A Simple, Velocity Dependent, Collision Probability Algorithm for Small Combined Hard Body Radii Close Approach Events

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Abstract

Over the last few years progress has been made in the velocity dependent collision probability problem. Recent progress in this problem has been approached via integration of the probability flux into the surface of the combined hard body object. The algorithm presented uses the surface flux approach and is designed to compute the collision probability rate between two objects given their time dependent states and state error covariance matrices. With regards to the computation of the collision probability (Pc) rate, two major differences exist between the development of this algorithm and the usual Pc algorithm. First, the shape of the at-risk volume is assumed to be a cube rather than a sphere. The size of the cube is chosen so that it circumscribes the usual hard body sphere chosen for the spherical Pc problem. This will result in half the length of a side of the cube being equal to the hard body radius (HBR) of the sphere. Second, it is assumed that the HBR of the cube is much smaller than the smallest combined position uncertainty (σ_{min} > 5*HBR) at each time point of evaluation. This is necessary as the actual collision probability calculation is based on a first order, small variable expansion in the position components of the Gaussian probability density function. This leads to the collision probability rate being to second order in the combined HBR. The collision probability rate is in a concise, closed form containing exponential and error functions.

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History of the Foster 2D Pc

- STS (Space Shuttle) orbital debris collision risk concern arose following the Challenger accident
  - No specific cause, just part of enhanced safety investigations
  - Problematic as GP state uncertainties for debris were not available
  - A simple, rectangular box centered on a Shuttle was selected for collision avoidance clearing
- About the same time, development for Space Station Freedom was in progress
  - Unlike with the Space Shuttle program, Freedom would be in orbit continuously for years
  - This elevated the potential collision risk with cataloged orbital debris
  - Lee Foster, Herb Estes and Mark Powell developed the 2D Pc concept

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2D Pc Assumptions

- State vectors and position vector error covariance matrices are available for both objects at the time of closest approach
  - The two solutions are uncorrelated
  - Uncertainties (position only) are multivariate Gaussian
- Duration of the close approach event is only a few seconds
  - Hypervelocity passes involving relative speeds around 10-15 kps
  - In the vicinity of the close approach trajectories are straight lines
  - Relative position uncertainty moves in the relative velocity direction
  - Combined relative position uncertainty passes the point of closest approach, $+n\sigma$ to $-n\sigma$, in a time too short to allow for the position error covariance matrix to change size, shape or orientation
- Position error sigmas are small compared to orbital curvature

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Standard algorithm for the 2D Pc

$$P_c = \iint \frac{e^{-0.5(r-\bar{r})^T C^{-1}(r-\bar{r})}}{2\pi\sqrt{|C|}} dA$$

Where:

- r: generalized relative position in the collision plane
- $\bar{r}$: mean (center) position of the probability density function
- C: 2D position error covariance matrix in the collision plane
- A: circular at risk area defined by the hard body radius, HBR

Closed form algorithm for the 2D Pc

$$P_c = \int_{-b}^{b} \frac{e^{-0.5\left[(x-\overline{x})^2 / \sigma_x^2\right]}}{\sqrt{2\pi}\sigma_x} dx \cdot \int_{-b}^{b} \frac{e^{-0.5\left[(y-\overline{y})^2 / \sigma_y^2\right]}}{\sqrt{2\pi}\sigma_y} dy$$

$$P_C = \frac{\left[Erf\left(\frac{(\overline{x}+b)}{\sqrt{2}\sigma_x}\right) - Erf\left(\frac{(\overline{x}-b)}{\sqrt{2}\sigma_x}\right)\right]}{2} \cdot \frac{\left[Erf\left(\frac{(\overline{y}+b)}{\sqrt{2}\sigma_y}\right) - Erf\left(\frac{(\overline{y}-b)}{\sqrt{2}\sigma_y}\right)\right]}{2}$$

Where

x, y : position components, 2D covariance principal axis frame

$\bar{x}, \bar{y}$: mean (center) positions, principal axis frame

σ_x, σ_y : standard deviations of the position error, principal axis frame

b : characteristic length (half length of the side of the square)
(if b=HBR, over estimate of Pc; b=0.5·HBR·$\sqrt{\pi}$ best on average)

A General 3D (Velocity Dependent) Pc Problem

- State vectors and state vector error covariance matrices are available for both objects during the interval of closest approach, which may be arbitrarily defined.
  - The two solutions are uncorrelated
  - Position and velocity uncertainties are multivariate Gaussian
  - Correlations between any two state elements may exist
- No restriction on relative velocity (speed)
- No restriction on linear/nonlinear nature of the individual or relative trajectories
- Position error sigmas are small compared to orbital curvature
- Small object restriction for the algorithm presented here: HBR < (smallest position sigma)/5

3D (Velocity Dependent) Pc Problem (approach)

- Solution is approached as a probability density flux problem
- Proceed very much like solving a control volume problem in fluid mechanics or aerodynamics
- Rate of accumulation of probability is determined by the probability flux through the surface (inward only, this goes directly to how a collision is defined)
- Make axis and control volume choices for convenience and to simplify the problem
- Consider how to approximate and solve the resulting problem

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The small object 3D Pc algorithm

<u>Step 1</u>: Advance the vectors and matrices to the next time step.

<u>Step 2</u>: Compute the relative position, velocity and state error covariance matrix:

$$\mathbf{R} = \mathbf{R}_2 - \mathbf{R_1} = [\mathrm{X} \quad \mathrm{Y} \quad \mathrm{Z}]^{\mathrm{T}}$$

$$\mathbf{V} = \mathbf{V_2} - \mathbf{V_1} = [\mathrm{U} \quad \mathrm{V} \quad \mathrm{W}]^{\mathrm{T}}$$

$$\mathbf{C} = \mathbf{C_2} + \mathbf{C_1} = \begin{bmatrix} \mathbf{C}_{\mathbf{RR}} & \mathbf{C}_{\mathbf{R}V} \\ \mathbf{C}_{\mathbf{VR}} & \mathbf{C}_{\mathbf{VV}} \end{bmatrix}$$

With $\mathbf{C}_{\mathbf{vv}}$ further detailed as:

$$\begin{bmatrix} \mathrm{c}_{\dot{x}\dot{x}} & \mathrm{c}_{\dot{x}\dot{y}} & \mathrm{c}_{\dot{x}\dot{z}} \\ \mathrm{c}_{\dot{x}\dot{y}} & \mathrm{c}_{\dot{y}\dot{y}} & \mathrm{c}_{\dot{y}\dot{z}} \\ \mathrm{c}_{\dot{x}\dot{z}} & \mathrm{c}_{\dot{y}\dot{z}} & \mathrm{c}_{\dot{z}\dot{z}} \end{bmatrix}$$

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The small object 3D Pc algorithm (continued)

Step 3: Determine the transformation, T, necessary to diagonalize the velocity partition, $\mathbf{C}_{VV}$, of the matrix $\mathbf{C}$.

$$\mathbf{C}^* = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{RR} & \mathbf{C}_{RV} \\ \mathbf{C}_{VR} & \mathbf{C}_{VV} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{\mathrm{T}} \end{bmatrix}$$

$$\mathbf{C}^* = \begin{bmatrix} c_{xx} & c_{xy} & c_{xz} & c_{xu} & c_{xv} & c_{xw} \\ c_{xy} & c_{yy} & c_{yz} & c_{yu} & c_{yv} & c_{yw} \\ c_{xz} & c_{yz} & c_{zz} & c_{zu} & c_{zv} & c_{zw} \\ c_{xu} & c_{yu} & c_{zu} & c_{uu} & 0 & 0 \\ c_{xv} & c_{yv} & c_{zv} & 0 & c_{vv} & 0 \\ c_{xw} & c_{yw} & c_{zw} & 0 & 0 & c_{ww} \end{bmatrix}$$

The transformed mean state vector is noted as:

$$\mathbf{S}_{\mu} = [\mu_x \quad \mu_{\mathrm{y}} \quad \mu_{\mathrm{z}} \quad \mu_u \quad \mu_v \quad \mu_w]^{\mathbf{T}}$$

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The small object 3D Pc algorithm (continued)

Step 4: A reduction in dimension by 2 may be made which is facilitated by the diagonalization of $\mathbf{C}_{\mathrm{VV}}$ and the choice of a cube as the hard body figure. This reduction results in three separate problems to be solved, one corresponding to each of the principal axis directions of the diagonalized velocity error covariance matrix. (The w component steps are presented here.)

$$\mathbf{S}_{\mathbf{w}} = [x \quad y \quad z \quad w]^{\mathbf{T}}$$

$$\mathbf{S}_{\mathbf{\mu w}} = [\mu_x \quad \mu_y \quad \mu_z \quad \mu_w]^{\mathbf{T}}$$

$$\mathbf{CW} = \begin{bmatrix} c_{xx} & c_{xy} & c_{xz} & c_{xw} \\ c_{xy} & c_{yy} & c_{yz} & c_{yw} \\ c_{xz} & c_{yz} & c_{zz} & c_{zw} \\ c_{xw} & c_{yw} & c_{zw} & c_{ww} \end{bmatrix}$$

The small object 3D Pc algorithm (continued)

<u>Step 4</u>: (continued) additional definitions of convenient terms

$$\mathrm{DCW} = |\mathbf{CW}|$$

$$\mathbf{BW} = \mathbf{CW}^{-1}$$

$$\mathbf{BW} = \begin{bmatrix} b_{xx} & b_{xy} & b_{xz} & b_{xw} \\ b_{xy} & b_{yy} & b_{yz} & b_{yw} \\ b_{xz} & b_{yz} & b_{zz} & b_{zw} \\ b_{xw} & b_{yw} & b_{zw} & b_{ww} \end{bmatrix}$$

$$\mathbf{Bw} = [b_{xw} \quad b_{yw} \quad b_{zw} \quad b_{ww}]^{\mathbf{T}}$$

$$\mathbf{S}_{\mu 0} = [\mu_x \quad \mu_y \quad \mu_z \quad 0]^{\mathbf{T}}$$

The small object 3D Pc algorithm (continued)

Intermediate forms of the probability density function that are used to create the axis by axis contributions to dPc/dt at any time t

$$pdf_x = \frac{e^{-0.5\left[(\boldsymbol{S}_u - \boldsymbol{S}_{\mu u})^T \boldsymbol{BU} (\boldsymbol{S}_u - \boldsymbol{S}_{\mu u})\right]}}{4\pi^2\sqrt{DCU}}$$

$$pdf_y = \frac{e^{-0.5\left[(\boldsymbol{S}_v - \boldsymbol{S}_{\mu v})^T \boldsymbol{BV} (\boldsymbol{S}_v - \boldsymbol{S}_{\mu v})\right]}}{4\pi^2\sqrt{DCV}}$$

$$pdf_z = \frac{e^{-0.5\left[(\boldsymbol{S}_w - \boldsymbol{S}_{\mu w})^T \boldsymbol{BW} (\boldsymbol{S}_w - \boldsymbol{S}_{\mu w})\right]}}{4\pi^2\sqrt{DCW}}$$

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3D Pc algorithm (intermediate discussion)

- So far, all is still general, no approximations
- Assume we would like a lower order approximation to the Pc
- Suppose we choose to do a small variable expansion of the 4D probability density function (pdf) in terms of only the position components
- What order do we choose?
  - When we integrate the pdf with respect to the position components we automatically get two orders
  - Do a zeroth order expansion of the pdf
  - Resulting rate of Pc accumulation will be to second order in the characteristic length, b, of the cube
- Integrate with respect to the x, y and w components of the relative state (this is for the w component specifically)

The small object 3D Pc algorithm (continued)

Step 5: The axis by axis parts of the time derivative of Pc at the current time of evaluation are given by:

$$\left(\dot{\mathrm{P}}_{\mathrm{C}}\right)_u = \frac{2\cdot \mathrm{b}^2 \cdot \mathrm{e}^{-0.5\cdot \mathbf{S}_{\mu 0}^{\mathrm{T}}\cdot \mathbf{BU}\cdot \mathbf{S}_{\mu 0}}}{\pi^2 \cdot \sqrt{\mathrm{DCU}}\cdot \mathrm{b}_{uu}} \cdot \left(1 + \frac{\sqrt{\pi}\cdot \mathbf{Bu}^{\mathrm{T}} \cdot \mathbf{S}_{\mu u} \cdot \mathrm{e}^{\left(\mathbf{Bu}^{\mathrm{T}}\cdot \mathbf{S}_{\mu 0}\right)^2 / (2\cdot \mathrm{b}_{uu})}}{\sqrt{2\cdot b_{uu}}} \cdot \mathrm{Erf}\left(\frac{\mathbf{Bu}^{\mathrm{T}}\cdot \mathbf{S}_{\mu 0}}{\sqrt{2\cdot \mathrm{b}_{uu}}}\right)\right)$$

$$\left(\dot{\mathrm{P}}_{\mathrm{C}}\right)_v = \frac{2\cdot \mathrm{b}^2 \cdot \mathrm{e}^{-0.5\cdot \mathbf{S}_{\mu 0}^{\mathrm{T}}\cdot \mathbf{BU}\cdot \mathbf{S}_{\mu 0}}}{\pi^2 \cdot \sqrt{\mathrm{DCV}}\cdot \mathrm{b}_{vv}} \cdot \left(1 + \frac{\sqrt{\pi}\cdot \mathbf{Bv}^{\mathrm{T}} \cdot \mathbf{S}_{\mu v} \cdot \mathrm{e}^{\left(\mathbf{Bv}^{\mathrm{T}}\cdot \mathbf{S}_{\mu 0}\right)^2 / (2\cdot \mathrm{b}_{vv})}}{\sqrt{2\cdot \mathrm{b}_{vv}}} \cdot \mathrm{Erf}\left(\frac{\mathbf{Bv}^{\mathrm{T}}\cdot \mathbf{S}_{\mu 0}}{\sqrt{2\cdot \mathrm{b}_{vv}}}\right)\right)$$

$$\left(\dot{\mathrm{P}}_{\mathrm{C}}\right)_w = \frac{2\cdot \mathrm{b}^2 \cdot \mathrm{e}^{-0.5\cdot \mathbf{S}_{\mu 0}^{\mathrm{T}}\cdot \mathbf{BW}\cdot \mathbf{S}_{\mu 0}}}{\pi^2 \cdot \sqrt{\mathrm{DCW}}\cdot \mathrm{b}_{ww}} \cdot \left(1 + \frac{\sqrt{\pi}\cdot \mathbf{Bw}^{\mathrm{T}} \cdot \mathbf{S}_{\mu w} \cdot \mathrm{e}^{\left(\mathbf{Bw}^{\mathrm{T}}\cdot \mathbf{S}_{\mu 0}\right)^2 / (2\cdot \mathrm{b}_{ww})}}{\sqrt{2\cdot \mathrm{b}_{ww}}} \cdot \mathrm{Erf}\left(\frac{\mathbf{Bw}^{\mathrm{T}}\cdot \mathbf{S}_{\mu 0}}{\sqrt{2\cdot \mathrm{b}_{ww}}}\right)\right)$$

$$\dot{\mathrm{P}}_{\mathrm{C}}(\mathrm{t}) = \left(\dot{\mathrm{P}}_{c}\right)_u + \left(\dot{\mathrm{P}}_{c}\right)_v + \left(\dot{\mathrm{P}}_{c}\right)_w$$

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The small object 3D Pc algorithm (continued)

Step 6: Integrate, by any appropriate technique, the time derivative of the probability over the time interval of interest

$$P_c(t_o, t_f) = \int_{t_o}^{t_f} \dot{\mathrm{P}}_{\mathrm{C}}(\mathrm{t})\,dt$$

Comments on the characteristic dimension, b

- Choosing b:
  - $b = HBR$ this is the most natural upper limit, circumscribed cube
  - $b = \frac{1}{\sqrt{3}} HBR$ this is the most natural lower limit, inscribed cube
  - $b = \frac{\sqrt{\pi}}{\sqrt{6}} HBR$ this may be the likely value, equivalent surface area
- Is b small enough?
  - For $b \leq 1m$, little doubt as to applicability
  - For $1\text{m} < b \leq 5m$, probably always applicable
  - For $b > 5\ m$, will go from maybe to no (like for the ISS: HBR=70m)
- What is the smallest position sigma to compare b against?
  - Preliminary interval integration over (t_1, t_2) to find best (t_o, t_f)
  - Save states and covariance matrices (ephemeris)

Future work

- Publish results on a higher order solution that removes the small HBR requirement
  - Actual higher order algorithm is complete
  - Would like to provide examples of application
  - Include this small HBR algorithm as a specific subcase
- Work on removing the assumption that position error sigmas are small compared to the orbital curvature

The End

- Thank you for your time and attention
- Any questions?