

An Improved Algorithm based on MOLS for CS

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Abstract. The purpose of sparse recovery based on compressed sensing is to reconstruct sparse signals from linear compressed measurements. Greedy algorithm is often used to solve the inverse problem of underdetermined equations. Both Orthogonal Matching Pursuit(OMP) and Orthogonal Least Squares(OLS) greedy algorithms have been widely applied. Unlike the OMP algorithm, the first task of the OLS algorithm is to find the support set dropping the residual fastest. Multiple Orthogonal Least Squares(MOLS) algorithm adds the idea of multiple support set selection to the OLS algorithm, which greatly reduces the time complexity of the OLS algorithm, but needs to take the sparsity K as a priori condition. Based on this, a novel sparse recovery algorithm called Changing Stage Orthogonal Least Squares(CSOLS) is proposed in this paper. Compared with the MOLS algorithm, the most innovative feature of the CSOLS algorithm is the signal reconstruction ability without a priori information sparsity, finishing sparse recovery by conditionally broadening the search step. Compared with the MOLS algorithm and the traditional greedy algorithms with regard to the Frequency of Exact Reconstruction(FER) under the different sparsity and measurements, the CSOLS algorithm shows terrific recovery performance.

1. Introduction

Compressed Sensing(CS)[1-4]theory has increasingly becoming a completely new sparse signal or compressible signal processing theory in recent years. After the famous Shanon sampling theory, the introduction of compressed sensing has become a major breakthrough in the field of signal processing. CS is put forward by Donoho, Cand s, Romberg and Tao in 2004, which also establish a mathematical foundation for it. The CS theory avoids the high-speed sampling, which means that the acquisition and processing of the signal can be carried out at a lower rate. The research on the basic theory of CS has made some significant achievements, and many variants of algorithms and algorithms have emerged.

In the greedy iterative algorithms, the Matching Pursuit (MP) [5] algorithm is the most fundamental algorithm, the basic procedure is to select the atom and iterative recursion through the matching judgment. On the basis of MP and OMP[6] algorithm, some improved algorithms, such as Stage-wise orthogonal matching pursuit algorithm(StOMP)[7] algorithm, regularized orthogonal matching pursuit algorithm(ROMP)[8] algorithm, subspace pursuit(SP) algorithm[9], compressed sampling matching pursuit(CoSAMP)[10] algorithm and so on, are proposed successively with the basic procedure based on the greedy iteration, which provide some recovery methods that have some advantages over MP and OMP, but the essence of all of them is to reconstruct the sparse signal by matching the atom library.

Another class of greedy algorithm similar to OMP is orthogonal least square(OLS)[11] algorithm, and OLS algorithm is proposed as a parameter estimation recognition algorithm for multivariable



nonlinear systems. This kind of algorithm has also been applied in compressed sensing recently. The main difference between OMP and OLS lies in updating the greedy support rules in each iteration. While OMP chooses the candidate set of strongest correlation with the signal residual, OLS seeks to catch the candidate set with the most significant reduction in residual power. It has been proved that OLS has better convergence, but the computational complexity is higher than that of OMP algorithm. Jian et al, in order to improve the more accurate recovery of OLS algorithm and reduce the computational cost of OLS algorithm, propose a new algorithm called multiple orthogonal least squares (MOLS)[12]. The MOLS algorithm has good performance and indicates that at most K iterations the algorithm can converge, greatly reducing the complexity of the OLS algorithm and improving the recovery performance. However, the disadvantage of this algorithm is that the algorithm needs to be carried out on the premise of knowing the sparsity K , but the real signal recovery is usually not known to the sparsity K . And inspired by [13-15], an idea appears that changing search step can be added. Based on those above, a novel algorithm called Changing stage orthogonal least squares (CSOLS) is proposed compressed sensing. In addition, CSOLS algorithm does not use sparsity K as a known condition. In each iteration, L index values can be selected, and L changes according to the decision condition.

2. Preliminaries

2.1 Notation

We briefly summarize notations in this paper. For $S \subseteq \Omega (\Omega = \{1, \dots, n\})$, $|S|$ is the cardinality of S . $x_S \in R^{m \times |S|}$ is the restriction of the vector x to the elements with indices in S . $\Phi_S \in R^{m \times |S|}$ is a submatrix of Φ that only contains columns indexed by S . If Φ_S has full column rank, then $\Phi_S^+ = (\Phi_S^T \Phi_S)^{-1} \Phi_S^T$ is the pseudoinverse of Φ_S where Φ_S^T denotes the transpose of Φ_S . $\text{Span}(\Phi_S)$ is the span of columns in Φ_S . $P_S = \Phi_S \Phi_S^+$ is the projection onto $\text{span}(\Phi_S)$. $P_S^\perp = I - P_S$ is the projection onto the orthogonal complement of $\text{span}(\Phi_S)$ where I is the identity matrix. S^k denotes the k th iteration of S . ϕ_i represents the i th column of Φ .

2.2 Compressed sensing theory

Assume a K -sparse signal $x \in R^N$ of N dimensions via a certain sparse matrix ($\Psi = \{\psi_1, \psi_2, \dots, \psi_N\}^T$) transform.

$$x = \sum_{i=1}^N \psi_i \theta_i = \Psi \Theta \quad (1)$$

where Ψ represents the orthogonal sparse matrix with $N \times N$, and the projection $\theta_i = \langle X, \psi_i \rangle$ of X on Ψ is denoted as $\theta_i \in R^N$. Assume Θ is the N dimensions signal coefficient with the sparsity of K ($K \ll N$), which is to say there are K nonzero values in Θ . Thus signal x is sparse on the base of the sparse matrix Ψ with the sparsity of K .

The nonzero coordinates of the sparse signal x have to be reduced while the signal needs compression sampling. This is because the signal x is often N -dimension (N is usually large) and the complexity is still relatively high. Therefore, a measurement matrix is needed to compress the dimension to reduce the computational complexity effectively. Assuming that Φ is a matrix with $M \times N$ ($K < M < N$), let Y be a one-dimension observation signal with the length of M , then it can be expressed as

$$Y = \Phi x = \Phi \Psi \Theta \quad (2)$$

Let the sensing matrix A with $M \times N$ be $A = \Phi\Psi$, then the equation above is equivalent to

$$Y = A\Theta \quad (3)$$

As the dimension of Y is much lower than the dimension of x , (3) will have infinite solutions. That is to say, the problem is underdetermined and it is difficult to reconstruct the original signal. However, if the signal x has only K nonzero values, and the observation matrix satisfies the Restricted Isometry Property (RIP) [8], and that is

$$(1 - \delta_k) \|\Theta\|_2^2 \leq \|\Phi\Psi\Theta\|_2^2 \leq (1 + \delta_k) \|\Theta\|_2^2 \quad (4)$$

where $0 < \delta_k < 1$, with the number of $M \geq cK \log(N/K)$ (c is a quiet small constant) the K coefficients can be reconstructed from the high probability of M measurements.

3. CSOLS algorithm

The CSOLS algorithm makes further improvement on the MOLS algorithm. The MOLS algorithm is devoted to reducing the complexity of the OLS algorithm by allowing multiple indices to be selected per iteration, and converges in fewer iterations at most K . However, The sparsity K needs to be known as the known condition in the MOLS algorithm, but the sparsity of signals in the reality is not known. To solve this problem, CSOLS algorithm is proposed in this paper. The key idea of the algorithm is that in each iteration, by comparing the adjacent two-step residual, when the residual energy is greater than the energy of the last residual, the algorithm broadens the length of the search step to be the suitable step, selects the suitable support set and then recovers the signal. The specific pseudo code of the CSOLS algorithm is Table 1.

Table 1. the CSOLS

Input :	sensing matrix $\Phi \in R^{m \times n}$,
	measurements vector $y \in R^m$, step size s ;
Output:	the estimated signal \hat{x} ;
Initialization:	
	Estimated signal $\hat{x} = 0$
	Residual vector $r^0 = y$
	Estimated support $T^0 = 0$
	The step size in the first stage
	Stage index $j = 1; L = s$
repeat	
	Preliminary Identity $S^k = \arg \max_{S: S =L} \sum_{i \in S} \langle \frac{P_{T^{k-1}}^\perp \phi_i}{\ P_{T^{k-1}}^\perp \phi_i\ _2}, r^k \rangle $
	Make Candidate List $C^k = T^{k-1} \cup S^k$
	Final List $T = \arg \max_{S: S =L} \Phi_{C^k}^+ y $
	Compute Residue $r = y - \Phi_T \Phi_T^+ y$

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    if halting condition true then
        quit the iteration;
    else if  $\|r\|_2 \geq \|r_{k-1}\|_2$  then
        Update the stage index  $j = j + 1$ 
        Update the size of finalist  $L = j \times s$ 
    else
        Update the finalist  $T^k = T$ 
        Update the residue  $r^k = r$ 
    end if
until halting condition true;

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Next, the specific steps of the CSOLS algorithm are discussed. The CSOLS algorithm no longer needs to refer to the sparsity as a known condition. What it needs is to initialize an a step length $L = s$, find the L indices corresponding to the maximum of the inner product of $\frac{P_{T^{k-1}}^\perp \phi_i}{\|P_{T^{k-1}}^\perp \phi_i\|^2}$ and r^k , and further get the candidate set C . Then, through the least square method to find the corresponding indices of the largest energy x , the residual will be calculated. Compared with the previous residual, if the residual is larger than the previous one, the step size is increased by s , if not, iteration is continued until the iteration stop requirement is met.

4. Simulation and results

In this part, we simulate the signal performance of the CSOLS algorithm in the absence of noise. The paper looks on the FER of the sparse signal as the criterion standard for the performance of the algorithms, and compares FER of CSOLS algorithm with those of MOLS algorithm and the traditional greedy algorithms. The measurement matrix used in this experiment is Gauss random matrix, and the input signal is Gauss sparse signal and Binary sparse signal both with the length of 128.

4.1 Performance comparison of algorithms for different sparsity K

In this experiment, it aims at fixing the number of $M = 128$ to study the recovery performance of all algorithms under different sparsity K (0–64). In each experiment, it generates a signal with the sparsity of K , and then takes $M = 128$ to measure the sparse signal. For each of the sparsity of K , it will repeat the measurement 1000 times, and then calculate the average of the 1000 FER as the final FER of the exact recovery of this K . Figure 1 (a) and figure 1 (b) depict the FER of these algorithms under the Gauss random signal and the Binary random signal. The x-axis represents different sparsity K , and the y-axis represents the FER.

It can clearly be seen that for Gauss sparse signal, the CSOLS algorithm performs far better than other algorithms. The CSOLS algorithm can guarantee exact recovery before $K = 48$, and the MOLS algorithm can guarantee exact recovery before $K = 42$, while other four algorithms can guarantee exact recovery with the sparsity K , which is lower than 38. The FER of CSOLS algorithm with different sparsity is higher than other algorithms. For example, when the sparsity is $K = 60$, the FER of the CSOLS algorithm can reach 0.8, while that of the MOLS ($S = 3$) algorithm only reaches 0.45, and the MOLS ($S = 5$) algorithm only reaches 0.38. For Binary sparse signal, the performance of CSOLS algorithm and MOLS algorithm is worse than that of SP algorithm and CoSaMP algorithm, but the performance of the CSOLS algorithm is better than that of the MOLS algorithm

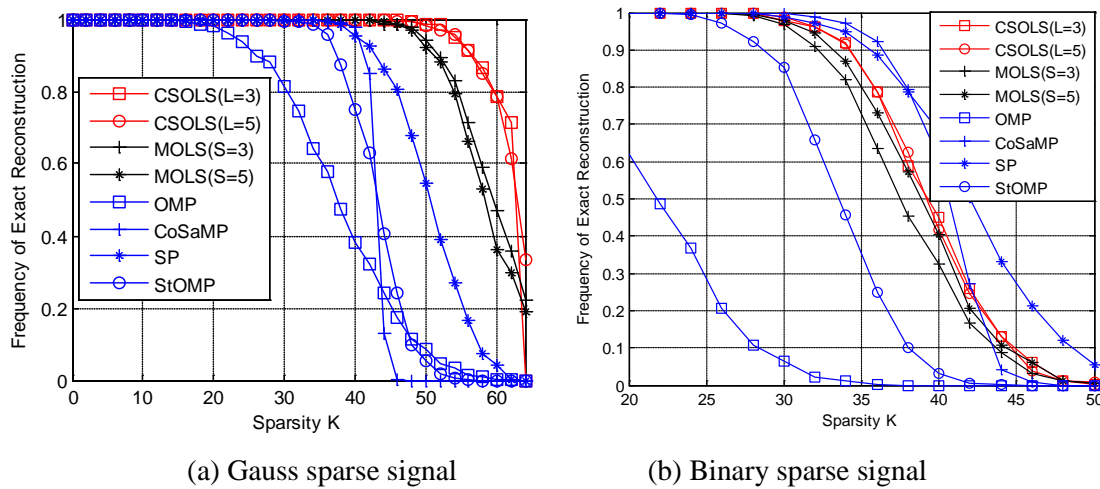


Figure 1. FER vs the signal sparsity K .

4.2 Performance comparison of the algorithms for different measurement M

In this experiment, its aim is to study the effect of different measurement M (50–140) on the FER of the algorithm in the case of a fixed sparsity of $K = 30$. For each measurement M , it generates a sparse signal with a sparsity of 30 as input to the recovery algorithm. It will repeat 1000 times for each measurement M , and then calculate the average value of the 1000 FER as the final FER of the this M . Figure 2 (a) and figure 2 (b) depict the FER of these algorithms under the Gauss sparse signal and the Binary sparse signal. The x-axis represents a different measurement value M , and the y-axis represents the FER.

As we can see, for the Gauss sparse signal, the FER of the CSOLS algorithm is much better than that of the MOLS algorithm and other four greedy algorithms. When the number of measurement is 80, FER of CSOLS ($L = 3, 5$) is almost 0.8, while FER of MOLS ($S = 3$) is only 0.6, and the FER of MOLS ($S = 5$) is only 0.45, and the FER of other four traditional recovery algorithms is less than 0.2. We can also observe that the performance of CSOLS ($L = 3$) algorithm is a little better than that of CSOLS ($L = 5$). For Binary sparse signal, the performance of the CSOLS algorithm and the MOLS algorithm is worse than SP algorithm and CoSaMP algorithm, but the performance of CSOLS algorithm is better than that of MOLS algorithm.

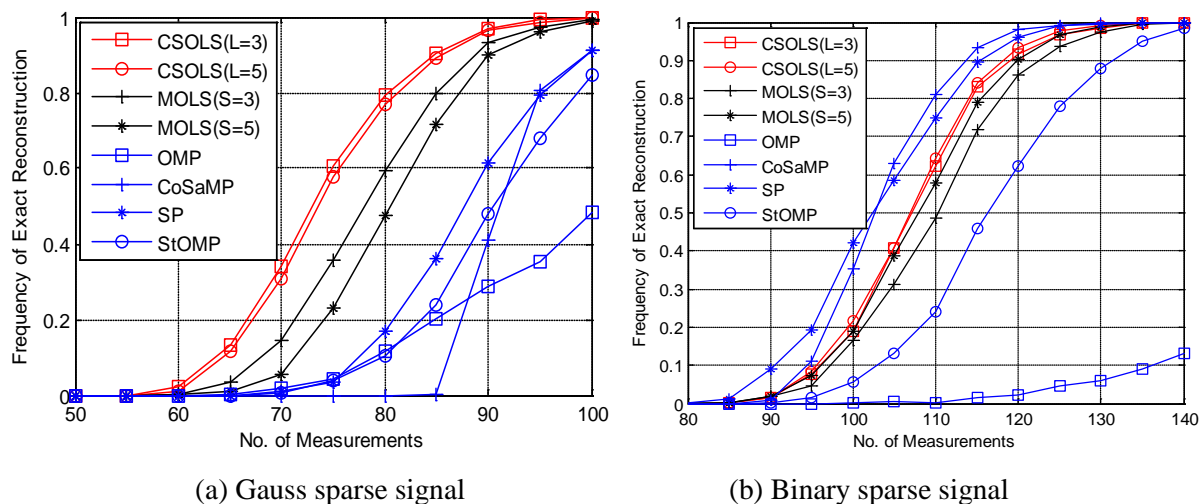


Figure 2. FER vs the measurements M .

5. Conclusions

In this paper, a novel greedy recovery algorithm called CSOLS is proposed compressed sensing. This algorithm is a variant of the MOLS algorithm. By constantly broadening the search step according to the condition of judgment, finally reaching the appropriate step, and then recovering the original signal, the sparsity can not be used as the known condition, which makes great progress in practicability of the algorithm, because the real signal recovery is not aware of the sparsity. Compared with the MOLS algorithm and the traditional compressed sensing algorithms, the CMOLS algorithm shows significant superiority, especially for the Gauss sparse signal.

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