

Analytical Solution of Optimization Problem of Stability of Frame Systems

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Abstract. The article deals with the problem of optimizing multi-story frame systems from the condition of stability. When solving the problem of optimizing the cross-sections of the elements of frame systems, the bearing capacity of which is determined by their stability, the known parameters are configuration, support fastenings, material and nodal loads. As an optimized parameter, the constant bending stiffness within each rod is assumed. The optimization problem is solved by the criterion of minimum volume of the frame material. The mathematical model compiled by the authors in a general form is a nonlinear mathematical programming task. To solve it, the method of Lagrange multipliers is used. The problem is solved in two stages. At the first stage of the stationary conditions, the authors obtained the relationships between the nodal displacements for free and non-free frames. In the second stage, at the found values of these relations, the determinant condition of the critical state of the frame structure can be represented in the form of a system of equations. In this case, the system of equations obtained by the method of Lagrange multipliers becomes analytically solvable with respect to unknown linear stiffnesses, since the equations entering into it do not contain reciprocal products of transcendental functions from unknowns. In this article, in closed form, equations are obtained that allow determining the optimal linear stiffnesses of columns and crossbars of free and non-free frames. The analytical solution obtained has been checked. To select the optimal stiffnesses from the stability condition, it is necessary to define the critical parameters using the constructive flexibility of the elements, using the functions presented in the paper, and then determine the linear stiffnesses of the posts and crossbars.

1. Introduction

At the enterprises of oil refining, chemical, gas, metallurgical, power industry, transportation of the product through pipelines laid above the ground by separately standing supports – multi-storey elevators is widely used. In connection with the extensive use of transport racks, the problem arises of their rational or optimal design. The design schemes of multistory overpasses are free and/or non-free frame systems, for which the exhaustion of load-bearing capacity in the form of loss of stability is characteristic. Thus, the problem of optimal design of a transport rack takes the form of the task of optimizing a multi-storey frame system from the stability condition. Solutions to optimization



problems are built on the basis of a series of numerical solutions [1], and by "groping" and discarding the scarcely used reserves of the constructive system [2], and by direct numerical solution of non-linear programming problems, etc. But the most important are closed solutions to optimization problems, which make it possible to obtain ready-made formulas for the design of individual structural classes [3, 4, 5].

2. Mathematical model of the optimization problem from the stability condition. Determination of the optimal ratio of the movement of frames in the critical state

When optimizing the cross-sections of the elements of frame systems, the bearing capacity of which is determined by their stability, the configuration, support fastenings, material and nodal loads are known. The optimized frame consists of n rods of unknown linear rigidity for bending ($i_j=x_j$), which is assumed to be constant within each rod. It is required to find the values of the variables $x_1, x_2, \dots, x_n$ for

which the weight (volume) function $G = \sum_{i=1}^m \bar{\lambda}_i^2 \cdot x_i$ takes a minimum value. Here λ_i is the constructive

flexibility of the i -th rod. In this case, it is necessary that the frame is in critical condition, and the necessary strength conditions are satisfied.

The equation of critical state by the displacement method in the matrix form has the form:

$$R \cdot Z = 0. \quad (1)$$

Where Z – a vector-column of unknown displacements of the frame nodes; R – the reaction matrix corresponding to the given system.

As is known, the non-trivial solution of the system of equations corresponds to the vanishing of the determinant of the system of equations (1):

$$|R| = \det[r_{ik}] = 0. \quad (2)$$

This equation expresses the condition of the critical state of the frame.

Here r_{ik} – reactions in additional connections from unit displacements of nodes, depending on the unknown linear stiffnesses of posts and crossbars and parameters

$$v_j = \sqrt{\frac{N_j \cdot h_j}{x_j}}, \quad (3)$$

where N_j – the longitudinal force in the rod; h_j – the length of the rod.

When solving the problem of optimizing the cross-sections and rigidities of the frame, the design loads at the nodes are known and are adopted as limiting (critical) loads. Therefore, the longitudinal forces N in individual elements can easily be found.

For multi-storey free and unfixed frames (figure 1), the equation of the displacement method (1) has a three-member structure, and the coefficients for unknowns form a Jacobian matrix of the order of $m \times m$:

$$R = \begin{pmatrix} r_{11} & r_{12} & 0 & 0 & 0 & \dots & 0 & 0 \\ r_{21} & r_{22} & r_{23} & 0 & 0 & \dots & 0 & 0 \\ 0 & r_{32} & r_{33} & r_{34} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & r_{m,m-1} & r_{mm} \end{pmatrix}. \quad (4)$$

Elements of this matrix have the form

$$r_{ik} = -x_k \frac{v_k}{\sin v_k}, \quad i \neq k$$

$$r_{kk} = x_{k-1} \frac{v_{k-1}}{\operatorname{tg} v_{k-1}} + x_k \frac{v_k}{\operatorname{tg} v_k} + 6x_{m+k} \quad (5)$$

for free frames, and

$$\begin{aligned}
 r_{ik} &= 2x_k \cdot \varphi_3(v_k), \quad i \neq k \\
 r_{kk} &= 4x_{k-1} \cdot \varphi_2(v_{k-1}) + 4x_k \cdot \varphi_2(v_k) + c_k x_{m+k}
 \end{aligned}
 \tag{6}$$

for non-free frames, where c_k are the coefficients characterizing the fixing of the crossbars; $\varphi(v)$ – special functions [6, 7].

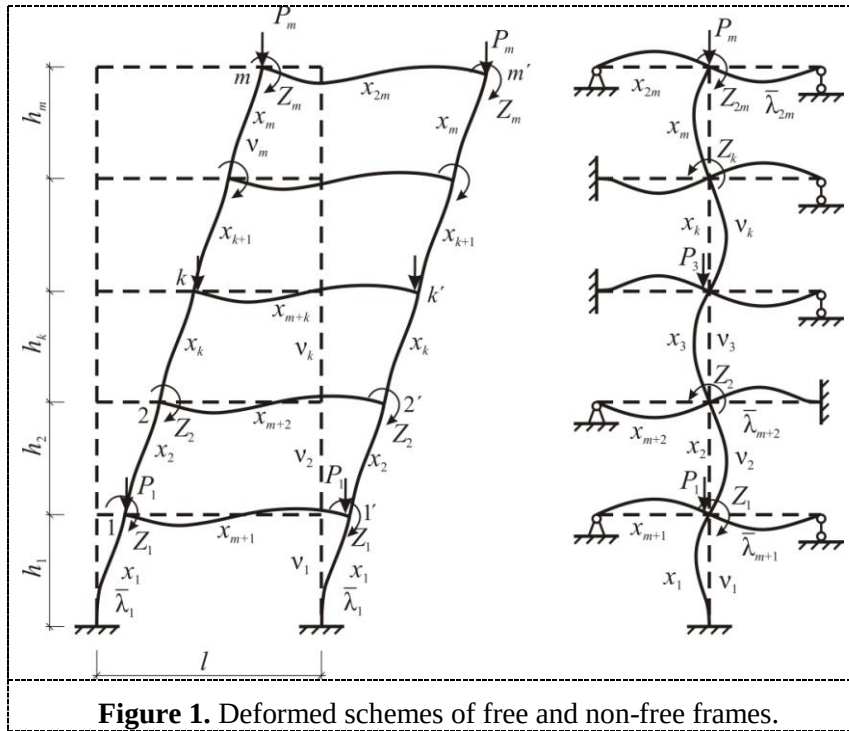


Figure 1. Deformed schemes of free and non-free frames.

Condition (2) is the main restriction of the optimization problem from the stability condition. Restrictions on the strength of compressed elements can be added after solving the main problem.

Thus, the mathematical model of the problem can be written in the form:

Find the minimum weight (volume) of the frame material

$$\left. \begin{aligned}
 f(x) &= V = \sum_{j=1}^{2m} \bar{\lambda}_j^2 \cdot x_j \\
 \text{on condition } R \cdot Z &= 0, \quad Z \neq 0, \\
 D(x) &= |R| = \det[r_{ik}] = 0, \\
 x_j &\geq 0.
 \end{aligned} \right\}
 \tag{7}$$

The mathematical model (7) compiled in a general form is a nonlinear mathematical programming problem. To solve it, we use the method of Lagrange multipliers. This method, as shown below, allows us to obtain a closed analytical solution and, in the solution process, reveals very important features of the optimization problem for multi-storey frames from the stability condition.

We compose the Lagrange function, i.e. we pass from the conditional problem (7) to the unconditional

$$L = V + u \cdot D(x), \tag{8}$$

where u is the indefinite Lagrange multiplier.

The necessary minimum conditions are obtained by computing and equating to zero the partial derivatives of the function (8) with respect to x_j and u :

$$\left. \begin{aligned} \frac{\partial L}{\partial x_j} &= \frac{\partial V}{\partial x_j} + u \cdot \frac{\partial D(x)}{\partial x_j} = 0, \\ \frac{\partial L}{\partial u} &= D(x) = \det[r_{ik}] = 0. \end{aligned} \right\} \quad j = 1, 2, \dots, 2m, \quad (9)$$

The vector of unknown string stiffnesses $\{x\}$ and the factor u can be determined directly from the solution of the system of equations (9). However, as noted in [8], even with the simplest calculation schemes, such a method proves to be practically unrealizable because of the need to solve very complex systems of transcendental equations. Apparently, for this reason, the Lagrange method did not find direct application in optimization problems. We will use system (9) for another purpose – we will try to determine the relationship between the unknowns of the displacement method. With the values of these relations found, the condition of the critical state can be represented not by the fact that the determinant (2) is zero, but by a system of equations of the form:

$$b_{k-1} \cdot r_{kk-1} + b_k \cdot r_{kk} + b_{k+1} \cdot r_{kk+1} = 0, \quad (10)$$

where $b_k = Z_k / Z_n$.

We consider the last $m+1$ equations of the system (9), where the differentiation is performed with respect to the rigidity of the crossbars and to the parameter u :

$$\left. \begin{aligned} \frac{\partial L}{\partial x_{m+k}} &= \bar{\lambda}_{m+k}^2 + u \cdot \frac{\partial D(x)}{\partial x_{m+k}} = 0, \quad k = 1, 2, \dots, m, \\ \frac{\partial L}{\partial u} &= \det[r_{ik}] = 0. \end{aligned} \right\} \quad (11)$$

The first m equations of system (11) can be written in the form:

$$\bar{\lambda}_{m+k}^2 + u \cdot c_k \cdot A_{kk} = 0, \quad (12)$$

where A_{kk} is the algebraic complement of r_{kk} .

We represent these algebraic complements in the form of a product of two minors:

$$A_{kk} = D_{k-1} \cdot D'_{m-k}, \quad (13)$$

where D_{k-1} are principal minors of order $k-1$, having the form:

$$D_0 = 1; \quad D_1 = r_{11}; \quad D_2 = \begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix}; \quad \dots;$$

D'_{m-k} are principal minors of order $m-k$ minor, having the form:

$$D_{m-1} = \begin{vmatrix} r_{22} & r_{23} & 0 & \dots \\ r_{32} & r_{33} & r_{34} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}; \quad D_{m-2} = \begin{vmatrix} r_{33} & r_{34} & 0 & \dots \\ r_{43} & r_{44} & r_{45} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}; \quad \dots$$

We now transform equations (12) to the form

$$\frac{c_k}{\bar{\lambda}_{m+k}^2} + D_{k-1} \cdot D'_{m-k} = -\frac{1}{u}. \quad (14)$$

The last equation of system (11) can be written in the form

$$\det[r_{ik}] = r_{11} \cdot D_{m-1} - r_{12}^2 \cdot D_{m-2} = r_{11} \cdot r_{22} \cdot D_{m-2} - r_{12}^2 \cdot D_{m-2} - r_{11} \cdot r_{23}^2 \cdot D_{m-3} = \dots = 0 \quad (15)$$

Consider the simultaneous solution of the equations $m+k$, $m+k+1$ and $2m+1$ of the system (11). Expressions for D_{k-1} will be written in expanded form. For $k=1$ we obtain

$$\left. \begin{aligned} \frac{c_1}{\bar{\lambda}_{m+1}^2} \cdot D'_{m-1} &= \frac{c_2}{\bar{\lambda}_{m+2}^2} \cdot r_{11} \cdot D'_{m-2} \\ r_{11} \cdot D'_{m-1} &= r_{12}^2 \cdot D'_{m-2} \end{aligned} \right\}. \quad (16)$$

Dividing the first equation by the second, we obtain

$$r_{11}^2 \cdot \frac{\bar{\lambda}_{m+1}^2}{c_1} = r_{12}^2 \cdot \frac{\bar{\lambda}_{m+2}^2}{c_2}, \quad (17)$$

then

$$r_{11} \cdot \bar{\lambda}_{m+1} \cdot c_1^{-1/2} = \pm r_{12} \cdot \bar{\lambda}_{m+2} \cdot c_2^{-1/2}. \quad (18)$$

Considering the scheme of deformation of a free and unfixed frame with loss of stability (figure 1), we obtain for a free frame ($c_k = \text{const}$):

$$\bar{\lambda}_{m+1} \cdot r_{11} + \bar{\lambda}_{m+2} \cdot r_{12} = 0, \quad (19a)$$

for a non-free frame:

$$\bar{\lambda}_{m+1} \cdot c_1^{-1/2} \cdot r_{11} - \bar{\lambda}_{m+2} \cdot c_2^{-1/2} \cdot r_{12} = 0. \quad (19b)$$

Consider $m+2$, $m+3$ and $2m+1$ of the equation ($k=2$):

$$\left. \begin{aligned} \frac{c_2}{\bar{\lambda}_{m+2}^2} \cdot r_{11} \cdot D'_{m-2} &= \frac{c_3}{\bar{\lambda}_{m+3}^2} \cdot (r_{11}r_{22} - r_{12}^2) \cdot D'_{m-3} \\ (r_{11}r_{22} - r_{12}^2) \cdot D'_{m-2} &= r_{11}r_{23}^2 \cdot D'_{m-3} \end{aligned} \right\}, \quad (20)$$

whence taking into account (19a) and (19b) we obtain for a free frame

$$\bar{\lambda}_{m+1} \cdot r_{21} + \bar{\lambda}_{m+2} \cdot r_{22} + \bar{\lambda}_{m+3} \cdot r_{23} = 0, \quad (21a)$$

for a non-free frame:

$$-\bar{\lambda}_{m+1} \cdot c_1^{-1/2} \cdot r_{21} + \bar{\lambda}_{m+2} \cdot c_2^{-1/2} \cdot r_{22} - \bar{\lambda}_{m+3} \cdot c_3^{-1/2} \cdot r_{23} = 0. \quad (21b)$$

The simultaneous solution of the equations $m+3$, $m+4$ and $2m+1$, taking into account the obtained expressions (21a) and (21b), gives

$$\bar{\lambda}_{m+2} \cdot r_{32} + \bar{\lambda}_{m+3} \cdot r_{33} + \bar{\lambda}_{m+4} \cdot r_{34} = 0$$

and

$$-\bar{\lambda}_{m+2} \cdot c_2^{-1/2} \cdot r_{32} + \bar{\lambda}_{m+3} \cdot c_3^{-1/2} \cdot r_{33} - \bar{\lambda}_{m+4} \cdot c_4^{-1/2} \cdot r_{34} = 0.$$

It is obvious that the k -th equation will have the form

$$\pm \bar{\lambda}_{m+k-1} \cdot c_{k-1}^{-1/2} \cdot r_{kk-1} + \bar{\lambda}_{m+k} \cdot c_k^{-1/2} \cdot r_{kk} \pm \bar{\lambda}_{m+k+1} \cdot c_{k+1}^{-1/2} \cdot r_{kk+1} = 0. \quad (22a)$$

Thus, the ratio of angular displacements has the form

$$\frac{Z_k}{Z_{k+n}} = \frac{\bar{\lambda}_{m+k} \cdot c_k^{-1/2}}{\bar{\lambda}_{m+k+n} \cdot c_{k+n}^{-1/2}}. \quad (23)$$

For frames of regular structure with the same values of constructive flexibility of crossbars, we obtain

$$Z_k = \text{const}. \quad (24)$$

In this case, equations (a) take the form

$$\pm r_{kk-1} + r_{kk} \pm r_{kk+1} = 0. \quad (22b)$$

Analogous (24) and (22b) dependences and for $n(m_1-2)$ equations in the case when n parts of the frame consist of m_1 floors with the same flexibility $\bar{\lambda}_{m+k}$.

Thus, the stationarity conditions (9) made it possible to establish relations between the unknown displacement methods of the original problem (7).

These relationships can be used to discretize the deterministic criterion of loss of stability.

3. Analytical solution of optimization problem

We use the obtained equations (22a) to form constraints in the form of a system of equations in place of the common determinant (2).

The problem under consideration (7) will have the form:

$$\left. \begin{aligned} &\text{Find} \\ &\min f(x) = V = \sum_{j=1}^{2m} x_j \cdot \bar{\lambda}_j^2 \\ &\text{with} \\ &\psi_k = \pm \bar{\lambda}_{m+k-1} \cdot c_{k-1}^{-1/2} \cdot r_{kk-1} + \bar{\lambda}_{m+k} \cdot c_k^{-1/2} \cdot r_{kk} \pm \bar{\lambda}_{m+k+1} \cdot c_{k+1}^{-1/2} \cdot r_{kk+1} = 0. \end{aligned} \right\} \begin{array}{l} \text{a)} \\ \text{b)} \end{array} \quad (25)$$

The Lagrange function takes the form:

$$L = \sum_{j=1}^{2m} x_j \cdot \bar{\lambda}_j^2 + \sum_{k=1}^m u_k \cdot \psi_k, \quad (26)$$

and the necessary stationary conditions are written in the form

$$\left. \begin{aligned} &\frac{\partial L}{\partial x_k} = \bar{\lambda}_k^2 + u_{k-1} \frac{\partial \psi_{k-1}}{\partial x_k} + u_k \frac{\partial \psi_k}{\partial x_k} = 0, \\ &\frac{\partial L}{\partial x_{m+k}} = \bar{\lambda}_{m+k}^2 + \bar{\lambda}_{m+k} \cdot c_k^{-1/2} \cdot u_k = 0, \\ &\frac{\partial L}{\partial u_k} = \pm \bar{\lambda}_{m+k-1} \cdot c_{k-1}^{-1/2} \cdot r_{kk-1} + \bar{\lambda}_{m+k} \cdot c_k^{-1/2} \cdot r_{kk} \pm \bar{\lambda}_{m+k+1} \cdot c_{k+1}^{-1/2} \cdot r_{kk+1} = 0. \end{aligned} \right\} \begin{array}{l} \text{a)} \\ \text{b)} \\ \text{c)} \end{array} \quad (27)$$

This system is already analytically solvable with respect to the unknown string rigidities x_j and the parameters u_k , since the equations contained in it do not contain reciprocal products of transcendental functions from unknowns.

From (27b) we find

$$u_k = -c_k^{-1/2} \cdot \bar{\lambda}_{m+k}. \quad (28)$$

Therefore, in order to determine the values $x_1, x_2, \dots, x_m$, corresponding to the linear stiffnesses of the racks, it is necessary to solve successively m equations (27a), each from one variable. The values of the variables $x_{m+1}, x_{m+2}, \dots, x_{2m}$ are found from the conditions (27c) into which the found $x_1, x_2, \dots, x_m$ are substituted.

For frames with shifting nodes, equations (27c) have the form

$$\psi_1 = \left(x_1 \frac{v_1}{\operatorname{tg} v_1} + x_2 \frac{v_2}{\operatorname{tg} v_2} + 6x_{m+1} \right) \bar{\lambda}_{m+1} - \bar{\lambda}_{m+1} x_2 \frac{v_2}{\sin v_2} = 0, \quad (29)$$

$$\psi_k = \bar{\lambda}_{m+k-1} \left(-x_k \frac{v_k}{\sin v_k} \right) + \bar{\lambda}_{m+k} \left(x_k \frac{v_k}{\operatorname{tg} v_k} + x_{k+1} \frac{v_{k+1}}{\operatorname{tg} v_{k+1}} + 6x_{m+k} \right) + \bar{\lambda}_{m+k+1} \left(-x_{k+1} \frac{v_{k+1}}{\sin v_{k+1}} \right) = 0. \quad (30)$$

We introduce the notation:

$$\gamma_1(v) = \left(x \frac{v}{\operatorname{tg} v} \right)' \cdot 2 = \frac{v}{\operatorname{tg} v} + \left(\frac{v}{\operatorname{tg} v} \right)^2 + v^2, \quad (31)$$

$$\gamma_2(v) = \left(x \frac{v}{\sin v} \right)' \cdot 2 = \frac{v}{\sin v} + \left(\frac{v}{\sin v} \right)^2 \cdot \cos v, \quad (32)$$

$$\gamma_3(v) = \left(x \frac{v}{\operatorname{tg} v} - x \frac{v}{\sin v} \right)' = \left(\frac{v}{2} \right)^2 + \left(\frac{v}{2} \operatorname{tg} \frac{v}{2} \right)^2 - \frac{v}{2} \operatorname{tg} \frac{v}{2}. \quad (33)$$

Now equations (27a) for free frames, taking into account (29) and (30), are obtained in the form

$$\left. \begin{aligned} \gamma_1(v_1) &= \frac{24\bar{\lambda}_1^2}{\bar{\lambda}_{m+1}^2} \\ \gamma_1(v_k)(\bar{\lambda}_{m+k-1}^2 + \bar{\lambda}_{m+k}^2) - 2\bar{\lambda}_{m+k-1}\bar{\lambda}_{m+k}\gamma_2(v_k) &= 24\bar{\lambda}_k^2 \end{aligned} \right\}. \quad (34)$$

For frames with non-displacing nodes, equations (27c) have the form

$$\left. \begin{aligned} \psi_1 &= \bar{\lambda}_{m+1}c_1^{-1/2} [4x_1\varphi_2(v_1) + 4x_2\varphi_2(v_2) + c_1x_{m+1}] - \bar{\lambda}_{m+2}c_2^{-1/2} [2x_2\varphi_3(v_2)] = 0 \\ \psi_k &= -\bar{\lambda}_{m+k-1}c_{k-1}^{-1/2} [2x_k\varphi_3(v_k)] + \bar{\lambda}_{m+k}c_k^{-1/2} [4x_k\varphi_2(v_k) + 4x_{k+1}\varphi_2(v_{k+1}) + c_kx_k] - \bar{\lambda}_{m+k+1}c_{k+1}^{-1/2} [2x_{k+1}\varphi_3(v_{k+1})] = 0 \end{aligned} \right\}. \quad (35)$$

Equations (27a) with allowance for (35) are obtained in the form

$$\left. \begin{aligned} \theta_1(v_1) &= \frac{\bar{\lambda}_1^2 c_1}{\bar{\lambda}_{m+1}^2}, \\ \left(\frac{\bar{\lambda}_{m+k-1}^2}{c_{k-1}} + \frac{\bar{\lambda}_{m+k}^2}{c_k} \right) \theta_1(v_k) - 2(c_{k-1}c_k)^{-1/2} \bar{\lambda}_{m+k-1}\bar{\lambda}_{m+k}\theta_2(v_k) &= \bar{\lambda}_k^2, \end{aligned} \right\} \quad (36)$$

where it is denoted:

$$\theta_1(v_k) = [4x\varphi_2(v_k)]' = \frac{1}{4}v_k \frac{1 - \left(\frac{v_k}{\sin v_k} \right)^2}{\operatorname{tg} \frac{v_k}{2} - \frac{v_k}{2}} + \frac{1}{2} \left(\frac{v_k}{2} \operatorname{tg} \frac{v_k}{2} \right)^2 \frac{1 - \frac{v_k}{\sin v_k}}{\left(\operatorname{tg} \frac{v_k}{2} - \frac{v_k}{2} \right)^2}, \quad (37)$$

$$\theta_2(v_k) = [2x\varphi_3(v_k)]' = \frac{1}{4}v_k \frac{\left(\frac{v_k}{\sin v_k} \right)^2 \cos v_k - 1}{\operatorname{tg} \frac{v_k}{2} - \frac{v_k}{2}} + \frac{1}{2} \left(\frac{v_k}{2} \operatorname{tg} \frac{v_k}{2} \right)^2 \frac{\frac{v_k}{\sin v_k} - 1}{\left(\operatorname{tg} \frac{v_k}{2} - \frac{v_k}{2} \right)^2}. \quad (38)$$

We introduce the notation:

$$\theta_3(v) = [4x\varphi_2(v) - 2x\varphi_3(v)]' = \frac{v/2}{\operatorname{tg} \frac{v}{2}} + \left(\frac{v/2}{\operatorname{tg} \frac{v}{2}} \right)^2 + \left(\frac{v}{2} \right)^2. \quad (39)$$

For frames of a regular structure with the same values of the structural flexibility of the crossbars, equations (34) and (36) take the form

$$\gamma_3(v) = \frac{6\bar{\lambda}_k^2}{\bar{\lambda}_{m+k}^2}, \quad (34a)$$

$$\theta_3(v) = \frac{\bar{\lambda}_k^2 c_k}{2\bar{\lambda}_{m+k}^2}. \quad (36a)$$

From equations (34) and (36), according to the known values of flexibilities, the linear stiffnesses of the racks $x_1, x_2, \dots, x_m$ are easily determined, and then the values of the running stiffnesses of the frame crossbars from equations (27c).

To select the optimal stiffnesses from the stability condition, it is necessary to determine the critical parameters using the constructional flexibility of the elements, using the functions $\gamma(v)$ and $\theta(v)$, so then determine the linear stiffness of the racks from the equations.

$$x_i = \frac{N_i \cdot h_i}{v_i^2} \quad (40)$$

The stiffener rigidity of the crossbars is calculated from equations

$$x_{m+k} = \frac{\bar{\lambda}_{m+k-1} x_k \frac{v_k}{\sin v_k} - \bar{\lambda}_{m+k} \left(x_k \frac{v_k}{\operatorname{tg} v_k} + x_{k+1} \frac{v_{k+1}}{\operatorname{tg} v_{k+1}} \right)}{6} + \frac{\bar{\lambda}_{m+k+1} x_{k+1} \frac{v_{k+1}}{\sin v_{k+1}}}{6} \quad (41)$$

for free frames, and

$$x_{m+k} = \frac{\bar{\lambda}_{m+k-1} c_{k-1}^{-1/2} 2x_{k-1} \varphi_3(v_{k-1})}{\bar{\lambda}_{m+k} c_k^{-1/2}} - \frac{\bar{\lambda}_{m+k} c_k^{-1/2} [4x_k \varphi_2(v_k) + 4x_{k+1} \varphi_2(v_{k+1})] \left(x_k \frac{v_k}{\operatorname{tg} v_k} + x_{k+1} \frac{v_{k+1}}{\operatorname{tg} v_{k+1}} \right) + \bar{\lambda}_{m+k+1} c_{k+1}^{-1/2} 2x_{k+1} \varphi_3(v_{k+1})}{\bar{\lambda}_{m+k} c_k^{-1/2}} \quad (42)$$

for non-free frames.

For regular frames, formulas for determining the rigidity of crossbars, taking into account expressions

$$x_k \frac{v_k}{\operatorname{tg} v_k} - x_k \frac{v_k}{\sin v_k} = -x_k v_k \operatorname{tg} \left(\frac{v_k}{2} \right), \quad (43)$$

$$4x_k \varphi_2(v_k) - 2x_k \varphi_3(v_k) = x_k \frac{v_k}{2} \operatorname{tg} \left(\frac{v_k}{2} \right), \quad (44)$$

it is more convenient to represent

$$\left. \begin{aligned} x_{m+1} &= -\frac{1}{6} x_1 \frac{v_1}{\operatorname{tg} v_1} + \frac{1}{6} x_2 v_2 \operatorname{tg} \left(\frac{v_2}{2} \right) \\ x_{m+k} &= -\frac{1}{6} (x_k + x_{k+1}) v \operatorname{tg} \left(\frac{v}{2} \right) \end{aligned} \right\} \quad (41a)$$

and

$$\left. \begin{aligned} x_{m+1} &= -\frac{1}{c} \left[4x_1 \varphi_2(v_1) + x_2 \frac{v/2}{\operatorname{tg} \left(\frac{v}{2} \right)} \right] \\ x_{m+k} &= -\frac{1}{c} (x_k + x_{k+1}) \frac{v/2}{\operatorname{tg} \left(\frac{v}{2} \right)} \end{aligned} \right\} \quad (42a)$$

for free and non-free frames, respectively. When the stability region is convex [9, 10, 11], the solution has a global character.

4. Conclusion

Thus, solving equations have been obtained that make it possible to easily determine the optimal conditions for the stability of the rigidity of plane frames. For free frames, these are equations (34), (40) and (41); for non-free frames with non-displacing nodes, these are equations (36), (40), (42).

5. References

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