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The response of the system “beam – foundation” on sudden changes of boundary conditions

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Abstract. The mathematic model of dynamic process into beam, which lies on the elastic foundation Winkler's type, at sudden change of boundary conditions caused by destruction of supporting is presented in the paper. Solution of static bending problem for such beam clamped at two ends is the initial state for dynamic process of it forced oscillations as cantilever rod at destruction of supports limiting edge displacements. The effect, which caused by sudden transformation of computational model of constructively nonlinear system 'beam - foundation' at different combinations of mechanical and geometrical parameters of beam and foundation, is investigated.

1. Introduction

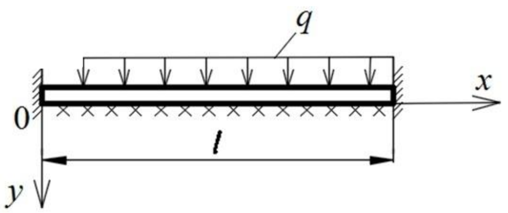
One of the important problems of Structural Mechanics is response analysis of new and operating building and structures to different changes of project solutions, variation of external impacts, structural transformations under load and other accidental impacts. [1, 2].

In recent years, the survivability and safety problems of designing, operating and reconstructing construction and structures become one of the most important problems [3–6]. At first, it determined by significant wear of buildings and structures, growth of different kind of technogenic and anthropogenic aggressive impacts, terrorism, not qualitative reconstruction etc. Analysis of Russian and abroad technical literature shows that there is small quantity of formulations of such problems and methods to solve it, which would take into account sudden changes of constructive and (or) computational models of structures and its elements. In addition, it should be noted that existing formulations of problem and computational methods are imperfect. It requires intensifying development of analytical and numerical methods, which allows solving constructively nonlinear problems for structures, the computational models with changes under load. Such methods would relate accidental impacts values with increments of internal forces in structures and kinematic factor of bearing capacity. Mathematic models of dynamic processes caused by instantaneous changes topology, cracking, local destruction, damage or destruction of supports are presented in the papers [7–11]. The mathematical models of studying objects in these papers are differential equations of different orders in combination with boundary and initial conditions. The common item for all these papers is using model analyzing algorithm, which based on the calculation static state as initial conditions for dynamic process caused by sudden distortion of static state by accidental impact; analysis of forced movements applying modal expansion of initial state and load by the damaged structure modes. Analogous techniques are applied in analysis of rod systems on the elastic foundation. For example, in the papers [12–19] the evaluations of dynamic load increasing in the 'beam - foundation' system are given for the case of damage of foundation. The article [14] deals with sudden transformation of boundary conditions and article [15] devoted evolutionary deformation of 'beam - foundation' system.

In the present paper, the mathematical model of dynamic process at sudden boundary condition transformation of clamped at the edges beam on the elastic foundation is given and analyzed.

2. Formulation of the problem

It is supposed, that beam with length l clamped at the ends and loaded by evenly distributed load q as it is shown in figure 1 (a). Flexural rigidity is EI. Beam lays on the elastic foundation Winkler's type with rigidity k. At the time $t = 0$, beam on two supports instantaneous transforms to cantilever as it is shown in Figure1 (b). This transformation is caused by damage of right support.



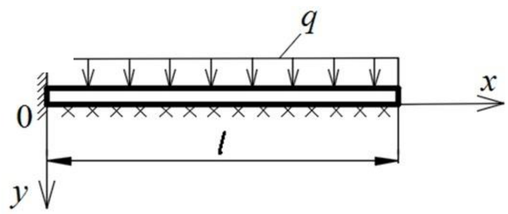


Figure 1. Computational models for static bending beam on two support without damages (a) and forced oscillations of damaged beam (b).

Sudden appear of damage in the form of right support destruction lead beam to movement, since common rigidity decrease of structural 'beam-foundation' system do not provide equilibrium of forces. Determination of character and parameters of movement is shown below in the paper in the following sequence:

1) formation of initial conditions by solving the static bending problem for beam on the elastic foundation with clamped ends without damages;
2) solving problem of eigenvalues for damaged cantilever beam on the elastic foundation: natural frequencies determination and modes of oscillations;
3) solving problem of forced oscillations for beam on the elastic foundation using modal expansion of load and static deflection by modes of natural oscillations, obtained before in p.2. Determination of internal forces.

3. Characteristics of initial static state

General solution of static bending of beam on the elastic foundation Winkler type with clamped both ends by evenly distributed load in dimensionless form with using Krylov function and initial parameters was obtained in the paper [12, 13, 14] in the form:

$$w_s = \frac{\overline{q}}{4\alpha^4}\left(1 - K_4\left(\alpha\xi\right)\right) + w_0'' K_2\left(\alpha\xi\right) + w_0''' K_1\left(\alpha\xi\right), \tag{1}$$

Where $\xi = \dfrac{x}{l}$, $w_s = \dfrac{\upsilon}{l}$, $\overline{q} = \dfrac{ql^3}{EI}$, $\alpha = \sqrt[4]{\dfrac{kl^4}{4EI}}$, $K_i\left(\alpha\xi\right)$ $\left(i = 1 \div 4\right)$ – Krylov function

$$K_1(\alpha\xi) = \frac{\sin\alpha\xi ch\alpha\xi - \cos\alpha\xi sh\alpha\xi}{4\alpha^3}, \quad K_2 = K_1', \quad K_3 = K_2', \quad K_4 = K_3', \quad K_4' = -4\alpha^4 K_1.$$

$w_0'' = w_s''(0),\ w_0''' = w_s'''(0)$ – initial parameters.

Distribution of bending moments at static state is characterized by function:

$$M_s = \overline{q}K_2(\alpha\xi) + w_0''K_4(\alpha\xi) + w_0'''K_3(\alpha\xi). \tag{2}$$

In Figure 2, bending moment diagrams for beam on the elastic foundation with clamped edges at different values general rigidity of 'beam - foundation' system $\lambda = 4\alpha^4$. is presented.

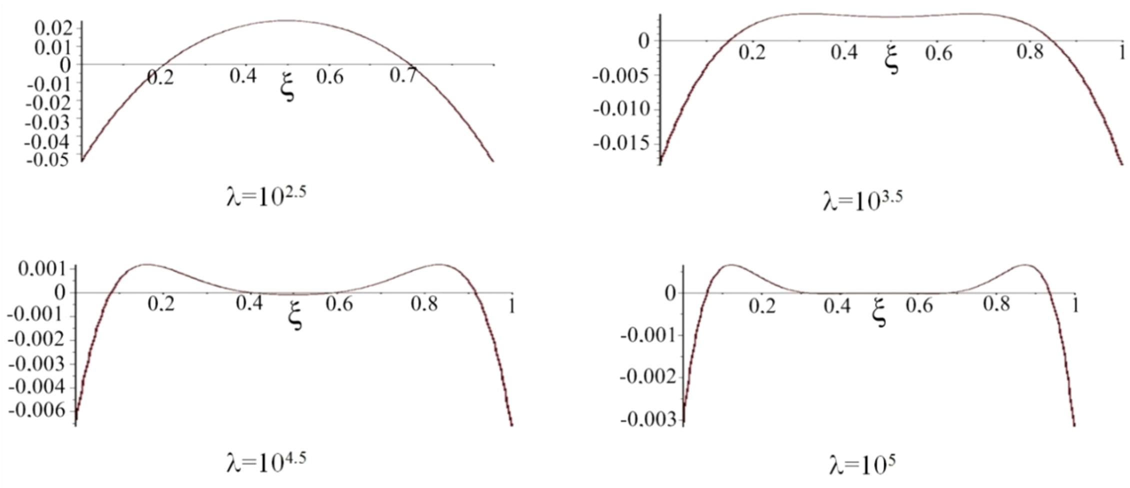


Figure 2. Bending moment diagrams for beams on the elastic foundation with clamped edges.

4. Natural bending oscillations of cantilever beam on the elastic foundation

Natural bending oscillations of cantilever beam on the elastic foundation are described by equation [12]:

$$\frac{\partial^4 w_d}{\partial\xi^4} + 4\alpha^4\left(w_d + \frac{\partial^2 w_d}{\partial\tau^2}\right) = 0. \tag{3}$$

In the equation (3) dimensionless variables w_d, τ and parameter ω_0 is introduced

$$w_d = \frac{\upsilon(\xi,t)}{l}, \quad \tau = \omega_0 t, \quad \omega_0 = \sqrt{\frac{k}{\rho A}}.$$

Parameter ω_0 has dimension of frequency and called conditional frequency.

Let us divide variables in the equation (3) using following substitution:

$$w_d = W(\xi)\sin\overline{\omega}\tau, \tag{4}$$

where $\overline{\omega} = \dfrac{\omega}{\omega_0}$ – comparative natural frequency of cantilever beam on the elastic foundation bending oscillations.

Formula of the mode of natural bending oscillation $W(\xi)$ is obtained by substitution of formula (4) to equation (3)

$$\frac{d^4W}{d\xi^4} + 4\alpha^4\left(1-\bar{\omega}^2\right)W = 0. \tag{5}$$

Using comparative frequency $\tilde{\omega}$ and 'conditional' frequency $\bar{\omega}_0$, which can be written as $\bar{\omega}_0 = \frac{\omega_0}{\omega_{1fr}}$; $\tilde{\omega} = \frac{\omega}{\omega_{1fr}}$, where $\omega_{1fr} = \left(\frac{1,875}{l}\right)^2\sqrt{\frac{EI}{\rho A}}$ is fundamental frequency of bending oscillation of cantilever beam without elastic foundation, transform equation (5) to the form:

$$\frac{d^4W}{d\xi^4} + (1,875)^4\left(\bar{\omega}_0{}^2 - \tilde{\omega}^2\right)W = 0. \tag{6}$$

Applying Euler's substitution to the equation (6)

$$W = Ae^{r\xi}, \tag{7}$$

it is obtained characteristic equation:

$$r^4 + (1,875)^4\left(\bar{\omega}_0^2 - \tilde{\omega}^2\right) = 0, \tag{8}$$

Roots, that can be presented in three variants in accordance to ratio of the frequencies $\bar{\omega}_0$ and $\tilde{\omega}$:

1. If $\tilde{\omega} > \bar{\omega}_0$, then roots of equation (8) are real and whole imaginary:

$$r_{1,2} = \pm\beta_1, \quad r_{3,4} = \pm i\beta_1, \quad \beta_1 = 1,875\sqrt[4]{\tilde{\omega}^2 - \bar{\omega}_0^2}. \tag{9}$$

2. If $\tilde{\omega} < \bar{\omega}_0$, then roots of the equation (8) are complex:

$$r_{1,2,3,4} = (\pm i \pm 1)\beta_2, \quad \beta_2 = \frac{1,875}{\sqrt{2}}\sqrt[4]{\bar{\omega}_0^2 - \tilde{\omega}^2}. \tag{10}$$

3. If $\tilde{\omega} = \bar{\omega}_0$, then we obtain quadruple root:

$$r_{1,2,3,4} = 0.$$

It is shown in the papers [16], that two simply supported ends, free ends and cantilever in the case of canonical boundary conditions for beam entirely laying on the elastic foundation Winkler's type, such as two clamped ends, the physically real result can be obtained only using (9):

$$\tilde{\omega} > \bar{\omega}_0.$$

Deflection function in this case in accordance first two conditions (11)

$$\begin{gathered} W_0 = W_0' = 0 \\ W''(1) = W'''(1) = 0. \end{gathered} \tag{11}$$

Takes the form:

$$W(\xi) = W_0''R_2(\beta_1\xi) + W_0'''R_1(\beta_1\xi). \tag{12}$$

In accordance with the second group of boundary conditions (11), we obtain system of algebraic equations with unknown initial parameters W_0'' и W_0'''

$$\begin{cases} W_0'' R_4(\beta_1) + W_0''' R_3(\beta_1) = 0 \\ W_0'' \beta_1^4 R_1(\beta_1) + W_0''' R_4(\beta_1) = 0. \end{cases} \tag{13}$$

Where $R_i\,(i = 1 \div 4)$ are Krylov functions, which have the from

$$R_1(\beta_1\xi) = \frac{sh\beta_1\xi - \sin\beta_1\xi}{2\beta_1^3}; \quad R_2 = R_1'; \quad R_3 = R_2'; \quad R_4 = R_3'; \quad R_4' = \beta_1^4 R_1.$$

Equaling determinant of the equation system (13) to zero and uncover it, we obtain frequencies equation

$$1 + ch\beta_1 \cos\beta_1 = 0, \tag{14}$$

which is analogous by the form with the equation of natural oscillations of cantilever beam. Roots of this equation give the row:

$$\beta_{11} = 1{,}875, \;\; \beta_{12} = 4{,}694, \;\; \beta_{1n} \approx \frac{2n-1}{2}\pi.$$

Frequencies of natural bending oscillation of cantilever beam on the elastic foundation can be found using formula (9):

$$\tilde{\omega}_n = \sqrt{\left(\frac{\beta_{1n}}{1{,}875}\right)^4 + \bar{\omega}_0^2}. \tag{15}$$

Each frequency $\tilde{\omega}_n$ corresponds mode of oscillations $W_n(\xi)$

$$W_n = W_0''\left(R_2(\beta_{1n}\xi) - \frac{R_4(\beta_{1n})}{R_3(\beta_{1n})} R_1(\beta_{1n}\xi)\right). \tag{16}$$

In this way, the modes of beam natural oscillations on the elastic foundation remain such as modes of beam without elastic foundation, and frequencies depend on the parameter $\bar{\omega}_0$ becoming in $\sqrt{\left(\frac{\beta_{1n}}{1{,}875}\right)^4 + \bar{\omega}_0^2}$ times higher by value, then corresponding frequencies of cantilever beam without elastic foundation.

5. Forced oscillation of cantilever beam on the elastic foundation

Solution of forced oscillation equation of cantilever beam laying on the elastic foundation

$$\frac{\partial^4 w_d}{\partial \xi^4} + 4\alpha^4\left(w_d + \frac{\partial^2 w_d}{\partial \tau^2}\right) = \bar{q} \tag{17}$$

can be obtained using expansion of the function $w_d(\xi,\tau)$ to the row by eigenfunctions $W_n(\xi)$(16) with coefficients in the form of time function $Q_n(\tau)$

$$w_d = \sum_{n=1}^{\infty} Q_n(\tau) W_n(\xi). \tag{18}$$

Let us derive differential equation to determine functions $Q_n(\tau)$, substituting row (18) and relation for $\frac{d^4W}{d\xi^4}$ from (6) to equation (17) and multiplying both parts of obtained equations to $W_n(\xi)$, integrating both part by ξ from 0 to 1 and applying orthogonality property of the natural oscillations modes $W_n(\xi)$

$$\frac{d^2Q_n}{d\tau^2} + \bar{\omega}_n^2 Q_n = R_n, \tag{19}$$

where

$$R_n = \frac{\bar{q}}{(1,875)^4 \bar{\omega}_0^2} \cdot \frac{\int_0^1 W_n(\xi)\,d\xi}{\int_0^1 W_n^2(\xi)\,d\xi}.$$

General solution of non-homogenous equations (19) has the form:

$$Q_n = D_{1n} cos\bar{\omega}_n\tau + D_{2n} sin\bar{\omega}_n\tau + R_n/\bar{\omega}_n^2, \tag{20}$$

where D_{1n} и D_{2n} are constants of integrating, which can be obtained from initial conditions of dynamic process

$$w_d(\xi,0) = w_s(\xi)$$
$$\left.\frac{\partial w_d}{\partial \tau}\right|_{\xi,0} = 0. \tag{21}$$

Using second condition of formula (21), which means that velocity of beam points equals to zero at $\tau = 0$, we obtain

$$D_{2n} = 0. \tag{22}$$

Using first condition of formula (21), we obtain

$$\sum_{n=1}^{\infty} \left(D_{1n} + \frac{R_n}{\bar{\omega}_n^2} \right) W_n(\xi) = w_s(\xi), \tag{23}$$

where by multiplying both part of formula (23) to $W_n(\xi)$ and integrating by ξ from 0 to 1 we obtain

$$D_{1n} = B_n - \frac{R_n}{\bar{\omega}_n^2}, \tag{24}$$

where $B_n = \dfrac{\int_0^1 w_s(\xi) W(\xi) d\xi}{\int_0^1 W_n^2(\xi) d\xi}$.

Substituting (22) and (24) to row

$$w_d = \sum_{n=1}^{\infty} (D_{1n} cos\bar{\omega}_n \tau + D_{2n} sin\bar{\omega}_n \tau + R_n/\bar{\omega}_n^2) W_n(\xi).$$

We obtain deflection function of beam-forced oscillations

$$w_d(\xi,\tau) = \sum_{n=1}^{\infty} \left(B_n \cos \bar{\omega}_n \tau + C_n \sin^2 \frac{\bar{\omega}_n}{2} \tau \right) W_n(\xi), \tag{25}$$

where $C_n = \dfrac{2\bar{q}}{(1,875)^4 \tilde{\omega}_n^2} \cdot \dfrac{\int_0^1 W_n(\xi) d\xi}{\int_0^1 W_n^2(\xi) d\xi}$.

Differentiating (25) by ξ two time, we obtain bending moments function $M_d(\xi,\tau)$

$$M_d(\xi,\tau) = \sum_{n=1}^{\infty} \left(B_n \cos \bar{\omega}_n \tau + C_n \sin^2 \frac{\bar{\omega}_n}{2} \tau \right) W_n''(\xi). \tag{26}$$

6. Numerical results

Dimensionless deflections $w(\xi,\tau)$ and bending moments $M(\xi,\tau)$ of beam laying on the elastic foundation Winkler type and loaded evenly distributed load $\bar{q} = 1$ are calculated at various combination of general rigidity of 'beam - foundation' system $\lambda = 4\alpha$. Deflection and moments are calculated for initial state at clamped ends of beam $w_s(\xi)$, $M_s(\xi)$, for quasi static transformation of clamping to free end $w_{qs}(\xi)$ and $M_{qs}(\xi)$; for instantaneous transformation to cantilever rod $w_d(\xi,\tau)$ and $M_d(\xi,\tau)$.

Computational results are shown in Figure 3 and 4, as well as in Table 1 Distribution of deflections $w_{qs}(\xi)$ and bending moments $M_{qs}(\xi)$ along beam after quasi static appear of damage at various values of general rigidity parameter of 'beam-foundation' system $\lambda = KL^4/EI$ are shown in Figure 3.

In Figure 2, the graphs of bending moment increasing in clamped edge $M_d(0,\tau)$ at the beginning of dynamic process after sudden beam transformation to cantilever $\tau = 0 \div 1$ as it is shown in Figure 4 (a) and graphs of stationary oscillations at $\tau > 1$ as it is shown in Figure 4 (b).

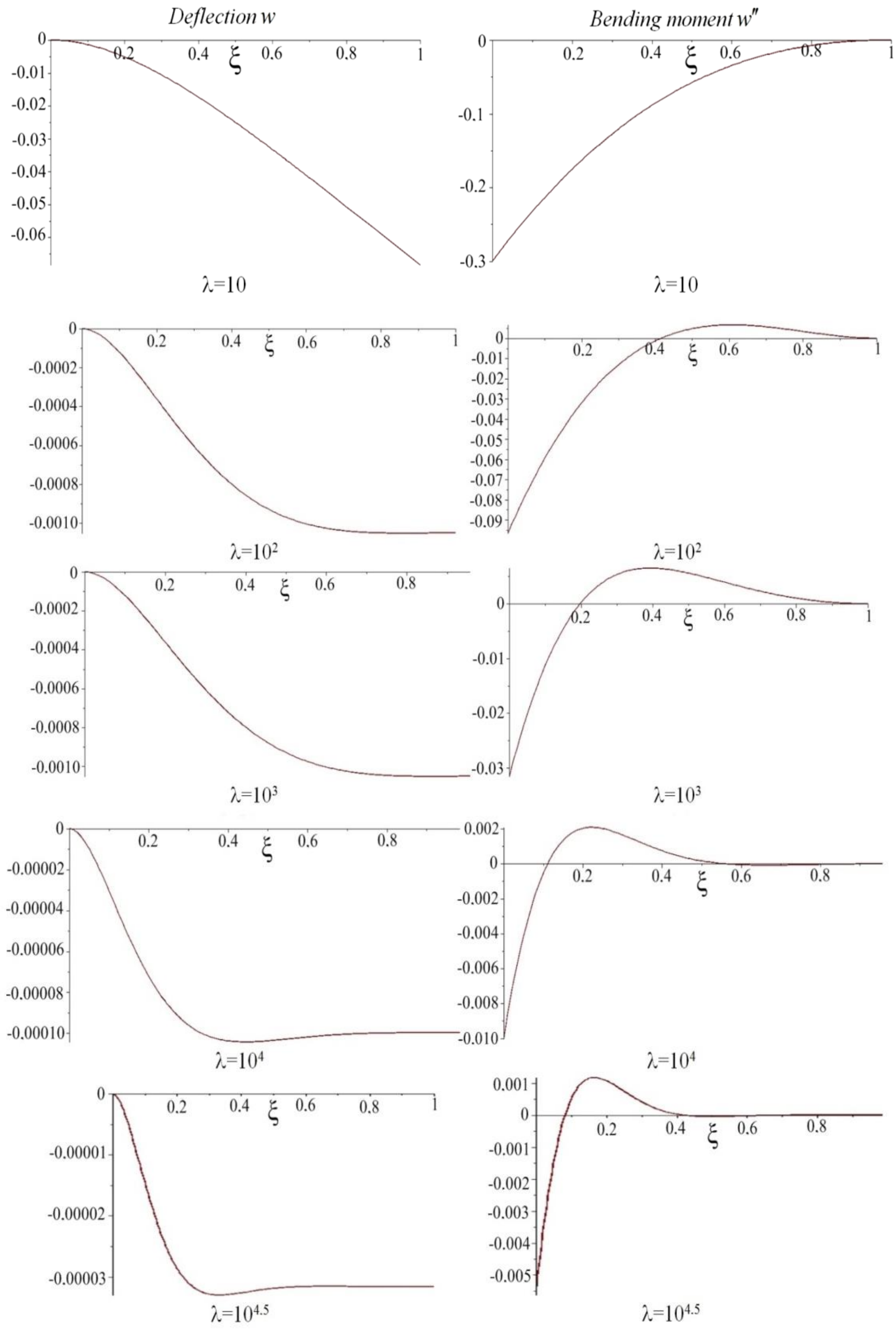


Figure 3. Distribution of deflection and bending moments along the beam length at various values general rigidity λ of "beam-foundation" system.

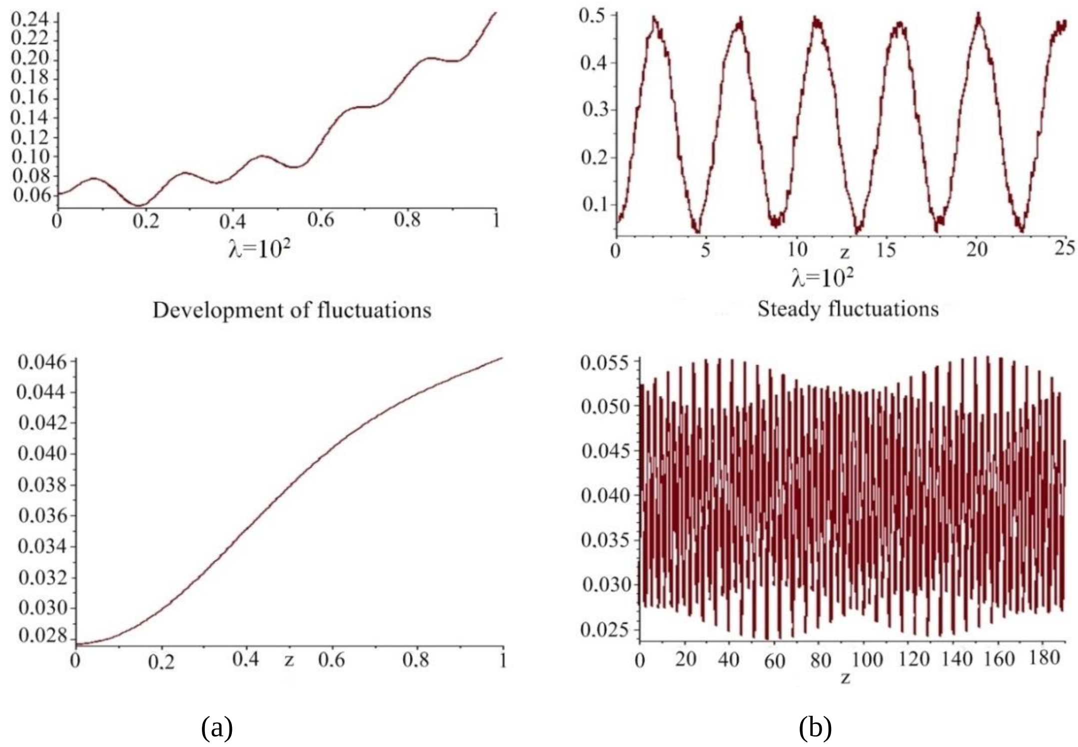


Figure 4. Bending moment diagrams at the beginning of dynamic process after sudden transformation (a) and for stationary oscillations (b).

Table 1. Maximum values of bending moments at initial state and at oscillations, caused by damage of beam.

λ	M_s	M_{qs}	M_d	K_{qs}	K_d
0	0.08	0.5	4.7	6.25	9.4
10^2	0.07	0.095	0.5	1.36	5.26
10^3	0.031	0.031	0.055	1	1.77
10^4	0.01	0.01	0.01	1	1

7. Conclusions

The calculation results shows, that instantaneous transformation of clamped on the both sides beam, which lays on the elastic foundation, to cantilever rod have significantly influence to stress-strain state of beam at low values general rigidity of 'beam – foundation' system $(\lambda \leq 3)$. It is obvious, that bending moment diagram changes however, it does not depend on speed of damages appearing.

At high values of general rigidity parameter $(\lambda > 3)$ changing of boundary condition lead to new distribution of deflection and bending moments with saving location $\xi = 0$ and value of maximum bending moment regardless of this process speed.

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