Determination of Load-bearing Capacity of Reinforced Concrete Slabs with the Aid of Computer Program PRINS

V P Agapov, R O Golovanov

Department of Applied Mechanics and Mathematics, National Research Moscow State University of Civil Engineering, 26, Yaroslavskoye Shosse, Moscow, 129337, Russia

E-mail: agapovpb@mail.ru

Abstract. The finite element method of calculation of reinforced concrete slabs strengthened with composite fabrics based on carbon fibers, implemented in the PRINS program, is considered. The method is designed for analyzing the stress-strain state of reinforced concrete structures when cracks in concrete and plastic deformations in the reinforcement arise. The calculation is carried out in increments, and at each step of loading a variable stiffness matrix is used. Its constant part represents the stiffness matrix at the beginning of the loading step, and the variable one is calculated taking into account the stress-strain state at the end of the current iteration. The variable part of the stiffness matrix, multiplied by the displacement vector found at the previous iteration, is transferred to the right side of the system of equations and is considered as an additional load. When cracks occur or when plastic strains appear, the stresses are corrected in accordance with the specified deformation diagrams. Therefore, at the end of the loading step the equilibrium conditions are checked. If necessary, the external and internal forces are balanced. When considering plastic deformations in concrete and reinforcement, the theory of plastic flow and the Huber-Mises yield criterion, modified taking into account the experimental studies of Kupfer et al, are used. An example of the reinforced concrete slab analysis with different variants of strengthening by composite and without strengthening is given. The results of the calculation are analyzed. The possibility of studying of the stress-strain state on the entire path of loading of reinforced concrete slabs up to destruction is shown.

1. Introduction

Modern construction norms and rules adopted in our country and abroad (see, for example, [1,2]) prescribe to carry out the calculations of reinforced concrete structures in a nonlinear formulation of the problem taking into account the real properties of concrete and reinforcement. The prerequisites for the successful implementation of such calculations were created by the development of computer technology, on the one hand, and numerical methods of structural mechanics, the finite element method first off all, on the other hand [3,4,5,6]. As a result of such development, nonlinear methods for structural analysis were implemented in a number of computer programs, such as NASTRAN [7], ANSYS [8], ABAQUS [9], ADINA[10], DIANA[11] and others. Common to all of these programs is the use of the step-by-step methods. The system of nonlinear algebraic equations is solved by the Newton-Raphson method in its full or modified form. The equilibrium equations at the loading step are written in the form:

$$\mathbf{K}_j^i \Delta \mathbf{u}_j^i = \mathbf{P}_j - \mathbf{F}_j^{i-1} \qquad (1)$$

where $\mathbf{K}-$ tangent stiffness matrix, $\Delta\mathbf{u}-$nodal displacements vector, $\mathbf{P}-$ vector of the externally applied nodal loads, $\mathbf{F}-$ vector of nodal point forces that are equivalent to the element stresses, j – step number, i –iteration number.

A feature of the Newton-Raphson method of solving of the equation (1) is the calculation and factorization of the tangent stiffness matrix on each iteration. In the case of large-order systems such calculations can be quite expensive.

When using the modified Newton method, the stiffness matrix is computed and factorized only once at the beginning of the step [3,4]. This simplifies calculations, but requires more iterations to achieve the specified accuracy. Therefore, to speed up convergence, different approaches are used, based on the correction of the displacement vector at the current iteration. This can be done with the help of energy relationships. Such methods are usually called quasi-Newton [12-19].

It should be noted, however, that the calculations of physically nonlinear constructions in the above mentioned programs are carried out using physical relationships based on certain experiments, and the resulting solving equations for the structure as a whole are solved by approximate methods. To increase the reliability of results, such calculations should be carried out using several programs. Therefore, designers should have in their arsenal several available calculation instruments. In connection with this, the development of alternative computational methods and corresponding programs remains to this day an urgent task.

In this paper the finite element method of calculating of reinforced concrete slabs, taking into account plastic deformations in the reinforcement and cracking in concrete is considered, the software implementation of the technique in the PRINS program is described and the example of the slab analysis is provided.

2. Methods

The analysis of physically nonlinear structures with the help of program PRINS is carried out by the finite element method in increments [20] using the equation:

$$\mathbf{K_{NL}}\Delta\mathbf{u}=\Delta\mathbf{P}, \tag{2}$$

where $\mathbf{K_{NL}}$ is the total nonlinear stiffness matrix connecting the increments of nodal forces $\Delta\mathbf{P}$ and displacements $\Delta\mathbf{u}$, related to the nodes of the finite element model.

The matrix $\mathbf{K_{NL}}$ varies continuously in the loading interval, therefore, in order to obtain an exact solution, it is necessary to go over to integration in (2):

$$\int_{\mathbf{u_0}}^{\mathbf{u_K}}\mathbf{K_{NL}}\mathbf{du}=\Delta\mathbf{P}, \tag{3}$$

where u_0 and u_K are the values of displacements at the beginning and at the end of the loading interval, respectively. However, it is practically impossible to calculate by formula (3), since there is no analytic expression for $\mathbf{K_{NL}}$, and the upper bound of the integration interval is unknown. Calculating the integral by the rule of trapezoids, we obtain

$$\frac{1}{2}\ \mathbf{K_0}+\mathbf{K_1}\ \ \Delta\mathbf{u}=\Delta\mathbf{P}, \tag{4}$$

where $\mathbf{K_0}$ and $\mathbf{K_1}$ are the stiffness matrices computed at the beginning and at the end of the loading step, respectively.

We write equation (4) in the form:

$$\mathbf{K_0}+\Delta\mathbf{K}\ \ \Delta\mathbf{u}=\Delta\mathbf{P}, \tag{5}$$

where $\Delta\mathbf{K}=\frac{1}{2}\ \mathbf{K_1}-\mathbf{K_0}\ $.

Equation (5) is solved by an iterative method:

$$\mathbf{K_0 \Delta u_i = \Delta P - \Delta K_{i-1} \Delta u_{i-1}} \tag{6}$$

where i is the iteration number.

When the convergence of the iterative process is reached, the total values of displacements and stresses are found by the formulas:

$$\mathbf{u = u_0 + \Delta u;\ \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \Delta\boldsymbol{\sigma}}. \tag{7}$$

Stress increments are found from the formula

$$\Delta\boldsymbol{\sigma} = \mathbf{C_{ep}}\Delta\boldsymbol{\varepsilon}. \tag{8}$$

where $\mathbf{C_{ep}}$ is the elasto-plastic matrix of material characteristics, defined below. This matrix does not remain constant at the loading step. Strictly speaking, the stresses must be found by the integration of expression (8), i.e.

$$\Delta\boldsymbol{\sigma} = \int_0^{\Delta\boldsymbol{\varepsilon}} \mathbf{C_{ep}}\Delta\boldsymbol{\varepsilon}. \tag{9}$$

However, when plastic deformations and cracks formation are taken into account, it is necessary to analyze the stress state at each loading step and at each iteration and to correct the stresses using the stress-strain diagrams. This requires the carrying out of the process of equilibration of the structure, therefore, the application of the approximate formula (8) is completely justified.

The stiffness matrix $\mathbf{K}$ for an individual finite element is given by the formula [3]:

$$\mathbf{K} = \int_V \mathbf{B^T C B dV}$$

where $\mathbf{B}$ is the matrix connecting the deformation components of the element with the components of the nodal displacements, $\mathbf{C}$ is the matrix connecting the stress components with the deformation components.

The vector of nodal loads of a finite element is found from the relation:

$$\mathbf{f} = -\int_V \mathbf{N^T p dV}$$

where $\mathbf{N}$ is the matrix of the form functions expressing the displacements of the interior points of the finite elements through the nodal displacements, $\mathbf{p}$ is the vector composed of the components of the distributed load.

The method for calculating the geometric matrix is well known (see, for example, [3]). When constructing a physical matrix the diagrams of deformation of concrete and reinforcement in the form shown in figure 1 are used.

It is assumed that the concrete is deformed linearly in the compressed zone until the yield point σ_T is reached, and in the tension zone - until the cracking pre-condition is reached. The type of the diagram in the tension zone is determined by the parameters σ_{cr}, α and ε_m, and in dependence on these parameters can be different.

Unloading occurs linearly with the initial modulus of elasticity in the compressed zone, and with the module $E_p = \sigma_\partial / \varepsilon_\partial$ in the tensile zone.

The stress-strain diagram of reinforcement in the tension and compressed zones are assumed to be identical.

For the two-dimensional state of stress, the graphics in figure 1 are treated as diagrams of the stresses intensity versus the deformations intensity.

In the interval from σ_T to σ_{cm} for a compressed concrete zone, the deformation law recommended by the European Commission for Concrete [21] is assumed. It has the form

$$\sigma = \frac{\frac{E_0}{E_1}\frac{\varepsilon}{\varepsilon_c} - \left(\frac{\varepsilon}{\varepsilon_c}\right)^2}{1 + \left(\frac{E_0}{E_1} - 2\right)\frac{\varepsilon}{\varepsilon_{cm}}}\sigma_{cm} \quad \text{при } |\varepsilon| < |\varepsilon_{cm}|, \tag{10}$$

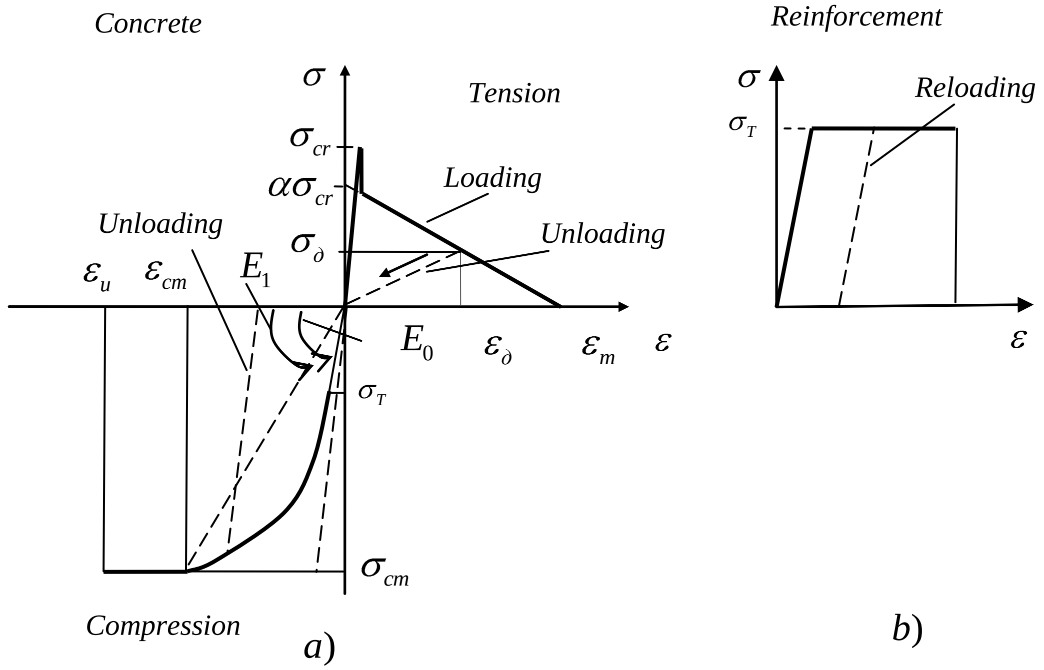

Figure 1. The stress-strain curves for the concrete (*a*) and reinforcement (*b*).

where σ and ε are the stresses and strains in the compressed zone of concrete, respectively, E_0 - the initial modulus of elasticity, E_1 - the secant modulus from the beginning to the peak value of the stresses σ_{cm}, ε_{cm} - the deformation corresponding to the peak value of the stress.

Multilayer finite elements of plates in bending, constructed using the Kirchhoff hypothesis and described in detail in [20] are used.

To obtain the relationship between the increments of stresses and deformations for the compressed and compressed - tensile zone, the theory of plastic flow is used. The yield criterion for concrete in a compressed zone is adopted in the form proposed in [22]:

$$f(I_1, J_2) = \left[\beta\ 3J_2\ + \alpha I_1\right]^{1/2} = \sigma_0, \tag{11}$$

where I_1 is the first invariant of the stress tensor, J_2 - the second invariant of the deviator of the stresses, α and β are the coefficients taken with allowance for the Kupfer et al. [23] experiments equal to $\alpha = 0.355\sigma_0$ and $\beta = 1.355$. We note that for $\alpha = 0$ and $\beta = 1$, condition (5) turns into the well-known Guber-Mises criterion [23].

The physical matrix is found from the relation (see, for example, [3], [20], [24]):

$$\mathbf{C_{ep}} = \mathbf{C} - \mathbf{Ca}\frac{\mathbf{a^T C}}{H' + \mathbf{a^T Ca}}$$

where $\mathbf{C}$ – the matrix of the coefficients of the generalized Hooke's law for a plane stress state, $\mathbf{a}$ – the flow vector, H' the tangent modulus of the stress intensity- increment of the plastic deformations intensity curve.

The flow vector is found by differentiating of the flow function along the stress components, i.e.

$$\mathbf{a} = \left\{ a_1 \; a_2 \; a_3 \right\}^T = \left\{ \frac{\partial f}{\partial \sigma_x} \quad \frac{\partial f}{\partial \sigma_y} \quad \frac{\partial f}{\partial \sigma_{xy}} \right\}^T$$

In the algorithm implemented in the PRINS program, the curve $\sigma(\varepsilon)$ is reconstructed by points to the diagram $\sigma(\varepsilon_p)$, by which the parameter H' is determined. The process of rebuilding is illustrated in figure 2. The same diagram also defines the hardening rule.

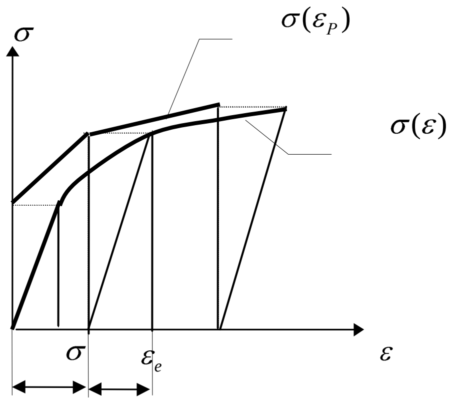

Figure 2. The construction of stress-plastic strain curve.

The relationship between stresses and strains in the tension zone is linear before the occurrence of a crack. The moment of occurrence of the crack is fixed by the main stresses. When a crack occurs, the modulus of elasticity in the direction perpendicular to the crack is assumed to be zero, and the shear moduli in the direction parallel to the crack are corrected in accordance with the recommendations given in [22]. The recommendations used take into account the aggregate interaction in the crack zone, the influence of longitudinal reinforcement and other factors affecting on the operation of the cracked concrete for shear. The normal stresses in the direction normal to the crack direction decrease abruptly to a value determined from the diagram in figure 1,a for the tension zone.

Physical equations in the event of a crack are formed first in the main axes, and then recalculated to the global axes.

The physical equations for the reinforcement and for the carbon fabric with unidirectional fibers are taken on the basis of the Prandtl diagram according to the procedure described in [20].

The use of linearized equations at the step of loading leads to a violation of the equilibrium conditions. Therefore, at the end of each loading step, the vector of nodal forces statically equivalent to the total values of internal stresses is calculated, the residual vector is found as the difference between the total vector of the external load and the static equivalent of internal stresses and the solution is corrected taking into account this discrepancy in accordance with equation (1). Static equilibration may require several iterations.

3. Results.

To illustrate the possibilities of the proposed methodology, a fragment of the reinforced concrete wall of one of the structures of the nuclear power plant in which cracks were found and which required to be strengthened was considered. Strengthening is supposed to be performed using a fabric based on carbon fibers. To convince ourselves of the reliability of the results obtained, in this work the fragment of the wall in the form of a strip that was strongly elongated in one direction and hingedly supported along short sides (figure 3) was calculated. For comparison, calculation of a plate without amplification by a composite was also performed.

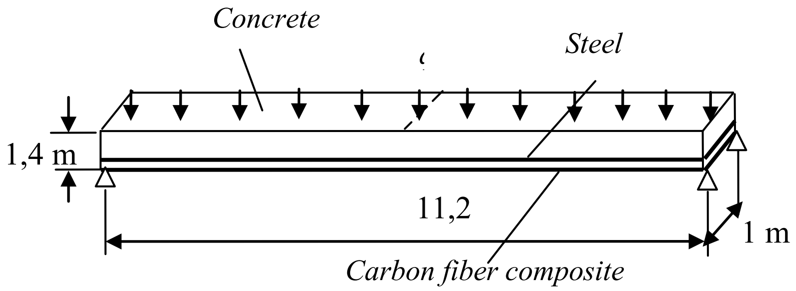

Figure 3. Fragment of reinforced concrete slab reinforced with composite fabric.

The finite element calculation scheme of the plate is shown in figure 4. A non-uniform grid of finite elements with condensation to the middle of the span was used. The grid contains three groups of elements 1, 2 and 3. The central group 3 consists of one line of elements.

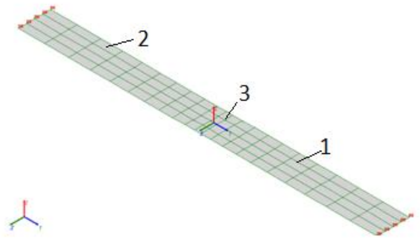

Figure 4. The finite element calculation scheme of the plate.

The partitioning of the slab into layers by thickness is shown in figure 5. Layers 1, 12 and 29 are of zero thickness. Layer 12 is basic, and layers 1 and 29 are fictitious. Fictitious layers are introduced to enable the output of the stresses on the lower and upper surfaces in the postprocessor. The thicknesses and materials of the layers are given in Table 1.

Table 1. Layer characteristics.

Layer number	Thickness, sm	Materials
2-11	7	Concrete
13-22	6.25	Concrete
23	0.21	Steel
24-25	3.5	Concrete
26-29	0.1	Composite fabric

The following materials were used: concrete of B20 class, reinforcement of A400 class and composite fabric with unidirectional fibers. The stress-strain diagram for compressed concrete under the uniaxial stressed state is shown in figure 6. The following characteristics were taken for the fabric: modulus of elasticity $E_K = 6{,}3 \times 10^7$ KPa, ultimate strength $R_K = 7 \times 10^5$ КПа, residual deformation $\varepsilon_m = 2$ %.

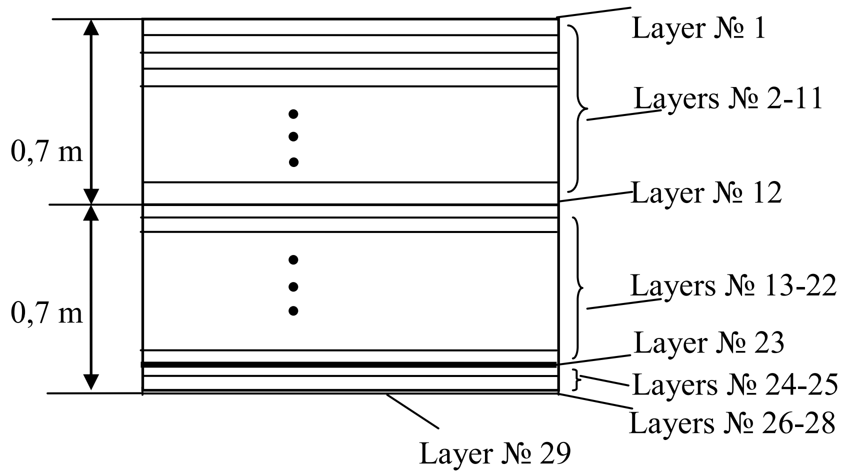

Figure 5. The partitioning of the slab into layers.

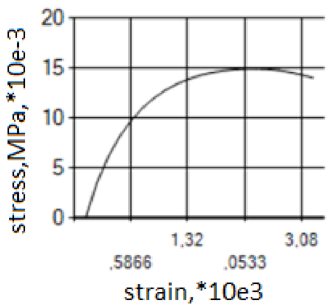

Figure 6. Stress-strain diagram for a compressed concrete zone under the uniaxial stress state.

The plate was loaded with a uniformly distributed load of intensity $q = 100$ KPa . The load factors are given in Table 2.

Table 2. Load distribution by steps.

Step numbers	1-16	17-32	33-40	41 and onwards
Load factors	0.1	0.05	0.025	0.01

Figures 7 and 8 show the equilibrium state curves for a plate without and with composite fabric, correspondingly.

Figures 9 and 10 show the values of the limiting moments for two variants of calculation. The limiting state for a plate without reinforcement by composite fabric was reached at a load $q = 53$ кПа , and for a reinforced plate – at $q = 114$ кПа . The theoretical value of the bending moments for such loads is 831 kNm / m and 1788 kNm / m, respectively.

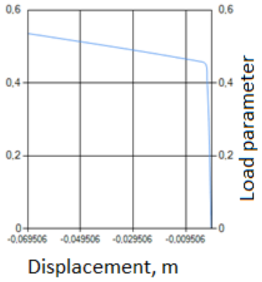

Figure 7. Equilibrium curve for the plate without composite fabric.

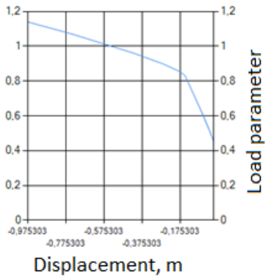

Figure 8. Equilibrium curve for the plate with composite fabric.

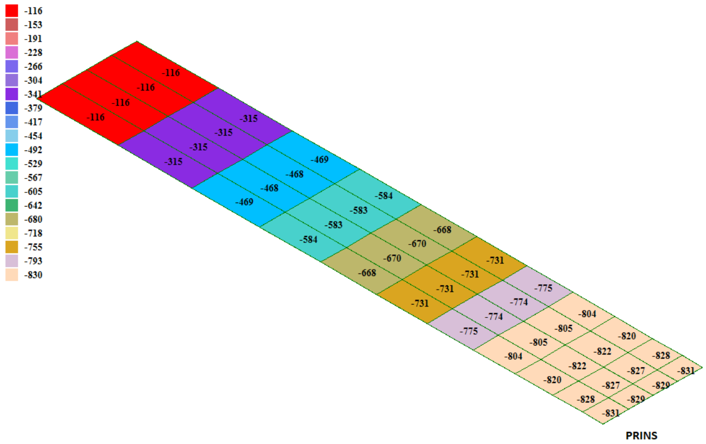

Figure 9. Limit bending moments for a plate without strengthening

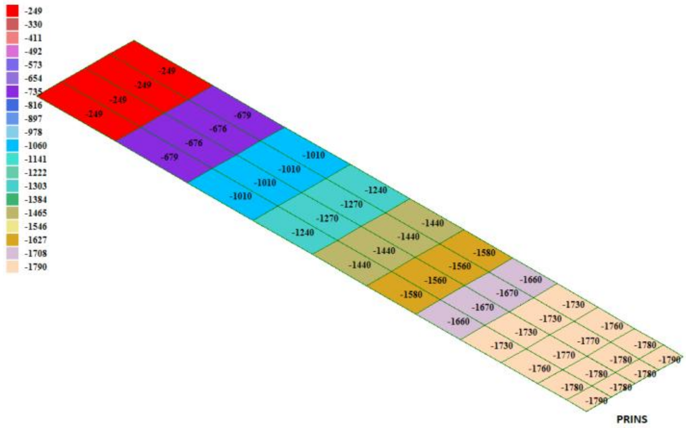

Figure 10. Limit bending moments for a plate with strengthening.

The ultimate values of the moments found by the PRINS program were 830 kNm / m and 1790 kNm / m, which is practically the same as the theoretical value.

With the adopted plate strengthening, the ultimate value of the bending moment increased by 116% compared with the original version.

Figure 11 shows the penetration depth of a crack for a plate reinforced with a composite in a state prior to fracture.

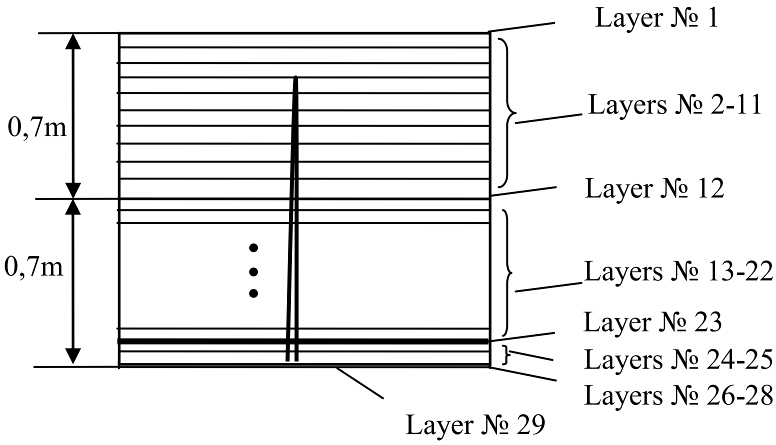

Figure 11. The depth of a crack for a plate reinforced with a composite.

When the number of composite layers increased twofold, the ultimate load reached into the value q = 221 KPa, i.e. increased by 93.8% compared with the original variant.

As a result of the calculation, the forces and stresses in the composite fabric were also determined, the processes of crack formation in concrete and the plastic deformation of the reinforcement, etc. were investigated. The size of the article does not allow us to bring the obtained results in full.

Conclusions

The investigations carried out in the present study have shown that the method of physically nonlinear calculation realized in the PRINS program gives the opportunity to analyze in detail the processes of deformation of reinforced concrete slabs with both traditional reinforcement and reinforcement with composite fabrics. Strict observance of the equilibrium conditions for the complex nature of the stressed state, noted in various ways of solving problems, attests to the reliability of the results obtained. The PRINS program is accessible to a wide range of specialists and can be useful in the calculation and design of reinforced concrete slabs.

References

[1] *Posobie po proektirovaniyu predvaritel'no napryazhennyh zhelezobetonnyh konstrukcij iz tyazhelogo betona* (k SP 52-102-2004) [Manual on the design of prestressed reinforced concrete structures from heavy concrete (to Building Regulation SP 52-102-2004)]. "Zhelezobeton" Association (TsNIIPROMZDANIY, NIIZhB). (rus)

[2] SP 63.13330.2012. *Betonnye i zhelezobetonnye konstrukcii. Osnovnye polozheniya. Aktualizirovannaya redakciya* SNiP 52-01-2003 [SP 63.13330.2012 Concrete and reinforced concrete structures. Basic provisions. Updated version of SNiP 52-01-2003]. Moscow: Ministry of Regional Development of Russia. 2012. (rus)

[3] Zienkiewicz O C, Taylor R L 2005 *The Finite Element for Solid and Structural Mechanics. Sixth edition 2005* (McGraw-Hill) p 631

[4] Bathe K J, Wilson E L 1976 *Numerical methods in finite element analysis (*N.J.: Prentice- Hall) p 528

[5] Crisfield M A 1977 *Non-linear finite element analysis of solids and structures (*John Wiley & Sons Ltd) p 488

[6] Oden J T 1972 *Finite elements in nonlinear continua (*New York: McGraw- Hill Book Company) p 464

[7] *MSC NASTRAN 2016. Nonlinear User’s Guide SOL 400* 2016 (MSC Software) p 790

[8] *ANSYS Theory Reference. Release 5.6* 1999 (Canonsburg, PAANSYS Inc)

IOP Conf. Series: Materials Science and Engineering 463 (2018) 042014 doi:10.1088/1757-899X/463/4/042014

[9] *ABAQUS 6.12. Theoretical manual* 2012 (DS Simulia)
[10] *ADINA Theory and Modeling Guide* (Elton Avenue Watertown, MA 02472 USA: ADINA R & D, Inc) p 705
[11] *DIANA FEA User's Manual. Release 10 2017* (DIANA FEA bv)
[12] Broyden C G 1967 *Math. Comp*. **19** 577
[13] Davidon W C 1975 *Math.Programming* **9** 1
[14] Fletcher R 1970 *A new approach to variable metric algorithms Ibid* **13** 317
[15] Fletcher R and Powell M J D (1963) *Comput. J*. **6** 163
[16] D. Goldfarb D 1969, Sufficient conditions for the convergence of a variable metric method *Optimization*, Fletcher R. ed. (London:Academic Press)
[17] Shanno D F 1970, Conditioning of quasi-Newton methods for function minimization *Math.Comp.* **24** 647
[18] Dennis J E, Jr. and More Jorge J (1977 Quasi-Newton Methods, Motivation and Theory *SIAM Review*, **19**, No. **1** 46
[19] Matthies H and Strang G. 1979 The Solution of Nonlinear Finite Element Equations *International Journal for Numerical Methods in Engineering* **14** 1613
[20] Agapov V.P. (2005) Metod konechnyih elementov v statike, dinamike i ustoychivosti konstruktsiy (Moscow: Izd-vo ASV) p 245
[21] *Comitee Euro-International De Beton.Ceb-Fip_Model Cod*, 1990 1993 (London: Thomas Telford House)
[22] Owen D R J, Figueiras J. and Damjanic F. 1983 *Computer Methods in Applied Mechanics and Engineering* **41** 323
[23] Kupfer H, Hilsdorf H K and Rusch H. (1969) *ACI Journal Proceedings* **66** No. **8**
[24] Chen W F (1982) *Plastisity in Reinforced Concrete* (New York: McGraw Hill) p 261