

General theory of cross-sections kinetics in calculating rod elements of structures. Cartesian coordinates

G F Sidorov

South Ural State University, 76, Lenin Avenue, Chelyabinsk 454080, Russia

E-mail: sidorovgf@susu.ru

Abstract. The applied mechanics of deformable bodies evolves to its more rigorously substantiated predominant empirical approaches, which significantly limit calculation possibilities. The most “severe” limitation is applying exclusively the hypotheses of non-warping cross-sections in calculating the structural rod elements. The research was carried out to expand the possibilities and to systematically generalize and complete a traditional approach within the theory of strength of materials, which is based on isolating a subsystem of points in a cross-sectional form. The presented theory is a synthesis of the theory of strength of materials and the applied theory of elasticity. The theory of strength of materials contributed with the paradigm of monitoring the behaviour of cross-sections as a subsystem of points of a deformable body. The theory of elasticity provided differential equations of internal constraints, modifiable according to the traditional assumptions of the theory of material strength, except for those relating to the behavior of cross-sections. The study touches upon the analytical issue of optimizing the forms of various rod junctions, such as threaded, toothed, and strained junctions, and also on the issue of stress concentration in the regions of curvilinear transitions.

1. Introduction

Let us mark the longitudinal axis of the rod “z”, and the displacement of the cross-section points in this direction “w” (Figure 1.) The end surfaces can be loaded with any forces distributed over them, and the lateral faces can be loaded with tangential forces distributed according to an arbitrary law (shown in Figure 1.). The coordinate axes xoy form a surface where the rod cross-sections is located in its initial condition. Let us denote the displacements that occur along such axes while loading the rod as “u” and “v”, respectively. In this case, the displacements “w” will outline a new shape that the cross-section obtains as a result of rod deformation. Let us set an immediate task of finding the function w (x, y), which determines its stress-strain state depending on the type of deformation (without the use of J. Bernoulli’s [1] and A.V. Verkhovsky’s [2] hypotheses). Let us use the system of Navier-Cauchy equations [3]:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0; \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0; \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (1)$$

In the case of rods with a load-free lateral face, two assumptions are traditionally used: the static one $\sigma_x = \sigma_y = 0$ and the kinematic one $\gamma_{xy} = 0$ ($\tau_{xy} = 0$). Taking into account the assumptions, the basic equations of the theory of rods will be the following:



$$\frac{\partial \tau_{zx}}{\partial z} = 0; \quad \frac{\partial \tau_{zy}}{\partial z} = 0; \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (2)$$

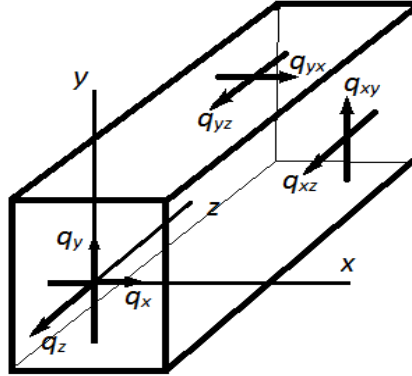


Figure1. Design diagram and legends.

2. Kinetics of cross-sections

Using Hooke's law [4], and then the Cauchy relations [5], let us rewrite the basic displacement equations (here μ is the Poisson coefficient [6]):

$$\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial z^2} = 0; \quad \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2} = 0; \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial z} + 2(1 + \mu) \frac{\partial^2 w}{\partial z^2} = 0 \quad (3)$$

Let us reduce the last equation (3) to one (required) parameter “w”. According to the static assumption $\varepsilon_x = \varepsilon_y = -\mu \varepsilon_z$. In this case we obtain: $\frac{\partial^2 u}{\partial x \partial z} = \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial z} (\varepsilon_x) = \frac{\partial}{\partial z} (-\mu \varepsilon_z) = -\mu \frac{\partial^2 w}{\partial z^2}$.

Similarly $\frac{\partial^2 v}{\partial y \partial z} = -\mu \frac{\partial^2 w}{\partial z^2}$.

After substituting it into equation (3), we obtain:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial z^2} = 0. \quad (4)$$

This equation defines the conditions and the law of cross-section warping. The other equations of system (3) fix the displacements of the cross-section as a stiff whole and depend on the types of rod deformation. Two conditions can be generated to determine the corresponding functions. The first follows from the kinematic assumption about the invariability of the cross-section in plan

$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, the second is a consequence of the same linear deformations in the cross-section

plane according to the static assumption $\varepsilon_x - \varepsilon_y = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$. Differentiating the first condition with

respect to x, and the second – with respect to y, we obtain a system of two differential equations that is sufficient to conclude about the structure of its two constituent functions:

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial y^2} = 0. \quad (5)$$

The difference of equations (5) gives: $\nabla^2 v = 0$. Similarly we obtain $\nabla^2 u = 0$. Here ∇^2 is the Laplace operator [7]. The set of functions that satisfy both the corresponding Laplace operator and equations (5) has the following form: $u(x, y, z) = u(z) - \varphi(z)y + f(z)x$, $v(x, y, z) = v(z) + \varphi(z)x + g(z)y$.

Each structural element of these functions has a clear kinetic interpretation. Thus, the meaning of the function $f(z)$ can be determined by deriving $\frac{\partial u}{\partial x} = f(z) = \varepsilon_x$. This is the lateral deformation in the cross-section plane in the x direction, which affects the distortion of the cross-section shape in plan. According to the kinematic assumption, the function $f(z)$ and the similar one $g(z)$ should be considered equal to zero, while assuming them to be values of small order in comparison with other components of the studied structures. Other components of the displacement formulas can be defined in terms of the theory of rods. Thus, $u(z)$ and $v(z)$ are the deflections of the rod axis in the planes xoz and yoz , respectively, and $\varphi(z)$ is the “stiff” component of the rotation angle of the cross-section when the rod is twisted. Now system of basic equations of the theory of rods (3) for a general case of rod loading can be presented in its complete form:

$$\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 \varphi(z)}{\partial z^2} y + \frac{\partial^2 u(z)}{\partial z^2} = 0; \quad \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 \varphi(z)}{\partial z^2} x + \frac{\partial^2 v(z)}{\partial z^2} = 0; \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial z^2} = 0 \quad (6)$$

The first two equations control the “stiff” part of longitudinal displacements of the cross-section points, connected with its rotations as a whole, and the third one controls its “deformation” component. The last equation can be rewritten to enable its clearer interpretation: $\frac{w_x + w_y}{2} + w_z = 0$, where the mean surface curvature of the warping section is the first term connected by the equation with the gradient of the longitudinal deformation in the perpendicular direction to the cross-section plane. Depending on the type of rod loading, these equations will have a particular form. Thus, when the rod is stretched or compressed, there will be no deflections and twisting. The system of equations in this case is the simplest:

$$\frac{\partial^2 w}{\partial x \partial z} = 0; \quad \frac{\partial^2 w}{\partial y \partial z} = 0; \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial z^2} = 0 \quad (7)$$

In case of torsion, we have to deal with a system of three equations of this type:

$$\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 \varphi(z)}{\partial z^2} y = 0; \quad \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 \varphi(z)}{\partial z^2} x = 0; \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial z^2} = 0 \quad (8)$$

Finally, in case of rod bending, the equations will have transverse displacements of the axis points (deflection functions):

$$\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u(z)}{\partial z^2} = 0; \quad \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 v(z)}{\partial z^2} = 0; \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial z^2} = 0 \quad (9)$$

To demonstrate the possibilities of the warping section method, let us consider several examples.

3. Warping of cross-sections

Example 1. A case of inhomogeneous distribution of external load on the rod end when it is stretched and compressed (Figure 2). Any load of such kind can be defined by a Fourier series [8]. In this case, the first two terms of this expansion are considered in the function of one of the two coordinates:

$q(x) = 0,5q_0 \left(1 + \cos\pi \frac{x}{h}\right)$. Here h is a half of the cross-section width. The stress distribution law σ_z has to be determined with respect to the normal cross-section of an arbitrary position. In this case, the differential equations of equilibrium are: $\frac{\partial^2 w}{\partial x \partial z} = 0$; $\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial z^2} = 0$. The first equation states the absence of the flat part of displacements caused by cross-section rotations around the y-axis (bending) and z-axis (torsion). The second equation, which takes into account the additional conditions (parity with respect to the argument x up to the first derivative with respect to z , decrease in warping with z increasing, and not vanishing to zero with $z = 0$), corresponds to the following function with accuracy up to the constant [9]:

$$w = cz + de^{-\omega z} \cos\sqrt{2}\omega x \quad (10)$$

Longitudinal deformation: $\varepsilon_z = \frac{\partial w}{\partial z} = c - \omega de^{-\omega z} \cos\sqrt{2}\omega x$.

The boundary condition with $z=0$: $\sigma_z = E \left(c - \omega d \cos\sqrt{2}\omega x \right) = \frac{q_0}{2} + \frac{q_0}{2} \cos\pi \frac{x}{h}$.

When the parts of the condition are contrasted, it follows: $\omega = \frac{\pi}{\sqrt{2}h}$; $c = \frac{q_0}{2E}$. Stresses in an arbitrary cross-section with the coordinate z are:

$$\sigma_z = \frac{q_0}{2} \left(1 + e^{-\frac{\pi z}{\sqrt{2}h}} \cos\pi \frac{x}{h} \right). \quad (11)$$

The example proves the principle of locality [10] (the principle of short-range interaction, and Saint-Venant's principle). Thus, the expression $e^{-\frac{\pi z}{\sqrt{2}h}}$ acts as the amplitude factor of the cross-section warping in formula (10). At the same time, it establishes the deviation of the value of normal stresses from their mean value (11). In the cross-section with the coordinate $z = 0$, this coefficient is equal to one. In the cross-section that is located $2h$ from the original one, it is equal to 0.0118 (1.2% of the original one). The size $2h$ in our case is the overall cross-section dimension in transverse direction. Farther than the overall distance from the rod edge there is practically no cross-section warping, and the stresses are distributed evenly over the entire cross-section area.

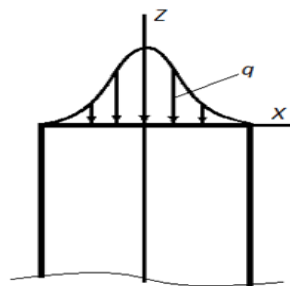


Figure 2. Inhomogeneous loading.

Example 2. The problem of non-circular rod torsion is of fundamental importance for engineering. In this case, the hypothesis of cross-section non-deformability leads to an absurd result. Let us show by the example of rectangular rod torsion (Figure 1.) that cross-section warping can be determined with a sufficient degree of accuracy and used to calculate stresses with a traditional technique of technical mechanics. Let us consider the case when the load is applied to the end section of the rod. To

exclude locality effects, we assume that its distribution over the cross-section area does not differ from that in the remote sections. In this case, warping will not depend on the coordinate z and equations (8) will have the following form:

$$\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 \varphi(z)}{\partial z^2} y = 0; \quad \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 \varphi(z)}{\partial z^2} x = 0; \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \quad (12)$$

The last equation is biharmonic. Traditionally, it is solved with the help of an additional concept of stress function [11]. Let us show that the problem can be solved according to its direct formulation, i.e. directly through the function of cross-section warping. The functions that satisfy the equation are [12]:

$$w = Axy + Bsh\pi \frac{y}{b} \sin\pi \frac{x}{b}. \quad (13)$$

It is noteworthy that the resulting formulas for calculating stresses and deformations were identical if the arguments x and y were permuted in formula (13). Integrating the first two equations (12) with respect to z , we obtain:

$$\frac{\partial w}{\partial x} - \theta y = \gamma_{zx}(x, y); \quad \frac{\partial w}{\partial y} + \theta x = \gamma_{zy}(x, y).$$

Let us place the warping function into the last equations and obtain expressions for angular deformations:

$$\gamma_{zx} = (A - \theta)y + B\frac{\pi}{b}sh\pi \frac{y}{b} \cos\pi \frac{x}{b}; \quad \gamma_{zy} = (A + \theta)x + B\frac{\pi}{b}ch\pi \frac{y}{b} \sin\pi \frac{x}{b}. \quad (14)$$

To determine the unknown constants A and B , we use the boundary condition of the corner points of the rectangular cross-section: with $x = 0.5b$, $y = 0.5h$, there are no angular deformations. Let us solve the system of two equations. The solution of this system of algebraic equations shows:

$$A = \theta; B = -\theta \frac{b^2}{\pi ch \frac{\pi h}{2b}}.$$

The subsequent substitution into equations (14) results in functions of angular deformations:

$$\gamma_{zx} = -\theta \frac{b}{ch \frac{\pi h}{2b}} sh\pi \frac{y}{b} \cos\pi \frac{x}{b}; \quad \gamma_{zy} = \theta \left(2x - \frac{b}{ch \frac{\pi h}{2b}} ch\pi \frac{y}{b} \sin\pi \frac{x}{b} \right) \quad (15)$$

The relationship between the specific twist angle θ and the torque T was established using the integral equilibrium condition [13, 14] within the range $x = \pm 0.5b$, $y = \pm 0.5h$:

$$T = \iint (\tau_{zy}x - \tau_{zx}y) dx dy = G\theta b^3 h \left(\frac{1}{6} + \frac{2}{\pi^2} - \frac{8bth \frac{\pi h}{2b}}{\pi^3 h} \right) = G\theta b^3 h \beta. \quad (16)$$

The geometric characteristic $J_{tor} = \beta hb^3$ is called the inertia moment of a cross-section under torsion [13]. The stiffness of the cross-section under torsion will thus be equal to $C_{tor} = GJ_{tor}$. The torsional stiffness of a prismatic rod with rectangular cross-section of the length L is:

$$C_{tor}^L = \frac{GJ_{tor}}{L} = \frac{G\beta hb^3}{L} \quad (17)$$

where the coefficient $\beta = 0,37 - 0,258 \frac{b}{h} \frac{\pi h}{2b}$.

The maximum stresses observed near the cross-section contour at a point in the middle of the long side of the rectangle are of great interest in calculating the strength. Let us derive a formula for their determination, expressing the relative twist angle θ through the torque:

$$\tau_{zy} = G\gamma_{zy} = G\theta \left(2x - \frac{b}{ch \frac{\pi h}{2b}} ch\pi \frac{y}{b} \sin\pi \frac{x}{b} \right) = \frac{T}{\beta hb^3} \left(2x - \frac{b}{ch \frac{\pi h}{2b}} ch\pi \frac{y}{b} \sin\pi \frac{x}{b} \right).$$

The coordinates of the point of maximum stress are $x = 0.5b$, $y = 0$. If we substitute them into the

last expression, we obtain: $\tau_{max} = \frac{T \left(1 - \frac{1}{ch \frac{\pi h}{2b}} \right)}{\beta hb^2} = \frac{T}{W_{tor}}$. Geometric values were grouped here into one,

which is called resistance modulus of section under torsion [15]. It is calculated using the dependence $W_{tor} = \alpha hb^2$, where, as follows from the previous calculations, $\alpha = \frac{\beta}{1 - \frac{1}{ch \frac{\pi h}{2b}}}$. The problem of

rectangular rod torsion has an exact solution obtained by the methods of elasticity theory [16]. Let us compare the approximate solution obtained in the study with the exact one. Comparison is made with the coefficients α and β , which have been widely used in engineering practice [17]. Here α_E u β_E – are exact values. The results are shown in Table 1.

Table 1. Comparison of results with exact solutions.

h/b	1	2	4
α	0,221	0,264	0,301
α_T	0,208	0,246	0,282
Error %	+6,25	+7,4	+6,7
β	0,133	0,241	0,300
β_T	0,141	0,229	0,281
Error %	-5,7	+5,2	+6,7

The results show that the approximate solution, obtained with purely an engineering method, is sufficiently accurate even for a two-term formula of cross-section warping (13). It is noteworthy that the dependencies obtained as a result of solving the problem of torsion of prismatic rectangular rods can be entirely applied to calculating thin-walled open rods [18, 19]. In this case, each of these sections must be considered as a set of long and narrow rectangles.

4. Conclusions

1. Traditional methods for calculating rod elements of structures are aimed at applying rigid-section hypotheses. Calculation possibilities of such approach are significantly limited, especially in the problem areas of rod junction or its junction with other structural elements.

2. The behavior patterns of rod cross-sections during rod deformation can be described basing on a theory that is similar to the theory of plate bending. Stiff behavior of a cross-section represents in this case a particular variant of the general theory of cross-section kinetics.

3. The general theory of cross-section kinetics provides possibilities for solving applied engineering problems by methods that do not require from the designer specialized education in physics and mathematics. The theory offers engineering solutions to such problems as, for example, torsion of rods with a non-circular cross-section, as well as problems with non-uniform axial loading of rods.

References

- [1] Bernoulli J 1705 *Veritable hypothese de la resistance des solides avec la demonstration de la courbure des corps qui font resort* (Histoire de l'Academy des scienses de Paris) pp 176–186
- [2] Verkhovskiy A V 1971 *Method of nonplane cross-sections* (Gorkiy: Volgo-Vyatscoe izd) p 248
- [3] Terebushko O I 1984 *Basic theory of elasticity and plasticity* (Moscow: Nauka) p 318
- [4] Feodosev V I 2004 *Strength of materials* (Moscow: MGTU im.N.E.Baumana) p 592
- [5] Rabotnov Yu N 1988 *Deformable body mechanics* (Moscow: Nauka) p 712
- [6] Procenko A M 1982 *The theory of elastic-perfectly-plastic systems* (Moscow: Nauka) p 287
- [7] Bronstein I N and Semendyaev K A 1956 *A handbook on mathematics for engineers and students of technical colleges* (Moscow: Izd. Tekniko-teoreticheskoy lit-ry) p 608
- [8] Bezukhov N I 1961 *Theory of elasticity and plastisity* (Moscow: Vyshaya shkola) p 537
- [9] Sidorov G F 1989 *Strength of materials. The case of deductive presentation of the course* (Chelyabinsk: Izd. CHVVAIU) p 177
- [10] Timoshenko S P 1972 *Course of the theory of elasticity* (Kiev: Naukova dumka) p 501
- [11] Birger I A 1986 *Strength of materials* (Moscow: Nauka) p 553
- [12] Zel'dovich Ya B and Myshcis A D 1968 *Elements of applied mathematics* (Moscow: Nauka) p 615
- [13] Abovsky N P, Endqievsky L B, Savchenkov V I, Deruga A P and Reitman M I 1978 *Selected problems of building mechanics and elasticity theory* (Moscow: Stroyizdat) p 188
- [14] Prudnikov A P, Brychkov U A and Marichev O I 1981 *Integrals and series* (Moscow: Nauka) p 793
- [15] Ikrin V A 2005 *Strength of materials with elements of the theory of elasticity* (Moscow: Izd. Assotsiatsii stroit. Vuzov) p 418
- [16] Tolokonnikov L A 1979 *Mechanics of deformable solids* (Moscow: Vyshaya shkola) p 315
- [17] Birger I A and Panovko Ya G 1968 *Durability, stability, oscillations* (Moscow: Mashinostroenie) p 831
- [18] Sidorov G F and Pozdnyshev E O 2011 *Expansion of static indeterminacy by the method of partial stiffnesses* (Gomel: BelGUT) pp 239–244
- [19] Eremeev P G 2006 *Features of design of unique large-span buildings vol 2* (Moscow: Sovremennoe promyshlennoe i grazhdanskoe stroitelstvo) pp 5–15

Acknowledgments

The research was conducted at the Department of building production and theory of structures of South Ural State University within the framework of the project 5/100 aimed to increase the competitiveness of leading Russian universities among the world's leading scientific and educational centers.