Linear stability of parallel flow of liquid metal in a rectangular duct driven by a constant pressure gradient under the influence of a uniform magnetic field

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Abstract
Linear stability of the plane Poiseuille flow of liquid metal under a uniform magnetic field is numerically studied. The liquid metal is driven by a constant pressure gradient in a rectangular duct. The induced magnetic field and the Joule heating are neglected in this analysis. The governing equations are the continuity of mass, momentum equation, Ohm's law and conservation of electric charge. The solution of basic state is well known as the Hartmann flow in the limit of the large aspect ratio of the cross-section of the duct. It is supposed that disturbance has periodicity along the direction of the basic flow. The linear stability of the basic flow that depends only on the Hartmann number is analyzed by a finite difference method discretized by a fourth order central difference scheme together with the use of HSMAC algorithm. We obtained the phase velocity of Tollmien-Schlichting wave and the Reynolds number as the neutral stability state when input parameters such as the aspect ratio, the Hartmann number and the wavenumber were given. It is predicted that the critical Reynolds number at the onset of instability is about 4.0 x 10^5 and the wavenumber is about 1.15 when the Hartmann number is 5.

Key words: Linear stability analysis, Hartmann number, liquid metal flow

Introduction
Use of magnetic field is one of the most promising method for controlling liquid metal flows, which are encountered in steel making process, semiconductor crystal growth and so on. The static magnetic field can be used to stabilize liquid metal flows, whereas the alternating magnetic field can be used to drive and stir liquid metal flows as well as to generate Joule heating. Our interest here is the stability of liquid metal flows when uniform static magnetic field is imposed. Among various magneto-hydrodynamic flows, the plane Poiseuille flow in a duct having rectangular cross section under the uniform magnetic field has been selected and focused. This kind of magnetohydrodynamic flow is call as Hartmann flow. The linear stability of such a parallel flow under the influence of a magnetic field was firstly studied by Lock [1]. It was concluded that the critical Reynolds number is found to rise very rapidly with increasing the Hartmann umber. Kakutani [2] investigated the stability of the modified plane Couette flow in the presence of a transverse magnetic field. It was found that the critical Reynold number takes its minimum value of 3.8 x 10^5 when Ha = 5.4. The stability of duct flow with rectangular cross-section depends on the aspect ratio of the cross-section of the duct. Tatsumi and Yoshimura [3] investigated the linear stability of the duct flow for the range of aspect ratio from 1 to 10 and the Reynold number up to 50000. They reported that the duct flow could be unstable in the range of aspect ratio more than 3.2. Priede *et al*. [4] analyzed linear stability of Hunt's flow which is a liquid metal flow in a square duct subjected to a transverse magnetic field. In this study, linear stability analyses were carried out for a duct flow of liquid metal under the influence of uniform magnetic field.

Configuration of problem and governing equations
Figure 1 shows the schematic model considered in this study as one of the simplest model among various magnetohydrodynamic duct flows. The duct is rectangular and the aspect ratio of the cross-section is assumed to be large since the duct flow for small aspect ratio is linearly stable [3]. The primary flow direction is z as shown in Fig. 1 and a uniform magnetic field is applied in x-direction. We consider the case that all the walls are electrically insulated.

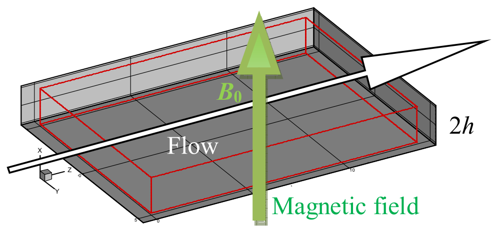


Fig. 1: Sketch of the duct flow problem with magnetic field

It is assumed that the fluid is an incompressible Newtonian fluid and the heat source such as viscous dissipation and Joule heating are neglected. In addition to this, since the magnetic Reynolds number is usually small in this kind of problems, the induced magnetic field is neglected in this analysis. The governing equations are written as follows.

$$\partial_j u_j = 0, \tag{1}$$

$$\rho\left(\partial_t u_i + u_j \partial_j u_i\right) = -\partial_i p + \mu \partial_j^{\,2} u_i + \varepsilon_{ijk} j_j b_k, \tag{2}$$

$$\partial_j j_j = 0, \tag{3}$$

$$j_i = \sigma\left(-\partial_i \phi + \varepsilon_{ijk} u_j b_k\right). \tag{4}$$

The boundary conditions are as follows. Here, $2h$ is the height of the duct and $2a$ is spanwise length of duct.

$$x = \pm h: \quad u = v = w = j_x = 0, \quad \text{and} \quad y = \pm a: \quad u = v = w = j_y = 0 \tag{5}$$

Basic state and disturbance equations

In the present analyses, we consider the case that the duct length along the primary flow is sufficiently long. The basic states without having any disturbance are determined by the force balance between the pressure gradient, viscous force and electromagnetic force by numerically solving the following three equations expressed in a dimensionless form.

$$\left(\partial_X^2 + \partial_Y^2\right)\overline{W} + 2 - Ha^2 \overline{J}_Y = 0, \tag{6}$$

$$\partial_X \overline{J}_X + \partial_Y \overline{J}_Y = 0, \tag{7}$$

$$\overline{J}_X = -\partial_X \overline{\Phi}, \quad \overline{J}_Y = -\partial_Y \overline{\Phi} + \overline{W}, \quad \overline{J}_Z = 0. \tag{8}$$

It is noted that this basic state is not related to the Reynolds number but to the Hartmann number, whose square value is the ratio between the electromagnetic force to the viscous force.

In order to examine the linear stability of the basic flow, it is assumed that disturbance has periodicity along the direction of basic flow and therefore the variables such as velocity, pressure, electric current density and potential are presumed as the summation of the basic state and the disturbance as follows:

$$\varphi(x,y,z,t) = \overline{\varphi}(x,y) + \tilde{\varphi}(x,y)\exp(i\alpha z + st), \quad \text{where} \quad \varphi = \left(u_i, p, j_i, \phi\right) \tag{9}$$

Here, α represents the wavenumber, and s represents the complex number, whose real part indicates the linear growth rate and imaginary part is the angular frequency. If $s_r > 0$ the disturbance grows and the basic flow is unstable while if $s_r < 0$ it attenuates and the basic flow is stable. After substituting eq. (9) into eqs. (1) to (5), and making non-dimensionalization, we have the following dimensionless disturbance equations linearized.

$$\partial_X \tilde{U} + \partial_Y \tilde{V} + ik\tilde{W} = 0, \tag{10}$$

$$S\tilde{U} + i\overline{W}k\tilde{U} = -\partial_X \tilde{P} + Re^{-1}\left(\partial_X^2 + \partial_Y^2 - k^2\right)\tilde{U}, \tag{11}$$

$$S\tilde{V} + i\overline{W}k\tilde{V} = -\partial_Y \tilde{P} + Re^{-1}\left(\partial_X^2 + \partial_Y^2 - k^2\right)\tilde{V} + Ha^2 Re^{-1}\tilde{J}_Z, \tag{12}$$

$$S\tilde{W} + \tilde{U}\partial_X \overline{W} + \tilde{V}\partial_Y \overline{W} + i\overline{W}k\tilde{W} = -ik\tilde{P} + Re^{-1}\left(\partial_X^2 + \partial_Y^2 - k^2\right)\tilde{W} - Ha^2 Re^{-1}\tilde{J}_Y, \tag{13}$$

$$\partial_X \tilde{J}_X + \partial_Y \tilde{J}_Y + ik\tilde{J}_Z = 0, \tag{14}$$

$$\tilde{J}_X = -\partial_X \tilde{\Phi}, \quad \tilde{J}_Y = -\partial_Y \tilde{\Phi} + \tilde{W}, \quad \tilde{J}_Z = -ik\tilde{\Phi} - \tilde{V}. \tag{15}$$

The definition of the characteristic velocity, dimensionless variables and non-dimensional number are shows as follows:

$$\begin{gathered} u_{ref} = -\frac{\left(d\overline{p}/dz\right)h^2}{2\rho\nu}, \quad (X,Y,Z) = \frac{(x,y,z)}{h}, \quad \left(\overline{W},\tilde{U},\tilde{V},\tilde{W}\right) = \frac{\left(\overline{w},\tilde{u},\tilde{v},\tilde{w}\right)}{u_{ref}}, \quad \left(\tilde{J}_X,\tilde{J}_Y,\tilde{J}_Z\right) = \frac{\left(\tilde{j}_x,\tilde{j}_y,\tilde{j}_z\right)}{\sigma u_{ref} B_0}, \\ P = \frac{p}{\rho u_{ref}^2}, \quad \left(\overline{\Phi},\tilde{\Phi}\right) = \frac{\left(\overline{\phi},\tilde{\phi}\right)}{u_{ref} B_0 h}, \quad Re = \frac{u_{ref} h}{\nu}, \quad Ha = B_0 h\sqrt{\frac{\sigma}{\rho\nu}}, \quad A = \frac{a}{h}, \quad k = \alpha h, \quad S = \frac{s}{u_{ref}/h} \end{gathered} \tag{16}$$

The characteristic length is the half value of the duct height, and the characteristic velocity is taken as the maximum velocity of the basic flow. Here, *Re* represents the Reynolds number, *Ha* the Hartmann number, *A* the aspect ratio, *k* the wavenumber and *S* the complex angular frequency. The system of the above equations and the boundary condition constitutes a generalized eigenvalue problem.

Numerical strategy and validation of the present code

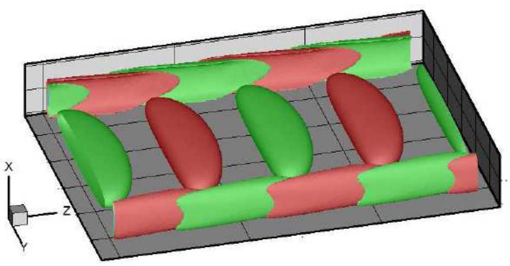
The simultaneous ordinary differential equations were discretized on a two-dimensional equidistant staggered mesh system within a cross-section of the duct and solved with a finite difference method using the fourth order central difference method. First, the basic state such as the velocity, potential and electric current density are obtained for a given Hartmann number. Second, after having given the wavenumber, the disturbance equations are solved by using the HSMAC method, meanwhile both growth rate (real part of S) and angular frequency (imaginary part of S) of the most unstable mode should be determined. For determining the complex eigenvalue, we have used a Newton-Raphson method. Before moving to the MHD flow instability, it would be necessary to examine the validity of the present numerical code. One of the most famous critical values for the onset of instability in fluid mechanics is that for the plane Poiseuille flow, of which critical Reynold number is known as 5772 with the critical wavenumber 1.02. Hence, we have implemented one dimensional computation neglecting y-directional meshing and the effect of electromagnetic field. Table 1 shows the neutral values of the Reynolds number, angular frequency and phase velocity when the critical wavenumber 1.0205474, which is believed to be exact with high accuracy, was given in the present analyses. It is judged that the present code can obtain the correct values when the number of meshes is sufficient.

Table 1: Dependency of number of meshes for the obtained neutral values (k = 1.0205474 and S_R = 0)

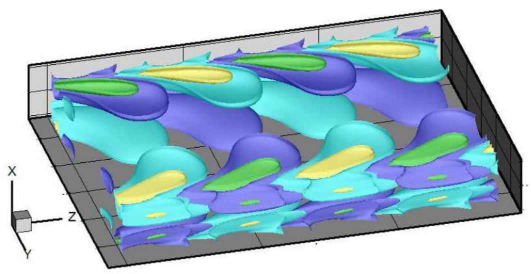
Number of meshes	Re	S_I	C_{ph}
100	5683.448	0.2701748	0.2647532
200	5756.642	0.2695493	0.2641222
400	5769.967	0.2694423	0.2640174
800	5771.919	0.2694271	0.2640025
Exact value	5772.222	0.2694248	0.2640003

Case in the finite aspect ratio

Figure 2 shows the example of visualization of the flow mode for two wavelengths at a neutral stability when the input parameters such as aspect ratio of cross-section, Hartmann number, wavenumber are given as shown in the figure caption. The uniform magnetic field is imposed in the X-direction, but its strength herein is limited to a very small case (Ha = 0). Figure 2 (a) indicates the Tollmien-Schlighting (T-S) wave in the core region far from the sidewalls. In the limit of large aspect ratio for the cross-section, the critical Reynolds number takes the value of 5772. For a finite aspect ratio, the critical Reynolds increases as decrease in the aspect ratio until its value is about 3.2 as indicated in the reference of Tatsumi & Yoshimura [3]. Figure 2 (b) indicates that high values of potential appear near the sidewall and almost zero in the core region. It is imagined that when the aspect ratio is very large, the effect of disturbed electric potential is negligible. Even in that situation, we take the insulating sidewall into account since the basic state of electric potential or electric field are crucial in this problem. In the next section, we will see the influence of the Hartmann number on the liquid metal flow driven by a constant pressure gradient when the aspect ratio of cross section is infinity.



(a) X-directional component of velocity



(b) Electric potential

Fig. 2: Iso-surfaces of each eigenfunction for the neutral stability condition at A = 5, Ha = 0, k = 0.91, Re_n = 9844 and S_I = 0.2135. The number of meshes employed for the computation is 70 x 350 for the cross section.

Influence of the strength of magnetic field

Figure 3 shows the velocity profile of the basic flows for several Hartmann numbers when the sidewalls are perfectly insulated. As increase in the Hartmann number, the core velocity tends to be flattened. This profile can be theoretically given by the equation shown below.

$$\overline{W} = 2\left\{Ha \cdot \tanh\left(Ha\right)\right\}^{-1}\left\{1-\cosh\left(Ha \cdot X\right)/\cosh\left(Ha\right)\right\} \quad (17)$$

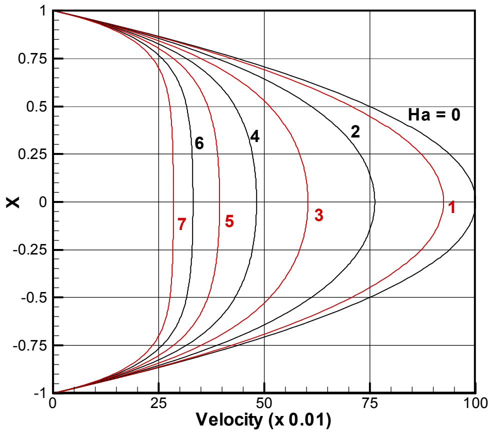


Fig. 3: The velocity profile of basic flows for several Hartmann numbers.

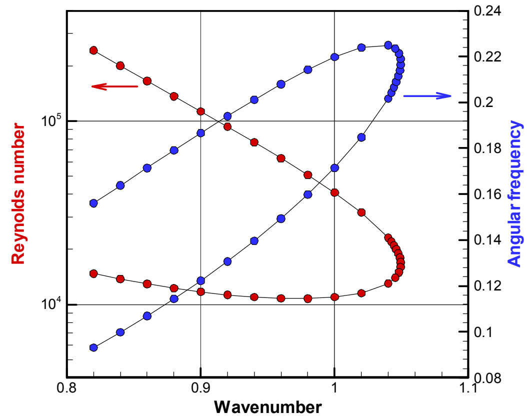
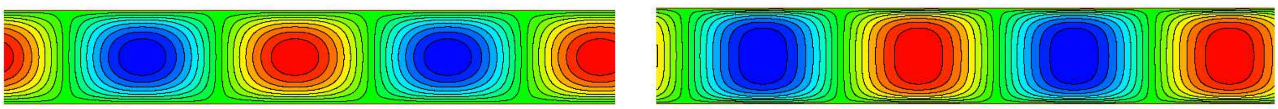


Fig. 4: Neutral stability curves in the vicinity of the critical point for Ha = 1.

Table 2 shows the summary of the values of critical point for several Hartmann numbers when the one-dimensional linear stability analysis is conducted with the number of meshes of 200. As increase in the Hartmann number, once the wavenumber decreases, but the Reynolds number increases monotonously. We confirmed that grid-size dependency occurs especially in the case of large Hartmann number. Up to Ha = 5, however, the discrepancy of the results between 200 meshes and 400 meshes is rather small. Figure 4 shows the neutral stability curves in the vicinity of the critical point for Ha = 1 together with the angular frequency. When the wavenumber exceeds the value of 1.05, the basic flow is always stable. Figure 5 shows the streamlines of the disturbed flow for two wavelengths at the critical state. The thickness of the boundary layer becomes thinner with increasing the Hartmann number. Therefore, finer mesh system is required to maintain the accuracy for the large Hartmann number.

Table 2: Summary of the computation results for the critical wavenumber, Reynold number, angular frequency and phase speed for several Hartmann numbers.

Ha	k	Re	S_I	C_{ph}
0	1.021059	5756.612	0.2697457	0.2641823
1	0.972707	10785.30	0.2119994	0.2179479
2	0.930397	37036.39	0.1366636	0.1468874
3	0.965232	104653.2	0.0993330	0.1029111
4	1.049365	222651.6	0.0819787	0.0781222
5	1.153168	400723.1	0.0721515	0.0625681



(a) Ha = 1 (b) Ha = 3

Fig. 5: The streamlines of flow for Ha = 1 and 3 at each critical point shown in Table 2.

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