Numerical solution of structural mechanics boundary problems with the use of wavelet-based boundary element method

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Abstract. The distinctive paper is devoted to theoretical foundations of boundary problems' numerical mechanics solution with the use of wavelet-based boundary element method (BEM). Particularly the simplest boundary problem for Laplace operator is under consideration. Initial continual formulation of boundary problems, simple-layer potential basics and double-layer potential (including their properties), numerical solutions of Dirichlet and Neumann problems (so-called "boundary" systems of linear equations) are presented.

1. Initial continual formulations of boundary problems

Let Ω be the given domain with boundary $\Gamma = \partial\Omega$; $x = (x_1, x_2)$ are Cartesian coordinates; $\overline{\Omega}$ is closure region of Ω; n is the external normal to the boundary Γ; $\partial_i = \partial / \partial x_i$, $i = 1, 2$; $u = u(x_1, x_2)$ is unknown function (harmonic (in Ω) and continuous (in $\overline{\Omega}$)); $\Delta = \partial_1^2 + \partial_2^2$ is Laplacian [1].

In Dirichlet problem, it is necessary to find u with the given boundary values of u:

$$\Delta u(x) = 0, \;\; x \in \Omega; \quad u(x) = g(x), \;\; x \in \Gamma. \tag{1}$$

In Neumann problem, it is necessary to find u with the given boundary values of normal derivative:

$$\Delta u(x) = 0, \;\; x \in \Omega; \quad [\partial u / \partial n](x) = f(x), \;\; x \in \Gamma. \tag{2}$$

It is necessary to find u with the given boundary values of u (at the part Γ_1 of the border Γ) in mixed boundary and normal derivative (at the part Γ_2 of the border Γ):

$$\Delta u(x)=0,\ x\in\Omega;\quad u(x)=g(x),\ x\in\Gamma_1;\quad [\partial u/\partial n](x)=f(x),\ x\in\Gamma_2. \tag{3}$$

2. Initial continual formulations of boundary problems

Simple-layer potential $V^{(0)}(x)$ and double-layer potential $V^{(1)}(x)$ [2] are defined by formulas:

$$V^{(0)}(x)=-\frac{1}{4\pi}\ln|x|^2 *(\delta_\Gamma v)=-\frac{1}{4\pi}\int_\Gamma v(y)\ln|x-y|^2 d\Gamma_y; \tag{4}$$

$$V^{(1)}(x)=-\frac{1}{4\pi}\ln|x|^2 *\frac{\partial}{\partial n}(\delta_\Gamma w)=\frac{1}{4\pi}\int_\Gamma w(y)\frac{\partial}{\partial n_y}\ln|x-y|^2 d\Gamma_y, \tag{5}$$

where δ_Γ is the delta-function of boundary Γ; v and w are continuous functions defined at boundary Γ; n_y is the external normal to the boundary Γ at point $y\in\Gamma$.

Let us now consider some properties of the simple potentials and a double layer.

1. For each continuous function at the boundary Γ simple-layer potential $V^{(0)}(x)$ is continuous at the extension of the entire space and for $|x|\to\infty$ we have

$$V^{(0)}(x)=-\frac{1}{4\pi}\int_\Gamma v(y)d\Gamma_y\cdot\ln|x|^2+\mathrm{O}\left(\frac{1}{|x|}\right). \tag{6}$$

2. For each sufficiently smooth Γ double-layer potential $V^{(1)}(x)$ has correct normal derivative. It is defined from the outside and from the inside by formulas

$$\left(\frac{\partial V^0}{\partial n}(x)\right)_+=-\frac{v(x)}{2}+\frac{\partial V^0}{\partial n}(x),\ x\in\Gamma;\quad \left(\frac{\partial V^0}{\partial n}(x)\right)_-=\frac{v(x)}{2}+\frac{\partial V^0}{\partial n}(x),\ x\in\Gamma; \tag{7}$$

$$\left(\frac{\partial V^0}{\partial n}(x)\right)_+-\left(\frac{\partial V^0}{\partial n}(x)\right)_-=v(x),\ x\in\Gamma. \tag{8}$$

3. For each continuous function at the boundary Γ double-layer potential $V^{(1)}(x)$ ispotential function outward the boundary Γ and for $|x|\to\infty$ we have

$$V^{(1)}(x)=\mathrm{O}(1/|x|). \tag{9}$$

4. Double-layer potential $V^{(1)}(x)$ is continuous function at the domains $\overline{\Omega}$, $\overline{\Omega}_1=\overline{R_2\setminus\Omega}$ and boundary Γ. Limit values from outside and inside are defined by formulas

$$(V^{(1)}(x))_+=0.5\cdot w(x)+V^{(1)}(x),\ x\in\Gamma;\quad (V^{(1)}(x))_-=-0.5\cdot w(x)+V^{(1)}(x),\ x\in\Gamma. \tag{10}$$

The double layer potential has a discontinuity of the first kind in passing through the boundary Γ

$$(V^{(1)}(x))_+-(V^{(1)}(x))_-=w(x),\ x\in\Gamma. \tag{11}$$

5. For each sufficiently smooth Γ double-layer potential $V^{(1)}(x)$ with unit density ($w(x)\equiv 1$) takes the following values:

$$V^{(1)}(x)=-\frac{1}{4\pi}\ln|x|^2 *\frac{\partial}{\partial n}\delta_\Gamma=\frac{1}{4\pi}\int_\Gamma\frac{\partial}{\partial n_y}\ln|x-y|^2 d\Gamma_y=\begin{cases}1,\ x\in\Omega\\ -0.5,\ x\in\Gamma.\\ 0,\ x\in\Omega_1.\end{cases} \tag{12}$$

3. Numerical solution of Dirichlet problem in unit basis

Let Γ be piecewise-linear boundary, which is divided into rectilinear interval (y_j,y_{j+1}), $j=1,...,m$ (thus we have mesh of boundary elements). Let $\rho_j=\rho(y_j^*)$, where y_j^* is the midpoint of the j - thinterval (boundary element). Then we have

$$\int_{\Gamma} \rho(y) R(x-y) d\Gamma_y = \sum_{j}^{m} P_j(x)\rho_j \;; \quad P_j(x) = \int_{\Gamma} R(x-y) d\Gamma_y \,. \tag{13}$$

We can seek a solution in the potential form of the a simple layer with an unknown density function v on the boundary (4):

$$u(x) = -\frac{1}{4\pi}\int_{\Gamma} v(y) \ln|x-y|^2 d\Gamma_y = -\frac{1}{4\pi}\sum_{j=1}^{m} P_j(x) v_j \;; \quad P_j(x) = \int_{y_j}^{y_{j+1}} \ln|x-y|^2 \, d\Gamma_y \,. \tag{14}$$

It can be shown that

$$P_j(x) = R_j(x, y_{j+1}) - R_j(x, y_j) \,; \quad R_j(x,y) = -2\left[(t_j, r)(\ln|r|-1) + (n_j, r) arctg \frac{(t_j, r)}{(n_j, r)}\right],$$

where $n_j = [n_{1j} \;\; n_{2j}]^T$ is the unit vector of the outward normal to the j-th boundary element; $t_j = [t_{1j} \;\; t_{2j}]^T$ is the unit vector of the tangent to the j-th boundary element; $r = x - y$.

Directly from the boundary condition of the Dirichlet problem, taking into account (14), a linear algebraic equations system of the following form is constructed

$$-\frac{1}{4\pi}\sum_{j=1}^{m} P_j(x) v_j = g(x), \;\; x \in \Gamma \,.$$

This condition should be considered at the midpoints of boundary elements, which leads to a linear algebraic equations system of order *m* relating to unknowns, v_i, $i = 1, ..., m$ i.e.

$$\sum_{j=1}^{m} a_{i,j} v_j = g_i, \;\; i = 1, ..., m \,; \quad a_{i,j} = -0.5 \cdot h_i P_j(x_i^*) \,; \quad g_i = 2\pi h_i g(x_i^*) \,; \quad h_i = |x_{i+1} - x_i| \,. \tag{15}$$

It can be shown that system (15) has a symmetric matrix, but it is not well-conditioned

$$a_{i,j} = a_{j,i} = -\frac{1}{2} h_i h_j \ln|x_i^* - y_j|^2, \;\; i \neq j \,; \quad a_{i,i} = -h_i^2\left(\ln\frac{h_i}{2} - 1\right). \tag{16}$$

We can also find a solution in the potential form of a double layer with an unknown density function w on the boundary (4):

$$u(x) = \frac{1}{4\pi}\int_{\Gamma} w(y) \frac{\partial}{\partial n_y} \ln|x-y|^2 d\Gamma_y = -\frac{1}{4\pi}\sum_{j=1}^{m} B_j(x) w_j \;; \; B_j(x) = \int_{y_j}^{y_{j+1}} \frac{\partial}{\partial n_j} \ln|x-y|^2 \, d\Gamma_y \,. \tag{17}$$

It can be shown that

$$B_j(x) = T_j(x, y_{j+1}) - T_j(x, y_j) \,; \quad T_j(x,y) = 2 arctg \frac{(t_j, r)}{(n_j, r)} \,. \tag{18}$$

Limit values (for $x \to \Gamma$) of $T_j(x,y)$ from the outside and from the inside of domain Ω are

$$2\left(arctg \frac{(t_j, r)}{(n_j, r)}\right)_- = -2 \cdot sign(t_j, r)\frac{\pi}{2} \,; \quad 2\left(arctg \frac{(t_j, r)}{(n_j, r)}\right)_+ = 2 \cdot sign(t_j, r)\frac{\pi}{2} \,. \tag{19}$$

Thus boundary condition of Dirichlet problem $u(x) = g(x)$, $x \in \Gamma$ goes into condition of $(V^{(1)}(x))_- = g(x)$, $x \in \Gamma$. We have the following system of linear algebraic equations

$$\frac{1}{4\pi}\sum_{j=1}^{m} B_j(x) w_j = g(x), \quad x \in \Gamma \,. \tag{20}$$

This condition should be considered at the midpoints of boundary elements, which leads to a linear algebraic equations system of order m relating to unknowns, w_i, $i = 1, ..., m$ i.e.

$$\sum_{j=1}^{m} a_{i,j} w_j = g_i, \;\; i=1,...,m; \;\; a_{i,j} = \frac{h_i}{2} B_j(x_i^*); \;\; g_i = 2\pi h_i g(x_i^*); \;\; h_i = | x_{i+1} - x_i |, \quad (21)$$

where it can be shown that

$$a_{i,j} = -h_i h_j \frac{(n_j, (x_i^* - y_j^*))}{| x_i^* - y_j^* |^2}, \;\; i \neq j; \;\; a_{i,i} = \pi h_i. \quad (22)$$

Therefore $a_{i,i} > | a_{i,j} |, \;\; j \neq i$. Generally matrix of the coefficients of the boundary equations system (22) is not symmetric and well-conditioned.

4. Numerical solution of Neumann problem in unit basis

We can find a solution in the potential form of a simple layer with an unknown density function v on the boundary (14). Boundary condition has the form

$$\frac{\partial u}{\partial n}(x) = f(x), \;\; x \in \Gamma \quad \text{or} \quad \left(\frac{\partial V^{(0)}}{\partial n}(x) \right)_- = f(x). \quad (23)$$

This condition should be considered at the midpoints of boundary elements, which leads to a system of linear algebraic equations of order m relating to unknowns, $v_i, \;\; i=1,...,m$ i.e.

$$\sum_{j=1}^{m} a_{ij} v_j = f_i, \;\; i=1,...,m; \;\; a_{ij} = -\frac{h_i}{2} \frac{\partial P_j(x_i^*)}{\partial n_i}; \;\; g_i = 2\pi h_i f(x_i^*); \;\; h_i = | x_{i+1} - x_i |, \quad (24)$$

where it can be shown that

$$a_{i,j} = h_i h_j \frac{(n_j, y_j^* - x_i^*)}{| x_i^* - y_j^* |^2} = -a_{j,i}, \;\; i \neq j; \;\; a_{i,i} = \pi h_i. \quad (25)$$

We can also seek a solution in the potential form of a double layer with an unknown density function w on the boundary (17). Boundary condition has the form (23). This condition should be considered at the midpoints of boundary elements, which leads to a linear algebraic equations system of order m relating to unknowns, $w_i, \;\; i=1,...,m$ i.e.

$$\sum_{j=1}^{m} a_{i,j} w_j = g_i, \;\; i=1,...,m; \;\; a_{i,j} = \frac{h_i}{2} \frac{\partial B_j(x_i^*)}{\partial n_i}; \;\; g_i = 2\pi h_i f(x_i^*); \;\; h_i = | x_{i+1} - x_i |, \quad (26)$$

where it can be shown thatwe have a system of equations with a symmetric matrix,

$$a_{ij} = -\frac{h_i h_j}{2} \frac{\partial^2}{\partial t_i \partial t_j} \ln | x_i^* - y_j^* |^2; \;\; | a_{ij} | \leq \frac{h_i h_j}{2} \frac{1}{(r_j^i)^2}. \quad (27)$$

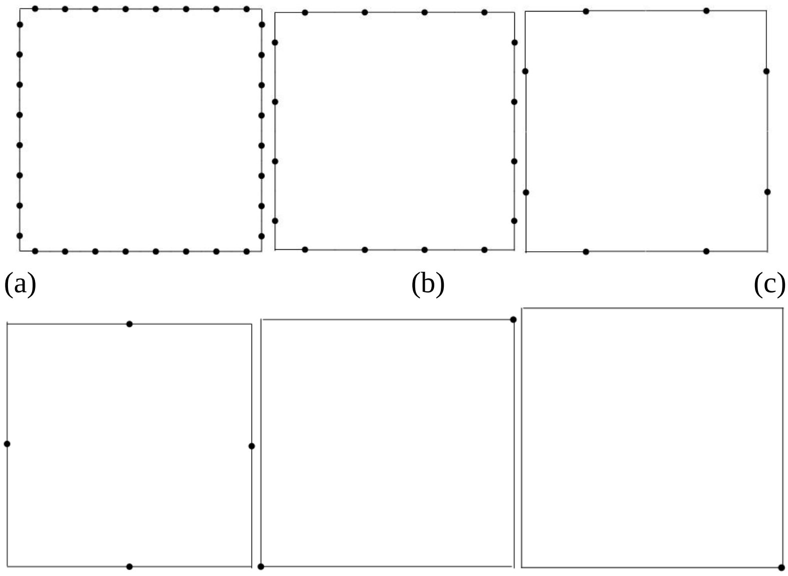
5. Proceeding to wavelet basis (Haar basis)

The construction of the discrete Haar basis, as well as the construction of matrix averaging operators (reduction), is described in [3-5].

Let us consider for instance quadratic domain $\Omega = \{ (x_1, x_2): \; -\ell_1 < x_1 < \ell_1 \wedge -\ell_2 < x_2 < \ell_2 \}$ and divide each side of domain into eight elements. The total number of unknowns $N = 32 = 2^5$, therefore, the maximum number of levels in the Haar basis $M = 5$. Locations of the nodes of each level of the Haar basis are presented in Figure 1. Let us consider the following problem:

$$\Delta u = F(x_1, x_2), \;\; (x_1, x_2) \in \Omega; \;\; u(x_1, x_2) = 0, \;\; (x_1, x_2) \in \Gamma,$$

where $F(x_1, x_2) = P\delta(x - x_P)$; $x_P \in \Omega$ is the given coordinate; $\delta(x)$ is the Dirac delta function [6].

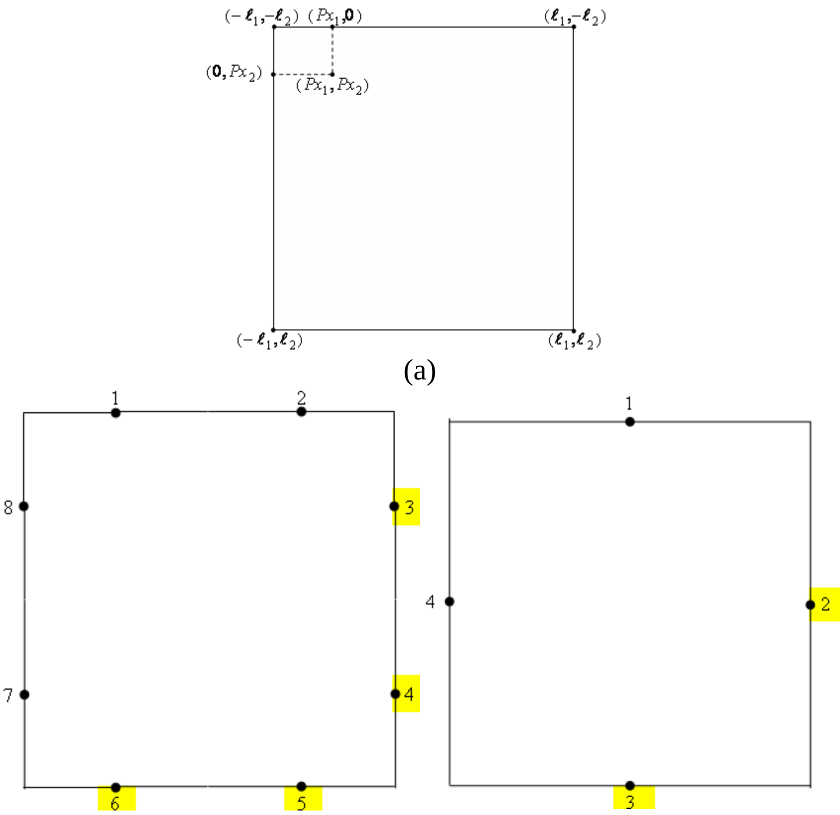


(a) (b) (c)

(d) (e) (f)

Figure 1. Locations of nodes at Haar levels basis: (a) zero level; (b) the first level; (c) the second level; (d) the third level; (e) the fourth level; (f) the fifth level.

An example of the reduction choice for a given problem is shown in Figure 2, where the nodes of the second and third levels are shown, with backgrounds for the averaging nodes being selected.



(a)

(b) (c)

Figure 2. Sample of reduction: (a) computational scheme; (b) the second level; (c) the third level.

6. Conclusion

The proposed wavelet-based boundary element method can be effectively used, when it is necessary to find only the most accurate solution in some pre-known domains. Generally the choice of these domains is a priori data with respect to the structure being modelled. Designers usually choose domains with the so-called edge effect and regions which are subject to specific operational requirements.

It is obvious that the stress-strain state in such domains is of paramount importance. Specified factors along with the obvious needs of the designer or researcher to reduce computational costs cause considerable urgency of constructing of special algorithms for obtaining local solutions of boundary problems. Wavelet analysis provides effective and popular tool for such researches.

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