LEAST DILATATION OF PURE SURFACE BRAIDS

BY

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DISSERTATION

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Abstract

This thesis finds its roots in the Nielsen-Thurston classification of the mapping class group, a result that is fundamental to the field of low dimensional topology. In particular, Thurston's work gives us a powerful normal form for mapping classes: up to taking powers and restricting to subsurfaces, every mapping class can be decomposed into pieces which are either the identity or pseudo-Anosov. Associated to each of these pseudo-Anosov mapping classes is a unique algebraic number called its dilatation or "stretch-factor". In this thesis, we build on work of Penner who introduced the study of the minimal dilatation of pseudo-Anosovs in subgroups of the mapping class group. We prove upper and lower bounds on the minimal dilatation of pseudo-Anosovs in the n-stranded pure surface braid group extending results of Aougab–Taylor and Dowdall for the 1-stranded pure surface braid group.

‘A ‘ohe loa i ka hana a ke aloha.

To my Ming for loving me across oceans and continents for so many years and for at long last standing by my side through this final year of our doctoral degrees.

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Prologue

I used to think the hardest part of grad school would be the math. I was wrong. Now I'm not saying math isn't hard. Trust me, math IS hard. It's way harder than I thought it was when I was a bright eyed and bushy tailed undergrad at a tiny school that no one has ever heard of. But even harder than the math is the doubt whispered (and sometimes shouted) at you from every side until it seeps cold and grey into your soul. I used to think there was nothing worse than failing, but once again I was wrong. If you fail, at least that means you had the courage to try. Doubt is relentlessly insidious. It makes you so certain of your own inability to succeed that you become convinced there is no use in trying before you have even got started.

I can't begin to tell you the number of times I have spent hours staring at a blank white sheet of paper afraid to start working because, let's be honest, no one thinks I can solve this problem, not even me. And before you tell me that all this doubt is just imagined, a conjuring of my own mind, born out of a long standing fear of failure (and a lack of practicing how to fail), which to be fair wouldn't be such a far fetched explanation, let me tell you a story. You see during my first year of grad school I realized I didn't have nearly the preparation that many of my classmates had and what I thought was a thorough knowledge of various topics which I had ambitiously sought out in my undergraduate years turned out to be little more than what would be covered in the first day of my graduate lectures. Now that's a difficult truth to press your hands up against, finding out that you are at the bottom of a ladder that you can't even see the top of when you used to think you were at least a fair ways up. But so what if I was a little behind, I was sure I could catch up before long.

Of course, the doubt was already whispering to me at this point, and had found a voice among a few of my classmates, but what did they know, they were only a few steps ahead of me. So I shook it off and kept my head down, plugging away. I found an area of research I liked, and a professor I liked and I thought things were starting to take shape. Till the beginning of my fourth semester, when my doubts materialized in the form of the following words spoken by my would-be advisor, “I don’t think you have it in you to write a good enough thesis to do research.”

My dreams evaporated right there in that dusty little office and unrelenting doubt took their place. I could hear the words playing over and over again in my head and all I could think was, “You can’t do this. You can’t get your PhD. You can’t do math. You can’t do anything. You are a failure.” I couldn’t take this opinion for granted. I couldn’t dismiss it, because it came from the mouth of one of my most respected professors, a leader in his field, a true mathematician. If he said it, then it must be so. Who could know better than he? That day the doubt wasn’t a whisper anymore. It was screaming into my ear, louder than any words of affirmation I have ever heard. Doubt said, “You are not enough!” And after that day, I believed it.

Chapter 1

Introduction

Let $S_{g,n}$ be a connected, oriented surface of genus $g \geq 2$ with $n \geq 1$ punctures and let $S_g = S_{g,0}$. We define the **mapping class group** of $S_{g,n}$, denoted by $\mathrm{Mod}(S_{g,n})$, to be the group of orientation preserving homeomorphisms of $S_{g,n}$ up to isotopy. The **pure mapping class group** of $S_{g,n}$, denoted $\mathrm{PMod}(S_{g,n})$, is the subgroup of $\mathrm{Mod}(S_{g,n})$ that fixes each puncture pointwise.

Consider the following short exact sequence

$$1 \longrightarrow \ker(\mathcal{F}orget) \longrightarrow \mathrm{PMod}(S_{g,n}) \longrightarrow \mathrm{Mod}(S_g) \longrightarrow 1, \tag{1}$$

where $\mathcal{F}orget : \mathrm{PMod}(S_{g,n}) \to Mod(S_g)$ is the forgetful map obtained by "filling in" the n punctures of $S_{g,n}$. This sequence is often refered to as the Birman Exact Sequence. The **n-stranded pure surface braid group** of a surface of genus g is defined to be the kernel of this forgetful map and is denoted by $\mathrm{PB}_n(S_g)$. Note that $\ker(\mathcal{F}orget)$ is isomorphic to the fundamental group of the configuration space of ordered n-tuples of points on S_g; see Section 2.2 for further discussion.

Given a **pseudo-Anosov** mapping class $f \in \mathrm{PB}_n(S_g)$ we denote its **dilatation** by $\lambda(f)$ and its **entropy** by $\log(\lambda(f))$, which is indeed the topological entropy of the pseudo-Anosov representative of f. In particular, we will be interested in the least entropy

$$L(\mathrm{PB}_n(S_g)) := \inf\{\log(\lambda(f)) \mid f \in \mathrm{PB}_n(S_g) \text{ is pseudo-Anosov}\}.$$

Main Theorem. *For a surface $S_{g,n}$ of genus $g \geq 2$ with $n \geq 1$ punctures there exist constants $c, c' > 0$ such that,*

$$c \log\left(\left\lceil \frac{\log g}{n} \right\rceil\right) + c \leq L(\mathrm{PB}_n(S_g)) \leq c' \log\left(\left\lceil \frac{g}{n} \right\rceil\right) + c'.$$

Explicit values for c and c' are obtained from the bounds given in Theorem 3.1, Theorem 4.1, and Theorem 4.2.

To put the Main Theorem in context, we recall the results of Penner [34] and Tsai [40], which give bounds on the least entropy in the full mapping class group (Penner for closed surfaces and Tsai for punctured surfaces).

Theorem 1.1 (Penner)**.** *For a surface S_g of genus $g \geq 2$,*

$$\frac{\log 2}{12g - 12} \leq L(\mathrm{Mod}(S_g)) \leq \frac{\log 11}{g}.$$

The constants in Penner's bounds have been improved by many authors; see Aaber–Dunfield [1], Bauer [6], Hironaka [19], Hironaka–Kin [20], Kin–Takasawa [25], and McMullen [31]. In particular, the best known upper bound, given by Hironaka [19], is

$$L(\mathrm{Mod}(S_g)) \leq \log\left(\frac{3+\sqrt{5}}{2}\right),$$

while the lower bound has also been sharpened by McMullen [31] to

$$\frac{\log 2}{6g-6} \leq L(\mathrm{Mod}(S_g)).$$

Theorem 1.2 (Tsai)**.** *For any fixed $g \geq 2$, there is a constant $c_g \geq 1$ depending on g such that, for all $n \geq 3$,*

$$\frac{\log n}{c_g n} < L(\mathrm{Mod}(S_{g,n})) < \frac{c_g \log n}{n}.$$

The constant c_g in Tsai's result was improved from an exponential dependence on genus to a polynomial one by Yazdi [41].

Theorem 1.1 shows that $L(\mathrm{Mod}(S_g))$ goes to 0 as g tends to infinity and Theorem 1.2 shows that, for fixed genus g, $L(\mathrm{Mod}(S_{g,n}))$ goes to 0 as n tends to infinity. Theorems 1.1 and 1.2 contrast sharply with the behavior of the least entropy in the pure surface braid group demonstrated by the Main Theorem, which shows that $L(PB_n(S_g))$ is bounded away from 0: in fact, for any fixed number of punctures n, $L(PB_n(S_g))$ tends to infinity as g tends to infinity.

In addition to studying the least entropy of the mapping class group, many people have studied the least entropy of various subgroups of the mapping class group. For example, Farb–Leininger–Margalit studied the minimal entropy of the Torelli group, the Johnson kernel, and congruence subgroups in [12] and Hirose–Kin studied the least entropy of hyperelliptic handlebody groups in [21]. The least entropy of classical pure braid groups, that is the fundamental group of the configuration space of ordered n-tuples of points in the complex plane, has also been an object of significant study. Song provided upper and lower bounds for the least entropy of the classical braid groups in [36]. Specific values of the least entropy were found when $n = 4$ and $n = 5$ by Song–Ko–Los [37] and Ham–Song [17], respectively. More recently, Lanneau–Thiffeault [27] gave simple constructions to realize the least entropy for $n = 4, 5$ and found the least entropy for braid groups of up to 8-strands. The entropy of pseudo-Anosovs in the point pushing subgroup was also studied extensively by Dowdall in [10]. Note that the point pushing subgroup coincides with the 1-stranded pure surface braid group $\mathrm{PB}_1(S_g)$. Combining the upper bound of Aougab and Taylor [5] and the lower bound of Dowdall [10] gives the following.

Theorem 1.3 (Aougab–Taylor, Dowdall)**.** *For the closed surface S_g of genus $g \geq 2$,*

$$\frac{1}{5}\log(2g) \leq L(\mathrm{PB}_1(S_g)) < 4\log(g) + 2\log(24).$$

For fixed genus, the upper bound in our Main Theorem interpolates between the $\log(g)$

upper bound in Theorem 1.3 in the case of a single puncture and a constant upper bound of $4\log(6)$ when $n > 2g$; see Theorem 3.1.

Dilatations of pseudo-Anosov mapping classes have been studied in a number of other situations; see [32, 22, 29, 35, 33]. In fact, an analogous problem to ours on small dilatation pseudo-Anosovs has been studied in the context of nonorientable surfaces by Liechti and Strenner [28].

Chapter 2

Background

Here we establish our notation for the remainder of this thesis and recall the necessary notions, definitions, and tools.

2.1 The Mapping Class Group

Let $S = S_{g,n}$ be a connected, oriented surface of genus $g \geq 2$ with $n \geq 0$ punctures and let $f : S \to S$ be a homeomorphism. Throughout the rest of the paper we will assume any surface we discuss is as described here.

Definition 2.1. *The homemorphism f is called* **periodic** *or* **finite order***, if f^k is isotopic to the identity for some $k > 0$.*

The most trivial example of a periodic homeomorphism is the identity. Another example is the rotation shown in Figure 2.1.

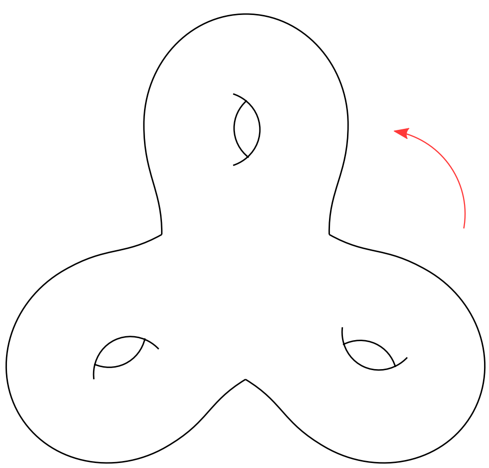

Figure 2.1: A rotation through $\frac{2\pi}{3}$ of this genus 3 surface is a periodic homeomorphism

Recall that a closed curve γ on S is a continuous map $\gamma : S^1 \to S$. However, we will often identify a curve with its image in S and simply denote $\gamma : S^1 \to S$ by $\gamma \subset S$. Furthermore, we will often abuse notation and conflate a curve with its unoriented homotopy class.

Definition 2.2. *If there is a collection $\mathcal{C}$ of disjoint, essential simple closed curves on S such that the homeomorphism f preserves $\mathcal{C}$, then f is said to be* **reducible**. *If f is not reducible, then it is said to be* **irreducible**.

Note that the rotation through $\frac{2\pi}{3}$ of the genus 3 surface shown in Figure 2.1 is not only an example of a periodic homeomorphism it is also an example of a reducible homeomorphism. We can see this by noting that it preserves the collection of three disjoint essential simple closed curves on the surface shown in blue in Figure 2.2.

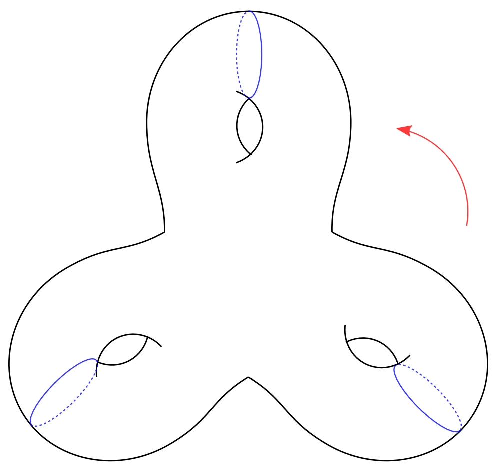

Figure 2.2: The rotation of this genus 3 surface through $\frac{2\pi}{3}$ preserves the collection of blue curves

Another important example of a reducible homeomorphism is a Dehn twist about a simple closed curve.

Definition 2.3. *Let α be a simple closed curve on a surface S. The (positive)* **Dehn twist** *about α is a homeomorphism $T_\alpha : S \to S$ with an annular neighborhood A of α homeomorphic to $\{re^{i\theta} \in \mathbb{C} \mid 1 \leq r \leq 2\}$ such that T_α acts by the identity outside of A and acts on A by $re^{i\theta} \mapsto re^{i(\theta+2\pi r)}$.*

From Definition 2.3 we can see that α is fixed by T_α for any simple closed curve α. Thus, T_α is indeed reducible. See Figure 2.3 for an example of a Dehn twist.

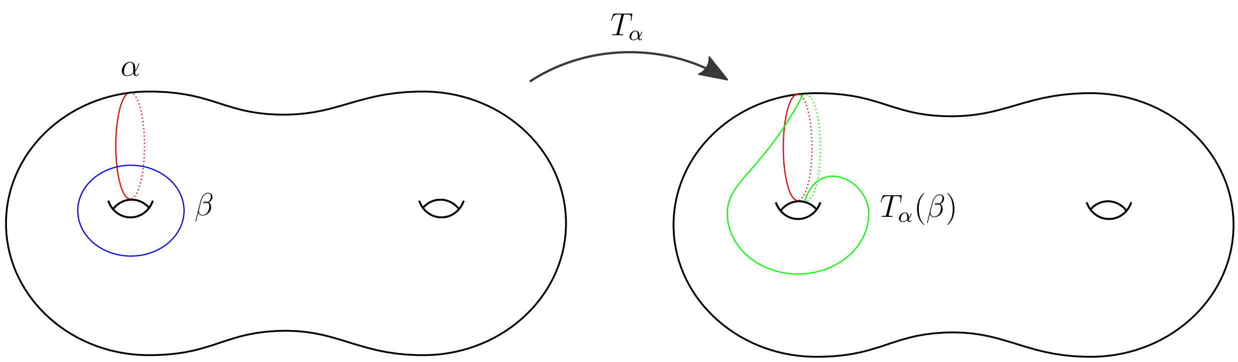

Figure 2.3: An example of a Dehn twist about a simple closed curve α (in red)

The final type of homeomorphism which we will discuss is called a pseudo-Anosov homeomorphism and it is central to this thesis. However, in order to define it we first need to introduce a few more concepts.

A **singular foliation** $\mathcal{F}$ on a surface S is a decomposition of S as a disjoint union of leaves (one dimensional injectively immersed submanifolds). Any point $x \in S$, outside a finite set of (singular) points, has a chart from a neighborhood of x to $\mathbb{R}^2$ that takes the leaves of $\mathcal{F}$ contained in U to horizontal intervals.

If y is a singular point, then x has a chart from a neighborhood of y to $\mathbb{R}^2$ that takes leaves to the level set of a k-prong singularity for $k \neq 2$. This is illustrated in Figure 2.4 for $k = 3$.

Two singular foliations are **transverse** if they have the same set of singular points and their leaves are transverse at every nonsingular point. A **transverse measure** μ on a singular foliation $\mathcal{F}$ is a function that assigns a positive real number to each arc transverse to $\mathcal{F}$ such that it is invariant under leaf preserving isotopy, and for each point, there is

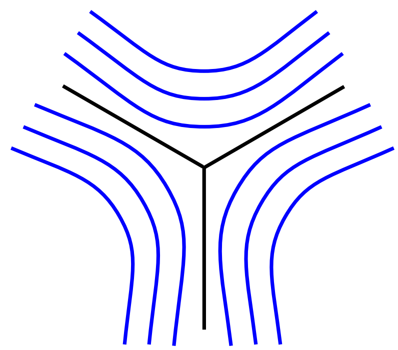

Figure 2.4: A 3-prong singularity

a smooth chart from a neighborhood to $\mathbb{R}^2$, so that the measure is induced by $|dy|$. We call a singluar foliation $\mathcal{F}$ equipped with a transverse measure μ a **transverse measured foliations**.

Definition 2.4. *If there exists a pair of transverse measured foliations* $(\mathcal{F}^s, \mu_s)$ *and* $(\mathcal{F}^u, \mu_u)$ *on* S *and a real number* $\lambda(f) > 1$ *such that*

$$f \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda(f)^{-1}\mu_s) \text{ and } f \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda(f)\mu_u),$$

then f *is called* **pseudo-Anosov**. *We call* $\lambda(f)$ *the* **stretch factor** *or* **dilatation** *of* f.

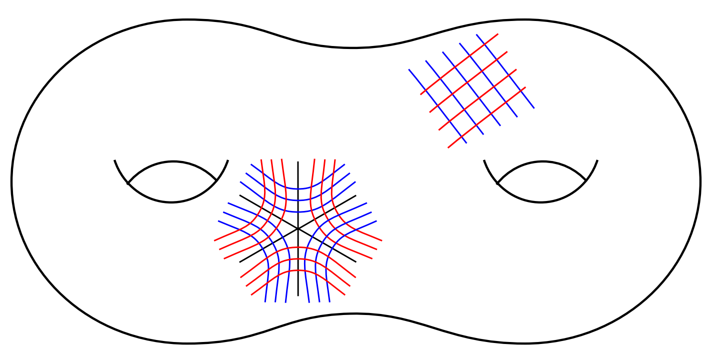

Figure 2.5: A pair of transverse measured foliations shown locally around at a singular and nonsingular point.

A mapping class $\varphi \in \mathrm{Mod}(S)$ is said to be **pseudo-Anosov**, **reducible**, or **periodic**, respectively, if there is a representative homeomorphism $f \in \varphi$ such that f is **pseudo-Anosov**, **reducible**, or **periodic**, respectively. Thurston proved the following classification of elements in $\mathrm{Mod}(S)$.

Theorem 2.1 (Nielsen–Thurston)**.** *A mapping class $\varphi \in \mathrm{Mod}(S)$ is pseudo-Anosov, reducible, or periodic. In addition, φ is pseudo-Anosov if and only if it is neither reducible nor periodic.*

A proof of this result can be found in [14], as well as a detailed discussion of the definitions above. The interested reader can also find an introduction to these topics in [13].

2.2 Surface Braids

Let X be a topological space. We define the **configuration space** of n distinct ordered points in X relative to a collection of m fixed but arbitrarily chosen distinct points $(y_1, \ldots, y_m)$ in X to be the subspace of X^n given by

$$\mathrm{Conf}(X^n, m) := \{(x_1, x_2, \ldots, x_n) \,:\, x_i \in X \setminus (y_1, \ldots, y_m) \text{ with } x_i \neq x_j \text{ for } i \neq j\}.$$

Note that the symmetric group, Σ_n, acts on $\mathrm{Conf}(X^n, m)$ on the left by

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots x_{\sigma(n)}).$$

Definition 2.5. *Let S be a surface and let $z_1^0, \ldots, z_n^0$ be a collection of n fixed but arbitrarily chosen points on S. The* **braid group** *of S on n-strands is*

$$\mathrm{B}_n(S) := \pi_1(\mathrm{Conf}(S^n, 0)/\Sigma_n, (z_1^0, \ldots, z_n^0)).$$

The **pure braid group** *of S on n-strands is*

$$\mathrm{PB}_n(S) := \pi_1(\mathrm{Conf}(S^n, 0), (z_1^0, \ldots, z_n^0)).$$

Note that $\mathrm{B}_n = \pi_1(\mathrm{Conf}(\mathbb{C}, n)/\Sigma_n)$ and $\mathrm{PB}_n = \pi_1(\mathrm{Conf}(\mathbb{C}, n))$ are the classical braid and pure braid groups, respectively; see [13]. Although at first glance Definition 2.5 appears different from the definition of $\mathrm{PB}_n(S_g)$ given in the introduction, Birman established that these definitions are equivalent in the following theorem, which first appeared in [7].

Theorem 2.2 (Birman). *For each pair of integers $g, n \geq 0$ let $\mathcal{F}orget : \mathrm{PMod}(S_{g,n}) \to \mathrm{Mod}(S_g)$ be the forgetful map. If $g \geq 2$, then $\ker(\mathcal{F}orget)$ is isomorphic to $\pi_1(\mathrm{Conf}(S_g, n))$.*

We include the proof of Theorem 2.2 here for the sake of completeness and refer the reader to [8] for a further discussion of braid groups. The proof of this result appeals to both a long and short exact sequence of homotopy groups, which at first glance can be a bit intimidating. However, the intuition is straightforward if we observe that for a homeomorphism representing a mapping class in the n-stranded pure braid group the isotopy on the closed surface from the homeomorphism back to the identity traces out a loop of n ordered point configurations and this defines the isomorphism in Theorem 2.2.

Proof. Let $z_1^0, \ldots, z_n^0$ be a collection of n fixed but arbitrarily chosen points on S_g, as in Definition 2.5. Consider the following portion of the long exact sequence of homotopy groups, where $\mathrm{Homeo}^+(S_{g,n})$ is the group of orientation preserving homeomorphisms of $S_{g,n}$ that fix the punctures pointwise:

$$\cdots \to \pi_1(\mathrm{Homeo}^+(S_{g,n})) \xrightarrow{\epsilon_*} \pi_1(\mathrm{Conf}(S_g^n, 0)) \xrightarrow{d_*} \pi_0(\mathrm{Homeo}^+(S_{g,n}))$$
$$\xrightarrow{\iota_*} \pi_0(\mathrm{Homeo}^+(S_{g,0})) \to \pi_0(\mathrm{Conf}(S_g^n, 0)) = 1.$$

The homomorphism ϵ_* is induced by the evaluation map $\epsilon : \mathrm{Homeo}^+(S_{g,n}) \to \mathrm{Conf}(S_g^n, 0)$ given by $f \mapsto (f(z_1^0), \ldots, f(z_n^0))$ and the homomorphism ι_* is induced by the inclusion $\iota : \mathrm{Homeo}^+(S_{g,n}) \to \mathrm{Homeo}^+(S_{g,n})$.

We will begin by constructing the homomorphism d_*. Consider a loop $\beta \in \pi_1(\mathrm{Conf}(S_g^n, 0))$ given by $\beta = (\beta_1, \ldots, \beta_n) : I \to \mathrm{Conf}(S_g^n, 0)$. It is straightforward to construct an isotopy $F_t : S_g \to S_g$ with $t \in (0,1)$ such that $F_0 = \mathrm{id}$ and $F_t(z_i^0) = \beta_i(t)$ and thus, $F_1 \in \pi_0(\mathrm{Homeo}^+(S_{g,n}))$. So we have that $[F_1] = d_*\beta$.

By exactness, we know that $\ker \iota_* = \operatorname{im} d_*$. Thus, it only remains to show that $\operatorname{im} d_* = \pi_1(\mathrm{Conf}(S_g^n, 0))$. In particular, we must show that d_* is injective. Consider $\ker d_*$. We will show that $\ker d_* = 1$ in two steps. First we show that $\ker d_* \subset \mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^n, 0)))$ and then show that $\mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^n, 0))) = 1$ for $g \geq 2$.

Suppose $\alpha \in \ker d_* = \operatorname{im} \epsilon_*$ and let $H \in \pi_1(\mathrm{Homeo}^+(S_{g,0}))$ such that $\epsilon_* H = \alpha$. The element H is represented by a loop $h = \{h_t | 0 \leq t \leq 1\}$ in $\mathrm{Homeo}^+(S_{g,0})$, where each h_t is in $\mathrm{Homeo}^+(S_{g,0})$ and $h_0 = h_1 = \mathrm{id}$. Then $\epsilon(h_t) = (h_t(x_1), \ldots, h_t(x_n))$ $(0 \leq t \leq 1)$ represents α. Let $\beta \in \pi_1(\mathrm{Conf}(S_g^n, 0))$ with β represented by $(\beta_1(s), \ldots, \beta_n(S))$. Define $G : I \times I \to \mathrm{Conf}(S_g^n, 0)$ by $G(t,s) = (h_t(\beta_1(s)), \ldots, h_t(\beta_n(s))$ $((t,s) \in I \times I)$. Then G is continuous and $G|_{\partial(I \times I)}$ represents the homotopy class $\alpha\beta\alpha^{-1}\beta^{-1}$. Since β was arbitrary in $\pi_1(\mathrm{Conf}(S_g^n, 0))$, then $\alpha \in \mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^n, 0)))$.

Next we will use induction to show that $\mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^n, 0))$ is trivial. Recall the Fadell–Neuwirth short exact sequence [11]:

$$1 \to \pi_1(\mathrm{Conf}(S_g^1, n-1)) \xrightarrow{j_*} \pi_1(\mathrm{Conf}(S_g^n, 0)) \xrightarrow{\pi_*} \pi_1(\mathrm{Conf}(S_g^{n-1}, 0)) \to 1.$$

The homomorphism j_* is induced by the inclusion $j : \mathrm{Conf}(S_g^1, n-1) \hookrightarrow \mathrm{Conf}(S_g^n, 0)$ given by $z_n \mapsto (z_1^0, \ldots, z_{n-1}^0, z_n)$ where $z_n \in S_g \setminus \{z_1^0, \ldots, z_{n-1}^0\}$ and the homomorphism π_* is induced by the projection $\pi : \mathrm{Conf}(S_g^n, 0) \to \mathrm{Conf}(S_g^{n-1}, 0)$ given by $(z_1, \ldots, , z_{n-1}, z_n) \mapsto (z_1, \ldots, z_{n-1})$. Note that when $n = 1$, we have that $\pi_1(\mathrm{Conf}(S_g^1, 0)) = \pi_1(S_g)$ and is thus centerless. Now assume that $\pi_1(\mathrm{Conf}(S_g^n, 0))$ is centerless. Since π_* is surjective, then $\pi_*(\mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^n, 0)))) \subset \mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^{n-1}, 0))) = 1$. Hence, $\mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^n, 0)))$ lies in $\operatorname{im} j_* = \ker \pi_*$. But $\pi_1(\mathrm{Conf}(S_g^{n-1}, 0)) \cong \operatorname{im} j_*$ is a free group of rank > 1, hence centerless. Thus, $\mathrm{Center}(\pi_1(\mathrm{Conf}(S_g^n, 0))) = 1$, as desired. $\square$

Note that the map $\iota_* : \pi_0(\mathrm{Homeo}^+(S_{g,n})) \to \pi_0(\mathrm{Homeo}^+(S_{g,0}))$ in the proof of Theorem 2.2

is precisely the "forgetful map" $\mathcal{F}orget : \mathrm{PMod}(S_{g,n} \to \mathrm{Mod}(S_g)$ in the short exact sequence given in (1).

2.3 Some Teichmüller Theory

A Teichmüller theoretic approach is employed in the proof of Theorem 4.2, which is part of the lower bound in the Main Theorem. Consequently, we will introduce several definitions and results which come from the study of Teichmüller theory and quasiconformal maps.

The Teichmüller space of a surface S, denoted $\mathcal{T}(S)$, can be defined equivalently as either the space of equivalence classes of complex structures on S or the space of equivalence classes of hyperbolic structures on S. We will focus on the former perspective. Note that a surface $S_{g,n}$ only admits a hyperbolic metric when $\chi(S_{g,n}) = 2 - 2g - n < 0$, hence our restriction to surfaces with genus at least 2.

Definition 2.6. *The Teichmüller space of a surface S is the collection of complex structures on S up to the following equivalence: two complex structures X and Y on S are equivalent if there exists a map $f : (S, X) \to (S, Y)$ that is isotopic to the identity and biholomorphic in the coordinate charts.*

In order to define the Teichmüller metric on $\mathcal{T}(S)$ we will recall the definition of a quasiconformal map; see [3] for more on quasiconformal mappings.

Definition 2.7. *Let $f : \Omega \to f(\Omega)$ be a homeomorphism between open sets $\Omega, f(\Omega) \subset \mathbb{C}$. Suppose f has locally integrable weak partial derivatives and let $D_f = \dfrac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1$. We say that f is* **quasiconformal** *if $\|D_f\|_\infty < \infty$ and K-***quasiconformal** *if $\|D_f\|_\infty \leq K$. The* **quasiconformal dilatation** *is $K(f) = \|D_f\|_\infty$.*

We can now define the Teichmüller metric on $\mathcal{T}(S)$ as

$$d_T(X, Y) := \frac{1}{2} \inf_{f \sim \mathrm{id}} \{\log K(f) \mid f : (S, X) \to (S, Y)\},$$

where f is a quasiconformal map.

One of the foundational results in Teichmüller Theory is the following theorem of Teichmüller which establishes that given any two complex structures there is a unique quasiconformal map which realizes their Teichmüller distance.

Theorem 2.3 (Teichmüller's Theorem)**.** *Given any $X, Y \in \mathcal{T}(S)$, there exists a unique quasiconformal map $f : (S, X) \to (S, Y)$ isotopic to the identity, called the* **Teichmüller map** *such that*

$$d_T(X, Y) = \frac{1}{2} \log K(f).$$

Furthermore, f has an explicit description in terms of holomorphic quadratic differentials on X and Y, respectively, and is affine in preferred coordinates.

We will briefly describe how to construct a Teichüller map given a Riemann surface (S, X), a holomorphic quadratic differential φ_X, and some $K > 1$. Note that the pair (S, X) describes a Riemann surface in terms of the underlying topological surface S and the complex structure X on S.

Let (S', X') be the complement of the zeros of φ_X. In fact, (S', X') is also a Riemann surface since X' is a complex structure with respect to a sufficiently large collection of preferred coordinates for φ_X, thought of as a holomorphic quadratic differential on X'.

Now compose each chart of X' with the affine map

$$f(x + iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y.$$

This new collection of charts defines a new complex structure, call it Y', on S'. The final step to obtain from Y' a complex structure on the closed surface S is to apply the removable singularities theorem to see that Y' extends uniquely to a complex structure Y on S.

So we have an induced homeomorphism $f : (S, X) \to (S, Y)$ and an induced holomorphic quadratic differential φ_Y on Y. By construction f is the unique Teichmüller map from X to Y as in Theorem 2.3.

An important component of our proof of the lower bound is a result of Teichmüller [38] and Gehring [16] which relates the dilatation of a quasiconformal map f on the hyperbolic plane $\mathbb{H}^2$ to the maximum distance a point of $\mathbb{H}^2$ is moved by f. We give a version of the statement which can be found in Kra [26].

Theorem 2.4 (Kra). *Consider $\mathbb{H}^2$ with Poincaré metric ρ. For $x, y \in \mathbb{H}^2$ there exists a unique self-mapping $f : \mathbb{H}^2 \to \mathbb{H}^2$ so that f is the identity on the boundary of $\mathbb{H}^2$, $f(x) = y$, and f minimizes the quasiconformal dilatation among all such mappings. Let $K(x, y)$ be the quasiconformal dilatation of such an extremal f. Then there exists a strictly increasing real-valued function $\varkappa : [0, \infty) \to [0, \infty)$ such that*

(i) $\log(1 + \frac{t}{2}) \leq \varkappa(t)$, *and*

(ii) $\frac{1}{2} \log K(x, y) = \varkappa(\rho(x, y))$.

The second important component of the proof of Theorem 4.2 is Theorem 2.5 below. The statement and proof of Theorem 2.5 in the case of $n = 2$ are due to Imayoshi–Ito–Yamamoto [23] with a weaker upper bound on the quasiconformal dilatation. The proof of Imayoshi–Ito–Yamamoto holds in the case of $n > 2$ punctures without any modification so we will omit the full argument and will instead provide a sketch of the proof.

Theorem 2.5 (Imayoshi–Ito-Yamamoto). *Let $\varphi : S_{g,n} \to S_{g,n}$ be a pseudo-Anosov homeomorphism representing an element of $\mathrm{PB}_n(S_g)$ and let $\widehat{\varphi} : S_g \to S_g$ be the extension of φ to the surface with the punctures filled in. There exists a conformal structure on S_g together with an isotopy $F_t : S_g \to S_g$ with $t \in [0, 1]$, through quasiconformal maps, between $\mathrm{id} : S_g \to S_g$ and $\widehat{\varphi}$ on the closed surface S_g. Furthermore, for each $t \in [0, 1]$ the quasiconformal dilatation K_t of F_t satisfies*

$$\log(K_t) \leq 3 \log(\lambda(\varphi)).$$

Sketch of Proof. We will begin by constructing F_t. Let S_g be given a conformal structure so that $[id] = [id : S_{g,n} \to S_{g,n}]$ lies on the axis for φ and $[0, 1] \ni t \mapsto [f_t] \in \mathcal{T}(S_{g,n})$ be the

Teichmüller geodesic connecting [id] and $\varphi^{-1}([\mathrm{id}])$. So for all $t \in [0,1]$, $f_t : S_{g,n} \to f_t(S_{g,n})$ is a Teichmüller mapping and

$$\frac{1}{2}\log(K(f_t)) \leq \frac{1}{2}\log(K(f_1)) = \log(\lambda(\varphi^{-1})) = \log(\lambda(\varphi)).$$

By filling in the punctures, we can extend f_t to $\widehat{f}_t : S_g \to \widehat{f}_t(S_g)$. Denote by $\widehat{\varphi}_t$ the Teichmüller map of S_g onto $\widehat{f}_t(S_g)$ isotopic to $\widehat{f}_t$ on S_g. Then we define the map $F_t : S_g \times [0,1] \to S_g$ by

$$F_t(x) = \widehat{\varphi}_t^{-1} \circ \widehat{f}_t(x) \text{ for } x \in S_g \text{ and } t \in [0,1].$$

The fact that F_t is an isotopy is proved in [23]. Note that

$$\log(K_t) = \log(K(\widehat{\varphi}_t^{-1} \circ \widehat{f}_t)) \leq \log(K(\widehat{\varphi}_t^{-1})) + \log(K(\widehat{f}_t)).$$

Furthermore, we have that $t \mapsto [\widehat{f}_t]$ is a closed loop of length at most $\log(\lambda(\varphi))$. So

$$\frac{1}{2}\log(K(\widehat{\varphi}_t^{-1})) = d_{\mathcal{T}(S_g)}([\widehat{f}_t], [\mathrm{id}]) \leq \mathrm{diam}_{\mathcal{T}(S_g)}(\{[\widehat{f}_s] \mid s \in [0,1]\}) \leq \frac{1}{2}\log(\lambda(\varphi)).$$

Thus,

$$\log(K_t) \leq 3\log(\lambda(\varphi)).$$ □

2.4 Perron–Frobenius Theory

Matrices play an important role in the proof and application of Thurston's construction. In particular, we will consider primitive integer matrices, where we call a matrix **primitive** if it has a power that is a positive matrix. Note that we call a matrix positive (respectively nonnegative) if all of its entries are positive (respectively nonnegative). A matrix is called **Perron–Frobenius** if it is both primitive and nonnegative. The following theorem is fundamental to the study of these matrices.

Theorem 2.6 (Perron–Frobenius)**.** *Let A be an $n \times n$ matrix with integer entries. If A is primitive, then A has a unique nonnegative unit eigenvector v. The vector v is positive and has a positive eigenvalue u that is larger in absolute value than all other eigenvalues.*

The eigenvector v in Theorem 2.6 is called the **Perron–Frobenius eigenvector** of A and the eigenvalue u in Thoerem 2.6 is called the **Perron–Frobenius eigenvalue** of A. The following is an important fact about Perron–Frobenius matrices that we will leverage frequently and can find [15].

Theorem 2.7 (Gantmacher)**.** *If an $n \times n$ matrix A is Perron–Frobenius, then its Perron–Frobenius eigenvalue is bounded above both by the maximal row sum and maximal column sum of A.*

2.5 Thurston's Construction

Here we will introduce a useful tool for constructing pseudo-Anosov mapping classes due to Thurston [39]. We say a collection C of essential simple closed curves **fills** our surface $S = S_{g,n}$ if the curves intersect transversely and minimally and the complement of C in S is a collection of disks and once-puncture disks. Equivalently, we could say that C fills S if any essential simple closed curve on S has nonzero geometric intersection number with at least one curve in our collection C.

Now suppose we have a collection $C = \{c_1, c_2, \ldots, c_m\}$ of pairwise disjoint, essential simple closed curves on S. We can define a **multi-twist** T_C about C to be the product of positive Dehn twists about each $c_i \in C$.

Theorem 2.8 (Thurston)**.** *Let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $B = \{\beta_1, \beta_2, \ldots, \beta_k\}$ be collections of pairwise disjoint, essential, simple closed curves on S such that $A \cup B$ fills S. There is a real number $\mu > 1$ and homomorphism*

$$\rho : \langle T_A, T_B \rangle \to \mathrm{PSL}(2, \mathbb{R}) \text{ given by}$$

$$T_A \mapsto \begin{pmatrix} 1 & -\mu^{1/2} \\ 0 & 1 \end{pmatrix} \textit{ and } T_B \mapsto \begin{pmatrix} 1 & 0 \\ \mu^{1/2} & 1 \end{pmatrix}.$$

Furthermore, for $f \in \langle T_A, T_B \rangle$, f is pseudo-Anosov if its image $\rho(f)$ is hyperbolic, in which case the dilatation of f is equal to the spectral radius of $\rho(f)$.

Consider a mapping class $T_A T_B^{-1} \in \langle T_A, T_B \rangle$ as given by Theorem 2.8. The image of $T_A T_B^{-1}$ under ρ is given by

$$\begin{pmatrix} 1 & -\mu^{1/2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mu^{1/2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mu+1 & -\mu^{1/2} \\ -\mu^{1/2} & 1 \end{pmatrix}.$$

The trace of this matrix is $2 + \mu$. Thus, by Theorem 2.8, $T_A T_B^{-1}$ is pseudo-Anosov and $\log(\lambda(T_A T_B^{-1}))$ is bounded above by $\log(2 + \mu)$.

The real number μ in Theorem 2.8 is the Perron–Frobenius eigenvalue of NN^T, where N is defined as $N_{i,j} = i(\alpha_i, \beta_j)$. If $A = \{\alpha\}$ and $B = \{\beta\}$, then $\mu = i(\alpha, \beta)^2$. This will be a useful fact to keep in mind for the following section. In general, μ cannot be computed in such a straightforward manner. Since NN^T is nonnegative and primitive we can bound μ from above by the maximum row sum of NN^T.

In order to compute the row sums of NN^T we will follow the method used in [2], which we describe here. Given N, we can build a labeled bipartite graph G with m red vertices and k blue vertices corresponding to the multicurves A and B, respectively. An edge from the ith red vertex to the jth blue vertex exists if $N_{i,j} \neq 0$, in which case it is labeled by $N_{i,j}$. We will define the **weight** of a path in G to be the product of edge labels in that path. The (i, j) entry of NN^T is equal to the sum of the weights of the paths of length 2 from the ith red vertex to the jth red vertex in G. To compute the row sum of NN^T corresponding to a particular curve we start at the vertex associated to that curve and sum the weights of all paths of length two, possibly with backtracking, beginning at that vertex.

2.6 The Point Pushing Subgroup

The construction given by Aougab–Taylor in [5] of point pushing homeomorphisms used to realize the upper bound in Theorem 1.3 will play an important role in our proof of the Main Theorem so we will recall it here. We also employ some further work of Aougab–Huang [4] to gain a more careful estimate of the upper bound than that provided in [5]. In particular, we will prove the following.

Theorem 2.9 (Aougab–Taylor)**.** *For the closed surface S_g of genus $g \geq 2$,*

$$L(\mathrm{PB}_1(S_g)) < 4\log(g) + 2\log(24).$$

Proof of Theorem 2.9. Let α and β be a minimally intersecting filling pair of curves on the closed surface S_g. By [4], we have that $i(\alpha, \beta) = 2g - 1$. Let β_1, β_2 be the boundary components of a small tubular neighborhood of β. Thus, β_1, β_2 are homotopic to β on S_g. Now place a marked point z at some point of $\beta \setminus \alpha$. We can puncture S_g at z to form the surface $S_{g,1}$.

Set $f_\beta = T_{\beta_1}^3 \circ T_{\beta_2}^{-3}$. This is a point pushing map in $S_{g,1}$ obtained by pushing the marked point z along β three times. Our goal is to show that $\{\alpha, f_\beta(\alpha)\}$ fills the punctured surface $S_{g,1}$, and then apply Theorem 2.8 to obtain a pseudo-Anosov mapping class in $\mathrm{PB}_1(S_g)$. We apply the following inequality of Ivanov found in [24] to show that any essential simple closed curve on $S_{g,1}$ must intersect either α or $f_\beta(\alpha)$.

Lemma 2.1 (Ivanov)**.** *Let $c_1, \ldots c_m$ be a collection of pairwise disjoint, pairwise non-homotopic simple closed curves on a surface S with negative Euler characteristic and let $(s_1, \ldots, s_m) \in$*

$\mathbb{Z}^m$. *For any simple closed curves* γ, ρ,

$$\sum_{i=1}^{m}(|s_i|-2)i(\rho,c_i)i(c_i,\gamma) - i(\rho,\gamma) \leq i(T_{c_1}^{s_1}\circ\cdots\circ T_{c_m}^{s_m}(\rho),\gamma)$$
$$\leq \sum_{i=1}^{m}|s_i|\, i(\rho,c_i)i(c_i,\gamma) + i(\gamma,\rho).$$

Suppose γ is an essential simple closed curve on $S_{g,1}$ such that $i(\gamma,\alpha)=0$. Now we can apply Lemma 2.1 with $\rho=\alpha$, $(s_1,s_2)=(3,-3)$, and $(c_1,c_2)=(\beta_1,\beta_2)$. Recall that α and β filled S_g, so $\{\alpha,\beta_1,\beta_2\}$ fill $S_{g,1}$. Thus, $i(\gamma,\beta_i)\neq 0$ for $i=1,2$, which implies that the lefthand side of the inequality in Lemma 2.1 is nonzero. Hence, $i(\gamma,f_\beta(\alpha))\neq 0$, as desired. Furthermore, we can use the fact that $i(\alpha,\beta)=2g-1$, together with Lemma 2.1, to calculate that $i(\alpha,f_\beta(\alpha))\leq 24g^2-24g+6$.

Since f_β is a point pushing map, we know that α and $f_\beta(\alpha)$ are homotopic on the closed surface S_g. Thus, $T_\alpha T_{f_\beta(\alpha)}^{-1}\in \mathrm{PB}_1(S_g)$ and, by Theorem 2.8, is also pseudo-Anosov. Recall that in the case of two filling curves Theorem 2.8 tells us that $\lambda(T_\alpha T_{f_\beta(\alpha)}^{-1})\leq i(\alpha,f_\beta(\alpha))^2+2$. Thus, $\lambda(T_\alpha T_{f_\beta(\alpha)}^{-1})<24^2g^4$ and we obtain the desired upper bound

$$L(\mathrm{PB}_1(S_g)) < 4\log(g)+2\log(24). \qquad \square$$

We will denote the curves α and $f_\beta(\alpha)$ which we constructed above by α and τ, respectively, and call them an **Aougab–Taylor pair**. Note that we can construct an Aougab–Taylor pair $\{\alpha,\tau\}$ on a surface of genus g with a single boundary component with the same bound of $24g^2-24g+6$ on intersection number, since on a surface of genus $g>2$ with a single boundary component there exists a pair of filling curves that intersect $2g-1$ times. In the case of a genus two surface with a single boundary component a minimally intersecting pair of filling curves will intersect 4, not 3, times. However we can still construct an Aougab–Taylor pair $\{\alpha,\tau\}$ with $i(\alpha,\tau)\leq 24$. When our surface is a torus with a single boundary component, we can construct an Aougab–Taylor pair $\{\alpha,\tau\}$ with $i(\alpha,\tau)=6$.

Chapter 3

The Upper Bound

We will begin by proving the Main Theorem's upper bound which depends on the genus g and number of punctures n of our surface. We state this upper bound with explicit constants in Theorem 3.1. To prove the upper bound it suffices to construct a pseudo-Anosov pure braid satisfying the desired upper bound for each g and n.

Theorem 3.1. *For a surface S_g of genus $g \geq 2$ with $1 \leq n \leq 2g$, we have*

$$L(\mathrm{PB}_n(S_g)) \leq 4\log\left(\left\lceil \frac{2g}{n} \right\rceil\right) + 4\log(7).$$

Fix a genus $g \geq 2$. Our main tool throughout this section will be leveraging Thurston's construction to build our desired pseudo-Anosov pure surface braids by building pairs of filling multicurves.

Proof of Theorem 3.1. The main strategy of our proof is to divide our surface into subsurfaces with a single boundary component, fill each of these subsurfaces with an Aougab–Taylor pair, and then add a few additional curves which bound twice punctured disks to combine these Aougab–Taylor pairs into a single pair of filling multicurves. We will employ this strategy in each of our three cases: when $n = 2, 3$, when $4 \leq n < 2g$, and when $n \geq 2g$.

Case 1. We begin our construction in the case of $n = 2$. Let A and B denote the multicurves marked in red and blue, respectively, in Figure 3.1 which are constructed in the following way. Consider two subsurfaces of $S_{g,n}$ given by cutting along a separating curve that divides $S_{g,n}$ into two subsurfaces of genus at most $\lceil \frac{g}{2} \rceil$ each containing a single puncture. On each of these subsurfaces we can construct an Aougab–Taylor pair as described in Section 2.6. We then add an additional curve bounding a twice-punctured disk containing the pair of punctures. We illustrate this construction in Figure 3.1 for the case of a genus 2 surface. In this situation our Aougab–Taylor pairs on each genus 1 subsurface intersect 6 times and our additional red curve, which bounds a twice-punctured disk containing the pair of punctures, intersects each blue curve 8 times. For $n = 3$ we can add an additional puncture, as shown on the right of Figure 3.1.

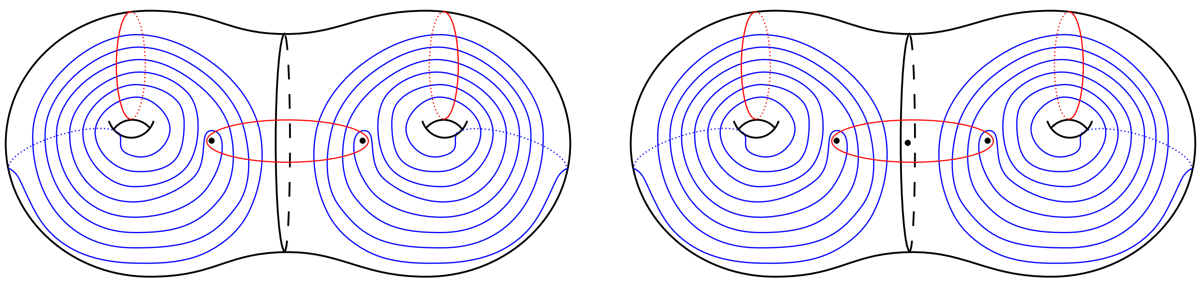

Figure 3.1: Construction of filling multicurves, A and B, for 2 and 3 punctures

Let $f = T_A T_B^{-1}$. Note that f is pseudo-Anosov by Thurston's Construction, since A and B jointly fill $S_{g,n}$. Furthermore, $f \in \mathrm{PB}_n(S_g)$, since the red curve bounding the twice punctured disk is trivial on the closed surface and the pairs of curves which fill each subsurface will be homotopic to each other on the closed surface. Thus, the composition of positive and negative multitwists about A and B is the trivial mapping class on the closed surface. As discussed in Section 2.5, we can bound $\lambda(f)$ from above by the Perron–Frobenius eigenvalue, μ, of NN^T. Since there are only 5 curves in $A \cup B$ as shown in Figure 3.1, we can explicitly compute μ. Note that the red curve which bounds a twice (or thrice) punctured disk intersects each blue curve at most $24\left(\left\lceil \frac{g}{2} \right\rceil\right)^2 - 24\left\lceil \frac{g}{2} \right\rceil + 8$ times. So we have that $\mu \leq 3(24\left(\left\lceil \frac{g}{2} \right\rceil\right)^2 - 24\left\lceil \frac{g}{2} \right\rceil + 8)^2 < 7^4\left(\left\lceil \frac{g}{2} \right\rceil\right)^4 - 2$, where μ is the Perron–Frobenius eigenvalue of NN^T as described in Theorem 2.8. Thus,

$$\log(\lambda(f)) \leq \log(\mu + 2) \leq \log\left(7^4\left(\left\lceil \frac{g}{2} \right\rceil\right)^4\right) = 4\log\left(\left\lceil \frac{g}{2} \right\rceil\right) + 4\log(7). \qquad \square$$

Case 2. Now consider the case when $4 \leq n < 2g$. We will illustrate our construction in Figure 3.2 in the case of a genus 4 surface. We will build our pair of filling multicurves on $S_{g,n}$ in the following way. We will paritition S_g into $\lfloor \frac{n}{2} \rfloor + 1$ subsurfaces, $\lfloor \frac{n}{2} \rfloor$ of which have genus at most $\lceil \frac{2g}{n} \rceil$ and one boundary component, and one of which is a sphere with $\lfloor \frac{n}{2} \rfloor$ holes. Puncture each non-planar subsurface once, and as before, we fill each of these subsurfaces with an Aougab–Taylor pair α and β, shown in red and blue, respectively, in Figure 3.2. We then add an additional puncture to each non-planar subsurface so that it is near the boundary component of that subsurface. This is illustrated in Figure 3.2. Let A be the union of the α curves and B be the union of the β curves from our Aougab–Taylor pairs. Now view the non-planar subsurfaces as being arranged cyclically around the sphere with boundary, as shown in Figure 3.2, and for consecutive pairs of punctures, one coming from the Aougab–Taylor pair and one a puncture added near the subsurface boundary, add a red curve to our multicurve A which bounds a twice punctured disc. We have now constructed a pair of filling multicurves A and B which fill our surface $S_{g,n}$.

Note that these additional bounding pair curves will each intersect with two blue curves. They will intersect with one blue curve twice and with the other blue curve at most $24 \left(\lceil \frac{2g}{n} \rceil \right)^2 - 24 \lceil \frac{2g}{n} \rceil + 8$ times. The picture on the left of Figure 3.2 illustrates the case of an even number of punctures and the picture on the right the case of an odd number of punctures where we add an additional puncture to the central sphere with boundary.

Let $f = T_A T_B^{-1}$. Note that f is a pseudo-Anosov pure braid for the same reasons given in Case 1. Thus, we can proceed immediately to computing the maximum row sum of NN^T in order to bound $\lambda(f)$. We can compute the maximum row sum of NN^T by considering the labeled bipartite graph in Figure 3.3 that describes the intersection pattern of red and blue curves.

Note that each blue vertex has valence 3 and each red vertex has valence at most 2. Furthermore, the dashed edges have label at most $24 \left(\lceil \frac{2g}{n} \rceil \right)^2 - 24 \lceil \frac{2g}{n} \rceil + 8$ and the solid edges have label 2.

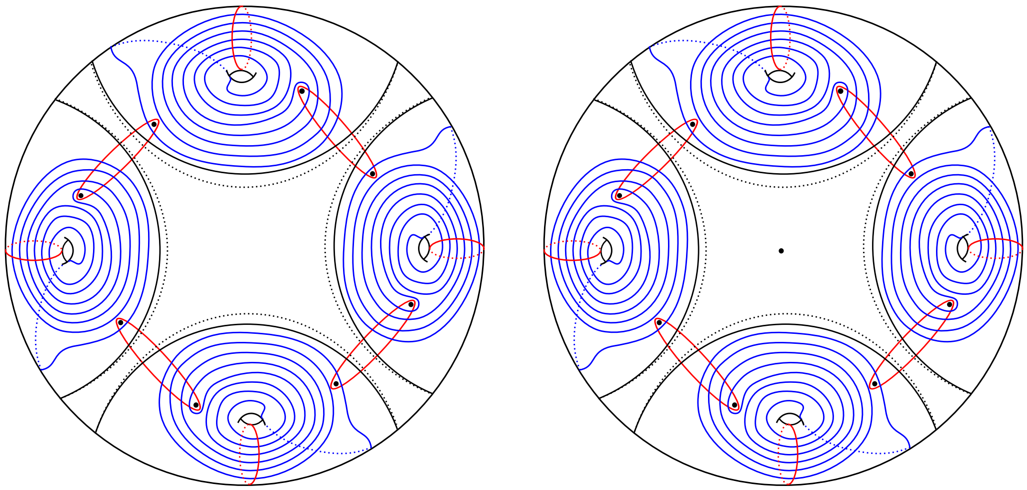

Figure 3.2: Examples of filling multicurves, A and B, for $4 \leq n < 2g$

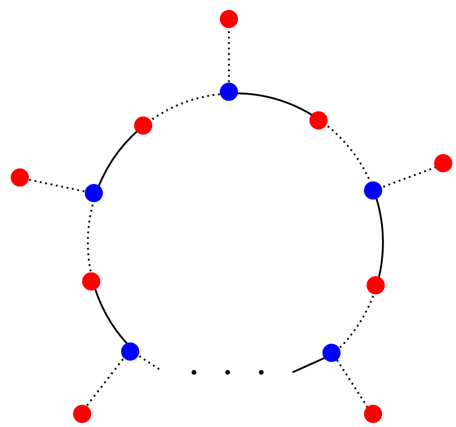

Figure 3.3: Bipartite graph for A and B when $4 \leq n < 2g$

Thus, for the red vertices of valence 2 we have a corresponding row sum of at most

$$2\left(24\left(\left\lceil\frac{2g}{n}\right\rceil\right)^2 - 24\left\lceil\frac{2g}{n}\right\rceil + 8\right)^2 + 6\left(24\left(\left\lceil\frac{2g}{n}\right\rceil\right)^2 - 24\left\lceil\frac{2g}{n}\right\rceil + 8\right) + 4.$$

For the red vertices of valence 1 we have a corresponding row sum of at most

$$2\left(24\left(\left\lceil\frac{2g}{n}\right\rceil\right)^2 - 24\left\lceil\frac{2g}{n}\right\rceil + 8\right)^2 + 2\left(24\left(\left\lceil\frac{2g}{n}\right\rceil\right)^2 - 24\left\lceil\frac{2g}{n}\right\rceil + 8\right).$$

Note that each of these is at most $1152\left(\left\lceil\frac{2g}{n}\right\rceil\right)^4 - 2 < 6^4\left(\left\lceil\frac{2g}{n}\right\rceil\right)^4 - 2$. Thus, the maximum row sum of NN^T is bounded above by $6^4\left(\left\lceil\frac{2g}{n}\right\rceil\right)^4 - 2$ and we have that

$$\log(\lambda(f)) \leq \log(\mu+2) \leq \log\left(6^4\left(\left\lceil\frac{2g}{n}\right\rceil\right)^4\right) = 4\log\left(\left\lceil\frac{2g}{n}\right\rceil\right) + 4\log(6). \qquad \square$$

Case 3. Note that when $n \geq 2g$ the inequality in Theorem 3.1 says that we have a constant upper bound on $L(\mathrm{PB}_n(S_g))$. The construction given above is for $n < 2g$, but can be extended to give a constant upper bound as we add additional punctures. Suppose we have $n \geq 2g$. We can divide S_g into g subsurfaces of genus 1 and one sphere with g boundary components. We then puncture each of the g non-planar subsurfaces and fill each one with an Aougab–Taylor pair, $\{\alpha, \tau\}$, such that $i(\alpha,\tau) = 6$ using the construction in Section 2.6 and continue to add punctures to the central sphere with boundary as shown in Figure 3.4 where the red curves belong to A and the blue curves belong to B. Note that this manner of adding additional punctures does not increase the number of pairwise intersections between red and blue curves nor does it introduce any curves that have nonzero intersection with more than two other curves.

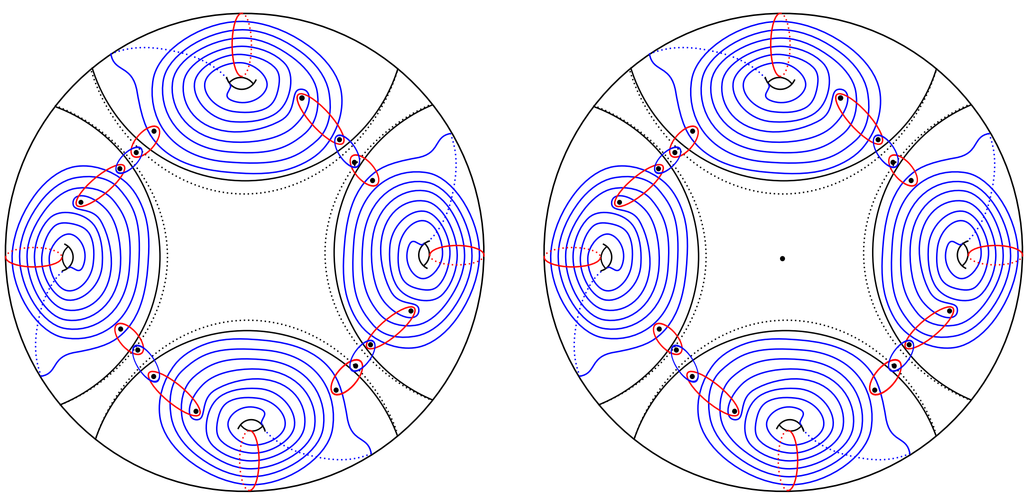

Figure 3.4: Examples of filling multicurves, A and B, for $n \geq 2g$

Let $f = T_A T_B^{-1}$. Note that f is a pseudo-Anosov pure braid by the same reasoning used previously. Thus, just as we did before, we can proceed directly to computing the maximum row sum of NN^T in order to bound $\lambda(f)$. We can compute the maximum row sum of NN^T

by considering the labeled bipartite graph in Figure 3.5 which is constructed in the same way as the bipartite graph in Figure 3.3.

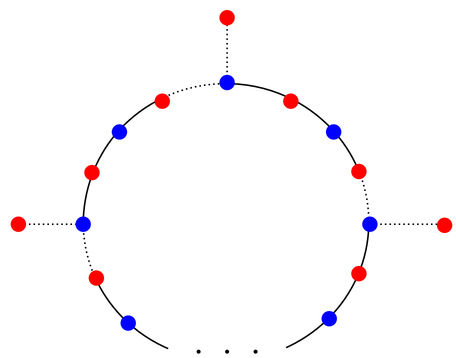

Figure 3.5: Bipartite graph for A and B when $n > 2g$

The dashed edges in Figure 3.5 are labeled by 8 and the solid edges are labeled by 2. Thus, we can compute that the maximum row sum of NN^T is 152 and we have that $\log(\lambda(f)) < 4\log(6)$. □

Thus, we have addressed each of our three cases and shown that

$$L(\mathrm{PB}_n(S_g)) \leq 4\log\left(\left\lceil \frac{2g}{n} \right\rceil\right) + 4\log(7). \qquad \square$$

Chapter 4

The Lower Bounds

4.1 A Constant Lower Bound

In this section we provide a constant lower bound on $L(\mathrm{PB}_n(S_g))$.

Theorem 4.1. *For a surface S_g of genus $g \geq 2$ with $n \geq 1$, we have*

$$.000155 \leq L(\mathrm{PB}_n(S_g)).$$

The proof of Theorem 4.1 relies on the following result of Agol–Leininger–Margalit which can be found in [2].

Proposition 4.1. *Let S be a surface and $f \in \mathrm{Mod}(S)$ pseudo-Anosov, then*

$$.00031 \left(\frac{\kappa(f) + 1}{|\chi(S)|} \right) \leq \log(\lambda(f)),$$

where $\kappa(f)$ is the dimension of the subspace of $H_1(S; \mathbb{R})$ fixed by f.

In order to make use of this result we must examine the action of a pure surface braid $f \in \mathrm{PB}_n(S_g)$ on $H_1(S_{g,n}; \mathbb{R})$. We can place the following lower bound on $\kappa(f)$.

Lemma 4.1. *If $f \in \mathrm{PB}_n(S_g)$, then*

$$\max\{2g, n-1\} \leq \kappa(f).$$

Proof of Lemma 4.1. Let M_f denote the mapping torus of f and let $b_1(M_f)$ denote the first

Betti number of M_f with coefficients in $\mathbb{R}$. Note that $b_1(M_f) = \kappa(f) + 1$. This can be obtained by an application of the Mayer–Vietoris long exact sequence, which we work out below.

Consider the following portion of the Mayer–Vietoris long exact sequence (Example 2.48, [18]), where coefficients are assumed to be in $\mathbb{R}$:

$$\cdots \to H_1(S_{g,n}) \to H_1(S_{g,n}) \to H_1(M_f) \to H_0(S_{g,n}) \to H_0(S_{g,n}) \to H_0(M_f) \to \cdots .$$

The maps from $H_k(S_{g,n}) \to H_k(S_{g,n})$ are given by $\mathrm{id} - f_*$ and the map from $H_k(S_{g,n}) \to H_k(M_f)$ is the map induced on homology by the inclusion $\iota : S_{g,n} \hookrightarrow M_f$. We will denote that map $H_1(M_f) \to H_0(S_{g,n})$ by T. Note that $H_1(S_{g,n}) \cong \mathbb{R}^{2g+n-1}$ and $H_0(S_{g,n}) \cong \mathbb{R}$.

By the exactness of this sequence we have that $\mathrm{im}(\mathrm{id} - f_*) = \ker(\iota_*)$ and $\mathrm{im}(\iota_*) = \ker(T)$. Thus, we have the following equalities:

$$\dim(H_1(M_f)) = \dim(\mathrm{im}(T)) + \dim(\ker(T)) = 1 + \dim(\mathrm{im}(\iota_*)) \tag{2}$$

$$\dim(H_1(S_{g,n}) = \dim(\mathrm{im}(\iota_*)) + \dim(\ker(\iota_*)) = \dim(\iota_*) + \dim(\mathrm{im}(\mathrm{id} - f_*)) \tag{3}$$

$$\dim(H_1(S_{g,n}) = \dim(\mathrm{im}(\mathrm{id} - f_*)) + \dim(\ker(\mathrm{id} - f_*)) \tag{4}$$

We can then solve (3) for $\dim(\mathrm{im}(\iota_*))$ and solve (4) for $\dim(\mathrm{im}(\mathrm{id} - f_*))$ to get

$$\dim(\mathrm{im}(\iota_*)) = 2g + n - 1 - \dim(\mathrm{im}(\mathrm{id} - f_*)) \tag{5}$$

$$\dim(\mathrm{im}(\mathrm{id} - f_*)) = 2g + n - 1 - \dim(\ker(\mathrm{id} - f_*)) \tag{6}$$

Lastly, we combine (2), (5), and (6).

$$\begin{aligned}\dim(H_1(M_f)) &= 1 + \dim(\mathrm{im}(\iota_*)) = 2g + n - \dim(\mathrm{im}(\mathrm{id} - f_*)) \\ &= 2g + n + \dim(\ker(\mathrm{id} - f_*)) - (2g + n - 1)) = \dim(\ker(\mathrm{id} - f_*)) + 1.\end{aligned}$$

Note that if $x \in H_1(S_{g,n})$ is fixed by f, then $(\mathrm{id} - f_*)(x) = \mathrm{id}(x) - f_*(x) = x - x = 0$. Conversely, if $x \in \ker(\mathrm{id} - f_*)$, then $(\mathrm{id} - f_*)(x) = 0$ implies that $f_*(x) = x$. So we have that $\kappa(f) = \dim(\ker(\mathrm{id} - f_*))$. Hence, we have that $b_1(M_f) = \dim(H_1(M_f)) = \dim(\ker(\mathrm{id} - f_*)) + 1 = \kappa(f) + 1$, as desired.

For $n - 1 < 2g$, we will show that $b_1(M_f) \geq 2g + 1$. Since $\widehat{f} : S_g \to S_g$, obtained by filling in the punctures of $S_{g,n}$ and extending f to S_g, is isotopic to the identity, then $M_{\widehat{f}} \cong M_{\mathrm{id}} \cong S_g \times S^1$. Thus, there exists a map from $M_f \to S_g \times S^1$ that induces a surjection on the fundamental groups. By the Hurewicz Theorem, we know that $H_1(M_f; \mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(M_f)$. Thus, we have that $\dim(H_1(M_f; \mathbb{R})) \geq \mathrm{rank}(\pi_1(S_g \times S^1)^{ab}) = 2g + 1$. Thus, $\kappa(f) \geq 2g$.

For $2g \leq n - 1$, observe that f fixes the subspace, $\mathcal{P}$, of $H_1(S_{g,n})$ generated by the peripheral curves bounding each puncture because f fixes each puncture. Thus, $\kappa(f) \geq n - 1$, since $\mathcal{P}$ has dimension $n - 1$. □

Proof of Theorem 4.1. By Lemma 4.1, for a pseudo-Anosov $f \in \mathrm{PB}_n(S_g)$, we have that $\dfrac{\kappa(f) + 1}{|\chi(S_{g,n})|} > \dfrac{1}{2}$. This, together with Proposition 4.1, gives our desired lower bound

$$.000155 \leq L(PB_n(S_g)).$$ □

4.2 A Lower Bound for Fixed Number of Punctures

We conclude with a proof of the lower bound which, for fixed n, goes to infinity as g does.

Theorem 4.2. *If $f \in \mathrm{PB}_n(S_g)$ is pseudo-Anosov and $g > 5$, then*

$$\frac{1}{3} \log \left(1 + \frac{\log \left(\frac{g-2}{3} \right) + 2}{160n} \right) \leq \log(\lambda(f)).$$

Proof of Theorem 4.2. By Theorem 2.5, we have a hyperbolic/conformal structure on S_g and an isotopy F_t through quasiconformal maps from the identity to f such that for each t the quasiconformal constant, K_t, satisfies

$$\log(K_t) \leq 3\log(\lambda(f)).$$

Choose a lift, $\widetilde{F}_t$, of F_t to the universal cover, $\mathbb{H}^2$, of S_g so that $\widetilde{F_0}$ is the identity. Therefore, $\widetilde{F}_t$ is the identity on the circle at infinity. Thus, we can apply Theorem 2.4 and Theorem 2.5 to see that

$$\varkappa\left(\max_{x\in\mathbb{H}^2}\rho(x,\widetilde{F}_t(x))\right) \leq \frac{1}{2}\log(K_t) \leq \frac{3}{2}\log(\lambda(f)).$$

Since this holds for all $t \in [0,1]$, we have

$$\varkappa\left(\max_{t\in[0,1]}\max_{x\in\mathbb{H}^2}\rho(x,\widetilde{F}_t(x))\right) \leq \frac{3}{2}\log(\lambda(f)).$$

Note that when measuring distance on the surface we are using the hyperbolic metric, denoted d_{S_g}, and in the hyperbolic plane we are using the Poincarè metric, denoted ρ, which is one-half the hyperbolic metric. Thus, the covering map $\pi : \mathbb{H}^2 \to S_g$ is 2-Lipschitz and for all $x \in \mathbb{H}^2$,

$$d_{S_g}(\pi(x), F_t(\pi(x))) \leq 2\rho(x, \widetilde{F}_t(x)).$$

So we have that

$$\varkappa\left(\max_{t\in[0,1]}\max_{x\in S_g} d_{S_g}(x, F_t(x))\right) \leq 3\log(\lambda(f)).$$

If $\{z_1, \ldots, z_n\}$ are the marked points of S_g such that $S_{g,n} = S_g \setminus \{z_1, \ldots, z_n\}$, then for each i, $\gamma_i : t \mapsto F_t(z_i)$, with $t \in [0,1]$, is a closed curve. Since f is pseudo-Anosov, $\gamma_1 \cup \cdots \cup \gamma_n$ fills S_g. These n curves define the 1-skeleton, Γ, of a cell decomposition of S_g. Thus, for some i,

$$\frac{\operatorname{diam}(\Gamma)}{n} \leq 2\max_{t\in[0,1]} d_{S_g}(z_i, F_t(z_i)).$$

By Theorem 5.1,

$$\frac{\log\left(\frac{g-2}{3}\right) - 2}{40n} \leq \frac{\operatorname{diam}(\Gamma)}{n}.$$

By Theorem 2.4, $\varkappa$ is strictly increasing, so we have that

$$\varkappa\left(\frac{\log\left(\frac{g-2}{3}\right)-2}{80n}\right) \leq \varkappa\left(\frac{\operatorname{diam}(\Gamma)}{2n}\right) \leq \varkappa\left(\max_{t\in[0,1]} d_{S_g}(z_i, F_t(z_i))\right) \leq 3\log(\lambda(f)).$$

Since, by Theorem 2.4, $\log\left(1+\frac{\log\left(\frac{g-2}{3}\right)-2}{160n}\right) \leq \varkappa\left(\frac{\log\left(\frac{g-2}{3}\right)-2}{80n}\right)$, then we have that

$$\frac{1}{3}\log\left(1+\frac{\log\left(\frac{g-2}{3}\right)-2}{160n}\right) \leq \log(\lambda(f)),$$

as desired. $\square$

Chapter 5

Bounding the Diameter of a Surface

joint work with Hugo Parlier

Let S be a closed genus $g \geq 2$ hyperbolic surface and let Γ be the 1-skeleton of a cell decomposition of S. Our goal in this appendix is to provide a lower bound on the diameter of Γ, which we define as

$$\operatorname{diam}(\Gamma) = \max_{x,y \in \Gamma} d_S(x, y).$$

This lower bound is a crucial piece of the proof of Theorem 4.2. For a result related to Theorem 5.1, see [30].

Theorem 5.1. *Let Γ be an embedded graph in S such that $S \setminus \Gamma$ is a collection of disks. If $g > 5$, then*

$$\frac{\log\left(\frac{g-2}{3}\right) - 2}{40} \leq \operatorname{diam}(\Gamma).$$

The first ingredient we will need for the proof of Theorem 5.1 is a type of generalized triangulation of S which consists of both geodesic triangles and a type of annular generalization of a triangle called a **trigon** as defined by Buser; see [9].

Definition 5.1. *Let S be a compact Riemann surface of genus ≥ 2. A closed domain $D \subset S$ is called a* **trigon** *if it is a simply connected, embedded geodesic triangle or if it is a doubly connected, embedded domain, with one boundary component a smooth closed geodesic and the other boundary component two geodesic arcs as shown in Figure 5.1. The closed geodesic and the two arcs are the* **sides** *of D.*

Buser proved that S admits such a triangulation into trigons of controlled size.

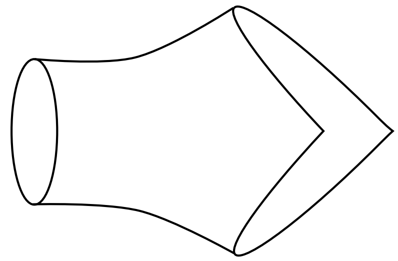

Figure 5.1: A trigon

Theorem 5.2 (Buser [9] Theorem 4.5.2)**.** *Any compact Riemann surface of genus* ≥ 2 *admits a triangulation such that all trigons have sides of length* $\leq \log 4$ *and area between* 0.19 *and* 1.36*. Furthemore, all geodesic triangles have sides of length at least* $\log(2)$.

Suppose we have a generalized triangulation T of S as in Theorem 5.2. We will extend our generalized triangulation to an even more general combinatorial model, T', for S in the following way. First, we note that a computation (which we omit) using equation (iii) of Theorem 2.3.1 in [9] shows that the width (i.e. minimal distance between non-adjacent boundary components) of a doubly connected trigon which occurs in T is at least $\frac{1}{4}$. Next, consider collars of closed geodesics in S_g formed by gluing together two doubly connected trigons along their closed geodesic sides as in Figure 5.2. Now we divide each collar along appropriately chosen simple closed curves (each an equidistant-curve to the closed geodesic) into annuli between simple closed curves and two **generalized trigons** on the ends, so that each annulus or generalized trigon has width between $\frac{1}{4}$ and $\log(2) > \frac{1}{2}$; see the right-hand side of Figure 5.2. Our combinatorial model T' consists of three types of **pieces**: geodesic triangles, generalized trigons, and annuli. Note that each of these pieces is of bounded size.

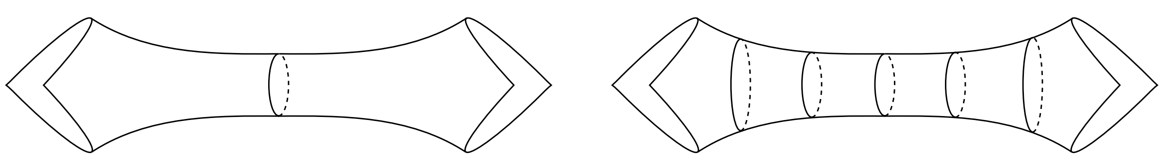

Figure 5.2: A collar formed by two trigons

We can now define the combinatorial length of a geodesic between two points $p, q \in S$ in terms of our combinatorial model T'. For a geodesic segment $\overline{pq} \subset S$ between p and q we define the combinatorial length of $\overline{pq}$, denoted by $\ell_C(\overline{pq})$, as the minimum number of pieces of T' that $\overline{pq}$ passes through. The following lemma establishes an explicit inequality between ℓ_C and the hyperbolic length ℓ_S.

Lemma 5.1. *Let $p, q \in S_g$, let $\overline{pq}$ be a geodesic segment between them, and let T' be the extended combinatorial model of S given above. Then $\ell_C(\overline{pq}) \leq 40 \cdot \ell_S(\overline{pq}) + 2$.*

Proof of Lemma 5.1. Note that $\overline{pq}$ can be subdivided into segments which each lie inside a single piece of T'. Our proof of Lemma 5.1 will consist mainly of analyzing which segments of $\overline{pq}$ are **short** and which are **good**. We will then show that segments of $\overline{pq}$ cannot be short too many times in a row.

There are three types of short segments we will consider, one in each of the three types of pieces. In order to define the first type, we add midpoints to each side of the geodesic triangles in T. A segment which has endpoints on adjacent subdivided pieces of a single geodesic triangle is called **short**. The second type of short segment occurs when $\overline{pq}$ enters and exits an annulus from a single side instead of passing through the entire width of the annulus. In this situation, a segment which has both endpoints on a single boundary component of an annulus will also be considered **short**. The third type of **short** segment occurs when a segment without self intersections has endpoints on adjacent subdivided pieces of the geodesic boundary arcs of a generalized trigon, cutting off a corner, as shown by the blue segment in Figure 5.3. If a segment is not short, then we will call it **good**.

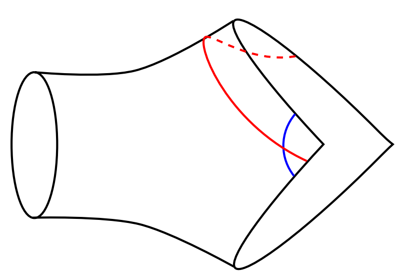

Figure 5.3: Short (blue) and good (red) segments in a generalized trigon

Recall the following formula for a geodesic triangle in the hyperbolic plane where a, b, c

are the sides of the triangle and α, β, γ are the respective opposite angles:

$$\cos(\gamma) = \frac{\cosh(c) - \cosh(a)\cosh(b)}{-\sinh(a)\sinh(b)}. \tag{7}$$

We can find a lower bound on the length of γ for a geodesic triangle in T' by maximizing the length of a and b and minimizing the length of c. Taking $a, b = \log(4)$ and $c = \log(2)$, equation (7) implies that $\gamma > \frac{\pi}{9}$. Thus, there is a lower bound of $\frac{\pi}{9}$ on the interior angles of the triangles in T. So we conclude that $\overline{pq}$ has no more than $\frac{\pi}{\pi/9} = 9$ short segments of the first type in a row. Next, we note that there cannot be two short segments of type two or three in a row. So we can assume that for every ten adjacent segments of $\overline{pq}$, at least one of them is good.

We now establish that good segments have length at least $\frac{1}{4}$. Once again we have three types of segments to consider, the shortest possible good segments within each of our three types of pieces in T'. Within a geodesic triangle in T' the shortest possible good segment is one that joins the midpoints of two sides of a triangle. Once again using equation (7), we see that the length of a good segment is bounded below by

$$\cosh^{-1}\Big(-\sinh(\log(2))\sinh(\log(2))\cos\left(\frac{\pi}{9}\right) + \cosh(\log(2))\cosh(\log(2))\Big) \geq \frac{1}{4}.$$

Within a generalized trigon the shortest possible good segment is a perpendicular segment going from the closed boundary component of the trigon to the midpoint of one of the geodesic arc boundary components. A segment of this type has length at least $\frac{1}{4}$ by our definition of T'. One might think that a shorter possible good segment in a generalized trigon is one passing from one geodesic arc boundary to the other as shown by the red arc in Figure 5.3. However, this red arc has length at least half of the length of the geodesic arc boundary and so it has length at least $\frac{\log(2)}{2} > \frac{1}{4}$. Lastly we have that within an annulus the shortest possible good segment is a perpendicular segment passing from one boundary component to the other, which has length at least $\frac{1}{4}$ since we defined our annuli to have width at least $\frac{1}{4}$.

Thus, at worst we have that $\frac{1}{4} \cdot \frac{\ell_C(\overline{pq})-2}{10} \leq \ell_S(\overline{pq})$, where the -2 comes from the fact that the initial and terminal segments of $\overline{pq}$ can be arbitrarily short depending on where they lie within a piece of T', but still add 2 to $\ell_C(\overline{pq})$. So we have that $\ell_C(\overline{pq}) \leq 40 \cdot \ell_S(\overline{pq}) + 2$, as desired. □

We now define the combinatorial distance, denoted d_C, between two points $p, q \in S_g$ as

$$d_C(p,q) = \inf\{\ell_C(\overline{pq}) \,:\, \overline{pq} \text{ is a geodesic segment between } p \text{ and } q\}.$$

Thus, by Lemma 5.1, we have that $d_C(p,q) \leq 40 \cdot d_S(p,q) + 2$.

Let T_Γ be the subset of T' that minimally covers Γ, where a piece $t \in T'$ belongs to T_Γ if $\Gamma \cap t \neq \varnothing$. We will denote by Γ' the 1-skeleton of T_Γ together with a geodesic arc for each generalized trigon and annulus in T' as shown by the dotted arc in Figure 5.4, which ensures Γ' is connected.

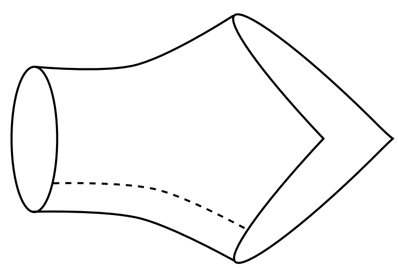

Figure 5.4: A generalized trigon's contribution to Γ'

We will need one more fact relating the length of Γ' to the genus of our surface before we continue to the proof of our main result.

Lemma 5.2. *Let* Γ' *be as above. Then*

$$\ell(\Gamma') = \sum_{e \in \Gamma'} \ell(e) > 2\pi(g-1).$$

Proof of Lemma 5.2. Note that if α is a simple closed curve that intersects Γ, then it must intersect Γ'. Thus, Γ' fills S_g, since Γ does. Now because Γ' fills, it cuts S_g into polygons. The sum of the lengths around these polygons is $2\ell(\Gamma')$, while the sum of their areas is $4\pi(g-1)$.

Now recall that the maximum area $A(p)$ enclosed by a loop of length p in the hyperbolic plane is at most the area of a circle of radius $r = \sinh^{-1}\left(\frac{p}{2\pi}\right)$. Therefore,

$$A(p) \leq 4\pi \sinh^2\left(\frac{\sinh^{-1}\left(\frac{p}{2\pi}\right)}{2}\right) \leq 4\pi \sinh^2\left(\frac{1}{2}\log\left(1+\frac{p}{\pi}\right)\right) = \frac{p^2}{p+\pi} < p.$$

Applying this inequality to each of the polygons and summing, we have $4\pi(g-1) \leq 2\ell(\Gamma')$, as desired $\square$

Lemma 5.2 implies that T_Γ contains at least $g-1$ pieces of T', since each piece contributes a length of at most $3\log(4) > 2\pi$ to $\ell(\Gamma')$. We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. Let T' and T_Γ be as described previously. Consider the graph G in S_g which is dual to T', that is the vertices of G each correspond to a piece of T' and edges in G correspond to shared boundary components. Note that each vertex of G has valence at most 3. Thus, we if we take a base piece $\Delta_0 \in T_\Gamma \subset T'$, then we know that at combinatorial distance d from Δ_0 there are at most $3 \cdot 2^{d-1} + 1$ pieces in T'. This is because a ball of radius d in G has size at most $3 \cdot 2^{d-1}$. So, unless $g - 1 < 3 \cdot 2^{d-1} + 1$, there is a piece of T_Γ not in this ball. Hence, the combinatorial diameter of T_Γ (within T') is at least $\log\left(\frac{g-2}{3}\right) < \log_2\left(\frac{g-2}{3}\right) < \mathrm{diam}_C(T_\Gamma)$ for $g > 5$ and we are done. $\square$

Epilogue

I don't believe that voice of doubt anymore. I *am* enough.

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