_1+k_2)(X^{k_2})_{a_2a_2}(X^{k_1})_{a_3a_1}\frac{\partial}{\partial p_{k+j}}) \\
& + \sum_{k_1,k_2\geq 1}(k_1k_2(X^{k_2})_{a_3a_2}(X^{k_1})_{a_2a_1})\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}).
\end{aligned}
$$

We use the notation FS'_α for the summation corresponding to $\alpha \in S_2$. We have

$$: D_{a_3a_2}D_{a_2a_1} := \sum_{\alpha \in S_2} FS'_\alpha.$$

Comparing with formula (2.3.1), the ranges of integers k_1, k_2 are the same in $: D_{a_3a_2}D_{a_2a_1} :$, i.e. from one to infinity (see Remark 2.1.8). Finally, let $a_3 = a_1$ and sum over a_1, a_2,

$$\begin{aligned}\sum_{a_1,a_2 \geq 1} : D_{a_1a_2}D_{a_2a_1} := &\sum_{a_1,a_2\geq 1}\sum_{k_1\geq 1, k_2 \geq 1}((k_1+k_2)(X^{k_2})_{a_2a_2}(X^{k_1})_{a_1a_1}\frac{\partial}{\partial p_{k_1+k_2}}) \\ &+ \sum_{a_1,a_2\geq 1}\sum_{k_1,k_2\geq 1}(k_1k_2(X^{k_2})_{a_1a_2}(X^{k_1})_{a_2a_1})\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}).\end{aligned}$$

We find that each summation can be written as some polynomial times a differential operator in variable p_i. We get the following formula

$$W([2]) = \frac{1}{2}\sum_{a_1,a_2\geq 1} : D_{a_1a_2}D_{a_2a_1} := \frac{1}{2}\sum_{k_1,k_2\geq 1}((k_1+k_2)p_{k_1}p_{k_2}\frac{\partial}{\partial p_{k_1+k_2}} + k_1k_2p_{k_1+k_2}\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}). \tag{2.3.2}$$

By induction on n, we can assume that $: D_{a_{n+1}a_n}...D_{a_2a_1} :$ can be written in the following way

$$: D_{a_{n+1}a_n}...D_{a_2a_1} := \sum_{\alpha \in S_n} FS'_\alpha,$$

where FS'_α is defined as

$$FS'_\alpha = \sum_{k_1,...,k_n\geq 1}\left(\prod_{r\in \hat{A}_\alpha}(X^{k_{t(r)}})_{a_{s(r)}a_{t(r)}}\right)DFS'_\alpha(k_1,...,k_n), \tag{2.3.3}$$

where $\hat{A}_\alpha$ is the set of arrows in $\hat{Q}_\alpha$, s is the source map, t is the target map (see Definition 2.2.3) and $DFS'_\alpha(k_1,...,k_n)$ is the differential part with constant coefficients depending on k_i, $1 \leq i \leq n$. The differential part $DFS'_\alpha(k_1,...,k_n)$ is uniquely determined by the permutation α and integers k_i, $1 \leq i \leq n$. Let's take $\alpha = (21)$ as an example, which is one of the summations in $: D_{a_3a_2}D_{a_2a_1} :$,

$$FS'_{(12)} = \sum_{k_1,k_2\geq 1}((X^{k_2})_{a_3a_2}(X^{k_1})_{a_2a_1})k_1k_2\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}).$$

$(X^{k_2})_{a_3a_2}(X^{k_1})_{a_2a_1}$ is the product of variables described by the arrows and the differential part is

$$DFS'_\alpha(k_1,...,k_n) = k_1k_2\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}.$$

Now we try to calculate the product $D_{a_{n+2}a_{n+1}}FS'_\alpha$. By the product rule, we have

$$\begin{aligned}&D_{a_{n+2}a_{n+1}}FS'_\alpha = \\ &\sum_{r'\in\hat{A}_\alpha}\sum_{k_1,\ldots,k_n\geq 1}(D_{a_{n+2}a_{n+1}}(X^{k_{t(r')}})_{a_{s(r')}a_{t(r')}})(\prod_{r\in\hat{A}_\alpha,r\neq r'}(X^{k_{t(r)}})_{a_{s(r)}a_{t(r)}})DFS'_\alpha(k_1,...,k_n) \\ &+\sum_{k_1,\ldots,k_n\geq 1}\left(\prod_{r\in\hat{A}_\alpha}(X^{k_{t(r)}})_{a_{s(r)}a_{t(r)}}\right)(D_{a_{n+2}a_{n+1}}\circ DFS'_\alpha(k_1,...,k_n)).\end{aligned}$$

We introduce another notation. If $j\neq 0$, there is a unique arrow $r'\in\hat{A}_\alpha$ such that $t(r')=j$. We define the operator $D_{a_{n+2}a_{n+1},j}$ acting on FS'_α as

$$\begin{aligned}&D_{a_{n+2}a_{n+1},j}FS'_\alpha := \\ &\sum_{k_1,\ldots,k_n\geq 1}(D_{a_{n+2}a_{n+1}}(X^{k_{t(r')}})_{a_{s(r')}a_{t(r')}})(\prod_{r\in\hat{A}_\alpha,r\neq r'}(X^{k_{t(r)}})_{a_{s(r)}a_{t(r)}})DFS'_\alpha(k_1,...,k_n).\end{aligned}$$

If $j=0$, we define $D_{a_{n+2}a_{n+1},0}FS'_\alpha$ as

$$\begin{aligned}&D_{a_{n+2}a_{n+1},0}FS'_\alpha := \\ &\sum_{k_1,\ldots,k_n\geq 1}\left(\prod_{r\in\hat{A}_\alpha}(X^{k_{t(r)}})_{a_{s(r)}a_{t(r)}}\right)(D_{a_{n+2}a_{n+1}}\circ DFS'_\alpha(k_1,...,k_n)).\end{aligned}$$

In terms of the new operators $D_{a_{n+2}a_{n+1},j}$, we have

$$D_{a_{n+2}a_{n+1}}FS'_\alpha = \sum_{j=0}^{n} D_{a_{n+2}a_{n+1},j}FS'_\alpha,$$

and

$$: D_{a_{n+2}a_{n+1}} FS'_\alpha := \sum_{j=0}^{n} : D_{a_{n+2}a_{n+1},j} FS'_\alpha : .$$

We can define $\widetilde{FS}_\beta$ inductively as

$$FS'_\beta =: D_{a_{n+2}a_{n+1},j} FS'_\alpha :, \tag{2.3.4}$$

where $\beta = [\alpha, j]$, $0 \le j \le n$. Recall the following two formulas in Lemma **??**

$$D_{a_{n+2}a_{n+1}} = \sum_{k=1}^{\infty} k(X^k)_{a_{n+2}a_{n+1}} \frac{\partial}{\partial p_k},$$

$$\sum_{k_j=1}^{\infty} D_{a_{n+2}a_{n+1}} (X^{k_j})_{a_i a_j} = \sum_{k_j=1}^{\infty} \sum_{k_n=0}^{\infty} (X^{k_n})_{a_i a_{n+1}} (X^{k_j})_{a_{n+2}a_j}.$$

With the above two formulas, we leave it for the reader to check that $\widetilde{FS}_\beta$ defined by Equation (2.3.4) can be written in the same form as $\widetilde{FS}_\alpha$ in Equation (2.3.3)

$$FS'_\beta = \sum_{k_1,\dots,k_{n+1}\ge 1} \left(\prod_{r\in \hat{A}_\beta} (X^{k_{t(r)}})_{a_{s(r)}a_{t(r)}} \right) DFS'_\beta(k_1, ..., k_n, k_{n+1}).$$

So, by induction, $: D_{a_{n+2}a_{n+1}} ... D_{a_2a_1} :$ can be written in the following way

$$: D_{a_{n+2}a_{n+1}} ... D_{a_2a_1} := \sum_{\beta\in S_n} FS'_\beta.$$

Finally, for each FS'_β, replace a_{n+2} by a_1 and take the sum over a_i, $1 \le i \le n+1$. Then, we will get a summation in variables p_i corresponding to FS'_β. $W([n+1])$ can be written as the sum of $(n+1)!$ summations, each of which corresponds to a unique permutation in S_{n+1}. □

Definition 2.3.2. *For any permutation $\beta \in S_{n+1}$, denote by FS_β the summation corresponding to FS'_β (or β) in the decomposition of $W([n+1])$.*

Remark 2.3.3. *Recall that p_k is defined as the trace of X^k. If we define the degree of p_k to be one, we claim that the degree of the polynomial part of FS_α is exactly the number of* *disjoint cycles of α. We will explain it in the rest of this remark.*

Given $\alpha \in S_n$, let $\alpha = \alpha_1 ... \alpha_l$ be the product of disjoint cycles. If we fix integers k_i, $1 \leq i \leq n$, the polynomial part of $\widetilde{FS}_\alpha$ with respect to k_i is

$$\left(\prod_{r \in \hat{A}_\alpha} (X^{k_{t(r)}})_{a_{s(r)} a_{t(r)}} \right).$$

Now replacing a_{n+1} by a_1 and taking the sum over $a_1, ..., a_n$, we have

$$\begin{aligned} \sum_{a_1,...,a_n \geq 1} \left(\prod_{r \in A_\alpha} (X^{k_{t(r)}})_{a_{s(r)} a_{t(r)}} \right) &= \prod_{i=1}^{l} \sum_{a_1,...,a_n \geq 1} \left(\prod_{r \in A_{\alpha_i}} (X^{k_{t(r)}})_{a_{s(r)} a_{t(r)}} \right) \\ &= \prod_{i=1}^{l} p_{\sum_{r \in A_{\alpha_i}} k_{t(r)}}. \end{aligned}$$

Hence, the degree of the polynomial part of FS'_α is the number of disjoint cycles of α.

Let's take $\alpha = (21)$ as an example.

$$FS_\alpha = \sum_{k_1, k_2 \geq 1} p_{k_1 + k_2} \left(k_1 k_2 \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}} \right).$$

The degree of the polynomial part is one, which is the number of disjoint cycles of α.

2.4 Degree of Summations FS_α

Consider the polynomial ring $\mathbb{C}[p_1, p_2, ...]$. In this section, we define the degree of each variable p_i to be one. In the previous section, we have shown that $W([n])$ can be written as the sum of $n!$ summations. Each summation is a formal differential operator. For example, the summation $FS_{(321)}$ in $W([3])$

$$\frac{1}{3} \sum_{i_1, i_2, i_3 \geq 1} (i_1 i_2 i_3 p_{i_1 + i_2 + i_3} \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}})$$

is an infinite sum of differential operators $i_1 i_2 i_3 p_{i_1+i_2+i_3} \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}}$, which has coefficients $i_1 i_2 i_3$, polynomial part $p_{i_1+i_2+i_3}$ and differential part $\frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}}$. Now we want to define the summation's degree, which depends on its polynomial part and differential part.

Definition 2.4.1. *Given any summation FS_α of $W([n])$, $dP(FS_\alpha)$ is the degree of its polynomial part and $dD(FS_\alpha)$ is the order of its derivative part. The degree of the summation FS_α is $d(FS_\alpha) = dP(FS_\alpha) + dD(FS_\alpha)$.*

Let's consider the example $W([3])$. There are 6 summations in $W([3])$,

$$\begin{aligned}
W([3]) = \frac{1}{3} \sum_{i_1,i_2,i_3 \geq 1} (i_1 i_2 i_3 p_{i_1+i_2+i_3} \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}} + & \qquad FS_{(321)} \\
+ i_1(i_2+i_3) p_{i_1+i_3} p_{i_2} \frac{\partial^2}{\partial p_{i_1} \partial p_{i_2+i_3}} + & \qquad FS_{(13)(2)} \\
+ i_2(i_1+i_3) p_{i_1+i_2} p_{i_3} \frac{\partial^2}{\partial p_{i_2} \partial p_{i_1+i_3}} + & \qquad FS_{(12)(3)} \\
+ i_3(i_1+i_2) p_{i_3+i_2} p_{i_1} \frac{\partial^2}{\partial p_{i_3} \partial p_{i_1+i_2}} + & \qquad FS_{(1)(23)} \\
+ (i_1+i_2+i_3) p_{i_1} p_{i_2} p_{i_3} \frac{\partial}{\partial p_{i_1+i_2+i_3}} + & \qquad FS_{(1)(2)(3)} \\
+ (i_1+i_2+i_3) p_{i_1+i_2+i_3} \frac{\partial}{\partial p_{i_1+i_2+i_3}}) & \qquad FS_{(123)} \ .
\end{aligned}$$

Five of them have degree 4 and the summation $FS_{(123)}$ is of degree 2. If we go back to $W([2])$ (Equation (2.3.2)), all summations are of degree 3. We know that the polynomial degree of FS_α is the number of disjoint cycles of α by Remark 2.3.3.

The following lemma describes the relation between the degree of FS_β and FS_α, when $\beta = [\alpha, i]$ (see Definition 2.2.11).

Lemma 2.4.2. *For any $\alpha \in S_n$,*

1. *If $[\beta] = [\alpha, 0]$, we have*

$$dP(FS_\beta) = dP(FS_\alpha), \quad dD(FS_\beta) = dD(FS_\alpha) + 1, \quad d(FS_\beta) = d(FS_\alpha) + 1.$$

2. *If $[\beta] = [\alpha, j]$ and j is a vertex in the chain of $\hat{Q}_\alpha$, then, we have*

$$dP(FS_\beta) = dP(FS_\alpha) + 1, \quad dD(FS_\beta) = dD(FS_\alpha), \quad d(FS_\beta) = d(FS_\alpha) + 1.$$

3. *If $[\beta] = [\alpha, j]$ and j is not a vertex in the chain of $\hat{Q}_\alpha$, we have*

$$dP(FS_\beta) = dP(FS_\alpha) - 1, \quad dD(FS_\beta) = dD(FS_\alpha), \quad d(FS_\beta) = d(FS_\alpha) - 1.$$

Proof. By the proof of Theorem 2.3.1, Equation (2.1.1) in Lemma 2.1.5 shows that the differential degree of FS'_β increases by one when $[\beta] = [\alpha, 0]$. The third formula in Lemma 2.1.5 implies when $j \neq 0$, the operator $D_{a_{n+1}a_n}$ fixes the differential degree. Now we consider the polynomial degree. If $[\beta] = [\alpha, 0]$, **Case 0** in Construction 2.2.8 tells us that the number of disjoint cycles of β is the same as that for α. By the proof of Theorem 2.3.1 and Remark 2.3.3, the number of disjoint cycles of α is the polynomial degree of FS_α. Hence, in **Case 0**, we have

$$dP(FS_\beta) = dP(FS_\alpha).$$

Case 1 in Construction 2.2.8 corresponds to $[\beta] = [\alpha, j]$, where j is a vertex in the chain of $\hat{Q}_\alpha$. β has one more disjoint cycle than α. So, we have

$$dP(FS_\beta) = dP(FS_\alpha) + 1.$$

Similarly, in **Case 2** in Construction 2.2.8, α has one more disjoint cycle than β. We have

$$dP(FS_\beta) = dP(FS_\alpha) - 1.$$

□

Remark 2.4.3. *From the above lemma, the highest degree of summations in $W([n])$ is $n+1$ and the other possible degrees are $n-1, n-3, \ldots$.*

Now we have the following question about the number of summations with highest degree.

Question 2.4.4. *Given a positive integer n, how many summations in $W([n])$ are of degree $n+1$? More precisely, what is the number of permutations in S_n such that the degree of the corresponding summation is $n+1$?*

In Section 2.5 and 2.6., we prove that this problem is equivalent to a special "perfect paring" problem in combinatorics (Theorem 2.5.10 and 2.6.4). We figure out this number in Section 2.8 (Theorem 2.8.2).

2.5 Ordinary Summations

In this section, we discuss the ordinary summations and prove a necessary and sufficient condition for the ordinary summations.

Definition 2.5.1 (Ordinary Summation). *Given $\alpha \in S_n$, FS_α is an ordinary summation (OS) of type (r, s), if $dP(FS_\alpha) = r$, $dD(FS_\alpha) = s$ and $r + s = n + 1$.*

Example 2.5.2.

$$FS_{(1)} = \sum_{k_1 \geq 1} p_{k_1} \frac{\partial}{\partial p_{k_1}}.$$

$FS_{(1)}$ is an OS of type $(1, 1)$.

$$FS_{(1)(2)} = \frac{1}{2} \sum_{k_1, k_2 \geq 1}^{\infty} p_{k_1} p_{k_2} \frac{\partial}{\partial p_{k_1+k_2}}.$$

So, $FS_{(1)(2)}$ is an OS of type $(2, 1)$.

Next we want to find a necessary and sufficient condition $(*)$ on permutations $\alpha \in S_n$ such that FS_α is an ordinary summation if and only if α satisfies the condition $(*)$.

Definition 2.5.3 (Condition $(*_1)$). *Let α be a permutation in S_n. Let $\alpha = \alpha_1 ... \alpha_r$ be the decomposition of α into disjoint cycles. We say α satisfies the condition $(*_1)$, if for each arrow a in the chain of $\hat{Q}_\alpha$, we have $t(a) < s(a)$, and there is only one arrow b in each loop of $\hat{Q}_\alpha$ such that $s(b) < t(b)$.*

Remark 2.5.4. *The above condition for $\hat{Q}_\alpha$ is equivalent to the condition for Q_α that there is only one arrow b in each loop of Q_α such that $s(b) < t(b)$. We use the definition in terms of $\hat{Q}_\alpha$ in the proof of Lemma 2.5.5, 2.5.8 and Theorem 2.5.10. We use the definition in terms of Q_α in the proof of Theorem 2.6.4.*

Lemma 2.5.5. *Given $\alpha \in S_n$, if FS_α is an OS, then α satisfies the condition $(*_1)$.*

Proof. We prove this lemma by induction on n. For the base step $n = 1$, $\hat{Q}_{(1)}$ is the only quiver and $FS_{(1)}$ is an OS. There is only one arrow $2 \to 1$ in the quiver $\hat{Q}_{(1)}$. Clearly, (1) satisfies the condition $(*_1)$.

Now we assume that for all $\alpha \in S_{k-1}$ if FS_α is an OS, then α satisfies $(*_1)$. Let $\beta \in S_k$ and assume $[\beta] = [\alpha, j]$ in the notation of Definition 2.2.11. FS_β is an OS implies that FS_α is also an OS. Indeed if FS_α is not an OS, then $d(FS_\alpha) < k$. By Lemma 2.4.2, $d(FS_\beta) < k+1$, contradicting the fact that β is an OS.

Let $\alpha = \alpha_1 ... \alpha_r$ be the decomposition of α into disjoint cycles with $1 \in \alpha_1$. By Lemma 2.4.2, j could be zero or the target of some arrow in the chain of $\hat{Q}_\alpha$. Now we discuss these two cases.

1. If $j = 0$, then $\beta = \beta_1\alpha_2...\alpha_r$, where $\hat{Q}_{\beta_1}$ is constructed from $\hat{Q}_{\alpha_1}$ by adding another arrow $k+1 \to k$. By induction, the statement is true.

2. If $j \neq 1$, then β is constructed from α by cutting the arrow $a : i \to j$, which is an arrow a in the chain of $\hat{Q}_\alpha$. We use the same notation as **Case 1** in Construction 2.2.8. Let $\beta = \beta_1\beta_2\alpha_2...\alpha_r$. The quiver $\hat{Q}_{\beta_1}$ of the cycle β_1 is

$$\hat{Q}_{\beta_1} : \quad k+2 \longrightarrow j \longrightarrow \cdots \longrightarrow 1 \ ,$$

where $j \to ... \to 1$ is a subquiver of α_1. Hence, all arrows in this chain satisfy that the source is larger than the target. The quiver $\hat{Q}_{\beta_2}$ is

$$\hat{Q}_{\beta_2} : \quad \begin{tikzcd} i \arrow[r] & k+1 \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] \end{tikzcd} \ ,$$

where $k+1 \to ... \to i$ is a subquiver of α_1 by construction. So the only arrow a in the cycle $\hat{Q}_{\beta_2}$ satisfying $s(a) < t(a)$ is $i \to k+1$. Hence, the statement is true for $n = k$.

□

The following condition is another condition of permutation α such that FS_α is an ordinary summation. Theorem 2.5.10 proves that FS_α is an OS if and only if α satisfies the following condition and the condition $(*_1)$.

Definition 2.5.6 (Condition $(*_2)$)**.** *α is a permutation in S_n. Let $\alpha = \alpha_1...\alpha_r$ be the decomposition of α into disjoint cycles. We say α satisfies the condition $(*_2)$, if any two distinct cycles α_i, α_j satisfy at least one of the following conditions,*

1. *pick an arbitrary element m in α_i, then we have $m > n$ for any n in α_j or $m < n$ for any n in α_j;*

2. *pick an arbitrary element m in α_j, then we have $m > n$ for any n in α_i or $m < n$ for any n in α_i.*

Remark 2.5.7. *This remark will give a brief explanation about the condition $(*_2)$. The two conditions in Definition 2.5.6 mean that any two cycles are "ordered" or one is "contained" in the other one. If the pair of cycles satisfies both these two conditions, then they are "ordered". If the pair only satisfies one of them, then one is contained in the other one.*

For instance, consider the following examples,

$$\tau_1 = (123)(45), \quad \tau_2 = (125)(34), \quad \tau_3 = (124)(35).$$

The two disjoint cycles in τ_1 satisfies both these two conditions. They are "ordered", since any integer in the second cycle is larger than any integer in the first cycle. The disjoint cycle $\alpha_i = (34)$ in τ_2 is contained in $\alpha_j = (125)$. They satisfy the second condition in Definition 2.5.6. We prefer to write it as

$$(\quad 5 \quad (\quad 4 \quad 3 \quad) \quad 2 \quad 1 \quad).$$

We will explain this notation in Construction 2.6.1. The last example τ_3 does not satisfy the condition $(*_2)$.

Lemma 2.5.8. *If FS_α is an OS, then α satisfies the condition $(*_2)$.*

Proof. Similar to the proof of Lemma 2.5.5, we prove this lemma by induction on the permutation group S_n. When $n = 1$, it is clear that the unique permutation (1) in S_1 satisfies the condition $(*_2)$.

Next, we assume that for all $\alpha \in S_{k-1}$ if FS_α is an OS, then α satisfies $(*_2)$. Let $\beta \in S_n$ and assume $[\beta] = [\alpha, j]$ in the notation of Definition 2.2.11. Let $\alpha = \alpha_1 ... \alpha_r$ be the decomposition of α into disjoint cycles. We will prove that if FS_β is an OS, then β satisfies the condition $(*_2)$. Before we give the proof, recall the property that if $[\beta] = [\alpha, j]$ and FS_β is an OS, then FS_α is also an OS by the proof of Lemma 2.5.5.

If $j = 0$, then $\beta = \beta_1\alpha_2...\alpha_r$, where $\hat{Q}_{\beta_1}$ is constructed from $\hat{Q}_{\alpha_1}$ by adding another arrow $k + 1 \to k$. In other words, we put another element k into the cycle α_1 (see Construction 2.2.8). By assumption that any two disjoint cycles of $\alpha \in S_{k-1}$ satisfy at least one of the conditions, we only have to check whether the pair (β_1, α_i) satisfies the condition $(*_2)$, $2 \le i \le r$. Since α_1 contains the smallest element 1, so if α_1 and α_i are "ordered", then any element in α_1 is smaller than any element in α_i. Since k is the largest element, so the statement is true for β_1 and α_i. Now we consider that α_1 and α_i are not "ordered". Since 1 is contained in α_1, so α_i is "contained" in α_1. Clearly, it still holds for β_1 and α_i. So, (β_1, α_i) satisfies the condition $(*_2)$.

Now let's consider the case that β is constructed from α by cutting the arrow $a : i \to j$ lying in the chain of α. We use the same notation as **Case 1** in Construction 2.2.8. Let $\beta = \beta_1\beta_2\alpha_2...\alpha_r$. So, we have to check whether the following three types of pairs satisfy the condition:

$$(\beta_1, \beta_2), \quad (\beta_1, \alpha_i), \quad (\beta_2, \alpha_i),$$

where $2 \leq i \leq r$.

- (β_1, β_2)

  Since FS_α is OS, so all arrows a in $\hat{Q}_{\alpha_1}$ satisfy $t(a) < s(a)$ by Lemma 2.5.5. Hence, when cutting the arrow $i \to j$, any elements in β_2 is larger than any elements in β_1. It is true in this case.

- (β_1, α_i)

  By induction, we know that the lemma is true for (α_1, α_i), $2 \leq i \leq r$. Since the elements of β_1 is a subset of the elements of α_1, so it is true for (β_1, α_i), $2 \leq i \leq r$.

- (β_2, α_i)

  If β_2 is a single disjoint "one cycle" (k), the statement is true. If $\beta_2 \neq (k)$, assume the largest element in β_2 except k is ϕ. If ϕ is smaller than the smallest element in α_i, then any element u except k in β_2 u is smaller than any element in α_i. Also, k is larger than any element in α_i. Hence, the statement is true in this case. Now let's consider the case that ϕ is larger than the smallest element in α_i. By construction, ϕ is an element in α_1, which contains 1. Hence, ϕ is larger than any elements in α_i by induction. Similarly, any other elements in β_2 is larger or smaller to all elements in α_i by induction. So, the statement is true.

In conclusion, the statement is true when $n = k$. □

Definition 2.5.9 (Condition $(*)$ and Non-crossing Permutation)**.** *Given $\alpha \in S_n$, we say that α satisfies the condition $(*)$ if α satisfies the conditions $(*_1)$ and $(*_2)$. We call such a permutation α non-crossing permutation in this paper.*

Theorem 2.5.10. *For $\alpha \in S_n$, FS_α is OS if and only if α satisfies the condition $(*)$.*

Proof. The "only if" part is exactly Lemma 2.5.5 and 2.5.8. So, we only have to prove the "if" part. We prove this theorem by induction on n.

When $n = 1$, it is easy to prove, since (1) is the only permutation. We assume that if $\alpha \in S_{k-1}$ satisfies the condition $(*)$, then FS_α is an OS. We will prove that if $\beta \in S_k$ satisfies the condition $(*)$, then FS_β is an OS. Assume $[\beta] = [\alpha, j]$ for some α in S_{n-1} and some nonnegative integer j. We claim that j is 0 or in the chain of $\hat{Q}_\alpha$ (**Claim 1**). Also, we claim that α also satisfies the condition $(*)$ (**Claim 2**). Since α satisfies the condition $(*)$, FS_α is an OS by induction. By **Claim 1**, j is 0 or in the chain of $\hat{Q}_\alpha$. By Construction 2.2.8 and Lemma 2.4.2, we know FS_β is an OS. Now we are going to prove these two claims.

Proof of **Claim 1**:

If not, β is constructed from α by cutting arrow $a : i \to j$ which is not the chain of $\hat{Q}_\alpha$. Hence, by **Case 2** in Construction 2.2.8, we will get a long chain

$$k+1 \longrightarrow j \longrightarrow \cdots \longrightarrow i \longrightarrow k \longrightarrow \cdots \longrightarrow 1 .$$

In this chain, we have $i < k$, which contradicts with our assumptions that β satisfies the condition (*). So, j must be in the chain of $\hat{Q}_\alpha$ or $j = 0$.

Proof of **Claim 2**:

By **Claim 1**, we know that $j = 0$ or j is in the chain of $\hat{Q}_\alpha$. If $j = 0$, it is easy to prove α satisfies the condition $(*)$. We leave it for the reader. Now we assume that j is in the chain of $\hat{Q}_\alpha$. With the same notation as in Construction 2.2.8, let $\beta = \beta_1\beta_2\alpha_2...\alpha_r$ with $1 \in \beta_1$.

First, we have to check α satisfies the condition $(*_1)$. By the assumption of β, there is exactly one arrow a in the quiver of α_i such that $t(a) > s(a)$, where $2 \leq i \leq r$. So, we have to show all arrows a in the chain of $\hat{Q}_\alpha$ satisfying $t(a) < s(a)$. We assume that there is an arrow a in the chain of $\hat{Q}_\alpha$ such that $s(a) < t(a)$. If $t(a) \neq j$, then this arrow will be in either β_1 or β_2, which contradicts with the assumption of β. If $t(a) = j$, then we get β_1

$$\hat{Q}_{\beta_1} : \quad k+1 \longrightarrow j \longrightarrow \cdots \longrightarrow 1 .$$

and β_2

$$\begin{tikzcd} \hat{Q}_{\beta_2} : & i \arrow[r] & k \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] \end{tikzcd} .$$

Since $k > j > i$, so (β_1, β_2) does not satisfy the second condition in the condition (*). Hence, we have $t(a) < s(a)$ for each arrow a in the chain of $\hat{Q}_\alpha$ and there is exactly one arrow b in each loop of $\hat{Q}_\alpha$ such that $s(b) < t(b)$.

Now, we are going to prove that α satisfies the condition $(*_2)$. The problem pair is

(α_1, α_i), $2 \le i \le r$. By assumption, β_1 contains the smallest element 1 and β_2 contains the element k. Hence, by Construction 2.2.8 and Lemma 2.5.8, we know that any element in β_1 is smaller than any element in β_2. Since β satisfies condition $(*)$, so for any cycle α_i, $2 \le i \le r$, there are three possible cases

- α_i is "contained" in β_1, i.e. if we pick an arbitrary element m in β_1, then we have $m > n$ for any n in α_i or $m < n$ for any n in α_i;
- α_i is "contained" in β_2, i.e. if we pick an arbitrary element m in β_2, then we have $m > n$ for any n in α_i or $m < n$ for any n in α_i;
- α_i is between β_1 and β_2, i.e. any element in α_i is larger than any element in β_1 and smaller than any element in β_2.

In the first case, if α_i is "contained" in β_1, then any element in β_2 is larger than any element in α_i, because the element in β_2 is always larger than the element in β_1. By the construction of α_1, the condition is true for (α_1, α_i). The same argument holds for the second case. For the third case, β_1 and β_2 are constructed from α_1 by cutting the arrow with target j and add another element k. Hence, α_i is "contained" in α_1. Hence, α satisfies the condition (2) of (*). □

2.6 Non-crossing Sequence

In this section, we prove that there is a bijective map between non-crossing sequences and ordinary summations.

In the previous section, we define the non-crossing permutation (Definition 2.5.9). The condition $(*)$ corresponds to the non-crossing partition [36]. In [28], Mingo and Nica define the non-crosing permutation. The non-crossing permutation in this paper is a little different from theirs but with similar idea. The following construction about the non-crossing sequence and Theorem 2.5.10 gave the idea for the definition of a non-crossing permutation (Definition 2.5.9) in this paper.

Construction 2.6.1 (Non-crossing Sequence)**.** *Given a positive integer n, we fix a standard sequence of n integers as follows*

$$n \quad n-1 \quad ... \quad 2 \quad 1 \ .$$

We insert r pairs of brackets into this sequence satisfying the following condition $(**)$

- *any integer is contained in at least one pair of brackets and any pair of brackets contains at least one integer,*
- *there can be at most one left bracket and at most one right bracket between two successive integers.*

We call the standard sequence with brackets satisfying $(**)$ *a non-crossing sequence.*

Now we use some examples to explain these conditions.

Example 2.6.2. *We consider the following three examples*

$$\begin{aligned}&(4) \quad 3 \quad 2 \quad (1),\\ &(4 \quad 3 \quad (2)) \quad (1),\\ &(4 \quad (3) \quad 2) \quad (1).\end{aligned}$$

The first one does not satisfy the first condition, since 3 *and* 2 *are not contained in any pair of brackets. The second one does not satisfy the second condition, since there are two right brackets between* 2 *and* 1*. The third one satisfies* $(**)$.

By the second point of the condition $(**)$, we can only have at most one left (right) between two successive integers. So, we use the following notation for the non-crossing sequence

$$\Box n \triangle \quad \Box n-1 \triangle \quad \ldots \quad \Box 1 \triangle\ ,$$

where $\Box$ is the place for left bracket and $\triangle$ is for right bracket. Each $\Box$ or $\triangle$ contains at most one bracket.

Before we construct the relation between permutations and the non-crossing sequences, we want to give an order to the r pairs of brackets. We order the r right brackets as follows: the right most right bracket is $)_1$, the next right most right bracket is $)_2$, etc. The order of the left brackets is the same as the corresponding right brackets. For example, let's consider the following non-crossing sequence with three paris of brackets

$$(\quad 4 \quad (\quad 3 \quad) \quad 2 \quad) \quad (\quad 1 \quad).$$

We first order the right brackets

$$(\quad 4 \quad (\quad 3 \quad)_3 \quad 2 \quad)_2 \quad (\quad 1 \quad)_1.$$

The order of the left brackets are the same as its corresponding right brackets. We have

$$(_2 \quad 4 \quad (_3 \quad 3 \quad)_3 \quad 2 \quad)_2 \quad (_1 \quad 1 \quad)_1.$$

Given two positive integers n, r such that $n \geq r$, we define three sets $Pmt(n,r)$, $Brk(n,r)$ and $OS(n,r)$ as follows.

Definition 2.6.3. *$Pmt(n,r)$ is the set of non-crossing permutations in S_n (see Definition 2.5.9) with r disjoint cycles. $Brk(n,r)$ is the set of all non-crossing sequences with r pairs of brackets (see Construction 2.6.1). $OS(n,r)$ is the set of all ordinary summations of type $(r, n-r+1)$ (see Definition 2.5.1).*

We want to remind the reader that we always insert brackets into the following sequence

$$n \quad n-1 \quad \dots \quad 2 \quad 1\ .$$

Theorem 2.6.4. *Given two positive integers n, r such that $n \geq r$, there is a bijective map $\phi_{n,r}$ between $Brk(n,r)$ and $Pmt(n,r)$.*

Proof. We want to construct a map

$$\phi_{n,r} : Brk(n,r) \to Pmt(n,r)$$

and show that this map is bijective.

First we will construct a permutation $\alpha \in S_n$ with r cycles from a non-crossing sequence in $Brk(n,r)$. Given a non-crossing sequence in $Brk(n,r)$, we start with the r-th pair of brackets

$$(_r \quad \dots \quad)_r.$$

By construction, the integers in this pair of brackets are not contained in any other pair of brackets, because r is the largest. Define α_r as the cycle with integers from this pair of brackets. Then, we delete this pair of brackets and the enclosed integers. We choose the next pair of brackets

$$(_{r-1} \quad \dots \quad)_{r-1}$$

from the remaining sequence and uses it to define another cycle α_{r-1}. Repeating this process, we get a unique permutation α in S_n with disjoint cycles $\alpha_r, ..., \alpha_1$. Now we have to prove that α satisfies the condition $(*)$ so that the image of $\phi_{n,r}$ is in $Pmt(n, r)$. In Construction 2.6.1, we first fix the base sequence

$$n \quad n-1 \quad ... \quad 2 \quad 1 \;.$$

So, the quiver of any cycle α_r only contains one arrow a such that $s(a) > t(a)$. Hence, α satisfies the condition $(*_1)$. The condition $(*_2)$ comes from the property of non-crossing sequence. Consider the following example

$$(_2 \quad (_3 \quad)_3 \quad)_2 \quad (_1 \quad)_1.$$

There are only two relations between two pairs of brackets: "ordered" or "contained". $(_3 \quad)_3$ is contained in $(_2 \quad)_2$ and $(_2 \quad)_2$, $(_1 \quad)_1$ are ordered. This property is exactly the condition $(*_2)$. In this way, we see that the image of $\phi_{n,r}$ is in $Pmt(n, r)$. Clearly, it is injective.

Now we are going to prove the map $\phi_{n,r}$ is surjective on $Pmt(n, r)$. For the base case $n = 1$, the only permutation $(1) \in S_1$ corresponds uniquely to the following non-crossing sequence

$$(_1 \quad 1 \quad)_1 \;.$$

We use induction on n and assume that $\phi_{k-1,r}$ is surjective for any positive integer r, $r \le k-1$. We will show that $\phi_{k,r}$ is surjective for any r. If $\beta \in S_k$ satisfies the condition $(*)$, we know that $[\beta] = [\alpha, j]$ where j is zero or is contained in the chain of $\hat{Q}_\beta$ and α satisfies the condition $(*)$ by the proof of Theorem 2.5.10. By induction, α corresponds to a unique sequence with brackets as following

$$(_m \quad k-1 \quad ... \quad v+1 \quad)_m (_1 \quad v \quad ... \quad 1 \quad)_1,$$

where m is some positive integer, $m \le r-1$. If $m = 1$, then $v = k-1$.

If $j = 0$, then $\alpha \in Pmt(k-1, r)$. In Construction 2.2.8, we construct the sequence with brackets corresponding to β as

$$(_1 \quad k \quad (_m \quad k-1 \quad ... \quad v+1 \quad)_m \quad ... \quad 1 \quad)_1.$$

Here, we add another integer k to the sequence and move the bracket $(_1$ to the left side of k.

If $j \neq 0$, then $\alpha \in Pmt(k-1, r-1)$. In Construction 2.2.8, to construct β, where $\beta = [\alpha, j]$, we cut the arrow $a : i \to j$ in the chain of $\hat{Q}_\alpha$. First, let's focus on the first pair of brackets $(_1 \quad v \quad ... \quad 1 \quad)_1$ more precisely,

$$(_{s_1} \quad k-1 \quad ... \quad)_{s_1} \quad ... \quad (_{s_2} \quad ... \quad v+1 \quad)_{s_2} \quad (_1 \quad v \quad ... \quad i \quad (_{s_3} \quad ... \quad)_{s_3} \quad j \quad ... \quad 1 \quad)_1,$$

where s_1, s_2, s_3 are the order of the pairs of brackets (if they exist). We construct the following sequence with bracket, which corresponds to β,

$$\begin{aligned}&(\quad k \quad (_{s_1+1} \quad k-1 \quad ... \quad)_{s_1+1} \quad ... \quad (_{s_2+1} \quad ... \quad v+1 \quad)_{s_2+1} \quad v \quad ... \quad i \quad) \\ &(_{s_3} \quad ... \quad)_{s_3} \quad (_1 \quad j \quad ... \quad 1 \quad)_1.\end{aligned}$$

This non-crossing sequence has one more pair of brackets (the unlabelled pair of brackets above) than α, because β has one more disjoint cycle than α by Construction 2.2.8. In fact, this non-crossing sequence maps to β under the map $\phi_{k,r}$. In conclusion, $\phi_{k,r}$ is surjective.

Combining with the first part of the proof, $\phi_{k,r}$ is bijective. $\square$

The following example will help the reader understand the proof above.

Example 2.6.5. *Consider the following non-crossing sequence*

$$(4)(321)\ .$$

By the construction of $\phi_{4,2}$ in the proof of Theorem 2.6.4, it corresponds to the element $\alpha = (4)(321)$ in S_4 which satisfies the condition $(*)$. $\hat{Q}_\alpha$ is

$$\hat{Q}_{(4)(321)}: \quad \begin{tikzcd} 5 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1, & & 4 \arrow[loop right] \end{tikzcd} \quad .$$

Now, consider the quiver

$$\hat{Q}_{(4)(5321)}: \quad \begin{tikzcd} 6 \arrow[r] & 5 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1, & & 4 \arrow[loop right] \end{tikzcd} \quad .$$

Clearly, it is $\hat{Q}_{\beta_1}$, where $[\beta_1] = [\alpha, 0]$, i.e. $\beta_1 = (4)(5321)$. The corresponding non-crossing *sequence of β_1 is*

$$(5(4)321),$$

which is the case when $j = 0$. Next, we consider another quiver

$$\begin{tikzcd} \hat{Q}_{(5)(4)(321)}: \quad 6 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1, & 4 \arrow[loop right] & , & 5 \arrow[loop right] & , \end{tikzcd}$$

which corresponds to β_2, where $[\beta_2] = [\alpha, 3]$. By calculation, $\beta_2 = (5)(4)(321)$. The corresponding non-crossing sequence is

$$(5)(4)(321),$$

which is the case $j \neq 0$ we discuss above.

Now we want to give some definitions about pairs of brackets.

Definition 2.6.6. *Given any non-crossing sequence in $Brk(n, r)$, $(_i \quad ... \quad)_i$ is of **top-level** if this pair of brackets is not contained in any other pair of brackets. $(_i \quad ... \quad)_i$ is **embedded** if $(_i \quad ... \quad)_i$ is not top-level. $(_i \quad ... \quad)_i$ is of **bottom-level** if there is no embedded pair of brackets in it. Two pairs of brackets are **adjacent** if there are no positive integers between these two pairs of brackets.*

Example 2.6.7. *Let $\alpha = (531)(2)(4)(6)$, then the corresponding non-crossing sequence is*

$$(_4 \quad 6 \quad)_4 (_1 \quad 5 \quad (_3 \quad 4 \quad)_3 \quad 3 \quad (_2 \quad 2 \quad)_2 \quad 1 \quad)_1.$$

$(_4 \quad 6 \quad)_4$ is both of bottom-level and top-level. $(_3 \quad 4 \quad)_3$ and $(_2 \quad 2 \quad)_2$ are embedded and of bottom-level. $(_3 \quad 4 \quad)_3$ and $(_2 \quad 2 \quad)_2$ are not adjacent, because 3 is between them. Finally, $(_1 \quad ... \quad)_1$ and $(_4 \quad ... \quad)_4$ are adjacent.

Remark 2.6.8. *Theorem 2.5.10 tells us that there is a bijective map between the non-crossing permutations and the ordinary summations. Theorem 2.6.4 tells us that there is a bijective map between the non-crossing permutations and the non-crossing sequences. In Section 2.8, we calculate the number of non-crossing sequences, which is exactly the answer to Question 2.4.4.*

2.7 Dual Non-crossing Sequence

In Construction 2.6.1, we construct the non-crossing sequence with r pairs of brackets. In this section, we construct the dual non-crossing sequence and prove that the number of (r, s)-type OS is the same as the number of (s, r)-type OS.

Construction 2.7.1. *Consider the following non-crossing sequence*

$$(_{i_1} \quad)_{i_1} \quad ... \quad (_{i_s} \quad)_{i_s},$$

where all pairs of brackets are top-leveled. There may be some embedded brackets in them. For each integer k in this sequence, there are at most four brackets "adjacent" to it,

$$k+1 \quad \triangle \quad \square k \triangle \quad \square \quad k-1,$$

the right bracket of $k+1$, the left bracket of $k-1$ and the two brackets of k. There are 16 *possibilities in these four positions. The following construction discuss these possibilities.*

	$\triangle\square k\triangle\square$	$\triangle\square k\triangle\square$
1	k	$)(k)($
2	$k)$	$)(k)$
3	$(k$	$(k)($
4	$k($	$)(k($
5	$)k$	$)k)($
6	(k)	(k)
7	$)k($	$)k($
8	$(k($	$(k($
9	$)k)$	$)k)$
10	$)(k$	$k)($
11	$k)($	$)(k$
12	$)(k)$	$k)$
13	$(k)($	$(k$
14	$)(k($	$k($
15	$)(k($	$k($
16	$)(k)($	k

The second column is all of the possible cases in the original non-crossing sequence, the third column is what we will get in the dual non-crossing sequence.

Given a non-crossing, we do the operations for all integers in the sequence simultaneously to get the dual sequence. It is easy to check that all operations are compatible with each other. We claim that the dual sequence we get is a non-crossing sequence, which we call the dual non-crossing sequence.

To prove that the dual sequence is a non-crossing sequence under the operations, we only have to check the dual sequence satisfies the first point of the condition (∗∗) in Construction 2.6.1, i.e. every integer in the dual sequence is contained in some pair of brackets. We leave it as an exercise for the reader to check.

From the construction, we see that the operations are dual in the following way

$$1 \Leftrightarrow 16, 2 \Leftrightarrow 12, 3 \Leftrightarrow 13, 4 \Leftrightarrow 14, 5 \Leftrightarrow 15,$$
$$6 \Leftrightarrow 6, 7 \Leftrightarrow 7, 8 \Leftrightarrow 8, 9 \Leftrightarrow 9, 10 \Leftrightarrow 11.$$

Hence, given a non-crossing sequence, the dual of its dual non-crossing sequence is itself. It is also easy to check that given any non-crossing sequence in $Brk(n, r)$, its dual non-crossing sequence is in $Brk(n, n-r+1)$.

Example 2.7.2. *Here is an example of Construction 2.7.1.*

Let $\alpha = (721)(65)(4)(3) \in Pmt(7, 4)$. *The corresponding sequence is*

$$(_1 \quad 7 \quad (_4 \quad 6 \quad 5 \quad)_4 \quad (_3 \quad 4 \quad)_3 \quad (_2 \quad 3 \quad)_2 \quad 2 \quad 1 \quad)_1 \ .$$

We see that

$$
\begin{aligned}
&7 \textit{ is of type } 8,\\
&6 \textit{ is of type } 3,\\
&5 \textit{ is of type } 11,\\
&4 \textit{ is of type } 16,\\
&3 \textit{ is of type } 12,\\
&2 \textit{ is of type } 5,\\
&1 \textit{ is of type } 2.
\end{aligned}
$$

So, the dual sequence is

$$(_3 \quad 7 \quad (_1 \quad 6 \quad)_1 \quad (_2 \quad 5 \quad 4 \quad 3 \quad)_2 \quad 2 \quad)_3 \quad (_4 \quad 1 \quad)_4 \ ,$$

which corresponds to the permutation $(72)(6)(543)(1)$.

Now, we are ready to prove the number of (r, s)-type OS in $: tr(D^n) :$ is the same as the number of (s, r)-type OS in $: tr(D^n) :$, where $r + s - 1 = n$.

Corollary 2.7.3. *Given two positive integers* n, r, *we have*

$$|OS(n, r)| = |OS(n, n-r+1)|.$$

Proof. By Theorem 2.5.10, we have

$$|OS(n,r)| = |Pmt(n,r)|, \quad |OS(n,n-r+1)| = |Pmt(n,n-r+1)|.$$

By Theorem 2.6.4, we know

$$|Pmt(n,r)| = |Brk(n,r)|, \quad |Pmt(n,n-r+1)| = |Brk(n,n-r+1)|.$$

By Construction 2.7.1, we have

$$|Brk(n,n-r+1)| = |Brk(n,r)|.$$

Hence,

$$|OS(n,r)| = |OS(n,n-r+1)|.$$

□

2.8 $|Brk(n,r)|$, Catalan Number and Narayana Number

In this section, we will calculate $|Brk(n,r)|$, the number of non-crossing sequences with r pairs of brackets in a sequence of length n, by using properties of the Catalan numbers and Narayana numbers. We use this to calculate the number of ordinary summations in the W-operator $W([n])$.

We first review some properties of the Catalan numbers and Narayana numbers [34]. The Catalan number C_n is

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad , \quad n \geq 0. \tag{2.8.1}$$

The generating function of Catalan numbers $c(x)$ is

$$c(x) = \sum_{n=0}^{\infty} C_n x^n,$$

which satisfies the following equation

$$c(x) = 1 + xc(x)^2. \tag{2.8.2}$$

Clearly, Equation (2.8.2) gives us two solutions for the generating function $c(x)$. But, if we know the initial value c_1, we will get a unique solution. If c_1=1, then $c(x)$ is exactly of the generating function of Catalan numbers C_n.

The Narayana number $N(n,r)$ is

$$N(n,r) = \begin{cases} \frac{1}{n+1}\binom{n+1}{r}\binom{n-1}{r-1}, 0 \leq r \leq n \\ 0, \text{ otherwise }. \end{cases}$$

The generating function of Narayana numbers is

$$n(x,y) = \sum_{n,r \geq 0} N(n,r)x^n y^r. \tag{2.8.3}$$

The Narayana number $N(n,r)$ satisfies the following condition

$$\sum_{r=1}^{n} N(n,r) = C_n,$$

i.e.

$$n(x,y) = 1 + n(x,y)^2 x.$$

Clearly, we have

$$c(x) = n(x,1).$$

We define a new set $\widetilde{Brk}(n,r)$, which contains all sequences in $Brk(n,r)$ with only one top-level pair of brackets. It means that any element in $\widetilde{Brk}(n,r)$ can be written in the following form

$$(_1 \quad ... \quad)_1,$$

where $(_1 \quad ... \quad)_1$ is the only top-level pair of brackets. Denote by $\widetilde{a}_n^r$ the number of elements in $\widetilde{Brk}(n,r)$. Also, we introduce the following notation

$$a_n^r = \begin{cases} |Brk(n,r)| & ,1 \le r \le n \\ 0 & ,\text{otherwise.} \end{cases} \quad \widetilde{a}_n^r = \begin{cases} |\widetilde{Brk}(n,r)| & ,1 \le r \le n \\ 0 & ,\text{otherwise.} \end{cases}$$

Lemma 2.8.1. *Given any positive integers n, r, $n \ge r \ge 1$, we have $\widetilde{a}_{n+1}^r = a_n^r$.*

Proof. We are going to construct a bijection between $\widetilde{Brk}(n+1,r)$ and $Brk(n,r)$. Take an element in $\widetilde{Brk}(n+1,r)$. It has only one top-level pair of brackets. So, the integer $n+1$ does not have right bracket and 1 does not have left bracket. The sequence in $\widetilde{Brk}(n+1,r)$ can be written in the following two cases

1. $$(_1 \quad n+1 \quad (_{j_1} \quad ... \quad)_{j_1} \quad ... \quad (_{j_k} \quad)_{j_k} \quad v \quad ... \quad 1 \quad)_1 \ ,$$

2. $$(_1 \quad n+1 \quad v \quad ... \quad (_{j_1} \quad ... \quad)_{j_1} \quad ... \quad (_{j_k} \quad)_{j_k} \quad ... \quad 1 \quad)_1 \ ,$$

   where the integer v is the largest integer smaller than $n+1$ in the top-level pair of brackets, i.e., not contained in any embedded brackets. In the second case, $v = n$.

We construct the sequence in $Brk(n,r)$ as follows

1. $$(_{j_1} \quad ... \quad)_{j_1} \quad ... \quad (_{j_k} \quad)_{j_k} \quad (_1 \quad v \quad ... \quad 1 \quad)_1 \ ,$$

2. $$(_1 \quad v \quad ... \quad (_{j_1} \quad ... \quad)_{j_1} \quad ... \quad (_{j_k} \quad)_{j_k} \quad v \quad ... \quad 1 \quad)_1 \ .$$

   Indeed, we get rid of the integer $n+1$ and move the bracket $(_1$ to the left side of the next integer not contained in any other pair of brackets. This gives a well defined element in $Brk(n,r)$.

Now let's consider how to construct elements in $\widetilde{Brk}(n+1,r)$ from elements in $Brk(n,r)$. In the proof of Theorem 2.6.4, we already gave the construction. Given an element in

$Brk(n,r)$, we assume it in the following form

$$(_{j_1} \quad n \quad ... \quad w \quad)_{j_1} \quad ... \quad (_1 \quad v \quad ... \quad 1 \quad)_1,$$

where $(_{j_1} \quad n \quad ... \quad w \quad)_{j_1}$ is the leftmost top-leveled pair of brackets. Now we give the construction as follow

$$(_1 \quad n+1 \quad (_{j_1} \quad n \quad ... \quad w \quad)_{j_1} \quad v \quad ... \quad 1 \quad)_1.$$

Indeed, if we consider the element in $Brk(n,r)$ corresponding to the permutation $\alpha \in S_n$, then the sequence we construct corresponds to the permutation $\beta \in S_{n+1}$, where $\beta = [\alpha, 0]$. It is easy to check that the above construction gives a one-to-one correspondence between $Brk(n,r)$ and $\widetilde{Brk}(n+1,r)$. Hence, $\widetilde{a}^r_{n+1} = a^r_n$. $\square$

Theorem 2.8.2. *The number of $(r, n-r+1)$-type OS in $W([n])$ is the Narayana number:*

$$|OS(n,r)| = \frac{1}{n+1}\binom{n+1}{r}\binom{n-1}{r-1}.$$

The number of all summations with degree $n+1$ in $W([n])$ is the Catalan number

$$\sum_{r\geq 1}^{n} \frac{1}{n+1}\binom{n+1}{r}\binom{n-1}{r-1} = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Any element in $Brk(n,r)$ can be written as

$$(_{i_1} \quad)_{i_1} \quad ... \quad (_{i_s} \quad)_{i_s},$$

where the pairs of brackets $(_{i_j} \quad)_{i_j}$, $1 \leq j \leq s$, are top-level. By construction, any integer k, $1 \leq k \leq n$, is contained in a unique top-level pair of brackets. Of course, $(_{i_j} \quad)_{i_j}$ can be considered as a non-crossing sequence with a unique top-level pair of brackets. Let n_j and t_j be the number of integers, respectively the number of pairs of brackets in $(_{i_j} \quad)_{i_j}$. Hence, we have

$$a^r_n = \sum_{s=1}^{r} \sum_{\substack{n_1+...+n_s=n \\ t_1+...+t_s=r}} \widetilde{a}^{t_1}_{n_1} ... \widetilde{a}^{t_s}_{n_s}.$$

By Lemma 2.8.1, we know $\widetilde{a}_{n+1}^r = a_n^r$. So, we have

$$a_n^r = \sum_{s=1}^{r} \sum_{\substack{n_1+\ldots+n_s=n \\ t_1+\ldots+t_s=r}} a_{n_1-1}^{t_1} \ldots a_{n_s-1}^{t_s}. \tag{2.8.4}$$

Now we consider the generating function

$$G(x,y) = \sum_{n,r\geq 0} a_n^r x^n y^r.$$

By (2.8.4), we have

$$\begin{aligned} & \sum_{s=1}^{\infty} G(x,y)^s x^s = G(x,y), \\ \Rightarrow \quad & \frac{1}{1-G(x,y)x} = G(x,y), \\ \Rightarrow \quad & G(x,y) = 1 + G(x,y)^2 x, \end{aligned}$$

which is the generating function of Narayana numbers. If we set $y=1$, we have

$$G(x,1) = 1 + G(x,1)^2 x,$$

which is the generating function for Catalan number C_n. But, the generating function is not enough to determine the value of a_n^r. We also have to check the initial value a_1^1. Clearly, $a_1^1 = 1$, which equals to the first Catalan number C_1. By the property of Catalan number and Narayama number we stated at the beginning of this section, we have

$$\begin{aligned} \sum_{r=1}^{n} a_n^r &= \frac{1}{n+1}\binom{2n}{n}, \\ a_n^r &= \frac{1}{n+1}\binom{n+1}{n+1-r}\binom{n-1}{r-1}. \end{aligned}$$

$\square$

2.9 A Formula about $W([d])$

Recall that $\Phi : \mathbb{C}S_n \to \mathbb{C}[p_1, p_2, ...]$ is the linear map defined as follows

$$\Phi(g) = p_\lambda,$$

where g is a permutation in S_n of type $\lambda = (\lambda_1, ..., \lambda_m)$ and $p_\lambda = p_{\lambda_1}...p_{\lambda_m}$. The variable p_{λ_i} is the trace of the infinite matrix X^i. The reader can also take p_i as independent variables. Also, if λ is a partition, $K_\lambda = \sum_{\substack{\sigma\in S_n \\ \sigma \text{ is of type}\lambda}} \sigma$ is a central element in the group ring $\mathbb{C}S_n$. As a special example, $K_{(1^{n-d}d)}$ is the sum of all d-cycles in S_n.

In this section, we will prove the following formula about $W([d])$. The applications of $W([d])$ to the Hurwitz number in the next chapter are based on this formula.

Theorem 2.9.1. *For any $g \in \mathbb{C}S_n$,*

$$\Phi(K_{(1^{n-d}d)}g) = W([d])\Phi(g), \tag{2.9.1}$$

where $K_{(1^{n-d}d)}$ is the central element in $\mathbb{C}S_n$ corresponding to the partition $(1^{n-d}d)$.

This theorem was known for $d = 2$, see [11]. We will use quivers to prove this theorem. Definitions about quivers can be found in Section 2.2.

Definition 2.9.2. *Denote by $\mathbb{FQ}$ the set of all quivers (V, A, s, t) with finite vertex set $\{1, ..., n\}$ for some positive integer n and finitely many arrows.*

Denote by $\mathbb{M}$ the set of all monomials with variables X_{ij}, $1 \leq i, j < \infty$.

Definition 2.9.3. *Let $Q = (V, A, s, t) \in \mathbb{FQ}$. We define the map $\beta : \mathbb{FQ} \to \mathbb{M}$ by $\beta(Q) = M_Q$, where $M_Q = \prod_{a\in A} X_{s(a)t(a)}$.*

Also, given any monomial $M = \prod_{k=1}^{l} = X_{i_k j_k}$, we can define the corresponding quiver Q_M as $Q_M = (V_M, A_M, s, t)$, where $V_M = \{1, ..., n\}, n = max\{i_k, j_k, 1 \leq k \leq l\}$ and $A_M = \{a_k : i_k \to j_k, 1 \leq k \leq l\}$.

Construction 2.9.4. *Given $\alpha \in S_n$, let $Q_\alpha = \Phi_n(\alpha)$ be the quiver corresponding to α (See Definition 2.2.1). Given two vertices $a_1, a_2 \in Q_\alpha$, we construct a new quiver denoted $(\bar{D}_{a_1a_2})Q_\alpha$ by replacing the unique arrow $a_2 \to b$ by $a_1 \to b$. So we get a new quiver denoted by $(\bar{D}_{a_1a_2})Q_\alpha$. More generally, if $a_1, ..., a_d$ are distinct vertices (or integers) of Q_α, we replace the arrows $a_i \to b_i$ with $a_{i-1} \to b_i$ simultaneously, $2 \leq i \leq k+1$, $a_{k+1} = a_1$. Denote by*

$(\prod_{i=1}^{d} \bar{D}_{a_i a_{i+1}})Q_\alpha$ *the new quiver. We introduce the notation (similar to Notation 2.1.10) as follows*

$$\bar{D}_{(a_1,...a_d)} = \prod_{i=1}^{d} \bar{D}_{a_i a_{i+1}},$$

where $(a_1, ..., a_d)$ *is an* n*-tuple of positive integers and* $a_{d+1} = a_1$*.*

Remark 2.9.5. *Given a* d*-tuple of positive integers* $(a_1, ..., a_d)$*, the quiver* $\bar{D}_{(a_1,...,a_d)}Q_\alpha$ *is obtained by doing the replacement operations simultaneously instead of consecutively, by composition of operations. For example, let* $\alpha = (123)$ *and* $\bar{D}_{(1,2,3)} = \bar{D}_{12}\bar{D}_{23}\bar{D}_{31}$*. If we do the operations simultaneously, the new quiver is*

$$\begin{tikzcd} 1 \arrow[r] & 3 \arrow[r] & 2 \arrow[ll, bend right] \end{tikzcd} .$$

But, if we do it as compositions, $\bar{D}_{31}Q_\alpha$ *is*

$$3 \longrightarrow 2\ , \qquad 2 \longrightarrow 3\ , \qquad 3 \longrightarrow 1\ .$$

This quiver has two arrows with source 3*. In this case,* $\bar{D}_{23}$ *cannot act on this quiver by Construction 2.9.4. This is the reason why we want to do all the operations simultaneously, otherwise, we don't know in general which arrow to replace.*

The new quiver $\bar{D}_{a_2a_1}Q_\alpha$ may not be of the form $\Phi_n(\beta)$, i.e not represent a well defined element β in the permutation group S_n under this operation. However, we have the following lemma.

Lemma 2.9.6. *Let* $\alpha \in S_n$ *and* $Q_\alpha = \Phi_n(\alpha)$ *is the corresponding quiver. Given* d *distinct positive integers* $a_1, ..., a_d$*, then* $\bar{D}_{(a_1,...,a_d)}Q_\alpha$ *represents a permutation in* S_n*.*

Proof. In the construction, this procedure only changes the source of each arrow and fixes the target. Therefore, we pick d arrows such that their sources are $a_1, ..., a_d$ respectively. By the construction, substitute the source a_i by a_{i+1}, where $i \leq d-1$ and a_1 by a_d, and get a new quiver $(\bar{D}_{(a_1,...,a_d)})Q_\alpha$. Clearly, this quiver still represents for an element in S_n, because each integer k $(k \leq n)$ appears once as a target and once as a source. □

Remark 2.9.7. *From the proof of the lemma, we have* $Q_{\alpha'} = (\bar{D}_{(a_1,...,a_d)})Q_\alpha$*, where* $\alpha' = (a_1\ a_2\ ...\ a_d)\alpha$*.*

Now, consider the monomial $\beta(\Phi_n((12...n))) = X_{12}X_{23}...X_{n1}$ which is a term in $tr(X^n)$. We use the permutation $(12...n)$ to represent this monomial or the quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, bend right] \end{tikzcd} .$$

We use D_{21} (refer to Definition 2.1.4) acting on this term, then we get

$$D_{21}X_{12}X_{23}...X_{n1} = X_{22}X_{23}...X_{n1}.$$

The new term $X_{22}X_{23}...X_{n1}$ can be represented by a quiver

$$\begin{tikzcd} 2 \arrow[loop above] \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & 1 \end{tikzcd} .$$

In this way, if we use quivers to represent the monomials, then $D_{a_1a_2}$ acting on monomials is the same as $\bar{D}_{a_1a_2}$ acting on the corresponding quivers. Hence, if $D_{a_1a_2}...D_{a_da_1}X$ is a nonzero monomial, then it can be represented by a permutation by Remark 2.9.7. With the discussion above, we have the following lemma.

Lemma 2.9.8. *Let $\alpha \in S_n$. Q_α is the corresponding quiver and M_α is the corresponding monomial. We have $\beta(\bar{D}Q_\alpha) = DM_\alpha$, where $D = D_{(a_1,...,a_d)}$ and $\bar{D} = \bar{D}_{(a_1,...,a_d)}$, where $(a_1, ..., a_d)$ is an d-tuple of positive integers.*

Definition 2.9.9. *Given a monomial $X \in \mathbb{C}[X_{11}, X_{12}, ..., X_{22}, ...]$ and a (formal) differential operator D. If $DX \neq 0$, then we say D is a non-trivial operator (with respect to X).*

In this section, we concentrate on the differential operator $D = D_{(a_1,...,a_n)}$.

Definition 2.9.10. *Let $T = \{t_i, 1 \leq i\}$ be a set of variables, define $\mathbb{M}_t$ is the set of all monomials with variables $X_{t_it_j}, i, j \geq 1$. Given an infinite sequence of positive integers $a = (a_1, a_2, ...)$, define the evaluation map $ev_a : \mathbb{M}_t \to \mathbb{M}$,*

$$ev_a(X_{t_it_j}) = X_{a_ia_j}.$$

If M_t is a monomial in $\mathbb{M}_t$, we define $M_t(a_1, a_n...) = ev_a(M_t)$.

Similar to 2.1.10, we introduce the following notation,

$$X_{(t_1,\dots,t_n)} = \left(\prod_{i=1}^{n-1} X_{t_i t_{i+1}}\right) X_{t_n t_1},$$
$$D_{(t_1,\dots,t_n)} =: \left(\prod_{i=1}^{n-1} D_{t_i t_{i+1}}\right) D_{t_n t_1} : .$$

Finally, we define $W_t([d]) = \frac{1}{d} : Tr((D_{t_i t_j})_{i,j\geq 1})^d :$

We are ready to prove Theorem 2.9.1.

Proof of Theorem 2.9.1. Let $g \in S_n$. We can write it in disjoint cycles

$$g = (c_1 \ \dots \ c_{\lambda_1})(c_{\lambda_1+1} \ \dots \ c_{\lambda_1+\lambda_2}) \ \dots \ (c_{n-\lambda_m+1} \ \dots \ c_n),$$

where λ is the partition corresponding to g, $\lambda = (\lambda_1, \dots, \lambda_m)$.

$W([d])$ is an infinite sum of operators $D_{(b_1,\dots,b_d)}$, b_i are positive integers, (see Definition 2.1.10) and $\Phi(g) = \prod_{i=1}^m p_{\lambda_i}$ is an infinite sum of monomials in the form

$$M(a_1, \dots, a_n) = X_{(a_1,\dots,a_{\lambda_1})} \dots X_{(a_{n-\lambda_m+1},\dots,a_n)}.$$

Given any monomial M, most of the operators $D_{(b_1,\dots,b_d)}$ in $W([d])$ will act by zero. Hence, $W([d])M$ is a finite sum of monomials. To analyze these monomials, we first consider the generic case M_t. Then, we go back to M as the evaluation of M_t,

$$M(a_1, \dots, a_n) = ev_a(M_t),$$

where $a = (a_1, \dots, a_n)$ is an n-tuple of positive integers.

We replace $W([d])$ by $W_t([d])$ (see Definition 2.9.10) and g by $\bar{g}$, where

$$\bar{g} = (t_1 \ \dots \ t_{\lambda_1})(t_{\lambda_1+1} \ \dots \ t_{\lambda_1+\lambda_2}) \ \dots \ (t_{n-\lambda_m+1} \ \dots \ t_n).$$

We consider a special case $M_t = X_{(t_1,\dots,t_{\lambda_1})} \dots X_{(t_{n-\lambda_m+1},\dots,t_n)}$. In this case, we prefer to use the notation M_{g_t} for M_t. Now we will calculate $W_t([d])M_t$. By Remark 2.9.7 and Lemma 2.9.8, let $i_1, \dots, i_d$ be distinct integers in $\{1, \dots, n\}$, we have

$$D_{(t_{i_1}, t_{i_2}, \dots, t_{i_d})} M_{g_t} = M_{\sigma_t g_t},$$

where σ_t is the d-cycle $(t_{i_1}....t_{i_d}) \in S_n^t = Aut\{t_1, ..., t_n\}$. Since $D_{(t_{i_1},...,t_{i_d})}M_{g_t}$ is nonzero if and only if $i_j \in \{1, ..., n\}, 1 \leq j \leq d$, we have

$$\sum_{\substack{(i_1,...,i_d),\\ i_j\in\{1,...,n\} \text{ and distinct}}} D_{(t_{i_1},...,t_{i_d})}M_{g_t} = d \sum_{\sigma_t \ d\text{-cycle in } \bar{S}_n} M_{\sigma_t g_t}.$$

Here we understand there are d d-tuples $(i_1, ..., i_d)$ giving rise to the same d-cycle. Hence, we have a coefficient at the right side of the above equation. We have the following formula

$$\begin{aligned}
W_t([d])M_{g_t} &= \frac{1}{d} \sum_{\substack{(i_1,...,i_d),\\ i_j\in\{1,...,n\} \text{ and distinct}}} D_{(t_{i_1},...,t_{i_d})}M_{g_t} \\
&= \frac{1}{d} \sum_{\substack{(i_1,...,i_d),\\ i_j\in\{1,...,n\} \text{ and distinct}}} M_{\sigma_t g_t} \\
&= \sum_{\sigma_t \ d\text{-cycle in } S_n^t} M_{\sigma_t g_t}.
\end{aligned}$$

Now we want to show for any d-tuple $(a_1, ..., a_d)$ (with maybe some a_i not distinct), we have

$$W([d])M_{g_t}(a_1, ..., a_n) = \sum_{\sigma_t \ d\text{-cycle in } S_n^t} M_{\sigma_t g_t}(a_1, ..., a_n). \tag{2.9.2}$$

We note that for any n-tuple $(a_1, ..., a_n)$, the right hand side of (2.9.2) is always a sum of $\frac{1}{d}\binom{n}{d}d!$ monomials, each of which corresponds an unique element in S_n^t, where $S_n^t = Aut\{t_1, ..., t_n\}$. But the left hand side is complicated. We hope that for any n-tuple $(a_1, ..., a_n)$, the left hand side is a sum of $\frac{1}{d}\binom{n}{d}d!$ monomials. We can find $\binom{n}{d}d!$ nontrivial operators in $W([d])$ with respect to $M_{g_t}(a_1, ...a_n)$. (Recall in the definition of $W([d])$, we have a coefficient $\frac{1}{d}$.) But the left hand side is not easy if the a_i are not distinct. Indeed, if a_i are not distinct, there are fewer nontrivial operators $D_{(a_{i_1},...,a_{i_d})}$ in $W([d])$ with respect to $M_{g_t}(a_1, ...a_n)$ than that in $W_t([d])$ with respect to M_{g_t}.

For example, consider

$$M = X_{(t_1,t_2,t_3)} = X_{t_1t_2}X_{t_2t_3}X_{t_3t_1}.$$

There are 6 nontrivial differential operators $D_{(t_{i_1},t_{i_2},t_{i_3})}$ in $W_t([3])$ with respect to M, where (i_1, i_2, i_3) is any 3-tuples such that $i_1, i_2, i_3 \in \{1, 2, 3\}$ and distinct. However, if we substitute $a_1 = a_2 = 1, a_3 = 2$, we get only 3 nontrivial operators in $W([3])$ with respect to

$X_{(1,1,2)}$. They are $D_{(1,1,2)}$, $D_{(1,2,1)}$, $D_{(2,1,1)}$. In this case, we have to check whether we can get enough monomials on the left hand side of the equation.

Before we discuss different cases, we first focus on some basic calculations. The number of d-cycles in S_n is $\frac{1}{d}\binom{n}{d}d!$. Given a monomial M_{g_t} of degree n, the number of non-trivial operators $D_{(t_{i_1},...,t_{i_d})}$ in $W_t([d])$ corresponding to M_{g_t} is $\binom{n}{d}d!$. Each differential operator $D_{(t_{i_1},...,t_{i_d})}$ corresponds to a unique d-tuple $(t_{i_1},...,t_{i_d})$, which corresponds to a unique d-cycle with integers $(t_{i_1}...t_{i_d})$. But, a d-cycle corresponds to d d-tuples.

Next, we will discuss how $W([d])$ acts on $M_{g_t}(a_1,...,a_n) = X_{(a_1,...,a_n)}$.

Case 1, a_i are distinct.

In this case, each "non-trivial operator" $D_{(a_{i_1},...,a_{i_d})}$ corresponds to a unique d-cycle in $\bar{S}_n$. But this correspondence is not injective, it is an d to 1 correspondence. For example,

$$: D_{(a_1,a_2,a_3)} :=: D_{(a_2,a_3,a_1)} :=: D_{(a_3,a_1,a_2)} : .$$

Hence, we get

$$W([d])M_{g_t}(a_1,...a_n) = \sum_{\sigma_t \ d\text{-cycle in } S_n} M_{\sigma_t g_t}(a_1,...,a_n).$$

The number of non-trivial operators with respect to $X_{g_t}(a_1,...a_n)$ in $W([d])$ is $\binom{n}{d}d!$.

Case 2, a_i are not all distinct and all $X_{a_i a_{i+1}}$ are distinct.

First, we consider a special case that only two numbers of $\{a_i\}_{1\leq i\leq n}$ are the same and we assume that $a_p = a_q$. In this case, we consider the operator $D_{(a_{i_1},...,a_{i_d})}$.

1. If all $a_{i_j} \neq a_p$, then each non-trivial differential operator $D_{(a_{i_1},...,a_{i_d})}$ with respect to $X_{(a_1,...,a_n)}$ corresponds to a unique d-tuple in t_i, which means it corresponds to a unique element in the permutation group $\bar{S}_n$. Under this condition, there are $\binom{n-2}{d}d!$ d-tuples $(a_{i_1},...,a_{i_d})$ satisfying this condition and each of them corresponds to a unique d-tuple $(t_{i_1},...,t_{i_d})$.

2. If only one of $\{a_{i_j}\}_{1\leq j\leq d}$ is a_p and we assume $a_{i_k} = a_p$, then each non-trivial differential operator $D_{(a_{i_1},...,a_{i_d})}$ corresponds to two elements in the permutation group $\bar{S}_n$. Indeed,

we have

$$
\begin{aligned}
D_{a_{i_{k-1}}a_{i_k}}X_{a_1a_2}...X_{a_na_1} &= D_{a_{i_{k-1}}a_p}X_{a_1a_2}...X_{a_na_1} = \\
&= \left(\sum_{c\geq 1} X_{a_{i_{k-1}}c}\frac{\partial}{\partial X_{a_pc}}\right)X_{a_1a_2}...X_{a_na_1} = \\
&= \left(X_{a_{i_{k-1}}a_{p+1}}\frac{\partial}{\partial X_{a_pa_{p+1}}} + X_{a_{i_{k-1}}a_{q+1}}\frac{\partial}{\partial X_{a_pa_{q+1}}}\right)X_{a_1a_2}...X_{a_na_1}.
\end{aligned}
$$

The last equality holds because only these two terms in $D_{a_{i_{k-1}}a_p}$ act non-trivially on $X_{(a_1,...,a_n)}$ with our assumptions $a_p = a_q$.

Compared with $(t_{i_1},...,t_{i_d})$, the differential operator $D_{(a_{i_1},...,a_{i_d})}$ now actually corresponds to two d-tuples. They are

$$
\begin{aligned}
&(t_{i_1},...,t_{i_{k-1}},t_p,t_{i_{k+1}},...,t_{i_d}),\\
&(t_{i_1},...,t_{i_{k-1}},t_q,t_{i_{k+1}},...,t_{i_d}).
\end{aligned}
$$

In this case, $D_{(a_{i_1},...,a_{i_d})}$ corresponds to two different elements in the permutation group $\bar{S}_n$.

Under this condition, there are $\frac{1}{2}\binom{n-2}{d-1}\binom{2}{1}$ d-tuples $(a_{i_1},...,a_{i_d})$ satisfying this condition and each of them corresponds to two d-tuples in $\bar{S}_n$.

3. If two of $\{a_{i_j}\}_{1\leq j\leq d}$ are a_p and we assume they are $a_{i_l} = a_{i_k} = a_p$, then each non-trivial differential operator $D_{(a_{i_1},...,a_{i_d})}$ corresponds to two elements in the permutation group $\bar{S}_n$. Indeed, we have

$$
:D_{a_{i_{l-1}}a_{i_l}}D_{a_{i_{k-1}}a_{i_k}}:X_{a_1a_2}...X_{a_na_1} =: D_{a_{i_{l-1}}a_p}D_{a_{i_{k-1}}a_p}:X_{a_1a_2}...X_{a_na_1}.
$$

Since we only care about the non-trivial terms, we have to calculate the differential operators $:D_{a_{i_{l-1}}a_p}D_{a_{i_{k-1}}a_p}:$ with differential part

$$
\frac{\partial^2}{\partial X_{a_pa_{p+1}}\partial X_{a_qa_{q+1}}}.
$$

By definition, we know

$$D_{a_{i_{l-1}}a_p} = \sum_{c\geq 1} X_{a_{i_{l-1}}c}\frac{\partial}{\partial X_{a_pc}},$$
$$D_{a_{i_{k-1}}a_p} = \sum_{d\geq 1} X_{a_{i_{k-1}}d}\frac{\partial}{\partial X_{a_pd}}.$$

So, we have

$$\begin{aligned} &: D_{a_{i_{l-1}}a_p}D_{a_{i_{k-1}}a_p} : X_{a_1a_2}...X_{a_na_1} = \\ =&(\sum_{c,d\geq 1} X_{a_{i_{l-1}}c}X_{a_{i_{k-1}}d}\frac{\partial}{\partial X_{a_pc}}\frac{\partial}{\partial X_{a_pd}})X_{a_1a_2}...X_{a_na_1} = \\ =&(X_{a_{i_{l-1}}a_{p+1}}X_{a_{i_{k-1}}a_{q+1}}\frac{\partial}{\partial X_{a_pa_{p+1}}}\frac{\partial}{\partial X_{a_pa_{q+1}}}+ \\ +&X_{a_{i_{l-1}}a_{q+1}}X_{a_{i_{k-1}}a_{p+1}}\frac{\partial}{\partial X_{a_pa_{q+1}}}\frac{\partial}{\partial X_{a_pa_{p+1}}})X_{a_1a_2}...X_{a_na_1}. \end{aligned}$$

The last equality holds because all $X_{a_ia_j}$ are distinct by the assumption of **Case 2**. Hence $a_{q+1} \neq a_{p+1}$.

Compared with $(t_{i_1},...,t_{i_d})$, the differential operator $D_{(a_{i_1},...,a_{i_d})}$ corresponds to two d-tuples. They are

$$(t_{i_1},...,t_{i_k},...,t_{i_l},...,t_{i_d}),$$
$$(t_{i_1},...,t_{i_l},...,t_{i_k},...,t_{i_d}).$$

Hence,in this case, $D_{(a_{i_1},...,a_{i_d})}$ corresponds to two different elements in the permutation group S_n^t.

Under this condition, there are $\frac{1}{2}\binom{n-2}{d-2}d!$ d-tuples $(a_{i_1},...,a_{i_d})$ satisfying this condition and each of them corresponds to two d-tuples in S_n^t.

Hence, in this special case, the number of d-tuples in S_n^t corresponding to the nontrivial differential operators with respect to the monomial $X_{(a_1,...,a_n)}$ is

$$\binom{n-2}{d}d! + 2\times\frac{1}{2}\binom{n-2}{d-1}\binom{2}{1}d! + 2\times\frac{1}{2}\binom{n-2}{d-2}d! = \binom{n}{d}d!.$$

By the discussion above, each tuple is counted for d times. Hence, in this case, we have

$$d \times W([d])M_{g_t}(a_1, ...a_n) = d \times \sum_{\sigma \text{ } d\text{-cycle in } S_n^t} M_{\sigma_t g_t}(a_, ..., a_n).$$

For the general case of s integers $a_{j_1} = a_{j_2} = ... = a_{j_2}$ but $X_{a_i a_{i+1}}$ all distinct, the same argument proves what we want. We leave it to the reader to check this.

Case 3, a_i are not all distinct, and some $X_{a_i a_{i+1}}$ are the same.

We still consider a special case that only two terms in $X_{(a_1,...,a_n)}$ are the same. We assume $X_{a_p a_{p+1}} = X_{a_q a_{q+1}}$, where $p \neq q$ and $p+1, q+1$ means the addition mod n. Under this condition, we consider some examples. First, we have $a_p = a_q$ and $a_{p+1} = a_{q+1}$ and the other a_i are distinct. Some examples are

$$X_{11}X_{11}, p = 1, q = 2,$$
$$X_{12}X_{21}X_{12}X_{23}X_{31}, p = 1, q = 3.$$

These are cases we want to study.

Of course, there are other examples. For instance,

$$X_{11}X_{11}X_{12}X_{21}.$$

In this example, we have X_{11}^2 and another term X_{12}, which means there are some other a_i such that $a_i = a_p$. To solve this type of question, it is a combination of **Case 2** and **Case 3**. We will not discuss it here.

Now, let's consider the problem that only two terms in $X_{(a_1,...,a_n)}$ are the same

$$X_{a_p a_{p+1}} = X_{a_q a_{q+1}}, \quad a_p = a_q, a_{p+1} = a_{q+1}, \quad p \neq q,$$

and the other a_i are distinct. In this case, we still consider the operator $D_{(a_{i_1},...,a_{i_d})}$.

1. If all $a_{i_j} \neq a_p$, then $D_{(a_{i_1},...,a_{i_d})}$ corresponds to a unique element in the permutation group.

   Under this condition, although $a_{i_j} \neq a_p$, a_{i_j} could be a_{p+1}. By our assumptions that only two terms in $X_{(a_1,...,a_n)}$ are the same, hence there are $\binom{n-2}{d}d!$ d-tuples $(a_{i_1}, ..., a_{i_d})$ satisfying this condition and each of them corresponds to a unique d-tuple in S_n^t by the conclusion of **Case 2**.

2. Only one integer in $\{a_{i_j}\}_{1\le j\le d}$ is a_p, say $a_{i_k}=a_p$.

   First, assume all a_{i_j} are not a_{p+1}. Then, we have

$$\begin{aligned}
&D_{a_{i_{k-1}}a_{i_k}}X_{(a_1,\dots,a_n)}=D_{a_{i_{k-1}}a_p}X_{a_1a_2}\dots X_{a_na_1}=\\
=&(\sum_{c\ge 1}X_{a_{i_{k-1}}c}\frac{\partial}{\partial X_{a_pc}})X_{a_1a_2}\dots X_{a_na_1}=\\
=&(X_{a_{i_{k-1}}a_{p+1}}\frac{\partial}{\partial X_{a_pa_{p+1}}})X_{a_1a_2}\dots X_{a_na_1}=\\
=&(X_{a_{i_{k-1}}a_{p+1}}\frac{\partial}{\partial X_{a_pa_{p+1}}})X^2_{a_pa_{p+1}}\dots.
\end{aligned}$$

   The last equality holds because we have $X_{a_pa_{p+1}}=X_{a_qa_{q+1}}$. We note there is a square $X^2_{a_pa_{p+1}}$ in the monomial $X_{(a_1,\dots,a_n)}$. Hence, we will get two (same) monomials at last. Compared with $(t_{i_1},\dots,t_{i_d})$, this differential operator $D_{(a_{i_1},\dots,a_{i_d})}$ corresponds to two d-tuples in S_n^t. They are

$$\begin{aligned}
&(t_{i_1},\dots,t_{i_{k-1}},t_p,t_{i_{k+1}},\dots,t_{i_d}),\\
&(t_{i_1},\dots,t_{i_{k-1}},t_q,t_{i_{k+1}},\dots,t_{i_d}).
\end{aligned}$$

   Hence, each differential operator in this type corresponds to two different elements in the permutation group S_n^t.

   Similarly, if some a_{i_j} are a_{p+1}, then the conclusion follows by the combination of the above argument and the argument in **Case 2**. (If it contains both a_p and a_q, then it corresponds to 4 permutations.)

   We conclude all non-trivial differential operators $D_{(a_{i_1},\dots,a_{i_d})}$ in the case correspond to $\binom{2}{1}\binom{n-2}{d-1}d!$ d-tuples in t_i.

3. Two of the integers $a_{i_j}, 1\le j\le d$ are a_p and we assume they are $a_{i_l}=a_{i_k}=a_p$.

Similarly, assume all a_{i_j} are not a_{p+1}. We have

$$\begin{aligned}
&: D_{a_{i_{l-1}}a_{i_l}} D_{a_{i_{k-1}}a_{i_k}} : X_{a_1a_2}...X_{a_na_1}\\
=&: D_{a_{i_{l-1}}a_p} D_{a_{i_{k-1}}a_p} : X_{a_1a_2}...X_{a_na_1}\\
=&: D_{a_{i_{l-1}}a_p} D_{a_{i_{k-1}}a_p} : X^2_{a_pa_{p+1}}...\\
=&(\sum_{c,d\geq 1} X_{a_{i_{l-1}}c} X_{a_{i_{k-1}}d} \frac{\partial}{\partial X_{a_pc}} \frac{\partial}{\partial X_{a_pd}}) X^2_{a_pa_{p+1}}...\\
=&(X_{a_{i_{l-1}}a_{p+1}} X_{a_{i_{k-1}}a_{p+1}} \frac{\partial^2}{\partial^2 X_{a_pa_{p+1}}}) X^2_{a_pa_{p+1}}....
\end{aligned}$$

Note we have a square $X^2_{a_pa_{p+1}}$. Hence, we will get two (equal) monomials.

Compared with $(t_{i_1}, ..., t_{i_d})$, this differential operator $D_{(a_{i_1},...,a_{i_d})}$ corresponds to two d-tuples. They are

$$\begin{aligned}
&(t_{i_1}, ..., t_{i_k}, ..., t_{i_l}, ..., t_{i_d}),\\
&(t_{i_1}, ..., t_{i_l}, ..., t_{i_k}, ..., t_{i_d}).
\end{aligned}$$

Hence, $D_{(a_{i_1},...,a_{i_d})}$ corresponds to two different elements in the permutation group S^t_n.

Similarly, if some a_{i_j} are a_{p+1}, then the conclusion follows by the combination of the above argument and **Case 2**. (If it contains both a_p and a_q, then it corresponds to 4 permutations.)

We conclude all non-trivial differential operators $D_{(a_{i_1},...,a_{i_d})}$ in the case correspond to $\binom{n-2}{d-2}d!$ d-tuples in t_i.

By the discussion above, this correspondence is unique. Hence, in this case, we have

$$d \times W([d])M_{g_t}(a_1, ...a_n) = d \times \sum_{\sigma\ d\text{-cycle in } S^t_n} M_{\sigma_t}(a, ..., a_n).$$

For the general case that there are k same factors in $X_{a_1a_2}...X_{a_na_1}$, the same argument proves what we want. We leave it to the reader to check.

Combining the above three cases, we get the following formula by summing over all monomials $M_{g_t}(a_1, ..., a_n) = X_{(a_1,...,a_n)}$ of $\Phi(g)$,

$$\Phi(K_{1^{n-d}d}g) = W([d])\Phi(g).$$

□

2.10 Another Definition of $W([d])$

In this section, we will consider $W([n])$ as a differential operator on the ring $\mathbb{C}[p_1, p_2, ...]$ or $\mathbb{C}[[p_1, p_2, ...]]$ by Theorem 2.3.1.

2.10.1 Definition of Δ_d

Consider the cut-and-join operator Δ [11],

$$\Delta = \frac{1}{2}\sum_{i\geq 1}\sum_{j\geq 1}(ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j} + (i+j)p_i p_j \frac{\partial}{\partial p_{i+j}}). \tag{2.10.1}$$

We have the following proposition.

Proposition 2.10.1. *For any $g \in \mathbb{C}S_n$,*

$$\Phi(K_{1^{n-2}2}g) = \Delta\Phi(g).$$

Proof. Goulden proves this in [11] Prop 3.1. □

Definition 2.10.2. *For any permutation $\delta \in S_d$, write $\delta = \delta_1 ... \delta_m$, which is the decomposition of δ into disjoint cycles. For a positive integer $N \leq d$, say $N \in \delta_i$ if $\delta_i(N) \neq N$. Fix d positive integers a_j, where $1 \leq j \leq d$. Define $\hat{p}_\delta(a_1, ..., a_d)$ to be the monomial*

$$\hat{p}_\delta(a_1, ..., a_d) = \prod_{i=1}^{m} p_{\sum_{j\in\delta_i} a_j}.$$

Similarly, define $\frac{\partial}{\partial \hat{p}_\delta}(a_1, ..., a_d)$ to be the operator on $\mathbb{C}[[p_1, p_2, ...]]$,

$$\frac{\partial}{\partial \hat{p}_\delta}(a_1, ..., a_d) = \prod_{i=1}^{m}((\sum_{j\in\delta_i} a_j)\frac{\partial}{\partial p_{\sum_{j\in\delta_i} a_j}}).$$

If we fix positive integers d and a_1,...,a_d, we abbreviate $\hat{p}_\delta(a_1, ..., a_d)$ by $\hat{p}_\delta$ and $\frac{\partial}{\partial \hat{p}_\delta}(a_1, ..., a_d)$ by $\frac{\partial}{\partial \hat{p}_\delta}$.

Example 2.10.3. *Let $\delta = (123)(4) \in S_4$, then we have*

$$\hat{p}_\delta(a_1, ..., a_4) = p_{a_1+a_2+a_3} p_{a_4},$$
$$\frac{\partial}{\partial \hat{p}_\delta}(a_1, ..., a_4) = (a_1 + a_2 + a_3) a_4 \frac{\partial^2}{\partial p_{a_1+a_2+a_3} \partial p_{a_4}}.$$

Remark 2.10.4. *Given $\delta \in S_d$, we consider $\hat{p}_\delta$ as a map from $\mathbb{Z}^d_{>0}$ to $\mathbb{C}[p_1, p_2, ...]$ and $\frac{\partial}{\partial \hat{p}_\delta}$ as a map from $\mathbb{Z}^d_{>0}$ to $\mathbb{C}[\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, ...]$. Generally, we can introduce variables t_i and we write $\hat{p}_\delta$ and $\frac{\partial}{\partial \hat{p}_\delta}$ in the following form similar to Definition 2.10.2,*

$$\hat{p}_\delta(t_1, ..., t_d) = \prod_{i=1}^{m} p_{\sum_{j \in \delta_i} t_j},$$
$$\frac{\partial}{\partial \hat{p}_\delta}(t_1, ..., t_d) = \prod_{i=1}^{m} ((\sum_{j \in \delta_i} t_j) \frac{\partial}{\partial p_{\sum_{j \in \delta_i} t_j}}).$$

Definition 2.10.5. *Consider the d-cycle $(d\ ...\ 2\ 1)$ in S_d. We define the bijective map ϕ_d of S_d as*

$$\phi_d(\delta) = (d\ ...\ 1)\delta, \quad \delta \in S_d.$$

If we fix d, we will use ϕ to represent this map.

Definition 2.10.6. *We define the differential operator Δ_d on the polynomial ring $\mathbb{C}[p_1, p_2, ...]$ as*

$$\Delta_d = \frac{1}{d} \sum_{\delta \in S_d} \sum_{a_1, ..., a_d \geq 1} \hat{p}_{\phi(\delta)}(a_1, ..., a_d) \frac{\partial}{\partial \hat{p}_\delta}(a_1, ..., a_d).$$

Remark 2.10.7. *The construction depends on the map $\phi_d(\sigma) = (d\ ...\ 1)\sigma$, where $(d\ ...\ 1)$ is the d-cycle. Actually, we can take any d-cycle in S_d to substitute $(d...1)$ to define the map, which will give the same operator Δ_d. We will obtain this property in the proof of the theorem in this section.*

Example 2.10.8.

$$\Delta_2 = \frac{1}{2} \sum_{i \geq 1} \sum_{j \geq 1} (ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}}),$$

where the first part corresponds to $(1)(2) \in S_2$ and the second part corresponds to $(12) \in S_2$. Clearly, Δ_2 is the cut-and-join operator Δ (2.10.1).

$$
\begin{aligned}
\Delta_3 = \frac{1}{3} \sum_{i_1,i_2,i_3 \geq 1} (i_1 i_2 i_3 p_{i_1+i_2+i_3} \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}} + & \qquad (1)(2)(3) \\
+ i_1(i_2+i_3) p_{i_1+i_3} p_{i_2} \frac{\partial^2}{\partial p_{i_1} \partial p_{i_2+i_3}} + & \qquad (1)(23) \\
+ i_2(i_1+i_3) p_{i_1+i_2} p_{i_3} \frac{\partial^2}{\partial p_{i_2} \partial p_{i_1+i_3}} + & \qquad (2)(13) \\
+ i_3(i_1+i_2) p_{i_3+i_2} p_{i_1} \frac{\partial^2}{\partial p_{i_3} \partial p_{i_1+i_2}} + & \qquad (3)(12) \\
+ (i_1+i_2+i_3) p_{i_1} p_{i_2} p_{i_3} \frac{\partial}{\partial p_{i_1+i_2+i_3}} + & \qquad (123) \\
+ (i_1+i_2+i_3) p_{i_1+i_2+i_3} \frac{\partial}{\partial p_{i_1+i_2+i_3}}) & \qquad (132).
\end{aligned}
$$

where the third column is the permutation, which the summation corresponds to.

Definition 2.10.9. *Let n and d be positive integers, $d \leq n$. $C_{n,d}$ is the set of all d-cycles in S_n and $\bar{C}_{n,d}$ is the set of all d-tuples $[a_1, ..., a_d]$ with positive integers a_i such that $1 \leq a_i \leq n$ and $a_i \neq a_j$ if $i \neq j$.*

Next, we define a map $\pi_{n,d} : \bar{C}_{n,d} \to C_{n,d}$ such that

$$\pi_{n,d}([a_1, ..., a_d]) = (a_1 \ ... \ a_d).$$

Clearly, this map is d-to-1.

Given an d-tuple $\bar{\sigma} \in \bar{C}_{n,d}$ and a permutation $g \in S_n$, we define the action of $\bar{C}_{n,d}$ on S_n as follows,

$$\bar{\sigma} g := \pi_{n,d}(\bar{\sigma}) g.$$

Define $\mathbb{C}\bar{C}_{n,d} = \oplus_{[a_1,...,a_d] \in \bar{C}_{n,d}} \mathbb{C}[a_1, ..., a_d]$ as the vector space with basis the elements of $\bar{C}_{n,d}$, define the element $\bar{K}_{1^{n-d}d} \in \mathbb{C}\bar{C}_{n,d}$ as the sum of all d-tuples in $\bar{C}_{n,d}$.

In this paper, given positive integers n and d, we abbreviate $\pi_{n,d}$ by π and consider π as a linear map from $\mathbb{C}\bar{C}_{n.d}$ to $\mathbb{C}C_{n,d}$.

We are going to use $\bar{K}_{1^{n-d}d}$ to show that $\Phi(K_{1^{n-d}d} g) = \Delta_d \Phi(g)$.

2.10.2 Case $d = 3$

Given $\bar{\sigma} \in \bar{C}_{n,3}$ and $g \in S_n$, we will calculate $\bar{\sigma}g$ and translate it into differential operators and polynomials.

Construction 2.10.10. *Write $\bar{\sigma} = [j_3, j_2, j_1]$. We are going to classify elements $g \in S_n$ according to the occurrence of j_1, j_2, j_3 in the disjoint cycles appearing in g. There are 6 cases with respect to $\bar{\sigma}$, one for each permutation of S_3,*

1. $g = (j_1 \ldots)(j_2 \ldots)(j_3 \ldots) \ldots$,
2. $g = (j_1 \ldots)(j_2 \ldots j_3 \ldots) \ldots$,
3. $g = (j_1 \ldots j_3 \ldots)(j_2 \ldots) \ldots$,
4. $g = (j_1 \ldots j_2 \ldots)(j_3 \ldots) \ldots$,
5. $g = (j_1 \ldots j_2 \ldots j_3 \ldots) \ldots$,
6. $g = (j_1 \ldots j_3 \ldots j_2 \ldots) \ldots$.

Clearly, for any element $g \in S_n$, it falls into one and only one case with respect to $\bar{\sigma}$. Now, consider case (4) $g = (j_1 \colorbox{red}{$\cdots$} j_2 \colorbox{blue}{$\cdots$})(j_3 \colorbox{green}{$\cdots$}) \ldots$, where the red dots represent the digits after j_1 before j_2, the blue dots represent the other digits after j_2 before j_1 (since it is a cycle, so the last element will go back to j_1) and the green points represent the other digits in the cycle of j_3. We use the following steps to calculate $\bar{\sigma}g$:

1. *Restrict $g = (j_1 \colorbox{red}{$\cdots$} j_2 \colorbox{blue}{$\cdots$})(j_3 \colorbox{green}{$\cdots$}) \ldots$ to the element $(j_1 j_2)(j_3)$ by forgetting all digits except j_1,j_2,j_3 but preserving the cycle structure. $(j_1 j_2)(j_3)$ can be considered as an element in $Aut\{j_1, j_2, j_3\}$. Let $g_{\bar{\sigma}} = (j_1 j_2)(j_3)$.*
2. *Calculate $[j_3, j_2, j_1]g_{\bar{\sigma}} = (j_1)(j_2 j_3)$.*
3. *Insert all numbers forgotten by the restriction into $\bar{\sigma}g_{\bar{\sigma}}$, then we have the consequence,*

$$\bar{\sigma}g = (j_1 \colorbox{red}{$\cdots$})(j_2 \colorbox{green}{$\cdots$} j_3 \colorbox{blue}{$\cdots$}) \ldots$$

Actually, this procedure works for all cases.

Remark 2.10.11. • *Let $\bar{\sigma} = [3,2,1]$ and $\bar{\sigma'} = [1,3,2]$. Although $\pi(\bar{\sigma}) = \pi(\bar{\sigma'}) = (132)$, $g_{\bar{\sigma}}$ and $g_{\bar{\sigma'}}$ are not in the same type in general. For instance, assume $g = (12)(3)$. Consider $\bar{\sigma} = [3,2,1]$, so that hence $g_{\bar{\sigma}} = (j_1 j_2)(j_3)$, which is in Case (4). Now, consider $\bar{\sigma'} = [132]$, so that $g_{\bar{\sigma'}} = (j_3 j_1)(j_2)$, which is in Case (3).*

- *Given different $\bar{\sigma}_1$, $\bar{\sigma}_2$, we can get $g_{\bar{\sigma}_1} = g_{\bar{\sigma}_2}$. For example, if $g = (321)$,$\bar{\sigma}_1 = [3,2,1]$ and $\bar{\sigma}_2 = [1,3,2]$, then we have $g_{\bar{\sigma}_1} = (321) = (213) = g_{\bar{\sigma}_2}$.*

Remark 2.10.12. *Let g be a permutation in S_n, $n \geq 3$. We consider two 3-tuples $\bar{\sigma} = [1,2,3]$ and $\bar{\sigma}' = [j_3, j_2, j_1]$, $j_1, j_2, j_3 \leq n$. Clearly, $g_{\bar{\sigma}'} \in Aut\{j_3, j_2, j_1\}$ and $g_{\bar{\sigma}} \in Aut\{1,2,3\}$. But, we want to compare the two permutations in the same permutation group $S_3 = Aut\{1,2,3\}$. Hence, we have to fix a bijective map between $\{1,2,3\}$ and $\{j_3, j_2, j_1\}$. We construct the map by sending the largest integer in $\{j_3, j_2, j_1\}$ to 3, smallest one to 1 and the last one to 2. This map will induce an isomorphism $\emptyset : Aut\{j_3, j_2, j_1\} \to Aut\{1,2,3\}$. Hence, by an abuse of notations, $g_{\bar{\sigma}'} \in S_3$ means $\emptyset(g_{\bar{\sigma}'}) \in S_3$.*

Definition 2.10.13 (Reduction Permutation)**.** *We say that β is the **reduction permutation** of $(g, \bar{\sigma})$ or β is the **reduction permutation** of g with respect to $\bar{\sigma}$, if $g_{\bar{\sigma}} = \beta$ as discussed in Construction .*

Definition 2.10.14 (Distance)**.** *Let $\sigma = (j_3\ j_2\ j_1)$ be a 3-cycle in S_n (or a 3-tuple $\bar{\sigma} = [j_3, j_2, j_1]$) and $\alpha = \alpha_1 ... \alpha_l$ be any permutation in S_n, where $\alpha_1 ... \alpha_l$ is the unique product of disjoint cycles. The set for fixed integer i, $1 \leq i \leq 3$,*

$$\{l \mid \alpha^l(j_i) \text{ is any } j_k, 1 \leq k \leq 3, l \geq 1\}$$

is nonempty, because $\alpha^{n!}$ is the identity map on the set $\{1, ..., n\}$, so $\alpha^{n!}(j_i) = j_i$ implies that $n!$ is contained in this set.

We define the "distance" between j_i and the set $\{j_1, ..., j_3\}$ with respect to the permutation α as

$$dist(j_i, \alpha, j_1, j_2, j_3) = min\{l \mid \alpha^l(j_i) \text{ is any } j_k, 1 \leq k \leq 3, l \geq 1\}.$$

Example 2.10.15. *We give some examples about the definition above. Consider Case (5) in Construction 2.10.10,*

$$\sigma = (j_3\ j_2\ j_1), \quad \alpha = (j_1 ... j_2 ... j_3 ...)\alpha_2 ... \alpha_l,$$

where $\alpha_1 = (j_1 ... j_2 ... j_3 ...)$*.* $dist(j_3, \alpha, j_1, j_2, j_3)$ *is the "distance" between* j_3 *and* j_1 *in the cycle* α_1*, because* j_1 *is the first element in* $\{j_1.j_2, j_3\}$ *after* j_3 *under the action of* α*. Similarly,* $dist(j_2, \alpha, j_1, j_2, j_3)$ *is the "distance" between* j_2 *and* j_3*. Clearly,* $\sum_{1 \leq i \leq 3} dist(j_i, \alpha, j_1, j_2, j_3)$ *is the length of the cycle* α_1*.*

Now, let's consider Case (1) in Construction 2.10.10. Here,

$$\alpha = (j_1 ...)(j_2 ...)(j_3 ...)\alpha_4 ... \alpha_l.$$

In this case, $dist(j_i, \alpha, j_1, j_2, j_3)$ *is the length of the cycle containing* j_i*.*

Remark 2.10.16. α, ω *are permutations in* S_n*, where* ω *is a d-cycles* $(j_d \ ... \ j_1)$*. Let* $\alpha' = \omega\alpha$*. Then, we have*

$$dist(j_i, \alpha, j_1, j_2, j_3) = dist(j_i, \alpha', j_1, j_2, j_3), \quad 1 \leq i \leq d.$$

This property comes from the calculation in Construction 2.10.10.

Definition 2.10.17. *Given any permutation* $\alpha \in S_n$*, we define the map*

$$\begin{aligned} &I_{\alpha,n,3} : \bar{C}_{n,3} \to \mathbb{Z}^3_{>0}, \\ &I_{\alpha,n,3}([j_3, j_2, j_1]) = (i_3, i_2, i_1), \end{aligned}$$

where $i_k = dist(j_k, \alpha, j_1, j_2, j_3), 1 \leq k \leq 3$*.*

Definition 2.10.18. *Let* α *be a permutation in* S_n *and let* i_k *be positive integers,* $1 \leq k \leq 3$*. Let* β *be a* 3*-cycle in* S_3*. Define the subset* $\bar{C}^{\beta}_{n,3}(\alpha, i_3, i_2, i_1)$ *of* $\bar{C}_{n,3}$ *as*

$$\begin{aligned} \bar{C}^{\beta}_{n,3}(\alpha, i_3, i_2, i_1) = \{[j_3, j_2, j_1] \mid i_k = dist(j_k, \alpha, j_1, j_2, j_3), 1 \leq k \leq 3, \\ \beta \text{ is the RP of } (\alpha, [j_3, j_2, j_1])\}. \end{aligned}$$

Remark 2.10.19. *Let* α *be a permutation in* S_n*. We have*

$$\bar{C}_{n,3} = \bigcup_{\beta \in S_3}^{6} \bigcup_{i_1, i_2, i_3 \geq 1} \bar{C}^{\beta}_{n,3}(\alpha, i_3, i_2, i_1).$$

Given any 3*-tuple* $[j_3, j_2, j_1]$*, the "distance"* $dist(j_i, \alpha, j_1, ..., j_3)$ *and the type of* $(\alpha, [j_3, j_2, j_1])$ *are uniquely determined. Hence, the union above is the disjoint union. Also, there are only finitely many nonempty sets* $\bar{C}_{n,3}(\alpha, i_3, i_2, i_1)$ *in the above union.*

Lemma 2.10.20. *Let α be a permutation in S_n and let i_1, i_2, i_3 be three positive integers. If $\bar{C}^{\beta}_{n,3}(\alpha, i_3, i_2, i_1)$ is nonempty, we have*

$$\Phi(\sum_{[j_3,j_2,j_1]\in\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)}[j_3, j_2, j_1]\alpha) == \hat{p}_{\phi(\beta)}(i_1, i_2, i_3)\frac{\partial}{\partial \hat{p}_{\beta}}(i_1, i_2, i_3)\Phi(\alpha).$$

Proof. We only give the proof when $\beta = (1)(2)(3)$,

$$\Phi(\sum_{[j_3,j_2,j_1]\in\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)}[j_3, j_2, j_1]\alpha) = i_1 i_2 i_3 p_{i_1+i_2+i_3}\frac{\partial^3\Phi(\alpha)}{\partial p_{i_1}\partial p_{i_2}\partial p_{i_3}}.$$

The other formulas can be proved similarly.

First, we make some assumptions and define some notations. Let α be a permutation in S_n and let $\bar{\sigma}$ be an element in $\bar{C}^{\beta}_{n,3}(\alpha, i_3, i_2, i_1)$. We use the same notations for α and $\bar{\sigma}$ as in Construction 2.10.10, i.e.

$$\alpha = (j_1...)(j_2...)(j_3...)\alpha_4...\alpha_l, \quad \bar{\sigma} = [j_3, j_2, j_1].$$

Also, by Definition 2.10.14, we have

$$i_k = dist(j_k, \alpha, j_1, j_2, j_3), \quad 1 \leq k \leq 3.$$

We assume the lengths of disjoint cycles α_v, $4 \leq v \leq l$, are not i_1, i_2, i_3.

By simple calculations, we have

$$\begin{aligned} &\alpha = (j_1...)(j_2...)(j_3...)\rho_4...\rho_l && \rightarrow \quad \bar{\sigma}\alpha = (j_3...j_2...j_1...)\rho_4...\rho_l \\ &\Phi(\alpha) = p_{i_1}p_{i_2}p_{i_3}\Phi(\rho_4...\rho_l) && \rightarrow \quad \Phi(\bar{\sigma}\alpha) = p_{i_1+i_2+i_3}\Phi(\rho_4...\rho_l)\ , \end{aligned}$$

and

$$p_{i_1+i_2+i_3}\frac{\partial^3}{\partial p_{i_1}\partial p_{i_2}\partial p_{i_3}}\Phi(\alpha) = \Phi(\bar{\sigma}\alpha).$$

Clearly, for any element $\bar{\sigma}'$ in $\bar{C}^{\beta}_{n,3}(\alpha, i_3, i_2, i_1)$, we have

$$\Phi(\bar{\sigma}'\alpha) = \Phi(\bar{\sigma}\alpha),$$

Hence, the differential operators should be the same, which means

$$p_{i_1+i_2+i_3}\frac{\partial^3}{\partial p_{i_1}\partial p_{i_2}\partial p_{i_3}}\Phi(\alpha)=\Phi(\bar{\sigma}\alpha)=\Phi(\bar{\sigma}'\alpha).$$

Now we want to find the number of elements in the set $\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)$. Since the lengths of the disjoint cycles $\rho_4,...,\rho_l$ are not i_1,i_2,i_3, so if we want to get the same differential operator, we only care about the cycles $(j_1...),(j_2...),(j_3...)$. We take one integer from each of the three cycles. Say we take j'_i from the cycle $(j_i...)$. They form a unique element $[j'_3,j'_2,j'_1]\in\bar{C}_{n,3}$. Clearly, $[j'_3,j'_2,j'_1]\in\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)$. There are $i_1i_2i_3$ possible choices. Hence, the number of elements in the set $\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)$ is $i_1i_2i_3$.

The discussion above gives the first formula

$$\Phi(\sum_{[j_3,j_2,j_1]\in\bar{C}^1_{n,3}(\alpha,i_3,i_2,i_1)}[j_3,j_2,j_1]\alpha)=i_1i_2i_3p_{i_1+i_2+i_3}\frac{\partial^3\Phi(\alpha)}{\partial p_{i_1}\partial p_{i_2}\partial p_{i_3}}.$$

For the general case that not all lengths of $\alpha_4,...,\alpha_l$ are different from i_1,i_2,i_3, the construction still works. We only consider a special case that the length of α_4 equal to the length of $(j_1...)$, which is i_1. In this case, the number of elements in $\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)$ is $2i_1i_2i_3$. Clearly, half of them come from the three cycles $(j_1...),(j_2...),(j_3...)$ and the others come from $\alpha_4,(j_2...),(j_3...)$. But, the formula in this case is still the same

$$\Phi(\sum_{[j_3,j_2,j_1]\in\bar{C}^1_{n,3}(\alpha,i_3,i_2,i_1)}[j_3,j_2,j_1]\alpha)=i_1i_2i_3p_{i_1+i_2+i_3}\frac{\partial^3\Phi(\alpha)}{\partial p_{i_1}\partial p_{i_2}\partial p_{i_3}}.$$

The reason is when we calculate the differential part $\frac{\partial^3\Phi(\alpha)}{\partial p_{i_1}\partial p_{i_2}\partial p_{i_3}}$, the degree of p_{i_1} in the monomial $\Phi(\alpha)$ is 2, so we will have a coefficient 2.

We leave the general case for the reader to check. $\square$

Remark 2.10.21. *The consequence of Lemma 2.10.20 works for any set* $\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)$*, which means if* $\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)=\emptyset$*, it also works. We only explain the reason for the case* $\beta=(1)(2)(3)$*. If* $\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)=\emptyset$*, it means one of* i_1,i_2,i_3 *is not the length of any disjoint cycle of* α*. Hence, we have*

$$\frac{\partial^3\Phi(\alpha)}{\partial p_{i_1}\partial p_{i_2}\partial p_{i_3}}=0,$$

which implies

$$0 = \Phi(\sum_{[j_3,j_2,j_1]\in\bar{C}^1_{n,3}(\alpha,i_3,i_2,i_1)} [j_3, j_2, j_1]\alpha) = i_1 i_2 i_3 p_{i_1+i_2+i_3} \frac{\partial^3 \Phi(\alpha)}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}} = 0.$$

Now we are ready to prove the theorem.

Theorem 2.10.22. *Let g be an element in $\mathbb{C}S_n$. We have*

$$3\Phi(K_{31^{n-3}}g) = \Phi(\bar{K}_{31^{n-3}}g) = 3\Delta_3\Phi(g).$$

Proof. We assume g is a permutation in S_n. Say $g = \alpha$. By Remark 2.10.19, we have

$$\bar{C}_{n,3} = \bigcup_{\beta\in S_3}^{6} \bigcup_{i_1,i_2,i_3\geq 1} \bar{C}^{\beta}_{n,3}(\alpha, i_3, i_2, i_1).$$

Then, we get

$$\begin{aligned}
\Phi(\bar{K}_{31^{n-3}}g) &= \Phi(\sum_{\beta\in S_3}^{6} \sum_{i_1,i_2,i_3\geq 1} \sum_{[j_3,j_2,j_1]\in\bar{C}^{\beta}_{n,3}(\alpha,i_3,i_2,i_1)} [j_3, j_2, j_1]\alpha) \\
&= \sum_{i_1,i_2,i_3\geq 1} (i_1 i_2 i_3 p_{i_1+i_2+i_3} \frac{\partial^3}{\partial p_{i_1} \partial p_{i_2} \partial p_{i_3}} \\
&+ i_1(i_2+i_3) p_{i_1+i_3} p_{i_2} \frac{\partial^2}{\partial p_{i_1} \partial p_{i_2+i_3}} \\
&+ i_2(i_1+i_3) p_{i_1+i_2} p_{i_3} \frac{\partial^2}{\partial p_{i_2} \partial p_{i_1+i_3}} \\
&+ i_3(i_1+i_2) p_{i_3+i_2} p_{i_1} \frac{\partial^2}{\partial p_{i_3} \partial p_{i_1+i_2}} \\
&+ (i_1+i_2+i_3) p_{i_1} p_{i_2} p_{i_3} \frac{\partial}{\partial p_{i_1+i_2+i_3}} \\
&+ (i_1+i_2+i_3) p_{i_1+i_2+i_3} \frac{\partial}{\partial p_{i_1+i_2+i_3}})\Phi(g) \\
&= 3\Delta_3\Phi(g),
\end{aligned}$$

where the second equality comes from Lemma 2.10.20 and the last equality comes from Definition 2.10.6 or Example 2.10.8.

□

We now give the extended definition of ϕ (Definition 2.10.5) and construction of Δ_d (Definition 2.10.6) if we choose arbitrary d-cycle.

Definition 2.10.23. *Given an d-cycle β in S_d, we define the map $\phi_\beta : S_d \to S_d$ as*

$$\phi_\beta(\delta) = \beta\delta, \quad \delta \in S_d.$$

Then, we construct Δ_β similar to definition 2.10.5, replacing ϕ by ϕ_β,

$$\Delta_\beta = \frac{1}{d} \sum_{\delta \in S_d} \sum_{a_1,...,a_d \geq 1} \hat{p}_{\phi_\beta(\delta)}(a_1, ..., a_d) \frac{\partial}{\partial \hat{p}_\delta}(a_1, ..., a_d).$$

Remark 2.10.24. *From this definition, it is clear $\Delta_{(321)} = \Delta_3$.*

Remark 2.10.25. *Recall the formula in Lemma 2.10.20,*

$$i_1 i_2 i_3 \Phi([j_3, j_2, j_1]\alpha) = \hat{p}_{\phi((1)(2)(3))}(i_1, i_2, i_3) \frac{\partial}{\partial \hat{p}_{(1)(2)(3)}}(i_1, i_2, i_3)\Phi(\alpha).$$

Similarly, we can prove

$$i_1 i_2 i_3 \Phi([j_1, j_2, j_3]\alpha) = \hat{p}_{\phi_\beta((1)(2)(3))}(i_1, i_2, i_3) \frac{\partial}{\partial \hat{p}_{(1)(2)(3)}}(i_1, i_2, i_3)\Phi(\alpha),$$

where $\beta = (1\ 2\ 3)$. Actually, the map ϕ_β corresponds to tuple $[j_1, j_2, j_3]$. We can prove the other cases similarly.

Corollary 2.10.26. *For any 3-cycle β, $\Delta_3 = \Delta_\beta$ as operators on the ring $\mathbb{C}[p_1, p_2, ...]$.*

Proof. Let $\beta = (123)$. We have

$$\begin{aligned}
\Delta_3\Phi(g) =& \frac{1}{3}\Phi(\sum_{[j_3,j_2,j_1] \in \bar{C}_{n,3}} [j_3, j_2, j_1]g) \\
&= \Phi(\bar{K}_{31^{n-3}}g) \\
&= \Phi(\sum_{[j_1,j_2,j_3] \in \bar{C}_{n,3}} [j_1, j_2, j_3]g) \\
&= \Delta_\beta\Phi(g),
\end{aligned}$$

where the last equality comes from Remark 2.10.25.

Hence, $\Delta_\beta = \Delta_3$ as operators on $\mathbb{C}[p_1, p_2, ...]$. $\square$

Remark 2.10.27. *The above argument can be extended to Δ_d, $d \geq 4$, i.e., for any d-cycle β, $\Delta_\beta = \Delta_d$. This will be shown in Corollary* **??**.

2.10.3 General Case

The proof of the general case is very similar to the case $d = 3$. First, we generalize Construction 2.10.10, Definition 2.10.13 and 2.10.14 to any positive integer d.

Construction 2.10.28 (Reduction Permutation)**.** *Let $\bar{\sigma} = [j_d, ..., j_1] \in \bar{C}_{n,d}$. We want to classify all permutations $g \in S_n$ according to the occurrence of $j_1, ..., j_d$ in the disjoint cycles appearing in g. There are $d!$ cases, one for each permutation in S_d. Here, S_d is the permutation group of $\{j_1, ..., j_d\}$. By an abuse the notation, we use the same notation. Restrict g to a permutation in S_d by forgetting all digits except for $j_1, ..., j_d$ but preserving the cycle structure. Denote by $g_{\bar{\sigma}}$ the permutation in S_d (similar to the construction of $g_{\bar{\sigma}}$ in Construction 2.10.10). We say that $\beta = g_{\bar{\sigma}}$ is the reduction permutation of $(g, \bar{\sigma})$ or β is the reduction permutation of g with respect to $\bar{\sigma}$. Clearly, for any element $g \in S_n$, g falls into one and only one case with respect to $\bar{\sigma}$.*

We want to explain the notation $\tau = g_{\bar{\sigma}} \in S_d$ in the above construction.

Remark 2.10.29. *Let g be a permutation in S_n, $n \geq d$. We consider two d-tuples $\bar{\sigma} = [d, d-1, ..., 2, 1]$ and $\bar{\sigma}' = [j_d, ..., j_1]$ in $\bar{C}_{n,d}$. Clearly, $g_{\bar{\sigma}'} \in Aut\{j_d, ..., j_1\}$ and $g_{\bar{\sigma}} \in S_d = Aut\{1, 2, ..., d\}$. But, we want to compare the two permutations in the same permutation group $S_3 = Aut\{1, 2..., d\}$. Recall the construction in Remark 2.10.12. Similarly, we construct the bijective map between $\{1, ..., d\}$ and $\{j_1, ..., j_d\}$ with respect to the order of the integers, which means small integer maps to the small one and larger integer goes to larger one. This map induces an isomorphism $\emptyset : Aut\{j_d, ..., j_1\} \to Aut\{1, ..., d\}$. Hence, by an abuse of notations, $g_{\bar{\sigma}'} \in S_d$ means $\emptyset(g_{\bar{\sigma}'}) \in S_d$.*

Definition 2.10.30 (Distance)**.** *Let $\sigma = (j_d \ ... \ j_1)$ be a d-cycle in S_n (or a d-tuple $\bar{\sigma} = [j_3, j_2, j_1]$) and $\alpha = \alpha_1 ... \alpha_l$ be any permutation in S_n, where $\alpha_1 ... \alpha_l$ is the unique product of disjoint cycles. We define the "distance" between j_i and the set $\{j_1, ..., j_d\}$ with respect to the permutation α as*

$$dist(j_i, \alpha, j_1, ..., j_d) = min\{l \mid \alpha^l(j_i) \text{ is any } j_k, 1 \leq k \leq d, l \geq 1\}.$$

Definition 2.10.31. *Given any permutation $\alpha \in S_n$ and a positive integer d such that $d \le n$, we define the map*

$$I_{\alpha,n,d} : \bar{C}_{n,d} \to \mathbb{Z}^3_{>0},$$
$$I_{\alpha,n,d}([j_d, ..., j_1]) = (i_d, ..., i_1),$$

where $i_k = dist(j_k, \alpha, j_1, ..., j_d), 1 \le k \le d$.

Definition 2.10.32. *Let α be a permutation in S_n. Let d be a positive integer such that $d \le n$. i_k are positive integers, $1 \le k \le d$. Let β be a permutation in S_d. We define the subset $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$ of $\bar{C}_{n,d}$ as*

$$\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d) = \{[j_d, ..., j_1] \mid i_k = dist(j_k, \alpha, j_1, ..., j_d), 1 \le k \le d, \\ (\alpha, [j_d, ..., j_1]) \text{ is of type } \beta\}.$$

Remark 2.10.33. *Let α be a permutation in S_n. We have*

$$\bar{C}_{n,d} = \bigcup_{\beta \in S_d} \bigcup_{i_1,...,i_d \ge 1} \bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d).$$

Given any d-tuple $[j_d, ..., j_1]$, the "distance" $dist(j_i, \alpha, j_1, ..., j_d)$ and the type of $(\alpha, [j_d, ..., j_1])$ are uniquely determined. Hence, the union above is the disjoint union. Also, there are only finitely many nonempty sets $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$ in the above union.

Lemma 2.10.34. *Let α be an n-cycle in S_n. $\bar{\sigma} = [j_d, ..., j_1]$ is a d-tuple and we assume $\bar{\sigma}$ is an element in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$ for some $\beta \in S_d$. Then, the number of all elements in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$ is n.*

Proof. If we want to use $\bar{\sigma}$ to construct some d-tuple $[j'_d, ..., j'_1]$ in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$, we have to pick d integers j'_i, $1 \le i \le d$, from α and we can assume the integers $i_1, ..., i_d$ imply

$$dist(j_k, \alpha, j_1, ..., j_d) = dist(j'_k, \alpha, j'_1, ..., j'_d).$$

At the same time, we know $j_1, ..., j_d$ are in the same disjoint cycle and

$$\sum_{k=1}^{d} dist(j_k, \alpha, j_1, ..., j_d) = \sum_{k=1}^{d} dist(j'_k, \alpha, j'_1, ..., j'_d) = n.$$

Hence, the choice of j'_1 will completely determine the d-tuple $[j'_d, ..., j'_1]$. So, there are n choices. It is easy to prove they are all of the elements in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$. We leave it as an exercise for the reader. $\square$

The next lemma is a generalization of Lemma 2.10.20.

Lemma 2.10.35. *Let α be a permutation in S_n and let $i_1, ..., i_d$ be d positive integers, where $d \leq n$. Let β be a permutation in S_d. If $\bar{C}^{\beta}_{n,d}(\alpha, i_d, ..., i_1)$ is nonempty, we have*

$$\Phi(\sum_{[j_d,...,j_1]\in \bar{C}^{\beta}_{n,d}(\alpha,i_1,...,i_d)} [j_d, ..., j_1]\alpha) \qquad = \hat{p}_{\phi_d(\beta)}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)\Phi(\alpha).$$

Proof. First, we make some assumptions and recall some notations. $\alpha = \alpha_1...\alpha_l$ is a permutation in S_n, where $\alpha_1...\alpha_l$ is the unique decomposition of α in disjoint cycles. Let $\bar{\sigma}$ be a d-tuple in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$, where $\bar{\sigma} = [j_d, ..., j_1]$. We assume $\beta = \beta_1...\beta_m$, which is the unique decomposition of β in disjoint cycles. β can be considered as the permutation β' in $Aut\{j_1, ..., j_d\}$ by the isomorphism in Remark 2.10.29, where β' is the permutation by forgetting all elements in α except for j_i, $1 \leq i \leq d$, in Construction 2.10.28. By abuse the notation, we assume the disjoint cycle β_i comes from the disjoint cycle α_i. Also, by definition, we have

$$i_k = dist(j_k, \alpha, j_1, ..., j_d), \quad 1 \leq k \leq d.$$

We assume the lengths of disjoint cycles α_i, $1 \leq i \leq l$, are different.

For any d-tuple $\bar{\sigma}' \in \bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$, by Definition 2.10.32 and Construction 2.10.28, we have

$$\frac{1}{\prod_{i=1}^{m}(\sum_{j\in\tau_i} i_j)}\hat{p}_{\phi_d(\beta)}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)\Phi(\alpha) = \Phi(\bar{\sigma}'\alpha).$$

We have to prove the number of d-tuples in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$ is $\prod_{i=1}^{m}(\sum_{j\in\beta_i} i_j)$.

We go back to the d-tuple $\bar{\sigma} = [j_d, ..., j_1] \in \bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$. If we want to use $\bar{\sigma}$ to construct some d-tuple $[j'_d, ..., j'_1]$ in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$, all integers j'_i, $1 \leq i \leq d$, should come from the first m disjoint cycles $\alpha_1, ..., \alpha_m$ and $|\beta_i|$ of them comes the disjoint cycle α_i, $1 \leq i \leq m$, where $|\beta_i|$ is the length of the cycle β_i. Also, we assume the integers $i_1, ..., i_d$

imply

$$dist(j_k, \alpha, j_1, ..., j_d) = dist(j'_k, \alpha, j'_1, ..., j'_d).$$

The choice of integers from different disjoint cycles is independent. Hence, by Lemma 2.10.34, the number of all elements in $\bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d)$ is $\prod_{i=1}^{m}(|\alpha_i|)$. By Example 2.10.15, we know

$$\prod_{i=1}^{m}(\sum_{j\in\beta_i} i_j) = \prod_{i=1}^{m}(|\alpha_i|).$$

Hence, we have

$$\begin{aligned}\hat{p}_{\phi_d(\beta)}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)\Phi(\alpha) &= \prod_{i=1}^{m}(\sum_{j\in\beta_i} i_j)\Phi(\bar{\sigma}\alpha)\\ &= \Phi(\sum_{\bar{\sigma}\in\bar{C}^{\beta}_{n,d}(\alpha,i_1,...,i_d)} \bar{\sigma}\alpha).\end{aligned}$$

□

Theorem 2.10.36. *For any $g \in \mathbb{C}S_n$,*

$$\Phi(K_{(1^{n-d}d)}g) = \Delta_d\Phi(g).$$

Proof. We assume g is a permutation in S_n. Say $g = \alpha$. By Remark 2.10.33, we have

$$\bar{C}_{n,d} = \bigcup_{\beta\in S_d}\bigcup_{i_1,...,i_d\geq 1} \bar{C}^{\beta}_{n,d}(\alpha, i_1, ..., i_d).$$

Then, we get

$$\begin{aligned}\Phi(\bar{K}_{(1^{n-d}d)}\alpha) &= \Phi(\sum_{\beta\in S_d}\sum_{i_1,...,i_d\geq 1}\sum_{[j_d,...,j_1]\in\bar{C}^{\beta}_{n,d}(\alpha,i_1,...,i_d)} [j_d, ..., j_1]\alpha)\\ &= \sum_{i_1,...,i_d\geq 1}\sum_{\beta\in S_d} \hat{p}_{\phi_d(\beta)}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)\Phi(\alpha)\\ &= d\Delta_d\Phi(\alpha),\end{aligned}$$

where the second equality comes from Lemma 2.10.35 and the last equality comes from

Definition 2.10.6. By Definition 2.10.9, we know the map $\pi_{n,d} : \bar{C}_{n,d} \to C_{n,d}$ is a d-to-1 map. So, we have

$$d\Phi(K_{1^{n-d}d}\alpha) = \Phi(\bar{K}_{1^{n-d}d}\alpha) = d\Delta_d\Phi(\alpha).$$

□

Theorem 2.10.37. *For any positive integer d, $\Delta_d = W([d])$ as an operator on $\mathbb{C}[p_1, p_2, ...]$.*

Proof. By Theorem 2.9.1 and Theorem 2.10.36, it is easy to get this consequence. □

Corollary 2.10.38. *For any $\beta \in S_d$, $\Delta_d = \Delta_\beta$ as operators on $\mathbb{C}[p_1, p_2, ...]$.*

Proof. Given any monomial $\prod_{i=1}^k p_{j_i}$ in $\mathbb{C}[p_1, p_2, ...]$, where $j_1 \leq j_2 \leq ... \leq j_k$, it corresponds to the partition $(j_1, ..., j_k)$. We pick a permutation g of type $(j_1, ..., j_k)$. Then, we have

$$\begin{aligned}
\Delta_d\Phi(g) =& \frac{1}{d}\Phi(\sum_{[j_d,...,j_1]\in\bar{C}_{n,d}} [j_d, ..., j_1]g) \\
=& \frac{1}{d}\Phi(\bar{K}_{1^{n-d}d}g) \\
=& \Phi(\sum_{[j_{\beta(d)},...,j_{\beta(1)}]\in\bar{C}_{n,d}} [j_{\beta(d)}, ..., j_{\beta(1)}]g) \\
=& \Delta_\beta\Phi(g).
\end{aligned}$$

□

Corollary 2.10.39. *Let d_1, d_2 be positive integers. $W([d_1]), W([d_2])$ commutes as operators on $\mathbb{C}[p_1, p_2, ...]$, i.e $W([d_1])W([d_2]) = W([d_2])W([d_1])$.*

Proof. We take any monomial $\prod_{i=1}^k p_{j_i}$ in the ring $\mathbb{C}[p_1, p_2, ...]$. We pick a permutation g corresponding to this monomial. We have

$$\begin{aligned}
&W([d_1])W([d_2])\Phi(g) \\
=&\Phi(K_{d_1 1^{n-d_1}} K_{d_2 1^{n-d_2}} g) \\
=&\Phi(K_{d_2 1^{n-d_2}} K_{d_1 1^{n-d_1}} g) \\
=&W([d_2])W([d_1])\Phi(g).
\end{aligned}$$

$K_{d_1 1^{n-d_1}}, K_{d_2 1^{n-d_2}}$ commutes, because they are central element in $\mathbb{C}S_n$. So, $W([d_1]), W([d_2])$ commutes. □

Chapter 3

Hurwitz Number

3.1 Simple Hurwitz Number

In this section, we give the definition of the Hurwitz number and review some known results about the simple Hurwitz number.

The Hurwitz enumeration problem aims at classifying all n-fold coverings of S^2 (or $\mathbb{C}P^1$) with k branch points $\{z_1, ..., z_k\}$. Given such a covering, each branch point z_i corresponds to a permutation σ_i in S_n. Denote by λ_i the partition corresponding to σ_i. The number of all connected n-coverings with k ordered branch points $z_i, 1 \leq i \leq k$, each of which corresponds to a permutation of type $\lambda_i, 1 \leq i \leq k$, is finite. This number is denoted by $\mathrm{Cov}_n(\lambda_1, ..., \lambda_k)$. Equivalently, $\mathrm{Cov}_n(\lambda_1, ..., \lambda_k)$ is the number of k-tuples $(\sigma_1, ..., \sigma_k) \in S_n^k$ satisfying the following conditions [1], [26],

(1) σ_i is of type λ_i,

(2) $\sigma_1 ... \sigma_k = 1$,

(3) The group generated by the elements $\{\sigma_1, ..., \sigma_k\}$ is transitive on the set $\{1, ..., n\}$.

Definition 3.1.1. *Given α a partition of n, the simple Hurwitz number is defined as*

$$h_k^{[2]}(\alpha) = \mathrm{Cov}_n(\overbrace{1^{n-2}2, ..., 1^{n-2}2}^{k}, \alpha).$$

It is the number of $(k+1)$-tuples $(\sigma_1, ..., \sigma_k, \sigma^{-1}) \in S_n^{k+1}$ satisfying the following conditions

(1) σ_i are transpositions (or of type $1^{n-2}2$), where $1 \leq i \leq k$, and σ^{-1} is of type α, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$,

(2) $\sigma_1 ... \sigma_k = \sigma$,

(3) the group generated by $\{\sigma_1, ..., \sigma_k\}$ is transitive on the set $\{1, ..., n\}$.

Simple means that all but one permutation are transpositions. The generating function $H^{[2]}(u,p)$ for simple Hurwitz numbers is

$$H^{[2]}(u,p) = H^{[2]}(u,p_1,p_2,...) = \sum_{n\geq 1}\frac{1}{n!}\sum_{k=1}^{\infty}\sum_{\alpha\vdash n} h_k^{[2]}(\alpha)\frac{u^k}{k!}p_\alpha \ ,$$

Proposition 3.1.2. *We have the following equation*

$$W([2])H^{[2]} = \frac{\partial H^{[2]}}{\partial u}.$$

We will prove this proposition in the next section.

Now we want that the number k of transpositions is minimal with respect to the given partition α. Denote by $\mu^2(\alpha)$ the minimal number. Sometimes we also use the notation

$$\mu^2(\sigma) := \mu^2(\alpha),$$

where σ is of type α.

Definition 3.1.3. *Given positive integers n, k, the minimal simple Hurwitz number $h^2(\alpha)$ is*

$$h^2(\alpha) := h^{[2]}_{\mu^2(\alpha)}(\alpha).$$

The minimal number $\mu^2(\alpha)$ can be computed by the Riemann-Hurwitz formula or by a combinatorial discussion [19]. It turns out that the minimal simple Hurwitz numbers counts coverings $X \to S^2$, where X is of genus 0.

Lemma 3.1.4. *Let $\alpha = (\alpha_1, ..., \alpha_l)$ be a partition of a positive integer n. Then, we have*

$$\mu^2(\alpha) = n - 2 + l.$$

Proof. The proof can be found in [11]. □

We define two generating functions for the minimal simple Hurwitz numbers as follows

$$\widetilde{H}_2^{min}(z,u,p_1,p_2,...) = \sum_{n\geq 1}\sum_{\alpha\vdash n} h^2(\alpha)\frac{z^n}{n!}\frac{u^{\mu^2(\alpha)}}{\mu^2(\alpha)!}\Phi(\alpha),$$

$$H_2^{min}(z,p_1,p_2,...) = \sum_{n\geq 1}\sum_{\alpha\vdash n} h^2(\alpha)\frac{z^n}{n!}\frac{1}{\mu^2(\alpha)!}\Phi(\alpha).$$

In fact, we have

$$H_2^{min}(z, p_1, p_2, ...) = \widetilde{H}_2^{min}(z, u, p_1, p_2, ...)|_{u=1}.$$

In the next section, we will prove the following theorem.

Proposition 3.1.5.

$$\frac{\partial \widetilde{H}_2^{min}}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}\left((i+j)p_i p_j \frac{\partial \widetilde{H}_2^{min}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial \widetilde{H}_2^{min}}{\partial p_i}\frac{\partial \widetilde{H}_2^{min}}{\partial p_j}\right).$$

Goulden and Jackson proved the above formula in [19]. We will give another proof, by using the W-operator, in the next section. In fact, we prove a generalization of Proposition 3.1.5 (see Theorem 3.3.6), where transpositions are replaced by d-cycles.

3.2 d-Hurwitz number and d-Frobenius Number

In this section, we review some results about the Hurwitz number and Frobenius number.

Definition 3.2.1. *Let G be a subgroup of the permutation group S_n. C_i, $1 \leq i \leq k$, (not necessarily distinct) are conjugacy classes of G. Denote by $\widetilde{\mathrm{Cov}}_G(C_1, ..., C_k)$ the number of all k-tuples $(g_1, ..., g_k)$ such that*

1. $g_i \in C_i$, $1 \leq i \leq k$,
2. $g_1...g_k = id$, *the identity element of G.*

We call $\widetilde{\mathrm{Cov}}_G(C_1, ..., C_k)$ the Frobenius number, which is known to be the number of ramified coverings (not necessarily connected) over $\mathbb{P}_1$.

Notation 3.2.2. *If G is the permutation group S_n, we prefer to use the notation*

$$\widetilde{\mathrm{Cov}}_n(\lambda_1, ..., \lambda_k) := \widetilde{\mathrm{Cov}}_G(C_1, ..., C_k),$$

where n corresponds to S_n and λ_i are partitions of n, which represent the conjugacy classes C_i, $1 \leq i \leq k$.

Theorem 3.2.3. *We have the Frobenius formula*

$$\widetilde{\mathrm{Cov}}_G(C_1, ..., C_k) = \frac{\prod_{i=1}^{k} |C_i|}{|G|} \sum_{\chi} \frac{\prod_{i=1}^{k} \chi(C_i)}{\chi(id)^{k-2}},$$

where $|A|$ denotes the number of elements in the set A and χ in the sum ranges over all irreducible complex characters of G.

Proof. Theorem 1.1.12 [26]. □

Definition 3.2.4. *Given k partitions λ_i of n, denote by $Cov_n(\lambda_1, ..., \lambda_k)$ the number of k-tuples $(\sigma_1, ..., \sigma_k) \in S_n^k$ satisfying the following conditions [1]:*

- *σ_i is of type λ_i for all i,*
- *$\sigma_1 ... \sigma_k = 1$,*
- *the subgroup generated by $\{\sigma_1, ..., \sigma_k\}$ acts transitively on the set $\{1, ..., n\}$ (transitivity).*

We call $Cov_n(\lambda_1, ..., \lambda_k)$ the Hurwitz number, which is known to be the number of connected ramfied covering over $\mathbb{P}_1$ with branch points described by λ_i, $1 \leq i \leq n$.

Remark 3.2.5. *If the group G is S_n, each conjugacy class can be represented uniquely by a partition of n. In this case, the definition of Hurwitz number has one more condition than that of Frobenius number, the transitivity condition. Although people can use the Frobenius formula to calculate Frobenius number, we still do not know a combinatorial formula to calculate the Hurwitz number.*

Now we consider a special type of Hurwitz and Frobenius number, the d-Hurwitz number and d-Frobenius number.

Definition 3.2.6. *Given positive integers d, n and k, where $d \leq n$, define the numbers $h_k^{[d]}(\alpha)$ and $f_k^{[d]}(\alpha)$ as follows*

$$h_k^{[d]}(\alpha) = Cov_n(\overbrace{1^{n-d}d, ..., 1^{n-d}d}^{k}, \alpha),$$
$$f_k^{[d]}(\alpha) = \widetilde{Cov}_n(\overbrace{1^{n-d}d, ..., 1^{n-d}d}^{k}, \alpha),$$

where α is a partition of n and there are k copies of the partition $(1^{n-d}d)$.

We define the generating functions for d-Hurwitz number and d-Frobenius number as follows

$$H^{[d]}(u,p) = H^{[d]}(u,p_1,p_2,...) = \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha \vdash n} h_k^{[d]}(\alpha) \frac{u^k}{k!} \Phi(\alpha) \ ,$$
$$F^{[d]}(u,p) = F^{[d]}(u,p_1,p_2,...) = \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha \vdash n} f_k^{[d]}(\alpha) \frac{u^k}{k!} \Phi(\alpha) \ ,$$

where the definition of Φ can be found at the beginning of Section 2.9.

By a combinatorial argument [1], we have

$$F^{[d]} = e^{H^{[d]}}.$$

An important application of the W-operator is the following differential equation for $F^{[d]}$.

Theorem 3.2.7. *$F^{[d]}$ is the unique formal series solution in u to the differential equation*

$$\frac{\partial F^{[d]}}{\partial u} = W([d])F^{[d]}$$

with initial condition

$$F^{[d]}(0,p) = e^{p_1}$$

The above theorem was first proved by Mironov et. al [18]. Here we give a different proof as an the application of Theorem 2.9.1.

The rest of this section is devoted to the proof of this theorem.

Notation 3.2.8. *Given a positive integer n, let α be a partition of n. We define the set $\mathcal{A}^{[d]}(\alpha,k)$ as $(k+1)$-tuples $(\sigma_1,...,\sigma_k,\sigma) \in S_n^{k+1}$ satisfying the two conditions about the Frobenius number in Definition 3.2.1, i.e.*

- *σ_i is of type $(1^{n-d}d)$ for all i and σ is of type α,*
- *$\sigma_1...\sigma_k = \sigma$ (the monodromy condition).*

Also, we define another set

$$\tilde{\mathcal{A}}^{[d]}(\alpha,k) = \{(\sigma_2,...,\sigma_k,\sigma) \mid (\sigma\sigma_k^{-1}...\sigma_2^{-1},\sigma_2,...,\sigma_k,\sigma) \in \mathcal{A}^{[d]}(\alpha,k)\}.$$

Remark 3.2.9. *By the definition of $h_k^{[d]}(\alpha)$, we have*

$$f_k^{[d]}(\alpha) = |\mathcal{A}^{[d]}(\alpha, k)| = |\tilde{\mathcal{A}}^{[d]}(\alpha, k)|.$$

Hence, we can write the generating function $F^{[d]}(z, p_1, p_2, ...)$ as

$$F^{[d]}(z, p) = F^{[d]}(z, p_1, p_2, ...) = \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha\vdash n} |\mathcal{A}^{[d]}(\alpha, k)| \frac{z^k}{k!} \Phi(\alpha) \ .$$

Remark 3.2.10. *Consider the generating series $F^{[d]}(u, p)$. Given a specific set $\mathcal{A}^{[d]}(\alpha, k)$, $\alpha \vdash n$, the elements in this set are $(k+1)$-tuples $(\delta_1, ..., \delta_k, \sigma)$. The parameter corresponding to this set is $\frac{z^k}{k!}\Phi(\alpha)$, where the exponent of z corresponds to the number of d-cycles k and $\Phi(\alpha)$ corresponds to the permutation σ. We take the sum over all partitions. We get the set-valued generating function*

$$\sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha\vdash n} \mathcal{A}^{[d]}(\alpha, k) \frac{z^k}{k!} \Phi(\alpha) \ .$$

Since every set is finite, we can take the cardinality of each set, and we get the generating function $\hat{H}^{[d]}(u, p)$.

Similarly, $\frac{\partial F^{[d]}}{\partial u}$ is the generating function for the sets $\tilde{\mathcal{A}}^{[d]}(\alpha, k)$, i.e.

$$\begin{aligned} \frac{\partial F^{[d]}}{\partial u} &= \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha\vdash n} |\tilde{\mathcal{A}}^{[d]}(\alpha, k)| \frac{u^{k-1}}{(k-1)!} \Phi(\alpha) \\ &= \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha\vdash n} f_k^{[d]}(\alpha) \frac{u^{k-1}}{(k-1)!} \Phi(\alpha) \ . \end{aligned}$$

Definition 3.2.11. *Let k, n, d be three positive integers, where $n \geq d$. We define the set $\mathcal{A}^{[d]}(k, n)$ as follows*

$$\mathcal{A}^{[d]}(k, n) = \bigcup_{\alpha\vdash n} \mathcal{A}^{[d]}(k, \alpha).$$

The union is disjoint.

Lemma 3.2.12. *Let k, n, d be three positive integers, where $n \geq d$. We have*

$$\sum_{\alpha \vdash n} f_k^{[d]}(\alpha)\Phi(\alpha) = \sum_{\alpha' \vdash n} f_{k-1}^{[d]}(\alpha)\Phi(K_{1^{n-d}d}\alpha').$$

Proof. We consider the sets $\mathcal{A}^{[d]}(k,n)$ and $\mathcal{A}^{[d]}(k-1,n)$. Given any element $(\sigma_1, ..., \sigma_k, \sigma) \in \mathcal{A}^{[d]}(k,n)$, it corresponds to a unique element $(\sigma_2, ..., \sigma_k, \sigma') \in \mathcal{A}^{[d]}(k-1,n)$, where $\sigma' = \sigma_1^{-1}\sigma$. Now given any element $(\sigma_2, ..., \sigma_k, \sigma') \in \mathcal{A}^{[d]}(k-1,n)$ and any d-cycle σ_1, we can construct an element $(\sigma_1, ..., \sigma_k, \sigma) \in \mathcal{A}^{[d]}(k,n)$, where $\sigma = \sigma_1\sigma'$. Indeed, we can construct different elements in $\mathcal{A}^{[d]}(k,n)$ by multiplying different d-cycles σ_1. The number of elements we construct from the element $(\sigma_2, ..., \sigma_k, \sigma')$ is $\frac{1}{d}\binom{n}{d}d!$, where $\frac{1}{d}\binom{n}{d}d!$ is the number of d-cycles in S_n. From the discussion, we can get all elements in $\mathcal{A}^{[d]}(k,n)$ by adding different d-cycles to elements in $\mathcal{A}^{[d]}(k-1,n)$. Also, we have

$$|\mathcal{A}^{[d]}(k,n)| = \frac{1}{d}\binom{n}{d}d!|\mathcal{A}^{[d]}(k-1,n)|.$$

Recall the definition of $\mathcal{A}^{[d]}(k,n)$,

$$\mathcal{A}^{[d]}(k,n) = \bigcup_{\alpha \vdash n} \mathcal{A}^{[d]}(k,\alpha).$$

Hence, we have the following formula

$$\sum_{\alpha \vdash n} f_k^{[d]}(\alpha)\Phi(\alpha) = \sum_{\alpha' \vdash n} f_{k-1}^{[d]}(\alpha')\Phi(K_{1^{n-d}d}\alpha').$$

$\square$

Proof of Theorem 3.2.7.

$$\begin{aligned}
\frac{\partial F[d]}{\partial z} &= \sum_{n \geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha \vdash n} \frac{z^{k-1}}{(k-1)!} f_k^{[d]}(\alpha)\Phi(\alpha) \\
&= \sum_{n \geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha' \vdash n} \frac{z^{k-1}}{(k-1)!} f_{k-1}^{[d]}(\alpha')\Phi(K_{1^{n-d}d}\alpha') \\
&= \sum_{n \geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha' \vdash n} \frac{z^{k-1}}{(k-1)!} f_{k-1}^{[d]}(\alpha')W([d])\Phi(\alpha') \\
&= W([d])F[d],
\end{aligned}$$

where the first equality comes from Remark 3.2.10, the second equality is the consequence of Lemma 3.2.12 and the last equality comes from Theorem 2.9.1. □

3.3 Minimal d-Hurwitz Number

Recall that $h_k^{[d]}(\alpha)$ is the number of $(k+1)$-tuples $(\delta_1, ..., \delta_k, \sigma)$ in S_n^{k+1} satisfying the following conditions (see Definition 3.2.4 and 3.2.6)

- δ_i is of type $(1^{n-d}d)$ (or a d-cycle), $1 \leq i \leq k$ and σ is of type α,
- $\delta_1 ... \delta_k = \sigma$,
- the subgroup generated by $\{\delta_1, ..., \delta_k\}$ acts transitively on the set $\{1, ..., n\}$.

Now we want that the number k of d-cycles is minimal with respect to given partition α (similar to the minimal simple Hurwitz number, see Definition 3.1.3). Denote by $\mu^d(\alpha)$ the minimal number. Sometimes we also use the notation

$$\mu^d(\sigma) := \mu^d(\alpha),$$

where σ is of type α.

As we mentioned in the introduction, minimal means that the genus of the covering space X is zero. If we want to calculate the Hurwitz number with the genus of the covering space greater than zero, we have a topological recursion formula which allows us to calculate the higher genus Hurwitz number in terms of lower genus ones. (See Section 3.5.)

Definition 3.3.1. *Given positive integers n, k, d, $d \leq n$, the minimal d-Hurwitz number $h^d(\alpha)$ is*

$$h^d(\alpha) := h^{[d]}_{\mu^d(\alpha)}(\alpha).$$

We can compute $\mu^d(\alpha)$ in two ways. It can be computed by the Riemann-Hurwitz Formula or by a combinatorial discussion [19].

Lemma 3.3.2. *Let $\alpha = (\alpha_1, ..., \alpha_l)$ be a partition of a positive integer n. Then, we have*

$$\mu^d(\alpha) = \frac{n - 2 + l}{d - 1}.$$

Proof. See [19]. □

We define two generating functions for the minimal d-Hurwitz numbers as follows

$$\widetilde{H}_d^{min}(z,u,p_1,p_2,...) = \sum_{n\geq 1}\sum_{\alpha\vdash n} h^d(\alpha)\frac{z^n}{n!}\frac{u^{\mu^d(\alpha)}}{\mu^d(\alpha)!}\Phi(\alpha),$$
$$H_d^{min}(z,p_1,p_2,...) = \sum_{n\geq 1}\sum_{\alpha\vdash n} h^d(\alpha)\frac{z^n}{n!}\frac{1}{\mu^d(\alpha)!}\Phi(\alpha).$$

In fact,

$$H_d^{min}(z,p_1,p_2,...) = \widetilde{H}_d^{min}(z,u,p_1,p_2,...)|_{u=1}.$$

Before we state the theorem we want to prove in this section, we first define the differential operator $\widetilde{HW}([d])$.

Construction 3.3.3. *Some free summations in $W([d])$ contain higher derivatives. For example, in $W([2])$, we have the summation*

$$FS_{(1)(2)} = \frac{1}{2}\sum_{i\geq 1}\sum_{j\geq 1} ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j},$$

which contains second derivatives. If we change the higher derivatives into the product of first derivatives, we will get a new nonlinear *operator. We take $FS_{(1)(2)}$ as an example,*

$$\widetilde{FS}_{(1)(2)} = \frac{1}{2}\sum_{i\geq 1}\sum_{j\geq 1}(ijp_{i+j}\frac{\partial}{\partial p_i}\times\frac{\partial}{\partial p_j}).$$

As an operator on the generating function F, it means

$$\widetilde{FS}_{(1)(2)}(F) = \frac{1}{2}\sum_{i\geq 1}\sum_{j\geq 1}(ijp_{i+j}\frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j}),$$

where $F\in\mathbb{C}[[p_1,p_2,...]]$. More generally, for any permutation β, $\widetilde{FS}_\beta$ is constructed by replacing all higher derivatives in FS_β by the products of first order derivative operators as mentioned above, i.e.

$$\widetilde{FS}_\beta = \frac{1}{d}\sum_{i_1,...,i_d\geq 1}\hat{p}_{\bar{d}\beta}(i_1,...,i_d)\widetilde{\frac{\partial}{\partial\hat{p}_\beta}(i_1,...,i_d)}, \tag{3.3.1}$$

where

$$\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1,...,i_d)} = (\sum_{j\in\beta_1} i_j)\frac{\partial}{\partial p_{\sum_{j\in\beta_1} i_j}} \times ... \times (\sum_{j\in\beta_d} i_j)\frac{\partial}{\partial p_{\sum_{j\in\beta_d} i_j}}. \tag{3.3.2}$$

In this paper, we prefer to use the following notation for $\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1,...,i_d)}$,

$$\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1,...,i_d)} = \times_{i=1}^{l}\left((\sum_{j\in\beta_i} i_j)\frac{\partial}{\partial p_{\sum_{j\in\beta_i} i_j}}\right). \tag{3.3.3}$$

Definition 3.3.4.

$$\widetilde{HW}([d]) := \sum_{\substack{\beta\in S_d \\ d(FS_\beta)=d+1}} \widetilde{FS}_\beta.$$

Remark 3.3.5. *Let* $\sigma' \in S_n$ *and let* $[j_d,...,j_1]$ *be a* d*-tuple of distinct integers smaller than* n. β *is the RP of* $(\sigma',[j_d,...,j_1])$ *and* $i_q = dist(j_q,\sigma',\{j_1,j_2,...,j_d\})$ *(see Construction 2.10.10 and Definition 2.10.13). Consider the operator* FS_β*. Clearly,* $dD(FS_\beta)$ *is the number of disjoint cycles of* β *and* $dP(FS_\beta)$ *is the number of disjoint cycles of* $\bar{d}\beta$*. Let* $\sigma = (j_d...j_1)\sigma'$*. Let* l, l' *be the number of disjoint cycles of* σ, σ' *respectively. We have*

$$l' = l - dD(FS_\beta) + dP(FS_\beta).$$

The following theorem was first proved by Goulden and Jackson [16]. Here we give another proof using the W-operators.

Theorem 3.3.6.

$$\frac{\partial \widetilde{H}_d^{min}}{\partial u} = \widetilde{HW}([d])(\widetilde{H}_d^{min}).$$

Goulden and Jackson used graph theory to construct the differential operator $\widetilde{HW}([d])$ [19].

Definition 3.3.7. *Let* $(\delta_1,...,\delta_k)$ *be a* k*-tuple of* d*-cycles in* S_n *and* $\sigma = \delta_1...\delta_k$*. We say* $(\delta_1,...,\delta_k)$ *is a* d*-minimal transitive factorization of* σ*, if* $(\delta_1,...,\delta_k)$ *satisfies the transitivity condition in Definition 3.2.4 and* $k = \mu^d(\sigma)$.

Since σ is uniquely determined by $\delta_1,...,\delta_k$, sometimes we omit σ and say $(\delta_1,...,\delta_k)$ is a d-minimal transitive factorization.

To prove Theorem 3.3.6, we have to change Definition 3.3.7 a little bit.

Definition 3.3.8. *Let $([\delta_1], \delta_2, ..., \delta_k)$ be a k-tuple, where $[\delta_1]$ is a d-tuple of distinct integers smaller than n and δ_i are d-cycles in S_n, $1 \leq i \leq k$. δ_1 is the corresponding d-cycle of $[\delta_1]$. We say $([\delta_1], \delta_2, ..., \delta_k)$ is a d-minimal transitive factorization, if the corresponding k-tuple $(\delta_1, ..., \delta_k)$ is a d-minimal transitive factorization.*

The d-tuple $[\delta_1]$ corresponds to a unique d-cycle δ_1 and this is a d-to-1 correspondence. The multiplication of a d-tuple $[\delta_1]$ and a permutation τ is defined as follows

$$[\delta_1]\tau = \delta_1\tau,$$

where δ_1 is the corresponding permutation of $[\delta_1]$.

Now we consider a general k-tuple of permutations $(\delta_1, ..., \delta_k)$. Let $\mathcal{S} = \{\delta_1, ..., \delta_k\}$ be the corresponding set and set $\sigma = \delta_1 ... \delta_k$. Let G be the subgroup of S_n generated by the permutations in $\mathcal{S}$. Let $X_1, ..., X_q$ be the connected components of $X = \{1, ..., n\}$ with respect to the action of G. For each connected component X_i, we define the subset $\mathcal{S}_i$ of $\mathcal{S}$ as

$$\mathcal{S}_i = \{\delta \in \mathcal{S} \mid \delta(j) \neq j \text{ for some } j \in X_i\}.$$

Denote by σ_i the product of the elements in $\mathcal{S}_i$ multiplied in the same order as in the tuple $(\delta_1, ..., \delta_k)$. Clearly, $\sigma = \sigma_1 ... \sigma_q$. We say that the set $\mathcal{S}_i$ corresponds to a transitive factorization of σ_i.

For example, consider the following tuple $((12), (34), (45))$. The corresponding set is $\mathcal{S} = \{(12), (34), (45)\}$ and the group generated by $\mathcal{S}$ is $G = \langle (12), (34), (45) \rangle$. We have

$$\sigma = \delta_1\delta_2\delta_3 = (12)(34)(45) = (12)(345).$$

G is a proper subgroup of S_5 acting on the set $X = \{1, 2, 3, 4, 5\}$. X has two connected components $X_1 = \{1, 2\}$ and $X_2 = \{3, 4, 5\}$. We have $\mathcal{S}_1 = \{(12)\}$, $\mathcal{S}_2 = \{(34), (45)\}$ and $\sigma_1 = (12)$, $\sigma_2 = (345)$, with $\sigma = (12)(345)$.

Lemma 3.3.9. *Let $([\delta_1], \delta_2, ..., \delta_k)$ be a d-minimal transitive factorization of σ and $\sigma' = \delta_2 ... \delta_k$. If β is the RP of $(\sigma', [\delta_1])$ (see Definition 2.10.13), then $X = \{1, ..., n\}$ has exactly $dD(FS_\beta)$ connected components X'_i, $1 \leq i \leq dD(FS_\beta)$, with respect to the group generated by $\{\delta_2, ..., \delta_k\}$. Denote by $\mathcal{S}'_i$ the set of δ_i's (in $\mathcal{S}' = \{\delta_2, ..., \delta_k\}$) that move at least one element of*

X'_i. Then, each $\mathcal{S}'_i$ corresponds to a minimal transitive factorization of σ'_i, $1 \leq i \leq dD(FS_\beta)$, where $\sigma' = \sigma'_1 ... \sigma'_{dD(FS_\beta)}$.

Proof. The lemma follows from Lemma 2.2 in [19], using the interpretation of $dD(FS_\beta)$ in Remark 3.3.5. $\square$

Lemma 3.3.10. *Let $([\delta_1], \delta_2, ..., \delta_k)$ be a d-minimal transitive factorization of σ and $\sigma' = \delta_2 ... \delta_k$. If β is the RP of $(\sigma', [\delta_1])$, we have*

$$d(FS_\beta) = d + 1.$$

Proof. By Remark 3.3.5, we know that $dD(FS_\beta)$ is the number of disjoint cycles in β and $dP(FS_\beta)$ is the number of disjoint cycles in $\bar{d}\beta$. Also, Lemma 3.3.2 tells us the following formula

$$\mu^d(\sigma) = \frac{n + l - 2}{d - 1}.$$

By Lemma 3.3.9, we know X is the disjoint union of X'_i, $1 \leq i \leq dD(FS_\beta)$. Let n_i be the cardinality of X'_i. Then,

$$\sum_{i=1}^{dD(FS_\beta)} n_i = n.$$

Also, we have $\sigma' = \sigma'_1 ... \sigma'_{dD(FS_\beta)}$ and each σ'_i is the product of permutations in $\mathcal{S}'_i$ in order (Lemma 3.3.9). The group generated by $\mathcal{S}'_i$ acts transitively on X'_i. Let l_i be the number of disjoint cycles of σ'_i. By Lemma 3.3.2, we have

$$\mu^d(\sigma'_i) = \frac{n_i + l_i - 2}{d - 1}.$$

By Remark 3.3.5, we have

$$\sum_{i=1}^{dD(FS_\beta)} l_i = l - dP(FS_\beta) + dD(FS_\beta).$$

Finally, we come to the following equation

$$\mu^d(\sigma) - 1 = \sum_{i=1}^{dD(FS_\beta)} \mu^d(\sigma'_i). \tag{3.3.4}$$

This equation holds, because we delete the first d-cycle δ_1 (or multiply δ_1^{-1} to σ) and $\mathcal{S}_i'$ corresponds to a minimal transitive factorization of σ_i' by Lemma 3.3.9. Equation (3.3.4) can be rewritten as

$$n + l - 2 = (d-1) + n + (l - dP(FS_\beta) + dD(FS_\beta)) - 2dD(FS_\beta).$$

We get

$$d + 1 = dP(FS_\beta) + dD(FS_\beta) = d(FS_\beta).$$

$\square$

Definition 3.3.11. *Given a positive integer n, let α be a partition of n. We define*

$$\begin{aligned}\mathcal{A}^d(\alpha) =&\{([\delta_1], \delta_2, ..., \delta_k, \sigma) \mid \sigma \textit{ is of type } \alpha, (\delta_1, ..., \delta_k)\\ &\textit{is a } d\textit{-minimal transitive factorization of } \sigma\}.\end{aligned}$$

Fixing a permutation $\beta \in S_d$, we define

$$\mathcal{A}^d_\beta(\alpha) = \{([\delta_1], \delta_2, ..., \delta_k, \sigma) \in \mathcal{A}^d(\alpha) \mid \ \beta \textit{ is the RP of } (\sigma', [\delta_1])\},$$

where $\sigma' = \delta_2 ... \delta_k$.

The generating function $\widetilde{H}_d^{min}$ can be rewritten as

$$\widetilde{H}_d^{min} = \sum_{n \geq 1} \sum_{\alpha \vdash n} \frac{|\mathcal{A}^d(\alpha)|}{d} \frac{z^n}{n!} \frac{u^{\mu^d(\alpha)}}{\mu^d(\alpha)!} \Phi(\alpha),$$

because

$$h^d(\alpha) = \frac{|\mathcal{A}^d(\alpha)|}{d}.$$

Also, given a d-tuple of distinct integers $[\delta_1]$, σ' corresponds to a unique β as we explained in Remark 3.3.5. Hence, $\mathcal{A}^d_\beta(\alpha)$ are pairwise disjoint, i.e.

$$\mathcal{A}^d_{\beta'}(\alpha) \bigcap \mathcal{A}^d_\beta(\alpha) = \emptyset, \quad \beta' \neq \beta.$$

By Definition 3.3.11, we have a disjoint union

$$\bigcup_{\beta\in S_d} \mathcal{A}^d_\beta(\alpha) = \mathcal{A}^d(\alpha).$$

By Lemma 3.3.10, $\mathcal{A}^d_\beta(\alpha)$ is nonempty if and only if $d(FS_\beta) = d+1$. With the discussion above, we can write the generating function $\widetilde{F}_d$ as following

$$\widetilde{H}^{min}_d(u,z,p) = \frac{1}{d}\sum_{n\geq 1}\sum_{\alpha\vdash n}\sum_{\substack{\beta\in S_d,\\ d(FS_\beta)=d+1}} |\mathcal{A}^d_\beta(\alpha)|\frac{z^n}{n!}\frac{u^{\mu^d(\alpha)}}{\mu^d(\alpha)!}p_\alpha,$$

Given a permutation $\beta \in S_d$, we define the generating function $(\widetilde{F}_d)_\beta$ as

$$(\widetilde{H}^{min}_d)_\beta(u,z,p) = \frac{1}{d}\sum_{n\geq 1}\sum_{\alpha\vdash n} |\mathcal{A}^d_\beta(\alpha)|\frac{z^n}{n!}\frac{u^{\mu^d(\alpha)}}{\mu^d(\alpha)!}p_\alpha.$$

Clearly,

$$\widetilde{H}^{min}_d = \sum_{\substack{\beta\in S_d,\\ d(FS_\beta)=d+1}} (\widetilde{H}^{min}_d)_\beta.$$

Now we have defined two different types of sets, $\mathcal{A}^d(\alpha)$ and pairwise disjoint sets $\mathcal{A}^d_\beta(\alpha)$, $\beta\in S_d$. But that's not enough to prove Theorem 3.3.6, since $|\mathcal{A}^d_\beta(\alpha)|$ is not easily computable. In this section, we break $\mathcal{A}^d_\beta(\alpha)$ into much smaller disjoint computable sets.

We will give a brief description of the sets which will be defined to get our target set $\mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))$.

- The first set is $\mathcal{B}^d_\beta(\alpha')$. This set is constructed from $\mathcal{A}^d_\beta(\alpha)$, contains k-tuples

$$(\delta_2,...,\delta_k,[\delta_1]^{-1}\sigma)$$

  such that $([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha)$ (see Definition 3.3.12).

- The second set is $\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)$. We fix d integers $i_1,...,i_d$ and use elements in $\mathcal{B}^d_\beta(\alpha')$ to construct pairwise disjoint subsets

$$\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)$$

of $\mathcal{A}^d_\beta(\alpha)$ such that

$$\bigcup_{i_1,...,i_d \geq 1} \mathcal{A}^d_\beta(\alpha, \alpha', i_1, ..., i_d) = \mathcal{A}^d_\beta(\alpha).$$

See Definition 3.3.13 and Remark 3.3.15.

- The last set is $\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$. Based on d integers $i_1, ..., i_d$, we use a single element $(\delta_2, ..., \delta_k, \sigma') \in \mathcal{B}^d_\beta(\alpha')$ to construct pairwise disjoint subsets

$$\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$$

such that

$$\bigcup_{(i_1,...,i_d) \in \mathbb{Z}^d_{>0}} \mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d)) = \mathcal{A}^d_\beta(\alpha, \alpha', i_1, ..., i_d),$$

(see the proof of Lemma 3.3.19).

The set $\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$ is our target set, since its cardinality is computable (see Lemma 3.3.18).

Definition 3.3.12. *Let α' be a partition of n. Let $\mathcal{B}^d_\beta(\alpha')$ be the set of k-tuples $(\delta_2, ..., \delta_k, \sigma')$, where δ_i, $2 \leq i \leq k$, are d-cycles and $\sigma' = \delta_2 ... \delta_k$ is of type α', such that there exist a d-tuple $[\delta_1]$, a partition α and a permutation σ satisfying $([\delta_1], \delta_2, ..., \delta_k, \sigma) \in \mathcal{A}^d_\beta(\alpha)$, i.e.*

$$\begin{aligned}\mathcal{B}^d_\beta(\alpha') = \{(\delta_2, ..., \delta_k, \sigma') \mid \sigma' = \delta_2 ... \delta_k, \sigma' \textit{ is of type } \alpha' \\ \textit{and } ([\delta_1], \delta_2, ..., \delta_k, \sigma) \in \mathcal{A}^d_\beta(\alpha) \textit{ for some partition } \alpha, \\ \textit{some permutation } \sigma \textit{ and some } d\textit{-tuple } [\delta_1]\}.\end{aligned}$$

Let $(\delta_2, ..., \delta_k, \sigma')$ be an element in $\mathcal{B}^d_\beta(\alpha')$. By Lemma 3.3.9, we have $dD(FS_\beta)$ connected components of $X = \{1, ..., n\}$ with respect to the action of the group $\{\delta_2, ..., \delta_k\}$. Usually, we say that $(\delta_2, ..., \delta_k, \sigma')$ has $dD(FS_\beta)$ connected components. Denote by $X_1, ..., X_{dD(FS_\beta)}$ the connected components. Let $\mathcal{S} = \{\delta_2, ..., \delta_k\}$. $\mathcal{S}_i$ is the subset of $\mathcal{S}$, which contains all nontrivial permutation in $\mathcal{S}$ on the connected component X_i, and σ'_i is the product of all permutations in $\mathcal{S}_i$ with respect to their order in the original tuple $(\delta_2, ..., \delta_k)$. Clearly, we have $\sigma' = \sigma'_1 ... \sigma'_{dD(FS_\beta)}$.

For example, consider $n = 8, d = 3$ and $\sigma = (1235)(4678)$, so that $\alpha = (4+4)$, where $(4+4)$ is a partition of 8. Clearly, $\mathcal{A}^3((4+4))$ contains the following tuple

$$([4,5,6],(123),(345),(678),\underbrace{(1235)(4678)}_{\sigma}).$$

Now $((123),(345),(678),\underbrace{(12345)(678)}_{\sigma'}) \in \mathcal{B}^3_\beta((5+3))$ for some β, where $\alpha' = (5+3)$ is a partition of 8.

Now we fix d positive integers $i_1, ..., i_d$.

Definition 3.3.13. *Denote by $\mathcal{A}^d_\beta(\alpha', i_1, ..., i_d)$ the set of $(k+1)$-tuples*

$$([\delta_1], \delta_2, ..., \delta_k, \sigma),$$

where $[\delta_1] = [j_d, ..., j_1]$, such that $([\delta_1], \delta_2, ..., \delta_k, \sigma)$ can be obtained from some element $(\delta_2, ..., \delta_k, \sigma') \in \mathcal{B}^d_\beta(\alpha')$ satisfying

$$\begin{aligned}
&\sigma = [\delta_1]\sigma', \\
&i_q = dist(j_q, \sigma', \{j_d, ..., j_1\}), 1 \le q \le d.
\end{aligned}$$

See Definition 2.10.30 for the definition of distance.

Lemma 3.3.14. *Assume $\mathcal{A}^d_\beta(\alpha', i_1, ..., i_d)$ is nonempty. Given any two elements*

$$([\delta_1], \delta_2, ..., \delta_k, \sigma), ([\widetilde{\delta}_1], \widetilde{\delta}_2, ..., \widetilde{\delta}_k, \widetilde{\sigma})$$

in the set $\mathcal{A}^d_\beta(\alpha', i_1, ..., i_d)$, σ and $\widetilde{\sigma}$ are of the same type, i.e. $\Phi(\sigma) = \Phi(\widetilde{\sigma})$.

Proof. Let $([\delta_1], \delta_2, ..., \delta_k, \sigma), ([\widetilde{\delta}_1], \widetilde{\delta}_2, ..., \widetilde{\delta}_k, \widetilde{\sigma})$ be two elements in $\mathcal{A}^d_\beta(\alpha', i_1, ..., i_d)$. Assume $([\delta_1], \delta_2, ..., \delta_k, \sigma)$ is constructed from the element $(\delta_2, ..., \delta_k, \sigma')$ by multiplying a d-tuple $[\delta_1]$ and $([\widetilde{\delta}_1], \widetilde{\delta}_2, ..., \widetilde{\delta}_k, \widetilde{\sigma})$ is constructed from $(\widetilde{\delta}_2, ..., \widetilde{\delta}_k, \widetilde{\sigma}')$ by multiplying a d-tuple $[\widetilde{\delta}_1]$. If the lengths of disjoint cycles are distinct, we have

$$\begin{aligned}
\left(\hat{p}_{(d...1)\beta}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)\right)\Phi(\sigma') = \Phi(\sigma), \\
\left(\hat{p}_{(d...1)\beta}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)\right)\Phi(\widetilde{\sigma}') = \Phi(\widetilde{\sigma}).
\end{aligned}$$

By Definition 3.3.12 and 3.3.13, we know $(\delta_2, ..., \delta_k, \sigma'), (\widetilde{\delta}_2, ..., \widetilde{\delta}_k, \widetilde{\sigma}') \in \mathcal{B}^d_\beta(\alpha')$, i.e. σ' and $\widetilde{\sigma}'$ both are of type α'. So $\Phi(\sigma) = \Phi(\widetilde{\sigma})$, which means σ and $\widetilde{\sigma}$ are of the same type.

The statement is true if the disjoint cycles are not necessarily distinct. We omit the proof here. □

Remark 3.3.15. *By Lemma 3.3.14, we know that given any element* $([\delta_1], \delta_2, ..., \delta_k, \sigma)$ *in the set* $\mathcal{A}^d_\beta(\alpha', i_1, ..., i_d)$*,* σ *is always of the same type. Denote by* α *the type of* σ*. Sometimes we use the notation* $\mathcal{A}^d_\beta(\alpha, \alpha', i_1, ..., i_d)$ *to emphasize the type* α*. Clearly,* $\mathcal{A}^d_\beta(\alpha, \alpha', i_1, ..., i_d)$ *is a subset of* $\mathcal{A}^d_\beta(\alpha)$ *and we have a disjoint union*

$$\bigcup_{\alpha' \vdash n} \bigcup_{i_1,...,i_d \geq 1} \mathcal{A}^d_\beta(\alpha, \alpha', i_1, ..., i_d) = \mathcal{A}^d_\beta(\alpha).$$

Let $(\delta_2, ..., \delta_k, \sigma')$ be an element in $\mathcal{B}^d_\beta(\alpha')$. Suppose that $([\delta_1], \delta_2, ..., \delta_k, \sigma)$ is an element in $\mathcal{A}^d_\beta(\alpha', i_1, ..., i_d)$ constructed from $(\delta_2, ..., \delta_k, \sigma')$ by adding the d-tuple $[\delta_1]$ as introduced in Definition 3.3.13. Clearly, the reduction permutation of σ' is β with respect to $[\delta_1]$. Similarly, the reduction permutation of σ'_i is β_i, $1 \leq i \leq dD(FS_\beta)$, where $\beta = \beta_1 ... \beta_{dD(FS_\beta)}$ is the decomposition in disjoint cycles.

Let $(\delta_2, ..., \delta_k, \sigma')$ be an element in $\mathcal{B}^d_\beta(\alpha')$. We define our target set as follows.

Definition 3.3.16. *Define* $\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$ *to be the set containing all elements* $([\delta_1], \delta_2, ..., \delta_k, \sigma)$ *in* $\mathcal{A}^d_\beta(\alpha', i_1, ..., i_d)$ *constructed from* $(\delta_2, ..., \delta_k, \sigma')$ *as in Definition 3.3.13.*

By the above definition, we have

$$\bigcup_{(\delta_2,...,\delta_k,\sigma') \in \mathcal{B}^d_\beta(\alpha')} \mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d)) = \mathcal{A}^d_\beta(\alpha', i_1, ..., i_d),$$

which is a disjoint union.

Now consider the differential operator (see Definition 2.10.2)

$$\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d) = \prod_{i=1}^{m} \left((\sum_{j \in \beta_i} i_j) \frac{\partial}{\partial p_{\sum_{j \in \beta_i} i_j}} \right).$$

We define a new operator $\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1, ..., i_d)}$ acting on $\Phi(\sigma') = \Phi(\sigma'_1)...\Phi(\sigma'_{dD(FS_\beta)})$ as follows

$$\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1, ..., i_d)}\,(\Phi(\sigma')) = \prod_{i=1}^{m}\left((\sum_{j\in\beta_i} i_j)\frac{\partial \Phi(\sigma'_i)}{\partial p_{\sum_{j\in\beta_i} i_j}}\right).$$

If the disjoint cycles of σ'_i are of distinct lengths, $1 \le i \le dD(FS_\beta)$, we have

$$\hat{p}_{\bar{d}\beta}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1, ..., i_d)}\,(\Phi(\sigma')) = \left(\prod_{i=1}^{m}(\sum_{j\in\beta_i} i_j)\right)\Phi(\sigma).$$

The operator $\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1, ..., i_d)}$ is different from $\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)$ by changing the higher order differential operator into the product of first derivatives as we did for $\widetilde{FS}_\beta$. Recall that we define $\widetilde{FS}_\beta$ as an operator on generating functions. We consider $\widetilde{FS}_\beta$ as the sum of $\hat{p}_{\bar{d}\beta}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1, ..., i_d)}$ (see Definition 2.10.3), i.e.

$$\widetilde{FS}_\beta = \frac{1}{d}\sum_{i_1,...,i_d\ge 1}\hat{p}_{(d...1)\beta}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1, ..., i_d)}.$$

Lemma 3.3.17. *With the same notation above, we have*

$$\begin{aligned}&\hat{p}_{(d...1)\beta}(i_1, ..., i_d)\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1, ..., i_d)}(\Phi(\sigma'))\\ &= \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))}\Phi(\sigma)\right),\end{aligned}$$

Proof. First, if we cannot find a disjoint cycle with length $\sum_{j\in\beta_v} i_j$ in σ'_v for some v, $1 \le v \le dD(FS_\beta)$, it means that $\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$ is empty. So, we have

$$\left(\sum_{(\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))}\Phi(\sigma)\right) = 0$$

Also, $\frac{\partial\Phi(\sigma_v)}{\partial p_{\sum_{j\in\beta_v} i_j}} = 0$. So, the formula is true in this special case.

Now we assume there is at least one disjoint cycle with length $\sum_{j\in\beta_v} i_j$ in σ'_v for all $1 \le v \le dD(FS_\beta)$ and c_v is the number of disjoint cycles with length $\sum_{j\in\beta_v} i_j$ in σ'_v. By the following lemma (Lemma 3.3.18), we know the number of elements in $\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$ is

$$\prod_{v=1}^{dD(FS_\beta)} c_v(\sum_{j\in\beta_v} i_j).$$

So, we have

$$\left(\sum_{(\delta_2,...,\delta_k,\sigma,\varepsilon)\in\mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))} \Phi(\sigma)\right) = \left(\prod_{v=1}^{dD(FS_\beta)} c_v(\sum_{j\in\beta_v} i_j)\right)\Phi(\sigma).$$

By assumption, we know there are c_v disjoint cycles with length $\sum_{j\in\beta_v} i_j$ in σ'_v. This means the degree of $p_{\sum_{j\in\beta_v} i_j}$ in the monomial $\Phi(\sigma'_v)$ is c_v. So, when we calculate $\frac{\partial\Phi(\sigma'_v)}{\partial p_{\sum_{j\in\beta_v} i_j}}$, we will have a coefficient c_v, i.e.

$$\hat{p}_{(d...1)\beta}(i_1, ..., i_d)\widetilde{\frac{\partial}{\partial\hat{p}_\beta}(i_1, ..., i_d)}(\Phi(\sigma')) = (\prod_{v=1}^{dD(FS_\beta)} c_v(\sum_{j\in\beta_v} i_j))\Phi(\sigma).$$

So, we have

$$\begin{aligned}&\hat{p}_{(d...1)\beta}(i_1, ..., i_d)\widetilde{\frac{\partial}{\partial\hat{p}_\beta}(i_1, ..., i_d)}(\Phi(\sigma'))\\ &= \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))} \Phi(\sigma)\right).\end{aligned}$$

□

Lemma 3.3.18. *With the same notation as in Definition 3.3.16, we have*

$$|\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))| = \prod_{v=1}^{dD(FS_\beta)} \left(c_v(\sum_{j\in\beta_v} i_j)\right).$$

Proof. If $c_v = 0$ for some $1 \le v \le dD(FS_\beta)$, then $\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$ is empty. It means

$$|\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))| = 0.$$

Also, $\prod_{v=1}^{dD(FS_\beta)}\left(c_v(\sum_{j\in\beta_v} i_j)\right) = 0$. Hence, the statement is true in this special case.

Now we assume that there is at least one disjoint cycle with length $\sum_{j\in\beta_v} i_j$ in σ'_v, $1 \le v \le dD(FS_\beta)$. We first pick disjoint cycle ρ'_v with length $\sum_{j\in\beta_v} i_j$ in σ_v, $1 \le v \le dD(FS_\beta)$. The number of the choices of ρ'_v is $\prod_{v=1}^{dD(FS_\beta)} c_v$. Now we fix a choice of the disjoint cycles ρ'_v, we claim that we can construct $\prod_{v=1}^{dD(FS_\beta)}\sum_{j\in\beta_v} i_j$ many $[\delta_1]$ such that $([\delta_1], \delta_2, ..., \delta_k, [\delta_1]\sigma') \in \mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$, which implies

$$|\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))| = \prod_{v=1}^{dD(FS_\beta)}\left(c_v(\sum_{j\in\beta_v} i_j)\right).$$

Now we will prove the claim. If we want to use these disjoint cycles ρ'_v to construct the other d-tuples $[\delta_1]$ such that β is the RP of σ' with respect to $[\delta_1]$, we have to pick $|\beta_v|$ ($|\beta_v|$ is the length of β_v) many integers m_j, $j \in \beta_v$, from ρ'_v such that $i_j = dist(m_j, \sigma', \{m_d, ..., m_1\}), 1 \le j \le d$. In fact, any integer k in ρ'_v uniquely determines the choices of all integers m_j, $j \in \beta_v$. Let $\beta_v = (j^v_1 ... j^v_{|\beta_v|})$. Let $k = m_{j^v_1}$. $m_{j^v_2}$ is uniquely determined by the distance $i_{j^v_1}$ in ρ'_v. Similarly, all m_j, $j \in \beta_v$, are uniquely determined. Hence, the number of choices of all possible integers from ρ'_v is the length of ρ'_v, i.e. $\sum_{j\in\beta_v} i_j$. Go through all of the disjoint cycles ρ'_v, $1 \le v \le dD(FS_\beta)$. We have $\prod_{v=1}^{dD(FS_\beta)}\sum_{j\in\beta_v} i_j$ many choices of $[\delta_1]$. $\square$

Lemma 3.3.19. *Let $i_1, ..., i_d$ be d positive integers. We have*

$$\sum_{(\delta_2,...,\delta_k,\sigma')\in\mathcal{B}^d_\beta(\alpha')} \hat{p}_{(d...1)\beta}(i_1, ..., i_d)\widetilde{\frac{\partial}{\partial\hat{p}_\beta}(i_1, ..., i_d)}(\Phi(\sigma'))$$
$$= \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)} \Phi(\sigma)\right).$$

Proof. Given any element $(\delta_2, ..., \delta_k, \sigma') \in \mathcal{B}^d_\beta(\alpha')$, we have the following formula (Lemma 3.3.17)

$$\hat{p}_{(d...1)\beta}(i_1, ..., i_d)\widetilde{\frac{\partial}{\partial\hat{p}_\beta}(i_1, ..., i_d)}(\Phi(\sigma'))$$
$$= \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))} \Phi(\sigma)\right).$$

If we sum over all elements in $\mathcal{B}^d_\beta(\alpha')$, we get the formula in the lemma

$$\begin{aligned}
&\sum_{(\delta_2,...,\delta_k,\sigma')\in\mathcal{B}^d_\beta(\alpha')} \hat{p}_{(d...1)\beta}(i_1,...,i_d)\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1,...,i_d)}(\Phi(\sigma')) \\
&= \sum_{\mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))} \sum_{\substack{([\delta_1],\delta_2,...,\delta_k,\sigma)\in \\ \mathcal{A}^d_\beta(\alpha',(\delta_2,...,\delta_k,\sigma'),(i_1,...,i_d))}} \Phi(\sigma) \\
&= \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha',i_1,...,i_d)} \Phi(\sigma) \right).
\end{aligned}$$

The first equality comes from Definition 3.3.13 and 3.3.16. From these two definitions, we know that an element $(\delta_2, ..., \delta_k, \sigma')$ in $\mathcal{B}^d_\beta(\alpha')$ corresponds uniquely to the set

$$\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d)).$$

So, summing over all elements in the set $\mathcal{B}^d_\beta(\alpha')$ is equivalent to sum over all possible sets $\mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d))$. The second equality comes from the following disjoint union

$$\bigcup_{(\delta_2,...,\delta_k,\sigma')\in\mathcal{B}^d_\beta(\alpha')} \mathcal{A}^d_\beta(\alpha', (\delta_2, ..., \delta_k, \sigma'), (i_1, ..., i_d)) = \mathcal{A}^d_\beta(\alpha', i_1, ..., i_d).$$

□

Now we are ready to prove the following key lemma.

Lemma 3.3.20.

$$\widetilde{FS}_\beta(\widetilde{H}^{min}_d) = \frac{\partial(\widetilde{H}^{min}_d)_\beta}{\partial u}.$$

Proof. We use the same notation as in Lemma 3.3.19. Recall the following formula in Lemma 3.3.10

$$\mu^d(\sigma) - 1 = \sum_{i=1}^{dD(FS_\beta)} \mu^d(\sigma'_i).$$

By Lemma 3.3.19, we have

$$\left(\sum_{(\delta_2,...,\delta_k,\sigma')\in\mathcal{B}^d_\beta(\alpha')} \hat{p}_{(d...1)\beta}(i_1,...,i_d)\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1,...,i_d)}(\Phi(\sigma')) \right) \frac{z^n}{n!}\frac{u^{\sum_{i=1}^{dD(FS_\beta)}\mu^d(\sigma'_i)}}{(\sum_{i=1}^{dD(FS_\beta)}\mu^d(\sigma'_i))!}$$
$$= \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)} \Phi(\sigma) \right) \frac{z^n}{n!}\frac{u^{\mu^d(\alpha)-1}}{(\mu^d(\alpha)-1)!}.$$

Summing over all partition α' of n (contribute to the generating function) and all positive integers $i_1,...,i_d$ (contribute to the operator), we have

$$\sum_{\substack{\alpha',\\ i_1,...,i_d}} \left(\sum_{(\delta_2,...,\delta_k,\sigma')\in\mathcal{B}^d_\beta(\alpha')} \hat{p}_{(d...1)\beta}(i_1,...,i_d)\frac{\partial}{\partial \hat{p}_\beta}\widetilde{(i_1,...,i_d)}(\Phi(\sigma')) \right) \frac{z^n}{n!}\frac{u^{\sum_{i=1}^{dD(FS_\beta)}\mu^d(\sigma'_i)}}{(\sum_{i=1}^{dD(FS_\beta)}\mu^d(\sigma'_i))!}$$
$$= \sum_{\substack{\alpha',\alpha,\\ i_1,...,i_d}} \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)} \Phi(\sigma) \right) \frac{z^n}{n!}\frac{u^{\mu^d(\alpha)-1}}{(\mu^d(\alpha)-1)!}.$$

The left hand of the equation is clear. We want to explain why we also take the sum over α on the right hand side of the equation. By Lemma 3.3.14, the data $\{i_1,...,i_d,\beta,\alpha'\}$ will uniquely determine the type α of the set $\mathcal{A}^d_\beta(\alpha',i_1,...,i_d)$. Hence, if α does not correspond to these data, we take $\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)$ as an empty set. Hence, we take the sum over all partitions α and α' of n.

By Remark 3.3.15, we have

$$\bigcup_{\alpha'\vdash n}\bigcup_{i_1,...,i_d\geq 1}\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)=\mathcal{A}^d_\beta(\alpha).$$

It means

$$\sum_{\substack{\alpha',\alpha\\ i_1,...,i_d}} \left(\sum_{([\delta_1],\delta_2,...,\delta_k,\sigma)\in\mathcal{A}^d_\beta(\alpha,\alpha',i_1,...,i_d)} \Phi(\sigma) \right) \frac{z^n}{n!}\frac{u^{\mu^d(\alpha)-1}}{(\mu^d(\alpha)-1)!}$$
$$= \sum_{n\geq 1}\sum_{\alpha\vdash n}|\mathcal{A}^d_\beta(\alpha)|\frac{z^n}{n!}\frac{u^{\mu^d(\alpha)-1}}{(\mu^d(\alpha)-1)!}\Phi(\alpha) = d\frac{\partial(\widetilde{H}^{min}_d)_\beta}{\partial u},$$

which gives the right hand side of the equation. The left hand side of the equation is exactly

$$\sum_{\substack{\alpha',\\ i_1,...,i_d}} \left(\sum_{(\delta_2,...,\delta_k,\sigma')\in\mathcal{B}^d_\beta(\alpha')} \hat{p}_{(d...1)\beta}(i_1,...,i_d) \frac{\partial}{\partial \hat{p}_\beta} \widetilde{(i_1,...,i_d)}(\Phi(\sigma')) \right) \frac{z^n}{n!} \frac{u^{\sum_{i=1}^{dD(FS_\beta)} \mu^d(\sigma'_i)}}{(\sum_{i=1}^{dD(FS_\beta)} \mu^d(\sigma'_i))!}$$
$$= d\widetilde{FS}_\beta(\widetilde{H}^{min}_d).$$

Hence, we have

$$\widetilde{FS}_\beta(\widetilde{H}^{min}_d) = \frac{\partial(\widetilde{H}^{min}_d)_\beta}{\partial u}.$$

□

Theorem 3.3.6 is a direct result of Lemma 3.3.20. Here is the proof.

Proof of Theorem 3.3.6. By Lemma 3.3.20, we have

$$\frac{\partial(\widetilde{H}^{min}_d)_\beta}{\partial u} = \widetilde{FS}_\beta(\widetilde{H}^{min}_d).$$

Take the sum over $\beta \in S_d$ such that $d(FS_\beta)$ is of degree $d+1$, we have

$$\frac{\partial \widetilde{H}^{min}_d}{\partial u} = \sum_{\substack{\beta\in S_d,\\ d(FS_\beta)=d+1}} \frac{\partial(\widetilde{H}^{min}_d)_\beta}{\partial u} = \sum_{\substack{\beta\in S_d,\\ d(FS_\beta)=d+1}} \widetilde{FS}_\beta(\widetilde{H}^{min}_d) = \widetilde{HW}([d])(\widetilde{H}^{min}_d).$$

□

Remark 3.3.21. *By Theorem 3.3.6, given any minimal transitive factorization*

$$([\delta_1], \delta_2, ..., \delta_{\mu^3(\alpha)}, \sigma),$$

it corresponds to a unique permutation $\beta \in S_d$ such that $dD(FS_\beta) = d+1$. This type of transitive factorization gives the construction of the operator $\widetilde{FS}_\beta$. All transitive factorizations of this type contribute to the generating function

$$\widetilde{FS}_\beta(\widetilde{H}^{min}_d),$$

more precisely,

$$\frac{\partial(\widetilde{H}_d^{min})_\beta}{\partial u} = \widetilde{FS}_\beta(\widetilde{H}_d^{min}).$$

With Lemma 3.3.20 and Theorem 3.3.6, we have the following corollary.

Corollary 3.3.22.

$$\frac{1}{d-1}\left(z\frac{\partial H_d^{min}}{\partial z} + \sum_{i\geq 1} p_i \frac{\partial H_d^{min}}{\partial p_i} - 2H_d^{min}\right) = \widetilde{HW}([d])(H_d^{min}).$$

Proof. Theorem 3.3.20 gives us the following equation,

$$\frac{\partial \widetilde{H}_d^{min}}{\partial u} = \widetilde{HW}([d])(\widetilde{H}_d^{min}). \tag{3.3.5}$$

Recall the definition of H_d^{min} and $\widetilde{H}_d^{min}$ (Construction 3.3.3). We know

$$H_d^{min}(z, p_1, p_2, ...) = \widetilde{H}_d^{min}(z, u, p_1, p_2, ...)|_{u=1}.$$

Let $u = 1$. The RHS of the equation (3.3.5) is

$$\text{Right Side} = \widetilde{HW}([d])(\widetilde{H}_d^{min})|_{u=1} = \widetilde{HW}([d])(H_d^{min}).$$

Now we want to calculate the LHS of (3.3.5) when $u = 1$. By simple calculations, we have

$$\begin{aligned}
\frac{\partial \widetilde{H}_d^{min}}{\partial u} &= \sum_{n\geq 1}\sum_{\alpha\vdash n} \mu^d(\alpha) h^d(\alpha) \frac{z^n}{n!} \frac{u^{\mu^d(\alpha)-1}}{\mu^d(\alpha)!} \Phi(\alpha), \\
z\frac{\partial H_d^{min}}{\partial z} &= \sum_{n\geq 1}\sum_{\alpha\vdash n} n h^d(\alpha) \frac{z^n}{n!} \frac{1}{\mu^d(\alpha)!} \Phi(\alpha), \\
\sum_{i\geq 1} p_i \frac{\partial H_d^{min}}{\partial p_i} &= \sum_{n\geq 1}\sum_{\alpha\vdash n} l(\alpha) h^d(\alpha) \frac{z^n}{n!} \frac{1}{\mu^d(\alpha)!} \Phi(\alpha).
\end{aligned}$$

where $l(\alpha)$ is the length for the partition α. By Lemma 3.3.2, we know

$$\mu^d(\alpha) = \frac{n + l(\alpha) - 2}{d-1}.$$

Hence, when $u=1$, LHS of the equation in Theorem 3.3.6 is

$$LHS = \frac{\partial \widetilde{H}_d^{min}}{\partial u} \mid_{u=1} = \frac{1}{d-1}(z\frac{\partial H_d^{min}}{\partial z} + \sum_{i\geq 1} p_i \frac{\partial H_d^{min}}{\partial p_i} - 2H_d^{min}).$$

Combining LHS and RHS, we prove this corollary. □

3.4 Generating Function of d-Hurwitz Number

Recall the d-Hurwitz number $h_k^{[d]}(\alpha)$ and the d-Frobenius number $f_k^{[d]}(\alpha)$ (see Definition 3.2.6). The Frobenius number $f_k^{[d]}(\alpha)$ counts the number of coverings of $\mathcal{P}^1$ with $k+1$ branch points (not necessarily connected), where k branch points correspond to d-cycles and the other one corresponds to a cycle of type α. We have an equation in Theorem 3.2.7 satisfied by the generating series for the Frobenius numbers. The Hurwitz number $h_k^{[d]}(\alpha)$ counts connected coverings. In this section, we will derive an equation satisfied by the generating functions

$$H^{[d]}(u,p) = \sum_{n\geq 1} \frac{1}{n!} \sum_{k=1}^{\infty} \sum_{\alpha \vdash n} h_k^{[d]}(\alpha) \frac{u^k}{k!} \Phi(\alpha) \ .$$

In the previous section, we discussed the minimal Hurwitz number $h^{[d]}(\alpha) = h_{\mu^d(\alpha)}^{[d]}(\alpha)$. For example, let α be the partition (1^2) of 2. Then $((12),(12),(1)(2))$ is a well-defined minimal transitive factorization of α contributing to $h_2^{[2]}(\alpha) = h^d(\alpha)$. In this section, we discuss all transitive factorizations (not necessarily minimal). For example, the transitive factorization $((12),(12),(12),(12),(1)(2))$ contributes to the 2-Hurwitz number $h_4^{[2]}(\alpha)$.

Let $([\delta_1], \delta_2, ..., \delta_{\mu^d(\sigma)}, \sigma) \in \mathcal{A}_\beta^d(\alpha)$ (See Definition 3.3.11). We found in Lemma 3.3.9 that the action of the group generated by $\{\delta_2, ..., \delta_{\mu^d(\sigma)}\}$ has exactly $dD(FS_\beta)$ connected components. This property gave us the idea to construct the operator $\widetilde{FS}_\beta$ (see Construction 3.3.3). We emphasize that the number of connected components is exactly the number of disjoint cycles of β (or $dD(FS_\beta)$) under the "minimal" condition (see Remark 3.3.5).

Now we are interested in all transitive factorizations $\delta_1 ... \delta_k = \sigma$, not necessarily minimal. In this case, if $\sigma' = [\delta_1]^{-1}\sigma$, with RP β with respect to $[\delta_1]$, the number of connected components of $(\delta_2, ..., \delta_k, \sigma')$ is between 1 and $dD(FS_\beta)$.

For example, let $\sigma = (1342)$. We have the following two transitive factorizations

$$\sigma = (23)(12)(34)(34)(34) = (23)(12)(23)(23)(34).$$

Clearly, (1)(2) is the RP of $\sigma' = (12)(34)$ with respect to $[\delta_1] = [(23)]$ in both cases. But the number of connected components is different:

$[\delta_1]$	σ'	$(\delta_2, \delta_3, \delta_4, \delta_5)$	# connected components
$[2, 3]$	$(12)(34)$	$((12), (34), (34), (34))$	2,
$[2, 3]$	$(12)(34)$	$((12), (23), (23), (34))$	1.

Let's take $d = 2$ and consider the generating function $H^{[2]}$. Details can be found in [13]. Consider a transitive factorization $([\delta_1], \delta_2, ..., \delta_k, \sigma)$. β is the RP of $\sigma' = \delta_2...\delta_k$ with respect to the 2-tuple $[\delta_1]$. There are three possible cases.

- $\beta = (12)$. In this case, $(\delta_2, ..., \delta_k, \sigma')$ always has one connected component. All transitive factorizations in this case contribute to the generating function

$$\frac{1}{2}\sum_{i,j\geq 1}(i+j)p_i p_j \frac{\partial H^{[2]}}{\partial p_{i+j}}.$$

- $\beta = (1)(2)$ and $(\delta_2, ..., \delta_k, \sigma')$ has one connected component. The generating function constructed from all transitive factorizations in this case is

$$\frac{1}{2}\sum_{i,j\geq 1} ijp_{i+j}\frac{\partial^2 H^{[2]}}{\partial p_i \partial p_j}.$$

- $\beta = (1)(2)$ and $(\delta_2, ..., \delta_k, \sigma')$ has two connected components. Similarly, this case contributes to following the generating function

$$\frac{1}{2}\sum_{i,j\geq 1} ijp_{i+j}\frac{\partial H^{[2]}}{\partial p_i}\frac{\partial H^{[2]}}{\partial p_j}.$$

Hence, we have

$$\frac{\partial H^{[2]}}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}\left((i+j)p_i p_j \frac{\partial H^{[2]}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 H^{[2]}}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial H^{[2]}}{\partial p_i}\frac{\partial H^{[2]}}{\partial p_j}\right).$$

In fact, the transitive factorization in the second case and third case have the same RP $\beta = (1)(2)$, but with different number of connected components. We introduce a new notation $\widehat{FS}_\beta$, $\beta \in S_2$, as follows

$$\widehat{FS}_{(12)} = \frac{1}{2}\sum_{i,j\geq 1}(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}},$$
$$\widehat{FS}_{(1)(2)} = \frac{1}{2}\sum_{i,j\geq 1}ijp_{i+j}\left(\frac{\partial^2}{\partial p_i\partial p_j} + \frac{\partial}{\partial p_i}\times\frac{\partial}{\partial p_j}\right).$$

We want to extend this construction to any permutation $\beta \in S_d$. With the same notation as above, we know the number of connected components of $(\delta_2, ..., \delta_k, \sigma')$ should be between 1 and $dD(FS_\beta)$. Assume that there are i connected components. We break the set $\{\beta_1, ..., \beta_{dD(FS_\beta)}\}$ into i nonempty disjoint sets. Denote by $SPar_{\beta,i}$ the set of all possible cases.

For example, let

$$\beta = \beta_1\beta_2\beta_3.$$

Then, $SPar_{\beta,2}$ has three elements

$$SPar_{\beta,2} = \{\quad \{\{\beta_1,\beta_2\},\{\beta_3\}\},\{\{\beta_1,\beta_3\},\{\beta_2\}\},\{\{\beta_2,\beta_3\},\{\beta_1\}\}\quad\}.$$

Now we return to the general case $\beta = \beta_1...\beta_{dD(FS_\beta)}$ and let $Par \in SPar_{\beta,i}$. Then, Par has i elements, Par_1,...,Par_i. Recall that for $i_1, ..., i_d \geq 1$ and $\beta \in S_d$, we define a polynomial $\hat{p}_\beta(i_1, ..., i_d)$ and a differential operator $\frac{\partial}{\partial \hat{p}_\beta}(i_1, ..., i_d)$ in Construction 3.3.3. Similarly, we define the operator $FPar_j(i_1, ..., i_d)$ as

$$FPar_j(i_1,...,i_d) = \prod_{\beta'\in Par_j}(\sum_{j\in\beta'} i_j)\frac{\partial}{\partial \hat{p}_{\sum_{j\in\beta'} i_j}}.$$

Then, we define the operator $\widehat{FS}_{\beta,i,Par}$ as follows

$$\widehat{FS}_{\beta,i,Par} = \frac{1}{d}\sum_{i_1,...,i_d\geq 1}\hat{p}_{(d...1)\beta}(i_1,...,i_d)\left(\times_{j=1}^{i}(FPar_j(i_1,...,i_d))\right). \qquad (3.4.1)$$

The operator $\left(\times_{j=1}^{i}(FPar_j(i_1,...,i_d))\right)$ means that each element in Par corresponds to a connected component, where $\times_{j=1}^{i}$ is the "product" of differential operators from different connected components, the same as the product defined in $\widetilde{FS}_\beta$, and $\prod_{\beta' \in Par_i} \frac{\partial}{\partial \hat{p}_{\beta'}}$ is the composition of differential operator as in Construction 3.3.3. Consider the example

$$Par = \{\{(2),(3)\},\{(1)\}\} \in SPar_{(1)(2)(3),2}.$$

For $Par_1 = \{(2),(3)\} \in Par$, we have

$$FPar_1(i_1,i_2,i_3) = \prod_{\beta' \in Par_1} \frac{\partial}{\partial \hat{p}_{\beta'}} = i_2 i_3 \frac{\partial^2}{\partial p_{i_2} \partial p_{i_3}}.$$

Similarly, $Par_2 = \{(1)\}$ gives

$$FPar_2(i_1,i_2,i_3) = \prod_{\beta' \in Par_2} \frac{\partial}{\partial \hat{p}_{\beta'}} = i_1 \frac{\partial}{\partial p_{i_1}}.$$

Finally, we take the "product" of these two differential operators

$$\left(\times_{j=1}^{2}(FPar_j(i_1,i_2,i_3)))\right) = i_2 i_3 \frac{\partial^2}{\partial p_{i_2} \partial p_{i_3}} \times i_1 \frac{\partial}{\partial p_{i_1}}.$$

Remark 3.4.1. *Let β be a permutation in S_d. Clearly, $SPar_{\beta,1}$ and $SP_{\beta,dD(FS_\beta)}$ have only one element. We have the following relations*

$$\begin{aligned} \widehat{FS}_{\beta,1} &= FS_\beta, \\ \widehat{FS}_{\beta,dD(FS_\beta)} &= \widetilde{FS}_\beta. \end{aligned}$$

If β has only one disjoint cycle (corresponding to a full length partition), we have

$$\widehat{FS}_\beta = \widetilde{FS}_\beta = FS_\beta.$$

Given an element $Par \in SPar_{\beta,i}$, we already defined the operator $\widehat{FS}_{\beta,i,Par}$ by Equation (3.4.1). Define $\widehat{FS}_{\beta,i}$ by summing over all elements in $SPar_{\beta,i}$, i.e.

$$\widehat{FS}_{\beta,i} = \sum_{Par \in SPar_{\beta,i}} \widehat{FS}_{\beta,i,Par}. \tag{3.4.2}$$

Similarly, the operator $\widehat{FS}_\beta$ is the sum of $\widehat{FS}_{\beta,i}$ for $1 \le i \le dD(FS_\beta)$, i.e.

$$\widehat{FS}_\beta = \sum_{i=1}^{dD(FS_\beta)} \widehat{FS}_{\beta,i}. \tag{3.4.3}$$

Finally, we give the definition of $\widehat{W}([d])$.

Definition 3.4.2. *Let d be a positive integer. Define $\widehat{W}([d])$ as the sum of $\widehat{FS}_\beta$ (Eq. (3.4.3)) over $\beta \in S_d$, i.e.*

$$\widehat{W}([d]) = \sum_{\beta \in S_d} \widehat{FS}_\beta.$$

Theorem 3.4.3.

$$\frac{\partial H^{[d]}}{\partial u} = \widehat{W}([d]) H^{[d]}.$$

Proof. We only give the idea of the proof. Details are similar to the proof of Theorem 3.3.6. Consider a transitive factorization $([\delta_1], \delta_2, ..., \delta_k, \sigma)$. β is the RP of $\sigma' = \delta_2 ... \delta_k$ with respect to the d-tuple $[\delta_1]$ and $(\delta_2, ..., \delta_k, \sigma')$ has i connected components. This gives an element $Par \in SPar_{\beta,i}$. In fact, all such transitive factorizations contributes to the generating function

$$\widehat{FS}_{\beta,i,Par}(H^{[d]}).$$

All transitive factorizations $([\delta_1], \delta_2, ..., \delta_k, \sigma)$ with RP β give the generating function

$$\widehat{FS}_\beta(H^{[d]}) = \left(\sum_{1 \le i \le dD(FS_\beta)} \sum_{Par \in SPar_{\beta,i}} \widehat{FS}_{\beta,i,Par} \right) (H^{[d]}).$$

By summing over all $\beta \in S_d$, we have

$$\frac{\partial H^{[d]}}{\partial u} = \widehat{W}([d])(H^{[d]}).$$

□

Example 3.4.4. *We already gave an example $d = 2$ as above. Here, we give another example, $d = 3$.*

1. *We begin with the easiest two operators* $\widehat{FS}_{(123)}$ *and* $\widehat{FS}_{(321)}$. *Since* (123) *and* (321) *have only one disjoint cycle, so*

$$\widehat{FS}_{(123)} = \widetilde{FS}_{(123)} = FS_{(123)},$$
$$\widehat{FS}_{(321)} = \widetilde{FS}_{(321)} = FS_{(321)}.$$

2. *Now let's consider the permutations* $\beta = (12)(3)$ *or* $(13)(2)$ *or* $(23)(1)$, *which have two disjoint cycles. It is easy to check the following relations*

$$\widehat{FS}_{\beta,1} = FS_\beta,$$
$$\widehat{FS}_{\beta,2} = \widetilde{FS}_\beta.$$

Hence, we have

$$\widehat{FS}_\beta = \widehat{FS}_{\beta,1} + \widehat{FS}_{\beta,2} = FS_\beta + \widetilde{FS}_\beta.$$

3. *Finally, let* $\beta = (1)(2)(3)$. *The number of connected components can be* $1, 2, 3$. *If we have only one connected component,* $SPar_{(1)(2)(3),1} = \{\{\{(1),(2),(3)\}\}\}$, *where there is only one element* $\{\{(1),(2),(3)\}\}$ *in the set* $SPar_{(1)(2)(3),1}$. *By calculation, we find*

$$\widehat{FS}_{\beta,1} = FS_\beta.$$

If the number of connected components is two, $SPar_{\beta,2}$ *has three elements*

$$SPar_{\beta,2} = \{\{\{(1),(2)\},\{(3)\}\},\{\{(1),(3)\},\{(2)\}\},\{\{(2),(3)\},\{(1)\}\}\}.$$

We take $Par = \{\{(1),(2)\},\{(3)\}\}$ *as an example as above. We have*

$$\widehat{FS}_{(1)(2)(3),i,Par} = \frac{1}{3}\sum_{i_1,\ldots,i_3\geq 1} i_1 i_2 i_3 p_{i_1+i_2+i_3} \frac{\partial^2}{\partial p_{i_1}\partial p_{i_2}} \times \frac{\partial}{\partial p_{i_3}}.$$

The reader can write down the operator $\widehat{FS}_{(1)(2)(3),i,Par}$ *for the other two elements in* $SPar_{\beta,2}$ *similarly. Now let's go to the case that we have three connected components. In this case,*

$$SPar_{(1)(2)(3),1} = \{ \quad \{\{(1)\},\{(2)\},\{(3)\}\} \quad \},$$

in which the unique element $\{\{(1)\},\{(2)\},\{(3)\}\}$ *has three element. By calculation, we have*

$$\widehat{FS}_{(1)(2)(3),3} = \widetilde{FS}_{(1)(2)(3)}.$$

With the discussion above, we have

$$\begin{aligned}
\widehat{W}([3]) &= \sum_{\beta \in S_3} \widehat{FS}_\beta \\
&= FS_{(123)} + FS_{(321)} + \sum_{\beta \in \{(12),(23),(13)\}} (FS_\beta + \widetilde{FS}_\beta) \\
&+ \widetilde{FS}_{(1)(2)(3)} + FS_{(1)(2)(3)} + \widehat{FS}_{(1)(2)(3),2} \\
&= W([3]) + \widetilde{HW}([3]) - FS_{(123)} + \widehat{FS}_{(1)(2)(3),2}.
\end{aligned}$$

3.5 Topological Recursion

In this section, we consider a ramified n-fold covering of $\mathbb{P}_1$ by a genus g smooth curve with $k+1$ ramified points, where k of them correspond to d-cycles and the last one corresponds to a permutation of type α. By Riemann-Hurwitz formula, we have

$$2g - 2 = n(-2) + ((d-1)k - l(\alpha) + n).$$

The number of ramified points corresponding to d-cycles is

$$k = \frac{n + l(\alpha) + 2g - 2}{d - 1}. \tag{3.5.1}$$

Denote this number by $\mu^{d,g}(\alpha)$. We emphasize that, given d, α, n, the genus g and the number of simple branched points k determine each other uniquely.

We define $h^{[d],g}(\alpha)$ to be the number of $(\mu^{d,g}(\alpha)+1)$-tuples $(\delta_1, ..., \delta_{\mu^{d,g}(\alpha)}, \sigma)$ satisfying the following conditions

- δ_i is of type $(1^{n-d}d)$ (or d-cycles), $1 \le i \le \mu^{d,g}(\alpha)$ and σ is of type α,
- $\delta_1 ... \delta_{\mu^{d,g}(\alpha)} = \sigma$,
- the subgroup generated by $\{\delta_1, ..., \delta_{\mu^{d,g}(\alpha)}\}$ acts transitively on the set $\{1, ..., n\}$,

Define the generating function of $h^{[d],g}(\alpha)$ as following

$$H^{[d],g}(u,p) = H^{[d],g}(u,p_1,p_2,...) = \sum_{n\geq 1} \frac{1}{n!} \sum_{\alpha \vdash n} h^{[d],g}(\alpha) \frac{u^{\mu^{d,g}(\alpha)}}{(\mu^{d,g}(\alpha))!} \Phi(\alpha) \ .$$

Clearly,

$$H^{[d]} = \sum_{g=1}^{\infty} H^{[d],g}.$$

We use another parameter y for the genus g and define the generating function as

$$H^{[d]}(y) = \sum_{g=1}^{\infty} H^{[d],g} y^g.$$

Given a transitive factorization $([\delta_1],...,\delta_k,\sigma)$ corresponding to a genus g covering, deleting the first d-tuple (or d-cycle), we get a factorization $(\delta_2,...,\delta_k,\sigma')$ of $\sigma' = \delta_2...\delta_k$. We assume β is the RP (see Definition 2.10.13) of σ' with respect to $[\delta_1]$. If $(\delta_2,...,\delta_k,\sigma')$ has i connected components $X_1,...,X_i$, each of which corresponds to a transitive factorization of a permutation σ_j, $1 \leq j \leq i$. Assume that the transitive factorization of σ_j (of type α_j) corresponds to a genus g_j covering. We have

$$k_j = \frac{n_j + l(\alpha_j) + 2g_j - 2}{d-1},$$
$$k = \frac{n + l(\alpha) + 2g - 2}{d-1},$$

where k_j is the number of permutations in the factorization of σ_j and n_j is the cardinality of X_j. Based on the following three equations

$$\sum_{j=1}^{i} k_j = k-1, \quad \sum_{j=1}^{i} n_j = n, \quad \sum_{j=1}^{i} l(\alpha_j) - dD(FS_\beta) + dP(FS_\beta) = l(\alpha),$$

we have

$$k - 1 = \frac{n + l(\alpha) - dP(FS_\beta) + dD(FS_\beta) + \sum_{j=1}^{i}(2g_j - 2)}{d-1}.$$

The above calculation gives us the relation for the genus,

$$\sum_{j=1}^{i} g_j = \frac{2g-1-d+dP(FS_\beta)-dD(FS_\beta)+2i}{2} = g - dD(FS_\beta)+i, \tag{3.5.2}$$

where the second equality comes from $d+1 = dD(FS_\beta)+dP(FS_\beta)$ by Lemma 3.3.10. This formula tells us that when we add a d-cycle δ_1 to a covering (may not be connected) corresponding to the factorization $(\delta_2, ..., \delta_k, \sigma')$, the genus of the corresponding transitive factorization $(\delta_1, ..., \delta_k, \sigma)$ will increase by

$$dD(FS_\beta)-i.$$

With this property, we add the parameter y to the operator $\widehat{FS}_{\beta,i}$ as follows

$$\widehat{FS}_{\beta,i}(y) := \widehat{FS}_{\beta,i} y^{dD(FS_\beta)-i}.$$

Similar to Definition 3.4.2, we define

$$\widehat{W}([d])(y) = \sum_{\beta\in S_d} \sum_{i=1}^{dD(FS_\beta)} \widehat{FS}_{\beta,i}(y). \tag{3.5.3}$$

With the same proof as Theorem 3.4.3, we have the following corollary.

Corollary 3.5.1.

$$\frac{\partial H^{[d]}(y)}{\partial u} = \widehat{W}([d])(y) H^{[d]}(y).$$

Recall that $\widehat{FS}_{\beta,i}(H^{[d]})$ is the "product" of differential operators acting on the same generating series $H^{[d]}$. Since the differential part of $\widehat{FS}_{\beta,i}$ is defined as the "product" of i differential operators, we can define $\widehat{FS}_{\beta,i}(H^{[d],g_1}, ..., H^{[d],g_i})$ as the i differential operators acting on $H^{[d],g_1}, ..., H^{[d],g_i}$ separately. For example,

$$\widehat{FS}_{(1)(2)(3),i,Par}(H^{[d],g_1}, H^{[d],g_2}) = \frac{1}{3} \sum_{i_1,...,i_3\geq 1} i_1 i_2 i_3 p_{i_1+i_2+i_3} \frac{\partial^2 H^{[d],g_1}}{\partial p_{i_1}\partial p_{i_2}} \times \frac{\partial H^{[d],g_2}}{\partial p_{i_3}},$$

where $i=2$ and $Par = \{\{(1),(2)\},\{(3)\}\}$.

Corollary 3.5.2 (Topological Recursion for Connected d-Hurwitz number)**.**

$$\frac{\partial H^{[d],g}}{\partial u} = \sum_{\beta\in S_d} \sum_{i=1}^{dD(FS_\beta)} \sum_{\substack{g_1+...+g_i=\\ g-dD(FS_\beta)+i}} \widehat{FS}_{\beta,i}(H^{[d],g_1}, ..., H^{[d],g_i}).$$

Proof. We give two proofs for this corollary. For the first one, given a power series $f(y) \in \mathbb{C}[[y]]$, $[y^n]f(y)$ means the coefficient of y^n in $f(y)$. Then, this corollary comes from Corollary 3.5.1 by taking the coefficient of y^g, i.e.

$$[y^g]\frac{\partial H^{[d]}(y)}{\partial u} = [y^g]\widehat{W}([d])(y)H^{[d]}(y). \tag{3.5.4}$$

The RHS of Eq. (3.5.4) is

$$\begin{aligned} [y^g]\widehat{W}([d])(y)H^{[d]}(y) &= [y^g]\sum_{\beta\in S_d} \sum_{i=1}^{dD(FS_\beta)} \widehat{FS}_{\beta,i}(y)H^{[d]}(y) \\ &= [y^g]\sum_{\beta\in S_d} \sum_{i=1}^{dD(FS_\beta)} \widehat{FS}_{\beta,i}(y)(\sum_{g'\geq 0} H^{[d],g'}y^{g'}) \\ &= [y^{g-dD(FS_\beta)+i}]\sum_{\beta\in S_d} \sum_{i=1}^{dD(FS_\beta)} \widehat{FS}_{\beta,i}(\sum_{g'\geq 0} H^{[d],g'}y^{g'}) \\ &= \sum_{\beta\in S_d} \sum_{i=1}^{dD(FS_\beta)} \sum_{\substack{g_1+...+g_i=\\ g-dD(FS_\beta)+i}} \widehat{FS}_{\beta,i}(H^{[d],g_1}, ..., H^{[d],g_i}). \end{aligned}$$

Now we give another method to prove this formula. Given a transitive factorization $([\delta_1], ..., \delta_k, \sigma)$ corresponding to a genus g covering, delete the first d-tuple (or d-cycle), we get a factorization $(\delta_2, ..., \delta_k, \sigma')$ of $\sigma' = \delta_2...\delta_k$. We assume that β is the RP of σ' with respect to $[\delta_1]$. If $(\delta_2, ..., \delta_k, \sigma')$ has i connected components, each of which corresponds to a transitive factorization of a permutation σ_j, $1 \leq j \leq i$. Assume that the transitive factorization of σ_j corresponds to a genus g_j covering. By Equation (3.5.2), we have

$$\sum_{j=1}^{i} g_j = g - dD(FS_\beta) + i.$$

All such transitive factorizations $([\delta_1], ..., \delta_k, \sigma)$ contribute to the generating function

$$\sum_{i=1}^{dD(FS_\beta)} \sum_{\substack{g_1+...+g_i=\\ g-dD(FS_\beta)+i}} \widehat{FS}_{\beta,i}(H^{[d],g_1}, ..., H^{[d],g_i}).$$

Taking the sum over β, we get the generating function. $\square$

Example 3.5.3. *Corollary 3.5.2 gives a recursion formula for $H^{[d],g}$. Let's consider the example $d = 2$. We get the following equation*

$$\frac{\partial H^{[2],g}}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}((i+j)p_ip_j\frac{\partial H^{[2],g}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 H^{[2],g-1}}{\partial p_i \partial p_j} + \sum_{g_1+g_2=g} ijp_{i+j}\frac{\partial H^{[2],g_1}}{\partial p_i}\frac{\partial H^{[2],g_2}}{\partial p_j}).$$

Taking the coefficient of $[y^0]$, i.e. $g = 0$, we have

$$\frac{\partial H^{[2],0}}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}((i+j)p_ip_j\frac{\partial H^{[2],0}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial H^{[2],0}}{\partial p_i}\frac{\partial H^{[2],0}}{\partial p_j}).$$

This equation is exactly the formula in Theorem 3.3.6 when $d = 2$.

Taking the coefficient of $[y^1]$, i.e. $g = 1$, we have

$$\frac{\partial H^{[2],1}}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}((i+j)p_ip_j\frac{\partial H^{[2],1}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 H^{[2],0}}{\partial p_i \partial p_j} + \sum_{g_1+g_2=1} ijp_{i+j}\frac{\partial H^{[2],g_1}}{\partial p_i}\frac{\partial H^{[2],g_2}}{\partial p_j}).$$

This formula only contains $H^{[2],0}$ and $H^{[2],1}$. Hence, if we solve the genus zero case, then we can plug the solution into this equation and solve for $H^{[2],1}$. In fact, Goulden and Jackson use this idea to calculate the genus one covering [13]. Using similar method, they calculate the genus two and three covering in [14] and make the polytonality conjecture, which was proved by Torsten Ekedahl, Sergei Lando, Michael Shapiro and Alek Vainshtein [5] [6] by the ESLV formula.

Example 3.5.4. *In this example, we study the case $d = 3$. By Corollary 3.5.2, we have*

$$\begin{aligned}\frac{\partial H^{[3],g}}{\partial u} &= \frac{1}{3}\sum_{i,j,k\geq 1}((i+j+k)p_ip_jp_k\frac{\partial H^{[3],g}}{\partial p_{i+j+k}} \\ &+ 3i(j+k)p_{i+j}p_k\frac{\partial^2 H^{[2],g-1}}{\partial p_i \partial p_{j+k}} + 3\sum_{g_1+g_2=g} i(j+k)p_{i+j}p_k \frac{\partial^2 H^{[2],g_1}}{\partial p_i}\frac{\partial^2 H^{[2],g_2}}{\partial p_{j+k}} \\ &+ ijkp_{i+j+k}\frac{\partial H^{[3],g-2}}{\partial p_i\partial p_j\partial p_k} + \sum_{g_1+g_2=g-1} ijkp_{i+j+k}\frac{\partial H^{[3],g_1}}{\partial p_i}\frac{\partial H^{[3],g_2}}{\partial p_j \partial p_k} \\ &+ \sum_{g_1+g_2+g_3=g} ijkp_{i+j+k}\frac{\partial H^{[3],g_1}}{\partial p_i}\frac{\partial H^{[3],g_2}}{\partial p_j}\frac{\partial H^{[3],g_3}}{\partial p_k} \\ &+ (i+j+k)p_{i+j+k}\frac{\partial H^{[3],g-1}}{\partial p_{i+j+k}}).\end{aligned}$$

Taking $g = 0$, we have

$$\begin{aligned}\frac{\partial H^{[3],0}}{\partial u} &= \frac{1}{3}\sum_{i,j,k\geq 1}((i+j+k)p_ip_jp_k\frac{\partial H^{[3],0}}{\partial p_{i+j+k}} + 3i(j+k)p_{i+j}p_k\frac{\partial^2 H^{[2],0}}{\partial p_i}\frac{\partial^2 H^{[2],0}}{\partial p_{j+k}} \\ &+ ijkp_{i+j+k}\frac{\partial H^{[3],0}}{\partial p_i}\frac{\partial H^{[3],0}}{\partial p_j}\frac{\partial H^{[3],0}}{\partial p_k},\end{aligned}$$

which is exactly the formula proved in Theorem 3.3.6. This formula first appears in Goulden and Jackson's paper [19]. But people do not know how to solve this formula and find the Hurwitz numbers in this case. Taking $g = 1$, we have

$$\begin{aligned}\frac{\partial H^{[3],1}}{\partial u} &= \frac{1}{3}\sum_{i,j,k\geq 1}((i+j+k)p_ip_jp_k\frac{\partial H^{[3],1}}{\partial p_{i+j+k}} + 3i(j+k)p_{i+j}p_k\frac{\partial^2 H^{[2],0}}{\partial p_i\partial p_{j+k}} \\ &+ 3\sum_{g_1+g_2=1} i(j+k)p_{i+j}p_k\frac{\partial^2 H^{[2],g_1}}{\partial p_i}\frac{\partial^2 H^{[2],g_2}}{\partial p_{j+k}} + ijkp_{i+j+k}\frac{\partial H^{[3],0}}{\partial p_i}\frac{\partial H^{[3],0}}{\partial p_j\partial p_k} \\ &+ \sum_{g_1+g_2+g_3=1} ijkp_{i+j+k}\frac{\partial H^{[3],g_1}}{\partial p_i}\frac{\partial H^{[3],g_2}}{\partial p_j}\frac{\partial H^{[3],g_3}}{\partial p_k} + (i+j+k)p_{i+j+k}\frac{\partial H^{[3],0}}{\partial p_{i+j+k}}).\end{aligned}$$

Chapter 4

References

[1] Carrell, S. (2009). Combinatorics and the KP Hierarchy (Master's thesis, University of Waterloo).

[2] Cavalieri, R., Johnson, P., Markwig, H. (2011). Wall crossings for double Hurwitz numbers. Advances in Mathematics, 228(4), 1894-1937.

[3] Chiodo, A. (2008). Stable twisted curves and their r-spin structures (Courbes champtres stables et leurs structures r-spin). Ann. Inst. Fourier, 58(5), 1635-1689.

[4] Crescimanno, M., Taylor, W. (1995). Large N phases of chiral QCD2. Nuclear Physics B, 437(1), 3-24.

[5] Ekedahl, T., Lando, S., Shapiro, M., Vainshtein, A. (1999). On Hurwitz numbers and Hodge integrals. Comptes Rendus de l'Acadmie des Sciences-Series I-Mathematics, 328(12), 1175-1180.

[6] Ekedahl, T., Lando, S., Shapiro, M., Vainshtein, A. (2001). Hurwitz numbers and intersections on moduli spaces of curves. Inventiones mathematicae, 146(2), 297-327.

[7] Fantechi, B., Pandharipande, R. (2002). Stable maps and branch divisors. Compositio Mathematica, 130(3), 345-364.

[8] Fulton, W. (2013). Intersection theory (Vol. 2). Springer Science Business Media.

[9] Fulton, W., Harris, J. (2013). Representation theory: a first course (Vol. 129). Springer Science Business Media.

[10] Graber, T., Pandharipande, R. (1999). Localization of virtual classes. Inventiones mathematicae, 135(2), 487-518.

[11] Goulden, I. P. (1994). A differential operator for symmetric functions and the combinatorics of multiplying transpositions. Transactions of the American Mathematical Society, 344(1), 421-440.

[12] Goulden, I., Jackson, D. (1997). Transitive factorisations into transpositions and holomorphic mappings on the sphere. Proceedings of the American Mathematical Society, 125(1), 51-60.

[13] Goulden, I. P., Jackson, D. M. (1999). A proof of a conjecture for the number of ramified coverings of the sphere by the torus. Journal of Combinatorial Theory, Series A, 88(2), 246-258.

[14] Goulden, I. P., Jackson, D. M. (1999). The number of ramified coverings of the sphere by the double torus, and a general form for higher genera. Journal of Combinatorial Theory, Series A, 88(2), 259-275.

[15] Goulden, I. P., Jackson, D. M., Vakil, R. (2005). Towards the geometry of double Hurwitz numbers. Advances in Mathematics, 198(1), 43-92.

[16] Goulden, I. P., Jackson, D. M., Vainshtein, A. (2000). The number of ramified coverings of the sphere by the torus and surfaces of higher genera. Annals of Combinatorics, 4(1), 27-46.

[17] Goulden, I. P., Guay-Paquet, M., Novak, J. (2012). Monotone Hurwitz numbers in genus zero. arXiv preprint arXiv:1204.2618.

[18] Goulden, I., Jackson, D. (l1997). Transitive factorisations into transpositions and holomorphic mappings on the sphere. Proceedings of the American Mathematical Society, 125(1), 51-60.

[19] Goulden, I. P., Jackson, D. M. (2000). Transitive factorizations in the symmetric group, and combinatorial aspects of singularity theory. European Journal of Combinatorics, 21(8), 1001-1016.

[20] Goulden, I. P., Guay-Paquet, M., Novak, J. (2012). Monotone Hurwitz numbers in genus zero. arXiv preprint arXiv:1204.2618.

[21] Harnad, J. (2016, June). Weighted Hurwitz numbers and hypergeometric -functions: an overview. In AMS Proc. Symp. Pure Math (Vol. 93, pp. 289-333).

[22] Hurwitz, A. (1891). über Riemann'sche Flächen mit gegebenen Verzweigungspunkten. Mathematische Annalen, 39(1), 1-60.

[23] Kac, V. G., Raina, A. K., Rozhkovskaya, N. (2013). Bombay lectures on highest weight representations of infinite dimensional Lie algebras (Vol. 29). World scientific.

[24] Kac, V. G., van de Leur, J. W. (2003). The n-component KP hierarchy and representation theory. Journal of Mathematical Physics, 44(8), 3245-3293.

[25] Kontsevich, M. (1995). Enumeration of rational curves via torus actions. In The moduli space of curves (pp. 335-368). Birkh?user Boston.

[26] Lando, S. K., Zvonkin, A. K. (2013). Graphs on surfaces and their applications (Vol. 141). Springer Science Business Media.

[27] Macdonald, I. G. (1998). Symmetric functions and Hall polynomials. Oxford university press.

[28] Mingo, J. A., Nica, A. (2004). Annular noncrossing permutations and partitions, and second-order asymptotics for random matrices. International Mathematics Research Notices, 2004(28), 1413-1460.

[29] Mironov, A., Morozov, A., Natanzon, S. (2012). Algebra of differential operators associated with Young diagrams. Journal of Geometry and Physics, 62(2), 148-155.

[30] Mironov, A. D., Morozov, A. Y., Natanzon, S. M. (2011). Complete set of cut-and-join operators in the Hurwitz-Kontsevich theory. Theoretical and Mathematical Physics, 166(1), 1-22.

[31] Okounkov, A. (2000). Toda equations for Hurwitz numbers. arXiv preprint math/0004128.

[32] Okounkov, A., Pandharipande, R. (2006). Gromov-Witten theory, Hurwitz theory, and completed cycles. Annals of mathematics, 517-560.

[33] Okounkov, A., Pandharipande, R. (2009). Gromov-Witten theory, Hurwitz numbers, and matrix models. In I, Proc. Symposia Pure Math (Vol. 80, pp. 325-414).

[34] Petersen, T. Kyle. "Eulerian numbers." Eulerian Numbers. Springer New York, 2015. 3-18.

[35] Shadrin, S., Spitz, L., Zvonkine, D. (2015). Equivalence of ELSV and BouchardCMarino conjectures for r-spin Hurwitz numbers. Mathematische Annalen, 361(3-4), 611-645.

[36] Simion, R. (2000). Noncrossing partitions. Discrete Mathematics, 217(1-3), 367-409.

[37] Shadrin, S., Spitz, L., Zvonkine, D. (2015). Equivalence of ELSV and BouchardCMari?o conjectures for r-spin Hurwitz numbers. Mathematische Annalen, 361(3-4), 611-645.

[38] Sun, H. (2016). W-Operators and Permutation Groups. arXiv preprint arXiv:1610.06624.

[39] Sun, H. (2016). W-Operator and Hurwitz Number. arXiv preprint arXiv:1611.04700.

[40] Sun, H. (2016). W-Operator and Differential Equation for 3-Hurwitz Number. arXiv preprint arXiv:1612.02884.

[41] Vershik, A. M., Okounkov, A. Y. (2005). A new approach to the representation theory of the symmetric groups. II. Journal of Mathematical Sciences, 131(2), 5471-5494.