

About choosing the form of perturbed body motion differential equations system

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Abstract. The article deals with determination of the second- and higher-order perturbations in Cartesian coordinates and body motion velocity constituents. A special perturbed motion differential equations system is constructed. The right-hand sides of this system are finite polynomials relative to an independent regularizing variable. This allows constructing a single algorithm to determine the second and higher order perturbations in the form of finite polynomials relative to some regularizing variables that are chosen at each approximation step. Following the calculations results with the use of the developed method, the coefficients of approximating polynomials representing rectangular coordinates and components of the regularized body speed were obtained. Comparison with the results of numerical integration of the equations of disturbed motion shows close agreement of the results. The developed methods make it possible to calculate, by the approximating polynomials, any intermediate point of the motion trajectory of the body.

1. Introduction

One of the crucial tasks associated with trajectory measurements is the determination of the partial derivatives of rectangular coordinates that make up the body motion speed with respect to the initial conditions. W. Goodyear in operation [1] added auxiliary functions, which are degree series with respect to the auxiliary variable. S. Herrick in operation [2] outlined ways of using universal variables in a number of tasks of mechanics to determine disturbances by the method of variation of arbitrary constants. In this case, it is convenient to consider the components of the initial values of the radius-vector and velocity as osculating variables. New methods for determining disturbances keep the standard features of the classical ones, while calculating the disturbances, the small parameter method is used, which makes it possible to obtain asymptotic decomposition of the solution. Recently, Picard method for integration of differential equations is more commonly used, which leads to a convergent process of successive approximations that gives a solution to a system of differential equations.

The error of the solution depends on the accuracy of the initial approximation of the perturbation function. General principles of the development of perturbation theory in coordinates were analyzed by R. Broucke in operation [3]. U. Szebehely, D. Pierce, S. Standish studied the use of regularizing variables for calculation of trajectories of motion. in operation [4]. The results of this research show that the use of regularizing variables increases the computer-based accuracy of calculations and significantly reduces the calculation time.



A crucial task of mechanics is to approximate the rectangular coordinates that make up the body speed and time in case of disturbed motion by algebraic polynomials of the lowest degree with respect to the auxiliary variable with a predetermined degree of accuracy.

One of the important problems in mechanics is the approximation of rectangular coordinates constituting the body velocity and time when the motion is perturbed by the lowest degree algebraic polynomials relative to the auxiliary variable with a predetermined degree of accuracy.

This paper describes a special system of differential equations of the perturbed moving body and this system is integrated through successive approximations method, which using the coordinates and constituents body velocity, take the form of polynomials in powers of some auxiliary variable. Its own independent variable is taken at each approximation step.

2. Problem specification and decision

An important problem of mechanics is the approximation, of rectangular coordinates body speed and time under perturbed motion by algebraic polynomials the lowest degree with respect to the auxiliary variable with a preassigned degree of accuracy.

The problem is set as follows. Spatial, almost parabolic, geocentric motion of a body is considered by a material point. The problem is considered as part of a limited three-body problem. The paper constructs an algorithm for calculating perturbations of any order in rectangular coordinates constituting body motion velocity, time and implements it on a computer to get the results in the form of corresponding polynomial coefficients. For the first time, the above functions are represented by finite polynomials at various stages of approximation with an accuracy high degree. An unperturbed parabolic or almost parabolic orbit is chosen as the initial approximation. The system of unperturbed two body problem regularized equations is chosen as the initial equations system [5, p. 36]:

$$\begin{aligned} x'' &= 2hx - f_1; \quad x''' = 2hx'; \quad y'' = 2hy - f_2; \quad y''' = 2hy'; \quad z'' = 2hz - f_3; \quad z''' = 2hz'; \\ r'' &= 2hz + \mu; \quad t' = \frac{dt}{d\psi} = r; \quad h = \frac{\xi^{12} + \eta^{12} + \zeta^{12}}{2\rho^2} - \frac{\mu}{\rho}; \quad \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2} \end{aligned} \quad (1)$$

where the μ is the product of gravitational constant by the Earth mass, h – energy integral constant,

f_1, f_2, f_3 are the Laplace integrals constants, ψ is the regularizing variable, ξ, η, ζ are the initial values of Cartesian coordinates.

Regularization is made to simplify the right-hand parts of spacecraft motion differential equations system and solution analytical properties. Let's set the Cauchy problem for the system (1). We need to find a solution of this system for the coordinates initial values ξ, η, ζ constituting the regularized velocity ξ', η', ζ' and the initial instant of time τ for a given value of the independent regularizing variable ψ . This value is chosen arbitrarily and considered a zero. We will find the general solution of the system (1) in the series form in powers of the following regularizing variable ψ .

$$x = \xi + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \xi^{(k)} \cdot \psi^k \quad (2)$$

$$x' = \xi' + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \xi^{(k+1)} \cdot \psi^k \quad (3)$$

$$x'' = \xi'' + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \xi^{(k+2)} \cdot \psi^k \quad (4)$$

Let's substitute the expressions (2), (4) in the first equation (1). In the left and right sides of the equation (1) we obtain series in powers of ψ . By equating the coefficients in both sides of the equation (1) with the same powers ψ , we obtain

$$\xi'' = 2 \cdot h \cdot \xi - f_1; \xi''' = 2 \cdot h \cdot \xi'; \xi^{IV} = 2 \cdot h \cdot \xi'', \dots, \xi^{(k+2)} = 2 \cdot h \cdot \xi^{(k)}, \text{ if } k=1,2,\dots \quad (5)$$

Here, it follows, that even coefficients are recurrently expressed in terms of even ones, odd coefficients – in terms of odd ones. Therefore, we can enter the following functions:

$$S_1 = \frac{\psi}{1!} + \frac{\sigma\psi^3}{3!} + \frac{\sigma^2\psi^5}{5!} + \dots$$

$$S_k = \frac{\psi^k}{k!} + \frac{\sigma \cdot \psi^{k+2}}{(k+2)!} + \dots = \sum_{i=0}^{\infty} \frac{\sigma^i \psi^{k+2i}}{(k+2i)!}, \quad (6)$$

$$\sigma = 2h \quad (7)$$

The functions S_k ($k=0,1,2,\dots$) are called generalized Stumpff functions. Please note that there are the following recurrence relations between functions S_k :

$$S_k = \frac{\psi^k}{k!} + \sigma \cdot S_{k+2}; \quad S'_k = S_{k-1}; \quad S_{k+1} = \int_0^\psi S_k d\psi \quad (k=2,3,\dots) \quad (8)$$

Thus, the expression (2) can be represented in the following form with the help of the functions S_k :

$$x = \xi + \xi' \cdot S_1 + \xi'' \cdot S_2 \quad (9)$$

The functions S_k are integers, they converge with absolutely any values s and ψ .

The rectangular coordinates constituting the regularized velocity and time of unperturbed motion in two-body problem are represented by convergent series in powers of some regularizing variable ψ for all initial values of the coordinates constituting the regularized velocity and time if the initial value of vector radius ρ is different from zero. General solution of the system (1) is written as:

$$\begin{aligned} x &= \xi + \xi'' \cdot S_1 + \xi''' \cdot S_2 & x' &= \xi' + \xi'' \cdot S_1 + \xi''' \cdot S_2 = \xi' \cdot S_0 + \xi'' \cdot S_1 & y &= \eta + \eta' \cdot S_1 + \eta'' \cdot S_2 \\ y' &= \eta' \cdot S_0 + \eta'' \cdot S_1 & z &= \zeta + \zeta' \cdot S_1 + \zeta'' \cdot S_2 & z' &= \zeta' \cdot S_0 + \zeta'' \cdot S_1 & r &= \rho + \rho' \cdot S_1 + \rho'' \cdot S_2 \\ t &= \tau + \rho \cdot \psi + \rho' \cdot S_2 + \rho'' \cdot S_3 \end{aligned} \quad (10)$$

Initial values of the regularized velocity ξ', η', ζ' are related to the normal velocity initial values $\dot{\xi}, \dot{\eta}, \dot{\zeta}$ as follows:

$$\xi' = \dot{\xi} \cdot \rho, \quad \eta' = \dot{\eta} \cdot \rho, \quad \zeta' = \dot{\zeta} \cdot \rho \quad (11)$$

Unperturbed motion differential equations system for the two-body problem can be written as follows:

$$x'' = \frac{r' \cdot x'}{r} - \frac{\mu \cdot x}{r}, \quad y'' = \frac{r' \cdot y'}{r} - \frac{\mu \cdot y}{r}, \quad z'' = \frac{r' \cdot z'}{r} - \frac{\mu \cdot z}{r} \quad (12)$$

equivalent to the first three equations system (1). Using the equations (12), we obtain the regularized acceleration initial values:

$$\xi'' = \frac{\rho' \cdot \xi'}{\rho} - \frac{\mu \cdot \xi}{\rho}, \quad \eta'' = \frac{\rho' \cdot \eta'}{\rho} - \frac{\mu \cdot \eta}{\rho}, \quad \zeta'' = \frac{\rho' \cdot \zeta'}{\rho} - \frac{\mu \cdot \zeta}{\rho} \quad (13)$$

Since the right-hand sides of the initial regularized equations (1) do not explicitly depend on the independent variable ψ , we can interchange the initial and current values, and change the sign of the variable ψ in any integral connecting the current and initial values of rectangular coordinates constituting velocity, time and variable ψ i.e. we make substitution:

$$\xi^{(k)}, \eta^{(k)}, \zeta^{(k)}, \tau, \psi \leftrightarrow x^{(k)}, y^{(k)}, z^{(k)}, t, -\psi \quad (14)$$

Given this substitution, we obtain the following equations connecting $\xi, \eta, \zeta, \xi', \eta', \zeta', \xi'', \eta'', \zeta''$ with $x, y, z, x', y', z', x'', y'', z''$

$$\begin{aligned}
\xi &= x - x' \cdot S_1 + x'' \cdot S_2, \quad \xi' = x' \cdot S_0 - x'' \cdot S_1 \\
\eta &= y - y' \cdot S_1 + y'' \cdot S_2, \quad \eta' = y' \cdot S_0 - y'' \cdot S_1 \\
\varsigma &= z - z' \cdot S_1 + z'' \cdot S_2, \quad \varsigma' = z' \cdot S_0 - z'' \cdot S_1 \\
\xi'' &= x'' \cdot S_0 - x' \cdot \sigma \cdot S_1, \quad \rho = r - r' \cdot S_1 + r'' \cdot S_2 \\
\eta'' &= y'' \cdot S_0 - y' \cdot \sigma \cdot S_1, \quad \rho' = r' \cdot S_0 + r'' \cdot S_1 \\
\varsigma' &= z'' \cdot S_0 - z' \cdot \sigma \cdot S_1, \quad \rho'' = \sigma \cdot \rho + \mu = r'' \cdot S_0 - r' \cdot \sigma \cdot S_1 \\
\tau &= t - r \cdot \psi + r' \cdot S_2 - r'' \cdot S_3
\end{aligned} \tag{15}$$

The equations (15) are the first integrals of unperturbed two-body problem regularized equations.

The perturbed motion regularized equations are written as:

$$\begin{aligned}
x'' &= \frac{x' \cdot r' - \mu \cdot x}{r} + r^2 \cdot X; \quad y'' = \frac{y' \cdot r' - \mu \cdot y}{r} + r^2 \cdot Y; \\
z'' &= \frac{z' \cdot r' - \mu \cdot z}{r} + r^2 \cdot Z
\end{aligned} \tag{16}$$

where X, Y, Z are the constituents of perturbing acceleration:

$$X = \mu_1 \cdot \left(\frac{x_1 - x}{\Delta_1^3} - \frac{x_1}{r_1^3} \right); \quad Y = \mu_1 \cdot \left(\frac{y_1 - y}{\Delta_1^3} - \frac{y_1}{r_1^3} \right); \quad Z = \mu_1 \cdot \left(\frac{z_1 - z}{\Delta_1^3} - \frac{z_1}{r_1^3} \right) \tag{17}$$

x_1, y_1, z_1 are the coordinates of the disturbing body, Δ_1 —mutual distance between the body and the perturbing body

$$\Delta_1 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} \tag{18}$$

r_1 is the geocentric distance to perturbing body.

Let's apply the main operation to the regularized equations (15). We consider the first group of equations (15). If the ξ is a constant of the unperturbed motion equations first integral, then

$$\frac{d\xi}{d\psi} = \frac{\partial \xi}{\partial t} \cdot t' + \frac{\partial \xi}{\partial x} \cdot x' + \frac{\partial \xi}{\partial y} \cdot y' + \frac{\partial \xi}{\partial z} \cdot z' + \frac{\partial \xi}{\partial x'} \cdot x'' + \frac{\partial \xi}{\partial y'} \cdot y'' + \frac{\partial \xi}{\partial z'} \cdot z'' = 0 \tag{19}$$

Let's substitute x'', y'', z'' for the expressions (16), then

$$\frac{d\xi}{d\psi} = r^2 \cdot \frac{\partial \xi}{\partial x'} \cdot X + r^2 \cdot \frac{\partial \xi}{\partial y'} \cdot Y + r^2 \cdot \frac{\partial \xi}{\partial z'} \cdot Z \tag{20}$$

To determine $\frac{d\xi}{d\psi}$, we need to find the derivatives $\frac{\partial \xi}{\partial x'}, \frac{\partial \xi}{\partial y'}, \frac{\partial \xi}{\partial z'}$

Given that x'', y'', z'' are determined by the unperturbed motion formulas, let's write the following correlations.

$$x'' = \frac{x' \cdot (x \cdot x' + y \cdot y' + z \cdot z')}{r^2} - \frac{\mu x}{r}; \quad r'' = \sigma r + \mu; \quad \sigma = \frac{(x')^2 + (y')^2 + (z')^2}{r^2} - \frac{2\mu}{r} \tag{21}$$

These equations result in:

$$\frac{\partial x''}{\partial x'} = \frac{x \cdot x' + y \cdot y' + z \cdot z'}{r^2} + \frac{x \cdot x'}{r^2} = \frac{r \cdot r' + x \cdot x'}{r^2}; \quad \frac{\partial x''}{\partial y'} = \frac{x' \cdot y}{r^2}; \quad \frac{\partial x''}{\partial z'} = \frac{x' \cdot z}{r^2}$$

$$\frac{\partial r''}{\partial x'} = r \frac{\partial \sigma}{\partial x'} = \frac{2x'}{r}; \quad \frac{\partial r''}{\partial y'} = \frac{2 \cdot y'}{r}; \quad \frac{\partial r''}{\partial z'} = \frac{2 \cdot z'}{r}; \quad \frac{\partial \sigma}{\partial x'} = \frac{2 \cdot x'}{r^2}; \quad \frac{\partial \sigma}{\partial y'} = \frac{2 \cdot y'}{r^2}; \quad \frac{\partial \sigma}{\partial z'} = \frac{2 \cdot z'}{r^2} \quad (22)$$

Using the correlations (22), we obtain an equation to determine the following variable σ

$$\frac{d\sigma}{d\psi} = r^2 \cdot \left(\frac{\partial \sigma}{\partial x'} \cdot X + \frac{\partial \sigma}{\partial y'} \cdot Y + \frac{\partial \sigma}{\partial z'} \cdot Z \right) = 2(x' \cdot X + y' \cdot Y + z' \cdot Z) = 2R' \quad (23)$$

where: $R' = x' \cdot X + y' \cdot Y + z' \cdot Z$; $R = x \cdot X + y \cdot Y + z \cdot Z$

Let's define the partial derivatives $\frac{\partial \xi}{\partial x'}, \frac{\partial \xi}{\partial y'}, \frac{\partial \xi}{\partial z'}$ using the first equation (15), equation (23) and

dependence of S_k on σ .

$$\begin{aligned} r^2 \frac{\partial \xi}{\partial x'} &= r^2 \cdot \left(-S_1 + S_2 \cdot \frac{\partial x''}{\partial x'} - x' \cdot \frac{\partial S_1}{\partial x'} + x'' \cdot \frac{\partial S_2}{\partial x'} \right) = -r^2 \cdot S_1 + S_2 (r \cdot r' + x \cdot x') - \\ &- x' \frac{\partial S_1}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial x'} + x'' \frac{\partial S_2}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial x'} = -r^2 \cdot S_1 + S_2 (r \cdot r' + x \cdot x') - x' \cdot \frac{\partial S_1}{\partial \sigma} \cdot 2x' + \\ &+ x'' \cdot 2x' \cdot \frac{\partial S_2}{\partial \sigma} = r(r' \cdot S_2 - r \cdot S_1) + x' \left(x \cdot S_2 - 2x' \cdot \frac{\partial S_1}{\partial \sigma} + 2x'' \cdot \frac{\partial S_2}{\partial \sigma} \right); \end{aligned} \quad (24)$$

$$r^2 \frac{\partial \xi}{\partial y'} = r^2 \cdot \left(S_2 \cdot \frac{\partial x''}{\partial y'} + x'' \cdot \frac{\partial S_2}{\partial y'} - x' \cdot \frac{\partial S_1}{\partial y'} \right) = x'y \cdot S_2 + \left(2x'' \cdot \frac{\partial S_2}{\partial \sigma} - 2x' \cdot \frac{\partial S_1}{\partial \sigma} \right) \cdot y' ; \quad (25)$$

$$r^2 \frac{\partial \xi}{\partial z'} = r^2 \cdot \left(S_2 \cdot \frac{\partial x''}{\partial z'} + x'' \cdot \frac{\partial S_2}{\partial z'} - x' \cdot \frac{\partial S_1}{\partial z'} \right) = x'z \cdot S_2 + \left(2x'' \cdot \frac{\partial S_2}{\partial \sigma} - 2x' \cdot \frac{\partial S_1}{\partial \sigma} \right) \cdot z'; \quad (26)$$

The Stumpff functions S_k ($k=0,1, 2, \dots$) have different analytic representations depending on the sign of constant energy integral h . If $h>0$, then the orbit of two bodies is a hyperbola and the functions S_k are as follows:

$$S_0 = chH, \quad S_1 = \frac{shH}{\sqrt{\sigma}}, \quad S_2 = \frac{chH-1}{\sigma}, \quad S_3 = \frac{shH}{\sigma\sqrt{\sigma}}, \dots \quad H = \sqrt{\sigma} \cdot \psi \quad (27)$$

If $h=0$, the orbit is a parabola and the functions S_k are represented with polynomials:

$$S_0 = 1, \quad S_1 = \psi, \quad S_2 = \frac{\psi^2}{2}, \quad S_3 = \frac{\psi^3}{6}, \dots \quad (28)$$

If $h<0$, the orbit is an ellipse and the functions S_k are as follows:

$$S_0 = \cos J, \quad S_1 = \frac{\sin J}{\sqrt{-\sigma}}, \quad S_2 = \frac{1-\cos J}{-\sigma}, \quad S_3 = \frac{J-\sin J}{-\sigma \cdot \sqrt{-\sigma}}, \dots \text{where } J = \sqrt{-\sigma} \cdot \psi \quad (29)$$

Let's define the partial derivatives $\frac{\partial S_k}{\partial \sigma}$. We differentiate the series for S_k (7) by σ resulting in:

$$2 \frac{\partial S_k}{\partial \sigma} = 2 \cdot \sum_{l=1}^{\infty} \frac{l \cdot \sigma^{l-1} \cdot \psi^{k+2l}}{(k+2l)!} = \psi \cdot \sum_{l=1}^{\infty} \frac{\sigma^{l-1} \cdot \psi^{k+2l-1}}{(k+2l-1)!} - k \cdot \sum_{l=1}^{\infty} \frac{\sigma^{l-1} \cdot \psi^{k+2l}}{(k+2l)!} ; \quad (30)$$

$$\begin{aligned}
& \sum_{l=1}^{\infty} \frac{2l \cdot \sigma^{l-1} \cdot \psi^{k+2l}}{(k+2l)!} + k \cdot \sum_{l=1}^{\infty} \frac{\sigma^{l-1} \cdot \psi^{k+2l}}{(k+2l)!} = \sum_{l=1}^{\infty} \frac{(2l+k) \cdot \sigma^{l-1} \cdot \psi^{k+2l}}{(k+2l)!} = \\
& = \sum_{l=1}^{\infty} \frac{(2l+k) \cdot \psi \cdot \sigma^{l-1} \cdot \psi^{k+2l-1}}{(k+2l-1)! \cdot (k+2l)} = \sum_{l=1}^{\infty} \frac{\psi^{k+2l} \cdot \sigma^{l-1}}{(k+2l-1)!};
\end{aligned} \tag{31}$$

By substituting the summation variable in the expression (30) $l=m+1$, we obtain:

$$2 \frac{\partial S_k}{\partial \sigma} = \psi \cdot S_{k+1} - k \cdot S_{k+2}, \quad 2 \frac{\partial S_1}{\partial \sigma} = \psi \cdot S_2 - S_3, \quad 2 \frac{\partial S_2}{\partial \sigma} = \psi \cdot S_3 - 2S_4 \tag{32}$$

The parentheses in the expressions (24), (25), (26) can be transformed as follows, using the expressions for the functions S_k (28) and their partial derivatives by σ (32).

$$r' \cdot S_2 - r \cdot S_1 = (\rho' \cdot S_0 + \rho'' \cdot S_1) \cdot S_2 - (\rho + \rho' \cdot S_1 + \rho'' \cdot S_2) \cdot S_1 = -\rho \cdot S_1 + \rho' \cdot (S_0 \cdot S_2 - S_1^2) = -\rho \cdot S_1 + \rho' \cdot S_2; \tag{33}$$

$$-x' \cdot 2 \frac{\partial S_1}{\partial \sigma} + x'' \cdot 2 \frac{\partial S_2}{\partial \sigma} = -(\xi' \cdot S_0 + \xi'' \cdot S_1) \cdot (\psi \cdot S_2 - S_3) + (\xi'' \cdot S_0 + \xi' \cdot \sigma \cdot S_1) \cdot (\psi \cdot S_3 - 2S_4) = \tag{34}$$

$$= -S_2 \cdot (\xi' \cdot S_1 + \xi'' \cdot S_2) + S_3 \cdot \xi' = S_2 \cdot (\xi - x) + S_3 \cdot \xi'$$

3. Main results

By substituting the obtained correlations (33), (34) to the expressions (24), (25), (26), and the substitution results to the expression (20), we obtain equations for determination ξ, η, ζ . Thus, we have obtained the equations, which are the first integrals of the unperturbed two-body problem regularized equations. A special system of disturbed motion regularized equations is constructed based on these equations and a series of transformations [6 pp. 254–255 formulas (30–34)]. The required functions in the constructed system of disturbed motion differential equations are the osculating initial values of the rectangular coordinates ξ, η, ζ constituting the regularized velocity ξ', η', ζ' of the regularized acceleration ξ'', η'', ζ'' , osculating initial values of the time instant τ , radius of the vector ρ and its derivatives ρ' and ρ'' by independent regularizing variable ψ . Normalization of the regularizing variable ψ gives a representation of the said perturbations in the form of lowest degree finite polynomials.

The main paper conclusions are as follows:

- A special system of the perturbed body motion differential equations is constructed, which allows representing the rectangular coordinates constituting the regularized velocity and time of motion in the second and higher approximations by finite algebraic polynomials relative to a specially introduced regularizing variable, which is chosen at each approximation step.
- Possibility of representing the above functions by polynomials in powers of the regularizing variable is rigorously proven.

References

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