

Generalized Nash Equilibrium Problem Involving Uncertain Random Variable

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Abstract. This paper investigates Nash equilibrium problem with uncertain random variable which is a tool to deal with a mixture of uncertainty and randomness. By using nonlinear complementarity problem function (denoted by NCP function), quasi-Monte Carlo method and inverse uncertain distribution method, the expected minimization model is put forward, the method of solution and convergence result of the corresponding solution are given.

1. Introduction

The Generalized Nash equilibrium is a vital problem which originated in the economic sciences but is being maturely applied in various fields. The generalized Nash equilibrium problem (GNEP for short) has attracted many scholars to research in different fields such as [1].

The generalized Nash equilibrium problem is formally said to be a non-cooperative game problem composed of N players in mathematical language. The composition of N players can be simply expressed in $v = 1, \dots, N$. Each player's V controlling variable satisfies $x^v \in R^{n_v}$. We use x which is expressed as $x := (x^1, \dots, x^N)^T \in R^n$ to represent a vector made up of all these decision variables, and

$n = \sum_{v=1}^N n_v$. In addition, the x^{-v} is used to represent the vector of all players other than the player's V decision variable, that is $x^{-v} = (x^1, \dots, x^{v-1}, x^{v+1}, \dots, x^N) \in R^{n-n_v}$. In order to emphasize the variable of the V special player role, we sometimes write it $x := (x^v, x^{-v})$ rather than x . It is important to note that x still satisfies the vector of $x = (x^1, \dots, x^v, \dots, x^N) \in R^n$. Each player has an objective function $\theta_v : R^n \rightarrow R$ depends not only on his own variable x^v , but also on all other player variables x^{-v} . Here θ_v has different forms of address in different applications of the generalized Nash equilibrium problem.

In addition, each participant's strategy must belong to a set $X_v(x^{-v}) \subseteq R^{n_v}$, it depends on the opponent's strategy, which we call a feasible set or strategy of the player's v space. Usually these sets have constraints and are recorded as $X_v(x^{-v}) := \{x^v \in R^{n_v} \mid g^v(x^v, x^{-v}) \leq 0\}$. The goal of player v is to select a strategy x^v to solve the x^v minimization problem on the premise of a given player's



strategy x^{-v} . The vector of $x^* \in X(x^*)$ is a generalized Nash equilibrium if and only if it satisfies the following $\theta_v(x^{*,v}, x^{*, -v}) \leq \theta_v(x^v, x^{*, -v}), \forall x^v \in X_v(x^{*, -v})$ are established, in other words it solve the optimization problem $\min_{x^v} \theta_v(x^{*,v}, x^{*, -v})$ s.t. $x^v \in X_v(x^{*, -v})$. If the constraint condition $g^v < 0$ does not exist for every player and other players, it will become a classical Nash equilibrium problem.

In reality, because it is inevitable that belief degree of the uncertainty needs to be estimated by well-known experts in the industry. In many practical problems, the effects of random problems inevitably include uncertain random factors, an uncertain random variable is a variable of the measurable function in the opportunity space to describe the mixed phenomenon of uncertainty and random. The uncertain random phenomena can be seen everywhere, so it is necessary to study it. Based on the uncertainty theory [2], the uncertain random generalized Nash equilibrium problem (URGNEP) is proposed in this paper.

2. Preliminaries

Theorem 1. [2] Let $\omega_1, \omega_2, L, \omega_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, L, \Psi_m$ and let $\gamma_1, \gamma_2, L, \gamma_n$ be independent uncertain variables with uncertainty distributions Y_1, Y_2, L, Y_n , respectively. Then the uncertain random variable $\xi = f(\omega_1, \omega_2, L, \omega_m, \gamma_1, \gamma_2, L, \gamma_n)$ has an expected value

$$E[\xi] = \int_{R^m} \int_0^1 f(y_1, y_2, L, y_m, Y_1^{-1}(\alpha), Y_2^{-1}(\alpha), L, Y_n^{-1}(\alpha)) d\alpha d\Psi_1(y_1) L d\Psi_m(y_m)$$

provided that $f(\omega_1, \omega_2, L, \omega_m, \gamma_1, \gamma_2, L, \gamma_n)$ is a strictly increasing function or a strictly decreasing function with respect to $\gamma_1, \gamma_2, L, \gamma_n$.

3. Main Results

In URGNEP, each player v controls the variable which solves the following optimization problem:

$\min_{x^v} \theta_v(x^{*,v}, x^{*, -v}, \xi)$ s.t. $g^v(x^{*,v}, x^{*, -v}) \leq 0, \xi \in Ba.s.$, where ξ is a uncertain random vector defined on chance space, “a.s.” is the abbreviation for “almost surely” under the given chance measure. By chance theory we know $\theta_v(x^{*,v}, x^{*, -v}, \xi) = \theta_v(x^{*,v}, x^{*, -v}, \omega, \gamma)$, we assume that the chance space is a nonempty compact set and the probability density function $\rho(\omega)$ is continuous on Ω . Both function $\theta_v(x^v, x^{*, -v}, \omega, \gamma)$ and $g^v(x^{*,v}, x^{*, -v})$ are twice continuously differentiable with respect to x^v and $\theta_v(x^v, x^{*, -v}, \omega, \gamma)$ is continuous with respect to ω and γ .

By introducing the NCP function [4] and KKT condition for URGNEP. We further use the expected residual minimization (ERM) model to get the determined reformulation in this section.

For $v = 1, L, N$, the KKT condition for URGNEP can be written as

$$\nabla_{x^v} \theta_v(x^{*,v}, x^{*, -v}, \xi) + \sum_{i=1}^{h_v} \mu_i^v \nabla g_i^v(x^{*,v}, x^{*, -v}) = 0, \quad (3.1)$$

$$g_i^v(x^{*,v}, x^{*, -v}) \leq 0, \mu_i^v \geq 0, \mu_i^v g_i^v(x^{*,v}, x^{*, -v}) = 0, i = 1, L, h_v, \quad (3.2)$$

here $\mu_i^v, i = 1, L, h_v$ are Lagrange multipliers. denote $\mu^v = (\mu_1^v, L, \mu_{h_v}^v)$

By using NCP function, (3.2) can be transformed equivalently into the equation $\Phi(x^{*,v}, x^{*, -v}, \mu^v) = 0$ (3.3)

$$\text{where } \Phi : R^{n_v} \times R^{h_v} \rightarrow R^{h_v} \text{ is defined by } \Phi(x^{*,v}, x^{*, -v}, \mu^v) = \begin{pmatrix} \phi(-g_1^v(x^{*,v}, x^{*, -v}), \mu_1^v) \\ M \\ \phi(-g_{h_v}^v(x^{*,v}, x^{*, -v}), \mu_{h_v}^v) \end{pmatrix}$$

Theorem 2 the expected minimization model for URGNEP, in which we try to find a vector x that minimizes an expected residual for (3.1) and (3.3), here we just consider $\theta(\omega, \gamma)$ is strictly increasing with respect to γ , then we have

$$\begin{aligned} & \min_{(x^v, \mu^v)} \Theta_v(x^v, x^{*, -v}, \mu^v) \\ &= E[\|\nabla_{x^v} \theta_v(x^v, x^{*, -v}, \omega, \gamma) + \sum_{i=1}^{h_v} \mu_i^v \nabla g_i^v(x^{*, v}, x^{*, -v})\|^2] + \|\Phi(x^{*, v}, x^{*, -v}, \mu^v)\|^2 \\ &= \int_{\Omega} \int_0^1 [\|\nabla_{x^v} \theta_v(x^v, x^{*, -v}, \omega, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \mu_i^v \nabla g_i^v(x^v, x^{*, -v})\|^2] d\alpha \rho(\omega) d\omega + \|\Phi(x^v, x^{*, -v}, \mu^v)\|^2 \end{aligned} \quad (3.4)$$

where E stands for the expectation with respect to the uncertain random variable $\xi \in B$ and $\rho(\omega) : \Omega \rightarrow [0, \infty)$ denotes the probability density function satisfying $\int_{\Omega} \rho(\omega) d\omega = 1$

Proof it is easy to get by theorem 1

Theorem 3 Let $\Omega_k = \{\omega^i \mid i = 1, 2, \dots, N_k\}$ be a set of observations generated by a quasi-Monte Carlo method such that $\Omega_k \subseteq \Omega$ and $N_k \rightarrow \infty$. For each $v = 1, \dots, N$, we then get the following approximation problem of (3.4)

$$\begin{aligned} & \min_{(x^v, \mu^v)} \Theta_v^k(x^v, x^{*, -v}, \mu^v) \\ &= \frac{1}{N_k} \sum_{\omega^i \in \Omega_k} [\int_0^1 \|\nabla_{x^v} \theta_v(x^v, x^{*, -v}, \omega, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \mu_i^v \nabla g_i^v(x^v, x^{*, -v})\|^2 d\alpha] + \|\Phi(x^v, x^{*, -v}, \mu^v)\|^2 \end{aligned} \quad (3.5)$$

Proof it is easy to get by quasi-Monte Carlo method of [3].

In the following we let S_v^* and $S_v^{k,*}$ denote the sets of optimal solutions of problems (3.4) and (3.5), respectively.

Theorem 4 Let $x^{k,v}, \mu^{k,v}$ satisfying (3.1) and (3.2) for each k, v and let $\lim_{k \rightarrow 0} x^{k,v} = \bar{x}^v$. Suppose that the Mangasarian-Fromovitz constraint qualification holds. Then, $\{\mu^{k,v}\}$ is bounded.

Proof denote $\sigma^{k,v} := \sum_{i=1}^{h_v} \mu_i^{k,v}$. For each v , suppose that $\{\mu^{k,v}\}$ is unbounded, which means $\lim_{k \rightarrow 0} \sigma^{k,v} = +\infty$. Assume that the limits $\beta_i^0 := \lim_{k \rightarrow 0} \frac{\mu_i^{k,v}}{\sigma^{k,v}}$ ($i = 1, 2, \dots, h_v$) exists. From (3.2) we know that $\beta_i^0 = 0$ for every $i \notin H_g(\bar{x}^v) := \{i \mid g_i(\bar{x}^v, x^{*, -v}) = 0, i = 1, 2, \dots, h_v\}$. We can also get that $\sum_{i \in H_g(\bar{x}^v)} \beta_i^0 = \sum_{i=1}^{h_v} \beta_i^0 = 1$. Owing to $\lim_{k \rightarrow 0} x^{k,v} = \bar{x}^v$, there must exist a compact set B containing the sequence $\{x^{k,v}\}$. Then, from the mean-value theorem and the twice continuous differentiability of $\theta_v(x^v, x^{*, -v}, \omega, Y^{-1}(\alpha))$ on B that, for each $\{x^{k,v}\}$ and each ω^i , there exists $y^{ki,v} = \beta_{ki,v} x^{k,v} + (1 - \beta_{ki,v}) \bar{x}^v \in B$ with $\beta_{ki,v} \in [0, 1]$ such that

$$\begin{aligned} & \left\| \int_0^1 \nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) d\alpha - \int_0^1 \nabla_{x^v} \theta_v(\bar{x}^v, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) d\alpha \right\| \\ & \leq \left\| \int_0^1 \nabla_{x^v}^2 \theta_v(y^{ki,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) d\alpha \right\| \|x^{k,v} - \bar{x}^v\| \leq C_1 \|x^{k,v} - \bar{x}^v\| \end{aligned}$$

Which means $\int_0^1 \nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) d\alpha \rightarrow \int_0^1 \nabla_{x^v} \theta_v(x^v, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) d\alpha$ as $k \rightarrow +\infty$. Then, dividing (3.1) by $\sigma^{k,v}$ and taking a limit, we obtain $\sum_{i \in H_g(\bar{x}^v)} \mu_i^v \nabla g_i^v(\bar{x}^v, x^{*, -v}) = 0$. Recall that the

Mangasarian-Fromovitz constraint qualification holds. So there exists a vector $y \in R^{n_v}$ such that

$$y^T \nabla g_i^v(\bar{x}^v, x^{*, -v}) < 0, i \in \Sigma_g(\bar{x}^v), \text{ we have } \mu_i^v = 0 \text{ for each } i \in H_g(\bar{x}^v).$$

which is contradict with $\sum_{i \in \Sigma(\bar{x}^v)} \mu_i^v = \sum_{i=1}^{h_v} \mu_i^v = 1$, hence $\{\mu^{k,v}\}$ is bounded for each v.

Theorem 5 For each k, assume that $(x^{k,v}, \mu^{k,v}) \in S_v^{k,*}, v=1, L, N$ and $(\bar{x}^v, \bar{\mu}^v)$ is an accumulation point of $\{(x^{k,v}, \mu^{k,v})\}$. Then we have $(\bar{x}^v, \bar{\mu}^v) \in S_v^*$.

Proof Supposing that $\lim_{k \rightarrow 0} \mu^{k,v} = \bar{\mu}^v, \lim_{k \rightarrow 0} x^{k,v} = \bar{x}^v$. Then there must exist a compact set B and S containing the sequence $\{x^{k,v}\}, \{\mu^{k,v}\}$ respectively. Considering the twice continuous differentiability of $\theta_v(x^v, x^{*, -v}, \xi), g_i^v(x^v, x^{*, -v})$ on B and the continuity of $\theta_v(x^v, x^{*, -v}, \xi)$ on compact set B, there is $C_2 > 0$ satisfying

$$\begin{aligned} & \left\| \int_0^1 (\nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v})) d\alpha \right\| \\ & + \left\| \int_0^1 (\nabla_{x^v} \theta_v(\bar{x}^v, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \bar{\mu}_i^v \nabla g_i^v(\bar{x}^v, x^{*, -v})) d\alpha \right\| \leq C_2 \end{aligned} \quad (3.6)$$

Further more, similar to theorem 4, we can get

$$\| \nabla g_i^v(x^{k,v}, x^{*, -v}) - \nabla g_i^v(\bar{x}^v, x^{*, -v}) \| \leq C_3 \| x^{k,v} - \bar{x}^v \| \quad (3.7) \quad \text{set}$$

$C_4 = \max \{ \sup S, \| g_i^v(\bar{x}^v, x^{*, -v}) \|, i=1, 2, L, l_v \}$, we have from (3.7) that

$$\begin{aligned} & \left\| \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v}) - \sum_{i=1}^{h_v} \bar{\mu}_i^v \nabla g_i^v(\bar{x}^v, x^{*, -v}) \right\| \\ & \leq \left\| \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v}) - \sum_{i=1}^{h_v} \bar{\mu}_i^v \nabla g_i^v(\bar{x}^v, x^{*, -v}) \right\| + \left\| \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(\bar{x}^v, x^{*, -v}) - \sum_{i=1}^{h_v} \bar{\mu}_i^v \nabla g_i^v(\bar{x}^v, x^{*, -v}) \right\| \\ & \leq \sum_{i=1}^{h_v} \mu_i^{k,v} \| \nabla g_i^v(x^{k,v}, x^{*, -v}) - \nabla g_i^v(\bar{x}^v, x^{*, -v}) \| + \left\| \sum_{i=1}^{h_v} (\mu_i^{k,v} - \bar{\mu}_i^v) \nabla g_i^v(\bar{x}^v, x^{*, -v}) \right\| \\ & \leq C_3 C_4 h_v \| x^{k,v} - \bar{x}^v \| + C_4 \sum_{i=1}^{h_v} |\mu_i^{k,v} - \bar{\mu}_i^v| \end{aligned} \quad (3.8)$$

Following from (3.6) and (3.8), we have

$$\begin{aligned} & \left\| \int_0^1 (\nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v})) d\alpha \right\|^2 \\ & - \left\| \int_0^1 (\nabla_{x^v} \theta_v(\bar{x}^v, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \bar{\mu}_i^v \nabla g_i^v(\bar{x}^v, x^{*, -v})) d\alpha \right\|^2 \\ & \leq \left\| \int_0^1 (\nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v})) d\alpha \right\|^2 \end{aligned}$$

$$\begin{aligned}
& + \|\int_0^1 (\nabla_{x^v} \theta_v(\bar{x}^{-v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \bar{\mu}_i^{-v} \nabla g_i^v(\bar{x}^v, x^{*, -v})) d\alpha\|^2] \\
& \cdot \|\int_0^1 (\nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v})) d\alpha\|^2 \\
& - \|\int_0^1 (\nabla_{x^v} \theta_v(\bar{x}^{-v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \bar{\mu}_i^{-v} \nabla g_i^v(\bar{x}^v, x^{*, -v})) d\alpha\|^2 | \\
& \leq C_2 \|\int_0^1 (\nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) - \nabla_{x^v} \theta_v(\bar{x}^{-v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) \\
& + \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v}) - \sum_{i=1}^{h_v} \bar{\mu}_i^{-v} \nabla g_i^v(\bar{x}^v, x^{*, -v})) d\alpha\| \\
& \leq C_1 C_2 \|x^{k,v} - \bar{x}^v\| + C_2 C_3 C_4 h_v \|x^{k,v} - \bar{x}^v\| + C_2 C_4 \sum_{i=1}^{h_v} |\mu_i^{k,v} - \bar{\mu}_i^{-v}| \quad (3.9)
\end{aligned}$$

By (3.9) and the continuity of $\|\Phi(x^v), \mu^v\|^2$ we have that

$$\begin{aligned}
& |\Theta_v^k(x^{k,v}, x^{*, -v}, \mu^{k,v}) - \Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v)| \\
& \leq \frac{1}{N_k} \sum_{\omega^i \in T_k} \rho(t) \|\int_0^1 (\nabla_{x^v} \theta_v(x^{k,v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \mu_i^{k,v} \nabla g_i^v(x^{k,v}, x^{*, -v})) d\alpha\|^2 \\
& - \|\int_0^1 (\nabla_{x^v} \theta_v(\bar{x}^{-v}, x^{*, -v}, \omega^i, Y^{-1}(\alpha)) + \sum_{i=1}^{h_v} \bar{\mu}_i^{-v} \nabla g_i^v(\bar{x}^v, x^{*, -v})) d\alpha\|^2 | \\
& + \|\Phi(x^{k,v}, \mu^{k,v})\|^2 - \|\Phi(\bar{x}^v, \bar{\mu}^v)\|^2 \xrightarrow{k \rightarrow \infty} 0 \quad (3.10) \text{ Owing that} \\
& |\Theta_v^k(x^{k,v}, x^{*, -v}, \mu^{k,v}) - \Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v)| \\
& \leq |\Theta_v^k(x^{k,v}, x^{*, -v}, \mu^{k,v}) - \Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v)| + |\Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v) - \Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v)|
\end{aligned}$$

According to quasi-Monte Carlo method of [3] and (3.10) that

$\lim_{k \rightarrow \infty} \Theta_v^k(x^{k,v}, x^{*, -v}, \mu^{k,v}) = \Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v)$ Owing to $(x^{k,v}, \mu^{k,v}) \in S_v^{k,*}$, which means

$\Theta_v^k(x^{k,v}, x^{*, -v}, \mu^{k,v}) \leq \Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v)$ for each $v = 1, L, N$. Letting $k \rightarrow +\infty$ in (3.10), we get from (3.9) and quasi-Monte Carlo method that

$\Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v) \leq \Theta_v^k(\bar{x}^v, x^{*, -v}, \bar{\mu}^v)$ for each $v = 1, L, N$. which means $(\bar{x}^v, \bar{\mu}^v) \in S_v^*$

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