Modelling uniform asymptotics of the filtration problem in a porous medium

L I Kuzmina[1], **Y V Osipov**[2] **and Y G Zheglova**[2]

[1] National Research University Higher School of Economics, 20, Myasnitskaya st., Moscow, 101000, Russia
[2] Moscow State University of Civil Engineering (National Research University), 26, Yaroslavskoe shosse, Moscow, 129337, Russia

yuri-osipov@mail.ru

Abstract. The solution of the filtration problem allows calculating the injection of the grout into a loose ground to create a solid foundation. The basic deep bed filtration model of a monodisperse suspension in a porous medium is considered. Based on the synthesis of the standard asymptotic solution for a short time and the asymptotics of the filtration problem for a long time, a uniform asymptotic solution is constructed. A numerical calculation shows that the asymptotics is close to the solution for any time.

1. Introduction

Before the construction of buildings and structures on loose grounds, it is necessary to create a reliable, solid foundation. Loose soils are a porous medium where a grout solution is pumped. The liquid solution fills the voids of the soil and strengthens it. Such methods are also used in the construction of underground storage facilities for toxic and radioactive wastes [1].

Filtration problems describe the transport and deposition of suspensions solid particles and colloids in a porous medium. The deposit formation depends on the chemical composition of the solid particles, fluid carrier and the frame of the porous medium. There are various filtration models that take into account the electrostatic and gravitational forces, diffusion and viscosity of the fluid [2–4]. If the particle and pore size distributions overlap, the deposit formation is determined by the size-exclusion mechanism of particle capture: solid suspended particles pass freely through pores larger than the particle diameter and get stuck in the throat of pores smaller than the particle size. It is assumed that one particle completely blocks one pore, and each pore can retain only one particle. The retained particle cannot be knocked out of the pore by other particles or a fluid flow, and permanently blocks the pore [5].

For some mathematical filtration models, exact solutions are obtained [6–8]. If the model is too complex and an exact solution is unknown, an asymptotics is constructed [9–12]. As a rule, asymptotic solutions are applicable only in a limited domain and do not determine the filtration process simultaneously at the initial moment and for a large time.

A uniform asymptotics of the one-dimensional filtration problem is constructed and shown in this paper. For a small blocking filtration coefficient, a standard asymptotics with respect to a small parameter is constructed, applicable for a limited time. For a large time, the asymptotic solution was obtained in [13] in the form of decreasing exponents series. Asymptotics uniform in time is constructed by re-expansion of the standard asymptotic terms in decreasing exponents.

In Section 2 the basic mathematical model of filtration, the governing equations and the properties of solutions are considered. Asymptotic solutions for a limited time and for a large time are constructed in Section 3. Section 4 is devoted to the construction of the uniform asymptotics. The results of numerical simulation are given in Section 5. The conclusion in Section 6 finalizes the paper.

2. Basic filtration model

In the domain $\Omega = \{(x,t): 0 < x < 1,\, t > 0\}$ it considers the one-dimensional filtration problem of a suspension in a homogeneous porous medium. The concentrations $C(x,t)$ of suspended particles and $S(x,t)$ retained particles satisfy the mass balance equation

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \frac{\partial S}{\partial t} = 0\,, \tag{1}$$

$$\frac{\partial S}{\partial t} = \varepsilon \Lambda(S) C\,. \tag{2}$$

Here ε is a small positive parameter.

For the uniqueness of the solution of the equations system (1), (2), the boundary and initial conditions are set

$$C\big|_{x=0} = 1\,; \tag{3}$$

$$C\big|_{t=0} = 0;\; S\big|_{t=0} = 0\,. \tag{4}$$

It is assumed that the blocking filtration coefficient $\Lambda(S)$ has a simple root

$$\Lambda(S) = (a - S)\lambda(S), \quad a > 0\,, \tag{5}$$

where the function $\lambda(S)$ is positive on the interval $[0; a]$ and can be written in the series form

$$\lambda(S) = \lambda_0 + \lambda_1 S + \lambda_2 S^2 + \ldots. \tag{6}$$

Conditions (3) and (4) mean that a suspension of constant concentration is injected at the porous medium inlet $x = 0$, and at the initial time $t = 0$ the porous medium does not contain any suspended and retained particles. So conditions (3) and (4) do not match at the origin, the problem solution (1) - (4) is discontinuous. The mobile boundary of the two phases, the concentrations front $t = x$ of the suspended and retained particles moves in a porous medium with a velocity $v = 1$ from the inlet to the outlet. The porous medium is empty before the concentrations front comes; suspended and retained particles are present behind the front.

Before the concentrations front comes in the domain $\Omega_0 = \{(x,t): 0 < x < 1,\, 0 < t < x\}$ the solution is zero; behind the front in the domain $\Omega_S = \{(x,t): 0 < x < 1,\, t > x\}$ the solution is positive.

On the front $t = x$, the solution $C(x,t)$ has a gap; the solution $S(x,t)$ is continuous in Ω. The continuity of the solution $S(x,t)$ on the concentrations front $t = x$ yields

$$S\big|_{t=x} = 0\,. \tag{7}$$

In the domain Ω_S, the solution of problem (1) - (3), (7) coincides with the solution (1) - (4).

In the characteristic variables $\tau = t - x,\; x = x$ in the domain $\Omega_S^\tau = \{(x,t): 0 < x < 1,\, \tau > 0\}$ the problem (1) - (3), (7) takes the form

$$\frac{\partial C}{\partial x}+\frac{\partial S}{\partial \tau}=0\,; \tag{8}$$

$$\frac{\partial S}{\partial \tau}=\varepsilon\Lambda(S)C\,; \tag{9}$$

$$C\big|_{x=0}=1;\quad S\big|_{\tau=0}=0\,. \tag{10}$$

3. Local asymptotic solutions

3.1. Asymptotics for small ε

An asymptotic solution is constructed in the series form in powers of the small parameter ε

$$C(x,\tau)=1+\varepsilon c_1+\varepsilon^2 c_2+...;\quad S(x,\tau)=\varepsilon s_1+\varepsilon^2 s_2+...\,. \tag{11}$$

Substitution of the expansions (11) into equations (8) and (9) and equating the coefficients for the same powers of ε yields a system of recurrent ordinary differential equations of the first order

$$\frac{\partial c_1}{\partial x}+\frac{\partial s_1}{\partial \tau}=0;\quad \frac{\partial s_1}{\partial \tau}=a\lambda_0;\quad \frac{\partial c_2}{\partial x}+\frac{\partial s_2}{\partial \tau}=0;\quad \frac{\partial s_2}{\partial \tau}=(a\lambda_1-\lambda_0)s_1+a\lambda_0 c_1\,. \tag{12}$$

From (10) the conditions for equations (12) are obtained

$$c_i\big|_{x=0}=0;\quad s_i\big|_{\tau=0}=0;\quad i=1,2\,. \tag{13}$$

The solution of the system (12), (13)

$$s_1=a\lambda_0\tau;\quad c_1=-a\lambda_0 x;\quad s_2=(a\lambda_1-\lambda_0)a\lambda_0\frac{\tau^2}{2}-a^2\lambda_0^2 x\tau;\quad c_2=-(a\lambda_1-\lambda_0)a\lambda_0\tau x+a^2\lambda_0^2\frac{x^2}{2}\,. \tag{14}$$

Substitution of the solution (14) into (11) yields

$$C_{\text{stand}}=1-\varepsilon a\lambda_0 x-\varepsilon^2\left((a\lambda_1-\lambda_0)a\lambda_0 x\tau-a^2\lambda_0^2\frac{x^2}{2}\right)+O(\varepsilon^3)\,; \tag{15}$$

$$S_{\text{stand}}=\varepsilon a\lambda_0\tau+\varepsilon^2\left((a\lambda_1-\lambda_0)a\lambda_0\frac{\tau^2}{2}-a^2\lambda_0^2 x\tau\right)+O(\varepsilon^3)\,. \tag{16}$$

In the variables x, t the standard asymptotics has the form

$$C_{\text{stand}}(x,t)=1-\varepsilon a\lambda_0 x-\varepsilon^2\left((a\lambda_1-\lambda_0)a\lambda_0 x(t-x)-a^2\lambda_0^2\frac{x^2}{2}\right)+O(\varepsilon^3)\,. \tag{17}$$

$$S_{\text{stand}}(x,t)=\varepsilon a\lambda_0(t-x)+\varepsilon^2\left((a\lambda_1-\lambda_0)a\lambda_0\frac{(t-x)^2}{2}-a^2\lambda_0^2 x(t-x)\right)+O(\varepsilon^3)\,. \tag{18}$$

3.2. Asymptotics for a large time

For a linear filtration coefficient $\Lambda(S)=\varepsilon\lambda_0(a-S),\ \ a>0,\ \lambda_0>0$ in the domain Ω_S^τ the problem (8) – (10) has the exact solution [14]

$$C=\frac{e^{\varepsilon\lambda_0\tau}}{e^{\varepsilon\lambda_0\tau}+e^{\varepsilon\lambda_0 ax}-1};\quad S=a\frac{e^{\varepsilon\lambda_0\tau}-1}{e^{\varepsilon\lambda_0\tau}+e^{\varepsilon\lambda_0 ax}-1}\,. \tag{19}$$

For large τ the asymptotics of the solution (19) has the form

$$C_\infty = 1-(e^{\varepsilon\lambda_0 ax}-1)e^{-\varepsilon\lambda_0\tau}+(e^{\varepsilon\lambda_0 ax}-1)^2 e^{-2\varepsilon\lambda_0\tau}+O(e^{-3\varepsilon\lambda_0\tau}); \tag{20}$$

$$S_\infty = a\left(1-e^{\varepsilon\lambda_0 ax}e^{-\varepsilon\lambda_0\tau}+e^{\varepsilon\lambda_0 ax}(e^{\varepsilon\lambda_0 ax}-1)e^{-2\varepsilon\lambda_0\tau}\right)+O(e^{-3\varepsilon\lambda_0\tau}). \tag{21}$$

In the general case, the asymptotic solution for the blocking filtration coefficient (5), (6) with a simple root $S=a$ is analogous to the expansions (20), (21) and can be written as a series in powers of the decreasing exponent $e^{-\varepsilon\lambda_0\tau}$ [13].

4. Uniform asymptotic solution

In order to construct an asymptotic solution that is uniform with respect to time, the constant and the powers τ, τ^2 of the standard asymptotics (15), (16) are replaced by linear combinations of exponents $e^{-\varepsilon\lambda_0\tau}$, $e^{-2\varepsilon\lambda_0\tau}$, $e^{-3\varepsilon\lambda_0\tau}$ with an accuracy $O(\varepsilon^3)$ so the limiting relations (8) are satisfied. Substitution of the asymptotic formulas

$$1=e^{-\varepsilon\lambda_0\tau}+O(\varepsilon);\; 1=2e^{-\varepsilon\lambda_0\tau}-e^{-2\varepsilon\lambda_0\tau}+O(\varepsilon^2);\; \varepsilon\lambda_0\tau=(e^{-\varepsilon\lambda_0\tau}-e^{-2\varepsilon\lambda_0\tau})+O(\varepsilon^2);$$

$$\varepsilon\lambda_0\tau=\left(1-\frac{1}{2}e^{-\varepsilon\lambda_0\tau}-e^{-2\varepsilon\lambda_0\tau}+\frac{1}{2}e^{-3\varepsilon\lambda_0\tau}\right)+O(\varepsilon^3);\; (\varepsilon\lambda_0)^2\tau^2=(e^{-\varepsilon\lambda_0\tau}-2e^{-2\varepsilon\lambda_0\tau}+e^{-3\varepsilon\lambda_0\tau})+O(\varepsilon^3);$$

in the terms of the standard asymptotics (15), (16) gives

$$C_{\text{unif}} = 1-\varepsilon ax\left((\lambda_0+a\lambda_1)e^{-\varepsilon\lambda_0\tau}-a\lambda_1 e^{-2\varepsilon\lambda_0\tau}\right)+\varepsilon^2 a^2\lambda_0^2\frac{x^2}{2}e^{-\varepsilon\lambda_0\tau}+O(\varepsilon^3); \tag{22}$$

$$S_{\text{unif}} = a\left(1-e^{-\varepsilon\lambda_0\tau}\right)+\frac{a^2\lambda_1}{2\lambda_0}\left(e^{-\varepsilon\lambda_0\tau}-2e^{-2\varepsilon\lambda_0\tau}+e^{-3\varepsilon\lambda_0\tau}\right)-\varepsilon a^2\lambda_0 x\left(e^{-\varepsilon\lambda_0\tau}-e^{-2\varepsilon\lambda_0\tau}\right)+O(\varepsilon^3). \tag{23}$$

In the variables x, t, the uniform asymptotics has the form

$$C_{\text{unif}}(x,t)=1-\varepsilon ax\left((\lambda_0+a\lambda_1)e^{-\varepsilon\lambda_0(t-x)}-a\lambda_1 e^{-2\varepsilon\lambda_0(t-x)}\right)+\varepsilon^2 a^2\lambda_0^2\frac{x^2}{2}e^{-\varepsilon\lambda_0(t-x)}+O(\varepsilon^3); \tag{24}$$

$$\begin{aligned} S_{\text{unif}}(x,t) &= a\left(1-e^{-\varepsilon\lambda_0(t-x)}\right)+\frac{a^2\lambda_1}{2\lambda_0}\left(e^{-\varepsilon\lambda_0(t-x)}-2e^{-2\varepsilon\lambda_0(t-x)}+e^{-3\varepsilon\lambda_0(t-x)}\right)- \\ &-\varepsilon a^2\lambda_0 x\left(e^{-\varepsilon\lambda_0(t-x)}-e^{-2\varepsilon\lambda_0(t-x)}\right)+O(\varepsilon^3). \end{aligned} \tag{25}$$

The uniform asymptotic solution (24), (25) coincides with the standard asymptotics (17), (18) with accuracy $O(\varepsilon^3)$ for small time.

5. Numerical modeling

Consider the quadratic filtration coefficient

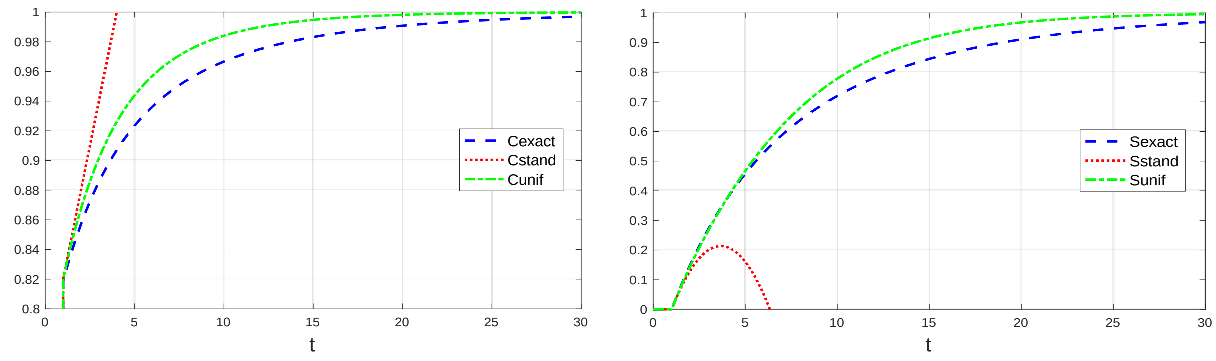
$$\Lambda(S)=\varepsilon(1-S)(2-S). \tag{26}$$

The exact solution of the problem (1) - (4) in the domain Ω_S has the form [15]

$$S_{\text{exact}}(x,t)=1-\frac{1}{D};\; C_{\text{exact}}(x,t)=\frac{2e^{-2\varepsilon x}(2e^{2\varepsilon(t-x)}-e^{\varepsilon(t-x)})}{D^2+D};\; D=\sqrt{1+4e^{-2\varepsilon x}e^{\varepsilon(t-x)}(e^{\varepsilon(t-x)}-1)}. \tag{27}$$

The calculation is performed for the parameters $\varepsilon=0.1;\; a=1;\; \lambda_0=2;\; \lambda_1=-1$.

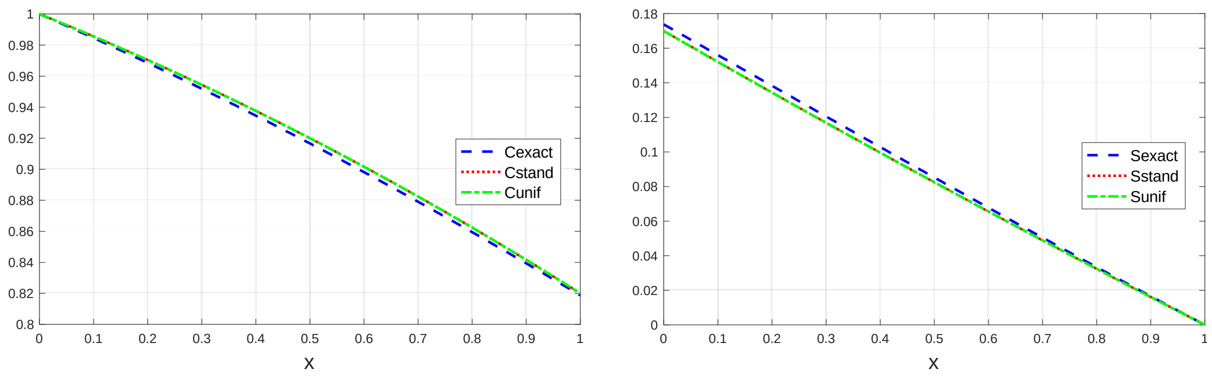
In Figure 1 (a), (b) graphs of suspended and retained particles concentrations at the porous medium outlet $x=1$ are presented: exact solution, standard and uniform asymptotics.



(a) (b)

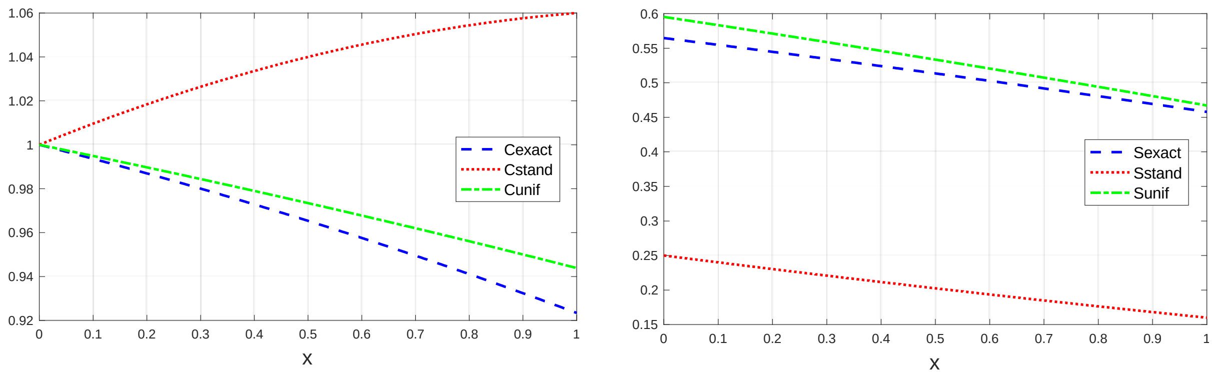
Figure 1. Outlet concentrations $C(x,t)\big|_{x=1}$ (a); $S(x,t)\big|_{x=1}$ (b).

Figures 2, 3 show the suspended and retained particles concentrations at fixed time $t=1$ and $t=5$.



(a) (b)

Figure 2. Suspended concentration $C(x,t)\big|_{t=1}$ (a); Retained concentration $S(x,t)\big|_{t=1}$ (b).



(a) (b)

Figure 3. Suspended concentration $C(x,t)\big|_{t=5}$ (a); Retained concentration $S(x,t)\big|_{t=5}$ (b).

Figures 1–3 show that at the outlet $x = 1$ of the porous medium, the maximum deviation of the uniform asymptotics from the exact solution is 2% for $C(x,t)|_{x=1}$ and 8% for $S(x,t)|_{x=1}$.

6. Conclusion

The uniform asymptotics of the one-dimensional filtration problem of a monodisperse suspension in a porous medium is constructed. In order to obtain the new asymptotics, the standard asymptotic solution for a small time and the asymptotics of the filtration problem in an almost stationary mode for large time were used. The standard asymptotics is transformed into a series of decreasing exponents - the asymptotic terms for a large time. It is shown that the new asymptotics is close to an exact solution for all values of time.

The greatest deviation of the new asymptotics from the exact solution is observed for "average" time values in the range from 5 to 20. To reduce the error, it is necessary to increase the terms number of the standard asymptotics for a short time and to increase the accuracy with substituting the decreasing exponents.

A uniform in time asymptotic solution makes it possible to cover all stages of the filtration process from an intense start at the initial moment to damping for a large time. The use of analytic solutions substantially reduces the amount and cost of laboratory research and field experiments [16].

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