BOUNDARIES AND HIERARCHICALLY HYPERBOLIC SPACES

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2019

Urbana, Illinois

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Abstract

Although the hierarchically hyperbolic space boundary is a generalization of the Gromov boundary, we will show there are fundamental differences between the two. First, we provide negative answers to questions posed by Durham, Hagen, and Sisto on the existence of boundary maps for some hierarchically hyperbolic spaces, namely maps from right-angled Artin groups to mapping class groups. We then answer another question of Durham, Hagen, and Sisto, proving that a Teichmüller geodesic ray does not necessarily converge to a unique point in the hierarchically hyperbolic space boundary of Teichmüller space. In fact, we prove that the limit set can be almost anything allowed by the topology.

To Kyle.

Acknowledgments

Above all, I am grateful to my advisor, Chris Leininger. It is under his mentorship that I've become a mathematician. In our weekly meetings (which he never canceled even when he was visibly exhausted and really did not have the time) he patiently listened to and encouraged my ideas. It was a privilege to be advised by someone so brilliant. It seemed like he always had a lecture ready to go in his back pocket on every topic that I needed to understand to continue my work. He helped me refine my mathematical writing skills, carefully reading draft after draft of my papers and providing detailed feedback. And of course, I'm thankful for Chris's genuineness and his comedy.

I am also thankful for Christina Ho (my eighth grade geometry teacher) and Dave Brown (my undergraduate mentor and professor) for identifying and encouraging my potential.

Thank you to my selfless parents, Kelly and David, for believing in me and prioritizing my education. Of course, I'm deeply grateful to my husband, Kyle Mackay who has been by my side literally every day of my graduate and undergraduate experience. Without his support, this thesis would not have been possible. I am also grateful for my graduate student cohort, in particular Marissa (my math sister), Vanessa, Itziar, and Alyssa.

I am grateful for all that comes with being a student at the University of Illinois. I've had access to wonderful classes, seminars and workshops, libraries, travel funds, professional development services, career fairs with renown companies, and a beautiful gym.

It was my privilege to be supported by the Department of Defense (DoD) through the National Defense Science and Engineering Graduate Fellowship (NDSEG) Program, by a Research Assistantship through NSF Grant number DMS-1510034, and by the Dr. Lois M. Lackner Mathematics Fellowship. During these periods of support, free from teaching duties, I was able to make significant progress on my research.

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List of Symbols

HHS	Hierarchically hyperbolic space
$\mathrm{Mod}(S)$	Mapping class group of surface S
$\mathrm{Teich}(S)$	Teichmüller space of surface S
$A(\Gamma)$	Right-angled Artin group determined by graph Γ
RAAG	Right-angled Artin group
$\partial_G X$	Gromov boundary of X
∂X	HHS boundary of X
$\mathbb{H}^n$	Hyperbolic space of dimension n
$i(\cdot,\cdot)$	Geometric intersection number
$(\cdot,\cdot)_z$	Gromov product with respect to z
$\mathcal{C}(Y)$	Curve graph of surface Y
$\mathcal{H}_\alpha$	Combinatorial horoball associated to curve α
$\tau_Y(\cdot)$	Translation length on $\mathcal{C}(Y)$
$\mathcal{PMF}$	Projectivized measured foliations
$\widetilde{\mathcal{M}}(S)$	Marking graph of S
T_α	Dehn twist about α
$\triangle^2$	Standard 2-simplex in $\mathbb{R}^3$
$\xi(S)$	Complexity of surface S
Ext_X	Extremal length with respect to X
Mod_X	Modulus with respect to X
Hyp_X	Hyperbolic length with respect to X
ℓ_q	Flat length with respect to q

CHAPTER 1

Introduction

In this thesis, I will present my results about hierarchically hyperbolic spaces and their boundaries. Hierarchically hyperbolic spaces were defined by Behrstock, Hagen and Sisto in [BHS2017b] and [BHS2014]. The full definition of a hierarchically hyperbolic space (HHS) is lengthy and technical, but the underlying spirit of the definition is simple:

> *A geodesic metric space is an HHS if its geometry can be approximately explained by projecting to a collection of associated spaces with negatively curved geometry.*

Perhaps the simplest, non-trivial example of an HHS is the Euclidean plane $\mathbb{R}^2$. It is an HHS because the geometry of the x-axis and the geometry of the y-axis are sufficient for coarsely describing the geometry of $\mathbb{R}^2$. In particular, the Euclidean distance between two points in $\mathbb{R}^2$ can be approximated to O(1) by projecting the points onto each axis and summing the distances between the projections (see Figure 1.1).

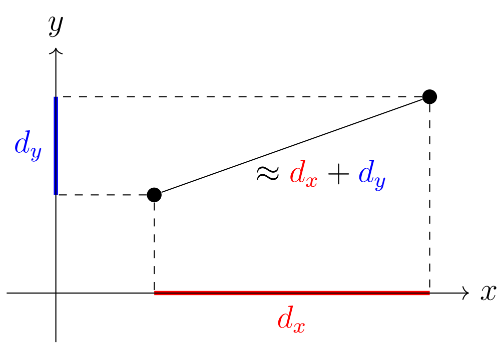

Figure 1.1: Because the ratio of the ℓ_1 and ℓ_2-norms of a vector in $\mathbb{R}^2$ is at most $\sqrt{2}$, the Euclidean distance between points in $\mathbb{R}^2$ can be approximated to $O(1)$ by projecting to the axes.

Of course, $\mathbb{R}^2$ is well-understood and an HHS structure is not needed to comprehend its geometry. However, the spaces we are primarily concerned have much more complicated geometries, and the negative curvature of the spaces we project to makes them preferable to deal with.

1.1 Motivation for hierarchically hyperbolic spaces

If a Riemannian manifold has constant negative sectional curvature, then its triangles are uniformly thin in the following sense:

> **Uniformly thin triangle property:** *There exists a constant $\delta \geq 0$ such that for every geodesic triangle, each side of the triangle is contained in the δ-neighborhood of the union of the other two sides.*

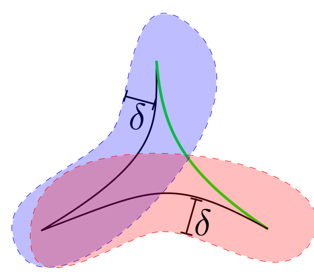

Figure 1.2: A δ-thin triangle.

We will call a geodesic metric space *Gromov hyperbolic* (and sometimes hyperbolic for short) if it has the uniformly thin triangle property. Unlike negative sectional curvature, Gromov hyperbolicity has the advantage of applying to spaces that are not manifolds; for example, we can talk about a graph being Gromov hyperbolic. Moreover, it turns out that there are many geometric consequences to assuming only uniformly thin triangles, and if the metric space is a group, then there are algebraic consequences as well (see Section 2.1).

Examples of Gromov hyperbolic spaces include hyperbolic space $\mathbb{H}^n$, graphs with no cycles (trees), and fundamental groups of orientable, connected, compact surfaces of negative Euler characeristic. In fact, in reasonable models of randomness, "most" finitely presented groups are hyperbolic [Os1992].

Nevertheless, many spaces arising naturally in low-dimensional topology are not Gromov hyperbolic, a primary example of interest being the mapping class group of an orientable surface S of negative Euler characteristic, denoted $\mathrm{Mod}(S)$. Masur and Minsky [MM2000] showed that the geometry of $\mathrm{Mod}(S)$ can be understood by projecting mapping classes to the curve complexes of the subsurfaces of S, spaces they proved are Gromov hyperbolic [MM1999]. The curve complex machinery has been deployed to answer important questions surrounding $\mathrm{Mod}(S)$ including computational complexity of the conjugacy problem [MM2000] and quasi-isometric rigidity [BKMM2012], as well as Thurston's Ending Lamination Conjecture on the geometry of 3-manifolds [BCM2012].

In [BHS2017b], the authors present analogous machinery for certain CAT(0) cube-complexes. They then defined the notion of hierarchically hyperbolic space structures to axiomatize the machinery. The axiomatization has the advantage that theorems about mapping class groups, right-angled Artin groups, Teichmüller space and more can now be proved simultaneously. HHS structures have been used to prove new theorems and strengthen existing theorems, a few of which we describe below.

- [BHS2017c] proved certain quasi-flats in HHSs are stable, answering conjectures of Farb in the mapping class group and Brock in Teichmüller space.
- [BHS2017a] proved HHSs have finite asymptotic dimension. In the mapping class group case, the bound on asymptotic dimension was reduced from exponential to quadratic in the complexity of the surface.
- [BHS2017b] proved every hierarchically hyperbolic group acts acylindrically on some hyperbolic space (new for certain cubical groups).

1.2 Motivation for HHS boundary

For each Gromov hyperbolic space X, there is an associated space "at infinity" called the *Gromov boundary*, which we will denote $\partial_G X$. If X is proper, then its Gromov boundary consists of equivalence classes of geodesic rays based at a given point. While the Gromov boundary is not formally a part of the space X, studying group actions on $X \cup \partial_G X$ (in particular the dynamics of actions) has historically proved useful for recovering information about X.

Every Gromov hyperbolic space is trivially an HHS, so it is natural to try to generalize the Gromov boundary to yield an equally fruitful tool for the HHS setting; and that was precisely the aim of [DHS2017]. There Durham, Hagen, and Sisto constructed a boundary for HHSs. They then took standard facts about the dynamics of actions on Gromov hyperbolic spaces and their boundaries and generalized them to the HHS setting. For example, analogues of loxodromic, elliptic, and parabolic isometries are defined, North-South dynamics are established, and dense orbits in the boundary are guaranteed under certain conditions. Their observations enabled them to generalize known theorems about mapping class groups (the Tits Alternative and Omnibus Subgroup Theorem) and CAT(0) cube complexes (Rank-rigidity) to all hierarchically hyperbolic groups.

1.3 Statements of main results

The results in [DHS2017] highlight commonalities between the Gromov hyperbolic and HHS settings. Our contribution is to expose fundamental differences between the two.

Our results in Section 1.3.1 are published in [Mou2018] and the results in Section 1.3.2 are to appear in Group, Geometry, and Dynamics (see [Mou2017]).

1.3.1 Boundary maps

Quasi-isometric embeddings between Gromov hyperbolic spaces always extend continuously to maps between Gromov boundaries. Durham, Hagen, and Sisto [DHS2017] asked if the same is true for hierarchically hyperbolic spaces. More broadly, they asked about extensions of the embedding maps of Clay, Leininger, and Mangahas [CLM2012] and Koberda [Kob2012] of right-angled Artin groups into mapping class groups of surfaces.

Question 1.1. *Let $A(\Gamma)$ be a right-angled Artin group embedded in $Mod(S)$ in the sense of either Clay, Leininger, and Mangahas [CLM2012] or Koberda [Kob2012]. Does the embedding $A(\Gamma) \to Mod(S)$ extend continuously to an injective map $\partial A(\Gamma) \to \partial Mod(S)$?*

We prove that in general the answer to Question 1.1 is no by providing, for each type of embedding, an explicit example where the embedding does not extend continuously.

Theorem 1.2. *There exists a surface S, a right-angled Artin group $A(\Gamma)$, a Clay, Leininger, and Mangahas embedding $\phi\colon A(\Gamma) \to Mod(S)$, and a Koberda embedding $\phi'\colon A(\Gamma) \to Mod(S)$ such that, regardless of the HHS structure on $A(\Gamma)$, neither ϕ nor ϕ' extends continuously to a map $\partial A(\Gamma) \to \partial Mod(S)$.*

Clay, Leininger, Mangahas embeddings are quasi-isometric embeddings (see Theorem 3.6). Thus Theorem 1.2 shows that the sufficient conditions for extendability of maps between hierarchically hyperbolic spaces are different than those for Gromov hyperbolic spaces.

Our Theorem 1.2 also contributes to the discussion of what it ought to mean for a subgroup of the mapping class group to be geometrically finite [Mos2006] (a well-defined concept in the Klennian group setting). Durham, Hagen, and Sisto [DHS2017] suggested it ought to mean that the inclusion map from the subgroup into the mapping class group extends continuously to a map on the HHSs boundaries, and thought the right-angled Artin subgroups were good candidates for geometric finiteness. Thus, our work shows that either these right-angled Artin groups are not geometrically finite or the proposed definition of geometric finiteness is incorrect.

We also prove the following result which gives a complete characterization of the Koberda embeddings of free groups sending all generators to powers of Dehn twists that have continuous extensions.

Theorem 1.3. *Let $\{\alpha_1, \ldots, \alpha_k\}$ be a collection of pairwise intersecting curves in S and Γ the graph with $V(\Gamma) = \{s_1, \ldots, s_k\}$ and no edges. For sufficiently large N, the homomorphism*

$$\phi : A(\Gamma) \to Mod(S) \quad \text{defined by} \quad \phi(s_i) = T_{\alpha_i}^N \text{ for all } i$$

is injective by the work of Koberda [Kob2012]. Moreover, ϕ extends continuously to a map $\partial A(\Gamma) \to \partial Mod(S)$ if and only if $\{\alpha_1, \ldots, \alpha_k\}$ pairwise fill S, where $A(\Gamma)$ is equipped with any HHS structure.

In fact, we prove something stronger than Theorem 1.3. We prove a non-existence result (Theorem 3.22) for a class of Koberda embeddings of right-angled Artin groups that are not necessarily free groups. We also prove an existence result (Theorem 3.23) for a class of embeddings of free groups that includes the Koberda embeddings described in Theorem 1.3 as well as a class of Clay, Leininger, Mangahas embeddings.

Remark. We call the embeddings that send generators of our right-angled Artin group to mapping classes that are pseudo-Anosov on subsurfaces Clay, Leininger, Mangahas embeddings and those that send generators to powers of Dehn twists Koberda embeddings, even though Koberda [Kob2012] proved that both these types of embeddings are injective. We do this primarily to distinguish the two types of embeddings, but also to emphasize that CLM embeddings have nice geometric properties (see Theorem 3.6).

1.3.2 Exotic limit sets in HHS boundary of Teichmüller space

Let $S = S_g$ be a connected, closed, orientable surface of genus $g \geq 2$ and let Teich(S) denote the Teichmüller space of S equipped with the Teichmüller metric.

Masur [Mas1975] proved that Teich(S) is not non-positively curved in the sense of Busemann, and Masur and Wolf [MW1995] showed that Teich(S) is not Gromov hyperbolic. Nevertheless, some properties of Teich(S) are hyperbolic-like (see Table 1.1 below and [Mas1992] for a detailed survey on the matter).

In this paper, we explore to what extent Teich(S) has features of negative curvature by studying the asymptotic behavior of geodesics. Working in the HHS paradigm, the question becomes how do the asymptotics of geodesic rays in the HHS boundary of Teich(S) compare to those of geodesic rays in the HHS boundary of a hyperbolic space? The identity map on a

Gromov hyperbolic space X	Teich(S)
Triangles are uniformly thin	Triangles in the thick part of Teich(S) are uniformly thin [KL2008, Raf2014]
Closed balls are quasi-convex	Closed balls are quasi-convex [LR2011]
Geodesics are contracting	Geodesics in the thick part of Teich(S) are contracting [Min1996b]
Any two geodesics with common endpoints fellow travel	Two geodesics with common endpoints both in the thick part of Teich(S) fellow travel [Raf2014]
Quasi-convex subgroups of hyperbolic groups are hyperbolic	Quasi-convex subsets contained in the thick part of Teich(S) are hyperbolic [Ham2010]
Every isometric, finite group action on X has a point whose orbit is bounded	Every isometric, finite group action on Teich(S) fixes a point [Ker1983]
Geodesic flow on quotient of $\mathbb{H}^2$ by any group of isometries acting properly discontinuously is ergodic [Hop1971]	Geodesic flow on the quotient of Teich(S) by its isometry group Mod(S) is ergodic [Mas1992]

Table 1.1: Many negative curvature features are present in Teich(S), especially in the *thick-part of Teich*(S); that is, in $\{X \in \text{Teich}(S) : \text{Ext}_X(\alpha) \geq \epsilon \text{ for all curves } \alpha \text{ in } S\}$, where ϵ is some predetermined constant. Facts in column one are standard (see for example [BH1999, Chapter III]).

hyperbolic space extends to a homeomorphism between its HHS and Gromov boundaries, so certainly in this case geodesic rays are well-behaved and limit to a unique boundary point. In [DHS2017] Durham, Hagen, and Sisto asked for a description of limit sets of Teichmüller geodesic rays in the HHS boundary. We provide an answer to the question.

Theorem 1.4. *Given a continuous map $\gamma\colon \mathbb{R} \to \triangle^2$ to the standard 2-simplex, there exists a Teichmüller geodesic ray $\mathcal{G}$ in Teich(S_3) and an embedding of $\triangle^2$ into the HHS boundary of Teich(S_3) such that the limit set of $\mathcal{G}$ in the HHS boundary is the image of $\overline{\gamma(\mathbb{R})}$.*

The study of limiting behaviors of Teichmüller geodesic rays began with Kerckhoff [Ker1980]. He proved that the Teichmüller boundary of Teich(S) (the collection of all geodesic rays emanating from a fixed basepoint) is basepoint dependent. Since then, the limit sets of geodesic rays in Thurston's compactification of Teich(S) by $\mathcal{PMF}$, the space of projectivized measured foliations, have received much attention. Masur [Mas1982] showed that almost all Teichmüller geodesic rays converge to a unique point in $\mathcal{PMF}$. Lenzhen [Len2008] provided the first example of a geodesic ray whose limit set in $\mathcal{PMF}$ is more than one point. The study of limit sets in $\mathcal{PMF}$ continued in [CMW2014] and [LLR2013], where the influence of the topological and dynamical properties of the associated vertical foliation was studied, and in [BLMR2016] and [LMR2016], where rays with limits sets homeomorphic to a circle and simplices of every dimension were constructed, respectively. It would be interesting to know whether the kind of behavior we produce in Theorem 1.4 can occur in $\mathcal{PMF}$.

1.4 Outline

In Chapter 2 we explain how to equip the mapping class group, Teichmüller space, and right-angled Artin groups with HHS structures and how the HHS structures allow us to construct boundaries for these spaces. Chapters 3 and 4 can be read independently of one another. Chapter 3 proves results on the non-existence and existence of boundary maps (Theorems 1.2 and 1.3) and Chapter 4 is concerned with limit sets of Teichmüller geodesics (Theorem 1.4).

CHAPTER 2

Background

We begin this chapter discussing implications of Gromov hyperbolicity and then formulate a definition of Gromov boundary that is most useful for our purposes. We then define the mapping class group and Teichmüller space of a surface and right-angled Artin groups, our primary objects of study. Next we discuss a framework for studying the coarse geometry of these spaces. Namely, we will describe how to equip them with hierarchically hyperbolic space structures and introduce the boundary construction proposed by [DHS2017] for hierarchically hyperbolic spaces.

Throughout this chapter, we let $S = S_{g,n}$ denote a connected, oriented surface of genus g with n punctures. Define the *complexity of* S to be $\xi(S) = 3g-3+n$. We will always assume $\xi(S) \geq 1$. Additionally, we fix a complete hyperbolic metric on S. That is, we assume that S is of the form $S = \mathbb{H}^2/\Lambda$, where $\Lambda \subseteq \mathrm{Isom}^+(\mathbb{H}^2)$ and Λ acts properly discontinuously and freely on $\mathbb{H}^2$.

2.1 Gromov hyperbolicity and the Gromov boundary

A geodesic metric space X is *(Gromov) hyperbolic* if there exists a $\delta \geq 0$ such that given any triangle in X, each side is contained in the δ-neighborhood of the union of the other two sides. (Here and throughout, triangle indicates geodesic sides). The geometry of a hyperbolic space is well-behaved. For example, if X is hyperbolic, then

- Every triangle in X has a center; that is, for each triangle, there is point in X that is uniformly close to all its three sides.
- Quasi-geodesic stability: For every $\lambda \geq 1$ and $\epsilon \geq 0$, there exists a constant K such that every (λ, ϵ)-quasi-geodesic is contained in the K-neighborhood of every geodesic between its endpoints.
- X satisfies a linear isoperimetric inequality.

See Table 1.1 for more geometric implications. Additionally, knowing X is hyperbolic reveals algebraic information about every group G that acts properly and cocompactly on X. For example, if X is hyperbolic, then

- Given $g_1, \ldots, g_r \in G$, there exists $n > 0$ such that $\langle g_1^n, \ldots, g_r^n \rangle$ is a free subgroup.
- If H is a finitely presented, one-ended subgroup of G, then up to conjugacy there are only finitely many subgroups in G isomorphic to H.
- For infinite order $g \in G$, $\langle g \rangle$ has finite index in the centralizer of g. This implies that all abelian subgroups of G are virtually cyclic.
- If G is torsion free, then it has a finite Eilenberg-MacLane complex $K(G, 1)$, which is useful for studying cohomology with coefficients in G.

Discussion of all the above geoemtric and algebraic properties can be found in [Gro1987] and [BH1999].

Gromov boundary. Given a Gromov hyperbolic space (X, d_X) and points $x, y, z \in X$, the *Gromov product* of x and y with respect to z is defined as

$$(x, y)_z = \frac{1}{2} \left(d_X(x, z) + d_X(y, z) - d_X(x, y) \right).$$

We say that a sequence (x_n) in X *converges at infinity* if $\liminf_{i,j \to \infty} (x_i, x_j)_z = \infty$ for some (every) $z \in X$. We define two such sequences (x_n) and (y_n) to be equivalent if $\liminf_{i,j \to \infty} (x_i, y_j)_z = \infty$ for some (every) $z \in X$. The *Gromov boundary of* X is the collection of all such sequences up to this equivalence, and is denoted $\partial_G X$ or just ∂X when it is clear from context that we are using the Gromov boundary.

Topology of the Gromov boundary. We give $X \cup \partial X$ the topology generated by the following basis of neighborhoods. Fix a basepoint $z \in X$. For $q \in X$, take the open metric balls centered at q (using the original metric on X) as the basis of neighborhoods at q. For $q \in \partial X$ and $M \geq 0$, define

$$U(q, M) = \left\{ \begin{array}{ccl} y \in X \cup \partial X & : & \begin{array}{l} \liminf_{i,j \to \infty} (q_i, y_j)_z \geq M \text{ for some } (q_n) \text{ and} \\ (y_n) \text{ representing } q \text{ and } y \text{ respectively.} \end{array} \end{array} \right\}.$$

Here if $y \in X$, the *sequence representing* y is the constant all y sequence. Take $\{U(q, M) : M \geq 0\}$ to be the basis of neighborhoods at q.

In the topology on $X \cup \partial X$ generated by this basis, a sequence (p_n) in $X \cup \partial X$ converges to a point $q \in \partial X$ if and only if the following holds. There exists a sequence (q_i) representing

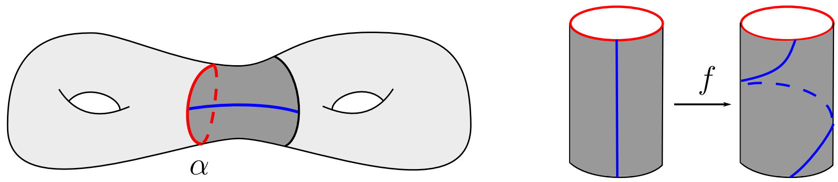

Figure 2.1: Dehn twist f about curve α.

q and for each p_n a representative sequence $(p_{n,j})_{j=1}^{\infty}$ such that

$$\liminf_{i,j\to\infty}(q_i, p_{n,j})_z \to \infty \quad \text{as} \quad n \to \infty.$$

Moreover, the topology is such that ∂X is closed, and the subspace topology on X as a subset of $X \cup \partial X$ is the same as the original topology on X. The topology is independent of the basepoint z. For further discussion of the Gromov boundary and its properties see [BH1999, III.H] and [KB2002].

2.2 Spaces of interest

2.2.1 Mapping class groups

The mapping class group of S, denoted $\mathrm{Mod}(S)$ is, roughly speaking, the group of symmetries of S. Formally,

$$\mathrm{Mod}(S) = \{f : S \to S : f \text{ is an orientation preserving homeomorphism}\}/\sim,$$

where $f, g : S \to S$ are equivalent if they are isotopic.

Examples:

- Each essential, simple, closed curve α on S yields an element of $\mathrm{Mod}(S)$ called a *Dehn twist about* α. First select a regular embedded neighborhood of α and associate it with $S^1 \times [0, 1]$. A *Dehn twist about* α is the identity outside of the neighborhood and inside maps $(\theta, x) \mapsto (\theta + 2\pi x, x)$. See Figure 2.1.

- Every matrix in $\mathrm{SL}_2(\mathbb{Z})$, viewed as a transformation of $\mathbb{R}^2$, descends to a homeomorphism of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$. In fact, $\mathrm{Mod}(T) = \mathrm{SL}_2(\mathbb{Z})$. For example, consider $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, whose eigenvectors $\left(\frac{1\pm\sqrt{5}}{2}, 1\right)$ are perpendicular. Observe that A

stretches in one eigendirection by some factor λ and compresses in the other eigendirection by a factor of $1/\lambda$. Each eigendirection yields a foliation of T, and the descension $T \to T$ of A inherits the dynamics of A, stretching one foliation and compressing the other. We call maps on T with such dynamics *Anosov*.

- Consider a non-annular subsurface $Y \subseteq S$. Because $\xi(Y) \geq 1$, it is not possible to foliate Y and so there are no Anosov transformations of Y. However, away from a finite set of points, called singular points, Y can be foliated. We call such almost-foliations *singular foliations*. Roughly speaking, $f \in \mathrm{Mod}(S)$ is defined to be *pseudo-Anosov on* Y if there exists a representative in the isotopy class of f that
  - Pointwise fixes the complement of Y (that is, f is *supported on* Y), and
  - Away from singular points, locally deforms a pair of transverse measured singular foliations of Y by stretching one foliation and compressing the other.

The Nielsen-Thurston classification of mapping classes implies that, after passing to a sufficiently high power, every infinite order element $f \in \mathrm{Mod}(S)$ can be expressed as a composition of mapping classes, each pseudo-Anosov on a subsurface with all the subsurfaces disjoint (see for example [CB1988]). (Here we define a *pseudo-Anosov on an annular subsurface* to be any non-zero power of a Dehn twist in S around the associated core curve.) So, if f is not pseudo-Anosov on S, then f can be reduced to studying pseudo-Anosovs on disjoint lower complexity surfaces.

Dehn [Deh1987] proved that, if S has no punctures, then a finite collection of Dehn twists generate $\mathrm{Mod}(S)$. If S has punctures, then $\mathrm{Mod}(S)$ is still finitely generated, but "half twists" are also needed [FM2012, Chapter 4]. Throughout, we fix a finite generating set for $\mathrm{Mod}(S)$ and associate $\mathrm{Mod}(S)$ with the corresponding Cayley graph, making $\mathrm{Mod}(S)$ a geodesic metric space. The mapping class group is not hyperbolic because it contains a subgroup isomorphic to $\mathbb{Z}^2$ (for example, the subgroup generated by Dehn twists around two disjoint curves.)

2.2.2 Teichmüller space

The *Teichmüller space of* S, denoted $\mathrm{Teich}(S)$, is the collection of equivalence classes of complex structures on S, where we define two complex structures to be equivalent if there is a map $S \to S$ isotopic to the identity that is biholomorphic when the domain is equipped with one of the complex structures and the range is equipped with the other. Throughout this paper, when it is convenient, given $X \in \text{Teich}(S)$, we will also use X to denote a structure in the equivalence class.

We equip $\text{Teich}(S)$ with a metric called the *Teichmüller metric*: for $X_1, X_2 \in \text{Teich}(S)$ the distance between them is

$$d_{\text{Teich}(S)}(X_1, X_2) = \frac{1}{2} \inf \log(K_f),$$

where the infimum is taken over all quasiconformal maps $f : (S, X) \to (S, Y)$ isotopic to the identity on S and K_f denotes the dilatation of f. Because we never explicitly compute distances between complex structures, we refer the reader to [FM2012, Chapter 11] for definitions of quasiconformal and dilatation.

Not only is $\text{Teich}(S)$ geodesic, but between every two points in $\text{Teich}(S)$, there is a *unique* geodesic. As we will discuss in Section 4.2.2, each geodesic in $\text{Teich}(S)$ can be described through deformations of a foliation of S associated to some quadratic differential.

Masur and Wolf [MW1995] proved that $\text{Teich}(S)$ is not hyperbolic by constructing, for every $\delta \geq 0$, a triangle that is not δ-thin.

2.2.3 Right-angled Artin groups

Let Γ be a finite graph with vertex set $V(\Gamma) = \{s_1, \ldots, s_k\}$. The right-angled Artin group (RAAG) determined by Γ, denoted by $A(\Gamma)$, is the group with the following presentation:

$$A(\Gamma) = \langle s_1, \ldots, s_k : [s_i, s_j] = 1 \Leftrightarrow s_i s_j \text{ is an edge in } \Gamma \rangle.$$

If Γ has no edges, then $A(\Gamma)$ is the free group of rank k, and if Γ is a complete graph, the $A(\Gamma)$ is the free abelian group $\mathbb{Z}^k$. We associate $A(\Gamma)$ with its Cayley graph built from the generating set $\{s_1, \ldots, s_k\}$, making $A(\Gamma)$ a metric space. Except in the free-group case, RAAGs are not hyperbolic because every edge in Γ yields a $\mathbb{Z}^2$ subgroup.

2.3 Tools for studying Mod(S) and Teich(S)

The purpose of this section is to define hyperbolic spaces associated to $\text{Mod}(S)$ and $\text{Teich}(S)$ and define projection maps to those hyperbolic spaces.

2.3.1 Curves and subsurfaces

By a *curve* in S, we will mean the geodesic representative in the homotopy class of an essential (non-null homotopic and non-peripheral), simple, closed curve in S. By a *multicurve* in S, we will mean a collection of pairwise disjoint curves in S. We write $i(\alpha, \beta)$ to denote the geometric intersection number of curves α and β. We say that a pair of curves α and β *fills* S if for every curve γ in S we have $i(\gamma, \alpha) > 0$ or $i(\gamma, \beta) > 0$.

A *non-annular subsurface* Y *of* S is a component of S after removing a (possibly empty) multicurve from S. Additionally, we require that Y satisfies $\xi(Y) \geq 1$; in particular, we do not consider a pair of pants to be a subsurface. We define ∂Y to be the collection of curves in S that are disjoint from Y and also are contained in the closure of Y, treating Y as a subset of S. When $Y \neq S$, the path metric completion of Y is a surface with boundary, and the image of this boundary under the map induced by the inclusion $Y \subseteq S$ is ∂Y.

An annular subsurface of S is defined as follows. Let α be a curve in S. Choose a component $\widetilde{\alpha}$ of the preimage of α in $\mathbb{H}^2$, and let $h \in \Lambda$ be a primitive isometry with axis $\widetilde{\alpha}$. Define

$$Y = (\overline{\mathbb{H}^2} - \{x, y\})/\langle h \rangle,$$

where x and y are the fixed points of h on $\partial\mathbb{H}^2$. Observe that Y is a compact annulus and $\mathrm{int}(Y) \to S$ is a covering. We say that Y is the *annular subsurface of* S *with core curve* α. We define ∂Y to be α.

2.3.2 Curve complex, combinatorial horoballs

Let Y be a subsurface of S. If Y satisfies $\xi(Y) \geq 1$, the *curve complex of* Y, denoted $\mathcal{C}(Y)$, is the simplicial complex whose vertices are curves contained in Y, and if $\xi(Y) > 1$, a set of vertices forms a simplex if and only if they are pairwise disjoint. If $\xi(Y) = 1$, then we define the simplices of $\mathcal{C}(Y)$ differently. In the case that Y is a once punctured torus, a set of vertices forms a simplex if and only if they pairwise intersect exactly once. If Y is a four times punctured sphere, a set of vertices forms a simplex if and only if they pairwise intersect exactly twice.

Now let Y be an annular subsurface with core curve α. Consider all embedded arcs in Y that connect one boundary component to the other. We define two arcs to be equivalent if one can be homotoped to the other, fixing the endpoints of the arcs throughout the homotopy. In this case, the *curve complex of* Y is the simplicial complex whose vertices are equivalence classes of arcs, and a set of vertices forms a simplex if and only if for each pair of vertices there exist representative arcs of each whose restrictions to $\mathrm{int}(Y)$ are disjoint. We let both

$\mathcal{C}(Y)$ and $\mathcal{C}(\alpha)$ denote the curve complex of Y.

The following simple formula will be useful to us: given inequivalent arcs γ, β in $\mathcal{C}(\alpha)$,

$$d_{\mathcal{C}(Y)}(\gamma, \beta) = |\gamma \cdot \beta| + 1, \tag{1}$$

where $\gamma \cdot \beta$ denotes the algebraic intersection number of γ and β.

Given a curve α in S, the *combinatorial horoball associated to* α, denoted $\mathcal{H}_\alpha$, is the following graph. Begin with the graph Cartesian product $\mathcal{C}(\alpha) \times \mathbb{Z}_{\geq 0}$ and then for each n add edges so that each vertex (x, n) is adjacent to every vertex in $\{(y, n) : d_{\mathcal{C}(\alpha)}(x, y) \leq e^n\}$.

The spaces $\mathcal{H}_\alpha$ and $\mathcal{C}(Y)$ for each subsurface Y are Gromov hyperbolic (see[GM2008] and [MM1999], respectively). We let $\partial\mathcal{H}_\alpha$ and $\partial\mathcal{C}(Y)$ denote their Gromov boundaries.

2.3.3 Extremal length

Consider $X \in \mathrm{Teich}(S)$ and equip S with a complex structure in X. Let A be an annulus embedded in S. We define the *modulus of A in X*, denoted $\mathrm{Mod}_X(A)$, to be the inverse of the circumference of the unique Euclidean cylinder of height one that is conformally equivalent to A. We define the *extremal length in X of a curve* α in S to be

$$\mathrm{Ext}_X(\alpha) = \inf \frac{1}{\mathrm{Mod}_X(A)},$$

where the infimum is taken over all annuli A embedded in S with core curve α.

2.3.4 Markings

A *marking μ on S* is a maximal collection of pairwise disjoint curves in S, denoted base(μ), together with another collection of associated curves called *transversals*: for each $\beta \in \mathrm{base}(\mu)$ its associated transversal γ_β is a curve that intersects β minimally (i.e. once or twice) and is disjoint from all other curves in base(μ).

Of course, there are infinitely many choices for markings on S. Given $X \in \mathrm{Teich}(S)$, we will typically select a marking μ_X as follows. For base(μ_X), first choose a curve α_1 with shortest extremal length in X, then of those curves that do not intersect α_1, choose one with shortest extremal length. Continue until a maximal collection of non-intersecting curves is obtained. Additionally, to each curve $\alpha \in \mathrm{base}(\mu_X)$ associate a transverse curve τ_α by selecting from those curves that intersect α but no other curves in base(μ_X) a curve with shortest length. We call μ_X a *short marking on X*.

In [Min1993], Minsky proved that for curves α and β,

$$i(\alpha,\beta)^2 \leq \mathrm{Ext}_X(\alpha)\mathrm{Ext}_X(\beta). \tag{2}$$

So if $\mathrm{Ext}_X(\alpha)$ is sufficiently small, then $\mathrm{Ext}_X(\beta) > \mathrm{Ext}_X(\alpha)$ for every curve β intersecting α, yielding the following fact.

Fact 2.1. *There exists a constant ϵ_0 such that for all $X \in Teich(S)$, if a curve α satisfies $\mathrm{Ext}_X(\alpha) \leq \epsilon_0$, then α is in the base of every short marking on X.*

Given $f \in \mathrm{Mod}(S)$ and a curve or simple bi-infinite geodesic γ in S, we define $f(\gamma)$ to be the curve or simple bi-infinite geodesic obtained as follows. Consider a component $\widetilde{\gamma}$ of the preimage of γ in $\mathbb{H}^2$. Choose a representative ψ in the isotopy class of f and lift it to a map $\widetilde{\psi} : \mathbb{H}^2 \to \mathbb{H}^2$. We define $f(\gamma)$ to be the image in S of the geodesic in $\mathbb{H}^2$ that connects the endpoints of $\widetilde{\psi}(\widetilde{\gamma})$ on $\partial\mathbb{H}^2$. Given a marking μ on S, we then define $f(\mu)$ to be the marking obtained by applying f to each base and transversal curve.

2.3.5 Subsurface projection

Let Y be a subsurface of S and β a multicurve in S. We will now define *the projection of β to Y*, which we will denote by $\pi_Y(\beta)$. Suppose Y is not an annulus and β is a single curve. If β is disjoint from Y, define $\pi_Y(\beta) = \emptyset$. If β is contained in Y, define $\pi_Y(\beta) = \beta$. Otherwise, $\beta \cap Y$ is a collection of essential arcs in Y with endpoints on ∂Y. For each such arc γ, take the geodesic representatives of the boundary components of a small regular neighborhood of $\gamma \cup \partial Y$ that are contained in Y. Define $\pi_Y(\beta)$ to be the collection of all such curves over all arcs γ in $\beta \cap Y$. If β is a multicurve, define $\pi_Y(\beta)$ to be the union of the projections to Y of each curve in β.

Now let Y be an annular subsurface with core curve α and $\mathrm{int}(Y) \to S$ the associated covering. Let β be a multicurve or a bi-infinite, simple geodesic in S. Consider the components of the full preimage of β in $\mathrm{int}(Y)$ that are arcs. We will view each such component as having endpoints on the boundary of Y. In this case, we define $\pi_Y(\beta)$ to be the (equivalence classes of) arcs in this collection that have an endpoint on each boundary component of Y. When convenient, we will write $\pi_\alpha(\beta)$ instead of $\pi_Y(\beta)$.

We now describe how to project a marking μ to a subsurface Y of S. If Y is non-annular or Y is an annulus whose core curve is not contained in $\mathrm{base}(\mu)$, we define $\pi_Y(\mu) = \pi_Y(\mathrm{base}(\mu))$. Otherwise, Y is an annulus with core curve $\alpha \in \mathrm{base}(\mu)$, and we define $\pi_Y(\mu)$ to be $\pi_Y(\gamma_\alpha)$, where γ_α is the transversal associated to α.

For each $X \in \mathrm{Teich}(S)$, fix a short marking μ_X. Abusing notation, we define the projection map

$$\pi_Y \colon \mathrm{Teich}(S) \to 2^{\mathcal{C}(Y)} \quad \text{via} \quad X \mapsto \pi_Y(\mu_X).$$

Additionally, for each curve α, we define a map $\pi_{\mathcal{H}_\alpha} \colon \mathrm{Teich}(S) \to 2^{\mathcal{H}_\alpha}$ as follows. For $X \in \mathrm{Teich}(S)$, if $\mathrm{Ext}_X(\alpha) > \epsilon_0$, define $n(X) = 0$. Otherwise, define $n = n(X) \in \mathbb{Z}_{\geq 0}$ so that $\frac{\epsilon_0}{e^{n+1}} < \mathrm{Ext}_X(\alpha) \leq \frac{\epsilon_0}{e^n}$. We then define

$$\pi_{\mathcal{H}_\alpha}(X) = \{(\tau, n(X)) : \tau \in \pi_\alpha(\mu_X)\}.$$

Here and throughout the remainder of this thesis, ϵ_0 denotes the minimum of the ϵ_0 constants from Fact 2.1 and [CRS2008] (our Theorems 4.2 and 4.6 state the results we require from [CRS2008]).

For every subsurface $Y \subseteq S$ and curve α define

$$d_Y(\cdot,\cdot) = \mathrm{diam}_{\mathcal{C}(Y)} \pi_Y(\cdot) \cup \pi_Y(\cdot) \quad \text{and} \quad d_{\mathcal{H}_\alpha}(\cdot,\cdot) = \mathrm{diam}_{\mathcal{H}_\alpha} \pi_{\mathcal{H}_\alpha}(\cdot) \cup \pi_{\mathcal{H}_\alpha}(\cdot).$$

Sometimes when Y is annular with core curve α we write d_α in place of d_Y.

2.3.6 Relations on subsurfaces

For every subsurface Y of S, we write $Y \subseteq S$ to indicate that Y is a subsurface of S, even though when Y is an annulus, Y is not a subset of S. We say that distinct subsurfaces X and Y are *disjoint* if $\pi_X(\partial Y) = \emptyset$ and $\pi_Y(\partial X) = \emptyset$. We say that X is a *proper subsurface of* Y, denoted $X \subsetneq Y$, if $\pi_Y(\partial X) \neq \emptyset$ and $\pi_X(\partial Y) = \emptyset$. We say that X and Y are *overlapping*, denoted $X \pitchfork Y$, if $\pi_Y(\partial X) \neq \emptyset$ and $\pi_X(\partial Y) \neq \emptyset$. In the case where X and Y are not annuli, these relationships, respectively, are disjointness, proper containment, and intersection without containment as subsets of S. We say X and Y *fill* S if for every curve γ in S we have $\pi_X(\gamma) \neq \emptyset$ or $\pi_Y(\gamma) \neq \emptyset$.

2.3.7 Pseudo-Anosovs and translation length

The information presented in this section is not needed to describe an HHS structure for $\mathrm{Mod}(S)$, but nevertheless illustrates how curve complexes can reveal information about mapping classes.

To determine if $f \in \mathrm{Mod}(S)$ is pseudo-Anosov on a non-annular subsurface $Y \subseteq S$, surprisingly it is not necessary to study foliations of Y. Instead, one can examine the asymptotic rate at which f moves curves through $\mathcal{C}(Y)$. Given $f \in \mathrm{Mod}(S)$ that is supported on Y, we define the *translation length of f on $\mathcal{C}(Y)$* to be

$$\tau_Y(f) = \lim_{n\to\infty} \frac{d_Y(\mu, f^n(\mu))}{n},$$

where μ is any marking on S. By the work of Masur and Minsky [MM1999],

$$\tau_Y(f) > 0 \quad \iff \quad f \text{ is pseudo-Anosov on } Y.$$

In Chapter 3, it will be convenient to have the following vocabulary. If $f \in \mathrm{Mod}(S)$ is a power of a Dehn twist about a curve α, we say that f is *supported on* the annular subsurface Y with core curve α, and define $\tau_Y(f)$ to be the absolute value of the power. In both the annular and non-annular case, we say that Y *fully supports* f if Y supports f and $\tau_Y(f) > 0$.

2.4 Hierarchically hyperbolic spaces

In [BHS2017b] Behrstock, Hagen, and Sisto defined hierarchically hyperbolic spaces (HHSs). Roughly, an HHS is a quasi-geodesic metric space $\mathcal{X}$, equipped with additional structure which we will call an HHS space structure. An HHS structure consists of an index set $\mathcal{G}$ endowed with binary relations called orthogonality, transversality, and nesting and for each $Y \in \mathcal{G}$ a Gromov hyperbolic space CY and a projection map $\pi_{CY} : \mathcal{X} \to 2^{CY}$. The elements of $\mathcal{G}$ and the projection maps must satisfy a long list of properties. See [BHS2017b] and [BHS2014].

Examples:

- **Mod(S)**. Let $\mathcal{G}$ be the collection of all subsurfaces of S. Define two subsurfaces to be orthogonal if they are disjoint, transverse if they overlap, and nested if one is a proper subsurface of the other. For $Y \in \mathcal{G}$, let CY be $\mathcal{C}(Y)$, the curve complex of Y. Fix a marking μ on S and for each $Y \in \mathcal{G}$, define the projection

  $$\pi_Y : \mathrm{Mod}(S) \to 2^{\mathcal{C}(Y)} \quad \text{via} \quad f \mapsto \pi_Y(f(\mu)).$$

  The works of Masur and Minsky [MM1999],[MM2000], Behrstock, [Beh2006], and Behrstock, Kleiner, Minsky, and Mosher [BKMM2012] imply that all the required axioms for an HHS structure are satisfied. See [BHS2014, Section 11] for details.

- **Teich(S)**. Let $\mathcal{G}$ be the collection of all subsurfaces of S, where orthogonality, transversality, and nesting are as in the above $\mathrm{Mod}(S)$ example. For $Y \in \mathcal{G}$, if Y is non-annular, let CY be $\mathcal{C}(Y)$, the curve complex of Y, and if Y is annular with core curve α, let CY be the combinatorial horoball $\mathcal{H}_\alpha$. Let

$$\{\pi_Y\colon \text{Teich}(S) \to 2^{\mathcal{C}(Y)} : Y \text{ non-annular}\} \cup \{\pi_{\mathcal{H}_\alpha}\colon \text{Teich}(S) \to 2^{\mathcal{H}_\alpha} : \alpha \text{ a curve in } S\}$$

be the associated projection maps. The results in [Dur2016], [EMR2014],[MM2000], [Raf2007] together establish that this is an HHS structure on Teich(S).

- **Right-angled Artin groups.** In the case of free groups, which are hypberbolic, there is always a trivial structure: let $\mathcal{G}$ consist of a single element whose associated hyperbolic space is the group itself and take the projection map to be the identity. In [BHS2017b], an HHS structure for every right-angled Artin group is constructed by considering interactions of convex subspaces in the universal cover of the associated Salvetti complex. Because we only deal with free groups (with the exception of Theorem 3.22), we refer the reader to [BHS2017b] for details on HHS structures for general RAAGs.

We emphasize that there is not a unique way to equip a space with an HHS structure. Nevertheless, throughout this thesis, we will regard Mod(S) and Teich(S) as HHSs equipped with the structures described above.

2.5 Boundary of hierarchically hyperbolic spaces

In [DHS2017] Durham, Hagen, and Sisto construct a boundary for HHSs, called the *hierarchically hyperbolic space boundary* which we now recall. We emphasize that the homeomorphism type of the HHS boundary may depend on the HHS structure taken [DHS2017, Question 1].

Let $\mathcal{X}$ be a hierarchically hyperbolic space with index set $\mathcal{G}$ and projection maps $\{\pi_{CY}\colon \mathcal{X} \to 2^{CY} : Y \in \mathcal{G}\}$. As a set, the HHS boundary of $\mathcal{X}$ is defined as follows:

$$\partial\mathcal{X} = \Bigg\{ \sum_{Y \in \mathcal{G}} c_Y \lambda_Y : c_Y \geq 0 \text{ and } \lambda_Y \in \partial_G CY \text{ for all } Y, \sum_{Y \in \mathcal{G}} c_Y = 1, \\ \text{and if } c_{Y'}, c_Y > 0, \text{ then } Y \text{ and } Y' \text{ are orthogonal or equal} \Bigg\}.$$

Every point in $\partial\mathcal{X}$ is a finite sum because every collection of pairwise orthogonal indices must be finite [DHS2017, Lemma 1.4].

Topology. We equip $\mathcal{X} \cup \partial\mathcal{X}$ with the Hausdorff topology described in [DHS2017]. The proof of [DHS2017, Theorem 3.4] reveals that if $\mathcal{X}$ is proper, then $\mathcal{X} \cup \partial\mathcal{X}$ is sequentially compact, implying that every infinite sequence has a non-empty limit set. Of particular interest to us, this means the limit set of a geodesic ray in Teich(S) is always non-empty.

In what follows, for $Y \in \mathcal{G}$, let

$$d_{CY}(\cdot,\cdot) = \mathrm{diam}_{CY}(\pi_{CY}(\cdot) \cup \pi_{CY}(\cdot)).$$

Consider a point $p = \sum_{Y \in \mathcal{G}} c_Y \lambda_Y \in \partial\mathcal{X}$ and let $Y_1, \ldots, Y_k$ be the collection of indices in $\mathcal{G}$ with $c_Y > 0$. By [DHS2017, Definition 2.10], a sequence of elements $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$ converges to p if and only if the following statements hold for some (every) $x \in \mathcal{X}$.

(I) $\lim_{n\to\infty} \pi_{CY_i}(x_n) = \lambda_{Y_i}$ for each $i = 1, \ldots, k$,

(II) $\lim_{n\to\infty} \dfrac{d_{CY_i}(x, x_n)}{d_{CY_j}(x, x_n)} = \dfrac{c_{Y_i}}{c_{Y_j}}$ for each $i, j = 1, \ldots, k$, and

(III) $\lim_{n\to\infty} \dfrac{d_{CW}(x, x_n)}{d_{CY_i}(x, x_n)} = 0$ for some (every) $i = 1, \ldots, k$ and every $W \in \mathcal{G}$ that is orthogonal to Y_j for all $j = 1, \ldots, k$.

Because $\sum_{Y \in \mathcal{G}} c_Y = 1$, we can replace (II) and (III) with the following equivalent statements.

(II′) $\lim_{n\to\infty} \dfrac{d_{CY_j}(x, x_n)}{\sum_{i=1}^{k} d_{CY_i}(x, x_n)} = c_{Y_j}$ for each $j = 1, \ldots, k$, and

(III′) $\lim_{n\to\infty} \dfrac{d_{CW}(x, x_n)}{\sum_{i=1}^{k} d_{CY_i}(x, x_n)} = 0$ for every $W \in \mathcal{G}$ that is orthogonal to Y_j for all $j = 1, \ldots, k$.

Examples:

- **Mod(S)**. Let Y and Z be two disjoint subsurfaces in S, for example as in Figure 2.2, and let $f, g \in \mathrm{Mod}(S)$ be pseudo-Anosov on Y and Z respectively. Then in $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$,

$$f^n g^n \;\to\; \left[\tau_Y(f)\lambda_Y + \tau_Z(g)\lambda_Z + \sum_{\alpha \in \partial Y \cup \partial Z} c_\alpha \lambda_\alpha\right] \quad \text{as} \quad n \to \infty$$

  for some $\lambda_Y \in \partial\mathcal{C}(Y), \lambda_Z \in \partial\mathcal{C}(Z), c_\alpha \geq 0$, and $\lambda_\alpha \in \partial\mathcal{C}(\alpha)$. Here the square brackets indicate that the coefficients should be scaled to sum to one.

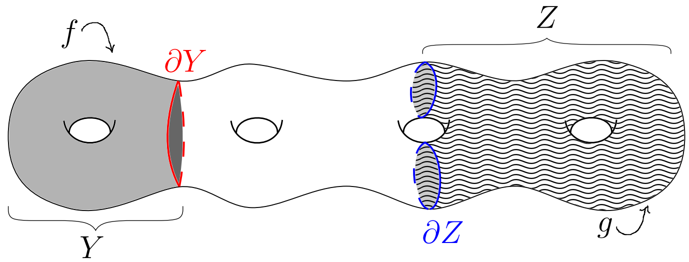

Figure 2.2: Pseudo-Anosov mapping classes f and g on disjoint subsurfaces Y and Z.

To see why, observe that $\tau_Y(f) > 0$ implies that for any curve γ in Y, the map

$$\mathbb{Z} \to \mathcal{C}(Y) \quad n \mapsto f^n(\gamma) = f^n g^n(\gamma)$$

is a quasi-isometric embedding. And so, because quasi-geodesics in $\mathcal{C}(Y)$ are stable, $(f^n g^n(\gamma))$ converges to a point $\lambda_Y \in \partial\mathcal{C}(Y)$ and thus $\pi_Y(f^n g^n) \to \lambda_Y$ as well. Similarly, $\pi_Z(f^n g^n)$ converges to some point $\lambda_Z \in \partial\mathcal{C}(Z)$.

For all subsurfaces W disjoint from Y and Z, except possibly the annular subsurfaces corresponding to curves in $\partial Y \cup \partial Z$, the sequence $(\pi_W(f^n g^n))$ is bounded in $\mathcal{C}(W)$. If W is non-annular, this is simply because $f^n g^n$ has a representative that fixes all curves in W. For the annular case, see Lemma 3.13.

- **Teich(S)**. Consider a geodesic ray $\mathcal{G}$ in Teich(S) such that the vertical foliation v of the quadratic differential associated to $\mathcal{G}$ is minimal; that is, no trajectory of v is a simple closed loop. Minimality of v guarantees that all points in the limit set of $\mathcal{G}$ in $\mathcal{PMF}(S)$ are topologically equivalent to v [CMW2014, Lemma 3.2]. From here, [Kla1999, Theorem 1.2] implies that the projection of $\mathcal{G}$ to $\mathcal{C}(S)$ limits to a unique point $\lambda_S \in \partial\mathcal{C}(S)$. Because there are no subsurfaces disjoint from S, condition (III) is trivially satisfied. Therefore, the limit set of $\mathcal{G}$ in ∂Teich(S) is $\{\lambda_S\}$.

- **Free groups.** Consider a hyperbolic space X (for example, a free group) equipped with any HHS structure. By [DHS2017, Theorem 4.3] the identity map on X extends to a homeomorphism
$$X \cup \partial_G X \to X \cup \partial X.$$
And so, convergence in the HHS boundary is equivalent to convergence in the Gromov boundary. When X is equipped with the trivial HHS structure, the HHS boundary is the Gromov boundary by definition.

CHAPTER 3

Non-existence of boundary maps

Throughout this chapter, we let $S = S_{g,n}$ denote a connected, oriented surface of genus g with n punctures and $\xi(S) \geq 1$. We fix a complete hyperbolic metric on S; that is, S is of the form $S = \mathbb{H}^2/\Lambda$, where $\Lambda \subseteq \mathrm{Isom}^+(\mathbb{H}^2)$ and Λ acts properly discontinuously and freely on $\mathbb{H}^2$. We equip $\mathrm{Mod}(S)$ with the HHS structure described in Section 2.4.

3.1 Introduction

The primary goal of this chapter is to answer Question 1.1 posed in [DHS2017] by proving the following theorem.

Theorem 1.2. *There exists a surface S, a right-angled Artin group $A(\Gamma)$, a Clay, Leininger, and Mangahas embedding $\phi\colon A(\Gamma) \to Mod(S)$, and a Koberda embedding $\phi'\colon A(\Gamma) \to Mod(S)$ such that, regardless of the HHS structure on $A(\Gamma)$, neither ϕ nor ϕ' extends continuously to a map $\partial A(\Gamma) \to \partial Mod(S)$.*

We also will prove an existence result.

Theorem 1.3. *Let $\{\alpha_1, \ldots, \alpha_k\}$ be a collection of pairwise intersecting curves in S and Γ the graph with $V(\Gamma) = \{s_1, \ldots, s_k\}$ and no edges. For sufficiently large N, the homomorphism*

$$\phi : A(\Gamma) \to Mod(S) \quad \textit{defined by} \quad \phi(s_i) = T_{\alpha_i}^N \textit{ for all } i$$

is injective by the work of Koberda [Kob2012]. Moreover, ϕ extends continuously to a map $\partial A(\Gamma) \to \partial Mod(S)$ if and only if $\{\alpha_1, \ldots, \alpha_k\}$ pairwise fill S, where $A(\Gamma)$ is equipped with any HHS structure.

By a map extending continuously we mean the following.

Definition 3.1. Let $\phi : A(\Gamma) \to \text{Mod}(S)$ be an injective homomorphism and let $A(\Gamma)$ be equipped with any fixed HHS structure. We say that ϕ *extends continuously to a map* $\partial A(\Gamma) \to \partial\text{Mod}(S)$ if there exists a function $\overline{\phi} : A(\Gamma) \cup \partial A(\Gamma) \to \text{Mod}(S) \cup \partial\text{Mod}(S)$ such that (1) $\overline{\phi}|_{A(\Gamma)} = \phi$, (2) $\overline{\phi}(\partial A(\Gamma)) \subseteq \partial\text{Mod}(S)$, and (3) $\overline{\phi}$ is continuous at each point in $\partial A(\Gamma)$.

Remark 3.2. When $A(\Gamma)$ is a free group, two sequences in $A(\Gamma)$ converge to the same point in $\partial_G A(\Gamma)$ if and only if they converge to the same point in $\partial A(\Gamma)$ (see free group example in Section 2.5). Consequently, given our goals, it will not be necessary to understand the HHS structure $A(\Gamma)$ is equipped with nor the boundary $\partial A(\Gamma)$ the structure yields.

Remark 3.3. To establish that $\phi : A(\Gamma) \to \text{Mod}(S)$ extends continuously, it is enough to show that for all $x \in \partial A(\Gamma)$, given any two sequences (x_n) and (y_n) in $A(\Gamma)$ that converge to x, we have that $(\phi(x_n))$ and $(\phi(y_n))$ converge to the same point in $\partial\text{Mod}(S)$. This follows from a diagonal sequence argument (see the end of the proof of Theorem 5.6 in [DHS2017] for details).

Idea behind non-existence proofs (Theorems 1.2, 1.3, and 3.22). All of the embeddings $\phi : A(\Gamma) \to \text{Mod}(S)$ we present that do not extend share the following key feature. For some pair of non-commuting generators a and b of $A(\Gamma)$, the subsurface Y filled by the full supports of $\phi(a)$ and $\phi(b)$ is a *proper* subsurface of S. For the embeddings we consider, this allows us to produce two sequences in $A(\Gamma)$ that converge to the same point in $\partial A(\Gamma)$, but whose images do not converge to the same point in $\partial\text{Mod}(S)$. We choose the n^{th} term of first sequence so that the annular projection of its image to some boundary component γ of Y is distance $O(n)$ from a basepoint, while the projection to γ of the image of the second sequence has bounded diameter. We then show that $O(n)$ is fast enough to conclude that every accumulation point in $\partial\text{Mod}(S)$ of the image of the first sequence has a term associated to γ. On the other hand, accumulation points of the image of the second sequence have no such term. Thus the images of the sequences do not converge to the same point in $\partial\text{Mod}(S)$.

The following opens question arises naturally from our non-existence proofs.

Question 3.4. *Let $A(\Gamma) \to Mod(S)$ be a Clay, Leininger, Mangahas embedding of a free group that sends some pair of generators of $A(\Gamma)$ to mapping classes whose full supports together do not fill S. Is it always the case that ϕ does not extend? In other words, does the forward direction of Theorem 1.3 hold for CLM embeddings? (Theorem 3.23 proves the backwards direction).*

Question 3.5. *Let Teich(S) denote the Teichmüller space of a surface S, equipped with the Weil-Petersson metric. There is an HHS structure on Teich(S), where the set of domains* *is all non-annular subsurfaces of S (see [Bro2003]). Given that we show annular subsurface projections can obstruct extendability, we wonder if an orbit map from $A(\Gamma)$ to Teich(S) corresponding to a CLM embedding $A(\Gamma) \to Mod(S)$ extends continuously to a boundary map. Note that this is clearly not the case for the Koberda embeddings described in Theorem 3.7 since applying powers of a Dehn twist to a point in Teich(S) can move it only a bounded amount.*

Chapter Outline. In Section 3.2 we will recall relevant definitions and theorems and introduce notation. Section 3.3 will establish a handful of lemmas that will be used for proving Theorem 1.2. Section 3.4 is devoted to proving Theorem 1.2 for a Clay, Leininger, Mangahas embedding, and in Section 3.5 we prove Theorem 1.2 for a Koberda embedding. Using similar techniques, we then prove that a more general class of Koberda embeddings of right-angled Artin groups do not extend continuously (Theorem 3.22), which will imply one direction of Theorem 1.3. In Section 3.6 we will prove Theorem 3.23, which will imply the other direction of Theorem 1.3.

3.2 Background

3.2.1 Embedding RAAGs in Mod(S)

Clay, Leininger, and Mangahas [CLM2012] proved the following result, which allows us to find quasi-isometrically embedded right-angled Artin subgroups inside Mod(S).

Theorem 3.6 ([CLM2012, Theorem 2.2])**.** *Let Γ be a finite graph with $V(\Gamma) = \{s_1, \ldots, s_k\}$, and let $\{X_1, \ldots, X_k\}$ be a collection of non-annular subsurfaces of S. Suppose $s_i s_j$ is an edge in Γ if and only if X_i and X_j are disjoint, and $s_i s_j$ is not an edge in Γ if and only if $X_i \pitchfork X_j$ or $i = j$. Then there exists a constant $C > 0$ such that the following holds. Let $\{f_1, \ldots, f_k\}$ be a set of mapping classes of S such that f_i is pseudo-Anosov on X_i and satisfies $\tau_{X_i}(f_i) \geq C$ for all i. Then the homomorphism*

$$\phi : A(\Gamma) \to Mod(S) \quad \text{defined by} \quad \phi(s_i) = f_i \quad \text{for all } i$$

is a quasi-isometric embedding, implying that ϕ is injective since $A(\Gamma)$ is torsion-free.

Koberda [Kob2012] also has a result which produces right-angled Artin subgroups of Mod(S). Below we give a special case of Koberda's result that we will use.

Theorem 3.7 ([Kob2012, Theorem 1.1])**.** *Let $\{\alpha_1, \ldots, \alpha_k\}$ be a collection of distinct curves in S. Let Γ be the graph with $V(\Gamma) = \{s_1, \ldots, s_k\}$ and with $s_i s_j$ an edge in Γ if and only if*

$i(\alpha_i, \alpha_j) = 0$. *Then for sufficiently large N, the homomorphism*

$$\phi : A(\Gamma) \to Mod(S) \quad \textit{defined by} \quad \phi(s_i) = T_{\alpha_i}^N \quad \textit{for all } \; i,$$

is injective, where T_{α_i} denotes a Dehn twist about α_i.

3.2.2 Subsurface projection theorems

In this section, we collect useful facts and theorems on subsurface projection.

Consider $f \in \mathrm{Mod}(S)$ and $Y \subseteq S$. Let μ and μ' be either markings on S, collections of curves, or bi-infinite simple geodesics (if Y is annular). It is a straight forward exercise to see that

$$d_Y(f(\mu), f(\mu')) = d_{f^{-1}(Y)}(\mu, \mu').$$

Here if Y is non-annular, $f(Y)$ denotes the non-annular subsurface in its isotopy class. If Y is an annulus with core curve α, then $f(Y)$ denotes the annular subsurface of S with core curve $f(\alpha)$.

Masur and Minsky [MM2000] define the *marking graph of* S, denoted $\widetilde{\mathcal{M}}(S)$, to be the graph whose vertices are markings and vertices are adjacent if one can be obtained from the other by an elementary move; see [MM2000] for a complete definition. Giving $\widetilde{\mathcal{M}}(S)$ the path metric $d_{\widetilde{\mathcal{M}}(S)}$ and $\mathrm{Mod}(S)$ a word metric $d_{\mathrm{Mod}(S)}$, there is an action of $\mathrm{Mod}(S)$ on $\widetilde{\mathcal{M}}(S)$ by isometries for which every orbit map is a quasi-isometry. The following theorem gives a relationship between distances in $\widetilde{\mathcal{M}}(S)$ and subsurface projections.

Theorem 3.8 ([MM2000, Lemma 3.5]). *For any subsurface Y of S and any markings μ and μ' on S, we have that $d_Y(\mu, \mu') \leq 4d_{\widetilde{\mathcal{M}}(S)}(\mu, \mu')$.*

Additionally, we will require the following theorems. The first theorem was proved in [Beh2006] and later a simpler proof with constructive constants appeared in [Man2013].

Theorem 3.9 (**Behrstock inequality** [Beh2006, Theorem 4.3], [Man2013, Lemma 2.13]). *Let X and Y be overlapping subsurfaces of S and μ a marking on S. Then*

$$d_X(\mu, \partial Y) \geq 10 \quad \textit{implies that} \quad d_Y(\mu, \partial X) \leq 4.$$

Theorem 3.10 ([MM2000, Lemma 2.3]). *For all subsurfaces Y of S, given any marking or multicurve μ such that $\pi_Y(\mu) \neq \emptyset$, we have that $diam_{\mathcal{C}(Y)}(\pi_Y(\mu)) \leq 2$. If Y is an annulus, then $diam_{\mathcal{C}(Y)}(\pi_Y(\mu)) \leq 1$.*

Theorem 3.11 (**Bounded Geodesic Image Theorem** [MM2000, Theorem 3.1]). *There exists a constant K_0 depending only on S such that the following is true. Let X and Y be subsurfaces of S with X a proper subsurface of Y. Let $v_1, \ldots, v_n$ be any geodesic segment in $\mathcal{C}(Y)$ satisfying $\pi_X(v_i) \neq \emptyset$ for all $1 \leq i \leq n$. Then*

$$diam_{\mathcal{C}(X)}(\pi_X(v_1) \cup \ldots \cup \pi_X(v_n)) \leq K_0.$$

We now establish a corollary of Theorem 3.11 that will be useful later.

Corollary 3.12. *Let X and Y be subsurfaces of S with X a proper subsurface of Y. Suppose $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of markings on S such that $\pi_Y(\mu_n) \to \lambda$ for some $\lambda \in \partial\mathcal{C}(Y)$. Then $diam_{\mathcal{C}(X)}(\pi_X(\mu_1) \cup \pi_X(\mu_2) \cup \ldots) < \infty$.*

Proof. For each n, choose $\alpha_n \in \pi_Y(\mu_n)$. Because $\pi_Y(\mu_n) \to \lambda \in \partial\mathcal{C}(Y)$, we can choose L large so that for all $n \geq L$ we have

$$(\alpha_n, \alpha_L)_{\alpha_1} \geq 2 + d_Y(\partial X, \alpha_1), \tag{3}$$

where the Gromov product is computed in $\mathcal{C}(Y)$. Consider $n \geq L$. Let γ_n be a geodesic in $\mathcal{C}(Y)$ with endpoints α_n and α_L. If there exists a vertex v on γ_n with $\pi_X(v) = \emptyset$, then v and ∂X form a multicurve in Y, which implies that

$$\begin{aligned}
(\alpha_n, \alpha_L)_{\alpha_1} &= \frac{1}{2}\Big(d_Y(\alpha_n, \alpha_1) + d_Y(\alpha_L, \alpha_1) - d_Y(\alpha_n, \alpha_L)\Big) \\
&\leq \frac{1}{2}\Big(d_Y(\alpha_n, v) + d_Y(v, \alpha_1) + d_Y(\alpha_L, v) + d_Y(v, \alpha_1) - (d_Y(\alpha_n, v) + d_Y(v, \alpha_L))\Big) \\
&= d_Y(v, \alpha_1) \leq d_Y(v, \partial X) + d_Y(\partial X, \alpha_1) \leq 1 + d_Y(\partial X, \alpha_1).
\end{aligned}$$

But this contradicts Inequality (3), so we conclude that $\pi_X(v) \neq \emptyset$ for all v on γ_n. We can now apply Theorems 3.10 and 3.11 to see that for all $n \geq L$

$$d_X(\mu_n, \mu_L) \leq \mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_n)) + d_X(\alpha_n, \alpha_L) + \mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_L)) \leq 2 + K_0 + 2,$$

where K_0 is an in Theorem 3.11. Therefore,

$$\mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_1) \cup \pi_X(\mu_2) \cup \ldots) \leq \mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_1) \cup \ldots \cup \pi_X(\mu_L)) + 2(K_0 + 4) < \infty.$$

□

3.2.3 Partial order on subsurfaces

Let μ, μ' be markings on S and $K \geq 20$. Let $\Omega(K, \mu, \mu')$ denote the collection of subsurfaces Y of S such that $d_Y(\mu, \mu') \geq K$. Behrstock, Kleiner, Minsky, and Mosher [BKMM2012] define the following partial order on $\Omega(K, \mu, \mu')$. Given $X, Y \in \Omega(K, \mu, \mu')$ such that $X \pitchfork Y$, define $X \prec Y$ if and only if one of the following equivalent conditions is satisfied:

$$d_X(\mu, \partial Y) \geq 10, \quad d_X(\partial Y, \mu') \leq 4, \quad d_Y(\mu, \partial X) \leq 4, \quad \text{or} \quad d_Y(\partial X, \mu') \geq 10.$$

That these conditions are equivalent is a consequence of Theorem 3.9; see Corollary 3.7 in [CLM2012].

3.2.4 Notation

Let $f, g : X \to \mathbb{R}$ be functions. Given constants $A \geq 1$ and $B \geq 0$, we write $f \overset{A,B}{\succ} g$ to mean $f(x) \geq \frac{1}{A} g(x) - B$ for all $x \in X$, and will just write $f \succ g$ when the constants are understood.

3.3 Lemmas on subsurface projections

The following lemmas are the heart of our proof of Theorem 1.2.

Lemma 3.13. *Suppose X and Y are disjoint subsurfaces of S, and if Y is an annulus, then the core of Y is not contained in ∂X. If μ and μ' are markings and $f \in Mod(S)$ a mapping class supported on X, then $|d_Y(\mu, f(\mu')) - d_Y(\mu, \mu')| \leq 4$.*

Proof. If Y is not an annulus, then $\pi_Y(f(\mu')) = \pi_Y(\mu')$ so the claim clearly holds. Assume then that Y is an annular subsurface of S with core α, and let $\text{int}(Y) \to S$ be the associated covering. If X is not an annulus, define Z to be the component of $S - X$ that contains α. If X is an annulus with core β, let Z be the component of S containing α after removing a small regular neighborhood of β. Let $\widetilde{\alpha}$ be the component of the preimage of α in $\text{int}(Y)$ that is a closed curve. Let $\widetilde{Z}$ be the component of the preimage of Z in $\text{int}(Y)$ that contains $\widetilde{\alpha}$.

Abusing notation, we let f denote a representative in the isotopy class of f that fixes Z pointwise. Consider the lift of f to $\text{int}(Y)$ that fixes a point on $\widetilde{\alpha}$, and thus fixes $\widetilde{Z}$ pointwise. Let $\widetilde{f} : Y \to Y$ denote the continuous extension of that lift. Consider $\beta' \in \pi_Y(\mu')$. We will show

$$d_{\mathcal{C}(Y)}(\beta', \widetilde{f}(\beta')) \leq 2. \tag{4}$$

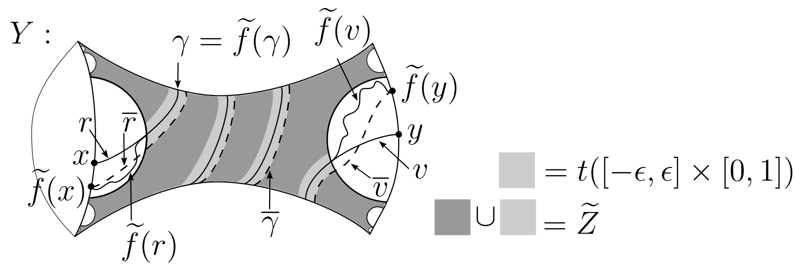

Figure 3.1: The arc β' is the concatenation of r, γ, and v. The concatenation of $\overline{r}, \overline{\gamma}$, and $\overline{v}$ is equivalent to $\widetilde{f}(\beta')$, and that representative of $\widetilde{f}(\beta')$ intersects β' at most once (drawn is the exactly once case). See the proof of Lemma 3.13.

This will complete the proof because (4), the triangle inequality, and Theorem 3.10 imply that

$$\begin{aligned} |d_Y(\mu, f(\mu')) - d_Y(\mu, \mu')| &\leq d_Y(\mu', f(\mu')) \\ &\leq \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\mu')) + d_{C(Y)}(\beta', \widetilde{f}(\beta')) + \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(f(\mu'))) \\ &\leq 1 + 2 + 1 = 4. \end{aligned}$$

Inequality (4) holds if β' is contained in $\widetilde{Z}$, because in that case $\widetilde{f}(\beta') = \beta'$. Thus, we assume β' is not contained in $\widetilde{Z}$. We break β' up into three parts. Let γ be the the largest subpath of β' contained in $\widetilde{Z}$. Let x and y denote the endpoints of β' on ∂Y. Removing γ from β' yields rays r and v that limit to x and y, respectively.

We now construct an arc equivalent to $\widetilde{f}(\beta')$ that intersects β' at most once. Figure 3.1 illustrates the construction. Let $t : [-\epsilon, \epsilon] \times [0, 1] \to Y$ be a small tubular neighborhood of γ so that $t|_{\{0\}\times[0,1]} = \gamma$ and $t([-\epsilon, \epsilon] \times \{0, 1\}) \subseteq \partial\widetilde{Z}$. Let R and V denote the components of $\operatorname{int}(Y) - \widetilde{Z}$ containing r and v, respectively. Because $\widetilde{f}$ fixes $\widetilde{Z}$ pointwise, $\widetilde{f}$ restricts to homeomorphisms of both R and V, implying that $\widetilde{f}(r)$ and $\widetilde{f}(v)$ are contained in R and V, respectively. Consequently, in R there exists a ray $\overline{r}$ based at $t(-\epsilon, 0)$ or $t(\epsilon, 0)$ that limits to $\widetilde{f}(x)$ and is disjoint from r. If $\overline{r}$ is based at $t(-\epsilon, 0)$, define $\bar{\gamma} = t|_{\{-\epsilon\}\times[0,1]}$. Otherwise, define $\bar{\gamma} = t|_{\{\epsilon\}\times[0,1]}$. Choose $\overline{v}$ to be an arc in V from $\overline{\gamma}(1)$ to $\widetilde{f}(y)$ that intersects v at most once. Observe that the arc obtained by concatenating $\overline{r}$, $\overline{\gamma}$, and $\overline{v}$ is equivalent to $\widetilde{f}(\beta')$ and intersects β' at most once.

Therefore, by Equation (1) we have $d_{\mathcal{C}(Y)}(\widetilde{f}(\beta'), \beta') = 1 + |\widetilde{f}(\beta') \cdot \beta'| \leq 2$, as desired.

□

Lemma 3.14. *Given a homomorphism $\phi : A(\Gamma) \to Mod(S)$ and a marking μ on S, there exists a constant $M \geq 1$ such that the following holds. Let $y_1 \dots y_n \in A(\Gamma)$, where each $y_i \in V(\Gamma)$. Then $d_W(\mu, \phi(y_1 \dots y_n)\mu) \leq Mn$ for all subsurfaces $W \subseteq S$.*

Proof. Define $M = 4\max\{d_{\widetilde{\mathcal{M}}(S)}(\mu, \phi(x)\mu) : x \in V(\Gamma)\}$. By the triangle inequality and Theorem 3.8,

$$\begin{aligned} d_W(\mu, \phi(y_1 \dots y_n)\mu) &\leq \sum_{i=1}^{n} d_W(\phi(y_1 \dots y_{i-1})\mu, \phi(y_1 \dots y_i)\mu) \\ &\leq \sum_{i=1}^{n} 4d_{\widetilde{\mathcal{M}}(S)}(\phi(y_1 \dots y_{i-1})\mu, \phi(y_1 \dots y_i)\mu) \\ &= \sum_{i=1}^{n} 4d_{\widetilde{\mathcal{M}}(S)}(\mu, \phi(y_i)\mu) \leq Mn. \end{aligned}$$

$\square$

Lemma 3.15. *Let $\phi : A(\Gamma) \to Mod(S)$ be a homomorphism. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements in $A(\Gamma)$ and μ a marking on S. Suppose for some subsurface $W \subseteq S$ there exist constants $A \geq 1$ and $B \geq 0$, that do not depend on n, such that $d_W(\mu, \phi(g_n)\mu) \overset{A,B}{\succ} ||g_n||$, where $||g_n||$ denotes the word length of g_n with respect to the standard generating set $V(\Gamma)$ for $A(\Gamma)$. Further suppose that $\lim\limits_{n\to\infty} ||g_n|| = \infty$ and that $(\pi_W(\phi(g_n)\mu))_{n\in\mathbb{N}}$ converges to some point λ_W in $\partial\mathcal{C}(W)$. Then all accumulation points of $(\phi(g_n))_{n\in\mathbb{N}}$ in $Mod(S) \cup \partial Mod(S)$ are in $\partial Mod(S)$ and are of the form $\sum\limits_{Y\subseteq S} c_Y\lambda_Y$, where $c_W > 0$.*

Proof. After passing to a subsequence, we may assume that $(\phi(g_n))_{n\in\mathbb{N}}$ converges. By assumption, $\lim\limits_{n\to\infty} d_W(\mu, \phi(g_n)\mu) = \infty$. Combine this with Theorem 3.8 to see that $\lim\limits_{n\to\infty} d_{\widetilde{\mathcal{M}}(S)}(\mu, \phi(g_n)\mu) = \infty$. Because $\widetilde{\mathcal{M}}(S)$ is quasi-isometric to $\mathrm{Mod}(S)$ via orbit maps, it follows that $\lim\limits_{n\to\infty} d_{\mathrm{Mod}(S)}(1, \phi(g_n)) = \infty$. Thus, it must be that $\lim\limits_{n\to\infty} \phi(g_n) \in \partial\mathrm{Mod}(S)$.

Suppose $\lim\limits_{n\to\infty} \phi(g_n) = \sum\limits_{Y\subseteq S} c_Y\lambda_Y$ for constants $c_Y \geq 0$ and $\lambda_Y \in \partial\mathcal{C}(Y)$. We will now argue that $c_W > 0$. Let $Z \subseteq S$ be such that $c_Z > 0$. If $W = Z$, we are done. So we assume $W \neq Z$. By definition of the topology on $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$, we have that $\lim\limits_{n\to\infty} \pi_Z(\phi(g_n)\mu) = \lambda_Z$. If $W \subsetneq Z$, then Corollary 3.12 implies that $\mathrm{diam}_{\mathcal{C}(W)}(\pi_W(\phi(g_1)\mu) \cup \pi_W(\phi(g_2)\mu) \cup \dots) < \infty$. But this cannot be since $\pi_W(\phi(g_n)\mu) \to \lambda_W \in \partial\mathcal{C}(W)$. Similarly, we cannot have $Z \subsetneq W$ for then Corollary 3.12 implies that $\mathrm{diam}_{\mathcal{C}(Z)}(\pi_Z(\phi(g_1)\mu) \cup \pi_Z(\phi(g_2)\mu) \cup \dots) < \infty$, contradicting that $\pi_Z(\phi(g_n)\mu) \to \lambda_Z \in \partial\mathcal{C}(Z)$. Now suppose that $Z \pitchfork W$. Then by Theorem 3.9, after passing to a subsequence, we have that

$$d_W(\partial Z, \phi(g_n)\mu) \leq 10 \quad \text{for all } n, \qquad \text{or} \qquad d_Z(\partial W, \phi(g_n)\mu) \leq 10 \quad \text{for all } n.$$

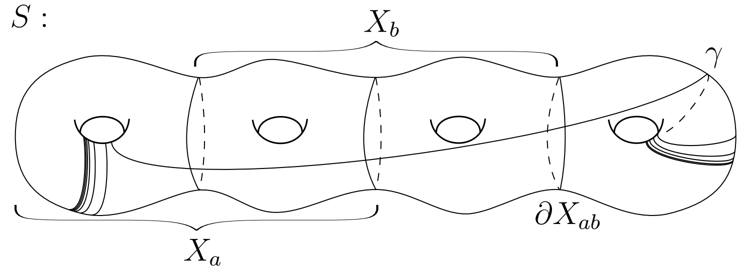

Figure 3.2: Overlapping subsurfaces X_a and X_b of surface S, curve ∂X_{ab}, and bi-infinite simple geodesic γ.

If $d_W(\partial Z, \phi(g_n)\mu) \leq 10$ for all n, then for all n

$$d_W(\mu, \phi(g_n)\mu) \leq d_W(\mu, \partial Z) + d_W(\partial Z, \phi(g_n)\mu) \leq d_W(\mu, \partial Z) + 10,$$

contradicting $\pi_W(\phi(g_n)\mu) \to \lambda_W \in \partial\mathcal{C}(W)$. Similarly, if $d_Z(\partial W, \phi(g_n)\mu) \leq 10$ for all n, then $d_Z(\mu, \phi(g_n)\mu)$ is bounded independent of n contradicting that $\pi_Z(\phi(g_n)\mu) \to \lambda_Z \in \partial\mathcal{C}(Z)$. So it is not the case that $Z \pitchfork W$. Therefore it must be that W and Z are disjoint for all $Z \subseteq S$ with $c_Z > 0$.

Fix $Z \subseteq S$ with $c_Z > 0$. Lemma 3.14 together with the fact that $d_W(\mu, \phi(g_n)\mu) \overset{A,B}{\succ} ||g_n||$ implies that

$$\frac{d_W(\mu, \phi(g_n)\mu)}{d_Z(\mu, \phi(g_n)\mu)} \geq \frac{\frac{1}{A}||g_n|| - B}{M||g_n||}, \tag{5}$$

where $M \geq 1$ is as in Lemma 3.14. Since $||g_n|| \to \infty$, Equation (5) implies

$$\lim_{n\to\infty} \frac{d_W(\mu, \phi(g_n)\mu)}{d_Z(\mu, \phi(g_n)\mu)} \geq \lim_{n\to\infty} \frac{\frac{1}{A}||g_n|| - B}{M||g_n||} > 0.$$

Therefore by definition of the topology of $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$, we have $c_W > 0$ as desired.

□

3.4 Clay, Leininger, Mangahas RAAGs

In this section, we prove the first part of Theorem 1.2. First, a definition.

For $i = 1, 2$ let $\widetilde{\gamma}_i$ be a bi-infinite path in $\mathbb{H}^2$ with ends limiting to distinct points x_i and y_i on $\partial\mathbb{H}^2$. We say that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ *link* if the geodesic connecting x_1 to y_1 intersects the geodesic connecting x_2 to y_2 in the interior of $\mathbb{H}^2$.

Embedding construction: We now give a description of a Clay, Leininger, Mangahas embedding $\phi : A(\Gamma) \to \mathrm{Mod}(S)$. Let Γ be the graph with vertex set $V(\Gamma) = \{a, b\}$ and no edges. Let $S = \mathbb{H}^2/\Lambda, X_a$, and X_b be the surfaces indicated in Figure 3.2. For short, let X_{ab}

denote $X_a \cup X_b$. Let $\widetilde{S - X_{ab}}$ be a component of the preimage of $S - X_{ab}$ in $\mathbb{H}^2$, and let $\widetilde{\partial X_{ab}}$ be a geodesic in $\mathbb{H}^2$ that is in the boundary of $\widetilde{S - X_{ab}}$.

Let $\widetilde{\gamma}$ be a geodesic in $\mathbb{H}^2$ that links with $\widetilde{\partial X_{ab}}$ and maps to a simple bi-infinite geodesic γ in S. Further suppose that $\widetilde{\gamma} \cap (\widetilde{S - X_{ab}})$ is an infinite ray and let p be its endpoint on $\partial\mathbb{H}^2$. For example, take γ to be the simple bi-infinite geodesic in S with one end spiraling around a curve essential in $S - X_{ab}$ and the other end spiraling around a curve in X_a as in Figure 3.2, and take $\widetilde{\gamma}$ to be an appropriate lift of γ. Choose $f_b \in \mathrm{Mod}(S)$ so that f_b is pseudo-Anosov on X_b. To simplify arguments, we abuse notation and let f_b denote a representative in the isotopy class of f_b that fixes all points outside X_b. This ensures that $\tilde{f}_b$ fixes $\widetilde{S - X_{ab}}$ pointwise, where $\tilde{f}_b : \mathbb{H}^2 \to \mathbb{H}^2$ is the lift of f_b fixing some point on $\widetilde{\partial X_{ab}}$. Thus, the extension of $\tilde{f}_b$ to $\partial\mathbb{H}^2$ fixes pointwise p and the endpoints x and y of $\widetilde{\partial X_{ab}}$. Additionally, we choose f_b to have the following properties:

(I) $\tilde{f}_b(\widetilde{\gamma})$ links with $h(\widetilde{\gamma})$, where $h \in \Lambda$ is a primitive isometry with axis $\widetilde{\partial X_{ab}}$, and

(II) $\tau_{X_b}(f_b) \geq C$, where C is as in Theorem 3.6.

We note that a pseudo-Anosov on X_b satisfying (I) can be obtained from any mapping class that is pseudo-Anosov on X_b by post-composing with some number of Dehn twists (or inverse Dehn twists) about ∂X_{ab}. Finally, a pseudo-Anosov on X_b satisfying (I) and (II) can be obtained from one satisfying (I) by passing to a sufficiently high power.

Let $f_a \in \mathrm{Mod}(S)$ be any mapping class that is pseudo-Anosov on X_a and satisfies $\tau_{X_a}(f_a) \geq C$. Theorem 3.6 says that the homomorphism

$$\phi : A(\Gamma) \to \mathrm{Mod}(S) \quad \text{defined by} \quad \phi(a) = f_a,\ \phi(b) = f_b$$

is a quasi-isometric embedding.

Equip $A(\Gamma)$ with any HHS structure. In the remainder of this section, we will prove the following theorem, which proves the first part of Theorem 1.2.

Theorem 3.16. *The sequences $(a^n)_{n\in\mathbb{N}}$ and $(a^nb^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$, but $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^nb^n))_{n\in\mathbb{N}}$ do not converge to the same point in $\partial Mod(S)$.*

We will divide the proof of Theorem 3.16 into two propositions.

Proposition 3.17. *The sequences $(a^n)_{n\in\mathbb{N}}$ and $(a^nb^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$.*

Proof. Let X be the Cayley graph of $A(\Gamma)$. By Remark 3.2, to show that $(a^n)_{n\in\mathbb{N}}$ and $(a^nb^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$, it is enough to show that they converge to the same point in $\partial_G X$. Now the Gromov product

$$(a^i, a^j b^j)_1 = \min(i,j) \to \infty \quad \text{as} \quad i,j \to \infty.$$

Thus, $\lim\limits_{n\to\infty} a^n = \lim\limits_{n\to\infty} a^n b^n$ in $\partial_G X$, as desired.

$\square$

Throughout the rest of this section μ will denote a fixed marking on S. To continue, we require the following lemma.

Lemma 3.18. *There exist constants $A \geq 1$ and $B \geq 0$ such that for all $n \geq 1$ we have $d_{\partial X_{ab}}(\mu, \phi(a^n b^n)\mu) \overset{A,B}{\succ} n$. Consequently, after passing to a subsequence, $(\pi_{\partial X_{ab}}(\phi(a^n b^n)\mu))_{n\in\mathbb{N}}$ converges to a point in $\partial\mathcal{C}(\partial X_{ab})$.*

Proof. We begin by establishing the following claim.

Claim 1: Let $n \geq 1$. Then $\widetilde{f_b}^n(\widetilde{\gamma})$ has endpoint p and links with $h^i(\widetilde{\gamma})$ for all $1 \leq i \leq n$.

Proof of Claim 1. By our choice of $\widetilde{f_b}$ and $\widetilde{\gamma}$, we know the claim holds for $n = 1$. Let $n \geq 2$. Inductively, suppose that $\widetilde{f_b}^{n-1}(\widetilde{\gamma})$ has endpoint p and links with $h^i(\widetilde{\gamma})$ for all $1 \leq i \leq n-1$. Let I be the interval in $\partial\mathbb{H}^2$ that connects the endpoints of $\widetilde{\partial X_{ab}}$ and does not contain p, oriented from the repelling fixed point of h to the attracting fixed point. We will use interval notation when speaking about connected subsets of I. Now $\widetilde{f_b}$ extends continuously to a homeomorphism of $\partial\mathbb{H}^2$, which we will also denote by $\widetilde{f_b}$, and because $\widetilde{f_b}$ fixes the endpoints of $\widetilde{\partial X_{ab}}$, this extension restricts to a homeomorphism of I. Let z be the endpoint of $\widetilde{\gamma}$ in I, and let $x \in \partial I$ be the attracting fixed point of h. Because $\widetilde{f_b}^{n-1}(\widetilde{\gamma})$ links with $h^i(\widetilde{\gamma})$ for all $1 \leq i \leq n-1$ and has endpoint p, we have

$$(\widetilde{f_b}^{n-1}(z), x] \subseteq (h^i(z), x] \quad \text{for all} \quad 0 \leq i \leq n-1. \tag{6}$$

Since $\widetilde{f_b}(\widetilde{\gamma})$ has endpoint p and links with $h(\widetilde{\gamma})$, it must be that $\widetilde{f_b}(z) \in (hz, x]$. It follows from this, the fact that $\widetilde{f_b}$ and h fix x, that $\widetilde{f_b}$ and h commute by uniqueness of map lifting, and (6), that for all $0 \leq i \leq n-1$

$$\widetilde{f_b}^n(z) = \widetilde{f_b}^{n-1}(\widetilde{f_b}(z)) \in \widetilde{f_b}^{n-1}(h(z), x] = h(\widetilde{f_b}^{n-1}(z), x] \subseteq h(h^i(z), x] = (h^{i+1}(z), x]. \tag{7}$$

Because $\widetilde{f_b}$ fixes p, we have $\widetilde{f_b}^n(p) = p$. This combined with (7) implies that $\widetilde{f_b}^n(\widetilde{\gamma})$ links with $h^{i+1}(\widetilde{\gamma})$ for all $0 \leq i \leq n-1$, proving Claim 1. $\square$

By Claim 1, after replacing $\widetilde{f_b}^n(\widetilde{\gamma})$ with the geodesic connecting its endpoints, the images of $\widetilde{f_b}^n(\widetilde{\gamma})$ and $\widetilde{\gamma}$ in $(\overline{\mathbb{H}}^2 - \{x, y\})/\langle h\rangle$ intersect each other at least n times, and all these intersections have the same sign. Now apply Equation (1) to see that

$$d_{\partial X_{ab}}(\gamma, \phi(b^n)\gamma) \geq n + 1.$$

It follows that

$$\begin{aligned} d_{\partial X_{ab}}(\mu, \phi(b^n)\mu) &\geq d_{\partial X_{ab}}(\gamma, \phi(b^n)\gamma) - d_{\partial X_{ab}}(\mu, \gamma) - d_{\partial X_{ab}}(\phi(b^n)\mu, \phi(b^n)\gamma) \\ &\geq n + 1 - 2d_{\partial X_{ab}}(\mu, \gamma) \end{aligned} \tag{8}$$

Lemma 3.13 says that $|d_{\partial X_{ab}}(\mu, \phi(a^n b^n)\mu) - d_{\partial X_{ab}}(\mu, \phi(b^n)\mu)| \leq 4$. This together with Equation (8) implies that

$$d_{\partial X_{ab}}(\mu, \phi(a^n b^n)\mu)) \succ n.$$

From this and the fact that $\mathcal{C}(\partial X_{ab})$ is quasi-isometric to $\mathbb{R}$ it is immediate that $(\pi_{\partial X_{ab}}(\phi(a^n b^n)\mu))_{n\in\mathbb{N}}$ has a subsequence converging to a point in $\partial\mathcal{C}(\partial X_{ab})$. $\square$

Proposition 3.19. *The sequences $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n b^n))_{n\in\mathbb{N}}$ do not converge to the same point in Mod$(S) \cup \partial$Mod(S).*

Proof. After passing to a subsequence, we may assume that $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n b^n))_{n\in\mathbb{N}}$ converge to points p and q respectively in $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$ and, by Lemma 3.18, that $(\pi_{\partial X_{ab}}(\phi(a^n b^n)\mu))_{n\in\mathbb{N}}$ converges to a point in $\partial\mathcal{C}(\partial X_{ab})$. Lemmas 3.15 and 3.18 imply that q is in $\partial\mathrm{Mod}(S)$. Say $q = \sum_{Y\subseteq S} c_Y^q \lambda_Y^q$, where $c_Y^q \geq 0$ and $\lambda_Y^q \in \partial\mathcal{C}(Y)$ for all $Y \subseteq S$. Then Lemmas 3.15 and 3.18 also imply that $c_{\partial X_{ab}}^q > 0$.

Now if p were in $\mathrm{Mod}(S)$, then we would be done since clearly then $p \neq q$. So we will assume that $p \in \partial\mathrm{Mod}(S)$, and let $p = \sum_{Y\subseteq S} c_Y^p \lambda_Y^p$. Now observe that by Lemma 3.13 and Theorem 3.10

$$d_{\partial X_{ab}}(\mu, \phi(a^n)\mu) \leq d_{\partial X_{ab}}(\mu, \mu) + 4 \leq 5.$$

Thus, $(\pi_{\partial X_{ab}}(\phi(a^n)\mu))_{n\in\mathbb{N}}$ does not limit to a point on $\partial\mathcal{C}(\partial X_{ab})$. So by definition of the topology of $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$, it must be that $c_{\partial X_{ab}}^p = 0$. Since $c_{\partial X_{ab}}^q > 0$, we see that $p \neq q$, which completes the proof. $\square$

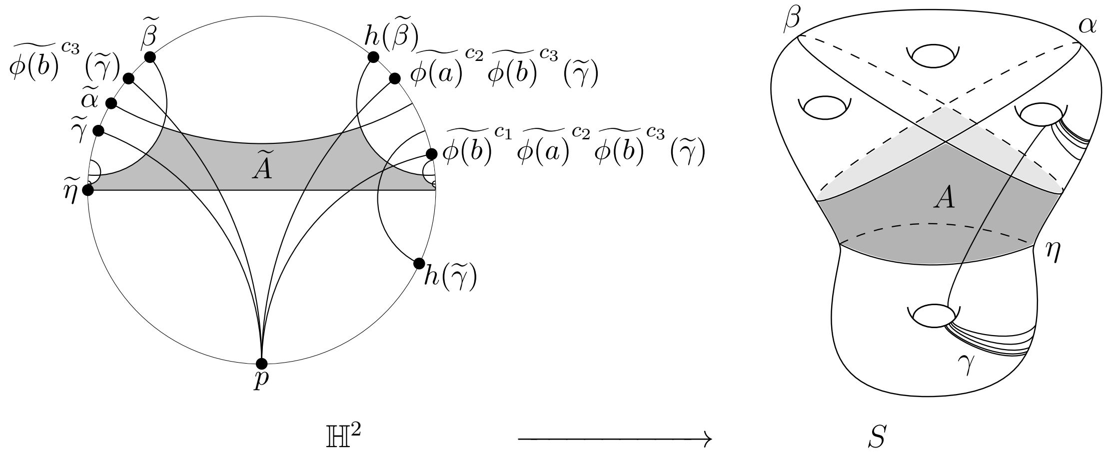

Figure 3.3: Curves α, β, and η, bounding an annulus A, and simple bi-infinite geodesic γ on surface S, and the universal cover $\mathbb{H}^2$ of S as in Lemma 3.21.

3.5 Koberda RAAGs

In this section we complete the proof of Theorem 1.2. Following this, we will discuss how to use similar techniques to prove a large class of Koberda embeddings do not extend.

Let α and β be the pair of intersecting curves on $S = \mathbb{H}^2/\Lambda$ depicted in Figure 3.3. Let Γ be the graph with $V(\Gamma) = \{a, b\}$ and no edges. For sufficiently large N, Theorem 3.7 says that the homomorphism

$$\phi : A(\Gamma) \to \mathrm{Mod}(S) \quad \text{defined by } \phi(a) = {T_\alpha}^N \text{ and } \phi(b) = {T_\beta}^N$$

is injective, where T_α and T_β denote Dehn twists about α and β respectively. Throughout this section, we let μ be a fixed marking on S. Equip $A(\Gamma)$ with an HHS structure.

In this section we prove the following theorem, which will complete the proof of Theorem 1.2.

Theorem 3.20. *There exists $g \in A(\Gamma)$ such that the sequences $(a^n)_{n\in\mathbb{N}}$ and $(a^n g^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$, but $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n g^n))_{n\in\mathbb{N}}$ do not converge to the same point in $\partial Mod(S)$.*

As a step towards proving Theorem 3.20, we prove the following lemma in which we construct $g \in A(\Gamma)$.

Lemma 3.21. *There exist constants $A \geq 1$ and $B \geq 0$ and a word $g \in A(\Gamma)$ such that for all $n \geq 1$ we have $d_\eta(\mu, \phi(a^n g^n)\mu) \overset{A,B}{\succ} n$, where η is the curve shown in Figure 3.3. Consequently, after passing to a subsequence, $(\pi_\eta(\phi(a^n g^n)\mu))_{n\in\mathbb{N}}$ converges to a point in $\partial\mathcal{C}(\eta)$.*

Proof. We will prove that there exist constants c_1, c_2, c_3 such that $g = b^{c_1}a^{c_2}b^{c_3}$ has the desired properties.

Let A be the annulus in Figure 3.3. Let $\widetilde{A}$ be a component of the preimage of A in $\mathbb{H}^2$. Let $\widetilde{\beta}$ be a component of the preimage of β such that a segment of $\widetilde{\beta}$ is in the boundary of $\widetilde{A}$, and let $\widetilde{\eta}$ denote the component of the preimage of η in the boundary of $\widetilde{A}$. Let $h \in \Lambda$ be a primitive isometry with axis $\widetilde{\eta}$. Let $\widetilde{\alpha}$ be the component of the preimage of α that links with $\widetilde{\beta}$ and $h(\widetilde{\beta})$ and contains a segment that is in the boundary of $\widetilde{A}$.

Let Y_α be the component of $S - \alpha$ that contain η. To simplify arguments, we let $\phi(a)$ denote a representative in its isotopy class that fixes Y_α pointwise. Let $\widetilde{\phi(a)} : \mathbb{H}^2 \to \mathbb{H}^2$ be the lift of $\phi(a)$ that fixes some point on $\widetilde{\alpha}$. Similarly define Y_β to be the component of $S - \beta$ containing η, choose a representative in the isotopy class of $\phi(b)$ that fixes Y_β pointwise, and let $\widetilde{\phi(b)}$ be the lift of $\phi(b)$ that fixes some point on $\widetilde{\beta}$. It then follows that

$$\widetilde{\phi(a)} = \mathbb{1} \text{ on } \widetilde{Y_\alpha} \quad \text{and} \quad \widetilde{\phi(b)} = \mathbb{1} \text{ on } \widetilde{Y_\beta},$$

where for $i \in \{\alpha, \beta\}$ we let $\widetilde{Y_i}$ denote the component of the preimage of Y_i in $\mathbb{H}^2$ whose boundary contains $\widetilde{i}$. Observe that for $i \in \{a, b\}$ we have that $\widetilde{\phi(i)}$ fixes the endpoints of $\widetilde{\eta}$.

Choose a geodesic $\widetilde{\gamma}$ in $\mathbb{H}^2$ that links with both $\widetilde{\beta}$ and $\widetilde{\eta}$ and maps to a simple bi-infinite geodesic in S. Further, suppose that $\widetilde{\gamma} \cap \widetilde{Y_\alpha} \cap \widetilde{Y_\beta}$ is an infinite ray, and let p denote its endpoint on $\partial\mathbb{H}^2$. For example, take γ to be the simple bi-infinite geodesic in S with one end spiraling around a curve essential in $Y_\alpha \cap Y_\beta$ and the other end spiraling around a curve essential in $S - Y_\beta$ as in Figure 3.3, and take $\widetilde{\gamma}$ to be an appropriate component of the preimage of γ. Observe that $\widetilde{\phi(a)}$ and $\widetilde{\phi(b)}$ must fix p.

Now choose $c_3 \in \mathbb{Z}$ so that $\widetilde{\phi(b)}^{c_3}(\widetilde{\gamma})$ links with $\widetilde{\alpha}$. Then pick $c_2 \in \mathbb{Z}$ so that $\widetilde{\phi(a)}^{c_2}\widetilde{\phi(b)}^{c_3}(\widetilde{\gamma})$ links with $h(\widetilde{\beta})$. Finally, choose $c_1 \in \mathbb{Z}$ so that $\widetilde{\phi(b)}^{c_1}\widetilde{\phi(a)}^{c_2}\widetilde{\phi(b)}^{c_3}(\widetilde{\gamma})$ links with $h(\widetilde{\gamma})$. See Figure 3.3.

To simplify notation, define

$$g = b^{c_1}a^{c_2}b^{c_3} \in A(\Gamma) \quad \text{and} \quad \widetilde{\phi(g)} = \widetilde{\phi(b)}^{c_1}\widetilde{\phi(a)}^{c_2}\widetilde{\phi(b)}^{c_3}.$$

As in Lemma 3.18, we have that $\widetilde{\phi(g)}^{n}(\widetilde{\gamma})$ has endpoint p and links with $h^i(\widetilde{\gamma})$ for all $1 \leq i \leq n$, implying that $d_\eta(\gamma, \phi(g^n)\gamma) \geq n+1$. It follows that

$$d_\eta(\mu, \phi(g^n)\mu) \geq d_\eta(\gamma, \phi(g^n)\gamma) - d_\eta(\mu, \gamma) - d_\eta(\phi(g^n)\mu, \phi(g^n)\gamma) \geq n + 1 - 2d_\eta(\mu, \gamma). \quad (9)$$

Now Lemma 3.13 says that $|d_\eta(\mu, \phi(a^n g^n)\mu) - d_\eta(\mu, \phi(g^n)\mu)| \leq 4$. This together with Equation (9) implies that $d_\eta(\mu, \phi(a^n g^n)\mu) \succ n$. From this and the fact that $\mathcal{C}(\eta)$ is quasi-isometric to $\mathbb{R}$, it is immediate that $(\pi_\eta(\phi(a^n g^n)\mu))_{n\in\mathbb{N}}$ has a subsequence converging to a point in $\partial\mathcal{C}(\eta)$. $\square$

We can now prove Theorem 3.20.

Proof of Theorem 3.20. Let $g \in A(\Gamma)$ be as in Lemma 3.21. By Remark 3.2, to show that $(a^n)_{n\in\mathbb{N}}$ and $(a^n g^n)_{n\in\mathbb{N}}$ converge to the same point $\partial A(\Gamma)$ it is enough to show that they converge to the same point in $\partial_G X$, where X is the Cayley graph of $A(\Gamma)$. Now the Gromov product

$$(a^i, a^j g^j)_1 = (a^i, a^j(b^{c_1}a^{c_2}b^{c_3})^j)_1 = \min(i,j) \to \infty \text{ as } i,j \to \infty.$$

Therefore $\lim_{n\to\infty} a^n = \lim_{n\to\infty} a^n g^n$ in $\partial_G X$, as desired.

To finish this proof, we mimic the proof of Proposition 3.19. Replacing b with g, and ∂X_{ab} with η, and Lemma 3.18 with Lemma 3.21, we find that $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n g^n))_{n\in\mathbb{N}}$ do not converge to the same point in $\partial\mathrm{Mod}(S)$. $\square$

Our techniques used to prove Theorem 3.20 can be used to prove a more general statement on non-existence of boundary maps for right-angled Artin groups that are not necessarily free groups. To prove this more general statement, one needs to understand HHS structures for all right-angled Artin groups. In the following theorem, by a *standard HHS structure on* $A(\Gamma)$, we mean one induced by a factor system generated by a rich family of subgraphs of Γ. We refer the reader to [BHS2017b], specifically Proposition 8.3 and Remark 13.2, for details. In the proof of the following theorem, we freely use definitions and notations used in [BHS2017b] and [DHS2017].

Theorem 3.22. *Let $\{\alpha_1, \ldots, \alpha_k\}$ be any collection pairwise distinct of curves in S. Let Γ be the graph with $V(\Gamma) = \{s_1, \ldots, s_k\}$ and $s_i s_j$ an edge in Γ if and only and $i(\alpha_i, \alpha_j) = 0$. Give $A(\Gamma)$ a standard HHS structure, or if $A(\Gamma)$ is a free group, any HHS structure. If there exist distinct intersecting curves α_i and α_j that do not fill S, then any corresponding Koberda embedding $\phi : A(\Gamma) \to Mod(S)$ does not extend continuously to a map $\partial A(\Gamma) \to \partial Mod(S)$.*

Proof. Consider the subgraph Λ of Γ with $V(\Lambda) = \{s_i, s_j\}$. Contained in the Salvetti complex S_Γ associated to Γ there is a subcomplex that is the Salvetti complex associated to $A(\Lambda)$. We let $\widetilde{S_\Lambda}$ denote the lift of this subcomplex to the universal cover $\widetilde{S_\Gamma}$ of S_Γ that contains 1. Let $\mathcal{R}$ be a rich family of induced subgraphs of Γ, and let $\mathcal{F}$ be the corresponding factor system in $\widetilde{S_\Gamma}$. Lemma 8.4 of [BHS2017b] tells us that

$$\mathcal{F}' = \{F \cap \widetilde{S_\Lambda} : F \in \mathcal{F}\}$$

is a factor system in $\widetilde{S_\Lambda}$. Associating $A(\Gamma)$ and $A(\Lambda)$ with $\widetilde{S_\Gamma}$ and $\widetilde{S_\Lambda}$ respectively, we equip each with the HHS structures corresponding to their respective factor systems. We first argue that the inclusion map $A(\Lambda) \to A(\Gamma)$ extends continuously to a map $\partial A(\Lambda) \to \partial A(\Gamma)$. If ϕ extends continuously to a map $\partial A(\Gamma) \to \partial \mathrm{Mod}(S)$, it will follow that $A(\Lambda) \to \mathrm{Mod}(S)$ extends continuously to a map $\partial A(\Lambda) \to \partial \mathrm{Mod}(S)$; we will show that this is impossible.

First, consider $A(\Lambda) \to A(\Gamma)$. Given $U \in \mathcal{F}'$ such that U is not a 0-cube, define $\pi(U)$ to be the parallelism class of the $\subseteq$-minimal $F \in \mathcal{F}$ such that $U = F \cap \widetilde{S_\Lambda}$. Observe that U and V are nested (respectively orthogonal) if and only if $\pi(U)$ and $\pi(V)$ are nested (respectively orthogonal). This together with Lemma 10.11 of [DHS2017] implies that $(A(\Lambda) \to A(\Gamma), \pi)$ is a hieromorphism. Theorem 5.6 of [DHS2017] gives a condition guaranteeing that a hieromorphism extends continuously. In our case, if the following claims are true, we can apply Theorem 5.6 to conclude that $A(\Lambda) \to A(\Gamma)$ extends continuously.

Claim 1: π is injective.

Proof of Claim 1. Suppose $U, V \in \mathcal{F}'$ and $\pi(U) = \pi(V)$. Then $\pi(U) \sqsubseteq \pi(V)$ and $\pi(V) \sqsubseteq \pi(U)$. Thus, $U \subseteq V$ and $V \subseteq U$, implying $U = V$, as desired. □

Claim 2: If $[F] \in \overline{\mathcal{F}}$ is not a class of 0-cubes and there exists no $U \in \mathcal{F}'$ satisfying $\pi(U) = [F]$, then $\mathrm{diam}_{\widehat{C}F}(\pi_F(\widetilde{S_\Lambda}))$ is bounded above uniformly for some (any) $F \in [F]$.

Proof of Claim 2. Let $[F] \in \overline{\mathcal{F}}$ be as in Claim 1. First, suppose there exists $F \in [F]$ such that $F \cap \widetilde{S_\Lambda} \neq \emptyset$. By Lemma 8.5 in [BHS2017b], we have $\mathfrak{g}_F(\widetilde{S_\Lambda}) \subseteq F \cap \widetilde{S_\Lambda}$. If $F \cap \widetilde{S_\Lambda}$ is a 0-cube, then $\mathrm{diam}_{\widehat{C}F}(\pi_F(\widetilde{S_\Lambda})) \leq 1$, so the claim holds. Otherwise, there must exists $\overline{F} \in \mathcal{F}$ such that $\overline{F} \subsetneq F$ and $\overline{F} \cap \widetilde{S_\Lambda} = F \cap \widetilde{S_\Lambda}$. It follows that $C\overline{F}$ is coned off in $\widehat{C}F$ and that $\mathfrak{g}_F(\widetilde{S_\Lambda}) \subseteq \overline{F}$. This implies that $\mathrm{diam}_{\widehat{C}F}(\pi_F(\widetilde{S_\Lambda})) \leq 4$.

Now assume $F \cap \widetilde{S}_\Lambda = \emptyset$ for all $F \in [F]$. An argument like that in the proof of Proposition 8.3 of [BHS2017b] shows that we can find $g \in A(\Gamma), \Gamma' \in \mathcal{R}$, and $x \in A(\Lambda)$ so that $g\widetilde{S}_{\Gamma'} \in [F]$ and

$$\mathfrak{g}_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda) \subseteq g(\widetilde{S}_{\Gamma' \cap \Lambda \cap \mathrm{Lk}\overline{g}}) \subseteq g(\widetilde{S}_{\Gamma' \cap \mathrm{Lk}\overline{g}}), \tag{10}$$

where $\overline{g} = g^{-1}x$ and $\mathrm{Lk}\overline{g}$ denotes the link of $\overline{g}$. Now if $\Gamma' \cap \Lambda \cap \mathrm{Lk}\overline{g} = \emptyset$, then $\mathfrak{g}_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda) = \{g\}$, implying that $\mathrm{diam}_{\widehat{C}(g\widetilde{S}_{\Gamma'})}(\pi_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda)) \leq 1$. Assume then that $\Gamma' \cap \Lambda \cap \mathrm{Lk}\overline{g} \neq \emptyset$. Then by definition of $\mathcal{R}$ and $\mathcal{F}$, we have that $\Gamma' \cap \mathrm{Lk}\overline{g} \in \mathcal{R}$ and $g(\widetilde{S}_{\Gamma' \cap \mathrm{Lk}\overline{g}}) \in \mathcal{F} - \{\text{0-cubes}\}$. If $g(\widetilde{S}_{\Gamma' \cap \mathrm{Lk}\overline{g}})$ is not a proper subcomplex of $g\widetilde{S}_{\Gamma'}$, then $\Gamma' \subseteq \mathrm{Lk}\overline{g}$, implying that $x\widetilde{S}_{\Gamma'}$ is parallel to $g\widetilde{S}_{\Gamma'}$ (see Lemma 2.4 in [BHS2017b]). But this cannot be because $(x\widetilde{S}_{\Gamma'}) \cap \widetilde{S}_\Lambda = x(\widetilde{S}_{\Gamma' \cap \Lambda}) \neq \emptyset$ and no factor parallel to $g\widetilde{S}_{\Gamma'}$ intersects $\widetilde{S}_\Lambda$ non-trivially. Therefore, $g(\widetilde{S}_{\Gamma' \cap \mathrm{Lk}\overline{g}})$ must be a proper subcomplex of $g\widetilde{S}_{\Gamma'}$. Thus, $Cg(\widetilde{S}_{\Gamma'\cap \mathrm{Lk}\overline{g}})$ is coned off in $\widehat{C}(g\widetilde{S}_{\Gamma'})$. This together with (10) implies that $\mathrm{diam}_{\widehat{C}(g\widetilde{S}_{\Gamma'})}(\pi_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda)) \leq 4$, completing the proof of Claim 2. □

We now argue that $A(\Lambda) \to \mathrm{Mod}(S)$ does not extend continuously. Let η denote a geodesic representative of an essential boundary component of a small regular neighborhood of $\alpha_i \cup \alpha_j$. Using the proof techniques of Lemma 3.21, we can construct $g \in A(\Lambda)$ so that $d_\eta(\mu, \phi(s_i^n g^n)\mu)$ grows linearly in n. For later convenience, we construct g so that when written in reduced form, the first letter of g is $s_j^{\pm 1}$. As in Proposition 3.19, we see that the sequences $(\phi(s_i^n))$ and $(\phi(s_i^n g^n))$ do not converge to the same point in $\mathrm{Mod}(S) \cup \partial \mathrm{Mod}(S)$. Now observe that (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial_G A(\Lambda)$. Therefore, by the discussion in Section 2.5, (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial A(\Lambda)$. We have now established that $A(\Lambda) \to \mathrm{Mod}(S)$ does not extend continuously to a map $\partial A(\Lambda) \to \partial \mathrm{Mod}(S)$. Therefore, $A(\Gamma) \to \mathrm{Mod}(S)$ does not extend continuously when $A(\Gamma)$ is equipped with a standard HHS structure.

Now suppose $A(\Gamma)$ is a free group equipped with any HHS structure. By Remark 3.2, because (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial_G A(\Gamma)$, we have that (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial A(\Gamma)$. Because $(\phi(s_i^n))$ and $(\phi(s_i^n g^n))$ do not converge to the same point in $\partial \mathrm{Mod}(S)$, it follows that $A(\Gamma) \to \mathrm{Mod}(S)$ does not extend continuously. □

3.6 Existence of boundary maps for some free groups

In this section, we show that a class of embeddings of free groups in $\mathrm{Mod}(S)$, which includes a class of Koberda embeddings and a class of CLM embeddings, extend continuously.

Throughout this section, let Γ be the graph with $V(\Gamma) = \{s_1, \ldots, s_k\}$ and no edges, and let $A(\Gamma)$ denote the corresponding right-angled Artin group (a rank k free group). Equip $A(\Gamma)$ with an HHS structure. Let $\{X_1, \ldots, X_k\}$ be a collection of distinct, pairwise overlapping, and pairwise filling subsurfaces of S and $\{f_1, \ldots, f_k\}$ a collection of mapping classes such that f_i is fully supported on X_i. Let μ be a fixed marking on S. The main theorem of this section is the following, which implies the remaining direction of Theorem 1.3.

Theorem 3.23. *Let $A(\Gamma)$ be the rank k free group equipped with any HHS structure. Let $\{X_1, \ldots, X_k\}$ be a collection of distinct, pairwise overlapping, and pairwise filling subsurfaces of S, and $\{f_1, \ldots, f_k\}$ a collection of mapping classes such that f_i is fully supported on X_i. There exists a $C > 0$ such that if $\tau_{X_i}(f_i) \geq C$ for all i, then the homomorphism*

$$\phi : A(\Gamma) \to Mod(S) \quad \textit{defined by} \quad \phi(s_i) = f_i \textit{ for all } i$$

is a quasi-isometric embedding and extends continuously to a map $\partial A(\Gamma) \to \partial Mod(S)$.

We emphasize the arguments we will use to establish that ϕ is a quasi-isometric embedding are essentially the same as those used by Clay, Leininger, and Mangahas to prove Theorem 3.6. In particular, when the the X_i are all non-annular, that ϕ is a quasi-isometric embedding is Theorem 3.6. To prove Theorem 3.23, we require the following proposition.

Proposition 3.24. *There exists $K > 0$ such that the following holds. For each $1 \leq i \leq k$, assume $\tau_{X_i}(f_i) \geq 2K$. Let $\phi : A(\Gamma) \to Mod(S)$ be the homomorphism defined by $\phi(s_i) = f_i$ for all i. Consider $g_1 \dots g_k \in A(\Gamma)$, where for each i we have $g_i = x_i^{e_i}$ for some $x_i \in \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ and $e_i > 0$, and $x_i \neq x_{i+1}$, and $x_1^{e_1} \dots x_k^{e_k}$ is a reduced word. Let Y_i be the subsurface of S that fully supports $\phi(x_i)$. Then*

(1) For each $1 \leq i \leq k$, we have $d_{\phi(g_1 \dots g_{i-1})Y_i}(\mu, \phi(g_1 \dots g_k)\mu) \geq Ke_i$,

(2) For all $1 \leq i < j \leq k$, we have $\phi(g_1 \dots g_{i-1})Y_i \prec \phi(g_1 \dots g_{j-1})Y_j$, where $\prec$ denotes the partial order on $\Omega(K, \mu, \phi(g_1 \dots g_k)\mu)$, and

(3) The homomorphism $\phi : A(\Gamma) \to Mod(S)$ is a quasi-isometric embedding.

Proof. Define $K = K_0 + 20 + 2\max\{d_{X_i}(\mu, \partial X_j) : 1 \leq i, j \leq k \text{ and } i \neq j\}$, where K_0 is maximum of the constants in Theorem 6.12 of [MM2000] and Theorem 3.11. Statements (1) and (2) of this proposition are essentially Theorem 5.2 in [CLM2012]. The difference is that Theorem 5.2 does not allow for the homomorphism to send a generator to a power of a Dehn twist. The only obstruction to Theorem 5.2 holding for homomorphisms ϕ of this type is the following. Suppose X_i is the subsurface that fully supports $\phi(s_i)$, and let $\sigma \in A(\Gamma)$ be a non-empty word in letters commuting with s_i, not including s_i. If X_i is non-annular, then $d_{X_i}(\phi(\sigma)\mu', \mu'') = d_{X_i}(\mu', \mu'')$ for any markings μ', μ''. This not necessarily true if X_i is an annulus. However, this issue does not arise for us because $A(\Gamma)$ a free group implies no such σ exists. Thus, the arguments used to prove Theorem 5.2 in [CLM2012] also prove our Statements (1) and (2). The proof of our Statement (3) is the same as the proof in [CLM2012] of Theorem 3.6, using our Statement (1) instead of their Theorem 5.2. □

The proof of the next lemma is essentially contained in the proof of Theorem 6.1 in [CLM2012]. We include a proof here for completeness.

Lemma 3.25. *Let $\phi : A(\Gamma) \to Mod(S)$, $g_1 \dots g_k \in A(\Gamma)$, and Y_i be as in Proposition 3.24. Let $\mathcal{G}$ be a geodesic in $\mathcal{C}(S)$ with one end in $\pi_S(\mu)$ and one end in $\pi_S(\phi(g_1 \dots g_k)\mu)$. Then for each $1 \leq i \leq k$, there exists a curve γ_i on $\mathcal{G}$ such that $\pi_{\phi(g_1 \dots g_{i-1})Y_i}(\gamma_i) = \emptyset$. If $|i - j| \geq 3$ and γ_i and γ_j are two such curves, then $\gamma_i \neq \gamma_j$.*

Proof. Fix $1 \le i \le k$. By way of contradiction, suppose for all curves v on $\mathcal{G}$, we have $\pi_{\phi(g_1 \dots g_{i-1})Y_i}(v) \neq \emptyset$. Then Theorems 3.10 and 3.11 together imply that

$$d_{\phi(g_1 \dots g_{i-1})Y_i}(\mu, \phi(g_1 \dots g_k)\mu) \le 4 + K_0.$$

But Proposition 3.24 says $d_{\phi(g_1 \dots g_{i-1})Y_i}(\mu, \phi(g_1 \dots g_k)\mu) \ge K > K_0 + 4$, a contradiction. Thus, there must exist a curve γ_i on $\mathcal{G}$ such that $\pi_{\phi(g_1 \dots g_{i-1})Y_i}(\gamma_i) = \emptyset$, as desired. Note that this implies that γ_i and $\partial\phi(g_1 \dots g_{i-1})Y_i$ form a multicurve.

Now consider γ_i and γ_j, where $1 \le i < j \le k$ and $|i - j| \ge 3$. We will show that γ_i and γ_j are distinct curves. To the contrary, suppose $\gamma_i = \gamma_j$. Because of the filling assumption on $\{X_1, \dots, X_k\}$, the pair of subsurfaces Y_{i+1} and Y_{i+2} fill S. Thus, the subsurface pair $\phi(g_1 \dots g_{i+1})Y_{i+1} = \phi(g_1 \dots g_i)Y_{i+1}$ and $\phi(g_1 \dots g_{i+1})Y_{i+2}$ also fill S. Thus, it must be that $\pi_{\phi(g_1 \dots g_{n-1})Y_n}(\gamma_i) \neq \emptyset$ for some $n \in \{i+1, i+2\}$. In any case, $i < n < j$.

In the remainder of this proof, to simplify notation, for each ℓ we define $\overline{Y_\ell} = \phi(g_1 \dots g_{\ell-1})Y_\ell$. By Proposition 3.24, we have

$$\overline{Y_i} \prec \overline{Y_n} \prec \overline{Y_j},$$

where $\prec$ is the partial order on $\Omega(K, \mu, \phi(g_1 \dots g_k)\mu)$. In particular, these three subsurfaces are pairwise overlapping. This together with the assumption that $\gamma_i = \gamma_j$ and Theorem 3.10 implies that

$$d_{\overline{Y_n}}(\partial\overline{Y_i}, \partial\overline{Y_j}) \le d_{\overline{Y_n}}(\partial\overline{Y_i}, \gamma_i) + d_{\overline{Y_n}}(\gamma_j, \partial\overline{Y_j}) \le 2 + 2 = 4.$$

It follows from this and the definition of $\prec$ that

$$d_{\overline{Y_n}}(\mu, \phi(g_1 \dots g_k)\mu) \le d_{\overline{Y_n}}(\mu, \partial\overline{Y_i}) + d_{\overline{Y_n}}(\partial\overline{Y_i}, \partial\overline{Y_j}) + d_{\overline{Y_n}}(\partial\overline{Y_j}, \phi(g_1 \dots g_k)\mu) \le 4 + 4 + 4 = 12.$$

But this cannot be, because $d_{\overline{Y_n}}(\mu, \phi(g_1 \dots g_k)\mu) \ge K \ge 20$ by Proposition 3.24. Therefore, γ_i and γ_j are distinct curves. $\square$

We have now developed the tools we will need to prove Theorem 3.23.

Proof of Theorem 3.23. Define $C = 2K$, where K is as in Proposition 3.24 and for each $1 \le i \le k$, assume that $\tau_{X_i}(f_i) \ge C$. By Proposition 3.24, ϕ is a quasi-isometric embedding.

Let X denote the Cayley graph of $A(\Gamma)$. Choose $x \in \partial_G X$. Let γ be the infinite geodesic ray in X based at 1 limiting to x in $\partial_G X$. We think of γ as an infinite word of the form $y_1 y_2 y_3 \dots$, where each $y_i \in \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ and the word $y_1 y_2 \dots y_i$ is a reduced word for all i. By construction, the sequence $(y_1 \dots y_n)$ converges to x in $X \cup \partial_G X$. Let (h_n) be another sequence in $A(\Gamma)$ that converges to x in $X \cup \partial_G X$. We will show that $(\phi(h_n))$ and $(\phi(y_1 \dots y_n))$ converge to the same point in $\partial\mathrm{Mod}(S)$. By Remark 3.3, this will prove the theorem. We will consider two cases: (1) There does not exist $N \geq 1$ such that $y_i = y_N$ for all $i \geq N$, and (2) such an N exists. In both cases, we will assume each h_n is written in the form $h_n = g_{n,1} \dots g_{n,N(n)}$, where for all i we have $g_{n,i} = x_{n,i}^{e_{n,i}}$ for some $e_{n,i} > 0$ and $x_{n,i} \in \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ satisfying $x_{n,i} \neq x_{n,i+1}$, and $x_{n,1}^{e,1} \dots x_{n,N(n)}^{e_{n,N(n)}}$ is a reduced word.

Case 1: Suppose there does not exist $N \geq 1$ such that $y_i = y_N$ for all $i \geq N$. Then we can think of γ as an infinite word of the form $g_1 g_2 g_3 \dots$, where $g_i = x_i^{e_i}$ for some $e_i > 0$ and $x_i \in \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ satisfying $x_i \neq x_{i+1}$, and $x_1^{e_1} \dots x_i^{e_i}$ is a reduced word for all i. Define Y_i to be the subsurface that fully supports $\phi(x_i)$. For short, we let $\overline{Y_i}$ denote $\phi(g_1 \dots g_{i-1})Y_i$.

Because (h_n) and $(y_1 \dots y_n)$ converge to the same point in $\partial_G X$ and X is a tree, h_n and $y_1 \dots y_n$ must agree on longer and longer initial segments as $n \to \infty$. In particular, given $L \geq 1$, there exists M such that for all $n \geq M$, we have $g_{n,1} \dots g_{n,L} = g_1 \dots g_L$. Consider $n \geq M$ and $k \geq e_1 + \dots + e_L$. Choose a curve $\beta \in \mathrm{base}(\mu)$. Given $\sigma \in A(\Gamma)$, let $\mathcal{G}(\sigma)$ denote some choice of geodesic in $\mathcal{C}(S)$ with endpoints β and $\phi(\sigma)\beta$. By Lemma 3.25, for all $1 \leq i \leq L$ there exist curves γ_i and γ_i' on $\mathcal{G}(y_1 \dots y_k)$ and $\mathcal{G}(h_n)$ respectively such that $\pi_{\overline{Y_i}}(\gamma_i) = \emptyset$ and $\pi_{\overline{Y_i}}(\gamma_i') = \emptyset$. Observe that

$$d_S(\gamma_i, \partial\overline{Y_i}) \leq 1 \quad \text{and} \quad d_S(\gamma_i', \partial\overline{Y_i}) \leq 1.$$

Choose γ_r to be the curve in $\{\gamma_i : 1 \leq i \leq L\}$ closest to $\phi(y_1 \dots y_k)\beta$. Lemma 3.25 tells us that if $|i - j| \geq 3$, then $\gamma_i \neq \gamma_j$. So necessarily $d_S(\beta, \gamma_r) \geq L/3$. Thus, the Gromov product, computed in $\mathcal{C}(S)$, is

$$\begin{aligned}
(\phi(y_1 \dots y_k)\beta, \phi(h_n)\beta)_\beta &= \frac{1}{2}\Big[d_S(\beta, \phi(y_1 \dots y_k)\beta) + d_S(\beta, \phi(h_n)\beta) - d_S(\phi(y_1 \dots y_k)\beta, \phi(h_n)\beta)\Big] \\
&\geq \frac{1}{2}\Big[d_S(\beta, \gamma_r) + d_S(\gamma_r, \phi(y_1 \dots y_k)\beta) + d_S(\beta, \gamma_r') + d_S(\gamma_r', \phi(h_n)\beta) - \\
&\quad \Big(d_S(\phi(y_1 \dots y_k)\beta, \gamma_r) + d_S(\gamma_r, \partial\overline{Y_r}) + d_S(\partial\overline{Y_r}, \gamma_r') + d_S(\gamma_r', \phi(h_n)\beta)\Big)\Big] \\
&\geq \frac{1}{2}\Big[d_S(\beta, \gamma_r) + d_S(\beta, \gamma_r') - 2\Big] \\
&\geq \frac{1}{2}(L/3 - 2).
\end{aligned}$$

It follows that

$$\liminf_{k,n \to \infty} (\phi(y_1 \dots y_k)\beta, \phi(h_n)\beta))_\beta = \infty. \tag{11}$$

Because (h_n) is an arbitrary sequence converging to x, we could have taken it to be $(y_1 \dots y_n)$. Thus, Equation (11) tells us two things: (1) $(\phi(y_1 \dots y_n)\mu)$ converges to a point in $\partial\mathcal{C}(S)$, and (2) $(\phi(y_1 \dots y_n)\mu)$ and $(\phi(h_n)\mu)$ converge to the same point in $\partial\mathcal{C}(S)$. By definition of the topology on $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$, this tells us that $(\phi(y_1 \dots y_n))$ and $(\phi(h_n))$ converge to the same point in $\partial\mathrm{Mod}(S)$.

Case 2: Assume there exists $N \geq 1$ such that $y_i = y_N$ for all $i \geq N$. Corollary 6.2 in [DHS2017] tells us that the action of $\mathrm{Mod}(S)$ by left multiplication extends to an action of $\mathrm{Mod}(S)$ on $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$ by homeomorphisms. Consequently, if we can show that $(\phi((y_1 \dots y_{N-1})^{-1}h_n))_{n\in\mathbb{N}}$ and $(\phi(y_N \dots y_n))_{n\in\mathbb{N}}$ converge to the same point in $\partial\mathrm{Mod}(S)$, then $(\phi(h_n))_{n\in\mathbb{N}}$ and $(\phi(y_1 \dots y_n))_{n\in\mathbb{N}}$ must converge to the same point in $\partial\mathrm{Mod}(S)$. Furthermore, $((y_1 \dots y_{N-1})^{-1}h_n)_{n\in\mathbb{N}}$ and $(y_N \dots y_n)_{n\in\mathbb{N}}$ converge to the same point in $\partial_G X$. Thus, without loss of generality we assume $N = 1$. By our assumption, $y_1 \dots y_n = y_1^n$ for all n.

Let Y be the subsurface that fully supports $\phi(y_1)$ and let $\partial Y = \{\beta_1, \dots, \beta_\ell\}$. Then

$$\lim_{n\to\infty} \frac{d_Y(\mu, \phi(y_1^n)\mu)}{n} > 0 \quad \text{and} \quad \pi_Y(\phi(y_1^n)\mu) \to \lambda_Y \text{ for some } \lambda_Y \in \partial\mathcal{C}(Y). \tag{12}$$

Further observe that for all i

$$\lim_{n\to\infty} \frac{d_{\beta_i}(\mu, \phi(y_1^n)\mu)}{n} \geq 0. \tag{13}$$

If (13) is an equality, let λ_i be any point in $\partial\mathcal{C}(\beta_i)$. Otherwise, define $\lambda_i \in \partial\mathcal{C}(\beta_i)$ to be $\lim\limits_{n\to\infty} \pi_{\beta_i}(\phi(y_1^n)\mu)$. For all subsurfaces W disjoint from Y and not an annulus with core curve in ∂Y, Lemma 3.13 and Theorem 3.10 imply that $d_W(\mu, \phi(y_1^n)\mu) \leq d_W(\mu, \mu) + 4 \leq 6$. Consequently,

$$\lim_{n\to\infty} \phi(y_1^n) = c_Y\lambda_Y + \sum_{i=1}^{\ell} c_i\lambda_i,$$

where

$$c_Y + \sum_{i=1}^{\ell} c_i = 1 \text{ and } \frac{c_i}{c_Y} = \lim_{n\to\infty} \frac{d_{\beta_i}(\mu, \phi(y_1^n)\mu)}{d_Y(\mu, \phi(y_1^n)\mu)}.$$

Because (h_n) and (y_1^n) converge to the same point in $\partial_G X$, given any $L \geq 1$, for all sufficiently large n we have $x_{n,1} = y_1$ and $e_{n,1} \geq L$. So by removing finitely many initial terms from (h_n), for convenience we may assume that $g_{n,1} = y_1^{e_{n,1}}$ for all n. Observe that $e_{n,1} \to \infty$ as $n \to \infty$. It is immediate from this and the definition of the topology of $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$ that $\lim\limits_{n\to\infty} \phi(g_{n,1}) = \lim\limits_{n\to\infty} \phi(y_1^n)$. Thus, to finish the proof, we must show $\lim\limits_{n\to\infty} \phi(g_{n,1}) = \lim\limits_{n\to\infty} \phi(h_n)$. By passing to subsequences, we may assume that either $N(n) = 1$ for all n or $N(n) \geq 2$ for all n. If the former holds, then $h_n = g_{n,1}$, and we are done. Assume then that $N(n) \geq 2$ for all n. To proceed, we require the following claims.

Claim 1: $d_Y(\phi(g_{n,1})\mu, \phi(h_n)\mu)$ is bounded above, independent of n.

Claim 2: Let W be a subsurface that is disjoint from Y. Then $d_W(\phi(g_{n,1})\mu, \phi(h_n)\mu)$ is bounded above, independent of n.

We postpone the proofs of these claims and for now assume they are true. First, observe that Claim 1 and (12) imply that $\pi_Y(\phi(h_n)\mu) \to \lambda_Y$. If Inequality (13) is strict, then Claim 2 implies that $\pi_{\beta_i}(\phi(h_n)\mu) \to \lambda_i$. Further observe that Claims 1 and 2 imply that for all W disjoint from Y

$$\lim_{n\to\infty} \frac{d_W(\mu, \phi(g_{n,1})\mu)}{d_Y(\mu, \phi(g_{n,1})\mu)} = \frac{\lim_{n\to\infty} \dfrac{d_W(\mu, \phi(g_{n,1})\mu)}{e_{n,1}}}{\lim_{n\to\infty} \dfrac{d_Y(\mu, \phi(g_{n,1})\mu)}{e_{n,1}}} = \frac{\lim_{n\to\infty} \dfrac{d_W(\mu, \phi(h_n)\mu)}{e_{n,1}}}{\lim_{n\to\infty} \dfrac{d_Y(\mu, \phi(h_n)\mu)}{e_{n,1}}} = \lim_{n\to\infty} \frac{d_W(\mu, \phi(h_n)\mu)}{d_Y(\mu, \phi(h_n)\mu)}.$$

It follows that $\lim\limits_{n\to\infty} \phi(g_{n,1}) = \lim\limits_{n\to\infty} \phi(h_n)$ as desired.

To finish the proof, we will now prove Claims 1 and 2. For each n, let Z_n denote the subsurface that fully supports $\phi(x_{n,2})$.

Proof of Claim 1. Fix $n \geq 1$. Because Y fully supports $\phi(x_{n,1})$, by Proposition 3.24, we know $Y \prec \phi(g_{n,1})Z_n$, where $\prec$ denotes the partial order on $\Omega(K, \mu, \phi(h_n)\mu)$. Thus, $d_Y(\partial\phi(g_{n,1})Z_n, \phi(h_n)\mu) \leq 4$. Therefore,

$$d_Y(\phi(g_{n,1})\mu, \phi(h_n)\mu) \leq d_Y(\phi(g_{n,1})\mu, \partial\phi(g_{n,1})Z_n) + d_Y(\partial\phi(g_{n,1})Z_n, \phi(h_n)\mu) \leq d_Y(\mu, \partial Z_n) + 4.$$

There are finitely many possibilities for Z_n, so this completes the proof of Claim 1. $\square$

Proof of Claim 2. Fix $n \geq 1$. Because Y and Z_n fill S and Y and W are disjoint, it must be that $\pi_{Z_n}(\partial W) \neq \emptyset$. There are two cases to consider: (1) $W \pitchfork Z_n$ and (2) $W \subsetneq Z_n$. First, suppose that $W \pitchfork Z_n$. It then follows from Proposition 3.24, Theorem 3.10, and the definition of K that

$$\begin{aligned} d_{Z_n}(\partial W, \phi(g_{n,2} \ldots g_{n,N(n)})\mu) &\geq d_{Z_n}(\mu, \phi(g_{n,2} \ldots g_{n,N(n)})\mu) - d_{Z_n}(\partial Y, \partial W) - d_{Z_n}(\mu, \partial Y) \\ &\geq K - 2 - K/2 \geq 10. \end{aligned}$$

Thus Theorem 3.9 implies that $d_W(\partial Z_n, \phi(g_{n,2} \ldots g_{n,N(n)})\mu) \leq 4$. From this and Theorem

3.8 we find that

$$\begin{aligned} d_W(\phi(g_{n,1})\mu, \phi(h_n)\mu) &= d_W(\mu, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \\ &\le d_W(\mu, \partial Z_n) + d_W(\partial Z_n, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \\ &\le 4 \max\{d_{\widetilde{\mathcal{M}}(S)}(\mu, \mu_i) : 1 \le i \le k\} + 4, \end{aligned} \tag{14}$$

where μ_i is a fixed choice of marking with $\partial X_i \subseteq \mathrm{base}(\mu_i)$ for each $1 \le i \le k$. This provides a uniform bound in the case that $W \pitchfork Z_n$.

Now suppose that $W \subsetneq Z_n$. First, observe that because Z_n fully supports $\phi(x_{n,2})$, the sequence $(\pi_{Z_n}(\phi(x_{n,2})^m \mu))_{m \in \mathbb{N}}$ converges to a point in $\partial \mathcal{C}(Z_n)$. Thus, by Corollary 3.12 there exists a constant M, that depends on W and $x_{n,2}$, such that $d_W(\mu, \phi(g_{n,2})\mu) \le M$ for all n. Note that there are only finitely many possibilities for $x_{n,2}$, so M can be chosen to be independent of n. This implies that

$$\begin{aligned} d_W(\phi(g_{n,1})\mu, \phi(h_n)\mu) &\le d_W(\mu, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \\ &\le d_W(\mu, \phi(g_{n,2})\mu) + d_W(\phi(g_{n,2})\mu, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \\ &\le M + d_{\phi(g_{n,2})^{-1}W}(\mu, \phi(g_{n,3} \dots g_{n,N(n)})\mu). \end{aligned}$$

Now if $N(n) = 2$, then we can apply Theorem 3.10 to see that

$$d_{\phi(g_{n,2})^{-1}W}(\mu, \phi(g_{n,3} \dots g_{n,N(n)})\mu) = d_{\phi(g_{n,2})^{-1}W}(\mu, \mu) \le 2,$$

and Claim 2 is established. Suppose then that $N(n) \ge 3$. Let V_n denote the subsurface that fully supports $\phi(x_{n,3})$. Observe that because $\tau_{Z_n}(\phi(x_{n,2})) \ge 2K$ and ∂Y and ∂W form a multicurve, we have

$$\begin{aligned} d_{Z_n}(\partial\phi(g_{n,2})^{-1}W, \partial V_n) &\ge d_{Z_n}(\partial W, \partial\phi(g_{n,2})^{-1}W) - d_{Z_n}(\partial W, \partial Y) - d_{Z_n}(\mu, \partial Y) - d_{Z_n}(\mu, \partial V_n) \\ &\ge 2K - 2 - K/2 - K/2 > 2. \end{aligned}$$

This together with Theorem 3.8 establishes that $\partial\phi(g_{n,2})^{-1}W$ and ∂V_n do not form a multicurve. Thus, $\phi(g_{n,2})^{-1}W \pitchfork V_n$. So to bound $d_{\phi(g_{n,2})^{-1}W}(\mu, \phi(g_{n,3} \dots g_{n,N(n)})\mu)$ from above independent of n, we can use the same techniques used above to bound $d_W(\mu, \phi(g_{n,2} \dots g_{n,N(n)})\mu)$ when $W \pitchfork Z_n$. This completes the proof of Claim 2, and thus the proof of Theorem 3.23. $\square$

$\square$

CHAPTER 4

Exotic limit sets of Teichmüller geodesics

4.1 Introduction

The goal of this chapter is to anwer the question of Durham, Hagen, and Sisto [DHS2017] on the uniqueness of accumulation points of Teichmüller geodesic rays in the HHS boundary by proving the following theorem.

Theorem 1.4. *Given a continuous map $\gamma\colon \mathbb{R} \to \triangle^2$ to the standard 2-simplex, there exists a Teichmüller geodesic ray $\mathcal{G}$ in Teich(S_3) and an embedding of $\triangle^2$ into the HHS boundary of Teich(S_3) such that the limit set of $\mathcal{G}$ in the HHS boundary is the image of $\overline{\gamma(\mathbb{R})}$.*

Strategy for proving Theorem 1.4. To build Teichmüller geodesic rays, we use a classical construction (see for example Masur and Tabachnikov [MT2002]) also used by Lenzhen [Len2008] and Lenzhen, Modami, Rafi [LMR2016] to study limit sets of Teichmüller geodesics in Thurston's compactification. Given irrational numbers $\theta_0, \theta_1, \theta_2$ and $0 < s < 1$, for each $i = 0, 1, 2$ cut a slit of length s and slope θ_i in a unit square R_i. Rotate R_i counterclockwise so that its slit is vertical. For each rotated R_i, identify its parallel sides to form a torus with one boundary component. Then identify the left side of the slit in R_i with the right side of the slit in R_{i-1} (indices mod 3). This produces a genus 3 translation surface, yielding a complex

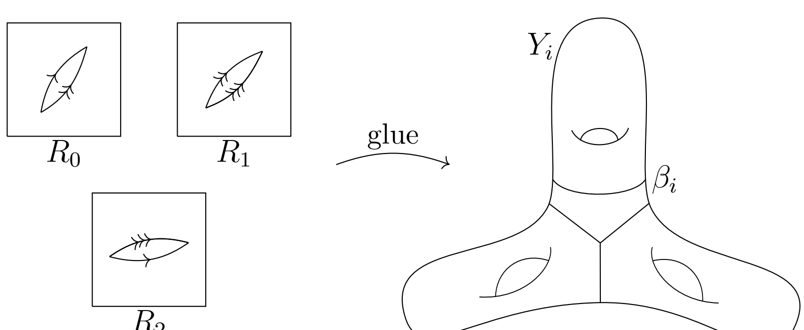

Figure 4.1: Three slitted unit squares glued to form a genus 3 translation surface.

structure X and a quadratic differential q with respect to X (see Figure 4.1). We consider the Teichmüller geodesic ray corresponding to (X, q). Given a continuous map $\gamma : \mathbb{R} \to \triangle^2$, we will show how to construct irrational numbers $\theta_0, \theta_1, \theta_2$ and an embedding of $\triangle^2$ into the HHS boundary of Teich$(S_{3,0})$ so that the limit set of the corresponding Teichmüller geodesic ray is the image of $\overline{\gamma(\mathbb{R})}$.

The vertical foliations of the geodesic rays used to prove Theorem 1.4 are not minimal. On the other hand, as mentioned in [DHS2017], if the vertical foliation is minimal, then the limit set in the HHS boundary consists of a single point (see the Teich(S) example in Section 2.5).

Chapter Outline. In Section 4.2 we define necessary terms and collect useful theorems. In Section 4.3 we give conditions on the irrational numbers to guarantee that the limit set in the HHS boundary of the corresponding Teichmüller geodesic ray is contained in a 2-simplex. Section 4.4 contains the proof of Theorem 1.4, rephrased there as Theorem 4.16. There we show how to carefully choose the entries of the continued fraction expansions of our irrational numbers to obtain fine control of the limit set.

4.2 Background

4.2.1 Notation and conventions

Throughout Section 4.2, let S denote a connected, closed, orientable surface of genus at least 2. Throughout this chapter , a *curve* in S means a homotopy class of an essential, simple, closed curve in S. Though when convenient, we will also call a representative in the homotopy class a curve.

Let $f, g\colon Y \to \mathbb{R}$ be functions. If there exist constants $A \geq 1$ and $B \geq 0$ that depend only on the topology of S, such that for all $y \in Y$, we have $\frac{1}{A}(g(y) - B) \leq f(y) \leq Ag(y) + B$, then we write $f \asymp g$. In the case that $B = 0$ we write $f \stackrel{*}{\asymp} g$, and if $A = 1$ we write $f \stackrel{+}{\asymp} g$. We define $\prec$, $\stackrel{*}{\prec}$, and $\stackrel{+}{\prec}$ similarly.

4.2.2 Extremal length and Teichmüller geodesics

Consider $X \in$ Teich(S). Every complex structure determines a collection of conformally equivalent Riemannian metrics on S, and in the collection there is a unique hyperbolic metric by the Uniformization Theorem. We let $\mathrm{Hyp}_X(\alpha)$ denote the length of the geodesic representative of α in the hyperbolic metric associated to $X \in$ Teich(S). The following theorem compares the hyperbolic and extremal lengths, showing that when the hyperbolic length of a curve is small, its extremal length and hyperbolic length are (coarsely) equal.

Theorem 4.1 (Maskit [Mas1985]). *Given $X \in Teich(S)$ and a curve α in S,*

$$\frac{1}{\pi} \leq \frac{\mathrm{Ext}_X(\alpha)}{\mathrm{Hyp}_X(\alpha)} \leq \frac{1}{2} e^{\mathrm{Hyp}_X(\alpha)/2}.$$

Let q be a (holomorphic) quadratic differential with respect to X. Local coordinates for q give S a singular flat structure, inducing a geodesic metric on S. We let $\ell_q(\gamma)$ denote the q-length of a geodesic representative of a curve γ in the metric induced by q. The collection of q-geodesic representatives of a curve α form a (possibly degenerate) Euclidean cylinder, which we will call F. An *expanding annulus with core α* is the largest one-sided regular neighborhood of a boundary component of F in a direction away from F that is an embedded annulus. Let E and G denote the two expanding annuli with core α. As a corollary to Minsky's work [Min1992], Choi, Series, and Rafi [CRS2008] deduce the following theorem that relates $\mathrm{Ext}_X(\alpha)$ to the moduli of F, E, and G. The subsequent theorem gives a way to estimate the modulus of an annulus that satisfies certain properties.

Theorem 4.2 (Minsky [Min1992, Theorems 4.5 and 4.6]; Choi, Series, Rafi [CRS2008, Corollary 5.4]). *There exists ϵ_0 depending only on S such that if $\mathrm{Ext}_X(\alpha) \leq \epsilon_0$, then*

$$\frac{1}{\mathrm{Ext}_X(\alpha)} \stackrel{*}{\asymp} Mod_X(E) + Mod_X(F) + Mod_X(G).$$

Theorem 4.3 (Rafi [Raf2005, Lemma 3.6]). *Let q be a quadratic differential with respect to $X \in Teich(S)$. Let A be an annulus in S such that with respect to the q-metric, A has equidistant boundary components and exactly one boundary component γ_0 a geodesic. Further suppose the interior of A does not contain any singularities of q. Then*

$$\mathrm{Mod}_X(A) \asymp \log\left(\frac{d}{\ell_q(\gamma_0)}\right),$$

where d is the q-distance between the boundary components of A.

Let q be a quadratic differential with respect to X. We now explain how the pair (X, q) determines a geodesic in the Teichmüller metric. Composing the natural coordinates of q away from its singularities with $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ yields a new complex structure $X_t \in \mathrm{Teich}(S)$ on S and a new quadratic differential q_t with respect to X_t. The map $\mathcal{G} : (-\infty, \infty) \to \mathrm{Teich}(S)$ given by $t \mapsto X_t$ is a geodesic. All geodesics in $\mathrm{Teich}(S)$ can be described in this way.

The *horizontal foliation* (respectively *vertical foliation*) associated to $\mathcal{G}$ is the collection of paths that are smooth with respect to X and whose tangent vectors are taken to positive (respectively negative) real numbers by q. Let α be a curve in S such that no representative of α is a leaf of the vertical or horizontal foliation of S corresponding to $\mathcal{G}$. Then we define the *balance time of* α *along* $\mathcal{G}$ to be the time t that minimizes $\ell_{q_t}(\alpha)$.

We define the *geodesic ray determined by* (X, q) to be $\mathcal{G}$ restricted to $[0, \infty)$. We will let Ext_t, Mod_t, and Hyp_t denote Ext_{X_t}, Mod_{X_t}, and Hyp_{X_t}, respectively.

4.2.3 Continued fractions for irrational numbers

Here we recall some elementary facts on continued fractions (see for example [RS1992]). Let θ be an irrational number with continued fraction expansion $[a_0; a_1, a_2, a_3, \ldots]$. That is,

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}\ .$$

We will always assume $a_0 \geq 0$ and all other a_n are strictly positive. We define the n^{th} *convergent of* θ to be the reduced fraction $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$. The numbers p_n and q_n are given recursively by

$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_{-1} = 0, \text{ and } \quad q_{-2} = 1 \tag{15}$$

and

$$p_n = a_n p_{n-1} + p_{n-2}, \quad p_{-1} = 1, \text{ and } \quad p_{-2} = 0$$

and satisfy

$$\frac{1}{q_n + q_{n+1}} \leq |p_n - \theta q_n| \leq \frac{1}{q_{n+1}} \tag{16}$$

and

$$|p_n q_{n+1} - q_n p_{n+1}| = 1. \tag{17}$$

A simple but useful observation is that θ and each p_n/q_n can be bounded as follows:

$$a_0 \leq \theta \leq a_0 + 1 \quad \text{and} \quad a_0 \leq p_n/q_n \leq a_0 + 1. \tag{18}$$

4.2.4 Teichmüller geodesic rays from irrational numbers

Let $\theta_0, \theta_1, \theta_2$ be irrational numbers and $0 < s < 1$ and consider the corresponding Teichmüller geodesic ray $\mathcal{G} : [0, \infty) \to \mathrm{Teich}(S)$, described in Section 4.1, parameterized by arc length. We let X_t denote $\mathcal{G}(t)$. For each i, let Y_i denote the subsurface of S that is the image of the slitted square R_i under the gluing map, and let β_i denote the boundary of Y_i. Let p_n^i/q_n^i denote the n^{th} convergent of θ_i. Let $\alpha_i(n)$ denote the curve in S corresponding to the trajectory in R_i with slope p_n^i/q_n^i. Define T_n^i to be the balance time along $\mathcal{G}$ of $\alpha_i(n)$. We shall use these notations throughout the paper. When we use them, it will be clear from context which irrational numbers and Teichmüller geodesic ray we are working with.

In [Len2008] Lenzhen gave an explicit formula for T_n^i and gave a useful bound for the extremal length of $\alpha_i(n)$ along $\mathcal{G}$.

Theorem 4.4 (Lenzhen [Len2008, Lemma 1, proof of Lemma 3])**.** *For all $n \geq 0$ and $i = 0, 1, 2$*

(1) $\mathrm{Ext}_t(\alpha_i(n)) \leq \left(\frac{\sqrt{1+\theta_i^2}}{\sqrt{1+\theta_i^2 - s|q_n^i\theta_i - p_n^i|}} \right) \ell_{q_t}^2(\alpha_i(n))$ *for $t \geq 0$.*

(2) The quadratic differential q_t induces a flat structure on the torus Y_i' obtained by ignoring the slit in Y_i. In that metric, for all $t \in [T_n^i, T_{n+1}^i]$, a shortest curve in Y_i' is $\alpha_i(n)$ or $\alpha_i(n+1)$. This statement also holds for the slitted torus Y_i using the q_t-metric. Moreover, the length of $\alpha_i(n)$ in the metric q_t induces on Y_i' is equal to $\ell_{q_t}(\alpha_i(n))$.

(3) $T_n^i = \frac{1}{2} \log \frac{p_n^i\theta_i + q_n^i}{|q_n^i\theta_i - p_n^i|}$.

Remark 4.5. There exists a constant K such that given any unit area flat structure on a torus, there is a curve of length less than K (Loewner's torus inequality). Thus, Statement (2) of Theorem 4.4 tells us that for $t \in [T_n^i, T_{n+1}^i]$, we have $\ell_{q_t}(\alpha_i(n))$ or $\ell_{q_t}(\alpha_i(n+1))$, that is the length of the q_t-shortest curve in Y_i, is bounded uniformly above.

Observe that β_i is a closed leaf in the vertical foliation associated to $\mathcal{G}$. The following theorem of Choi, Rafi, and Series gives us useful information about how the projection of $\mathcal{G}$ to $\mathcal{C}(\beta_i)$ moves through $\mathcal{C}(\beta_i)$.

Theorem 4.6 ([CRS2008, Theorem 5.13])**.** *There exists a constant ϵ_0 depending only on S such that the following holds. Let $\mathcal{G}$ be a Teichmüller geodesic with horizontal and vertical foliation ν^+ and ν^-, respectively. Suppose α is a closed leaf in ν^- and $\mathrm{Ext}_t(\alpha) \leq \epsilon_0$. Then*

$$d_\alpha(\nu^+, X_t) \prec \frac{1}{\mathrm{Hyp}_t(\alpha)}.$$

4.3 Form of accumulation points of Teichmüller geodesics

Throughout this section, for each $i = 0, 1, 2$ we fix sequences $(\theta_i(j))_{j=1}^{\infty}$ and $(n_i(j))_{j=1}^{\infty}$, where $\theta_i(j) \geq 2$ for all i, j. We then define

$$\theta_i = [0; \underbrace{\theta_i(1), \ldots, \theta_i(1)}_{n_i(1)}, \ldots, \underbrace{\theta_i(j), \ldots, \theta_i(j)}_{n_i(j)}, \ldots].$$

We fix a slit length s. We let S denote the genus 3 surface, and let $\mathcal{G} : [0, \infty) \to \mathrm{Teich}(S)$ denote the Teichmüller geodesic ray associated to $(\theta_0, \theta_1, \theta_2)$ with slit length s. Define $N_i(0) = 0$ and for $k \geq 1$ define $N_i(k) = \sum_{j=1}^{k} n_i(j)$.

In this section, through a sequence of lemmas, we will show that if the sequences $(n_i(j))_{j=1}^{\infty}$ grow sufficiently fast, then there exists $\eta_i \in \partial\mathcal{C}(Y_i)$ such that every point in the limit set of $\mathcal{G}$ is of the form $\sum_{i=0}^{2} c_{Y_i}\eta_i$ for some $c_{Y_i} \geq 0$.

We begin with Lemma 4.7, where we establish that the projection of $\mathcal{G}$ to $\mathcal{C}(Y_i)$ converges to a unique point $\eta_i \in \partial\mathcal{C}(Y_i)$. This is almost immediate from Theorem B of Rafi [Raf2014], which says that the projection of any Teichmüller geodesic to $\mathcal{C}(Y)$ is an unparameterized quasi-geodesic for every non-annular subsurface Y, but we provide a direct proof in our setting that will also reveal information about when $\mathcal{G}$ makes progress in $\mathcal{C}(Y_i)$ that will be useful later. From Lemma 4.7, it will follow that every point in the limit set of $\mathcal{G}$ is of the form

$$\sum_{i=0}^{2} \left(c_{Y_i}\eta_i + c_{\beta_i}\eta_{\beta_i}\right),$$

where η_{β_i} is the point in $\partial\mathcal{H}_{\beta_i}$. To determine what the constants c_{Y_i} and c_{β_i} can be, we must understand how fast the projection of $\mathcal{G}$ moves through each of the $\mathcal{C}(Y_i)$ and $\mathcal{H}_{\beta_i}$ for $i = 0, 1, 2$ relative to one another. In Lemma 4.7, we will see that $d_Y(X_0, X_{T_n^i}) \stackrel{+}{\asymp} n$. From here, we use Theorem 4.4 to provide useful estimates for balance times T_n^i (Lemma 4.8). We will then prove Lemma 4.10, which puts an upper bound on how fast the projection of $\mathcal{G}$ can move through a horoball $\mathcal{H}_{\beta_i}$. We use this upper bound to prove that if the sequence $(n_i(j))_{j=1}^{\infty}$ grows fast enough, then $c_{\beta_i} = 0$ (Lemma 4.12).

To simplify the notation, throughout the rest of this section, we will fix $i \in \{0, 1, 2\}$ and suppress i in all the associated notations. In particular, $Y, T_n, \theta(j), \beta$, and q_n will denote $Y_i, T_n^i, \theta_i(j), \beta_i$, and q_n^i, respectively.

Lemma 4.7. *For all $n \geq 1$,*

$$t \in [T_{n-1}, T_n] \;\Rightarrow\; d_Y(X_0, X_t) \stackrel{+}{\asymp} n.$$

Thus, the projection of $\mathcal{G}$ to $\mathcal{C}(Y)$ is an unparameterized quasi-geodesic converging to a unique point $\eta_i \in \partial\mathcal{C}(Y)$.

Proof. Let $n \geq 1$. By (17) the curves $\alpha(n-1)$ and $\alpha(n)$ are adjacent in $\mathcal{C}(Y)$. In fact, because the convergent p_n/q_n has a depth n continued fraction expansion with all but the zeroth coefficient at least 2, we have

$$d_Y(\alpha(0), \alpha(n)) = n. \tag{19}$$

(See [Ser1985]). Fix $t \in [T_{n-1}, T_n]$. By the triangle inequality, we have

$$|d_Y(X_0, X_t) - n| = |d_Y(X_0, X_t) - d_Y(\alpha(0), \alpha(n))| \leq d_Y(X_0, \alpha(0)) + d_Y(\alpha(n), X_t).$$

We now show that $d_Y(X_0, \alpha(0))$ and $d_Y(\alpha(n), X_t)$ are each bounded above by a constant depending only on S.

Consider the Euclidean cylinder A in S with core $\alpha(0)$ that is the union of the q_0-geodesic representatives of $\alpha(0)$. Because $\theta, s \in (0,1)$, we have that $\mathrm{Mod}_0(A) \geq \frac{1}{4}$. Thus, $\mathrm{Ext}_0(\alpha(0)) \leq 4$. Now for every curve β in the base of the short marking μ_0 on X_0, we also have that $\mathrm{Ext}_0(\beta)$ is bounded uniformly above by a constant depending only on S (see [Min1996a] and Theorem 2.3 in [Raf2014]). So by Inequality (2), the intersection of $\alpha(0)$ with every curve in $\mathrm{base}(\mu_0)$ is bounded above uniformly. Therefore, $d_Y(X_0, \alpha(0))$ is bounded above uniformly.

Observe that (16) together with the fact that $\theta(j) \geq 2$ for all j and $\theta, s \in (0,1)$ imply that $\frac{\sqrt{1+\theta^2}}{\sqrt{1+\theta^2} - s|q_n\theta - p_n|}$ is bounded above by a uniform constant. So, Theorem 4.4 tells us

$$\mathrm{Ext}_t(\alpha(m)) \prec \ell^2_{q_t}(\alpha(m)) \quad \text{for all} \quad m \geq 0. \tag{20}$$

Combining this with Remark 4.5, we find that $\mathrm{Ext}_t(\alpha(n-1))$ or $\mathrm{Ext}_t(\alpha(n))$ is bounded above uniformly. An argument similar to that used above for $d_Y(X_0, \alpha(0))$ together with the fact that $d_Y(\alpha(n-1), \alpha(n)) = 1$ implies $d_Y(\alpha(n), X_t)$ is bounded above by a uniform constant, as desired. Therefore,

$$d_Y(X_0, X_t) \stackrel{+}{\asymp} n.$$

Because this coarse equality is true for all n, the projection of $\mathcal{G}$ to $\mathcal{C}(Y)$ is an unparameterized quasi-geodesic. Consequently, $\{\pi_Y(X_t)\}_{t \geq 0}$ accumulates on a unique point in $\partial\mathcal{C}(Y)$. $\square$

Remark: Theorem 4.4 of Lenzhen gives us an exact formula for T_n, but this formula is insufficient for our purposes because it requires us to know θ exactly. In Lemma 4.8 we use Lenzhen's formula as a starting point to show that an initial segment of length $n+1$ of the continued fraction expansion of θ is all that is required to obtain a coarse estimate for T_n. We remark that Lenzhen, Modami, and Rafi [LMR2016] also provided a coarse estimate with this property. The estimates we present in Lemma 4.8 are more useful to us because the continued fractions we consider will have long stretches of the same number.

Before stating the lemma, for $x \in \mathbb{R}$ we define

$$\lambda(x) = \frac{x + \sqrt{x^2+4}}{2} \quad \text{and} \quad \overline{\lambda}(x) = \frac{x - \sqrt{x^2+4}}{2}.$$

Lemma 4.8. *There exists a uniform additive error such that for all $j \geq 1$ the following hold.*

1. *For all $0 \leq \ell \leq n(j)$ we have*

$$\log q_{N(j-1)+\ell} \stackrel{+}{\asymp} \log q_{N(j-1)} + \ell \log \lambda(\theta(j)).$$

2. *For all $0 \leq \ell \leq n(j) - 1$, we have*

$$T_{N(j-1)+\ell} \stackrel{+}{\asymp} \log q_{N(j-1)} + (\ell + 1/2) \log \lambda(\theta(j)).$$

Proof. Fix $j \geq 1$.

Proof of 1. Equation (15) says the q_n are given recursively by

$$q_{N(j-1)+\ell} = \theta(j) q_{N(j-1)+\ell-1} + q_{N(j-1)+\ell-2} \quad \text{when } 1 \leq \ell \leq n(j). \tag{21}$$

The solution to this recursion is

$$q_{N(j-1)+\ell} = A(j)\lambda(\theta(j))^\ell + B(j)\overline{\lambda}(\theta(j))^\ell \qquad 0 \leq \ell \leq n(j),$$

where we define

$$A(j) = \frac{q_{N(j-1)+1} - \overline{\lambda}(\theta(j)) q_{N(j-1)}}{\lambda(\theta(j)) - \overline{\lambda}(\theta(j))} \quad \text{and} \quad B(j) = \frac{q_{N(j-1)}\lambda(\theta(j)) - q_{N(j-1)+1}}{\lambda(\theta(j)) - \overline{\lambda}(\theta(j))}. \tag{22}$$

If $\ell = 0$, statement 1 is clearly true. So assume $1 \leq \ell \leq n(j)$. Observe that $\lambda(\theta(j)) > 1$ and $-1 < \overline{\lambda}(\theta(j)) < 0$. This with Equation (15) and our assumption that $\theta(j) \geq 2$ implies

$$
\begin{aligned}
\left|\frac{B(j)\overline{\lambda}(\theta(j))^{\ell}}{A(j)\lambda(\theta(j))^{\ell}}\right| &= \left|\frac{q_{N(j-1)}\lambda(\theta(j)) - q_{N(j-1)+1}}{q_{N(j-1)+1} - \overline{\lambda}(\theta(j))q_{N(j-1)}}\left(\frac{\overline{\lambda}(\theta(j))^{\ell}}{\lambda(\theta(j))^{\ell}}\right)\right| \\
&\le \left|\frac{q_{N(j-1)}\lambda(\theta(j)) - q_{N(j-1)+1}}{q_{N(j-1)+1}}\left(\frac{\overline{\lambda}(\theta(j))^{\ell}}{\lambda(\theta(j))^{\ell}}\right)\right| \\
&\le \left|\frac{\overline{\lambda}(\theta(j))^{\ell}}{\lambda(\theta(j))^{\ell-1}}\right| + \left|\frac{\overline{\lambda}(\theta(j))^{\ell}}{\lambda(\theta(j))^{\ell}}\right| \\
&\le 2|\overline{\lambda}(\theta(j))| \ \le \ 2|\overline{\lambda}(2)|.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|\log q_{N(j-1)+\ell} - \log(A(j)\lambda(\theta(j))^{\ell})| &= |\log[A(j)\lambda(\theta(j))^{\ell} + B(j)\overline{\lambda}(\theta(j))^{\ell}] - \log(A(j)\lambda(\theta(j))^{\ell})| \\
&= \left|\log\left(1 + \frac{B(j)\overline{\lambda}(\theta(j))^{\ell}}{A(j)\lambda(\theta(j))^{\ell}}\right)\right| \\
&\le |\log(1 + 2\overline{\lambda}(2))|. \qquad (23)
\end{aligned}
$$

To complete the proof of statement (1), we now show $\log A(j) \overset{+}{\asymp} \log q_{N(j-1)}$. It follows directly from (21) and (22) that

$$
\log A(j) \le \log \frac{2q_{N(j-1)+1}}{\lambda(\theta(j))} = \log \frac{2(\theta(j)q_{N(j-1)} + q_{N(j-1)-1})}{\lambda(\theta(j))} \le \log 4q_{N(j-1)},
$$

and

$$
\log A(j) \ge \log \frac{q_{N(j-1)+1}}{\lambda(\theta(j)) - \overline{\lambda}(\theta(j))} \ge \log \frac{\theta(j)q_{N(j-1)}}{2\theta(j)} \overset{+}{\succ} \log q_{N(j-1)}.
$$

Proof of 2. Let $n \ge 0$. Theorem 4.4 (Lenzhen) tells us $T_n = \frac{1}{2}\log\frac{p_n\theta+q_n}{|q_n\theta-p_n|}$. We will use this to first show that T_n is coarsely $\frac{1}{2}\log q_nq_{n+1}$. We remark that Lenzhen, Modami, and Rafi [LMR2016] obtain this same coarse estimate for the sequences they consider. Because our sequences do not fit their form, we derive the estimate for sequences in our setting.

By (16), we have

$$
\frac{p_n\theta + q_n}{|q_n\theta - p_n|} \ge (p_n\theta + q_n)q_{n+1} \ge q_nq_{n+1},
$$

and applying (16) and (18) and the fact that $q_{n+1} > q_n$, we find

$$
\frac{p_n\theta + q_n}{|q_n\theta - p_n|} \le (p_n + q_n)(q_n + q_{n+1}) \le (q_n)^2 + q_nq_{n+1} + (q_n)^2 + q_nq_{n+1} \le 4q_nq_{n+1}.
$$

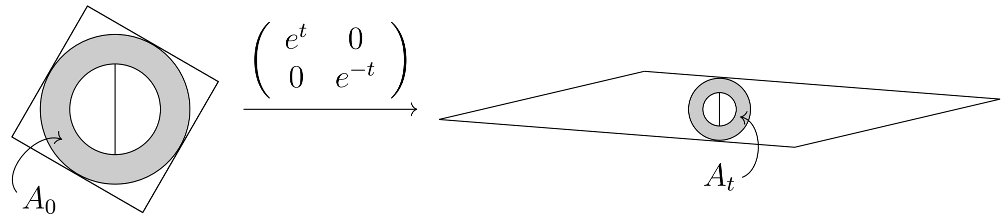

Figure 4.2: Annulus A_t in Y with core curve β_i and large modulus in $X_t \in \mathrm{Teich}(S)$.

These inequalities show that

$$T_n \stackrel{+}{\asymp} \frac{1}{2} \log q_n q_{n+1} \quad \text{for all } n \geq 0. \tag{24}$$

This together with statement 1 of the lemma implies that for $0 \leq \ell \leq n(j) - 1$

$$T_{N(j-1)+\ell} \stackrel{+}{\asymp} \log q_{N(j-1)} + (\ell + 1/2) \log(\lambda(\theta(j))).$$

□

The projection of $\mathcal{G}$ to the horoball $\mathcal{H}_\beta$ depends on whether or not the extremal length of β is small. Thus, we now show that if the slit length s happens to be small enough, then the extremal length of β is small at every time along $\mathcal{G}$.

Lemma 4.9. *If s is sufficiently small, then* $\mathrm{Ext}_t(\beta) \leq \frac{\epsilon_0}{e}$ *for all* $t \geq 0$.

Proof. Let $t \geq 0$. Because $\mathrm{Ext}_t \beta = \inf \frac{1}{\mathrm{Mod}_t A}$, where the infimum is taken over all annuli A in S with core β, to show $\mathrm{Ext}_t \beta$ is small, we exhibit such an annulus with large modulus.

Let A_t be the annulus contained in Y with core curve β and boundary components Euclidean circles in the flat q_t-metric as pictured in Figure 4.2. Let $r(t)$ and $R(t)$ denote the flat q_t-length of the inner and outer radii of A_t respectively. As we move along $\mathcal{G}$, the flat q_t-length of a segment in S shrinks at most exponentially. Thus, $R(t) \geq e^{-t} R(0)$. Observe that

$$\mathrm{Mod} A_t = \frac{1}{2\pi} \log \frac{R(t)}{r(t)} \geq \frac{1}{2\pi} \log \frac{e^{-t} R(0)}{\frac{1}{2} s e^{-t}} = \frac{1}{2\pi} \left(\log 2R(0) + \log \frac{1}{s} \right).$$

Therefore, provided that s is sufficiently small, we have $\mathrm{Ext}_t(\beta) \leq \frac{\epsilon_0}{e}$. □

Notice that how small s must be for the conclusion of Lemma 4.9 to hold is independent of θ. Thus, throughout the remainder of this paper, we can and do assume the slit length s is small enough to satisfy Lemma 4.9.

Lemma 4.10. *For all $t \geq 1$ we have*

$$d_{\mathcal{H}_\beta}(X_0, X_t) \overset{+}{\prec} \log t,$$

where the error constant depends only on s.

Proof. For each $t \geq 0$, define $h(t)$ and $v(t)$ so that $\pi_{\mathcal{H}_\beta}(X_t) = (h(t), v(t))$. Recall that two vertices at height n in $\mathcal{H}_\beta$ are adjacent if their horizontal components are within e^n of each other in $\mathcal{C}(\beta)$. By Lemma 4.9, $\mathrm{Ext}_t(\beta) \leq \frac{\epsilon_0}{e}$ for all $t \geq 0$. So the construction of $\mathcal{H}_\beta$ implies $v(t) \geq 1$ and $v(t) \overset{+}{\asymp} \log \frac{1}{\mathrm{Ext}_t \beta}$ for all $t \geq 0$. These observations together with the triangle inequality imply

$$d_{\mathcal{H}_\beta}(X_0, X_t) \overset{+}{\prec} v(0) + \log\left(\frac{1}{\mathrm{Ext}_t \beta}\right) + \frac{d_\beta(X_0, X_t)}{e^{v(t)} - 1}.$$

To establish the desired bound on $d_{\mathcal{H}_\beta}(X_0, X_t)$, our strategy is to prove the following claims, which for now we assume are true.

Claim 1: $\frac{1}{\mathrm{Ext}_t(\beta)} \prec t$ for $t \geq 0$.

Claim 2: $d_\beta(X_0, X_t) \prec \frac{1}{\mathrm{Ext}_t(\beta)}$ for $t \geq 0$.

Claim 1 implies that $v(0)$ is bounded above uniformly and if $t \geq 1$, it implies that $\log\left(\frac{1}{\mathrm{Ext}_t \beta}\right) \overset{+}{\prec} \log t$ (this is because t is bounded uniformly away from 0). Claim 2 together with the fact that $\mathrm{Ext}_t(\beta) \leq \frac{\epsilon_0}{e}$ implies that $\frac{d_\beta(X_0, X_t)}{e^{v(t)} - 1}$ is bounded above by a uniform constant. Combining these observations, we have

$$d_{\mathcal{H}_\beta}(X_0, X_t) \overset{+}{\prec} \log t \quad \text{for all} \quad t \geq 1.$$

Thus, all that remains is to prove the claims.

Proof of Claim 1. Let $t \geq 0$. We consider the flat structure determined by q_t. The flat annulus with core β is degenerate. Observe that $\ell_{q_t}(\beta) = 2se^{-t}$ and that the distance between the boundary components of the expanding annulus in the direction opposite Y is at most $\frac{1}{2}se^{-t}$. So by Theorem 4.3, the modulus of that expanding annulus is uniformly bounded above. It then follows by Theorems 4.2 and 4.3 that

$$\frac{1}{\mathrm{Ext}_t(\beta)} \asymp \log \frac{d_t}{\ell_{q_t}(\beta)} = t + \log \frac{d_t}{2s},$$

where d_t is the q_t-distance between the boundary components of the expanding annulus in the direction of Y at time t. Now d_t is at most half the length of the shortest q_t-length curve in Y at time t, which is bounded above uniformly (see Remark 4.5). So, we have $\frac{1}{\mathrm{Ext}_t(\beta)} \prec t$, establishing Claim 1. $\square$

Proof of Claim 2. For $t \geq 0$ let μ_t be a short marking on X_t. Because $\mathrm{Ext}_t(\beta) \leq \epsilon_0$, we know $\beta \in \mathrm{base}(\mu_t)$. This tells us $\pi_\beta(X_t)$ is the projection to $\mathcal{C}(\beta)$ of the transversal in μ_t associated to β. Let ν^+ denote the horizontal foliation of $\mathcal{G}$. Because β is a leaf of the vertical foliation of $\mathcal{G}$, by Theorem 4.6

$$d_\beta(\nu^+, X_t) \prec \frac{1}{\mathrm{Hyp}_t(\beta)}. \tag{25}$$

Further observe that because $\mathrm{Ext}_t(\beta) \leq \epsilon_0$, Theorem 4.1 tells us that $\frac{1}{\mathrm{Hyp}_t(\beta)} \overset{*}{\asymp} \frac{1}{\mathrm{Ext}_t(\beta)}$. So (25) and Claim 1 imply that

$$d_\beta(X_0, X_t) \leq d_\beta(\nu^+, X_0) + d_\beta(\nu^+, X_t) \prec \frac{1}{\mathrm{Ext}_0(\beta)} + \frac{1}{\mathrm{Ext}_t(\beta)} \prec \frac{1}{\mathrm{Ext}_t(\beta)},$$

proving Claim 2 and thus completing the proof of the lemma. $\square$

$\square$

Convention 4.11. Throughout the rest of this paper, when we say the sequence $(n_i(j))_{j=1}^\infty$ *grows sufficiently fast* we shall mean that for each k we have $n_i(k)$ is larger than some function f_k of the numbers in $(n_\ell(j))_{j=1}^{k-1}$ and $(\theta_\ell(j))_{j=1}^{k+1}$ for each $\ell \in \{0, 1, 2\}$, where the functions vary based on the context in which this phrase is used.

Lemma 4.12. *If the sequence $(n(j))_{j=1}^\infty$ grows sufficiently fast, then*

$$\frac{d_{\mathcal{H}_\beta}(X_0, X_t)}{d_Y(X_0, X_t)} \to 0 \quad as \quad t \to \infty.$$

Proof. Consider $t \geq T_{N(1)} \geq 1$. For some $k \geq 2$ and $0 \leq \ell \leq n(k) - 1$ we have that

$$T_{N(k-1)+\ell-1} < t \leq T_{N(k-1)+\ell}.$$

Regardless of how fast the $n(j)$ are growing, the following will be true. Lemmas 4.10 and

4.8 imply that

$$\begin{aligned}
d_{\mathcal{H}_\beta}(X_0, X_t) \overset{+}{\prec} \log t \le \log T_{N(k-1)+\ell} \\
\overset{+}{\prec} \log\left(\log q_{N(k-1)} + \left(\ell + \frac{1}{2}\right)\log\lambda(\theta(k))\right) \\
\overset{+}{\prec} \log\left(\log q_{N(k-2)} + n(k-1)\log\lambda(\theta(k-1)) + \left(\ell + \frac{1}{2}\right)\log\lambda(\theta(k))\right),
\end{aligned} \tag{26}$$

and Lemma 4.7 implies that

$$d_Y(X_0, X_t) \overset{+}{\succ} N(k-1) + \ell = N(k-2) + n(k-1) + \ell. \tag{27}$$

Observe that $N(k-2)$, $q_{N(k-2)}$, $\lambda(\theta(k-1))$, and $\lambda(\theta(k))$ are completely determined by $(\theta(j))_{j=1}^{k}$ and $(n(j))_{j=1}^{k-2}$, and thus are completely independent of $n(k-1)$. Further observe that if $n(k-1)$ is sufficiently large relative to the numbers in $(\theta(j))_{j=1}^{k}$ and $(n(j))_{j=1}^{k-2}$, then the ratio of the upper bound of (26) to the lower bound of (27) is arbitrarily small, implying that $\frac{d_{\mathcal{H}_\beta}(X_0,X_t)}{d_Y(X_0,X_t)}$ is also small. This proves the lemma. $\square$

Remark 4.13. The conclusion of Lemma 4.12 holds under a weaker hypothesis. With only a little more work, the result can be obtained by only assuming that $n(k)$ is larger than some function f_k of $(\theta(j))_{j=1}^{k+1}$. In fact, if $(\theta(j))_{j=1}^{\infty}$ is a bounded sequence, then $(n(j))_{j=1}^{\infty}$ need not grow at all. However, when proving Lemmas 4.14 and 4.15 more is required of $(n(j))_{j=1}^{\infty}$. It is with those lemmas and the simpler proof of Lemma 4.12 that we make our definition of sufficiently fast growth.

4.4 Teichmüller geodesics with exotic limit sets

Throughout this section, we fix s sufficiently small in the sense of Lemma 4.9. We fix a continuous map $\gamma\colon \mathbb{R} \to \triangle^2$ to the standard 2-simplex in $\mathbb{R}^3$, and let γ_i denote the i^{th} component function of γ. In this section, we will show how to carefully choose infinite sequences $(\theta_i(j))_{j=1}^{\infty}$ and $(n_i(j))_{j=1}^{\infty}$ for $i = 0, 1, 2$ and an embedding $\triangle^2 \to \partial\mathrm{Teich}(S)$ so that the limit set of the associated Teichmüller geodesic ray is the image of $\overline{\gamma(\mathbb{R})}$, proving Theorem 1.4.

We also fix a sequence $(t_j)_{j=1}^{\infty}$ in $\mathbb{R}$ so that $(\gamma(t_j))_{j=1}^{\infty}$ is dense in $\gamma(\mathbb{R})$ and

$$|\gamma_i(t_{j-1}) - \gamma_i(t_j)| < \epsilon_j \quad \text{for each} \quad i = 0, 1, 2 \quad \text{and } j \ge 2, \tag{28}$$

where (ϵ_j) is some decreasing sequence, $\epsilon_j < \frac{1}{2}$, and $\lim_{j\to\infty} \epsilon_j = 0$.

Let L denote the additive error in the coarse estimates of Lemma 4.8. Now for $i = 0, 1, 2$ choose the sequence $(\theta_i(j))_{j=1}^{\infty}$ so that the following hold for all $j \geq 1$ and $i, \ell = 0, 1, 2$:

$$\log \lambda(\theta_i(j)) \geq 4L, \tag{29}$$

$$\frac{\log \lambda(\theta_\ell(j))}{\log \lambda(\theta_i(j+1))} < \epsilon_{j+1}, \tag{30}$$

and

$$\left| \frac{[\log \lambda(\theta_i(j))]^{-1}}{\sum_{\ell=0}^{2} [\log \lambda(\theta_\ell(j))]^{-1}} - \gamma_i(t_j) \right| < \epsilon_{j+1}. \tag{31}$$

Note that it is always possible to find sequences $(\theta_i(j))_{j=1}^{\infty}$ satisfying (29), (30), and (31) by choosing $\theta_\ell(k)$ for all $\ell = 0, 1, 2$ and $1 \leq k \leq j-1$ before choosing $\theta_i(j)$.

Given a sequence $(n_i(j))_{j=1}^{k}$, $i = 0, 1, 2$, we define $N_i(0) = 0$ and $N_i(k) = \sum_{j=1}^{k} n_i(j)$ for $k \geq 1$. When we say the Teichmüller geodesic ray corresponding to $(n_i(j))_{j=1}^{\infty}$, $i = 0, 1, 2$, we shall mean the Teichmüller geodesic ray $\mathcal{G}$ with slit length s associated to $(\theta_0, \theta_1, \theta_2)$, where

$$\theta_i = [0; \underbrace{\theta_i(1), \ldots, \theta_i(1)}_{n_i(1)}, \ldots, \underbrace{\theta_i(j), \ldots, \theta_i(j)}_{n_i(j)}, \ldots].$$

We then use $\mathcal{G}$ to define a map $\phi\colon [0, \infty) \to \triangle^2$ given by

$$t \;\mapsto\; \frac{1}{\sum_{i=0}^{2} d_{Y_i}(X_0, X_t)} (d_{Y_0}(X_0, X_t), d_{Y_1}(X_0, X_t), d_{Y_2}(X_0, X_t)),$$

where as usual X_t is the point on $\mathcal{G}$ distance t from the base of the ray. We will write ϕ_i to denote the i^{th} component of ϕ.

We will show how to pick the $(n_i(j))_{j=1}^{\infty}$ so that if t is between the balance times $T^0_{N_0(k-1)-1}$ and $T^0_{N_0(k)-1}$, then for each i we have $\phi_i(t)$ is close to

$$\frac{[\log \lambda(\theta_i(k-1))]^{-1}}{\sum_{\ell=0}^{2} [\log \lambda(\theta_\ell(k-1))]^{-1}} \quad \text{or} \quad \frac{[\log \lambda(\theta_i(k))]^{-1}}{\sum_{\ell=0}^{2} [\log \lambda(\theta_\ell(k))]^{-1}}$$

and thus close to $\gamma_i(t_{k-1})$ or $\gamma_i(t_k)$, which are close to each other by (28). As a first step toward this goal, we prove the following lemma. (See Convention 4.11 for our definition of sufficiently fast growth.)

Lemma 4.14. *For each $i = 0, 1, 2$ suppose $(n_i(j))_{j=1}^{\infty}$ grows sufficiently fast and that the balance times along the corresponding Teichmüller geodesic ray satisfy*

$$T^i_{N_i(k)-3} < T^0_{N_0(k)-1} \leq T^i_{N_i(k)-1} \qquad \textit{for all } \; k \geq 1 \;\; \textit{and} \;\; i = 0, 1, 2.$$

Then for all $k \geq 2$ and $i = 0, 1, 2$

$$t \in [T^0_{N_0(k-1)-1}, T^0_{N_0(k)-1}) \quad \Longrightarrow \quad |\phi_i(t) - \gamma_i(t_k)| \leq 11\epsilon_k. \tag{32}$$

Proof. Let $\mathcal{G}\colon [0, \infty) \to \mathrm{Teich}(S)$ be the Teichmüller geodesic associated to $(n_i(j))_{j=1}^{\infty}$, $i = 0, 1, 2$. Fix $k \geq 2$ and $t \in [T^0_{N_0(k-1)-1}, T^0_{N_0(k)-1})$. For each $i = 0, 1, 2$, define $m(i)$ so that

$$T^i_{N_i(k-1)+m(i)-1} < t \leq T^i_{N_i(k-1)+m(i)}. \tag{33}$$

Fix $\ell \in \{0, 1, 2\}$. Then Lemma 4.7 implies that $\phi_\ell(t)$ is bounded above and below as follows:

$$\frac{N_\ell(k-1) + m(\ell) - R}{\sum_{i=0}^{2}(N_i(k-1) + m(i) + R)} \leq \frac{d_{Y_\ell}(X_0, X_t)}{\sum_{i=0}^{2} d_{Y_i}(X_0, X_t)} \leq \frac{N_\ell(k-1) + m(\ell) + R}{\sum_{i=0}^{2}(N_i(k-1) + m(i) - R)}, \tag{34}$$

where $R > 0$ denotes the additive error from Lemma 4.7. Thus, to bound $\phi_\ell(t)$ we must compare $m(\ell)$ to $m(i)$ for each $i = 0, 1, 2$.

First, observe our assumption that

$$T^i_{N_i(k-1)-3} < T^0_{N_0(k-1)-1} \quad \text{and} \quad T^0_{N_0(k)-1} \leq T^i_{N_i(k)-1}$$

implies $-2 \leq m(i) \leq n_i(k) - 1$. This means that $m(i) + 2 \geq 0$. Now by the definition of the $m(i)$, we know that

$$T^\ell_{N_\ell(k-1)+m(\ell)-1} \leq T^i_{N_i(k-1)+m(i)} \leq T^i_{N_i(k-1)+m(i)+2}. \tag{35}$$

So provided that $m(\ell) - 1 \geq 0$, (35) together with Lemma 4.8 part 2 implies that

$$\log q^\ell_{N_\ell(k-1)} + \left(m(\ell) - \frac{1}{2}\right) \log \lambda(\theta_\ell(k)) - L \leq \log q^i_{N_i(k-1)} + \left(m(i) + \frac{5}{2}\right) \log \lambda(\theta_i(k)) + L. \tag{36}$$

Now (36), Lemma 4.8 part 1, and (30) imply

$$
\begin{aligned}
m(i) &\geq -\frac{5}{2} - \frac{\log q^i_{N_i(k-1)} + 2L + \frac{1}{2}\log\lambda(\theta_\ell(k))}{\log\lambda(\theta_i(k))} + \frac{\log\lambda(\theta_\ell(k))}{\log\lambda(\theta_i(k))} m(\ell) \\
&\geq -\frac{5}{2} - \frac{\log q^i_{N_i(k-2)} + 3L + \frac{1}{2}\log\lambda(\theta_\ell(k))}{\log\lambda(\theta_i(k))} - \epsilon_k N_i(k-1) + \frac{\log\lambda(\theta_\ell(k))}{\log\lambda(\theta_i(k))} m(\ell) \\
&= -H_{i,\ell}(k) - \epsilon_k N_i(k-1) + \frac{\log\lambda(\theta_\ell(k))}{\log\lambda(\theta_i(k))}(m(\ell)+2), \qquad (37)
\end{aligned}
$$

where $H_{i,\ell}(k)$ is defined precisely so that the equality holds. Notice that $H_{i,\ell}(k)$ is completely determined by the finite sequences $(n_i(j))_{j=1}^{k-2}$ and $(\theta_i(j))_{j=1}^{k}$ and $\theta_\ell(k)$. Further observe that the lower bound (37) for $m(i)$ still holds even if $m(\ell) \leq 0$ since $m(i) \geq -2$ and $\log q^i_{N_i(k-2)}$, L, $\log\lambda(\theta_i(k))$, and $\log\lambda(\theta_\ell(k))$ are all greater than 0.

We assume $n_i(j) \geq 3$ for all i, j (as part of our sufficiently fast growth assumption). Lemma 4.8 part 2 and our assumption on the balance times tell us

$$
\frac{N_i(k-1)+R}{T^0_{N_0(k-1)-1}} \leq \frac{N_i(k-1)+R}{T^i_{N_i(k-1)-3}} \leq \frac{N_i(k-2)+n_i(k-1)+R}{\log q^i_{N_i(k-2)} + (n_i(k-1)-\frac{5}{2})\log\lambda(\theta_i(k-1)) - L} \qquad (38)
$$

and

$$
\begin{aligned}
\frac{N_i(k-1) - 2H_{i,\ell}(k) - 2R}{T^0_{N_0(k-1)-1}} &\geq \frac{N_i(k-1) - 2H_{i,\ell}(k) - 2R}{T^i_{N_i(k-1)-1}} \\
&\geq \frac{N_i(k-2)+n_i(k-1) - 2H_{i,\ell}(k) - 2R}{\log q^i_{N_i(k-2)} + (n_i(k-1)-\frac{1}{2})\log\lambda(\theta_i(k-1)) + L}. \qquad (39)
\end{aligned}
$$

If $n_i(k-1)$ is sufficiently large, the right hand sides of Inequalities (38) and (39) are both greater than 0 and very close to $1/\log\lambda(\theta_i(k-1))$. The crucial part of this proof is to notice that how large $n_i(k-1)$ must be to guarantee a certain prescribed closeness is determinable from $\theta_\ell(k)$ and the numbers in the finite sequences $(n_i(j))_{j=1}^{k-2}$ and $(\theta_i(j))_{j=1}^{k}$. Thus, if $(n_i(j))_{j=1}^{\infty}$ grows sufficiently fast for each $i = 0, 1, 2$, then

$$
0 < \frac{N_\ell(k-1)+R}{\sum_{i=0}^{2}(N_i(k-1) - 2H_{i,\ell}(k) - 2R)} \leq \frac{[\log\lambda(\theta_\ell(k-1))]^{-1}}{\sum_{i=0}^{2}[\log\lambda(\theta_i(k-1))]^{-1}} + \epsilon_k,
$$

which together with (37), (31), and (28) implies that

$$\begin{aligned}
\frac{d_{Y_\ell}(X_0,X_t)}{\sum\limits_{i=0}^{2} d_{Y_i}(X_0,X_t)} &\leq \frac{N_\ell(k-1)+m(\ell)+R}{\sum\limits_{i=0}^{2}(N_i(k-1)+m(i)-R)} \\
&\leq \left(\frac{1}{1-\epsilon_k}\right) \frac{N_\ell(k-1)+R+(m(\ell)+2)}{\sum\limits_{i=0}^{2}(N_i(k-1)-2H_{i,\ell}(k)-2R)+\left(\sum\limits_{i=0}^{2}\frac{\log\lambda(\theta_\ell(k))}{\log\lambda(\theta_i(k))}\right)(m(\ell)+2)} \\
&\leq \left(\frac{1}{1-\epsilon_k}\right) \max\left\{\frac{N_\ell(k-1)+R}{\sum\limits_{i=0}^{2}(N_i(k-1)-2H_{i,\ell}(k)-2R)}, \frac{[\log\lambda(\theta_\ell(k))]^{-1}}{\sum\limits_{i=0}^{2}[\log\lambda(\theta_i(k))]^{-1}}\right\} \\
&\leq \left(\frac{1}{1-\epsilon_k}\right) \max\left\{\frac{[\log\lambda(\theta_\ell(k-1))]^{-1}}{\sum\limits_{i=0}^{2}[\log\lambda(\theta_i(k-1))]^{-1}}+\epsilon_k, \frac{[\log\lambda(\theta_\ell(k))]^{-1}}{\sum\limits_{i=0}^{2}[\log\lambda(\theta_i(k))]^{-1}}\right\} \\
&\leq \left(\frac{1}{1-\epsilon_k}\right)(\gamma_\ell(t_k)+3\epsilon_k) \leq \gamma_\ell(t_k)+11\epsilon_k.
\end{aligned}$$

Note that the last inequality follows because $\gamma(t_k) \in \triangle^2$ implies $\gamma_\ell(t_k) \leq 1$, and $\epsilon_k < \frac{1}{2}$ implies $\frac{1}{1-\epsilon_k} \leq 1+2\epsilon_k$.

A similar argument shows that if $(n_i(j))_{j=1}^{\infty}$ grows sufficiently fast for all $i = 0,1,2$, then

$$\frac{d_{Y_\ell}(X_0,X_t)}{\sum\limits_{i=0}^{2} d_{Y_i}(X_0,X_t)} \geq \gamma_\ell(t) - 11\epsilon_k.$$

□

The goal of the next lemma is to show that sequences satisfying the hypotheses of Lemma 4.14 can actually be constructed.

Lemma 4.15. *Let $k \geq 1$. Given $(n_i(j))_{j=1}^{k-1}$, $i = 0,1,2$ and any number N, for each i there exists $n_i(k) \geq N$ so that the following holds. If θ_i is any irrational number whose continued fraction expansion begins with*

$$0, \underbrace{\theta_i(1),\ldots,\theta_i(1)}_{n_i(1)}, \ldots, \underbrace{\theta_i(k),\ldots,\theta_i(k)}_{n_i(k)}$$

for each i, then the balance times on the Teichmüller geodesic ray corresponding to $(\theta_0,\theta_1,\theta_2)$ satisfy

$$T^i_{N_i(k)-3} < T^0_{N_0(k)-1} \leq T^i_{N_i(k)-1} \qquad \textit{for} \;\; i = 0,1,2.$$

Proof. Suppose we are given $(n_i(j))_{j=1}^{k-1}$ for each i and a number N, which we may assume is at least 3. Choose $n_0(k) \geq N$ so that for each $i = 1, 2$ if ℓ is an integer satisfying

$$\log q^0_{N_0(k-1)} + \left(n_0(k) - \frac{1}{2}\right) \log \lambda(\theta_0(k)) + L \leq \log q^i_{N_i(k-1)} + \left(\ell - \frac{1}{2}\right) \log \lambda(\theta_i(k)) - L, \quad (40)$$

then $\ell \geq N$. Now for $i = 1, 2$ choose $n_i(k)$ to be the smallest integer ℓ satisfying Inequality (40). Then we have $n_i(k) \geq N \geq 3$ for all $i = 0, 1, 2$.

For each $i = 0, 1, 2$, let θ_i be any irrational number whose continued fraction expansion begins with

$$0, \underbrace{\theta_i(1), \ldots, \theta_i(1)}_{n_i(1)}, \ldots, \underbrace{\theta_i(k), \ldots, \theta_i(k)}_{n_i(k)}.$$

Consider the Teichmüller geodesic ray corresponding to $(\theta_0, \theta_1, \theta_2)$. Lemma 4.8 part 2 and (40) tells us that $T^0_{N_0(k)-1} \leq T^i_{N_i(k)-1}$ for each i.

We now show that because we chose $\theta_i(k)$ to be large relative to L, then necessarily $T^i_{N_i(k)-3} < T^0_{N_0(k)-1}$. Observe that

$$\begin{aligned}
T^i_{N_i(k)-3} &\leq \log q^i_{N_i(k-1)} + \left(n_i(k) - \frac{5}{2}\right) \log \lambda(\theta_i(k)) + L && \text{by Lemma 4.8 part 2} \\
&\leq \log q^i_{N_i(k-1)} + \left(n_i(k) - \frac{3}{2}\right) \log \lambda(\theta_i(k)) - 3L && \text{by Eq.(29)} \\
&< \log q^0_{N_0(k-1)} + \left(n_0(k) - \frac{1}{2}\right) \log \lambda(\theta_0(k)) - L && \text{by def. of } n_i(k) \\
&\leq T^0_{N_0(k)-1} && \text{by Lemma 4.8 part 2.}
\end{aligned}$$

□

We now use Lemmas 4.14 and 4.15 to prove our main result, Theorem 1.4, which we rephrase as Theorem 4.16 below.

Theorem 4.16. *There exists a triple of irrational numbers such that the limit set in $\partial\,Teich(S)$ of the associated Teichmüller geodesic ray is*

$$\{c_0\eta_0 + c_1\eta_1 + c_2\eta_2 : (c_0, c_1, c_2) \in \overline{\gamma(\mathbb{R})}\}$$

for some $\eta_i \in \partial\mathcal{C}(Y_i)$, $i = 0, 1, 2$.

Proof. By Lemma 4.15, we can choose sequences $(n_i(j))_{j=1}^{\infty}$ growing sufficiently fast in the sense of both Lemmas 4.12 and 4.14 such that the corresponding geodesic ray

$\mathcal{G} : [0, \infty) \to \mathrm{Teich}(S)$ satisfies

$$T^i_{N_i(k)-3} < T^0_{N_0(k)-1} \leq T^i_{N_i(k)-1} \quad \text{for all} \quad k \geq 1.$$

(See Convention 4.11 for our definition of sufficiently fast growth.) We can now apply Lemmas 4.12 and 4.14 to conclude that for each $i = 0, 1, 2$ and $k \geq 2$

$$\lim_{t \to \infty} \frac{d_{\mathcal{H}_{\beta_i}}(X_0, X_t)}{d_{Y_i}(X_0, X_t)} = 0 \tag{41}$$

and

$$t \in [T^0_{N_0(k-1)-1}, T^0_{N_0(k)-1}) \implies |\phi_i(t) - \gamma_i(t_k)| \leq 11\epsilon_k. \tag{42}$$

Let $\mathcal{L}$ denote the limit set of $\mathcal{G}$ in $\partial\mathrm{Teich}(S)$. By the definition of the topology of $\mathrm{Teich}(S) \cup \partial\mathrm{Teich}(S)$, Equation (41) and Lemma 4.7 imply that for some $\eta_i \in \partial\mathcal{C}(Y_i)$

$$\mathcal{L} = \{c_0\eta_0 + c_1\eta_1 + c_2\eta_2 : (c_0, c_1, c_2) \in \mathcal{L}_\phi\},$$

where $\mathcal{L}_\phi$ denotes the set of accumulation points of ϕ in $\triangle^2$. So, to complete this proof, we must show that $\mathcal{L}_\phi = \overline{\gamma(\mathbb{R})}$.

Throughout the rest of the proof, we think of $\triangle^2$ as a subset of $\mathbb{R}^3$ equipped with the ℓ_1-norm. Consider a point $P \in \overline{\gamma(\mathbb{R})}$. Because $(\gamma(t_j))_{j=1}^\infty$ is a sequence dense in $\gamma(\mathbb{R})$, some subsequence $\gamma(t_{j_n}) \to P$ as $n \to \infty$. Observe that (42) tells us that $\phi(T^0_{N_0(j_n-1)-1})$ is within $33\epsilon_{j_n}$ of $\gamma(t_{j_n})$. Since $\epsilon_{j_n} \to 0$ as $n \to \infty$, it must be that $\phi(T^0_{N_0(j_n-1)-1}) \to P$ as $n \to \infty$. Therefore, $P \in \mathcal{L}_\phi$, which establishes that $\overline{\gamma(\mathbb{R})} \subseteq \mathcal{L}_\phi$.

We now establish that $\mathcal{L}_\phi \subseteq \overline{\gamma(\mathbb{R})}$. For each $p \in \triangle^2 \setminus \overline{\gamma(\mathbb{R})}$, there exists $\epsilon > 0$ such that p is not contained in the closed 33ϵ-neighborhood of $\gamma(\mathbb{R})$, denoted by $N_{33\epsilon}(\gamma(\mathbb{R}))$. Now choose $K \geq 2$ so that $\epsilon_j < \epsilon$ for all $j \geq K$. It follows from (42) that

$$\phi[T^0_{N_0(K-1)-1}, \infty) = \bigcup_{j=K}^\infty \phi[T^0_{N_0(j-1)-1}, T^0_{N_0(j)-1}) \subseteq \bigcup_{j=K}^\infty N_{33\epsilon_j}(\gamma(t_j)) \subseteq N_{33\epsilon}(\gamma(\mathbb{R})).$$

Therefore, $p \notin \mathcal{L}_\phi$, which establishes that $\mathcal{L}_\phi \subseteq \overline{\gamma(\mathbb{R})}$. $\square$

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