

# Nonlinear Dynamics Behaviors of a Composite Laminated Cantilevered Plate Under Transverse Excitation

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**Abstract.** The nonlinear dynamics analyses of a composite laminated cantilevered rectangular plate are studied, which is forced by the transverse excitation. We use the Hamilton's principle and establish the nonlinear partial differential governing equations of motion for the composite laminated cantilevered rectangular plate. Numerical simulations are presented to investigate the effects of the transverse excitation on the steady-state responses of the cantilevered plate. The bifurcation diagrams of the composite laminated cantilevered plate for  $\omega_1$  via the base excitation amplitude  $F$  is obtained. From the bifurcation diagram, it is found that the motions of the system are as follows: from periodic motion to multiple periodic motion, then to chaotic motion. Based on the above bifurcation diagrams and using the same parameters, the base excitation amplitude  $F$  are changed to obtain the waveforms, the two-dimensional phase portraits, the three-dimensional phase portraits and the Poincare maps of the system. The results of numerical simulation demonstrate that there exist the periodic and chaotic motions of the composite laminated cantilevered rectangular plate.

## 1. Introduction

With the increased use of composite laminated cantilevered plates in engineering fields, such as aeronautic and astronautic engineering, the research on the nonlinear dynamics of the composite laminated cantilevered plate plays a key role in engineering applications [1,2]. However, only a few studies on the bifurcations and chaotic dynamics of composite laminated cantilevered plate have been conducted.

Bhimaraddi [3] studied the large amplitude nonlinear vibrations of imperfect antisymmetric angle-ply laminated plates. Zhang et al. [4] investigated the nonlinear oscillations and chaotic dynamics of a parametrically excited simply supported symmetric cross-ply laminated composite rectangular thin plate with the geometric nonlinearity and nonlinear damping.

## 2. Equations of motion

As shown in Figure 1, a composite laminated cantilevered rectangular plate clamped at edge ob and subjected to the transverse excitation is considered, whose edge length and width in the  $x$  and  $y$  directions are, respectively,  $a$  and  $b$  and the thickness is  $h$ . The composite laminated cantilevered rectangular plate is considered as symmetric laminates with  $n$  layers. These layers are made of fiber-reinforced composite materials. It is assumed that different layers are perfectly bonded to each other. A Cartesian coordinate  $Oxyz$  is located in the middle surface of composite laminated cantilevered rectangular plate. The transverse excitation is represented by  $F_0 + F \cos(\Omega t)$ .



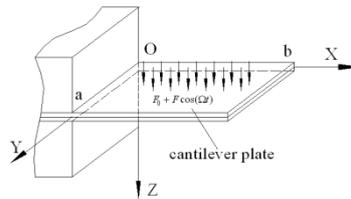


Figure1 The simplified model of a cantilevered rectangular plate structure

According to the Reddy’s classic deformation plate theory and the von Karman type equations for the geometric nonlinearity, the displacement field of the composite laminated cantilevered plate is assumed to be[5]:

$$u_1 = u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w_0}{\partial x}, u_2 = v(x, y, z, t) = v_0(x, y, t) - z \frac{\partial w_0}{\partial y}, u_3 = w(x, y, z, t) = w_0(x, y, t) \quad (1)$$

where  $(u_1, u_2, u_3)$  are the displacement components along the  $(x, y, z)$  directions,  $(u_0, v_0, w_0)$  is the deflection of a point on the middle plane  $(z = 0)$ .

The nonlinear strain-displacement relations are given as follows

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \epsilon_{zz} = \frac{\partial w}{\partial z}, \\ \epsilon_{xy} &= \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right), \epsilon_{yz} = \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right), \epsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \end{aligned} \quad (2)$$

The stress–strain relationship of the composite laminated cantilevered plate is given

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} \quad (3)$$

where  $Q_{ij}$  is the elastic stiffness coefficient, shown as follows

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, Q_{21} = Q_{12}, Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, Q_{44} = G_{23}, Q_{55} = G_{13}, Q_{66} = G_{12} \quad (4)$$

where  $E_i (i = 1, 2)$  is the elastic modulus,  $G_{12}$  is the shear modulus,  $\nu_{12}$  and  $\nu_{21}$  are Poisson’s ratio for single layer materials .

Using the Hamilton’s principle the nonlinear governing equations of motion for the composite laminated cantilevered plate are obtained as

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x}, \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_0 \ddot{v}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial y}, \\ \frac{\partial N_{xx}}{\partial x} \frac{\partial w_0}{\partial x} + \frac{\partial N_{xy}}{\partial x} \frac{\partial w_0}{\partial y} + \frac{\partial N_{xy}}{\partial y} \frac{\partial w_0}{\partial x} + \frac{\partial N_{yy}}{\partial y} \frac{\partial w_0}{\partial y} + \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} \\ + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + F_0 + F \cos(\Omega t) - c_3 \dot{w}_0 &= I_0 \ddot{w}_0 + I_1 \left( \frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y} \right) - I_2 \left( \frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2} \right) \end{aligned} \quad (5)$$

where, the dot represents partial differentiation with respect to time  $t$ ,  $c_3$  is the damping coefficient,

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \end{Bmatrix} dz, (\alpha, \beta = x, y), I_i = \sum_{k=1}^N \int_{z_k}^{z_{k+1}} \rho_k(z) z^i dz, (i = 0, 1, 2) \quad (6)$$

The stress-strain relations are given as follows

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{Bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}, \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{Bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (7)$$

Where, because the ply mode of the composite laminated plate is symmetric,  $B_{ij} = 0$  ( $i, j = 1, 2, 6$ ).

Substituting equation (7) into equation (5), the governing equations of motion in terms of generalized displacements are obtained as

$$\begin{aligned} & A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial^2 u_0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} + A_{11} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + A_{66} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} \\ & = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^3 w_0}{\partial x \partial t^2}, \end{aligned} \quad (8a)$$

$$\begin{aligned} & A_{22} \frac{\partial^2 v_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} + (A_{21} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} + A_{22} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} + A_{66} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x^2} + (A_{21} + A_{66}) \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} \\ & = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^3 w_0}{\partial y \partial t^2} \end{aligned} \quad (8b)$$

$$\begin{aligned} & -D_{11} \frac{\partial^4 w_0}{\partial x^4} + (-D_{21} - 4D_{66} - D_{12}) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} - D_{22} \frac{\partial^4 w_0}{\partial y^4} + \frac{3A_{11}}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \frac{\partial^2 w_0}{\partial x^2} + \frac{3}{2} A_{22} \left( \frac{\partial w_0}{\partial y} \right)^2 \frac{\partial^2 w_0}{\partial y^2} \\ & + \left( \frac{1}{2} A_{21} + A_{66} \right) \left( \frac{\partial w_0}{\partial x} \right)^2 \frac{\partial^2 w_0}{\partial y^2} + \left( \frac{1}{2} A_{12} + A_{66} \right) \left( \frac{\partial w_0}{\partial y} \right)^2 \frac{\partial^2 w_0}{\partial x^2} + 2 \left( \frac{1}{2} A_{21} + 2A_{66} + \frac{1}{2} A_{12} \right) \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} \\ & + A_{11} \frac{\partial u_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + A_{22} \frac{\partial v_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} + A_{21} \frac{\partial u_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} + A_{12} \frac{\partial v_0}{\partial y} \frac{\partial^2 w_0}{\partial x^2} + 2A_{66} \frac{\partial u_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} + 2A_{66} \frac{\partial v_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} \\ & + (A_{21} + A_{66}) \frac{\partial w_0}{\partial y} \frac{\partial^2 u_0}{\partial y \partial x} + (A_{12} + A_{66}) \frac{\partial w_0}{\partial x} \frac{\partial^2 v_0}{\partial y \partial x} + A_{11} \frac{\partial w_0}{\partial x} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial w_0}{\partial x} \frac{\partial^2 u_0}{\partial y^2} + A_{66} \frac{\partial w_0}{\partial y} \frac{\partial^2 v_0}{\partial x^2} \\ & + A_{22} \frac{\partial w_0}{\partial y} \frac{\partial^2 v_0}{\partial y^2} + F_0 + F \cos(\Omega t) = c_3 \dot{w}_0 + I_0 \ddot{w}_0 - I_2 \frac{\partial^2 \dot{w}_0}{\partial x^2} - I_2 \frac{\partial^2 \dot{w}_0}{\partial y^2} + (I_1 \frac{\partial \ddot{u}_0}{\partial x} + I_1 \frac{\partial \ddot{v}_0}{\partial y}) \end{aligned} \quad (8c)$$

where  $(A_{ij}, D_{ij})$  respectively are the stiffness elements of the composite laminated cantilevered piezoelectric rectangular plate, which are denoted as

$$(A_{ij}, D_{ij}) = \sum_{k=1}^N \int_{z_k}^{z_{k+1}} Q_{ij}^k(1, z^2) dz, (i, j = 1, 2, 6) \quad (9)$$

The nonlinear dynamics of the composite laminated cantilevered plate in the first mode of  $u_0, v_0$  and the first four modes of  $w_0$  are considered [6]. We write  $u_0, v_0, w_0$  in the following forms:

$$\begin{aligned} & u_0(x, y, t) = u_1(t) \sin\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi y}{b}\right), v_0(x, y, t) = v_1(t) \sin\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi y}{b}\right), \\ & w_0(x, y, t) = w_1(t) \alpha_1 \beta_1, \alpha_1 = \cosh \frac{k_1 x}{a} - \cos \frac{k_1 x}{a} - \varphi_1 \left( \sinh \frac{k_1 x}{a} - \sin \frac{k_1 x}{a} \right), \\ & \cos k_i \cosh k_i + 1 = 0, k_i^4 = \omega^2 \frac{\rho A}{EJ} (i = 1), \varphi_i = \frac{\sinh k_i - \sin k_i}{\cosh k_i + \cos k_i} (i = 1), \beta_1 = \sqrt{3} \left( 1 - \frac{2y}{b} \right) \end{aligned} \quad (10)$$

By means of the Galerkin method, neglecting all inertia terms in equation (9a), equation (9b) and equation (9c) about  $u_0, v_0$ , substituting equation (10) into equation (9), we obtain the expressions  $w_1, w_2$  as follows

$$\begin{aligned} \ddot{w}_1 + \mu_1 \dot{w}_1 + \omega_1^2 w_1 + m_1 w_1^3 + m_2 w_2^3 + m_3 w_1 w_2^2 + m_4 w_2 w_1^2 &= m_5 F \cos \Omega t, \\ \ddot{w}_2 + \mu_2 \dot{w}_2 + \omega_2^2 w_2 + n_1 w_1^3 + n_2 w_2^3 + n_3 w_1 w_2^2 + n_4 w_2 w_1^2 &= n_5 F \cos \Omega t \end{aligned} \tag{11}$$

**3. Numerical simulation**

The fourth-order Runge-Kutta algorithm is employed to analyze numerically the nonlinear dynamic responses of the composite laminated cantilevered plate [7].

The equation (11) is chosen for numerical simulation. The transverse excitation amplitude  $F$  is used as the controlling parameter to investigate the periodic and chaotic responses of the system. The physical parameters are chosen respectively as following in table 1.

When the transverse excitation amplitude  $F$  is located in the interval 22N–40N, the bifurcation diagram is obtained, as shown in Figure 2, in which the abscissa denotes the amplitude of the base excitation  $F$ , while the ordinate denotes the transverse deflection of the plate. Analyzing the Figure 2, the nonlinear dynamic responses of the composite laminated cantilevered plate change from the periodic motion to the multiple periodic motion, and then to the chaotic motions with the increase of the transverse excitation amplitude  $F$ . When the base excitation amplitude  $F$  exceed 36N, the multiple periodic motion and the chaotic motion appear alternately.

Table 1 The physical parameters of the composite laminated cantilevered plate

physical quantity	value	physical quantity	value
$a$	2m	$G_{12}$	6.0Gpa
$b$	1.2m	$\nu_{12}$	0.3
$h$	0.002m	$\rho$	1850kg/m <sup>3</sup>
$E_1$	106Gpa	$E_2$	4.5Gpa

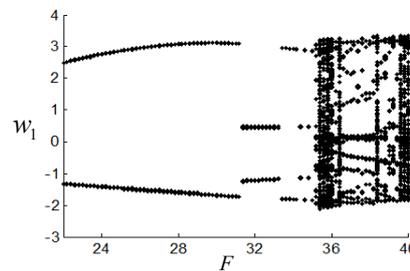


Figure2 The bifurcation diagram of the first mode of  $w_0$  for transverse excitation amplitude

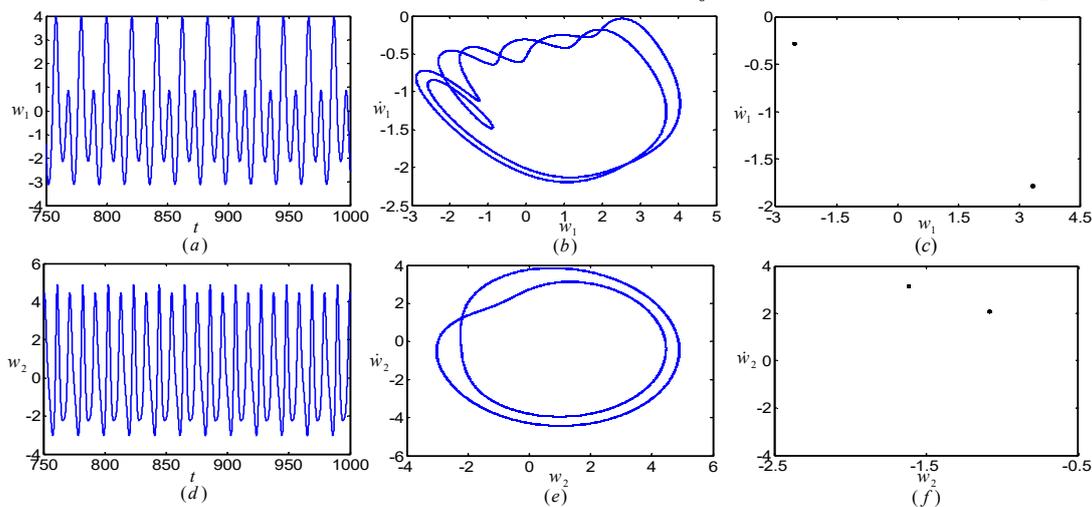


Figure3 The periodic motion of the system when  $F=28N$

In the following investigation, we use the same parameters and change the transverse excitation

amplitude  $F$  to obtain the waveforms, the two-dimensional phase portraits, and the Poincare maps of the composite laminated cantilevered plate based on Figure 2. When the transverse excitation  $F$  is 28N, Figure 3 shows the existence of the periodic motion for this system. When the transverse excitation  $F$  is 36N, the chaotic motion of the system is observed, as shown in Figure 4.

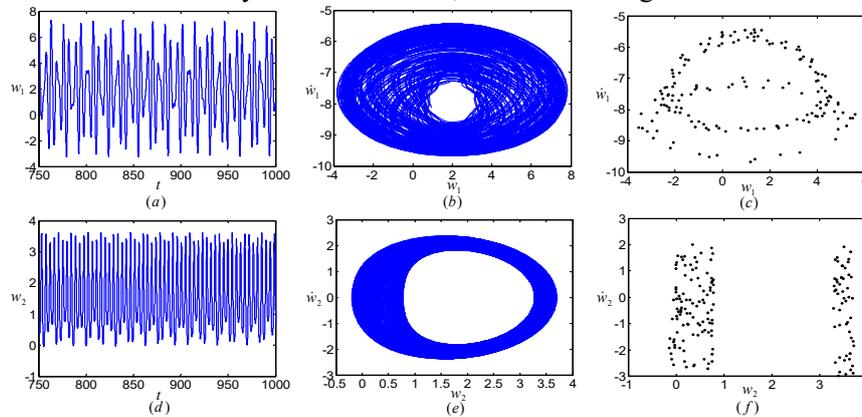


Figure 4 The chaotic motion of the system when  $F=36\text{N}$

#### 4. Conclusion

The nonlinear vibrations and chaotic responses of a composite laminated cantilevered plate subjected to the transverse excitation are investigated. Based on the von Karman type equations and the Reddy's classic plate theory, we establish the governing equations of the system. Analytical study is given by using the fourth-order Runge-Kutta algorithm. The numerical results show that there exist the periodic, multiple periodic and chaotic motions of the system. From bifurcation diagram, it is found that transverse excitation amplitude  $F$  has significant influence on the system. Therefore, we can control the nonlinear dynamic responses of the composite laminated cantilevered plate by changing the value of transverse excitation amplitude  $F$ .

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