

L(j, k)-labeling Number of Generalized Petersen Graph

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Abstract. For $j \leq k$, the $L(j, k)$ -labeling arose from code assignment problem in the computer wireless network. For positive real numbers j and k , an $L(j, k)$ -labeling f of G is an assignment of numbers to vertices of G such that $|f(u) - f(v)| \geq j$ if u, v are adjacent, and $|f(u) - f(v)| \geq k$ if u, v are distance two apart. The span of f is the maximum difference among the numbers assigned by f . The $L(j, k)$ -labeling number of G , denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j, k)$ -labeling of G . The generalized Petersen graph, denoted by $G(n, k)$, is a graph with vertex set $\{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$ and edge set $\{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+k}) : i = 0, \dots, n-1\}$, where subscripts are to be taken modulo n and $k \leq \lfloor \frac{n}{2} \rfloor$. In this paper, the author determines the $L(j, k)$ -labeling numbers of generalized Petersen graphs $G(n, 1)$, $G(n, 2)$ and $G(n, \frac{n}{2})$, where n is even and $2j \leq k$.

1. Introduction

In a large network, the number of transmission codes is smaller than the number of network nodes. The time overlap of two or more packet receptions at the destination station may take place. That is called collision or interference. The collision or interference causes damaged useless packets at destination. Collided packets must be retransmitted, thus the time delay of transmission increases, and the system throughput lowers. In order to reduce or eliminate the collision or interference, several protocols have been devised. For example, using code division multiple accesses (CDMA) in packet radio network permits multiple stations within range of the same receivers to transmit simultaneously, without collision or interference. Two stations are called adjacent if two stations can directly transmit to each other. If two stations are nonadjacent but they are adjacent to same station, they are called at distance two. There exist two types of interference in a PRN using CDMA. Direct interference is due to two adjacent stations transmitting to each other concurrently. Secondary interference can occur when two transmitters or receivers at distance 2 have the same code.

In order to avoid secondary interference, Bertossi and Bonuccelli [1] introduced a kind of code assignment, that is, assign disparate codes to each pair of vertices at distance two and use the minimum number of different codes. By corresponding codes to labels, the above problem is equivalent to an $L(0, 1)$ -labeling problem, where vertices at distance two must be assigned different labels. Based on this premise, Jin and Yeh [2] generalized the code assignment problem to $L(j, k)$ -labeling problem, where $j \leq k$. It means that any two adjacent stations are required to be assigned j -separated codes to avoid direct interference, then any two stations at distance two need to be assigned k -separated codes to avoid secondary interference, as well as to avoid direct interference. The network can be modelled as an undirected graph $G = (V, E)$, such that the set of vertices $V = \{v_0, v_1, \dots, v_{n-1}\}$



represents the set of stations, and the set of edges E represents the relationship between two adjacent stations. That is, two vertices v_i and v_j in V are joined by an undirected edge e_{ij} in E if and only if the stations corresponding to vertices v_i and v_j are adjacent. Note that there is a one-to-one mapping of the stations onto the vertices in V . Then, the $L(j, k)$ -labeling of graph G can be defined as follows.

For positive real numbers j and k , an $L(j, k)$ -labeling f of G is an assignment of numbers to vertices of G such that $|f(u) - f(v)| \geq j$ if u, v are adjacent, and $|f(u) - f(v)| \geq k$ if u, v are distance two apart. The span of f is the maximum difference among the numbers assigned by f . The $L(j, k)$ -labeling number of G , denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j, k)$ -labeling of G . All notation not defined in this article can be found in the book [3]. Recently, the $L(j, k)$ -labeling problem for $j \leq k$ has been paid attention to study since it can be applied in computer network to solve the problem of the scarcity of available codes for communication with minimum interference, please refer to [1, 2, 4, 5, 6, 7, 8, 9, 10, 11] for $j \leq k$.

In 1950, the generalized Petersen graph was introduced by Coxeter[12] and Watkins[13] gave names to these graphs in 1969. The generalized Petersen graph, denoted by $G(n, k)$, is a graph with vertex set $\{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$ and edge set $\{(u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+k}) : i = 0, \dots, n-1\}$, where subscripts are to be taken modulo n and $k \leq \lfloor \frac{n}{2} \rfloor$. Two labels are t -separated if the difference between them is at least t . A set S is t -separated if any two members of S are t -separated. The length of interval $[a, b]$, denote by $[a, b]$, is equal to the number $b - a$. And $[a]_n$ is defined as the remainder when a is divided by n .

Lemma 1.1. Let j and k be two positive numbers with $j \leq k$. Suppose G is a graph and H is an induced subgraph of G . Then $\lambda_{j,k}(G) \geq \lambda_{j,k}(H)$.

Note that Lemma 1.1 is not true if H is not an induced subgraph.

2. $L(j, k)$ -Labelling number of $G(n, 1)$

This section introduces the $L(j, k)$ -Labelling number of $G(n, 1)$. Since the graph $G(n, 1)$ is isomorphic to the Cartesian product of path P_2 and cycle C_n , denoted by $P_2 \square C_n$, then $\lambda_{j,k}(P_2 \square C_n) = \lambda_{j,k}(G(n, 1))$. Fortunately, we have determined the $L(j, k)$ -Labeling number of $P_2 \square C_n$ for $k \geq 2j$ in 2016 [10], then we can obtain the following results directly.

Theorem 2.1. [10] Let n be a positive integer and j, k be two positive real numbers. For $k \geq 2j$,

$$\lambda_{j,k}(G(n, 1)) = \begin{cases} 3j + k, & \text{if } n = 3; \\ 2k + j, & \text{if } n \equiv 0 \pmod{6}; \\ 2k + 2j, & \text{if } n \equiv 3 \pmod{6} \text{ and } n \geq 9; \\ 3k, & \text{if } n \not\equiv 0 \pmod{3}. \end{cases}$$

3. $L(j, k)$ -Labelling number of $G(n, 2)$

In this section, let n be a positive integer and j, k be two positive real numbers. For $n \geq 4$ and $k \geq 2j$, we have the following results.

Theorem 3.1. For $k \geq 2j$, $\lambda_{j,k}(G(4, 2)) = 2j + 2k$.

Proof. For $k \geq 2j$, given a labeling f for graph $G(4, 2)$ as follows.

Let $f(u_0) = 0$, $f(u_1) = j$, $f(u_2) = k$, $f(u_3) = j + k$, $f(v_0) = f(v_3) = j + 2k$ and $f(v_1) = f(v_2) = 2j + 2k$. It is easy to verify that f satisfies the conditions of $L(j, k)$ -labeling and the span is $2j + 2k$, that is, $\lambda_{j,k}(G(4, 2)) \leq 2j + 2k$.

On the other hand, suppose $\lambda_{j,k}(G(4, 2)) < 2j + 2k$. Let $\lambda_{j,k}(G(4, 2)) = \lambda$ and f be a λ - $L(j, k)$ -labeling of $G(4, 2)$. At first, we consider the vertices u_0, u_2, v_1, v_3 . Since vertices u_0, v_1, u_2 are at

distance two mutually, then labels $\{f(u_0), f(v_1), f(u_2)\}$ should be k -separated. By the symmetry of the graph H , without loss of generality, let $f(u_0) < f(v_1) < f(u_2)$. Similarly, v_3 is also distance two apart from vertices u_0 and u_2 , and $d(v_1, v_3) = 1$. According to definition of $L(j, k)$ -labeling, we have following cases.

- If $f(v_3) < f(u_0)$ or $f(v_3) > f(u_2)$, then $\lambda \geq 3k$.
- If $f(u_0) < f(v_3) < f(v_1)$ or $f(v_1) < f(v_3) < f(u_2)$, then $\lambda \geq j + 2k$.

That is, $\lambda \geq j + 2k$.

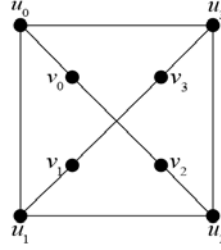


Figure 1: Graph $G(4, 2)$

For convenience, let $f(u_0) = f_0$, $f(v_1) = f_1$, $f(v_3) = f_2$, $f(u_2) = f_3$ and label interval $[0, \lambda] = \bigcup_{i=0}^5 I_i$, where subintervals $I_0 = [0, j)$, $I_1 = [j, k)$, $I_2 = [k, j+k)$, $I_3 = [j+k, 2k)$, $I_4 = [2k, j+2k)$ and $I_5 = [j+2k, \lambda]$. According to above discussion, without loss of generality, let $0 \in \{f_0, f_1, f_2, f_3\}$ and $f_1 < f_2$, then the order of labels has the following cases.

- If $f_0 = 0 < f_1 < f_2 < f_3$, it implies that $f_1 \in I_2$, $f_2 \in I_3$ and $f_3 \in I_5$. Since u_1, u_3 are distance two apart from or adjacent to vertices u_0, u_2, v_1, v_3 , then $f(u_1), f(u_3)$ should be j -separated from labels f_i for $i = 0, 1, 2, 3$. It means that $f(u_1), f(u_3)$ lie in subintervals I_1 and I_4 . Since two adjacent vertices v_0, v_2 are at distance two from vertices u_1, u_3 , then labels $f(v_0), f(v_2)$ should be k -separated from $f(u_1), f(u_3)$ at same time, it means that $f(v_0), f(v_2) \in [j+k, \lambda] \cap [0, j+k) = \emptyset$ which is a contradiction.
- If $f_0 = 0 < f_3 < f_1 < f_2$, then $f_3 \in I_2$, $f_1 \in I_4$ and $f_2 \in I_5$. Similar to above case, since u_1, u_3 are distance two apart from or adjacent to vertices u_0, u_2, v_1, v_3 and two adjacent vertices v_0, v_2 are at distance two from vertices u_1, u_3 , then $f(u_1), f(u_3)$ should lie in subintervals I_1 and I_3 and $f(v_0), f(v_2) \in [j+2k, \lambda]$. It is a contradiction since $f(v_0), f(v_2)$ are j -separated but $|[j+2k, \lambda]| < j$.
- If $f_1 = 0 < f_2 < f_0 < f_3$, it means that $f_2 \in I_1$, $f_0 \in I_3$ and $f_3 \in I_5$. Similar to the reasons in the second case, we have $f(v_0), f(v_2) \in [0, j)$. It is a contradiction since $f(v_0), f(v_2)$ are j -separated but $|[0, j)| < j$.

Thus, $\lambda \geq 2j + 2k$ for $k \geq 2j$. Hence, $\lambda_{j,k}(G(4, 2)) = 2j + 2k$ for $k \geq 2j$.

Lemma 3.2. [5] Let C_n be a cycle with n vertices ($n \geq 4$). For $k \geq 2j$,

$$\lambda_{j,k}(C_n) = \begin{cases} j+k, & \text{if } n \equiv 0 \pmod{4} \\ 2k, & \text{otherwise} \end{cases}.$$

Theorem 3.3. For $k \geq 2j$, $\lambda_{j,k}(G(5, 2)) = j + 4k$.

Proof. For $k \geq 2j$, given a labeling f for graph $G(5, 2)$ as follows.

Let $f(u_0) = 0$, $f(u_1) = j$, $f(u_2) = k$, $f(u_3) = j+k$, $f(u_4) = 2k$, $f(v_0) = j+3k$, $f(v_1) = j+4k$,

$f(v_2) = 3k$, $f(v_3) = 4k$, and $f(v_4) = j + 2k$.

It is easy to verify that f satisfies the conditions of $L(j, k)$ -labeling and the span is $j + 4k$, that is, $\lambda_{j,k}(G(5, 2)) \leq j + 4k$.

On the other hand, let f be a λ - $L(j, k)$ -labeling of graph $G(5, 2)$. Each vertex set of $\{u_0, u_1, \dots, u_4\}$ and $\{v_0, v_1, \dots, v_4\}$ induces a cycle C_5 , respectively, by Lemma 3.2, $\lambda_{j,k}(C_5) = 2k$. It means that the span of labels of C_n is at least $2k$. And each pair of the vertices are adjacent or distance two apart, that is, $|f(v_s) - f(u_k)| \geq j$ or $s, t \in \mathbb{Z}_5$. Thus, $\lambda \geq 2k + j + 2k = j + 4k$. Hence, $\lambda_{j,k}(G(5, 2)) = j + 4k$ for $k \geq 2j$.

Theorem 3.4. For $k \geq 2j$, $\lambda_{j,k}(G(6, 2)) = 2j + 2k$.

Proof. For $k \geq 2j$, given a labeling f for graph $G(6, 2)$ follows.

Let $f(u_0) = f(u_3) = j$, $f(u_1) = f(u_4) = 2j + 2k$, $f(u_2) = f(u_5) = j + k$, $f(v_0) = f(v_3) = 0$, $f(v_1) = f(v_4) = j + 2k$, $f(v_2) = f(v_5) = 2j + k$.

It is easy to verify that f satisfies the conditions of $L(j, k)$ -labeling and the span is $2j + 2k$, that is, $\lambda_{j,k}(G(6, 2)) \leq 2j + 2k$.

On the other hand, suppose $\lambda_{j,k}(G(6, 2)) = \lambda < 2j + 2k$. Let f be a λ - $L(j, k)$ -labeling of graph $G(6, 2)$. According to the symmetry of the graph, without loss of generality, let $f(v_0) = 0$ or $f(u_0) = 0$.

- (1) If $f(u_0) = 0$, then $f(u_2), f(u_4) \in [k, \lambda]$ since vertices u_2, u_4 are distance two apart from vertex u_0 . By the symmetry of the graph, without loss of generality, let $f(u_2) > f(u_4)$. Table 1 is the following procedure of the discussion. According to the conclusion, we have $f(u_2), f(v_2) \in [j + 2k, \lambda]$, it is a contradiction since $|[j + 2k, \lambda]| < j$ but $d(u_2, v_2) = 1$.

Table 1. The procedure of the induction

Order	Conclusion	Reason
1	$f(u_2) \in [2k, \lambda], f(u_4) \in [k, \lambda - k]$	$d(u_2, u_4) = 2, f(u_2) > f(u_4)$
2	$f(v_0) \in [j, \lambda - 2k]$	1, $ [2k, \lambda] < j, [k, \lambda - k] < j$
3	$f(u_4) \in [j + k, \lambda - k], f(u_2) \in [j + 2k, \lambda]$	1, 2, $d(v_0, u_4) = 2, d(v_0, u_2) = 2$
4	$f(v_2) \in [j + 2k, \lambda]$	2, 3 $d(v_2, u_0) = 2, d(v_2, u_4) = 2$

- (2) If $f(v_0) = 0$, similar to above case, $f(u_2), f(u_4) \in [k, \lambda]$ since $d(v_0, u_4) = 2, d(v_0, u_2) = 2$. By the symmetry of the graph, without loss of generality, let $f(u_2) > f(u_4)$. Table 2 is the following procedure of the discussion.

Table 2. The procedure of the induction

Order	Conclusion	Reason
1	$f(u_2) \in [2k, \lambda], f(u_4) \in [k, \lambda - k]$	$d(u_2, u_4) = 2, f(u_2) > f(u_4)$
2	$f(u_0) \in [j, \lambda - 2k]$	1, $ [2k, \lambda] < j, [k, \lambda - k] < j$
3	$f(u_4) \in [j + k, \lambda - k], f(u_2) \in [j + 2k, \lambda]$	1, 2, $d(u_0, u_4) = 2, d(u_0, u_2) = 2$
4	$f(v_2) \in [j + 2k, \lambda]$	2, 3 $d(v_2, u_0) = 2, d(v_2, u_4) = 2$

It is a contradiction since $f(u_2), f(v_2) \in [j + 2k, \lambda]$, $|[j + 2k, \lambda]| < j$ and $d(u_2, v_2) = 1$.

That is, $\lambda_{j,k}(G(6, 2)) \geq 2j + 2k$ for $k \geq 2j$. Hence, $\lambda_{j,k}(G(6, 2)) = 2j + 2k$ for $k \geq 2j$.

Theorem 3.5. For $n \equiv 0 \pmod{3}$ and $n \geq 9$, $\lambda_{j,k}(G(n,2)) \leq j+3k$.

Proof. For $n \equiv 0 \pmod{3}$ and $n \geq 9$, define an $L(j,k)$ -labeling f for the graph $G(n,2)$ as follows.

$$\text{Let } f(u_s) = \begin{cases} 0, & \text{if } s \equiv 0 \pmod{3}; \\ j+k, & \text{if } s \equiv 1 \pmod{3}; \\ 3k, & \text{if } s \equiv 2 \pmod{3}, \end{cases} \quad f(v_t) = \begin{cases} j, & \text{if } t \equiv 0 \pmod{3}; \\ 2k, & \text{if } t \equiv 1 \pmod{3}; \\ j+3k, & \text{if } t \equiv 2 \pmod{3}. \end{cases}$$

It is not difficult to verify that f is a $(j+3k)$ - $L(j,k)$ -labeling of graph $G(n,2)$, where $n \equiv 0 \pmod{3}$ and $n \geq 9$, that is, $\lambda_{j,k}(G(n,2)) \leq j+3k$. for $n \equiv 0 \pmod{3}$ and $n \geq 9$.

4. $L(j,k)$ -Labelling number of $G(n, \frac{n}{2})$ for even n

In this section, let n be an even positive integer and j, k be two positive real numbers. For $n \geq 6$ and $k \geq 2j$, we have the following results.

Theorem 4.1. For $k \geq 2j$, $\lambda_{j,k}(G(6,3)) = j+3k$.

Proof. For $k \geq 2j$, given a labeling f for graph $G(6,3)$ as Figure 2 shows.

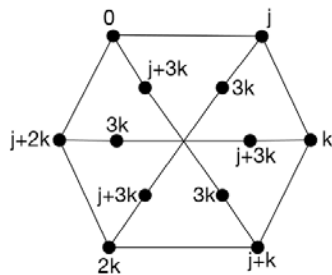


Figure 2: A $(j+3k)$ - $L(j,k)$ -labeling for $G(6,3)$.

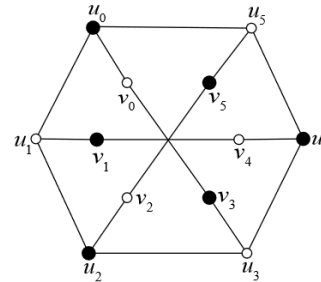


Figure 3: Graph $G(6,3)$.

It is easy to verify that f is a $(j+3k)$ - $L(j,k)$ -labeling for $G(6,3)$, that is, $\lambda_{j,k}(G(6,3)) \leq j+3k$.

Moreover, suppose $\lambda_{j,k}(G(6,3)) = \lambda < j+3k$. Let f be a λ - $L(j,k)$ -labeling of graph $G(6,3)$. As Figure 3 shows, the black vertices u_0, u_2, u_4, v_s are distance two apart from each other, where $s \in \{1,3,5\}$, it means that their labels should be k -separated, without loss of generality, let $f(u_0) = 0$, $f(u_2) \in [k, \lambda - 2k]$, $f(u_4) \in [2k, \lambda - k]$, and $f(v_s) \in [3k, \lambda]$ for $s \in \{1,3,5\}$. Similarly, white vertices u_1, u_3, u_5, v_1 are at distance two mutually, and they are adjacent to one vertex of v_1, v_3, v_5 . It implies that $f(u_1), f(u_3), f(u_5), f(v_1) \in [0, \lambda - j]$ since $|[3k, \lambda]| < j$. It means that $|[0, \lambda - j]| \geq 3k$ since $f(u_1), f(u_3), f(u_5), f(v_1)$ are k -separated mutually, that is, $\lambda \geq 3k + j$ which contradicts to the hypothesis. Thus, $\lambda_{j,k}(G(6,3)) = \lambda \geq j+3k$. Hence, $\lambda_{j,k}(G(6,3)) = j+3k$ for $k \geq 2j$.

Theorem 4.2. For $n \equiv 0 \pmod{8}$, $\lambda_{j,k}(G(n, \frac{n}{2})) = j+2k$.

Proof. Let $f_0 = j, f_1 = 2k, f_2 = j+k, f_3 = k$ and $g_0 = j+2k, g_1 = 0, g_2 = j+k, g_3 = k$. Then, given a labeling f for graph $G(n, \frac{n}{2})$ for $n \equiv 0 \pmod{8}$ as follows.

$$\text{Let } f(u_s) = \begin{cases} f_{[s]_4}, & \text{if } 0 \leq s \leq \frac{n}{2} - 1 \\ g_{[s]_4}, & \text{if } \frac{n}{2} \leq s \leq n - 1 \end{cases}, \quad f(v_t) = \begin{cases} 0, & \text{if } t \equiv 0 \pmod{2} \text{ and } 0 \leq t \leq \frac{n}{2} - 2; \\ 0, & \text{if } t \equiv 3 \pmod{4} \text{ and } \frac{n}{2} - 1 \leq t \leq n - 2; \\ 2k, & \text{if } t \equiv 0 \pmod{4} \text{ and } \frac{n}{2} - 1 \leq t \leq n - 2; \\ j, & \text{if } t \equiv 1 \pmod{4} \text{ and } \frac{n}{2} - 1 \leq t \leq n - 2; \\ j+2k, & \text{if } t \equiv 2 \pmod{4} \text{ and } \frac{n}{2} - 1 \leq t \leq n - 2; \\ j+2k, & \text{if } t = n - 1. \end{cases}$$

It is not difficult to verify that f is a $(j+2k)$ - $L(j,k)$ -labeling of graph $G(n, \frac{n}{2})$, that is, $\lambda_{j,k}(G(n, \frac{n}{2})) \leq j+2k$, where $n \equiv 0 \pmod{8}$.

On the other hand, suppose $\lambda_{j,k}(G(n, \frac{n}{2})) = \lambda < j+2k$. Let f be a λ - $L(j,k)$ -labeling of graph $G(n, \frac{n}{2})$ for $n \equiv 0 \pmod{8}$. Without loss of generality, let $f(u_0) = 0$ or $f(v_0) = 0$.

- (1) If $f(u_0) = 0$, then $f(u_{n-1}), f(v_0), f(u_1) \in [j, \lambda]$ since u_{n-1}, v_0, u_1 are adjacent to vertex u_0 , then $\lambda \geq j+2k$ since u_{n-1}, v_0, u_1 are distance two apart from each other and $\{f(u_{n-1}), f(v_0), f(u_1)\}$ should be k -separated.
- (2) If $f(v_0) = 0$, then $f(u_{n-1}), f(u_1)$ lie in intervals $[k, \lambda-k]$ and $[2k, \lambda]$, respectively. Since u_{n-1}, v_0, u_1 are adjacent to vertex u_0 , then $f(u_0) \in [j, \lambda-k-j]$ or $f(u_0) \in [j+k, \lambda-j]$. Moreover, u_0 is at distance two from u_2 and v_1 which are of distance two apart, then $\max\{f(u_2), f(v_1)\} \geq j+2k$. That is, $\lambda \geq j+2k$.

Thus, $\lambda_{j,k}(G(n, \frac{n}{2})) \geq j+2k$, where $n \equiv 0 \pmod{8}$. Hence, $\lambda_{j,k}(G(n, \frac{n}{2})) = j+2k$ for $n \equiv 0 \pmod{8}$.

Theorem 4.3. For $n \equiv 4 \pmod{8}$ and $n \geq 12$, $\lambda_{j,k}(G(n, \frac{n}{2})) = 2j+2k$.

Proof. Let $f_0 = 0, f_1 = j, f_2 = j+2k, f_3 = 2j+2k$ and $g_0 = g_3 = j+k, g_1 = k, g_2 = 2j+k$. Then, given a labeling f for graph $G(n, \frac{n}{2})$ for $n \equiv 4 \pmod{8}$ and $n \geq 12$ as follows.

Let $f(u_s) = f_{[s]_4}$, $f(v_t) = g_{[t]_4}$, where $s, t \in \mathbb{Z}_n$.

It is not difficult to verify that f is a $(2j+2k)$ - $L(j, k)$ -labeling of graph $G(n, \frac{n}{2})$, that is $\lambda_{j,k}(G(n, \frac{n}{2})) \leq 2j+2k$, where $n \equiv 4 \pmod{8}$ and $n \geq 12$.

On the other hand, suppose $\lambda_{j,k}(G(n, \frac{n}{2})) = \lambda < 2j+2k$. Then, let f be a λ - $L(j, k)$ -labeling of graph $G(n, \frac{n}{2})$ for $n \equiv 4 \pmod{8}$ and $n \geq 12$. Similar to the proof of Theorem 4.2, we have $\lambda \geq j+2k$.

For convenience, let interval $[0, \lambda] = \bigcup_{i=0}^5 I_i$, where $I_0 = [0, j), I_1 = [j, k), I_2 = [k, j+k), I_3 = [j+k, 2k), I_4 = [2k, j+2k)$, and $I_5 = [j+2k, \lambda]$.

Claim 1: For $s, t \in \mathbb{Z}_n$, $f(u_s) \in I_0 \cup I_2 \cup I_4$ if $s \equiv 0 \pmod{2}$ and $f(u_t) \in I_1 \cup I_3 \cup I_5$ if $t \equiv 1 \pmod{2}$.

According to the above discussion, by the symmetry of the graph, without loss of generality, let $f(u_0) = 0$ and $f(u_1) \in I_1$ or $f(u_1) \in I_5$.

- (1) If $f(u_1) \in I_1$, then $f(u_3) \in I_3$ or $f(u_3) \in I_5$ since u_1, v_2, u_3 are distance two apart from each other. Similarly, u_0, v_1, u_2 are also distance two apart mutually, then $f(u_2) \in I_2$ or $f(u_2) \in I_4$. Similar to the above discussion, we can induce that $f(u_s) \in I_0 \cup I_2 \cup I_4$ and $f(u_t) \in I_1 \cup I_3 \cup I_5$, where $s \equiv 0 \pmod{2}, t \equiv 1 \pmod{2}$ and $s, t \in \mathbb{Z}_n$.
- (2) If $f(u_1) \in I_5$, then $f(u_2) \in I_2$ or $f(u_2) \in I_4$ since vertices u_0, v_1, u_2 are also distance two apart mutually and they are adjacent to u_1 . Similarly, $\{f(u_2), f(v_3), f(u_4)\}$ should be k -separated, it means that $f(u_4) \in I_0$ or $f(u_4) \in I_2$ or $f(u_4) \in I_4 \cup I_5$. Moreover, u_1, v_2, u_3 are also at distance two from each other, then $f(u_3) \in I_1$ or $f(u_3) \in I_3$. Similar to the above discussion, we can conclude that $f(u_s) \in I_0 \cup I_2 \cup I_4$ and $f(u_t) \in I_1 \cup I_3 \cup I_5$, where $s \equiv 0 \pmod{2}, t \equiv 1 \pmod{2}$ and $s, t \in \mathbb{Z}_n$.

Note that Claim 1 holds when $f(u_0) \in I_2$ or I_4 , here we omit the details.

Next, we consider the vertices $u_{i-2}, u_i, u_{\frac{n}{2}+i-2}, u_{\frac{n}{2}+i}$, for convenience, the vertex set $\{u_{i-2}, u_i, u_{\frac{n}{2}+i-2}, u_{\frac{n}{2}+i}\}$ is called i -th term, denoted by T_i , where subscripts are taken modulo n and $i \in \mathbb{Z}_n$. Note that

$T_i = T_{\frac{n}{2}+i}$. By the symmetry of the graph, without loss of generality, we can consider the vertex set T_1 instead. According to the Claim 1 and symmetry of the graph, the labels of vertices in T_1 only have the following four cases:.

(1) If $f(u_{n-1}), f(u_1) \in I_1 \cup I_3, f(u_{\frac{n}{2}-1}), f(u_{\frac{n}{2}+1}) \in I_1 \cup I_5$, by the symmetry of the graph, without loss of generality, let $f(u_{n-1}) \in I_1, f(u_1) \in I_3$.

(a) If $f(u_{\frac{n}{2}-1}) \in I_1, f(u_{\frac{n}{2}+1}) \in I_5$, then the label intervals cover the vertices as Figure 4 shows. That is, the labels of vertices $u_{n-1}, u_0, u_1, \dots, u_{\frac{n}{2}-2}$ must lie in the interval sequence $I_1, I_0, I_3, I_4, I_5, I_2$ in turn and circularly, and the labels of vertices $u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, \dots, u_{n-2}$ must lie in the interval sequence $I_1, I_2, I_5, I_4, I_3, I_0$ in turn and circularly. It is a contradiction since the vertices u_0, u_1, \dots, u_{n-1} induce a cycle but the two interval sequences are different.

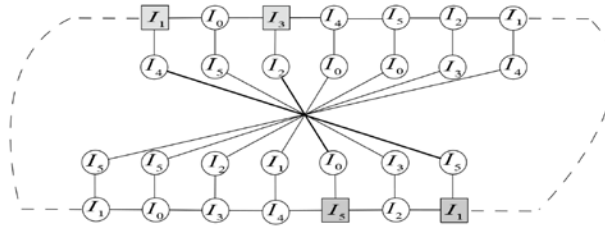


Figure 4: The distribution of label intervals for the graph.

(b) If $f(u_{\frac{n}{2}-1}) \in I_5, f(u_{\frac{n}{2}+1}) \in I_1$, then the label intervals cover the vertices as Figure 5 shows. That is, the labels of vertices $u_{n-1}, u_0, u_1, \dots, u_{\frac{n}{2}-2}$ must lie in the interval sequence $I_1, I_0, I_3, I_2, I_5, I_4$ in turn and circularly, and the labels of vertices $u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, \dots, u_{n-2}$ must lie in the interval sequence $I_5, I_2, I_1, I_4, I_3, I_0$ in turn and circularly. It is a contradiction since the vertices u_0, u_1, \dots, u_{n-1} induce a cycle but the two interval sequences are also different.

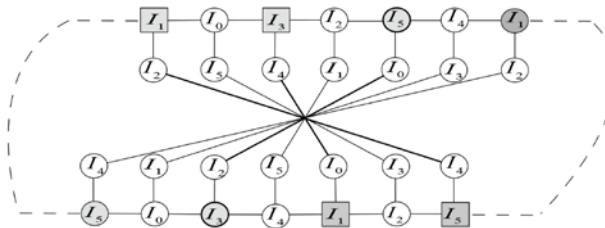


Figure 5: The distribution of label intervals for the graph.

(2) If $f(u_{n-1}), f(u_1) \in I_3 \cup I_5, f(u_{\frac{n}{2}-1}), f(u_{\frac{n}{2}+1}) \in I_1 \cup I_5$, by the symmetry of the graph, without loss of generality, let $f(u_{n-1}) \in I_3, f(u_1) \in I_5$.

(a) If $f(u_{\frac{n}{2}-1}) \in I_5, f(u_{\frac{n}{2}+1}) \in I_1$, this case has included in the case (1)(b). That is, this case is equivalent to the labels of vertices in $T_{\frac{n}{2}+5}$ (See the grey circular vertices in Figure 5). Then, it is also a contradiction.

(b) If $f(u_{\frac{n}{2}-1}) \in I_5, f(u_{\frac{n}{2}+1}) \in I_1$, similar to the last case, we can obtain that $f(u_{n-4}), f(v_{n-3}), f(u_{n-2})$ lie in the intervals I_4, I_0, I_2 , respectively, and $f(u_{\frac{n}{2}-4}), f(v_{\frac{n}{2}-3}), f(u_{\frac{n}{2}-2})$ lie in the intervals I_4, I_0, I_2 , respectively, it implies a contradiction since $|f(v_{n-3}) - f(v_{\frac{n}{2}-3})| \geq j$ but $|I_0| < j$.

(3) If $f(u_{n-1}), f(u_1), f(u_{\frac{n}{2}-1}), f(u_{\frac{n}{2}+1}) \in I_3 \cup I_5$, similar to above case, let $f(u_{n-1}) \in I_3, f(u_1) \in I_5$.

- (a) If $f(u_{\frac{n-1}{2}}) \in I_5, f(u_{\frac{n+1}{2}}) \in I_3$, this case has included in the case (1)(a). That is, this case is equivalent to the labels of vertices in T_3 (See Figure 4). It is also a contradiction.
- (b) If $f(u_{\frac{n-1}{2}}) \in I_3, f(u_{\frac{n+1}{2}}) \in I_5$, we can obtain that $f(u_0), f(u_{\frac{n}{2}}) \in I_4$ and $f(v_{n-1}), f(v_{\frac{n}{2}-1}) \in I_0$ directly, it is a contradiction since $|f(v_{n-1}) - f(v_{\frac{n}{2}-1})| \geq j$ but $|I_0| < j$.
- (4) If $f(u_{n-1}), f(u_1) \in I_3 \cup I_5, f(u_{\frac{n-1}{2}}), f(u_{\frac{n+1}{2}}) \in I_1 \cup I_3$, by the symmetry of the graph, without loss of generality, let $f(u_{n-1}) \in I_3, f(u_1) \in I_5$.
- (a) If $f(u_{\frac{n-1}{2}}) \in I_1, f(u_{\frac{n+1}{2}}) \in I_3$, this case also has included in the case (1)(b). That is, this case is equivalent to the labels of vertices in T_3 (See Figure 5). Then it implies a contradiction.
- (b) If $f(u_{\frac{n-1}{2}}) \in I_3, f(u_{\frac{n+1}{2}}) \in I_1$, according to the above cases, we have $f(u_{i-2}), f(u_i) \in I_3 \cup I_5$, $f(u_{\frac{n}{2}+i-2}), f(u_{\frac{n}{2}+i}) \in I_1 \cup I_3$ or $f(u_{i-2}), f(u_i) \in I_1 \cup I_3, f(u_{\frac{n}{2}+i-2}), f(u_{\frac{n}{2}+i}) \in I_3 \cup I_5$, where i is odd and $i \in \mathbb{Z}_n$. Since $f(u_{n-1}) \in I_3$, then $f(u_s) \in I_3$ if $s \equiv 3 \pmod{4}$ and $s \in \mathbb{Z}_n$ according to the above discussion. It mean that $f(u_{\frac{n}{2}}) \in I_3$ since $n \equiv 4 \pmod{8}$. It contradicts to the hypothesis $f(u_{\frac{n}{2}}) \in I_1$.

Thus, $\lambda_{j,k}(G(n, \frac{n}{2})) \geq 2j + 2k$, where $n \equiv 4 \pmod{8}$ and $n \geq 12$. Hence, $\lambda_{j,k}(G(n, \frac{n}{2})) = 2j + 2k$ for $n \equiv 4 \pmod{8}$ and $n \geq 12$.

Theorem 4.4. For $n \equiv 2 \pmod{4}$ and $n \geq 10$, $\lambda_{j,k}(G(n, \frac{n}{2})) = 3k$.

Proof. Suppose $\lambda_{j,k}(G(n, \frac{n}{2})) = \lambda < 3k$. Then, let f be a λ - $L(j, k)$ -labeling of $G(n, \frac{n}{2})$ for $n \equiv 2 \pmod{4}$ and $n \geq 10$. Since vertices u_0, u_2, v_1 are distance two apart from each other, then their labels should lie in intervals $[0, k), [k, 2k), [2k, \lambda]$, respectively. For convenience, let $I_0 = [0, k), I_1 = [k, 2k), I_2 = [2k, \lambda]$, and $f(u_0) \in I_a, f(u_2) \in I_b, f(v_1) \in I_c$, where a, b, c are different numbers and $\{a, b, c\} = \{1, 2, 3\}$. Since u_2, v_3, u_4 are also distance two apart mutually, then their labels should be k -separated mutually, that is, $f(u_4) \in I_a$ or $f(u_4) \in I_c$. Similar to the above discussion, we can induce that $f(u_s) \in I_i$ for $s \equiv 0 \pmod{2}$, where $i \in \{1, 2, 3\}$.

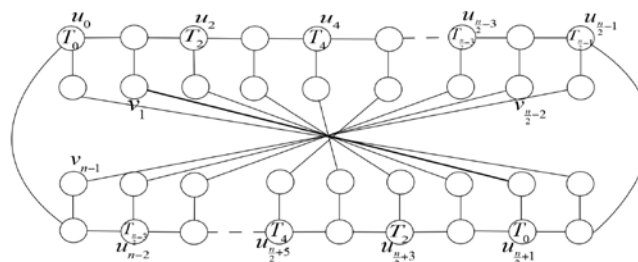


Figure 6: The distribution of label intervals for the graph.

Furthermore, since $f(v_1) \in I_c$ and $d(v_1, u_{\frac{n}{2}+1}) = 2$, then $f(u_{\frac{n}{2}+1}) \in I_a$ or $f(u_{\frac{n}{2}+1}) \in I_b$. By the symmetry of the graph, without loss of generality, let $f(u_{\frac{n}{2}+1}) \in I_a$. Similarly, since $v_{\frac{n}{2}+2}$ is at distance two from u_2 and $u_{\frac{n}{2}+1}$ simultaneously, then $f(v_{\frac{n}{2}+2}) \in I_c$. It implies that $f(u_{\frac{n}{2}+3}) \in I_b$. Similar to the above discussion, we can obtain that $f(u_i), f(u_{\frac{n}{2}+i})$ lie in the same one interval T_i , where $i \equiv 0 \pmod{2}$ and $T_i \in \{I_1, I_2, I_3\}$ (See Figure 6). Since $n \equiv 2 \pmod{4}$, then we have $f(u_{\frac{n}{2}-3}), f(u_{n-2}) \in T_{\frac{n}{2}-3}$ and $f(u_{\frac{n}{2}-1}) \in T_{\frac{n}{2}-1}$. Since $d(u_0, u_{n-2}) = d(u_{\frac{n}{2}+1}, u_{\frac{n}{2}-1}) = d(u_{\frac{n}{2}-3}, u_{\frac{n}{2}-1}) = 2$, then $T_{\frac{n}{2}-3}, T_{\frac{n}{2}-1}, T_0$ are different

intervals mutually. However, vertex v_{n-1} is at distance two from u_0 and u_{n-2} , then $f(v_{n-1}) \in T_{\frac{n}{2}-1}$. It means that $f(v_{n-1}), f(u_{\frac{n}{2}-1}) \in T_{\frac{n}{2}-1}$. It is a contradiction since $d(v_{n-1}, u_{\frac{n}{2}-1}) = 2$ and $|T_{\frac{n}{2}-1}| < k$. Thus,

$$\lambda_{j,k}(G(n, \frac{n}{2})) \geq 3k \text{ for } n \equiv 2 \pmod{4}.$$

On the other hand, define a $3k - L(j, k)$ -labeling f for the graph $G(n, \frac{n}{2})$ as the following shows.

- (1) If $n \equiv 2 \pmod{8}$ and $n \geq 10$, let $f_0 = j + k, f_1 = k, f_2 = j + 2k, f_3 = 0, g_0 = 0, g_1 = j + k, g_2 = k, g_3 = 3k, m_0 = j + 2k, m_1 = 0, m_2 = 3k, m_3 = j, n_0 = j, n_1 = n_3 = 2k, n_2 = 0$. For u_s , let $f(u_0) = f(u_4) = 0, f(u_1) = f(u_{\frac{n}{2}+1}) = k, f(u_2) = f(u_{\frac{n}{2}+2}) = j + k, f(u_3) = j + 2k, f(u_{\frac{n}{2}}) = j, f(u_{\frac{n}{2}+3}) = 2k, f(u_{\frac{n}{2}+4}) = 3k, f(u_s) = f_{[s-1]_4}$ if $5 \leq s \leq \frac{n}{2} - 1$, and $f(u_s) = g_{[s-2]_4}$ if $\frac{n}{2} + 5 \leq s \leq n - 1$; For v_t , let $f(v_0) = 2k, f(v_{\frac{n}{2}-1}) = j + k, f(v_{\frac{n}{2}}) = f(v_{\frac{n}{2}+1}) = f(v_{\frac{n}{2}+2}) = 3k, f(v_{\frac{n}{2}+3}) = 0, f(v_{\frac{n}{2}+4}) = k, f(v_t) = m_{[t-1]_4}$ if $1 \leq t \leq \frac{n}{2} - 2$, and $f(v_t) = n_{[t-2]_4}$ if $\frac{n}{2} + 5 \leq t \leq n - 1$, where $s, t \in \mathbb{Z}_n$.
- (2) If $n \equiv 6 \pmod{8}$ and $n \geq 14$, let $f_0 = 0, f_1 = j, f_2 = k, f_3 = j + k$. For u_s , let if $0 \leq s \leq \frac{n}{2} - 4, f(u_s) = f_{[s-2]_4}$ if $\frac{n}{2} \leq s \leq n - 4, f(u_{\frac{n}{2}-3}) = f(u_{n-3}) = 2k, f(u_{\frac{n}{2}-2}) = f(u_{n-2}) = j + 2k$, and $f(u_{\frac{n}{2}-1}) = f(u_{n-1}) = 3k$; For v_t , let $f(v_0) = 2k, f(v_t) = j + 2k$ if $1 \leq s \leq \frac{n}{2} - 5, f(v_{\frac{n}{2}-4}) = 3k, f(v_{\frac{n}{2}-3}) = j, f(v_{\frac{n}{2}-2}) = k, f(v_{\frac{n}{2}-1}) = f(v_{\frac{n}{2}}) = j + k, f(v_t) = 3k$ if $\frac{n}{2} + 1 \leq s \leq n - 5, f(v_t) = 0$ if $n - 4 \leq s \leq n - 2, f(v_{n-1}) = k$, where $s, t \in \mathbb{Z}_n$.

It is not difficult to verify that f is a $3k - L(j, k)$ -labeling of graph $G(n, \frac{n}{2})$, then, $\lambda_{j,k}(G(n, \frac{n}{2})) \leq 3k$, where $n \equiv 2 \pmod{4}$ and $n \geq 10$. Hence, $\lambda_{j,k}(G(n, \frac{n}{2})) = 3k$ for $n \equiv 2 \pmod{4}$ and $n \geq 10$.

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