

Wavelet-based two-grid numerical method of structural analysis with the use of discrete Haar basis

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Abstract. The distinctive paper is devoted to so-called two-grid method of structural analysis based on discrete Haarbasis (three-dimensional problems are under consideration). Approximations of the mesh functions in discrete Haar bases of zero and first levels are described (the mesh function is represented as the sum where one term is its approximation of the first level, and the second term is so-called complement (up to the initial state) on the grid of the first level). Special projectors are constructed for the spaces of the original grid vector functions to the space of their approximation on the first-level grid and its complement (the detailing component) to the initial state. Basic scheme of the two-grid method is presented. This method allows boundary problems solution of structural mechanics with the matrix operators' use of significantly smaller dimension. It should be noted that discrete analogue of the initial operator equation (defined on a given interval) is a system of linear algebraic equations which is constructed with the use of finite element method or finite difference method. Block Gauss method can be used for direct solution.

1. Introductory remarks

Let the initial three-dimensional domain be given as a rectangular prism [1]. Let L_1 , L_2 and L_3 be lengths of this prism sides in directions corresponding to Cartesian coordinates x_1 , x_2 and x_3 . We can use simple rectangular mesh (grid) for approximation of domain and divide each side of the rectangular prism (the initial domain) into $(N-1)$ equal parts. The corresponding mesh (grid) width are defined by formulas

$$h_1 = L_1 / (N-1); \quad h_2 = L_2 / (N-1); \quad h_3 = L_3 / (N-1). \quad (1)$$

Thus, the resulting mesh contains N^3 nodes.

Let us introduce the mesh function



$$\bar{u} = [u_{1,1,1} \quad \dots \quad u_{1,N,1} \quad \dots \quad u_{N,N,1} \quad \dots \quad u_{N,N,N}]^T. \quad (2)$$

We can represent the mesh function (1.2) in the form

$$\bar{u} = \sum_{j_3=1}^{N_0} \sum_{j_2=1}^{N_0} \sum_{j_1=1}^{N_0} u_{j_2,j_1,j_3}^0 \bar{\Phi}_{j_2,j_1,j_3}^0, \quad (3)$$

where $\bar{\Phi}_{j_2,j_1,j_3}^0(i_2, i_1, i_3)$ is the (j_2, j_1, j_3) -th vector of a unit basis or a discrete zero-level Haar basis

$$\bar{\Phi}_{j_2,j_1,j_3}^0(i_2, i_1, i_3) = \begin{cases} 1, & (i_1 = j_1) \cap (i_2 = j_2) \cap (i_3 = j_3) \\ 0, & (i_1 \neq j_1) \cup (i_2 \neq j_2) \cup (i_3 \neq j_3), \end{cases} \quad (4)$$

$$1 \leq j_2, j_1, j_3 \leq N_0 = N; \quad 1 \leq i_2, i_1, i_3 \leq N_0 = N;$$

$$u_{j_2,j_1,j_3}^0 = u_{j_2,j_1,j_3}, \quad 1 \leq j_1, j_2, j_3 \leq N; \quad (5)$$

$$N_0 = N. \quad (6)$$

The mesh function (2) can also be represented in the form of an expansion in the Haar basis [2-18] of the first level:

$$\bar{u} = \sum_{j_3=1}^{N_1} \sum_{j_2=1}^{N_1} \sum_{j_1=1}^{N_1} u_{j_2,j_1,j_3}^1 \bar{\Phi}_{j_2,j_1,j_3}^1 + \sum_{k=1}^7 \sum_{j_3=1}^{N_1} \sum_{j_2=1}^{N_1} \sum_{j_1=1}^{N_1} v_{k,j_2,j_1,j_3}^1 \bar{\Psi}_{k,j_2,j_1,j_3}^1, \quad (7)$$

where $\bar{\Phi}_{j_2,j_1,j_3}^1$ is (j_2, j_1, j_3) -th approximating vector of the discrete Haar basis of the first level,

$$\begin{aligned} \bar{\Phi}_{j_2,j_1,j_3}^1 = & \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 + \\ & + \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \end{aligned} \quad (8)$$

$\bar{\Psi}_{k,j_2,j_1,j_3}^1$ - (k, j_2, j_1, j_3) -th refining vector of the discrete Haar basis of the first level,

$$\begin{aligned} \bar{\Psi}_{1,j_2,j_1,j_3}^1 = & \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 + \\ & + \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \end{aligned} \quad (9)$$

$$\begin{aligned} \bar{\Psi}_{2,j_2,j_1,j_3}^1 = & \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 + \\ & + \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \end{aligned} \quad (10)$$

$$\begin{aligned} \bar{\Psi}_{3,j_2,j_1,j_3}^1 = & \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 + \\ & + \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \end{aligned} \quad (11)$$

$$\begin{aligned} \bar{\Psi}_{4,j_2,j_1,j_3}^1 = & \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 - \\ & - \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \end{aligned} \quad (12)$$

$$\begin{aligned} \bar{\Psi}_{5,j_2,j_1,j_3}^1 = & \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 - \\ & - \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \end{aligned} \quad (13)$$

$$\bar{\Psi}_{6,j_2,j_1,j_3}^1 = \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 - \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \quad (14)$$

$$\bar{\Psi}_{7,j_2,j_1,j_3}^1 = \alpha(\bar{\Phi}_{2j_2-1,2j_1-1,2j_3-1}^0 - \bar{\Phi}_{2j_2-1,2j_1,2j_3-1}^0 - \bar{\Phi}_{2j_2,2j_1-1,2j_3-1}^0 + \bar{\Phi}_{2j_2,2j_1,2j_3-1}^0 - \bar{\Phi}_{2j_2-1,2j_1-1,2j_3}^0 + \bar{\Phi}_{2j_2-1,2j_1,2j_3}^0 + \bar{\Phi}_{2j_2,2j_1-1,2j_3}^0 - \bar{\Phi}_{2j_2,2j_1,2j_3}^0); \quad (15)$$

$$u_{j_2,j_1,j_3}^1 = (\bar{u}, \bar{\Phi}_{j_2,j_1,j_3}^1); \quad v_{k,j_2,j_1,j_3}^1 = (\bar{u}, \bar{\Psi}_{k,j_2,j_1,j_3}^1), \quad k=1, \dots, 7; \quad (16)$$

$$\alpha = 1/(2\sqrt{2}); \quad N_1 = N/2; \quad (17)$$

α is normalizing factor.

In accordance with (1.7), the mesh function \bar{u} is represented as a sum where the first summand (the first sum) is its approximation on the grid of the first level including N_1^3 nodes, and the second term (the second sum) is called the refinement (the complement to the initial state) on the grid the first level. The representation (6) can be written in the form

$$\bar{u} = \bar{u}_1^0 + \sum_{k=1}^7 \bar{v}_{k,1}^0, \quad (18)$$

$$\bar{u}_1^0 = \sum_{j_3=1}^{N_1} \sum_{j_2=1}^{N_1} \sum_{j_1=1}^{N_1} u_{j_2,j_1,j_3}^1 \bar{\Phi}_{j_2,j_1,j_3}^1 = \Phi_1(\Phi_1, \bar{u}) = \Phi_1 \Phi_1^T \bar{u}; \quad (19)$$

$$\bar{v}_{k,1}^0 = \sum_{j_3=1}^{N_1} \sum_{j_2=1}^{N_1} \sum_{j_1=1}^{N_1} v_{k,j_2,j_1,j_3}^1 \bar{\Psi}_{k,j_2,j_1,j_3}^1 = \Psi_{k,1}(\Psi_{k,1}, \bar{u}) = \Psi_{k,1} \Psi_{k,1}^T \bar{u}, \quad k=1, \dots, 7; \quad (20)$$

$$\Phi_1 = [\bar{\Phi}_{1,1,1}^1 \quad \dots \quad \bar{\Phi}_{1,N_1,1}^1 \quad \dots \quad \bar{\Phi}_{1,N_1,N_1,1}^1 \quad \dots \quad \bar{\Phi}_{N_1,N_1,N_1}^1]; \quad (21)$$

$$\Psi_{k,1} = [\bar{\Psi}_{k,1,1,1}^1 \quad \dots \quad \bar{\Psi}_{k,1,N_1,1}^1 \quad \dots \quad \bar{\Psi}_{k,N_1,N_1,1}^1 \quad \dots \quad \bar{\Psi}_{k,N_1,N_1,N_1}^1], \quad k=1, \dots, 7; \quad (22)$$

Due to the orthonormality of the Haar basis [2,3,5,6], the operators

$$P_\Phi = \Phi_1 \Phi_1^T, \quad P_{\Psi,k} = \Psi_{k,1} \Psi_{k,1}^T, \quad k=1, \dots, 7 \quad (23)$$

are space projectors of vector functions of the original grid to their approximation space on the first-level grid and its complement (the refining component) to the initial state, respectively.

2. Basic scheme of the two-grid method

Let systems of linear algebraic equations

$$A\bar{u} = \bar{f} \quad (24)$$

are discrete analogs of some operator equation defined on a given rectangular prism of order N^3 .

We can substitute in (22) the expression for \bar{u} in the form (16). Then we can multiply, in turn, both sides of the equality on the left by the matrices Φ_1^T and $\Psi_{k,1}^T$, $k=1, 2, 3, 4, 5, 6, 7$. Thus we have

$$\Phi_1^T A \bar{u}_1^0 + \sum_{k=1}^7 \Phi_1^T A \Psi_{k,1} \bar{v}_k^1 = \Phi_1^T \bar{f}, \quad (25)$$

$$\Psi_{k,1}^T A \bar{u}_1^0 + \sum_{k=1}^7 \Psi_{k,1}^T A \Psi_{k,1} \bar{v}_k^1 = \Psi_{k,1}^T \bar{f}, \quad k=1, \dots, 7. \quad (26)$$

Equations (25) and (26) can be combined in the system

$$\sum_{i=1}^4 A_{i,j} \bar{w}_j = \bar{f}_i, \quad i=1, \dots, 8. \quad (27)$$

where $A_{i,j}$ are block matrices of size $N_1^3 \times N_1^3$; \bar{f}_i and \bar{w}_j are block vectors of size N_1^3 ,

$$A_{1,1} = \Phi_1^T A \Phi_1; \quad A_{1,k+1} = \Phi_1^T A \Psi_{k,1}, \quad A_{k+1,1} = \Psi_{k,1}^T A \Phi_1, \quad k=1, \dots, 7; \quad (28)$$

$$A_{k+1,p+1} = \Psi_{k,1}^T A \Psi_{p+1}, \quad k=1, 2, 3, 4, 5, 6, 7, \quad k=1, \dots, 7; \quad (29)$$

$$\bar{f}_1 = \Phi_1^T \bar{f}, \quad \bar{f}_{k+1} = \Psi_{k,1}^T \bar{f}, \quad k=1, \dots, 7; \quad (30)$$

$$\bar{w}_1 = \bar{u}^1, \quad \bar{w}_{k+1} = v_k^1, \quad k=1, \dots, 7. \quad (31)$$

We can find the solution of the system (27) using the block Gaussian method. The expanded block matrix has the form:

$$\left[\begin{array}{cccccccc|c} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & \bar{f}_1 \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & \bar{f}_2 \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & \bar{f}_3 \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} & A_{4,8} & \bar{f}_4 \\ A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & A_{5,7} & A_{5,8} & \bar{f}_5 \\ A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} & A_{6,8} & \bar{f}_6 \\ A_{7,1} & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} & A_{7,8} & \bar{f}_7 \\ A_{8,1} & A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & A_{8,7} & A_{8,8} & \bar{f}_8 \end{array} \right]. \quad (32)$$

We have the following result of forward algorithm:

$$\left[\begin{array}{cccccccc|c} A_{1,1}^0 & A_{1,2}^0 & A_{1,3}^0 & A_{1,4}^0 & A_{1,5}^0 & A_{1,6}^0 & A_{1,7}^0 & A_{1,8}^0 & \bar{f}_1^0 \\ 0 & A_{2,2}^1 & A_{2,3}^1 & A_{2,4}^1 & A_{2,5}^1 & A_{2,6}^1 & A_{2,7}^1 & A_{2,8}^1 & \bar{f}_2^1 \\ 0 & 0 & A_{3,3}^2 & A_{3,4}^2 & A_{3,5}^2 & A_{3,6}^2 & A_{3,7}^2 & A_{3,8}^2 & \bar{f}_3^2 \\ 0 & 0 & 0 & A_{4,4}^3 & A_{4,5}^3 & A_{4,6}^3 & A_{4,7}^3 & A_{4,8}^3 & \bar{f}_4^3 \\ 0 & 0 & 0 & 0 & A_{5,5}^4 & A_{5,6}^4 & A_{5,7}^4 & A_{5,8}^4 & \bar{f}_5^4 \\ 0 & 0 & 0 & 0 & 0 & A_{6,6}^5 & A_{6,7}^5 & A_{6,8}^5 & \bar{f}_6^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{7,7}^6 & A_{7,8}^6 & \bar{f}_7^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{8,8}^7 & \bar{f}_8^7 \end{array} \right], \quad (33)$$

where

$$A_{i,j}^k = A_{i,j}^{k-1} - C_{i,k} A_{k,j}^{k-1}, \quad k=1, \dots, 7, \quad i=k+1, \dots, 8, \quad j=k+1, \dots, 8; \quad (34)$$

$$\bar{f}_i^k = \bar{f}_i^{k-1} - C_{i,k} \bar{f}_k^{k-1}, \quad k=1, \dots, 7, \quad i=k+1, \dots, 8; \quad (35)$$

$$C_{i,k} = A_{i,k}^{k-1} (A_{k,k}^{k-1})^{-1}, \quad k=1, \dots, 7, \quad i=k+1, \dots, 8. \quad (36)$$

We have the following result of backward algorithm

$$\bar{w}_8 = (A_{8,8}^7)^{-1} \bar{f}_8^7; \quad (37)$$

$$\bar{w}_i = (A_{i,i}^{i-1})^{-1} \left(\bar{f}_i^{i-1} - \sum_{j=i+1}^4 A_{i,i}^{i-1} \bar{w}_j \right), \quad i = 7, \dots, 1. \quad (38)$$

In accordance with formulas (18)-(20) solution of the considering problem (24) has the form

$$\bar{u} = \bar{u}_1^0 + \sum_{k=1}^7 \bar{v}_{k,1}^0 = \Phi_1 \bar{u}^1 + \sum_{k=1}^7 \Psi_{k,1} \bar{v}_k^1 = \Phi_1 \bar{w}_1 + \sum_{k=1}^7 \Psi_{k,1} \bar{w}_{k+1}. \quad (39)$$

It should be noted that the proposed method makes it possible to obtain a solution with the use of matrices of size $N_1^3 \times N_1^3$ and vectors of size N_1^3 .

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