IOP Conf. Series: Materials Science and Engineering 466 (2018) 012045 doi:10.1088/1757-899X/466/1/012045

Model Reduction Method based on Rational Canonical form of System Matrix and Krylov Subspace

Zunhai Gao[a] and Zhuo Chen[b]

[a] School of Mathematics and Computer Science, Wuhan Polytechnic University, Wuhan, 430048, China; [b] School of Economics and Management, Wuhan Polytechnic University, Wuhan 430048, China.

Email: haigao007@whpu.edu.cn

Abstract. For single input and single output time-invariant linear system, a new projection method to obtain reduced models is presented by making use of the rational canonical form of system matrix and the Krylov subspace. At first, the system matrix is transformed to its rational canonical form by use of linear transformation. And then both projection method and Krylov subspace method are used to reduce model. The advantage of this method is the poles of the reduced system are same as those of the original. Thus the reduced system remains the stability when the original system is. This method is more effective than simple Krylov subspace method. Numerical examples demonstrate the effectiveness of the method.

1. Introduction

Model reduction is a technique to simplify the simulation of dynamical systems described by differential equations. The idea is to project the original, high-dimensional, state-space onto a properly chosen low-dimensional subspace to arrive at a (much) smaller system having properties similar to the original system. Complex systems can thus be approximated by simpler systems involving fewer equations and unknown variables, which can be solved much more quickly than the original problem.

Consider the linear, time-invariant system described by the state–space equations

$$\begin{cases} \dot{X} = AX + Bu \\ Y = CX \end{cases} \tag{1}$$

where $X(t) \in R^n$, A, B and C are real matrices of appropriate sizes. $u(t)$ and $Y(t)$ are appropriate real function matrices. and The system (1) will be expressed as $\{A,B,C\}$ in short. Let $X_0 = Z^T X$ and $Z^T V = E_m$, where Z and V are all $n \times m$ matrices, called projection matrices, E_m is $m \times m$ unit matrix. If for Z and V, there exist an $m \times m$ real matrix A_0 , an $m \times n_1$ real matrix B_0 and an $n_2 \times m$ real matrix C_0 such that

$$\begin{cases} \dot{X}_0 = A_0 X_0 + B_0 u \\ Y_0 = C_0 X_0 \end{cases} \tag{2}$$

with $A_0 = Z^T AV$, $B_0 = Z^T B$, $C_0 = CV$, then (2) is a reduced model of (1).

The model reduction methods based on projection matrix got a wide range of applications in the last decade. Those gradually resolve the problems of numerical stability, matching accuracy, and maintaining RLC network passivity. These methods essentially make use of the greater redundancy in observability and controllability of the system.

This paper focus on linear time-invariant state-space models with only single input and single output system in state space form

$$\begin{cases} \dot{X} = AX + bu \\ Y = cX \end{cases} \tag{3}$$

where $A \in R^{n\times n}$, $X(t), b, c \in R^n$ and $u(t), Y(t) \in R$.

There have been several kinds of model reduction methods. They can be classified into three categories according to [1-5]:

(a) Singular value decomposition (SVD) based methods,
(b) Krylov (moment matching) based methods, and
(c) SVD-Krylov based methods.

In recent years, Krylov subspace methods have become popular tools for computing reduced models of high order linear time-invariant systems ([4-9]). The reduction can be done by applying a projection from high order to lower order space using the bases of some subspaces called input and output Krylov subspaces.

A Krylov subspace is a subspace spanned by a sequence of vectors generated by a given matrix and a vector as follows. Given a matrix A and a starting vector b, the kth Krylov subspace $\kappa_k(A,b)$ is spanned by a sequence of k column vectors:

$$\kappa_k(A,b) = span\left\{b, Ab, A^2b, \cdots, A^{k-1}b\right\}$$

where A is a constant $n\times n$-matrix, b is a constant $n\times 1$-vector (the so-called starting vector) and k is some given positive integer. The vectors $b, Ab, A^2b, \cdots, A^{k-1}b$ constructing the subspace are called basic vectors.

In this paper, a new method of model reduction for SISO large-scale dynamical systems is put forward. It is a kind of projection method, whose projection matrix depends on the rational canonical form of system matrix and the Krylov subspace. This method is more effective than simple Krylov subspace method.

For the use of the rational canonical form of square matrix to model reduction, we will explain it as the following theorem.

Theorem For any *n*×*n* matrix *A* over a field *F*, there exists a nonsingular matrix *P* over the field *F* such that *A* is similar to the unique following quasi-diagonal matrix

$$PAP^{-1} = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix},$$

where A_i $(i = 1, 2, \cdots, r)$ are companion matrices of the polynomials $d_i(\lambda)$ correspondingly, and satisfying $d_r(\lambda) | \cdots | d_2(\lambda) | d_1(\lambda)$, in which the sum of the degrees of these polynomials $d_1(\lambda), d_2(\lambda), \cdots, d_r(\lambda)$ is *n*.

It is important to note that these A_i $(i = 1, 2, \cdots, r)$ in the quasi diagonal matrix are of reverse order compared with conventional rational canonical form, which is still called a rational canonical form for convenience.

2. Model reduction based on rational canonical form of system matrix and Krylov subspace

For the model (3), doing nonsingular linear transformation $\overline{X} = PX$, it is transformed to the following

$$\begin{cases} \dot{\overline{X}} = \overline{A}\overline{X} + \overline{b}u \\ Y = \overline{c}\overline{X} \end{cases} \tag{4}$$

where

$$\overline{A} = PAP^{-1} = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix},\ \overline{b} = Pb \text{ and } \overline{c} = cP^{-1}.$$

Next we do order reduction to the above model.

Let $m\times n$ matrix $Z = V$, $Z^T = \begin{pmatrix} E & O & \cdots & O \end{pmatrix}$ satisfying $Z^T V = \begin{pmatrix} E & O & \cdots & O \end{pmatrix}\begin{pmatrix} E \\ O \\ \vdots \\ O \end{pmatrix} = E$ and

$$Z^T \overline{A} V = \begin{pmatrix} E & O & \cdots & O \end{pmatrix}\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}\begin{pmatrix} E \\ O \\ \vdots \\ O \end{pmatrix} = A_1 \cdot$$

Let $X_1 = Z^T \overline{X}$, $b_1 = Z^T \overline{b}$ and $c_1 = \overline{c}V$, where Z and V (V=Z) are two $m\times n$ matrices, called aggregation matrices. So the model (4) is transformed to the following reduced model

$$\begin{cases} \dot{X}_1 = A_1 X_1 + b_1 u \\ Y_1 = c_1 X_1 \end{cases} \tag{5}$$

Since the minimal polynomial of A_1 is $d_1(\lambda)$, which is also the minimal polynomial of the original system matrix A. It means that all the poles of the original system (3) are included in the system (5). That is, the reduced model will remain to be stable if the original one is.

Now we go on to the further model reduction to the model (5) by the use of Krylov subspace.

For linear system (5), the probability that it is controllable is 1[10], which is means that the controllability matrix $Q_c = \begin{pmatrix} b\ Ab\ \cdots\ A^{m-2}b\ A^{m-1}b \end{pmatrix}$ is nonsingular.

For any positive integer k , $(1< k \le m-1)$, column vectors of the matrix $\begin{pmatrix} b\ Ab\ \cdots\ A^{k-2}b\ A^{k-1}b \end{pmatrix}$ are linearly independent. These vectors can be transformed into a set of normal orthogonal vector sets and then compose to matrix V and matrix $Z = V$ satisfying $Z^T V = E$.

Let $\overline{A}_1 = Z_1^T A_1 V_1$, $\overline{b}_1 = Z_1^T b_1$ and $\overline{c}_1 = c_1 V_1$, the projection matrices are Z_1 and V_1, then we obtain an order reduced model as (6).

$$\left\{ \overline{A}_1, \overline{b}_1, \overline{c}_1 \right\}. \tag{6}$$

3. Model Simulations

Suppose a state-space model is

$$A = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{pmatrix},\ b = \begin{pmatrix} 0 & -1 & 2 & -1 & 3 & 0 \end{pmatrix}^T,\ c = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \tag{7}$$

where $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 \end{pmatrix}$.

Let $Z = V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T$, we get $A_1 = Z^T AV$, $b_1 = Z^T b$, $c_1 = cV$, then the reduced model is

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix},\ b_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix},\ c_1 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \tag{8}$$

Now we reduce the model (8) to order 2 model, the column vectors in matrix $\begin{pmatrix} b_1 & A_1 b_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \\ 2 & -3 \end{pmatrix}$ can be transformed into a set of orthogonal vector sets and then compose to matrix $V_1 = \begin{pmatrix} 0 & -5/6 \\ -1 & 1/3 \\ 2 & 1/6 \end{pmatrix}$, let $Z_1 = 1/5 \begin{pmatrix} 0 & -5 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$, which is satisfying $Z_1^T V_1 = E$. We get the following reduced model

$$\bar{A}_1 = Z^T A_1 V = \begin{pmatrix} -1.6 & -0.3 \\ 1.2 & -0.4 \end{pmatrix},\ \bar{b}_1 = Z^T b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \bar{c}_1 = c_1 V = \begin{pmatrix} 2 & -2/3 \end{pmatrix}, \tag{9}$$

If we reduce the model (7) to order 2 model by the use of Krylov subspace directly, we get model (10).

$$A_{k2} = \begin{pmatrix} -2.0534 & 0.9124 \\ -0.2903 & -1.0537 \end{pmatrix},\ b_{k2} = \begin{pmatrix} -3.8026 \\ 0.7348 \end{pmatrix},\ c_{k2} = \begin{pmatrix} -0.1422 & 0.6249 \end{pmatrix}, \tag{10}$$

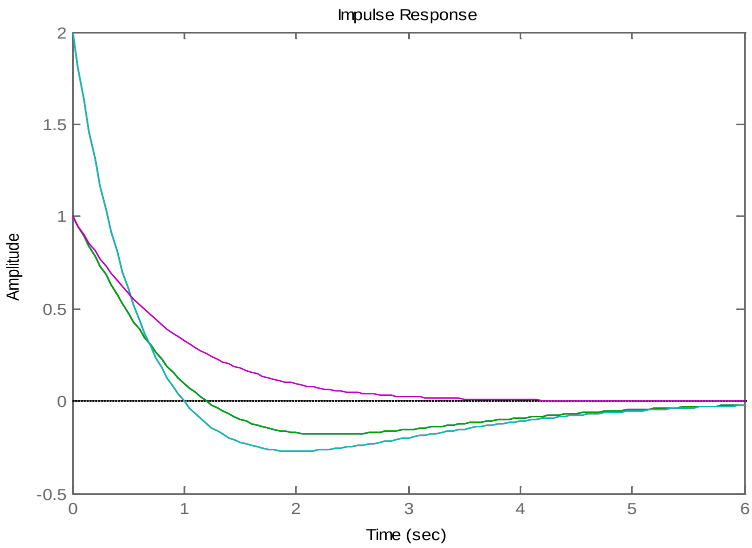

Figure 1. The unit response waveforms of three systems

The unit response waveforms of the original system (7) and of the reduction system models (8), (9) and (10) show as above. As can be seen from the figure, the graphics of reduction model (8) and of the original system (7) are entirely overlapping (the middle line shown in the Fig). The graphic of the reduction model (9) (the below line shown in the Fig) and of the original system (7) have error in a small range. The graphic of the reduction model (10) (the above line shown in the Fig) and of the original system (7) have error in a certain range.

4. Conclusions

This paper presents a new reduction models for SISO linear system by using the rational canonical form of system matrix and the Krylov subspace. Firstly, the system matrix is transformed to its rational canonical form by use of linear transformation. And then both projection method and Krylov subspace method are used to reduce model. The advantage of this method is the poles of the reduced system are same as those of the original. Thus the reduced system remains the stability when the original system is. This method is more effective than simple Krylov subspace method. Simulation results are show to verify the validity and feasibility of the methods.

5. Acknowledgements

This work is supported by the National Social Science Fund of China under grant No. 14BGL195.

6. References

[1] Wilhelmus H.A. Schilders, Henk A. van der Vorst and Joost Rommes 2008 *Model Order Reduction: Theory, Research Aspects and Applications* (Springer)
[2] Peter Benner, Volker Mehrmann and Danny C Sorensen 2005 *Dimension Reduction of Large-Scale Systems* (Springer)
[3] K Mohamed, A Mehdi, M Abdelkader 2015 SVD-AORA method for Dynamic Linear Time Invariant Model Order Reduction *IFAC-PapersOnLine* 48(1) pp 695-696
[4] Yan Zhe, Lu Fangming and Zhou Lin 2016 A Large Scale System Model-Order Reduction Method Based on SVD-Krylov *International Journal of Grid and Distributed Computing* 9 (10) pp119-128
[5] S Gugercin 2008 An iterative SVD-Krylov based algorithm for model reduction of large-scale dynamical systems *Linear Algebra and Its Applications* 428(8/9) pp 1964-1986
[6] Jbilou, K. A 2016 Survey of Krylov-Based Methods for Model Reduction in Large-Scale MIMO Dynamical Systems *Applied and Computational Mathematics* 15(2) pp 117-148
[7] Bazaz, Mohammad Abid, Mashuq-un-Nabi and Janardhanan S 2013 Automated and efficient order selection in Krylov-based model order reduction *International Journal of Modelling, Identification and Control* 18(4) pp 332-340
[8] Tobias Breiten and Tobias Damm 2010 Krylov subspace methods for model order reduction of bilinear control systems *Systems & Control Letters* 59(8) pp 443-450
[9]Michiels Wim, Jarlebring Elias and Meerbergen Karl 2011 Krylov-Based Model Order Reduction of Time-Delay Systems *SIAM Journal on Matrix Analysis & Applications* 32(4) pp 1399-1421
[10] Lin Huang 1984 *Linear Algebra in Systems and Control Theory* (Beijing: Science Press)