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Analytical kinematics for direct coupled shafts using a point-surface contact

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Abstract. The purpose of the paper is to accomplish the kinematics study of a direct coupling between two shafts with a high pair joint, of point-surface type. Due to Hartenberg and Denavit, the kinematics analysis of spatial mechanisms can be made by a well-known method, named the "homogenous operators method". This manner is applicable only for spatial mechanisms containing cylindrical pairs with particular solutions: prismatic pair and revolute pair. In order to apply the Hartenberg-Denavit procedure for the two shafts, the contact between the two shafts must be previously replaced by a succession of 5 prismatic and revolute pairs. It results a system of six trigonometric equations with six unknowns that requires a numerical methodology for solving it. The paper aims to obtain an analytical dependency between the motions of the two shafts. To this end, the geometrical condition that defines the connection between the two shafts is directly used. Thus, after all positional parameters are expressed in the same frame of reference an equation between the positional parameters of the two shafts is obtained.

1. Introduction

One of the major requirements of Mechanisms and machines theory [1-3] is to provide constructive solutions capable of transforming the given motion of a driving element into a desired motion of a final element. In this regard, is to be underlined the observation made by Hunt [4] about the structural optimization of a mechanism, showing that there isn't a possibility of attaining the optimum structural solution by continuous passing through intermediate mechanisms; but, at the moment when a mechanism isn't satisfying firm structural constraints it must be abandoned and a new solution should be adopted, the ingenuity of the designer being decisive in the new structural option. One of the most widespread challenges refers to complete mechanisms as simple as possible capable of transmitting rotation motion between two shafts with crossed axes. Dynamical and economical requirements impose that in the structure of the mechanism should be encountered the smallest number of elements, when feasible. In [5-6] it is shown that, when a direct contact between two shafts with crossed axes is envisaged to transmit the rotation motion, the only possible structural solution is represented by a class one pair. The respective pair is obtained in the mentioned work [5-6] by the contact between two straight lines, each one attached to the respective shaft. The weakness of this constructive solution resided from the fact that the curve-curve contact cannot be practically built using only two elements.

As shown in [7-10], the constructive assembly imposes an intermediate element and therefore the transmission is not actually a direct contact between the two shafts.

The present work intends o solution for direct coupling of the two shafts by employing a class one pair of point-surface type. This pair can be materialized by the direct contact, as shown in figure 1, via introducing a sphere attached to one of the shafts into a prismatic groove having the width equal to the diameter of the sphere, cut into the second shaft. This ensures that the centre of the ball will permanently be placed in the plane of symmetric of the groove, figure 1.

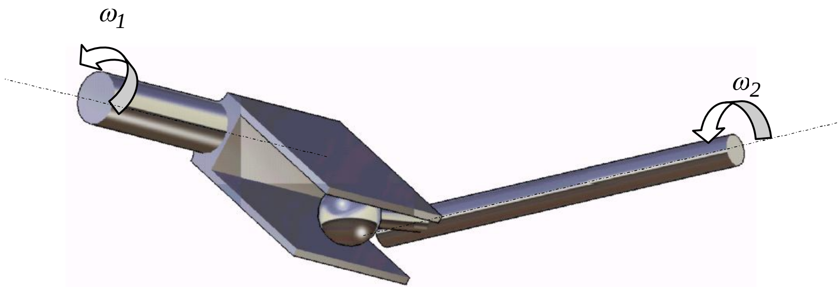

Figure 1. The design of point-plane pair

2. Theoretical considerations

Since in the structure of the mechanism from figure 1 only two mobile elements exist, the kinematics of the mechanism will be completely identified when the following dependence is known:

$$\theta_2 = \theta_2(\theta_1) \tag{1}$$

where θ_2 is the position angle of the driven element 2 and θ_1 is the input angle of the driving element.

The kinematical analysis of spatial kinematical chains can be made by accepted methodologies presented in technical literature. The most known method is the homogenous operators method proposed by Hartenberg and Denavit, [11] that in principle involves the transformation relations of coordinates of a point when the reference system is changed. Considering a series of coordinate systems attached to the elements of kinematical chain, conveniently oriented and positioned, the coordinate transformation relations for a point are expressed when passing from a frame to another, following the natural order of linking the elements of the kinematical chain until the initial element is reached again. According to McCarthy [12] the coordinate transformation expressed in tridimensional space when changing from frame $"1"$ to frame $"2"$ must describe the axis translation using the displacement vector $\boldsymbol{d}_{12}$ and the orientation of the new axis with respect to the old ones using the rotation matrix $\boldsymbol{R}_{12}$. Denoting by $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ the position vectors of the same point with respect to the two systems, the following relation can be written:

$$\boldsymbol{x}_1 = \boldsymbol{R}_{12}\boldsymbol{x}_2 + \boldsymbol{d}_{12} \tag{2}$$

The drawback of relation (2) consists in the fact that it has a inhomogeneous character due to the displacement vector $\boldsymbol{d}_{12}$. To surpass this disadvantage, Hartenberg and Denavit propose writing the relation into the next form:

$$\begin{bmatrix} \boldsymbol{x}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{12} & \boldsymbol{d}_1 \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_2 \\ 1 \end{bmatrix} \tag{3}$$

where $\boldsymbol{0} = (0, 0,0)$. The equation (3) can be written in contracted form:

$$\boldsymbol{X}_1 = \boldsymbol{T}_{12}\boldsymbol{X}_2 \tag{4}$$

where:

$$\boldsymbol{X}_1 = [x_1, y_1, z_1, 1]^T,$$
$$\boldsymbol{X}_2 = [x_2, y_2, z_2, 1]^T \quad (5)$$

and $\boldsymbol{T}_{12}$ is the operator characterizing the displacement of the new frame $"1"$ over the old frame $"2"$. The form (4) of coordinate transformation attests the homogenous character of the transformation relation. Next, a closed kinematical chain consisting of n elements is considered, each element having an attached coordinate system, and with the convention that the $"n+1"$ system is the same as $"1"$, by applying successively the relation 4, it results the following equation:

$$\boldsymbol{X}_1 = \boldsymbol{T}_{12}\boldsymbol{T}_{23}...\boldsymbol{T}_{n-1,n}\boldsymbol{T}_{n,n+1}\boldsymbol{X}_{n+1} \quad (6)$$

Since:

$$\boldsymbol{X}_{n+1} \equiv \boldsymbol{X}_1 \quad (7)$$
$$\boldsymbol{T}_{n,n+1} \equiv \boldsymbol{T}_{n,1}$$

and the relation (6) is valid for any point, it results:

$$\boldsymbol{T}_{12}\boldsymbol{T}_{23}...\boldsymbol{T}_{n-1,n}\boldsymbol{T}_{n,1} = \boldsymbol{I}_4 \quad (8)$$

where $\boldsymbol{I}_4$ is the unit matrix of fourth order. The equation (8) is the matrix closing equation for the kinematical chain. For the situation when in the structure of the kinematical chain only cylindrical pairs of the fourth class exist, with particular forms as revolute or prismatic, the convenient selection of the frames substantially simplifies the calculations.

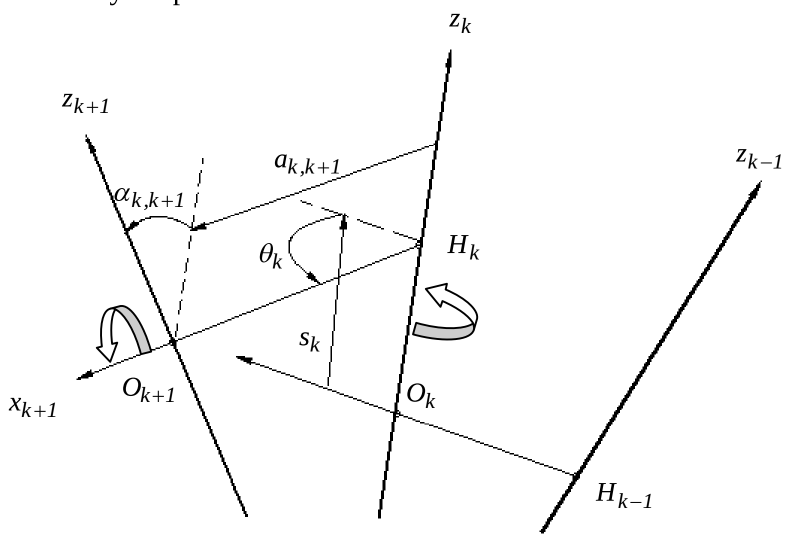

Figure 2. Hartenberg-Denavit convention

To this end, as a step in applying the Hartenberg Denavit methodology, after denoting the elements according to the order of jointing, the z_k axis are chosen to be oriented along the pairs and the x_k xis to be oriented along the common normal of the axis z_{k-1} and z_k, figure 2. The transformation from k to $k+1$ frame can be obtained via a roto-translation of parameters θ_k, s_k around the axis z_k described by the matrix:

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$$\boldsymbol{Z}(\theta_k, s_k) = \begin{bmatrix} cos\theta_k & -sin\theta_k & 0 & 0 \\ sin\theta_k & cos\theta_k & 0 & 0 \\ 0 & 0 & 1 & s_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

followed by a roto-translation around the x_{k+1} axis expressed by the matrix:

$$\boldsymbol{X}(\alpha_{k,k+1}, a_{k,k+1}) = \begin{bmatrix} 1 & 0 & 0 & a_{k,k+1} \\ 0 & cos\alpha_{k,k+1} & -sin\alpha_{k,k+1} & 0 \\ 0 & sin\alpha_{k,k+1} & cos\alpha_{k,k+1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

3. Kinematical analysis of the mechanism

A proposed solution for direct coupling of two shafts using a higher pair of first class is presented in figure 3 and a review of the possible solutions for obtaining this specific higher pair presented in [5-6]. For the present work, the point-surface alternative offered in [13-14] is preferred. The kinematical study of the transmission where the driving element is denoted 1 and the driven one 2, aims finding the following dependence:

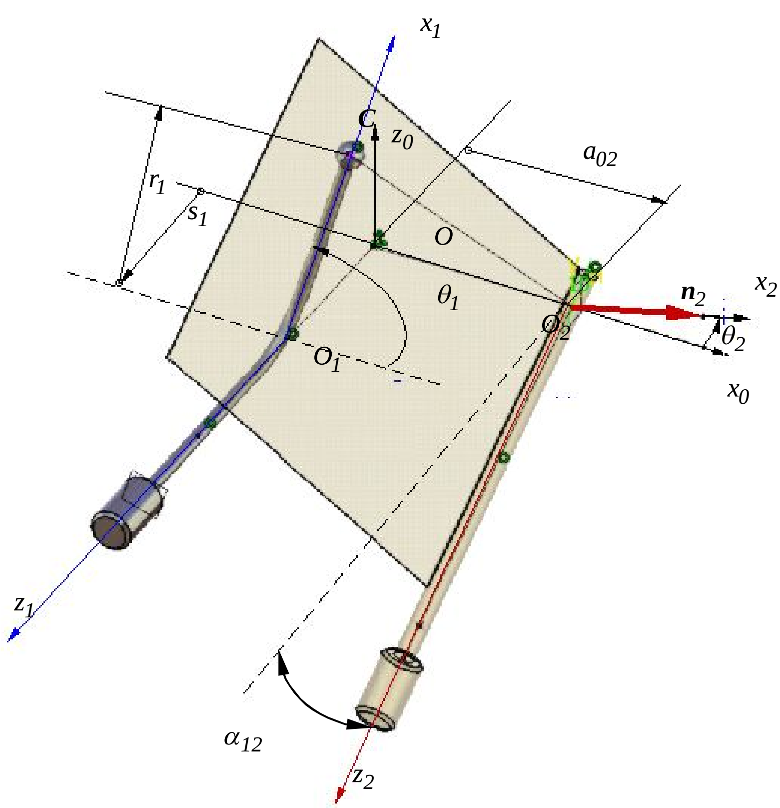

Figure 3. Two shafts coupled via a point-surface pair of first class

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$$\theta_2 = \theta_2(\theta_1) \tag{11}$$

In order to apply the Hartenberg-Denavit method for the transmission shown in figure 2, the replacement of the mechanism with an equivalent one - having in structure only lower pairs with well defined axes, is required. Considering the fact that between the mobile elements a pair of first class is obtained, this is equivalent to a kinematical chain consisting in four elements and five pairs: three revolute and two prismatic pairs. Applying the Hartenberg-Denavit method to the replacing mechanism, the dependence between the position angle of the driven element and θ_1, the position angle of the driving element is found. The disadvantage of the methodology consists in the fact that, for obtaining the relation (11), all the motions from the pairs of the replacing mechanism must be established even if these dependencies have no physical significance.

Next it will be proved that the study of the mechanism can be simplified considerably by direct employment of the constraint imposed by the kinematical pair between the mobile elements of the mechanism. To this end, the constructive and kinematical parameters of the mechanism are specified, as in figure 1. Three coordinate systems are considered: the frame $"0"$ attached to the ground, $"1"$ fixed to the driving element and $"2"$ fixed to the driven element. The z_1, z_2 axes are chosen according to Hartenberg-Denavit convention, along the axes of revolute pairs made with the ground. The x_0 axis of the ground is directed along the common normal of z_1 and z_2 axes. The origin O is the foot of the common normal from the z_1 axis. The Oz_0 axis is normal to the plane made by Oz_1 and Ox_0. The origin O_1 of the frame 1 is placed on the axis z_1 at a distance s_1 (with sign) from the origin of the immobile frame. The O_1x_1 axis is defined by the O_1 point and the contact point C where the higher pair is completed. The distance from the rotation axis r_1 z_1 to the C point, r_1 is a constant given that it is a constructive parameter of the transmission. Finally, the last frame has the O_2 origin positioned in the other foot of common normal and the z_2 axis is directed along $\boldsymbol{n}_2$ - the normal to the plane involved in the higher pair construction. The existence of the higher pair in the structure of the mechanisms imposes the geometrical constraint that the contact point C positioned on the element 1 is simultaneously placed in the plane attached to the element 2, permanently. An arbitrary point having the vector of position $\boldsymbol{r}$ in the plane of normal $\boldsymbol{n}$ passing through the point with the position vector $\boldsymbol{r}_0$ is considered. The equation of the plane in vectorial form is:

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0 \tag{12}$$

The plane attached to the driven element always passes through the point O_2 . The condition that the point C is contained into the plane P is written, according to relation (12) as:

$$\boldsymbol{n} \cdot (\boldsymbol{r}_C - \boldsymbol{r}_{O_2}) = 0 \tag{13}$$

The relation (13) can be used only if all the vectors are expressed by their projections on the axes of the same frame. In the current case, the vectors are expressed in the $"0"$ system, attached to the ground. The contact point C is known by the components in the frame $"1"$ and the position vector is:

$$\boldsymbol{X}C_1 = \begin{bmatrix} r_1 & 0 & 0 & 1 \end{bmatrix}^T \tag{14}$$

The operator that overlaps the system $"0"$ over the system $"1"$ has the following matrix:

$$\boldsymbol{T}_{01} = \boldsymbol{X}(\pi / 2, 0)\boldsymbol{Z}(\theta_1, s_1) \tag{15}$$

The position vector of the point C in the immobile system after the transformation is:

$$XC_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\pi/2 & -\sin\pi/2 & 0 \\ 0 & \sin\pi/2 & \cos\pi/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta & 1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_1\cos\theta_1 \\ -s_1 \\ r_1\sin\theta_1 \\ 1 \end{bmatrix} \tag{17}$$

In order to express the components of the $\boldsymbol{n}_2$ normal in the frame $"0"$, the point N is chosen on the axis O_2x_2, with the coordinates:

$$XN_2 = [r_2\, 0\, 0\, 1]^T \tag{18}$$

The superposition of the $"0"$ frame over $"2"$ frame is described by the matrix:

$$\boldsymbol{T}_{02} = \boldsymbol{X}(\alpha_{02}, a_{02})\boldsymbol{Z}(\theta_2, s_2) \tag{19}$$

$$XN_0 = \begin{bmatrix} 1 & 0 & 0 & a_{02} \\ 0 & \cos\alpha_{02} & -\sin\alpha_{02} & 0 \\ 0 & \sin\alpha_{02} & \cos\alpha_{02} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_2\cos\theta_2 + a_{02} \\ r_2\cos\alpha_{02}\sin\theta_2 \\ r_2\sin\alpha_{02}\sin\theta_2 \\ 1 \end{bmatrix} \tag{20}$$

Based on figure 3, it can be noticed from relation (20) that:

$$\alpha_{02} = \alpha_{12} + \pi/2 \tag{21}$$

In the fixed system, the O_2 point has the coordinates:

$$XO2_0 = [a_{02}\, 0\, 0\, 1] \tag{22}$$

Now, using the above expressions, the relation 13 becomes:

$$(\boldsymbol{XN}_0 - \boldsymbol{XO2}_0)\cdot(\boldsymbol{XC}_0 - \boldsymbol{XO2}_0) = 0 \tag{23}$$

and expressing it by explicit manner it takes the form:

$$(\cos\theta_2\cos\theta_1 + \sin\alpha_{02}\sin\theta_2\sin\theta_1)r_1 - a_{02}\cos\theta_2 - s_1\cos\alpha_{02}\sin\theta_2 = 0 \tag{24}$$

The equation (24) allows for finding the sought after dependency $\theta_2 = \theta_2(\theta_1)$, specifically:

$$\theta_2 = a\tan\frac{r_1\cos\theta_1 - a_{02}}{s_1\cos\alpha_{02} - r_1\sin\alpha_{02}\sin\theta_1} \tag{25}$$

The occurrence of the function $a\tan(x)$ in relation (25) leads to discontinuities in the graphical representation of the θ_2 function. To obtain a continuous variation of θ_2 the following function is used to replace it:

$$\Theta_2 = \int_0^{\theta_1} \frac{d\theta_2(\theta_1)}{d\theta_1} d\theta_1 + \theta_2(0) \tag{26}$$

The plots of the two functions (25) and (26) are presented in figure 4.

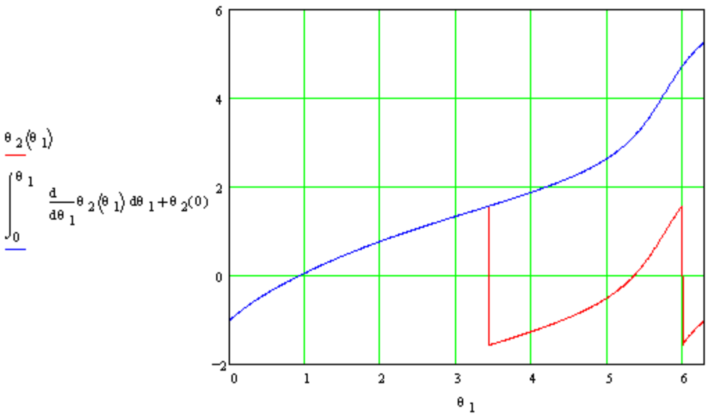

Figure 4. The position angle of the driven element

The validity of relation (26) was tested by modeling the mechanism in CAD software, with the following constructive data: $a_{02} = 60$ (mm); $\alpha_{02} = 120^o$; $r_1 = 70$ (mm); $s_1 = 50$ (mm).

The figure 5 presents in comparative manner the variations of θ_2 angle versus the position angle of the driving element.

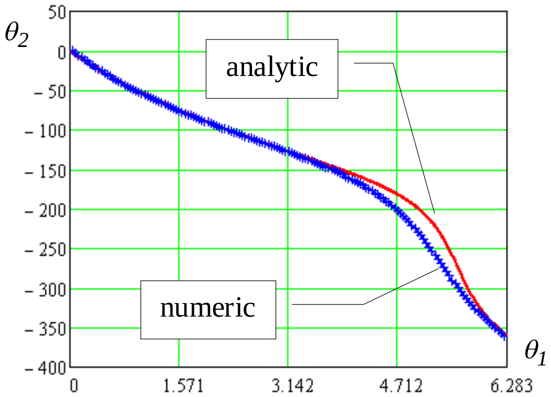

Figure 5. Comparison between numerical and analytical results

The relation (25) is differentiated with respect to time and divided by $\dot{\theta}_1$; the expression of the transmission ratio of the mechanism is obtained:

$$i_{12} = -\frac{r_1}{a_{02}} \frac{\frac{s_1}{a_{02}} \sin\theta_1 \cos\alpha_{02} - \frac{r_1}{a_{02}} + \sin\alpha_{02} \cos\theta_1}{\left[1 - \frac{r_1}{a_{02}} \cos\theta_1\right]^2 + \left[\frac{s_1}{a_{02}} \cos\alpha_{02} - \frac{r_1}{a_{02}} \sin a_{02} \sin\theta_1\right]^2} \tag{27}$$

Two comments should be done about the expression (27): the denominator is positive or equal to zero, the equality to zero assumes both parenthesis are simultaneously null:

$$\begin{cases} 1 - \frac{r_1}{a_{02}} \cos\theta = 0 \\ \frac{s_1}{a_{02}} \cos\alpha_{02} - \frac{r_1}{a_{02}} \sin a_{02} \sin\theta_1 = 0 \end{cases} \tag{28}$$

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Solving the system (28) with respect to r_1 and s_1, it results:

$$r_1(\theta_1) = a_{02} / \cos\theta_1 \qquad (29)$$

$$s_1(\theta_1) = a_{02} \tan\alpha_{02} \tan\theta_1$$

representing the parametric equations of the self-locking curve from (r_1, s_1) plane. An actual case of self-locking curve is presented in figure 6.

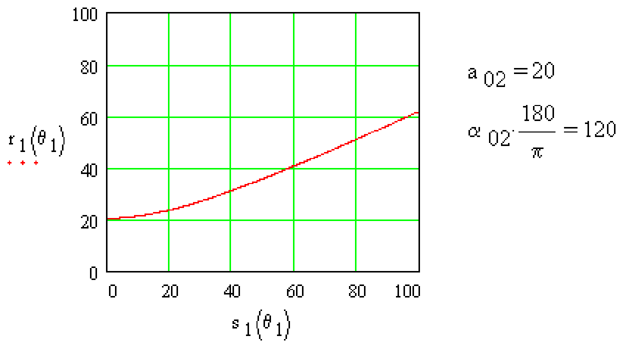


Figure 6. Self-locking curve

The shape of the self-locking curve leads to the conclusion that, for any value of the parameter s_1 there is a value of parameter r_1 for which the mechanism is locking and the reverse is also valid.

The second comment refers to the sign of the transmission ratio. It can be observed that the entire kinematics of the mechanism is ruled by the values of: α_{02}, r_1 / a_{02}, s_1 / a_{02}. In figure 7 is presented the variation of the transmission ration for a constant angle α_{02} and all possible situations for the values of the parameters $s1$ and r_1 versus a_{20}. From figure 7 it can be remarked that, for a given pair of values for parameters r_1, s_1, the motion of the driven element is oscillatory when the transmission ratio changes its sign or of continuous rotation for constant sign of the transmission ratio.

Following this observation, for a designer is useful to identify which type of the motions of the mechanism is obtained when the constructive parameters r_1 and s_1 are adopted. To this purpose, for a finite set of numbers b with $N+1$ elements, the following function is defined:

$$\Sigma = \sum_{k=0}^{N} sign(b_k)/(N+1) \qquad (30)$$

Obviously, when all the elements of the set are positive, $\Sigma = 1$ and when all elements are negative, $\Sigma = -1$. If at least two elements have dissimilar signs, it results:

$$|\Sigma| < 1 \qquad (31)$$

For the considered mechanism, the transmission ratio is regarded as a function of θ_1, s_1 and r_1 parameters:

$$i_{12} = i_{12}(\theta_1, s_1, r_1) \qquad (32)$$

The variation domains of the three variables are discretised and the following matrix is defined:

$$\Sigma_{i,j} = \left(\sum_{k=0}^{N} sgn[\, i_{12}(\theta_{1_k}, s_{1_i}, r_{1_j})] \right) \Big/ (N+1) \qquad (33)$$

where N is the number of sub-intervals of a complete rotation of the input shaft.

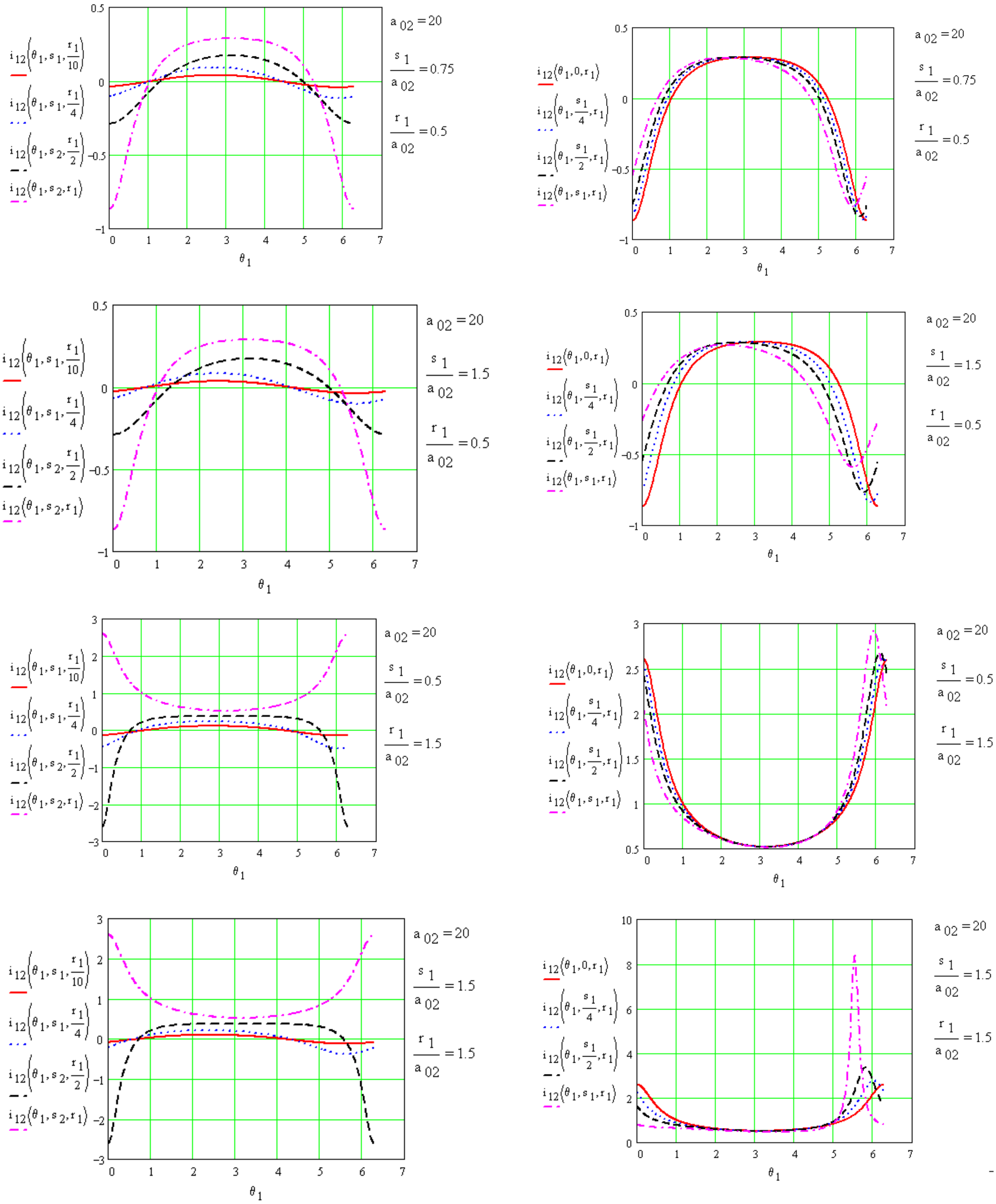
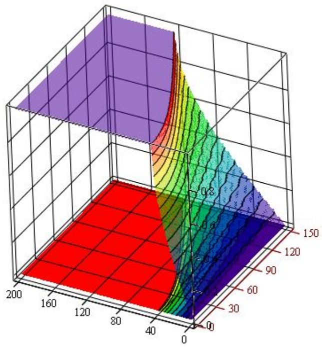
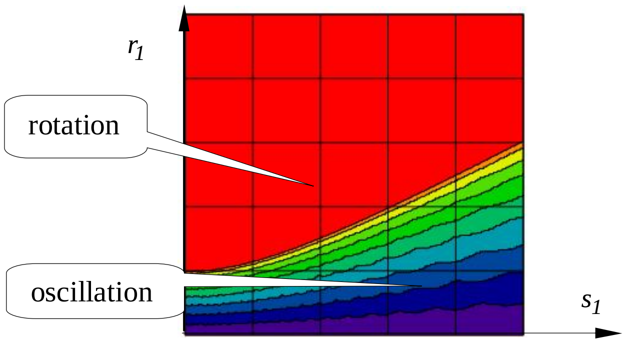


Figure 7. The effect of constructive parameters upon the transmission ratio of the mechanism

The motion of the output element will be a continuous rotation motion in all the situations when the elements of the matrix Σ are equal to unit. In all the other points the motion will be an oscillatory one. In figure 8.a there are presented the spatial plot of the Σ matrix and the level curves and in figure 8.b, there are evidenced the domains of rotation and oscillatory motion. The boundary between the two domains is the self-locking curve from figure 6.



a)



b)

Figure 8. Finding the domains of oscillation and rotation of the driven element

4. Conclusions

The paper proposes a mechanism designed to transmit the motion between two shafts with crossed axes. The proper motion is accomplished by means of a class one pair of point-surface type. In contrast to other constructive solutions where the higher pair is obtained via two contacting curves, the present case materializes the higher pair without the employment of a intermediate element. The Hartenberg-Denavit transformation relations are applied to express the geometrical constraint that defines the higher pair in the ground frame and the analytical dependence of between the rotations of the two shafts is obtained. Subsequently the self-locking condition of the mechanism is established and there are identified the variation domains of constructive parameters where the driven element performs oscillatory or rotational motions.

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