IOP Conf. Series: Materials Science and Engineering 456 (2018) 012116 doi:10.1088/1757-899X/456/1/012116

Nonlinear differential equations piecewise continuous linear approximation in computational structural analysis

R L Leibov

National Research Moscow State University of Civil Engineering, 26, Yaroslavskoe Highway, 129337, Moscow, Russia

r_leibov@mtu-net.ru

Abstract. This paper presents a piecewise-continuous approach to nonlinear differential equations system approximation problem. In many cases, the nonlinear differential equations right-hand sides' linearization is extremely difficult or even impossible. Then piecewise continuous and piecewise linear approximations of nonlinear differential equations can be used. The piecewise continuous linear differential equations can improve piecewise linear differential equations by increasing the number of separate linear differential equations systems areas and thereby reducing errors on the borders of these areas. The matrices of piecewise continuous linear differential equations system for steady-state points are estimated using nonlinear model time responses, nonlinear programming and random search method. The proposed approach application results are presented.

1. Introduction: linear, piecewise-linear and piecewise-continuous linear approximation of nonlinear differential equations

Let's consider nonlinear differential equations system [1].

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x},\mathbf{u}\right) \tag{1}$$

There are $\mathbf{x}$ — state vector, $\mathbf{u}$ — input vector.

The constant input vector ${}_{\mathrm{s}}\mathbf{u}$ determines steady-state vector of state variables ${}_{\mathrm{s}}\mathbf{x}$.

In a small neighborhood of r-th steady-state point nonlinear differential equations are approximated by linear differential equations [2].

$$\dot{\mathbf{x}}^{\mathrm{n}} = {}^{r}\mathbf{A}\left(\mathbf{x}^{\mathrm{n}} - {}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right) + {}^{r}\mathbf{B}\left(\mathbf{u}^{\mathrm{n}} - {}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}\right) \tag{2}$$

There are $\mathbf{x}^{\mathrm{n}}$ — vector of normalized state variables, $\mathbf{u}^{\mathrm{n}}$ — vector of normalized input variables,

$$x_i^{\mathrm{n}} = \frac{x_i}{\max\limits_{r=1,\ldots,R}\left|{}^{r}_{\mathrm{s}}x_i\right|}, \quad i=1,\ldots,n, \qquad u_i^{\mathrm{n}} = \frac{u_i}{\max\limits_{r=1,\ldots,R}\left|{}^{r}_{\mathrm{s}}u_i\right|}, \quad i=1,\ldots,m \tag{3}$$

${}^{r}\mathbf{A}$, ${}^{r}\mathbf{B}$ are linear differential equations system matrices for r-th steady-state point,
${}^{r}\mathbf{A}$, ${}^{r}\mathbf{B}$, $r = 1, 2,\ldots, R$ are piecewise-linear differential equations system matrices,
${}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}$, ${}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}$ are vectors of normalized input and state variables for r-th steady-state point,
${}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}$, ${}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}$, $r = 1, 2,\ldots, R$ are vectors of normalized input and state variables for piecewise-linear differential equations R steady-state points.

The vectors of normalized input and state variables and piecewise-continuous linear differential equations system matrices are defined as follows

$$ {}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}} = {}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}} + \theta\left({}^{r+1}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}} - {}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}\right), \quad {}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}} = {}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}} + \theta\left({}^{r+1}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}} - {}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right), \quad r = 1,\dots,R-1 \tag{4} $$

$$ \mathbf{A} = {}^{r}\mathbf{A} + \theta\left({}^{r+1}\mathbf{A} - {}^{r}\mathbf{A}\right), \quad \mathbf{B} = {}^{r}\mathbf{B} + \theta\left({}^{r+1}\mathbf{B} - {}^{r}\mathbf{B}\right), \quad r = 1,\dots,R-1 \tag{5} $$

$$ \theta = \left(\left\|\mathbf{x}^{\mathrm{n}}\right\| - \left\|{}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\|\right) \Big/ \left(\left\|{}^{r+1}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| - \left\|{}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\|\right). \tag{6} $$

There is

$$ r = 1 \left|\ \left\|\mathbf{x}^{\mathrm{n}}\right\| / \left\|{}^{R}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| \le \left\|{}^{2}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| / \left\|{}^{R}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| \right. $$

$$ r = 2,\dots,R-2 \left|\ \left\|{}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| / \left\|{}^{R}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| < \left\|\mathbf{x}^{\mathrm{n}}\right\| / \left\|{}^{R}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| \le \left\|{}^{r+1}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| / \left\|{}^{R}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| \right. $$

$$ r = R-1 \left|\ \left\|{}^{R-1}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| / \left\|{}^{R}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| < \left\|\mathbf{x}^{\mathrm{n}}\right\| / \left\|{}^{R}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| \right. $$

2. Piecewise-continuous linear differential equations system matrices estimation

The matrices of piecewise-continuous linear differential equations system for steady-state points can be equal to piecewise-linear differential equations system matrices or can be estimated using nonlinear model time responses, nonlinear programming and random search method.

Nonlinear differential equations time responses ${}^{\mathrm{NL}}\mathbf{u}^{\mathrm{n}}(t_k)$, $k = 0,\dots,N-1$, ${}^{\mathrm{NL}}\mathbf{x}^{\mathrm{n}}(t_k)$, $k = 0,\dots,N$, where $t_{k+1} = t_k + \Delta t$, $k = 0,\dots,N-1$, can be used for estimation of piecewise-continuous linear differential equations system matrices for steady-state points. These matrices and known vectors of normalized input and state variables ${}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}$, ${}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}$, r = 1, 2,…, R correspond to R neighboring steady-state points. The ${}^{\mathrm{NL}}\mathbf{u}^{\mathrm{n}}(t_k)$, $k = 0,\dots,N-1$ represent several consecutive stepwise time responses which connect with ${}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}$, r = 1, 2,…, R.

Such estimation can be reduced to nonlinear programming problem

$$ {}^{r}\mathbf{A}, {}^{r}\mathbf{B}, \ r = 1,2,\dots,R: $$

$$ \min \left\{ \sqrt{\frac{\sum_{k=1}^{N}\left[\mathbf{x}^{\mathrm{n}}(t_k) - {}^{\text{НЛ}}\mathbf{x}^{\mathrm{n}}(t_k)\right]^T {}_{\mathbf{x}}\mathbf{W}(t_k)\left[\mathbf{x}^{\mathrm{n}}(t_k) - {}^{\text{НЛ}}\mathbf{x}^{\mathrm{n}}(t_k)\right]}{\sum_{k=1}^{N}\left[{}^{\text{НЛ}}\mathbf{x}^{\mathrm{n}}(t_k) - {}^{\text{НЛ}}\mathbf{x}^{\mathrm{n}}(t_0)\right]^T {}_{\mathbf{x}}\mathbf{W}(t_k)\left[{}^{\text{НЛ}}\mathbf{x}^{\mathrm{n}}(t_k) - {}^{\text{НЛ}}\mathbf{x}^{\mathrm{n}}(t_0)\right]}} \right| $$

$$ \mathbf{x}^{\mathrm{n}}(t_{k+1}) \approx \mathbf{x}^{\mathrm{n}}(t_k) + \Delta t\,\mathbf{A}\left[\mathbf{x}^{\mathrm{n}}(t_k) - {}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right] + \Delta t\,\mathbf{B}\left[{}^{\text{НЛ}}\mathbf{u}^{\mathrm{n}}(t_k) - {}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}\right] $$

$$ {}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}} = {}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}} + \theta\left({}^{r+1}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}} - {}^{r}_{\mathrm{s}}\mathbf{u}^{\mathrm{n}}\right), \quad {}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}} = {}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}} + \theta\left({}^{r+1}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}} - {}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right), \quad r = 1,\dots,R-1 $$

$$ \mathbf{A} = {}^{r}\mathbf{A} + \theta\left({}^{r+1}\mathbf{A} - {}^{r}\mathbf{A}\right), \quad \mathbf{B} = {}^{r}\mathbf{B} + \theta\left({}^{r+1}\mathbf{B} - {}^{r}\mathbf{B}\right), \quad r = 1,\dots,R-1 $$

$$ \left. \theta = \left(\left\|\mathbf{x}^{\mathrm{n}}(t_k)\right\| - \left\|{}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\|\right) \Big/ \left(\left\|{}^{r+1}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\| - \left\|{}^{r}_{\mathrm{s}}\mathbf{x}^{\mathrm{n}}\right\|\right), \quad k = 0,1,\dots,N-1 \right\} \tag{7} $$

There are ${}_{\mathbf{x}}\mathbf{W}(t_k)$, $k = 1,\dots,N$ positive definite diagonal weight matrices, $\mathbf{x}^{\mathrm{n}}(t_0) = {}^{\text{НЛ}}\mathbf{x}^{\mathrm{n}}(t_0)$. The nonlinear programming problem is solved by the effective random search method [3]. The initial values of piecewise-continuous linear differential equations system matrices for steady-state points are piecewise-linear differential equations system matrices.

3. Piecewise-continuous linear differential equations system matrices estimation example

In this example m = 4, n = 5, the vectors of normalized steady-state values of input and state variables ${}^{6}_{s}\mathbf{u}^{n}$, ${}^{5}_{s}\mathbf{u}^{n}$, …, ${}^{1}_{s}\mathbf{u}^{n}$, ${}^{6}_{s}\mathbf{x}^{n}$, ${}^{5}_{s}\mathbf{x}^{n}$, …, ${}^{1}_{s}\mathbf{x}^{n}$ are

$$\begin{bmatrix}1\\-0.452\\0\\0\end{bmatrix},\begin{bmatrix}0.944\\0.444\\-0.089\\0\end{bmatrix},\begin{bmatrix}0.771\\0.444\\-0.374\\0\end{bmatrix},\begin{bmatrix}0.572\\0.444\\-0.740\\-0.107\end{bmatrix},\begin{bmatrix}0.288\\0.444\\-1\\-0.226\end{bmatrix},\begin{bmatrix}0.186\\0.444\\-1\\-0.402\end{bmatrix},$$

$$\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\begin{bmatrix}0.974\\0.983\\0.966\\0.978\\0.979\end{bmatrix},\begin{bmatrix}0.918\\0.941\\0.836\\0.862\\0.923\end{bmatrix},\begin{bmatrix}0.845\\0.899\\0.672\\0.713\\0.851\end{bmatrix},\begin{bmatrix}0.723\\0.840\\0.451\\0.479\\0.690\end{bmatrix},\begin{bmatrix}0.620\\0.793\\0.335\\0.379\\0.611\end{bmatrix}.$$

The piecewise continuous linear differential equations system matrices for steady-state points ${}^{6}\mathbf{A}$, ${}^{6}\mathbf{B}$, ${}^{5}\mathbf{A}$, ${}^{5}\mathbf{B}$, …, ${}^{1}\mathbf{A}$, ${}^{1}\mathbf{B}$ are estimated through (7).

$$\begin{bmatrix}-1.821 & 1.177 & 0 & 0 & 0\\0.070 & -1.651 & 0 & 0 & 0\\9.739 & 11.519 & -20.117 & -13.187 & 0\\9.671 & 3.512 & -5.592 & -19.357 & 0\\2.987 & -6.816 & -3.416 & -3.117 & -30.042\end{bmatrix},\begin{bmatrix}0.609 & 0.820 & -0.139 & 0.179\\0.445 & 0.574 & -0.039 & -0.220\\10.833 & -40.906 & 2.621 & 1.720\\7.185 & -39.004 & 2.150 & 0.506\\16.083 & 3.773 & -0.805 & -1.232\end{bmatrix},$$

$$\begin{bmatrix}-2.099 & 1.032 & 0 & 0 & 0\\0.187 & -1.856 & 0 & 0 & 0\\9.791 & 11.573 & -20.030 & -13.150 & 0\\9.784 & 3.676 & -5.572 & -19.289 & 0\\2.862 & -6.960 & -3.429 & -3.224 & -29.951\end{bmatrix},\begin{bmatrix}0.565 & 2.272 & -0.424 & 0.251\\0.600 & 0.574 & 0.119 & -0.201\\14.272 & -47.260 & 2.237 & -1.549\\9.231 & -44.307 & 2.486 & -2.137\\17.901 & 1.825 & -0.412 & -1.688\end{bmatrix},$$

$$\begin{bmatrix}-1.627 & 1.292 & 0 & 0 & 0\\0.065 & -2.018 & 0 & 0 & 0\\11.240 & 12.657 & -19.717 & -11.576 & 0\\10.433 & 4.674 & -5.034 & -18.721 & 0\\2.885 & -6.947 & -3.449 & -3.284 & -29.981\end{bmatrix},\begin{bmatrix}0.505 & 0.926 & -0.012 & 0.179\\0.589 & 0.545 & -0.051 & -0.282\\13.795 & -31.678 & 1.716 & 3.674\\11.471 & -32.176 & 1.743 & 2.328\\16.704 & -3.878 & 0.996 & 2.309\end{bmatrix},$$

$$\begin{bmatrix}-2.352 & 1.730 & 0 & 0 & 0\\0.108 & -2.840 & 0 & 0 & 0\\13.277 & 14.630 & -18.606 & -10.389 & 0\\9.737 & 4.263 & -4.263 & -15.641 & 0\\2.828 & -3.545 & -3.545 & -3.261 & -30.003\end{bmatrix},\begin{bmatrix}0.598 & 1.851 & -0.130 & 0.125\\0.725 & -0.100 & -0.195 & -0.327\\13.279 & -20.524 & 0.053 & 1.807\\9.133 & -19.341 & -0.245 & 1.419\\25.591 & -9.012 & -1.012 & -3.229\end{bmatrix},$$

$$\begin{bmatrix}-1.728 & 1.940 & 0 & 0 & 0\\0.569 & -2.511 & 0 & 0 & 0\\12.826 & 14.995 & -18.752 & -11.988 & 0\\11.496 & 6.471 & -4.582 & -17.737 & 0\\2.854 & -6.802 & -3.466 & -3.252 & -29.992\end{bmatrix},\begin{bmatrix}0.933 & 0.455 & 0.191 & 0.099\\0.575 & -0.018 & -0.140 & -0.135\\11.456 & -10.619 & -0.112 & 2.167\\6.719 & -12.075 & -0.253 & 1.049\\32.449 & -3.523 & 2.077 & -2.286\end{bmatrix},$$

$$\begin{bmatrix}-1.994 & 0.935 & 0 & 0 & 0\\1.271 & -2.625 & 0 & 0 & 0\\12.538 & 14.767 & -18.630 & -11.976 & 0\\11.452 & 6.631 & -4.550 & -17.685 & 0\\2.078 & -7.221 & -4.949 & -3.468 & -29.472\end{bmatrix},\begin{bmatrix}0.774 & 1.115 & -0.221 & -0.073\\1.430 & -0.569 & 0.275 & -0.006\\14.364 & -10.691 & -2.141 & 1.297\\8.724 & -12.451 & -0.811 & 0.842\\38.991 & -2.471 & -0.167 & 1.549\end{bmatrix}.$$

There are $k = 1,\ldots,2001$, $\Delta t = 0.025$, and the weight matrices are identity matrices.

The nonlinear, piecewise continuous linear, piecewise linear and improved piecewise linear differential equations systems state variable x_5^{n} and its derivative time responses are presented in Figures 1, 2. These improved piecewise-linear differential equations system matrices and vectors of normalized input and state variables are calculated by piecewise-continuous linear differential equations system matrices and vectors of normalized input and state variables using equations (4), (5), (6) every 0.25 sec.

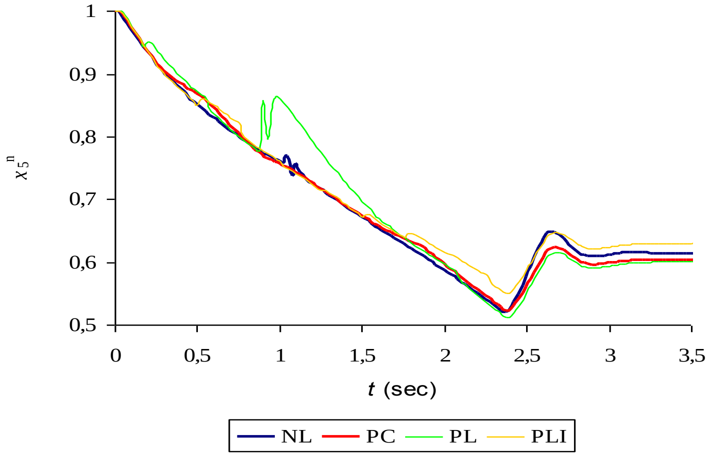

Figure 1. Nonlinear (NL), piecewise continuous linear (PC), piecewise linear (PL) and improved piecewise linear (PL1) differential equations systems variable x_5^n time responses.

The piecewise continuous linear approximation errors are less than the errors of both piecewise linear approximations. Besides the errors of improved piecewise linear approximation are much smaller than those of piecewise linear approximation.

The differences between the errors of the derivative time responses are the most significant. The piecewise linear differential equations system state variable x_5^n derivative time response errors are very large. The improved piecewise linear differential equations system state variable x_5^n derivative time response errors are much less.

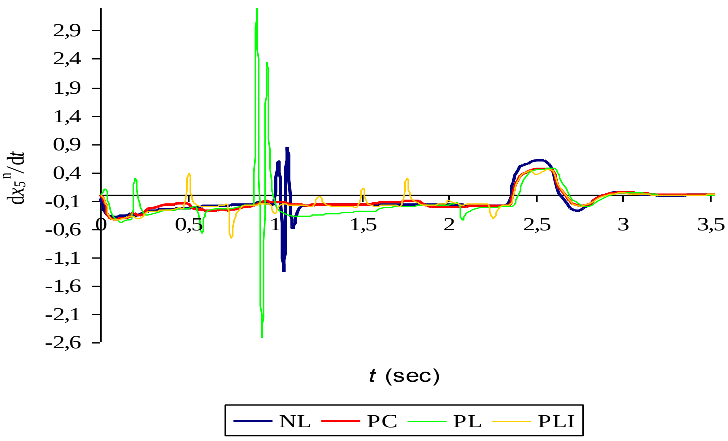

Figure 2. Nonlinear, piecewise continuous linear, piecewise linear and improved piecewise linear differential equations systems variable x_5^n derivative time responses.

References

[1] Leibov R L 2015 Int. J. Comp. Civil Struct. Eng. **11** 2 107–30
[2] Leibov R L 2017 Int. J. Comp. Civil Struct. Eng. **13** 3 77–5
[3] Chao Yang and Mrinal Kumar 2018 Automatica **87** 1 301–09