

# Analysis of thin nonlinear plates' forced vibrations in a viscoelastic medium under the conditions of the additive combinational internal resonance

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**Abstract.** The force driven dynamic response of a nonlinear plate embedded in a viscoelastic medium, damping features described by the Kelvin-Voigt fractional derivative model, is studied in the present paper. The motion of the plate is described by three coupled nonlinear differential equations with due account for the fact that the plate is being under the conditions of the internal combinational resonance accompanied by the external resonance, resulting in three modes corresponding interaction to the mutually orthogonal displacements. The displacement functions are determined in terms of linear vibrations' eigenfunctions. The solution of motion nonlinear governing equations was obtained by the utilization of the multiple scales method, so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. The influence of viscosity on the energy exchange mechanism between interacting modes was analyzed. A comparative analysis of numerical calculations for the free and forced vibrations cases was carried out.

## 1. Introduction

During the intense development of the modern industry, a reduction in the materials consumption of machine structures and various infrastructure objects is one of the main problems in the civil engineering. For the purpose of material saving, the need arises to manufacture thin-walled structures. The thinner is the element and the more flexible it is, the stronger its susceptibility to buckling, resulting in the loss of stability and/or the destruction of the structure in general.

The plates are widely used in engineering structures such as constructive elements in different mechanical, civil, and aerospace systems. Vibrations of viscoelastic plates are considered in the literature [1, 2]. The Kelvin-Voigt viscoelastic model was used in [3, 4] for modelling free nonlinear vibrations of sandwich rectangular plates with simply-supported moveable edges. Such plates are often subjected to nonstationary and harmonic mechanical loads. The resonant harmonic vibrations, occurring when the frequency of the harmonic time-dependent force becomes equal to the natural frequency of the plate, it is very dangerous. This raises the problem of damping the stationary and nonstationary vibrations of thin rectangular plates. For this purpose, widespread use is made of the passive-damping technique where elements with high hysteresis losses are incorporated into structures to be damped [3, 4].

Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in multimode response with a strong interaction of the modes involved [5]. In the



majority of the research papers, the internal resonance has been investigated within some natural frequencies combination of one and the same type of vibrations [6-8].

Another type of the internal resonance was studied in [9], when one frequency of the in-plane vibrations was twice as large as (a 1:2 internal resonance) some frequency of the out-of-plane vibrations, the damping features of the surrounding medium were modeled via the Riemann-Liouville fractional derivative. Nonlinear free damped vibrations of a rectangular plate described by three nonlinear ordinary differential equations have been analyzed in [10], where in the procedure resulting in decoupling, linear parts of equations has been proposed with the further utilization of the multiple scales method for solving nonlinear governing equations of motion, the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. All possible types of the internal resonance were revealed in [10], and it was shown that the type of the resonance depends on the smallness order of the fractional derivative entering in the equations of the plate motion. Further this approach has been extended to the analysis of the force driven vibrations with weak fractional damping of a nonlinear oscillator [11] and suspension bridge [12].

In the present paper, the procedures proposed in [10-12] were generalized for the force driven vibrations analysis of a thin plate under the additive combinational resonance when the force frequency is approximately equal to a certain natural frequency of vertical vibrations.

## 2. Problem formulation

Let us consider the dynamic behavior of a free supported nonlinear thin rectangular plate, which vibrations in a viscoelastic fractional derivative medium are described by the following three differential equations in the dimensionless form (free damped equations presented in [10] are supplemented herein by the vertical harmonic force  $F$  applied at the point with the coordinates  $x_0, y_0$ ):

$$u_{xx} + \frac{1-\nu}{2}\beta_1^2 u_{yy} + \frac{1+\nu}{2}\beta_1 v_{xy} + w_x \left( w_{xx} + \frac{1-\nu}{2}\beta_1^2 w_{yy} \right) + \frac{1+\nu}{2}\beta_1^2 w_y w_{xy} = \ddot{u} + \alpha_1 D_{0+}^\gamma u, \quad (1)$$

$$\beta_1^2 v_{yy} + \frac{1-\nu}{2}v_{xx} + \frac{1+\nu}{2}\beta_1 u_{xy} + \beta_1 w_y \left( \beta_1^2 w_y + \frac{1-\nu}{2}w_{xx} \right) + \frac{1+\nu}{2}\beta_1 w_x w_{xy} = \ddot{v} + \alpha_2 D_{0+}^\gamma v, \quad (2)$$

$$\begin{aligned} & \frac{\beta_2^2}{12}(w_{xxxx} + 2\beta_1^2 w_{xxyy} + \beta_1^4 w_{yyyy}) - w_{xx}(u_x + \nu\beta_1 v_y) - w_x(u_{xx} + \nu\beta_1 v_{xy}) \\ & - \frac{1-\nu}{2}\beta_1 [w_{xy}(\beta_1 u_y + v_x) + w_y(\beta_1 u_{xy} + v_{xx})] - \beta_1^2 [w_{yy}(\nu u_x + \beta_1 v_y) + w_y(\nu u_{xy} + \beta_1 v_{yy})] \\ & - \frac{1-\nu}{2}\beta_1 [w_{xy}(\beta_1 u_y + v_x) + w_x(\beta_1 u_{yy} + v_{xy})] - \hat{F}\delta(x-x_0)\delta(y-y_0)\cos(\Omega_F t) = -\ddot{w} - \alpha_3 D_{0+}^\gamma w; \end{aligned} \quad (3)$$

where  $u = u(x, y, t)$ ,  $v = v(x, y, t)$ , and  $w = w(x, y, t)$  are the displacements of points located in the plate's middle surface in the  $x$ -,  $y$ -, and  $z$ - directions, respectively,  $\nu$  is Poisson's ratio,  $\beta_1 = a/b$  and  $\beta_2 = h/a$  are the parameters defining the dimensions of the plate,  $a$  and  $b$  are the plate's dimensions along the  $x$ - and  $y$ - axes, respectively,  $h$  is the thickness,  $t$  is the time, an overdot denotes the time-derivative, lower indices label the derivatives with respect to the corresponding coordinates,  $F = \hat{F}\delta(x-x_0)\delta(y-y_0)\cos(\Omega_F t)$  is the harmonic force,  $\hat{F}$  is its amplitude,  $\Omega_F$  is the frequency,  $\delta$  is the Dirac delta function,  $\alpha_i = \varepsilon\mu_i\tau_i^\gamma$  ( $i=1,2,3$ ) are damping coefficients,  $\varepsilon$  is a small dimensionless parameter of the same order of magnitude as the amplitudes,  $\mu_i$  are finite values,  $\tau_i$  is the relaxation time of the  $i$ th generalized displacement, and  $D_{0+}^\gamma$  is the Riemann-Liouville fractional derivative of the  $\gamma$ -order [13].

For solving nonlinear governing equations of motion (1)–(3), the procedure resulting in decoupling linear parts of equations was proposed with the further utilization of the multiple scales

method [10], in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. It is shown that the phenomenon of internal resonance could be very critical, since in the thin plate under consideration the internal resonance is always present. Moreover, its type depends on the smallness order of the viscosity involved into consideration. The following types of the internal resonance have been revealed:

order of  $\varepsilon$  :

the two-to-one internal resonance

$$\omega_1 = 2\omega_3 \quad (\omega_2 \neq \omega_1, \omega_2 \neq 2\omega_3), \quad \omega_2 = 2\omega_3 \quad (\omega_1 \neq \omega_2, \omega_1 \neq 2\omega_3); \quad (4)$$

the one-to-one-to-two internal resonance

$$\omega_1 = \omega_2 = 2\omega_3 \quad (1:1:2) \quad (5)$$

order of  $\varepsilon^2$  :

the one-to-one internal resonance

$$\omega_1 = \omega_2 \quad (\omega_3 \neq \omega_1, \omega_3 \neq \omega_2), \quad \omega_1 = \omega_3 \quad (\omega_2 \neq \omega_1, \omega_2 \neq \omega_3), \quad \omega_2 = \omega_3 \quad (\omega_1 \neq \omega_2, \omega_1 \neq \omega_3); \quad (6)$$

the one-to-one-to-one internal resonance

$$\omega_1 = \omega_2 = \omega_3 \quad (1:1:1) \quad (7)$$

the combinational resonance of the additive-difference type

$$\omega_1 = \omega_2 + 2\omega_3, \quad \omega_1 = 2\omega_3 - \omega_2, \quad \omega_1 = \omega_2 - 2\omega_3, \quad (8)$$

where  $\omega_1$  and  $\omega_2$  are the certain modes frequencies of in-plane vibrations in the  $x$ -and  $y$ -axes, respectively, and  $\omega_3$  is the frequency of a certain mode of vertical vibrations.

Now let us consider the case of the additive internal resonance (8) accompanied by the external resonance, when  $\omega_1 + \omega_2 = 2\omega_3 = 2\Omega_F$ ,  $\Omega_F$  is external force frequency. Using the set of solvability equations to eliminate secular terms similarly to the case of free vibrations [10] and adding the external resonance, we obtain the following solvability equations for the case of force driven vibrations:

$$\begin{aligned} 2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1 \tau_1)^\gamma A_1 + 2\zeta_1 k_5 A_1 A_3 \bar{A}_3 + 2\zeta_1 k_8 \bar{A}_2 A_3^2 &= 0, \\ 2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2 \tau_2)^\gamma A_2 + 2\zeta_2 k_6 A_2 A_3 \bar{A}_3 + 2\zeta_2 k_7 \bar{A}_1 A_3^2 &= 0, \\ 2i\Omega_F D_2 A_3 + \mu_3 (i\Omega_F \tau_3)^\gamma A_3 + (\zeta_{13} k_2 + \zeta_{23} k_4) A_3^2 \bar{A}_3 + \zeta_{13} k_7 A_1 A_3 \bar{A}_1 & \\ + \zeta_{23} k_8 A_2 A_3 \bar{A}_2 + (\zeta_{13} k_8 + \zeta_{23} k_7) A_1 A_2 \bar{A}_3 - k_F &= 0, \end{aligned} \quad (9)$$

where  $D_2 = \partial / \partial T_2$  is the time-derivative due to the utilization of the multiple time scales method [5, 10],  $A_j(T_2)$  ( $j=1,2,3$ ) are unknown complex functions,  $\zeta_1, \zeta_2, \zeta_{13}, \zeta_{23}$  are coefficients depending on the plate dimensions and numbers of excited modes [10],  $k_p$  ( $p=1,2,\dots,8$ ) are coefficients depending on the natural frequencies of plate,  $k_F = 2\hat{f}(\omega_3^2 - \Omega_F^2)^{-1}$ , and  $\hat{f}$  is a finite value.

Representing the functions  $A_i(T_1, T_2)$  in equations (9) in the polar form

$A_i(T_1, T_2) = a_i(T_1, T_2) \exp[i\varphi_i(T_1, T_2)]$  ( $i=1,2,3$ ) and separating real and imaginary parts yield

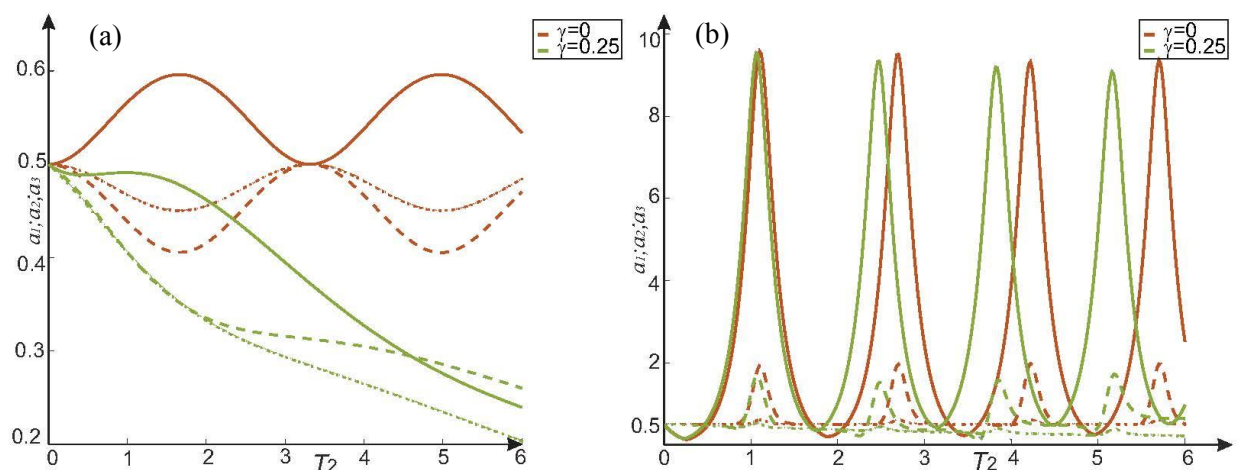
$$\begin{aligned}
(\dot{a}_1^2) + s_1 a_1^2 + 2\omega_1^{-1} \zeta_1 k_8 a_1 a_2 a_3^2 \sin \delta &= 0, \\
(\dot{a}_2^2) + s_2 a_2^2 + 2\omega_2^{-1} \zeta_2 k_7 a_1 a_2 a_3^2 \sin \delta &= 0, \\
(\dot{a}_3^2) + s_3 a_3^2 = \Omega_F^{-1} (\zeta_{13} k_8 + \zeta_{23} k_7) a_1 a_2 a_3^2 \sin \delta + 2k_F \Omega_F^{-1} a_3 \sin \varphi_3, \\
\dot{\varphi}_1 - \frac{1}{2} \sigma_1 - \zeta_1 \omega_1^{-1} k_8 a_3^2 - \zeta_1 \omega_1^{-1} k_8 a_3^2 a_1^{-1} a_2 \cos \delta &= 0, \\
\dot{\varphi}_2 - \frac{1}{2} \sigma_2 - \zeta_2 \omega_2^{-1} k_7 a_3^2 - \zeta_2 \omega_2^{-1} k_7 a_3^2 a_2^{-1} a_1 \cos \delta &= 0, \\
\dot{\varphi}_3 - \frac{1}{2} \sigma_3 - \frac{1}{2} \zeta_{13} \Omega_F^{-1} a_1^2 k_7 - \frac{1}{2} \zeta_{23} \Omega_F^{-1} a_2^2 k_8 - \frac{1}{2} \Omega_F^{-1} (\zeta_{13} k_2 + \zeta_{23} k_4) a_3^2 \\
- \frac{1}{2} \Omega_F^{-1} (\zeta_{13} k_8 + \zeta_{23} k_7) a_1 a_2 \cos \delta + k_F (\Omega_F a_3)^{-1} \cos \varphi_3 &= 0,
\end{aligned} \tag{10}$$

where a dot denotes differentiation with respect to  $T_2$ ,  $a_i, \varphi_i$  are amplitudes and phases, respectively,  $\delta = 2\varphi_3 - \varphi_2 - \varphi_1$  is the phase difference,  $s_i = \mu_i \tau_i^\gamma \omega_i^{\gamma-1} \sin \psi$ ,  $\sigma_i = \mu_i \tau_i^\gamma \omega_i^{\gamma-1} \cos \psi$ , and  $\psi = \pi\gamma / 2$ .

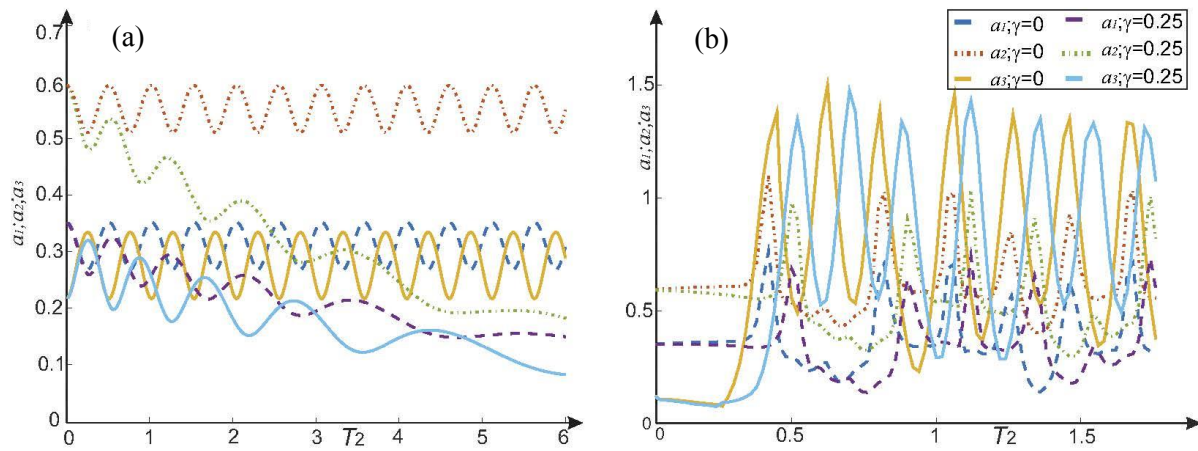
### 3. Numerical calculations

The influence of the external force on the amplitudes of vibrations  $a_i(T_2)$  calculated using the Runge-Kutta fourth-order method according to equations (10) at different magnitudes of the fractional parameter  $\gamma$  is traced in Figures 1-3, and hence its impact on the energy exchange between three interacting modes coupled by the additive combinational resonance ( $\omega_1 + \omega_2 = 2\omega_3 = 37.43946$ ).

Figures 1 and 2 reveal the damping of the energy exchange between three subsystems, but Figure 2 shows how much force can affect the energy exchange subjected to the initial conditions. From Figure 3 it is seen the viscosity influence on vibrations amplitudes at different levels of exciting force amplitude.



**Figure 1.** Free vibrations (a); forced vibrations (b), when  $a_{i0} = 0.5$  are initial amplitudes,  $\nu = 0.27$ ,  $\hat{f} = 0.15$ ,  $\beta_1 = 1.77$  and  $\beta_2 = 0.119$ : solid line— $a_3(T_2)$ ; dotted line— $a_2(T_2)$ ; dashed line— $a_1(T_2)$ .



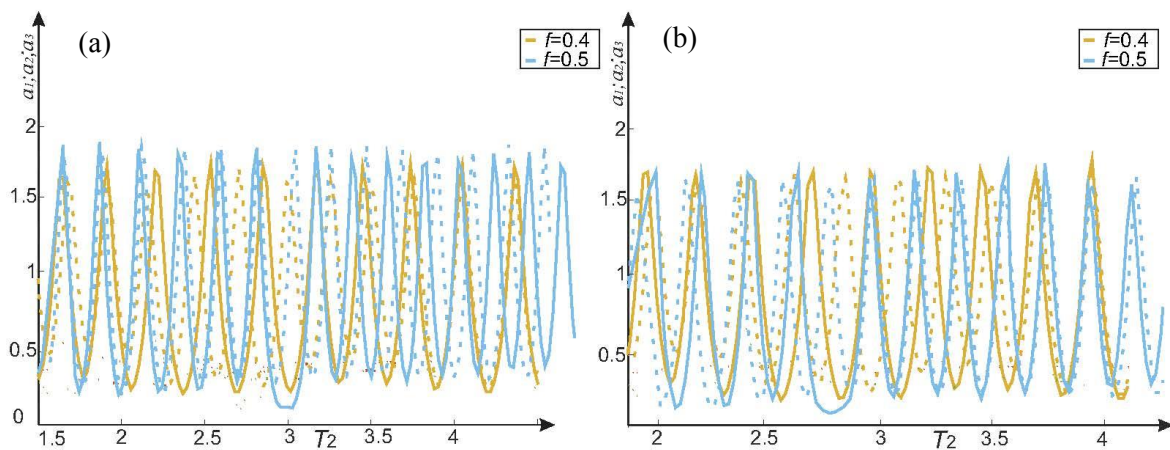
**Figure 2.** Free vibrations (a); forced vibrations (b), when  $a_{10} = 0.352$ ;  $a_{20} = 0.594$ ;  $a_{30} = 0.216$ ;

$$\zeta_1 = 165.25, \zeta_2 = -182.61, \zeta_{13} = 165.26, \zeta_{23} = -165.85, \nu = 0.3, \varphi_{10} = -\frac{\pi}{2}; \varphi_{20} = \frac{\pi}{2}; \varphi_{30} = -\frac{\pi}{2};$$

$$\hat{f} = 0.4, \beta_1 = 4, \text{ and } \beta_2 = 0.176.$$

#### 4. Conclusion

This paper describes the nonlinear force driven vibrations of thin plates in a viscoelastic medium, when the motion of the plate is described by a set of three coupled nonlinear differential equations subjected to the condition of the combinational resonance accompanied by the external resonance, resulting in the interaction of three modes corresponding to the mutually orthogonal displacements. Nonlinear sets of resolving equations in terms of amplitudes and phase differences were obtained. The influence of viscosity on the energy exchange mechanism was analyzed.



**Figure 3.** Forced vibrations at (a)  $\alpha_i = 0.15$  and (b) at  $\alpha_i = 0.25$  : solid line –  $\gamma = 0$  ; dashed –  $\gamma = 0.25$  .

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**Reference**

- [1] Amabili M 2016 *J. Sound Vibration* **362** 142–56
- [2] Aboudi J, Cederbaum G and Elishakoff I 1990 *J. Sound Vibration* **139** 459–67
- [3] Xia Z Q and Lukasiewicz S 1995 *Nonlinear Dynamics* **8** 417–33
- [4] Xia Z Q and Lukasiewicz S 1994 *J. Sound Vibration* **175** 219–32
- [5] Nayfeh A H 2000 *Nonlinear Interaction: Analytical, Computational, and Experimental Methods* (New York: Wiley)
- [6] Ribeiro P and Petyt M 2000 *International Journal of Non-Linear Mechanics* **35** 2 263–78
- [7] Chang S I, Bajaj A K and Krousgrill C M 1993 *Nonlinear Dynamics* **4** 5 433–60
- [8] Anlas G and Elbeyli O 2002 *Nonlinear Dynamics* **30** 1 1–28
- [9] Rossikhin Yu A and Shitikova M V 2003 *Materials Science Forum* **440–441** 29–36
- [10] Rossikhin Yu A Shitikova M V and Ngenzi J C 2015 *Shock and Vibration* **2015** 795606 1-28
- [11] Rossikhin Yu A, Shitikova M V and Shcheglova T A 2009 *J. Mech. Mater. Struct.* **4** 1619–36
- [12] Rossikhin Yu A and Shitikova M V 2013 *Journal of Mechanics* **7** 3 155–63
- [13] Rossikhin Yu A and Shitikova M V 2010 *Applied Mechanics Reviews* **63** 010801 1-52