About the solution of a structural class optimization problems. Part 1: Formulation of theoretical foundations problems of the solution procedure

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Abstract. Earlier, the criterion of minimum material consumption was formulated within the outline design of the I-shaped bar width and the stability constraints or restriction to the value of the first natural frequency in one principal plane of the cross-section inertia. In the distinctive paper, we formulate a criterion for the minimum material capacity of the I-shaped bar with a variation in its thickness and outline of the width, with restrictions on the value of the critical force or restriction to the value of the first natural frequency in two principal planes of the section inertia.

1. Introduction

I-shaped bars are under consideration. The adopted coordinate system is shown in Figures 1 and 2.

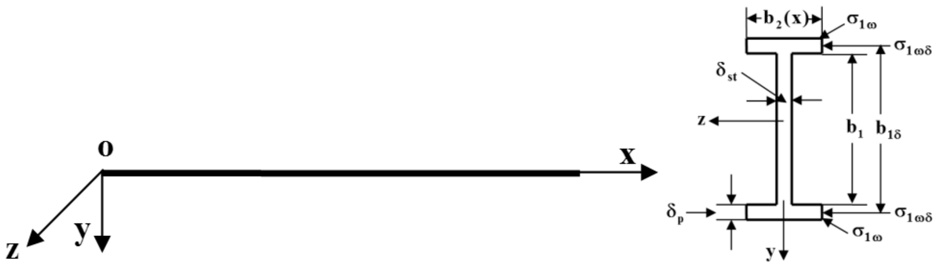

Figure 1. Adopted coordinate system.

The criterion of the minimum material consumption within the design of the width outline of the I-shapedbars flange and the stability constraints or restriction to the value of the first natural frequency in one principal plane of the cross-section $x-0-y$ inertia is presented in [1]. In some cases, it becomes necessary to introduce restrictions on the value of the first critical force or natural frequency in the second principal plane of inertia $x-0-z$. In these cases, it becomes possible to vary within optimizing not only the width outline of the flange, but also its thickness.

2. Formulation of problem

The function of the flange width ($b_2(x)$) and the thickness (δ_p) of the flange are varied. The web height (b_1) and web thickness (δ_{st}) are not varied. The values of the first critical force or the natural frequency in two principal planes of inertia are limited. The influence of the longitudinal (axial) force on the value of the natural frequency and the possibility influence of a given frequency vibrations on the value of the critical force are taken into account. Target functional has the form:

$$V_0 = 2\int_0^l b_2(x)\delta_p dx, \tag{1}$$

where V_0 is flange material volume; $2b_2(x)\delta_p$ is cross-sectional area of flanges.

Let us derive, under the conditions set, the criterion of the flange minimum material consumption, with restrictions to the value of the first natural frequency with allowance for influence of longitudinal force. The criterion formulated in this way can also be used for stability constraints if we define zero value of the natural frequency in it.

Restrictions to the value of the first natural frequency have the form

$$\omega_0 \le \omega 1[1]; \tag{2}$$

$$\omega_0 \le \omega 2[1], \tag{3}$$

where ω_0 is the number, which limits the value of the lowest natural frequency; $\omega 1[1]$ is the lowest natural frequency in the plane $x-0-y$; $\omega 2[1]$ is the lowest natural frequency in the plane $x-0-z$.

If we take the constraints in the form

$$\omega_0 = \omega 1[1] = \omega 2[1]; \tag{4}$$

then we have

$$\Phi_{\omega 1} = \frac{1}{2}\int_0^l \{EI_1(x)(v''_\omega)^2 - P(x)(v'_\omega)^2 - (\omega_O)^2[m(x) + \rho F(x)v_\omega^2]\}dx = 0; \tag{5}$$

$$\Phi_{\omega 2} = \frac{1}{2}\int_0^l \{EI_2(x)(w''_\omega)^2 - P(x)(w'_\omega)^2 - (\omega_O)^2[m(x) + \rho F(x)w_\omega^2]\}dx = 0, \tag{6}$$

where $\Phi_{\omega 1}$ and $\Phi_{\omega 2}$ are energy functional of natural oscillations in the principal planes of inertia; $I_1(x)$ and $I_2(x)$ are the corresponding moments of the cross-section inertia; E and ρ are respectively, the modulus of elasticity and the specific mass of the bar material; $P(x)$ is longitudinal force; $m(x)$ is external mass intensity; $F(x)$ is cross-sectional area of the bar; v_ω and w_ω are ordinates of the forms of natural oscillations, respectively, in the principal planes of inertia $x-0-y$ and $x-0-z$.

Thus, it is required to find such a function of variation the flange width $b_2(x)$ and such a value of the flange thickness δ_p, which provide minimum of (1) flange volume functional and fulfilment of the conditions (5) and (6).

3. Theoretical foundation of solution procedure

A functional which extremum ensures the minimum of the functional (1) and the fulfilment of conditions (5) and (6) can be written in the form

$$V_{0\omega}=\int_0^l\{[2b_2(x)\delta_p]-\lambda_1[EI_1(x)(v''_\omega)^2-P(x)(v'_\omega)^2-(\omega_O)^2[m(x)+\rho F(x)v_\omega^2]]-\\-\lambda_2[EI_2(x)(w''_\omega)^2-P(x)(w'_\omega)^2-(\omega_O)^2[m(x)+\rho F(x)w_\omega^2]]\}dx, \quad (7)$$

where λ_1 and λ_2 are undetermined coefficients.

Taking into account the relations (5) and (6), considering problem is isoperimetric; λ_1 and λ_2 are constants. It is obvious that variations of the functional $V_{0\omega}$ with respect to v and w lead to the equations of natural oscillations in the principal planes of inertia. Variations of the functional $V_{0\omega}$ with respect to λ_1 and λ_2 lead to fulfilment of the conditions (5) and (6).

The moments of the cross-section inertia can be written in the form

$$I_1(x)=\frac{b_2(x)}{12}(b_1+2\delta_p)^3-\frac{b_2(x)-\delta_{st}}{12}(b_1)^3=\frac{1}{12}[(b_2(x)(b_1+2\delta_p)^3-(b_2(x)-\delta_{st)})b_1^3]; \quad (8)$$

$$I_2(x)=\frac{2[b_2(x)]^3\delta_p+[b_{st}]^3b_1}{12}. \quad (9)$$

The functional (7) takes the form

$$V_{0\omega}=\int_0^1\{[2b_2(x)\delta_p]-\lambda_1[\frac{E}{12}[(b_2(x)(b_1+2\delta_p)^3-(b_2(x)-\delta_{st})b_1^3](v''_\omega)^2-P(x)(v'_\omega)^2-\\-(\omega_0)^2[m(x)+\rho(2b_2(x)\delta_p+b_1\delta_{st})]v_\omega^2]-\lambda_2[E\frac{2[b_2(x)]^3\delta_p+[b_{st}]^3b_1}{12}(w''_\omega)^2-\\-P(x)(w'_\omega)^2-(\omega_0)^2[m(x)+\rho(2b_2(x)\delta_p+b_1\delta_{st})]w_\omega^2]\}dx. \quad (10)$$

The extremum of the functional (10) is determined by solution of the system of equations [2]

$$\frac{\partial V_{0\omega}}{\partial\delta_p}=0,\quad \delta(V_{0\omega})_{b_2(x)}=0 \quad (11)$$

or in expanded form

$$\frac{\partial V_{0\omega}}{\partial\delta_p}=\int_0^1\{2b_2(x)-\lambda_1\{[\frac{E}{2}(b_2(x)(b_1+2\delta_p)^2)](v''_\omega)^2-2(\omega_0)^2\rho b_2(x)v_\omega^2\}-\\-\lambda_2[\frac{E}{6}b_2(x)^3(w''_\omega)^2-2(\omega_0)^2\rho b_2(x)w_\omega^2]\}\mathrm{d}x=0; \quad (12)$$

$$\delta(V_{0\omega})_{b_2(x)}=2\delta_p-\lambda_1\{\frac{E}{12}[b_1+2\delta_p)^3-b_1^3](v''_\omega)^2-2(\omega_0)^2\rho\delta_p v_\omega^2\}-\\-\lambda_2[E\frac{b_2^2(x)\delta_p}{2}(w''_\omega)^2-2(\omega_0)^2\rho\delta_p w_\omega^2=0. \quad (13)$$

We transform some expressions from equations (12) and (13). We consider the following expression from the second equation the expression:

$$\frac{E}{12}[b_1+2\delta_p)^3-b_1^3](v''_\omega)^2=\frac{E^2(b_1^3+6b_1^2\delta_p+12b_1\delta_p^2+8\delta_p^3-b_1^3)}{12E}(v''_\omega)^2=\\=\frac{2E^2\delta_p(3b_1^2+6b_1\delta_p+4\delta_p^2)}{12E}(v''_\omega)^2=\frac{E^2\delta_p[(b_1^2+4b_1\delta_p+4\delta_p^2)+2b_1^2+2b_1\delta_p)}{6E}(v''_\omega)^2=\\=\frac{E^2\delta_p[(b_1+2\delta_p)^2+2b_1(b_1+\delta_p)]}{6E}(v''_\omega)^2. \quad (14)$$

Let us multiply the numerator and denominator of (14) by $[I_1(x)]^2$. As is known

$$EI_1(x)v''_{\omega} = M_1(x) \quad . \tag{15}$$

Is a bending moment in the plane $x-0-y$. It is obvious that

$$\frac{M_1(x)(b_1+2\delta_p)}{2I_1(x)} = \sigma_{1\omega} \tag{16}$$

is normal stress in the outer fibers of the bar cross-section. Let us introduce the notation

$$b_{1\delta} = \sqrt{b_1(b_1+\delta_p)} \ . \tag{17}$$

Then (Figure 1)

$$\frac{M_1(x)b_{1\delta}}{2I_1(x)} = \sigma_{1\omega\delta} \tag{18}$$

is normal stress in the fibers of the bar cross-section spaced from the neutral axis by a distance

$$\frac{1}{2}b_{1\delta} = \frac{1}{2}\sqrt{b_1(b_1+\delta_p)} \ . \tag{19}$$

Expression (14) can be written in the form

$$\frac{2\delta_p}{3E}\sigma^2_{1\omega}(x) + \frac{4\delta_p}{3E}\sigma^2_{1\omega\delta}(x) \, . \tag{20}$$

Analogously we have

$$E\frac{b_2^2(x)\delta_p}{2}(w''_{\omega})^2 = \frac{\delta_p E^2 b_2^2(x)}{2E}\frac{[I_2(x)]^2}{[I_2(x)]^2}(w''_{\omega})^2 \, , \tag{21}$$

where

$$EI_2(x)w''_{\omega} = M_2(x) \quad . \tag{22}$$

Is a bending moment in the plane $x-0-z$;

$$\frac{M_2(x)b_2(x)}{2I_2(x)} = \sigma_{2\omega}(x) \, . \tag{23}$$

Is normal stress in the outer fibers of the bar cross-section, caused by bending moment $M_2(x)$.

Taking into account transformations carried out, equations (12), (13) can be rewritten in the form

$$\begin{aligned}\frac{\partial V_{0\omega}}{\partial \delta_p} = \int_0^1 \{2b_2(\mathrm{x}) - \lambda_1[\frac{2}{E}(b_2(x)\sigma^2_{1\omega}(x) - 2(\omega_0)^2\rho b_2(x)v^2_{\omega}] - \\ - \lambda_2[\frac{2}{3E}b_2(x)\sigma^2_{2\omega}(x) - 2(\omega_0)^2\rho b_2(x)w^2_{\omega}]\}dx = 0;\end{aligned} \tag{24}$$

$$\begin{aligned}\delta(V_{0\omega})_{b_2(x)} = 2\delta_p - \lambda_1[\frac{2\delta_p}{3E}\sigma^2_{1\omega}(x) + \frac{4\delta_p}{3E}\sigma^2_{1\omega\delta}(x) - 2(\omega_0)^2\rho\delta_p v^2_{\omega}] - \\ - \lambda_2[\frac{2\delta_p}{E}\sigma^2_{2\omega}(x) - 2(\omega_0)^2\rho\delta_p w^2_{\omega}] = 0.\end{aligned} \tag{25}$$

We multiply all the terms of the equations (24) and (25) by E, the terms of the first equation are also multiplied by δ_p and terms of the second equation are also multiplied by $b_2(x)$. Let us denote the cross-section area of the flange

$$b_2(x)\delta_p = F_p(x) \tag{26}$$

and integrate the second equation in the range from 0 to 1. Then we have

$$\frac{\partial V_{0\omega}}{\partial \delta_p} = \int_0^1 \{2EF_p(x) - \lambda_1[2\sigma_{1\omega}^2(x)F_p(x) - 2(\omega_0)^2 \rho F_p(x)Ev_\omega^2] - \\ - \lambda_2[\frac{2}{3}\sigma_{2\omega}^2(x)F_p(x) - 2(\omega_0)^2 \rho F_p(x)Ew_\omega^2]\}dx = 0; \tag{27}$$

$$\delta(V_{0\omega})_{b_2(x)} = \int_0^1 \{\ 2EF_p(x) - \lambda_1[\frac{2}{3}\sigma_{1\omega}^2(x)F_p(x) + \frac{4}{3}\sigma_{1\omega\delta}^2(x)F_p(x) - \\ - 2(\omega_0)^2 \rho F_p(x)Ev_\omega^2] - \lambda_2\left[2\sigma_{2\omega}^2(x)F_p(x) - 2(\omega_0)^2 \rho F_p(x)Ew_\omega^2\right]\ \}dx = 0. \tag{28}$$

The difference between the equations (27) and (28) is defined by formula

$$\int_0^1 \left\{ \lambda_1\left[\frac{4}{3}(\sigma_{1\omega}^2(x) - \sigma_{1\omega\delta}^2(x))F_p(x)\right] - \lambda_2\left[\frac{4}{3}\sigma_{2\omega}^2(x)]F_p(x)\right]\ \right\}dx = 0\,. \tag{29}$$

From (29) it follows that

$$\lambda_1\int_0^1\left[\frac{4}{3}F_p(x)(\sigma_{1\omega}^2(x) - \sigma_{1\omega\delta}^2(x)\right]dx = \lambda_2\int_0^l\left[\frac{4}{3}F_p(x)\sigma_{2\omega}^2(x)\right]dx\,; \tag{30}$$

$$\lambda_2 = \lambda_1\frac{\int_0^1 \frac{4}{3}F_p(x)(\sigma_{1\omega}^2(x) - \sigma_{1\omega\delta}^2(x))dx}{\int_0^l\left[\frac{4}{3}F_p(x)\sigma_{2\omega}^2(x)\right]dx} = \mu\lambda_1\,; \quad \mu = \frac{\int_0^1 \frac{4}{3}F_p(x)(\sigma_{1\omega}^2(x) - \sigma_{1\omega\delta}^2(x))dx}{\int_0^l\left[\frac{4}{3}F_p(x)\sigma_{2\omega}^2(x)\right]dx}\,. \tag{31}$$

Equation (25) can be rewritten in the form

$$2\delta_p - \lambda_1\left[\frac{2\delta_p}{3E}\sigma_{1\omega}^2(x) + \frac{4\delta_p}{3E}\sigma_{1\omega\delta}^2(x) - 2(\omega_0)^2\rho\delta_p v_\omega^2\right] - \\ - \mu\lambda_1\left[\frac{2\delta_p}{E}\sigma_{2\omega}^2(x) - 2(\omega_0)^2\rho\delta_p w_\omega^2\right] = 0. \tag{32}$$

Multiplying the terms of this equation by E and dividing by $2\delta_p$ we get

$$E - \lambda_1\left[\frac{1}{3}\sigma_{1\omega}^2(x) + \frac{2}{3}\sigma_{1\omega\delta}^2(x) - (\omega_0)^2\rho E v_\omega^2\right] - \mu\lambda_1\left[\sigma_{2\omega}^2(x) - (\omega_0)^2\rho E w_\omega^2\right] = 0\,. \tag{33}$$

Hence we obtain

$$\left[\frac{1}{3}\sigma_{1\omega}^2(x) + \frac{2}{3}\sigma_{1\omega\delta}^2(x) - (\omega_0)^2\rho E v_\omega^2\right] + \mu\left[\sigma_{2\omega}^2(x) - (\omega_0)^2\rho E w_\omega^2\right] = \frac{E}{\lambda_{1\omega}} = const \tag{34}$$

or

$$\overline{\sigma}_{1\omega}(x) = \sqrt{\left[\frac{1}{3}\sigma_{1\omega}^2(x) + \frac{2}{3}\sigma_{1\omega\delta}^2(x) - (\omega_0)^2\rho E v_\omega^2\right] + \mu\left[\sigma_{2\omega}^2(x) - (\omega_0)^2\rho E w_\omega^2\right]} = const\,. \tag{35}$$

Thus, it is shown that the criterion of the minimum material consumption of the *I*-shaped bar, when the shape and thickness of the flange vary, and the height and thickness of the web do not vary with the stability constraints or restriction to the value of the first natural frequency, will be the constancy along the length of the bar of the reduced stresses $\overline{\sigma}_{1\omega t}(x)$ arising in the corresponding natural mode under natural oscillations or loss of stability.

If the value of the first natural frequency is limited and longitudinal force exists, criterion (35) doesn't take it into account. Under the action of only stability constraints $\omega_0 = 0$ is substituted in (35).

The reduced stresses are normalized so that the largest value of each of $\overline{\sigma}_{1\omega}(x)$ along the length of the bar is equal to unity. Then the closeness of the solution obtained to the minimum material-intensive solution is estimated by the closeness of the value $\overline{\sigma}_{1\omega}(x)$ to unity along the entire length of the bar.

The criterion obtained in this paper, as well as those obtained earlier (presented, for instance, in [3 - 6]), can also be used for problems solution of structures optimal reinforcement and generally for corresponding problems solution of structural analysis [7-10].

References

[1] Chudnovsky V G 1952 *Method for Calculating the Oscillations and Stability of Rod Systems* (Kiev: Publishing House of the Academy of Sciences of USSR)
[2] Lyakhovich L S 2017 *Int. J. Comp. Civil Struct. Eng.* **13** Issue 2 103–24
[3] Lyakhovich L S 2009 *Special Properties of Optimal Systems and the Main Directions of their Implementation in Methods of Calculating Structures* (Tomsk: Publishing House of Tomsk State University of Architecture and Building)
[4] Lyakhovich L S and Malinovsky A P 2015 *J. Constr. Architec.* Issue 5 41–50
[5] Lyakhovich L S, Tukhfatullin B A, Puteeva L E and Grigoryev A I *J. Constr. Architec.* Issue 6 57–70
[6] Lyakhovich L S, Malinovsky A P and Tukhfatullin B A 2016 *Int. J. Comp. Civil Struct. Eng.* **12** Issue 2 118–25
[7] Lyakhovich L S and Perelmuter A V 2014 *Int. J. Comp. Civil Struct. Eng.* **10** Issue 2 14–23
[8] Lyakhovich L S, Akimov P A and Tukhfatullin B A 2017 *Int. J. Comp. Civil Struct. Eng.* **13** Issue 1 9–22
[9] Lyakhovich L S 2015 *Proc. Eng.* **111** 510–5
[10] Belostotsky A M and Akimov P A 2016 *Int. J. Comp. Civil Struct. Eng.* **12** Issue 1 8–45