Mathematical model, the solution of boundary value problems for alloys with shape memory effect by analytical, numerical-analytical and numerical methods

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Abstract. In this paper we consider the application of analytical, numerical-analytical and numerical methods for solving boundary value problems for structures made from alloys with shape memory. As the defining relations, the equations of structural-analytic mesomechanics were applied.

Introduction

Materials with the shape memory effect (SME) have unique properties that are absent in most traditional materials used in engineering applications. They are called intellectual materials of the XXI century, as well as smart materials. Accordingly, their use provides new design capabilities, which makes it possible both to improve the characteristics of devices and to offer innovative solutions. Alloys with shape memory (ASM) are widely used in medicine, engineering, aircraft building, construction, etc. To calculate the stress-strain state and the engineering solution of problems, it is necessary to develop an appropriate mathematical apparatus that allows to reflect the real mechanical properties of ASM. The development of a mathematical model of ASM that takes into account the features of their behavior in simple and complex deformations, as well as the creation of methods for the analytical and numerical solution of boundary value problems for structures and products from SPF are relevant.

The basic equations of structural-analytic mesomechanics are given in [1-3]. They make it possible to perform a forecast of the basic deformation phenomena in alloys possessing shape memory effects. In this paper, the defining relations of the deformation type are presented, and therefore it becomes possible to apply the mathematical apparatus of the theory of plasticity for solving boundary value problems with elements made of ASM. Within the framework of this approach, some boundary-value problems that admit an analytical solution are solved-the problem of deformation of a long thick-walled tube under internal pressure; numerical-analytical solution by the method of boundary elements - calculation of the truss structure; numerical methods - rod system, bending of the beam, and also plates with different support of the ends. Only isothermal martensitic transformations are considered. A detailed derivation of the defining relations is given in [3].

1.Determined relations for an alloy with shape memory effect

at the load stage:

$$\varepsilon_i^{\Sigma} = \frac{\sigma_i}{E} + \left(\frac{2}{3} B_\phi \sigma_i \frac{k\left(\sigma_i - \sigma_{н\uparrow}^{A \to M} \right)}{M_н - M_к} \right) H\left(\sigma_i - \sigma_{н\uparrow}^{A \to M} \right) H\left(\sigma_{к\uparrow}^{A \to M} - \sigma_i \right) H(\dot{\sigma}_i) \qquad (1)$$

At the unloading stage:

$$\varepsilon_i^{\Sigma} = \frac{\sigma_i}{E} + \left(\frac{2}{3}B_\phi\sigma_i^0 \frac{k\left(\sigma_i + \sigma_{\Pi\gamma} - \sigma_{\text{н}\uparrow}^{A\to M}\right)}{M_\text{н} - M_\text{к}}\right) H\left(\sigma_i - \sigma_{\text{к}\uparrow}^{A\to M}\right) H\left(\sigma_{\text{н}\uparrow}^{A\to M} - \sigma_i\right) H(\dot{\sigma}_i) \qquad (2)$$

where σ_i – voltage intensity; E – elastic modulus; B_Φ – material constant; σ_i^0 – intensity of voltage

accumulated during the loading phase; $k = \dfrac{T_0}{q_0}D_i$; T_0 – thermodynamic equilibrium temperature; q_0 –

thermal reaction effect; D_i – distortion of phase transformation; $\sigma_{\text{н}\uparrow}^{A\to M} = \dfrac{T_D - M_\text{н}}{k}$ – the voltage of the

onset of direct martensitic transformation; T_D – deformation start temperature; $M_\text{н}$, $M_\text{к}$ – the temperature of the beginning and the end of the direct martensitic transformation, respectively; $\sigma_{\text{к}\uparrow}^{A\to M} = \dfrac{T_D - M_\text{к}}{k}$ – endurance of martensitic transformation, $\sigma_{\Pi\gamma} = \dfrac{\Delta}{k}$ – limit of pseudoelasticity at

unloading; Δ – width of hysteresis of the martensitic transformation; $\dot{\sigma}_i = \dfrac{d\sigma_i}{dt}$; $H(\dots)$ – Heaviside

function.

The diagram at the stage of loading in the regime of austenitic pseudoelasticity (ferroelasticity), obtained by the formula (1) is shown in Fig. 1.
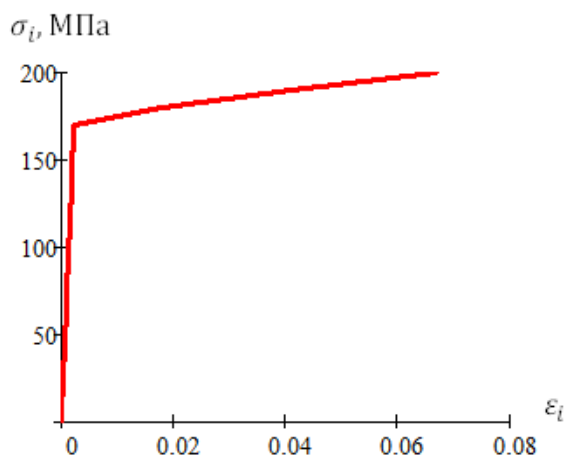

Figure 1. Deformation diagram $\sigma_i - \varepsilon_i$ under active loading.

2. Boundary value problems for ASM

Initial data for the alloy TiNi are presented in Table 1.

Table 1. The initial data for Ti-Ni alloy.

Name	Formula
Characteristic temperatures of forward and reverse martensitic transformation	$M_\text{н} = 330K$, $M_\text{к} = 320K$
	$A_\text{н} = 370K$, $A_\text{к} = 380K$
Crystallographic constant k	$k = 0{,}29 \text{ KMPa}^{-1}$
Constant B_Φ	$B_\Phi = 0{,}06\cdot10^{-2} \text{ MPa}^{-1}$
Elastic modulus E	$E = 7{,}42\cdot10^4 \text{ MPa}$

Deformation temperature T_{D1}	$T_{D1} = A_\kappa = 380\text{K}$
Deformation temperature T_{D2}	$T_{D2} = (A_\kappa + A_\text{н})/2 = 375\text{K}$
Voltage of the beginning of a direct martensitic transformation at deformation temperature T_{D1} (phase yield strength) $\sigma_{\text{н}1\uparrow}^{A\to M} = \sigma_{T1}$	$\sigma_{\text{н}1\uparrow}^{A\to M} = \sigma_{T1} = 172\,\text{MPa}$
Voltage of the beginning of a direct martensitic transformation at deformation temperature T_{D2} (phase yield strength) $\sigma_{\text{н}2\uparrow}^{A\to M} = \sigma_{T2}$	$\sigma_{\text{н}2\uparrow}^{A\to M} = \sigma_{T2} = 155\,\text{MPa}$
Voltage of the end of a direct martensitic transformation at deformation temperature T_{D1} $\sigma_{\kappa1\uparrow}^{A\to M} = \sigma_{\max 1}$	$\sigma_{\kappa1\uparrow}^{A\to M} = \sigma_{\max 1} = 207\,\text{MPa}$
Voltage of the end of a direct martensitic transformation at deformation temperature T_{D2} $\sigma_{\kappa2\uparrow}^{A\to M} = \sigma_{\max 2}$	$\sigma_{\kappa2\uparrow}^{A\to M} = \sigma_{\max 2} = 190\,\text{MPa}$
The module of ferroelasticity $E_{\Phi\gamma1} = E_{\kappa1}$ (tangent module) at the deformation temperature T_{D1}	$E_{\Phi\gamma1} = E_{\kappa1} = 414\,\text{MPa}$
The module of ferroelasticity $E_{\Phi\gamma2} = E_{\kappa2}$ (tangent module) at the deformation temperature T_{D2}	$E_{\Phi\gamma2} = E_{\kappa2} = 452\,\text{MPa}$

2.1. Analytical calculation of a thick-walled pipe from ASm under internal pressure, under conditions of direct martensitic transformation

Formulation of the problem. Determine the voltage distribution in the cross section of a thick-walled pipe $r_1 = 10$ мм, $r_2 = 20$ мм under internal pressure, made of alloy *TiNi*, having a shape memory effect, under conditions of direct martensitic transformation.

Assumptions:
1) the material characteristics are calculated at different but constant temperatures;
2) the material undergoes elastic and inelastic (phase) deformations;
3) we take a bilinear model of material deformation.

The results of the calculation for the deformation temperature $T_D = A_\kappa$ are presented in table 2.

Table 2. Results of calculation of pipe yield radius
depending on internal pressure.

N	Yield radius r_T, мм	Load p, MPa
1	10	75
2	13	110
3	15	125
4	18	137
5	20	139

Graphs of voltage distribution normalized to $\sigma_{\text{н}1\downarrow}^{A\to M}$ for different pressure values are shown in the figure 2.
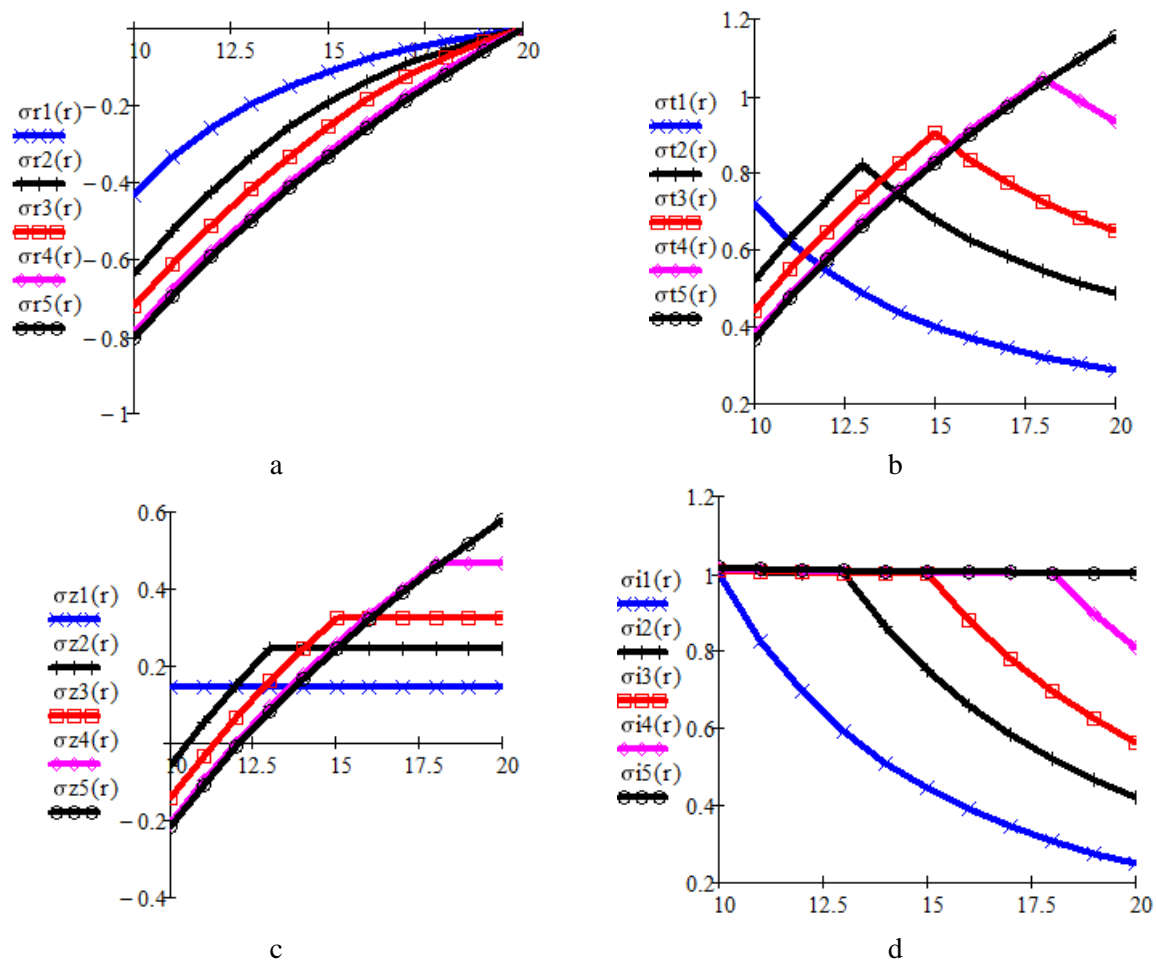
Fogure 2. The distribution of voltage normalized to $\sigma_{H\downarrow}^{A\to M}$ by pipe radius:
a) diagram of radial voltage; b) the diagram of the circumferential voltage;
c) diagram of axial voltage; d) an equivalent voltage diagram.

2.2. Core systems
Calculation of a statically indeterminate rod system.

Formulation of the problem. The rod system consists of two rods made of ASM with different mechanical characteristics (Figure 3). Both materials have the same elastic moduli E. Tangent modules and voltage of the onset of direct martensitic transformation $\sigma_{H\uparrow}^{A\to M}$ are different. The external horizontal force is $F = 2 \cdot A \cdot \sigma_{H\uparrow}^{A\to M}$ and is applied at the junction of two rods, where A is the cross-sectional area of both rods. Deformation temperature of the first rod T_{D1}, of the second T_{D2}. It is required to determine the stresses in the rods by the finite element method, using the iterative Newton-Raphson algorithm with the following initial data:

Table 3. Initial data.

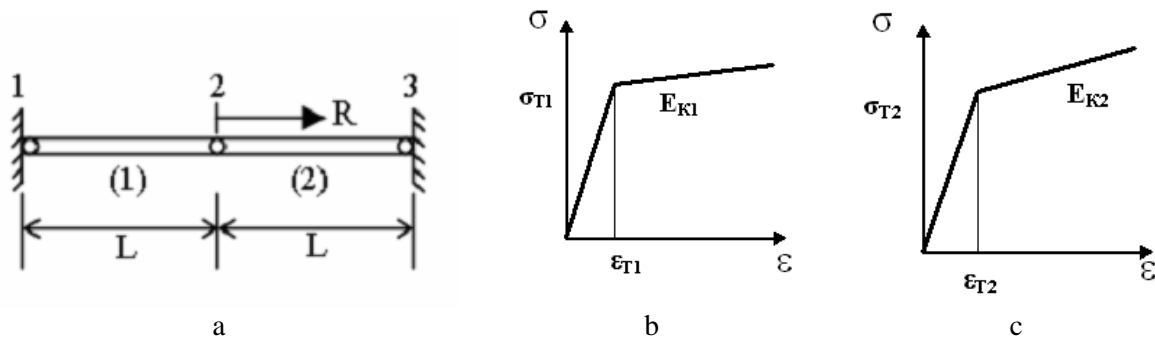|	
Cross-sectional areas of rods A	$A = 314{,}159$ мм^2
Length of each rod L	$L = 50$ мм
Load	$R = 2 \cdot A \cdot \sigma_{T1} = 1{,}083 \cdot 10^5$ Н

Figure 3. The calculation scheme. Deformation diagrams.

Table 4. Results of the calculation of the rod system.

Voltage in the first rod:	Voltage in the second rod:
$\sigma_1 = \sigma_{T1} + E_{k1}(\Delta_1 \cdot \varepsilon_{T1}) = 180\,\text{MPa}$	$\sigma_2 = -\sigma_{T2} - E_{k2}(\Delta_2 \cdot \varepsilon_{T2}) = -164\,\text{MPa}$

Stresses in the first rod are tensile, in the second - compressive.

2.3. Calculation of a statically indeterminate farm
For calculation, the numerical-analytical method of boundary elements [4] is used in conjunction with the methods of elastic solutions and variable elasticity parameters.

Formulation of the problem. Determine the internal forces in the rods of a statically indeterminate shape made from ASM, figure 4.

Table 5. Initial data.

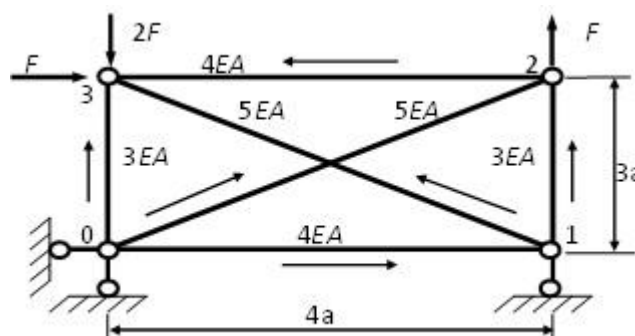| Name	Formula
Tangent module, Pa	$E_{\kappa} = \dfrac{15q_0}{T_D \cdot D_{31}^2}(M_{\text{н}} - M_{\kappa}) = 2{,}469 \cdot 10^{11}$
Phase yield strength, Pa	$\sigma_T = 6 \cdot 10^8$
Deformation temperature, К	$T_D = 300$
Cross-sectional areas of rods	$A = 3{,}142 \cdot 10^{-4}\,\text{м}^2$
Load	$F = EA \cdot 0{,}002$
Geometric parameter	$a = 1\,\text{м}$

Figure 4. The calculation scheme. Boundary element grid.

Table 6. Results of the calculation of the shape.

Elastic solution	The solution of the nonlinear problem
$N^{0-1}=2{,}727\cdot10^5\,\text{H}$; $N^{0-2}=2{,}089\cdot10^5\,\text{H}$;	$N^{0-1}=1{,}827\cdot10^5\,\text{H}$; $N^{0-2}=3{,}214\cdot10^5\,\text{H}$;
$N^{0-3}=-6{,}751\cdot10^5\,\text{H}$; $N^{1-2}=3{,}145\cdot10^5\,\text{H}$;	$N^{0-3}=7{,}307\cdot10^3\,\text{H}$; $N^{1-2}=2{,}470\cdot10^5\,\text{H}$;
$N^{1-3}=-3{,}409\cdot10^5\,\text{H}$; $N^{2-3}=-1{,}671\cdot10^5\,\text{H}$.	$N^{1-3}=-2{,}284\cdot10^5\,\text{H}$; $N^{2-3}=-2{,}571\cdot10^5\,\text{H}$.

2.4. Calculation of a beam on two supports loaded with a distributed load
A single-span beam of square cross-section, made of ASM with the following parameters, presented in Table 7, was calculated.

Table 7. Initial data.

Load, MPa	$q = 10$
Deformation temperature, к	$T_D = A_\kappa = 380$
Length, мм	$a = 10$
Thickness, мм	$h = 1$
Width, мм	$b = 1$
The axial moment of inertia, мм^4	$J_z = b\cdot h^3/12 = 0{,}083$

The calculation was carried out by three methods: Ritz, elastic solutions, variable elastic moduli. For approximation of the deformed beam axis, the Vlasov method was used. The problem was solved with two boundary conditions: 1) both edges are hinged-supported, 2) both edges are rigidly embedded.

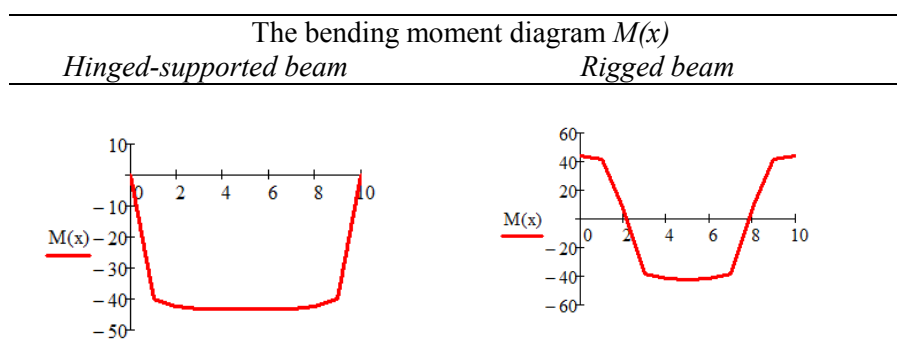**Table 8.** Results of the calculation of the beam by the Ritz method.

The bending moment diagram *M(x)*	
Hinged-supported beam	*Rigged beam*

Table 9. Maximum voltage.

The Ritz method	The method of elastic solutions	The method of variable elasticity parameters
Hinged-supported beam cross-section $\sigma\,(a/2;h/2)$, MPa		
177	176	176
Rigged beam cross-section $\sigma\,(0;h/2)$ and $\sigma\,(a;h/2)$, MPa		
178	174	174

2.5. Plates

Formulation of the problem. A square plate of ASM loaded with a distributed load with two anchors along the contour: articulated and rigidly fixed. The calculation was carried out by three methods: Ritz, Vlasov-Kantorovich together with the method of elastic solutions and the finite difference method together with the method of elastic solutions.

Table 10. Initial data.

Load, MPa	$q = 10$
Deformation temperature, к	$T_D = A_к = 380$
Length, мм	$a = 10$
Thickness, мм	$h = 1$

Table 11. The results of calculating the plate by the Ritz method.

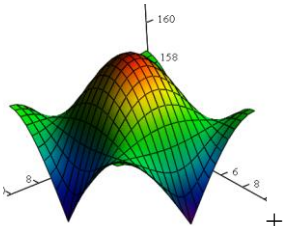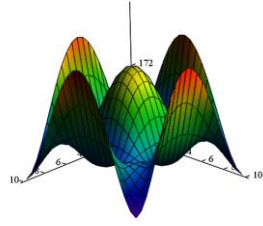| Voltage intensity σ_i, MPa	
Hinged-supported plate	*Rigged plate*

In the center of the plate $\sigma_{i,0} = 179$

In the center of the plate and in the middle of the sides
$\sigma_{i,0} = \sigma_{i,C} = 173$

Table 12. Results of calculating the maximum voltage intensity for a hinged-supported plate by various methods, MPa

The Ritz method	The Vlasov – Kantorovich method together with the method of elastic solutions	The finite difference method together with the method of elastic solutions
178	173	174

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