L(j, k)-labeling Number of Cactus Graph

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Abstract. For $j \le k$, the $L(j,k)$-labelling problem arose from code assignment problem of computer wireless networks. That is, let *j*, *k* and *m* be positive numbers, an *m*-$L(j,k)$-labelling of a graph *G* is a mapping $f: V(G) \to [0,m]$ such that $|f(u)-f(v)| \ge j$ if $d(u,v)=1$, and $|f(u)-f(v)| \ge k$ if $d(u,v)=2$. The span of *f* is the difference between the maximum and the minimum numbers assigned by *f*. The $L(j,k)$-labelling number of *G*, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labellings of *G*. In this paper, we introduce the $L(j,k)$-labelling numbers of Cactus graph for any positive real numbers j,k with $j \le k$.

1. Introduction

The rapid growth of computer wireless networks highlighted the scarcity of available codes for communication with minimum interference. For example, the Packet Radio Network (PRN) first displayed in 1969 at University of Hawaii [1] is a computer network that used radio frequencies to transmit data among computers. It consists of computer stations which include computers and transceivers. Two stations are called adjacent if two stations can directly transmit to each other. If two stations are nonadjacent but they are adjacent to same station, they are called at distance two.

There exist two types of interference in a PRN using CDMA. Direct interference is due to two adjacent stations transmitting to each other concurrently. Secondary interference can occur in two scenarios. The first case occurs when two stations at distance two transmit to the same receiving station at the same time.

The network can be modelled as an undirected graph $G=(V,E)$, such that the set of vertices $V=\{v_0,v_1,\cdots,v_{n-1}\}$ represents the set of stations, and the set of edges *E* represents the relationship between two adjacent stations. That is, two vertices v_i and v_j in *V* are joined by an undirected edge e_{ij} in *E* if and only if the stations corresponding to vertices v_i and v_j are adjacent. Note that there is a one-to-one mapping of the stations onto the vertices in *V*.

Suppose direct interference is so weak that we can ignore it, and two stations can generate a secondary interference if and only if they are at distance two. In order to avoid secondary interference, Bertossi and Bonuccelli [2] introduced a kind of code assignment, that is, two at distance two stations transmit on different codes. Thus, the secondary interference avoidance problem can be formulated as follows: assign disparate codes to each pair of vertices at distance two and use the minimum number of different codes.

In general, it seems reasonable to consider the direct interference. Based on this premise, Jin and Yeh [3] generalized the code assignment problem to $L(j,k)$-labelling problem, where $j \le k$.

For positive numbers *j* and *k*, an $L(j,k)$-labelling f of G is an assignment of numbers to vertices of *G* such that such that $|f(u)-f(v)|\geq j$ if $d(u,v)=1$, and $|f(u)-f(v)|\geq k$ if $d(u,v)=2$. The span of *f* is the difference between the maximum and the minimum numbers assigned by *f*. The $L(j,k)$-*labelling number* of *G*, denoted by $\lambda_{j,k}(\mathrm{G})$, is the minimum span over all $L(j,k)$-labellings of *G*. All notation not defined in this article can be found in the book [4].

Cactus graph is a connected graph in which every block is a cycle or an edge, in other words, no edge belongs to more than one cycle. It has many applications in various fields like computer scheduling, radio communication system etc. Let $G(m_1,m_2,\cdots,m_p)$ be a *p-Cactus graph* which contains *p* induced cycles: $C_{m_1},C_{m_2},\cdots, C_{m_p}$, and *v* be the cut-vertex. Let $v,v_1^i,v_2^i,\cdots,v_{m_i-1}^i$ be the vertices of induced cycle C_{m_i}, where $i=1,2,\cdots,p$ and $p\geq 2$.

Suppose *p*-Cactus graph $G(m_1,m_2,\cdots,m_p)$ contains *q* induced circles C_3 's, according to the property of Cactus graph, without loss of generality, let $m_1=\cdots=m_q=3$ and $m_{q+1},\cdots,m_p\geq 4$, where $0\leq q\leq p$ and $p\geq 2$.

Cactus graphs have been extensively studied and used as discrete mathematical structure for modelling problem arising in the real world. For example, Khan determined the $L(0,1)$)-labelling number of Cactus graph in [5]. Recently, the $L(j,k)$-labelling problem for $j\leq k$ has been paid attention to study since it can be applied in computer network to solve the problem of the scarcity of available codes for communication with minimum interference, please refer to [6-9] for $j\leq k$.

In this paper, we introduce the $L(j,k)$-labelling numbers of Cactus graph for any positive real numbers *j, k* with $j\leq k$.

Lemma 1.1 *Let j and k be two positive numbers with* $j\leq k$. *Suppose G is a graph and H is an induced subgraph of G. Then* $\lambda_{j,k}(G)\geq\lambda_{j,k}(H)$.

Note that Lemma 1.1 is not true if *H* is not an induced subgraph.

2. Upper Bounds on $L(j,k)$-Labelling Numbers of p-Cactus Graph

In this section, we deal with the upper bounds on $\lambda_{j,k}(G(m_1,m_2,\cdots,m_p))$ basing to the value of *p* and m_p, where $p\geq 2$ and $m_p\geq 3$.

At first, we consider the upper bound of $\lambda_{j,k}(G(m_1,m_2))$.

Lemma 2.1 *Let j, k be two positive numbers with* $j\leq k$, *then* $\lambda_{j,k}(G(3,3))\leq\max\{2j+k,4j\}$.

Proof: Given graph $G(3,3)$ a labelling f as following shows.

For $j\leq k\leq 2j$, let $f(v)=2j, f(v_1^1)=0, f(v_2^1)=j;\quad f(v_1^2)=3j, f(v_2^2)=4j.$

For $k\geq 2j$, let $f(v)=2j, f(v_1^1)=0, f(v_2^1)=j;\ f(v_1^2)=j+k, f(v_2^2)=2j+k.$

It is obviously that f is a $\max\{2j+k,4j\}$-$L(j,k)$-labelling of graph $G(3,3)$ for $j\leq k$.

Hence, $\lambda_{j,k}(G(3,3))\leq\max\{2j+k,4j\}$.

Lemma 2.2 *Let j, k be two positive numbers and* m_2 *be a positive integer with* $m_2\geq 4$. *For* $j\leq k$, $\lambda_{j,k}(G(3,m_2))\leq\max\{j+2k,3j+k\}$.

Proof: Define a labelling f for graph $G(3,m_2)$ as follows.

For the induced cycle C_3, let $f(v)=2j, f(v_1^1)=0, f(v_2^1)=j.$

For the induced cycle C_{m_2}, we have the following two cases according to the relationship between *j* and *k*.

- If $j\leq k\leq 2j$, the labelling f is defined as follows.

1. If $m_2 \equiv 0(\bmod 4)$ and $m_2 \geq 4$, let $f(v)=2j$, $f(v_1^2)=3j$, $f(v_t^2)=[\mathrm{t}-2]_4\, j$ if $2 \leq t \leq m_2-2$, and $f(v_{m_2-1}^2)=3j+k$.
2. If $m_2=5$, let $f(v)=2j+k$, $f(v_1^2)=j+k$, $f(v_2^2)=0, f(v_3^2)=j$ and $f(v_4^2)=3j+k$. If $m_2 \equiv 1(\bmod 4)$ and $m_2 \geq 9$, let $f(v_t^2)=(2+t)j$ if $1 \leq t \leq 2$, $f(v_t^2)=[\mathrm{t}-3]_4\, j$ if $3 \leq t \leq m_2-2$, and $f(v_{m_2-1}^2)=3j+k$.
3. If $m_2 \equiv 2(\bmod 4)$ and $m_2 \geq 6$, let $f(v)=2j$, $f(v_t^2)=[\mathrm{t}+2]_4\, j$ if $1 \leq t \leq m_2-5$, $f(v_{m_2-4}^2)=4j$, $f(v_{m_2-3}^2)=j$, $f(v_{m_2-2}^2)=0$, and $f(v_{m_2-1}^2)=3j+k$.
4. If $m_2 \equiv 3(\bmod 4)$ and $m_2 \geq 7$, let $f(v)=2j$, $f(v_t^2)=[\mathrm{t}+2]_4\, j$ if $1 \leq t \leq m_2-5$, $f(v_{m_2-4}^2)=3j+k$, $f(v_{m_2-3}^2)=2j$, $f(v_{m_2-2}^2)=0$, and $f(v_{m_2-1}^2)=3j+k$.

- If $k \geq 2j$, let $f_0=0$, $f_1=j$, $f_2=k$, $f_3=j+k$, then the labelling f is defined as follows.
  1. If $m_2 \not\equiv 1(\bmod 4)$ and $m_2 \geq 4$, let $f(v)=k$, $f(v_t^2)=f_{[\mathrm{t}+2]_4}$ if $1 \leq t \leq m_2-3$, $f(v_{m_2-2}^2)=2k$, and $f(v_{m_2-1}^2)=j+2k$.
  2. If $m_2 \equiv 1(\bmod 4)$ and $m_2 \geq 5$, let $f(v)=k$, $f(v_t^2)=f_{3-[\mathrm{t}-1]_4}$ if $1 \leq t \leq m_2-2$ and $f(v_{m_2-1}^2)=j+2k$.

It is not difficult to verify that f is a $\max\{j+2k, 3j+k\}$-$L(j,k)$-labelling of graph $G(3,m_2)$ for $j \leq k$. Hence, we have $\lambda_{j,k}(G(3,m_2)) \leq \max\{j+2k, 3j+k\}$ for $j \leq k$. □

Lemma 2.3 *Let j,k be two positive numbers and m_1, m_2 be two positive integers with $m_1, m_2 \geq 4$. For $j \leq k$, $\lambda_{j,k}(G(m_1,m_2)) \leq \max\{2j+2k, 3k\}$.*

Proof: When $m_1, m_2 \geq 4$, define a labelling f for graph $G(m_1,m_2)$ as follows.

- If $k \geq 2j$, let $f_0=0$, $f_1=j$, $f_2=k$, $f_3=j+k$.

  For the induced subgraph C_{m_1}, the labelling f is given as following two cases.
  1. If $m_1 \equiv 1(\bmod 4)$ and $m_1 \geq 5$, let $f(v_1^1)=0, f(v_2^1)=3k$, $f(v_s^1)=f_{3-[\mathrm{s}+1]_4}$ if $3 \leq s \leq m_1-3$, $f(v_{m_1-2}^1)=j+2k$ and $f(v_{m_1-1}^1)=2k$.
  2. If $m_1 \not\equiv 1(\bmod 4)$ and $m_1 \geq 4$, let $f(v_s^1)=f_{3-[\mathrm{s}+2]_4}$ if $1 \leq s \leq m_1-3$, $f(v_{m_1-2}^1)=j+2k$ and $f(v_{m_1-1}^1)=2k$.

  For the induced subgraph C_{m_2}, let $f(v)=j$, $f(v_t^2)=f_{[\mathrm{t}+1]_4}$ if $1 \leq t \leq m_2-3$, $f(v_{m_2-2}^2)=j+2k$, and $f(v_{m_2-1}^2)=3k$.

- If $j \leq k \leq 2j$, let $f(v)=j+k$, $f(v_{m_1-1}^1)=k$, $f(v_1^2)=2j+k$, and $f(v_{m_2-1}^2)=2j+2k$.

For the remaining vertices in the induced subgraph C_{m_1}, the labelling f is given by the following two cases.

1. If $m_1 \equiv 0(\bmod 4)$ and $m_1 \geq 4$, let $f(v_s^1)=j[\mathrm{s}-1]_4$ if $1 \leq s \leq m_1-3$, and $f(v_{m_1-2}^1)=2j+2k$.
2. If $m_1 \not\equiv 0(\bmod 4)$ and $m_1 \geq 5$, let $f(v_s^1)=j[\mathrm{s}-1]_4$ if $1 \leq s \leq m_1-4$, $f(v_{m_1-3}^1)=2j+2k$ and $f(v_{m_1-2}^1)=j+2k$.

For the remaining vertices in the induced subgraph C_{m_2}, the labelling f is given by the following two cases.

1. If $m_2 \equiv 0(\bmod 4)$ and $m_2 \geq 4$, let $f(v_t^2)=j[\mathrm{t}-1]_4$ if $2 \leq t \leq m_2-2$.
2. If $m_2 \equiv 1(\bmod 4)$ and $m_2 \geq 5$, let $f(v_t^2)=j[\mathrm{t}-2]_4$ if $2 \leq t \leq m_2-2$.
3. If $m_2 \equiv 2,3(\bmod 4)$ and $m_2 \geq 6$, let $f(v_t^2)=j[\mathrm{t}-2]_4$ if $2 \leq t \leq m_2-3$, and $f(v_{m_2-2}^2)=j+2k$.

It is not difficult to verify that f is a $\max\{2j+2k,3k\}$-$L(j,k)$-labelling of graph $G(m_1,m_2)$ with $m_1,m_2\geq 4$ for $j\leq k$. Hence, we have $\lambda_{j,k}(G(m_1,m_2))\leq\max\{2j+2k,3k\}$ for $j\leq k$. □

Next, we consider the upper bound on $L(j,k)$-labelling number of p-Cactus Graphs with $p\geq 3$.

Lemma 2.4 *Let j,k be two positive numbers with $j\leq k$ and $m_1,m_2,\cdots,m_p$ be positive integers. Suppose p-Cactus graph $G(m_1,m_2,\cdots,m_p)$ contains q induced cycles C_3's, then $\lambda_{j,k}(G(m_1,m_2,\cdots,m_p))$ $\leq(2p-q-2)k+qj+\max\{k,2j\}$, where $0\leq q\leq p$ and $p\geq 3$.*

Proof: According to the property of p-Cactus graph, without loss of generality, let $m_1=\cdots=m_q=3$ and $m_{q+1},\cdots,m_p\geq 4$.

Define a labelling f for p-Cactus graph $G(m_1,m_2,\cdots,m_p)$ as follows.

For induced cycle C_{m_s},

- if $m_s=3$, let $f(v_1^s)=(s-1)(j+k)$, $f(v_2^s)=s(j+k)-k$, where $s=1,2,\cdots,q$.
- if $m_s\geq 4$, let $f(v)=(2p-q-2)k+(q+1)j$, $f(v_1^p)=(2p-q-2)k+qj$, $f(v_{m_p-1}^p)=(2p-q-2)k$ $+qj+\max\{k,2j\}$, and $f(v_1^s)=qj+(2s-q-2)k$, $f(v_{m_s-1}^s)=qj+(2s-q-1)k$, where $q+1\leq s\leq$ $p-1$, then we define the labels for the remaining vertices in induced cycle C_{m_s} as follows.

1. If $k\geq 2j$, let $f_0=0,\ f_1=j,\ f_2=k,\ f_3=j+k$. We have the following two cases.

   <u>Case 1</u>: $q=0$.

   For induced cycle C_{m_1}, the labelling f is given by the following three cases.

   (a) If $m_1=4$, let $f(v_2^1)=(2p-3)k+j$.

   (b) If $m_1=5$, let $f(v_3^1)=(2p-3)k+j$ and $f(v_2^1)=(2p-3)k$.

   (c) If $m_1\geq 6$, let $f(v_{m_1-2}^1)=(2p-3)k+j$, $f(v_{m_1-3}^1)=(2p-3)k$, and $f(v_i^1)=f_{[i-1]_4}$, where $2\leq i\leq$ m_1-4.

   For induced cycle C_{m_s}, let $f(v_i^s)=f_{[i-1]_4}$, where $2\leq i\leq m_1-2$, and $2\leq s\leq p$.

   <u>Case 2</u>: $q\neq 0$.

   For induced cycle C_{m_s}, let $f(v_i^s)=f_{[i-2]_4}$, where $2\leq i\leq m_1-2$, and $q+1\leq s\leq p$.

2. If $j\leq k\leq 2j$, we also have two cases.

   <u>Case 1</u>: When $q=0$ and $s=1$, the labelling f is given to the remaining vertices of C_{m_1} by the following two cases.

   (a) If $m_1=4$, let $f(v_2^1)=(2p-3)k+j$.

   (b) If $m_1=5$, let $f(v_3^1)=(2p-3)k+j$ and $f(v_2^1)=(2p-3)k$.

   (c) If $m_1\geq 6$, let $f(v_{m_1-2}^1)=(2p-3)k+j$, $f(v_{m_1-3}^1)=(2p-3)k+2j$, and $f(v_i^1)=j[i-1]_4$ if $2\leq i\leq$ m_1-4.

   <u>Case 2</u>: When $q=0$, $2\leq s\leq p$ or $q\neq 0$, $q+1\leq s\leq p$, the labelling f is given to the remaining vertices of C_{m_s} by the following two cases.

   (a) If $4\leq m_s\leq 6$, let $f(v_i^s)=j[i-2]_4$, where $2\leq i\leq m_s-2$.

   (b) If $m_s\geq 7$, let $f(v_{m_s-2}^s)=(2p-q-3)k+(q+1)j$, $f(v_{m_s-3}^s)=(2p-q-3)k+(q+2)j$ and $f(v_i^s)$ $=j[i-2]_4$, where $2\leq i\leq m_s-4$.

It is not difficult to verify that f is a $[(2p-q-2)k+qj+\max\{k,2j\}]$-$L(j,k)$-labelling of graph p-Cactus graph for $j\leq k$. That is, $\lambda_{j,k}(G(m_1,m_2,\cdots,m_p))$ $\leq(2p-q-2)k+qj+\max\{k,2j\}$ for $j\leq k$. □

3. Lower Bound on $L(j,k)$-Labelling Number of *p*-Cactus Graph with $p \geq 2$

Two labels are *t-separated* if the difference between them is at least *t*. According to the definition of $L(j,k)$-labelling, the labels of adjacent vertices are *j*-separated and the labels of vertices at distance two are *k*-separated.

The graph $K_{1,p}^q$ is induced by vertices $\{v_1^1, v_{m_1-1}^1, \cdots, v_1^q, v_{m_q-1}^q, v_1^{q+1}, v_{m_{q+1}-1}^{q+1}, \cdots, v_1^p, v_{m_p-1}^p\}$ from *p*-Cactus graph (See figure 1). Note that vertices $v_1^s, v_{m_s-1}^s$ are adjacent if $0 \leq s \leq q$.

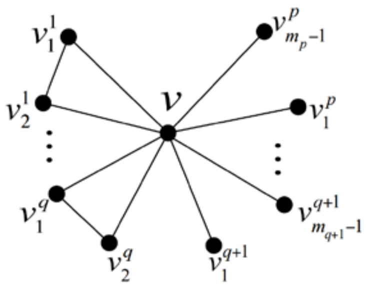

Figure 1. Graph $K_{1,p}^q$

Lemma 3.1 *Let j,k be two positive numbers and p be a positive integers. For* $j \leq k$, $\lambda_{j,k}(K_{1,p}^q) \geq (2p-q-2)k + qj + \max\{k, 2j\}$, *where* $0 \leq q \leq p$, *and* $p \geq 2$.

Proof: Let f be a λ-$L(j,k)$-labelling of $K_{1,p}^q$. According to the symmetry of the graph, without loss of generality, let $f(v_1^s) < f(v_{m_s-1}^s)$ for $1 \leq s \leq p$, and $\mathcal{F} = \{f(v_1^1), f(v_1^2), \cdots, f(v_1^q), f(v_1^{q+1}), f(v_{m_{q+1}-1}^{q+1}), \cdots, f(v_1^p), f(v_{m_p-1}^p)\}$, $\mathcal{V} = \{v_1^1, v_1^2, \cdots, v_1^q, v_1^{q+1}, v_{m_{q+1}-1}^{q+1}, \cdots, v_1^p, v_{m_p-1}^p\}$. Since any two vertices in set $\mathcal{V}$ are distance two apart, then each pair of labels in set $\mathcal{F}$ should be *k*-separated. Without loss of generality, let $f(v_1^1) = 0$ and $\mathcal{F}$ be an increasing sequence. Then, $f(v_{m_p-1}^p) \geq (q-1)k + 2(p-q)k = (2p-q-1)k$. Next, we add the label $f(v_2^s)$ to the sequence $\mathcal{F}$, then the new sequence should be non-decreasing, where $0 \leq s \leq q$. Since v_2^s is at distance two from vertices in $\mathcal{V} / \{v_1^s\}$ and adjacent to vertex v_1^s, then $f(v_2^s)$ should be *k*-separated from labels in $\mathcal{F} / \{f(v_1^s)\}$ and *j*-separated from $f(v_1^s)$.

At first, we add $f(v_2^1)$ into $\mathcal{F}$ to obtain the new increasing sequence $\mathcal{F}_1$.

1. If $f(v_2^1)$ lies in between any two adjacent members in $\mathcal{F} / \{f(v_1^1)\}$, then $f(v_2^1)$ should be k-separated from labels in $\mathcal{F} / \{f(v_1^1)\}$ since v_2^s is at distance two from vertices in $\mathcal{V} / \{v_1^1\}$, that is, $\lambda \geq f(v_{m_p-1}^p) \geq (2p-q-1)k + k$.
2. If $f(v_2^1)$ is larger than $f(v_{m_p-1}^p)$, then $\lambda \geq f(v_2^1) \geq (2p-q-1)k + k$.
3. If $f(v_2^1)$ lies in between $f(v_1^1)$ and $f(v_1^2)$, then $\lambda \geq f(v_{m_p-1}^p) \geq (2p-q-1)k + j$.

Thus, in order to obtain the minimum span of the labelling f, $f(v_2^1)$ should lie in between $f(v_1^1)$ and $f(v_1^2)$. That is, $\mathcal{F}_1 = \{f(v_1^1), \underline{f(v_2^1)}, f(v_1^2), \cdots, f(v_1^q), f(v_1^{q+1}), f(v_{m_{q+1}-1}^{q+1}), \cdots, f(v_1^p), f(v_{m_p-1}^p)\}$. Next, we add the label $f(v_2^2)$ to $\mathcal{F}_1$ to obtain the new sequence $\mathcal{F}_2$ which should be also increasing.

1. If $f(v_2^2)$ lies in between $\{f(v_1^1), f(v_2^1)\}$ or any two adjacent members in $\mathcal{F} / \{f(v_1^1), f(v_1^2)\}$, then $f(v_2^2)$ should be *k*-separated from labels in $\mathcal{F} / \{f(v_1^2)\}$ since v_2^2 is at distance two from vertices v_1^1, v_2^1 and $\mathcal{V} / \{v_1^1, v_1^2\}$. That is, $\lambda \geq f(v_{m_p-1}^p) \geq (2p-q-1)k + 2k$.
2. If $f(v_2^2)$ is larger than $f(v_{m_p-1}^p)$, then $\lambda \geq f(v_2^2) \geq (2p-q-1)k + j + k$.
3. If $f(v_2^2)$ lies in between $f(v_1^2)$ and $f(v_1^3)$, then $\lambda \geq f(v_{m_p-1}^p) \geq (2p-q-1)k + 2j$.

Thus, in order to obtain the minimum span of the labelling f , $f(v_2^2)$ should lie in between $f(v_1^2)$ and $f(v_1^3)$ .That is, $\mathcal{F}_2 = \{f(v_1^1), \underline{f(v_2^1)}, f(v_1^2), \underline{f(v_2^2)}, f(v_1^3), \cdots, f(v_1^q), f(v_1^{q+1}), f(v_{m_{q+1}-1}^{q+1}), \cdots, f(v_1^p), f(v_{m_p-1}^p)\}$.

Similar to above discussion, we can add labels $\{f(v_2^3), \cdots, f(v_2^q)\}$ into $\mathcal{F}_2$ one by one to obtain a new increasing sequence

$$\mathcal{F}_q = \{f(v_1^1), \underline{f(v_2^1)}, f(v_1^2), \underline{f(v_2^2)}, \cdots, f(v_1^q), \underline{f(v_2^q)}, f(v_1^{q+1}), f(v_{m_{q+1}-1}^{q+1}), \cdots, f(v_1^p), f(v_{m_p-1}^p)\},$$

Then we have $\lambda \geq f(v_{m_p-1}^p) \geq (2p-q-1)k + qj$.

Moreover, when $j \leq k \leq 2j$, we also need to consider cut-vertex v . Since v is adjacent to vertices $v_1^s, v_{m_s-1}^s$, then $f(v)$ should be j-separated from $f(v_1^s), f(v_{m_s-1}^s)$, where $1 \leq s \leq p$.

1. If $f(v_1^s) < f(v) < f(v_{m_s-1}^s)$ for some integer $s \in [1, q]$, then $\lambda \geq f(v_{m_p-1}^p) \geq (q+1)j + (2p-q-1)k$.
2. If $f(v_1^s) < f(v) < f(v_{m_s-1}^s)$ for some integer $s \in [q+1, p]$, then $\lambda \geq f(v_{m_p-1}^p) \geq (q+2)j + (2p-q-2)k$.
3. If $f(v_{m_s-1}^s) < f(v) < f(v_1^{s+1})$ for some integer $s \in [1, p-1]$, then $\lambda \geq f(v_{m_p-1}^p) \geq (q+2)j + (2p-q-2)k$.
4. If $f(v) > f(v_{m_p-1}^p)$, then $\lambda \geq f(v) \geq f(v_{m_p-1}^p) + j \geq (2p-q-1)k + qj + j = (q+1)j + (2p-q-1)k$.

Thus, $\lambda \geq (q+2)j + (2p-q-2)k$ for $j \leq k \leq 2j$ and $\lambda \geq (2p-q-1)k + qj$ for $k \geq 2j$.

Hence, $\lambda_{j,k}(K_{1,p}^q) \geq (2p-q-2)k + qj + \max\{k, 2j\}$, where $j \leq k$, $0 \leq q \leq p$ and $p \geq 2$. □

Lemma 3.2 *Let j,k be two positive numbers with $j \leq k$ and $m_1, m_2, \cdots, m_p$ be positive integers. Then, $\lambda_{j,k}(G(m_1, \cdots, m_p)) \geq (2p-q-2)k + qj + \max\{k, 2j\}$, where $0 \leq q \leq p$ and $p \geq 2$.*

Proof: Let f be a λ - $L(j,k)$-labelling of p-Cactus graph $G(m_1, m_2, \cdots, m_p)$. Since graph $K_{1,p}^q$ is an induced subgraph of $G(m_1, m_2, \cdots, m_p)$, by Lemma 1, we have $\lambda_{j,k}(G(m_1, \cdots, m_p)) \geq \lambda_{j,k}(K_{1,p}^q) \geq (2p-q-2)k + qj + \max\{k, 2j\}$ for $j \leq k$, where $0 \leq q \leq p$ and $p \geq 2$. □

By Lemma 2.1-3.2, we can obtain the following conclusion.

Theorem 3.3 *Let j,k be two positive numbers with $j \leq k$ and $m_1, m_2, \cdots, m_p$ be positive integers. Suppose p-Cactus graph $G(m_1, m_2, \cdots, m_p)$ contains q circles C_3 , then $\lambda_{j,k}(G(m_1, \cdots, m_p)) = qj + (2p-q-2)k + \max\{k, 2j\}$, where $0 \leq q \leq p$ and $p \geq 2$.*

4.Acknowledgments

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5.References

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