

# Comparison of the one-equation LWR models for density and for speed

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**Abstract.** The macroscopic models describe road traffic flow without consideration of individual vehicles, but through aggregated traffic parameters, i.e. the flow rate, density and speed. The one-equation models are various LWR models that the most important component is the continuity equation. This equation must be supplemented by an additional equation that defines a specific relationship between two variables (traffic parameters), usually between speed and density. However, it is possible to eliminate one of those variables from the system, as a result of which the problem is limited only to one equation and one variable. In this article, two formulations of the one-equation LWR model will be considered, i.e. the density-dependent equation, and the speed-dependent equation. The question therefore arises, which of two formulations is more advantageous for a road traffic solution? Comparison of those two formulations of the LWR model is carried out only on a continuous level. Due to similar, quasi-linear form of both equations, the comparison has been carried out using their eigenvalues (characteristic speeds) for various static relationships between speed and density. Final evaluation of both one-equation formulations used spectral radiuses expressed as functions of the variables used in particular models.

## 1. Introduction

Theoretical formulations describing traffic flow are usually classified according to the level of detail of their description. In general, four basic categories are distinguished for the description of traffic flow: macroscopic, mesoscopic, microscopic and submicroscopic (nanoscopic) models. The macroscopic model class captures traffic flow in the most general way, i.e. by describing it through the basic traffic parameters (flow rate, density and speed), without referring to the behaviour of vehicle groups, individual vehicles or particular vehicle subassemblies. Departure from the description of individual vehicles (drivers) and the recognition that traffic flow is a flow of some idealized, fictitious continuous medium results in some similarity to the description of fluid flow (the Euler or Navier–Stokes equations), where the behaviour of individual molecules is also not considered.

The consequence of assuming that the medium is continuous, is describing its behaviour by the differential and integral calculus. The road traffic flow in macroscopic approach is therefore described similarly to the behaviour of a compressible fluid, i.e. via hyperbolic (convection) equations or hyperbolic-parabolic (convection–diffusion) equations, generally in the two-dimensional space of a road network, but locally as a one-dimensional flow. In order to solve the problem formulated in this way, it is necessary to apply mathematical analysis methods, as a result of which the flow rates, densities and speeds can be obtained in the entire area of the considered road system. On this basis, a traffic state at any point of space and time can be unambiguously determined. However, this is only possible for the



simplest configuration of the road system. The use of analytical methods for more complex systems and traffic flows is practically unfeasible.

The only way out in this situation is to discretize the problem (i.e. to describe its behaviour not in relation to all, infinitely many points of the computational domain, but only in relation to selected points of the system usually defined through a computational grid) and its approximation (to determine the traffic parameters in any point in the domain, as well as for the expression of differential operators via corresponding differentiation operation on functions, usually polynomial functions). This results in the creation of a set of algebraic equations approximating the initial system of differential equations. Such approximate solutions of discretized systems are much more effective than attempts to directly solve continuous problems with analytical methods, especially in relation to complex computational configurations.

Two factors determine the quality of the results obtained from macroscopic simulations. The first is the quality of the continuous model describing the traffic flow, and the second – the quality of used discretization and approximation of this model, and the accuracy and stability of numerical algorithms used to solve the resulting systems of algebraic equations. The main goal of this paper is the evaluation of traffic model formulations, without referring to the issue of their discretization and numerical implementation.

Macroscopic traffic flow models can be divided in general into:

- one-equation models, i.e. first-order models, usually referred to as LWR models.
- two-equation models often referred to as second-order models.

This publication applies only to one-equation models, while two-equation models will be considered in the accompanying publication: On the conditioning of two-equation road traffic models.

The course of considerations on the evaluation of one-equation traffic models (the LWR models) will be carried out in the following steps:

- presentation and discussion of the basic relationships between selected variables (traffic parameters) included in static models,
- discussion and characterization of the generalized LWR model,
- transformation of the generalized LWR model into a one-equation form with one variable, density or speed,
- illustration of normalized static models, and eigenvalues and spectral radiuses of the LWR models,
- comparison and quality evaluation of individual LWR models (for different static models), formulated for density and speed.

## 2. Static models

Static models define statistical relationships between particular traffic parameters (flow rate, density and speed), and are determined on the basis of traffic observations and measurements. They are the basis for the development of a fundamental diagram (flow – density) and related diagrams (flow – speed and density – speed). These completely statistical relationships are often written in the form of algebraic relationships between two selected traffic parameters. Most often, in this way, the dependence on the speed as a function of density (or vice versa) is defined. Most of these relationships are one-range, but sometimes two-range models (for free and congested flow) and three-range models (additionally taking into account the intermediate state) can be found.

The most popular one-range models are described as follows:

- the Greenshields model [1]

$$u = u_f \left(1 - \frac{k}{k_j}\right), \quad (1)$$

where specific quantities mean:  $u_f$  – the free flow speed, and  $k_j$  – the jam density. This linear model is one of the simplest, but nevertheless correctly satisfies the conditions for zero speed and zero density. Its reliability is rather low, especially for highway traffic.

- the Greenberg model [2]

$$u = u_m \ln\left(\frac{k_j}{k}\right), \quad (2)$$

where  $u_m$  is the optimal speed (corresponding to the road capacity). This model is logarithmic and generally performs better than the model described above.

- the Underwood model [3]

$$u = u_f \exp\left(\frac{-k}{k_m}\right), \quad (3)$$

where  $k_m$  is the optimal density (corresponding to the road capacity). This model is distinguished by the exponential connection of density and speed. It performs better than the models mentioned above in the case of free flow, but for congested traffic does not show an adequate quality.

- the Northwestern model [4]

$$u = u_f \exp\left[-\frac{1}{2}\left(\frac{k}{k_m}\right)^2\right], \quad (4)$$

This model is characterized by an exponential relationship of speed and density. With regard to free flow, it works better than the models mentioned earlier, but it is not appropriate for congested flow.

- the Pipes–Munjal model [5]

$$u = u_f \left[1 - \left(\frac{k}{k_j}\right)^n\right], \quad (5)$$

where  $n$  is the exponent determined during calibration. This model is the next version of the Greenshields model, trying to remove some of the deficiencies of the original.

- the Drew model [6]

$$u = u_f \left[1 - \left(\frac{k}{k_j}\right)^{n+\frac{1}{2}}\right]. \quad (6)$$

This model can be considered as a modified version of the Greenshields model, devoid of its most essential disadvantages.

- the Newell model [7]

$$u = u_f \left\{1 - \exp\left[-\frac{\varphi}{u_f}\left(\frac{1}{k} - \frac{1}{k_j}\right)\right]\right\}, \quad (7)$$

where  $\varphi$  is the parameter from a microscopic model (the slope of the distance–speed curve). This model has been developed on the basis of a microscopic car-following model, but it exhibits too fast a decrease in speed in relation to the density increase.

- the modified Greenshields model [8]

$$u = u_j + (u_f - u_j) \left(1 - \frac{k}{k_j}\right). \quad (8)$$

This model modifies the original form of the Greenshields model to somewhat generalize it.

- the modified Greenberg model [9]

$$u = u_m \ln\left(\frac{k_j + k_0}{k + k_0}\right), \quad (9)$$

where  $k_0$  is an average non-zero minimum density introduced into the original model.

- in addition to the above-mentioned, there are also less popular one-range models, such as: the Kerner–Konhäuser model [10], the Del Castillo model [11, 12], the Van Aerde model [13] or the MacNicholas model [14].

The number of multi-range models is still small. Two-range models attempt to describe the above-mentioned relationships via usually two or three “curves”. Probably the first two-range model was the Edie model [15] using the Underwood model for free flow, and the Greenberg model for congested flow. Other two-range models were presented by May [16] (including the two-range Greenberg model) and Lu Sun [17]. The three-range model, which additionally takes into account dependencies for the transitional flow (between free and congested), was presented by May [16]. Its elements are respectively three linear “curves” defined on the basis of the original Greenshields model. The inconvenience of

multi-range models is the difficulty in determining the appropriate inter-range transition points that can change under the influence of many different factors.

The static models presented and discussed above, describing the relationships between aggregated traffic parameters, are not in itself suitable to stand-alone describe a traffic flow, let alone simulating it. This is because these identity relationships concern only to “point” properties of traffic, i.e. they mutually relate traffic parameters in specific road cross-sections. In order to describe the traffic variability, even in the simplest configuration of the road system, at least the LWR models are needed, which also include (as a significant element) any static model.

### 3. The LWR models

The first macroscopic models of traffic flow, also called first-order models, were formulated by Lighthill and Whitham [18] and independently by Richards [19]. In its basic form, this model boiled down to a simple differential equation defining the equilibrium equation, i.e. the continuity equation for the traffic flow. The continuity equation corresponds to the “mass flow” conservation, which in discrete terms means that the change in the number of vehicles in the system depends only on the declared boundary conditions. The continuity equation in the most popular form is expressed by:

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = g, \quad (10)$$

where:  $t$  – time,  $x$  – space (distance),  $q$  – flow rate,  $k$  – density,  $g$  – source (off the boundary). In further consideration, for its simplicity, it was assumed that  $g = 0$ . As can be seen, the LWR equation (10) is only one and has two variables (two unknowns):  $q$  and  $k$ . To solve the problem, it is necessary to supplement it with an additional equation.

Because the flow rate  $q$  is dependent on the other two variables, i.e.  $k$  and  $u$ , through the following fundamental relationship:

$$q = ku, \quad (11)$$

where  $u$  is the speed, the continuity equation can be written in an alternative form

$$\frac{\partial k}{\partial t} + u \frac{\partial k}{\partial x} + k \frac{\partial u}{\partial x} = 0, \quad (12)$$

with two variables, the density  $k$  and the speed  $u$ . In order to solve the problem, the equation also needs to be supplemented with an additional equation.

If the additional equation is a relationship of two variables:

- for equation (10): flow  $q$  and density  $k$ ,
- in the case of equation (12): density  $k$  and speed  $u$ ,

it is always possible to reduce these two equations to a one equation. This additional relation between variables can be assumed in the form of any static model from those presented previously, or other. Assuming for further considerations the continuity equation (12), the one-equation LWR model can be written in function of density only or speed only.

Assuming that the introduced static model expresses a relationship in the general form  $u = u_e(k)$ , the LWR equation will be defined as a function of the density  $k$ , where  $u_e$  is the equilibrium speed dependent on the density  $k$  in the manner defined by the given static model. Thus, by introducing  $u = u_e(k)$  to the second and third terms on the left side of the equation (12), they take the form:

$$u \frac{\partial k}{\partial x} + k \frac{\partial u}{\partial x} = \left( u_e + k \frac{du_e}{dk} \right) \frac{\partial k}{\partial x}, \quad (13)$$

which results in the one-equation LWR model for density in the form:

$$\frac{\partial k}{\partial t} + \left( u_e + k \frac{du_e}{dk} \right) \frac{\partial k}{\partial x} = 0, \quad (14)$$

or in a short quasi-linear form:

$$\frac{\partial k}{\partial t} + \lambda \frac{\partial k}{\partial x} = 0, \quad (15)$$

where  $\lambda$  is an eigenvalue or a characteristic speed.

In turn, assuming that the static model has the general form  $k = k_e(u)$ , then the LWR equation will depend on the speed  $u$ , where  $k_e$  is the equilibrium density being a function of the speed  $u$  corresponding to the assumed static model. By inserting  $k = k_e(u)$  into the second and third terms on the left side of the equation (12), it takes the form:

$$u \frac{\partial k}{\partial x} + k \frac{\partial u}{\partial x} = \left( k_e + u \frac{dk_e}{du} \right) \frac{\partial u}{\partial x}, \quad (16)$$

which results in the following form of the one-equation LWR model for speed:

$$\frac{\partial u}{\partial t} + \left( \frac{k_e}{\frac{dk_e}{du}} + u \right) \frac{\partial u}{\partial x} = 0, \quad (17)$$

which can also be written in a quasi-linear form:

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0, \quad (18)$$

where  $\lambda$  is the eigenvalue (characteristic speed).

Now it is possible to compare the magnitude of eigenvalues determined for both general formulations of one-equation LWR models. These values are expressed respectively for the formulated models:

- as a function of density

$$\lambda(k) = u_e + k \frac{du_e}{dk}, \quad (19)$$

- as a function of speed

$$\lambda(u) = \frac{k_e}{\frac{dk_e}{du}} + u. \quad (20)$$

Accordingly, for each one-equation model with only one variable ( $k$  or  $u$ ), there exists only one eigenvalue, and only one spectral radius defined  $\rho = |\lambda|$ , which means respectively that

$$\rho(k) = \left| u_e + k \frac{du_e}{dk} \right|, \quad (21)$$

$$\rho(u) = \left| \frac{k_e}{\frac{dk_e}{du}} + u \right|. \quad (22)$$

Concrete comparisons of eigenvalues and spectral radiuses, however, require the introduction of specific mathematical relationships defining each of the considered static models.

#### 4. Normalized relationships for static models and the LWR models

A characteristic feature of road traffic is a large range of variability of traffic parameters, which makes it extremely difficult to carry out reliable comparisons of traffic models in relation to different roads. A helpful solution is to normalize the quantities describing the traffic, by assuming the range of variation on a scale from 0 to 1. Reasoning for normalization is the unification of the traffic description for all roads, regardless of the given permissible speed and the highest possible density. In our case, normalization concerns two basic variables: density  $k$  and speed  $u$ . The normalization transformations are based on the following relationships:

$$k = \frac{k_{real} - k_{min}}{k_{max} - k_{min}} = \frac{k_{real}}{k_{max}} \quad (\text{for } k_{min} = 0), \quad (23)$$

$$u = \frac{u_{real} - u_{min}}{u_{max} - u_{min}} = \frac{u_{real}}{u_{max}} \quad (\text{for } u_{min} = 0), \quad (24)$$

where  $k_{max}$  is the density for jammed flow, and  $u_{max}$  is a permissible speed. The flow rate  $q$  is not subject to direct normalization, but is its indirect effect by normalizing the variables  $k$  and  $u$ .

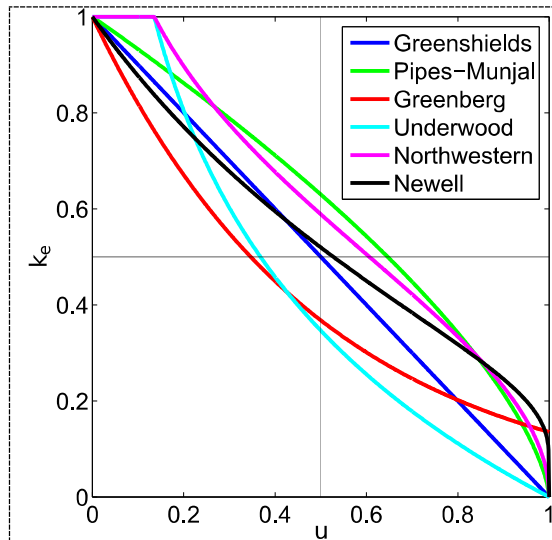
#### 4.1. Parameters of models as functions of density

For static models, the variability of speed, and for the corresponding LWR models, the variability of eigenvalues, are functions of normalized density (in the range from 0 to 1).

The normalized static models under consideration are here:

- the Greenshields model (GS),
- the Pipes–Munjial model (PM),
- the Greenberg model (GG),
- the Underwood model (UD),
- the Northwestern model (NN),
- the Newell model (NL).

In figure 1, the static relationships  $u = u_e(k)$  for these models are shown. Accordingly, the eigenvalues for the LWR models  $\lambda = \lambda(k)$  (one for one model) corresponding to above mentioned static models, are presented in figure 2 as a function of (normalized) density  $k$ .



**Figure 1.** The speed versus density diagram for the following normalized static models ( $c$  is specific for each model):

$$\text{GS: } u_e = 1 - k$$

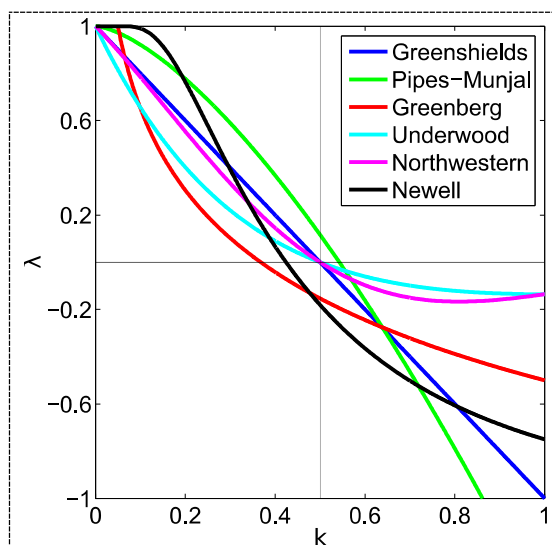
$$\text{PM: } u_e = 1 - k^c; \quad c = 1.5$$

$$\text{GG: } u_e = c \ln\left(\frac{1}{k}\right); \quad c = 0.5$$

$$\text{UD: } u_e = \exp\left(-\frac{k}{c}\right); \quad c = 0.5$$

$$\text{NN: } u_e = \exp\left[-\frac{1}{2}\left(\frac{k}{c}\right)^2\right]; \quad c = 0.5$$

$$\text{NL: } u_e = 1 - \exp\left[-c\left(\frac{1}{k} - 1\right)\right]; \quad c = 0.75$$



**Figure 2.** The eigenvalue versus density graph for the following normalized static models ( $c$  is specific for each model):

$$\text{GS: } \lambda = 1 - 2k$$

$$\text{PM: } \lambda = 1 - (1 + k^c); \quad c = 1.5$$

$$\text{GG: } \lambda = c \left[ \ln\left(\frac{1}{k}\right) - 1 \right]; \quad c = 0.5$$

$$\text{UD: } \lambda = \left(1 - \frac{k}{c}\right) \exp\left(-\frac{k}{c}\right); \quad c = 0.5$$

$$\text{NN: } \lambda = \left[1 - \left(\frac{k}{c}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{k}{c}\right)^2\right]; \quad c = 0.5$$

$$\text{NL: } \lambda = 1 - \left(1 + \frac{c}{k}\right) \exp\left[-c\left(\frac{1}{k} - 1\right)\right]; \quad c = 0.75$$

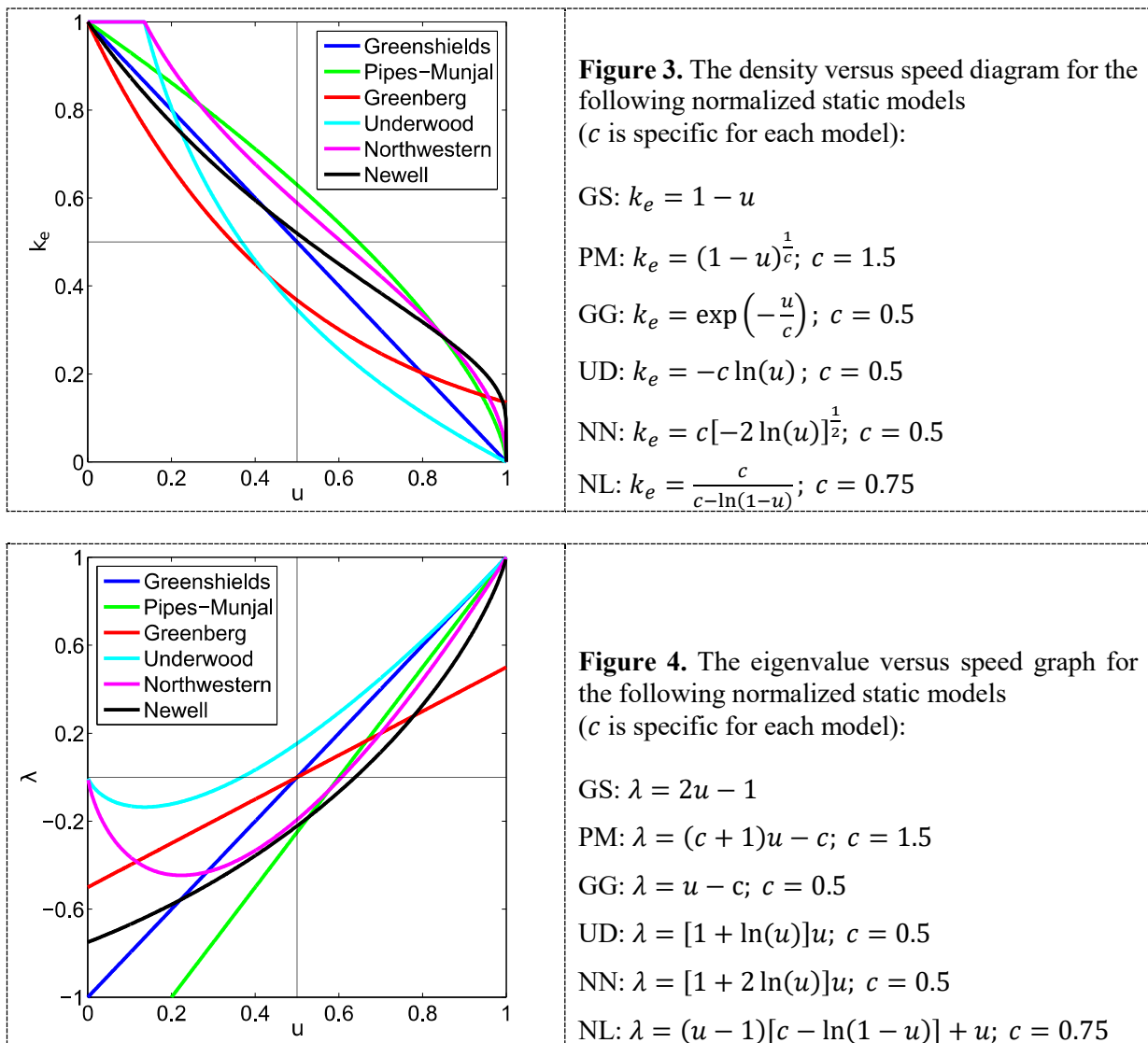
#### 4.2. Parameters of models as a function of speed

For static models, the variability of density, and for the corresponding LWR models, the variability of eigenvalues, are functions of normalized speed (in the range from 0 to 1).

These normalized static models are as follows:

- the Greenshields model (GS),
- the Pipes–Munjal model (PM),
- the Greenberg model (GG),
- the Underwood model (UD),
- the Northwestern model (NN),
- the Newell model (NL).

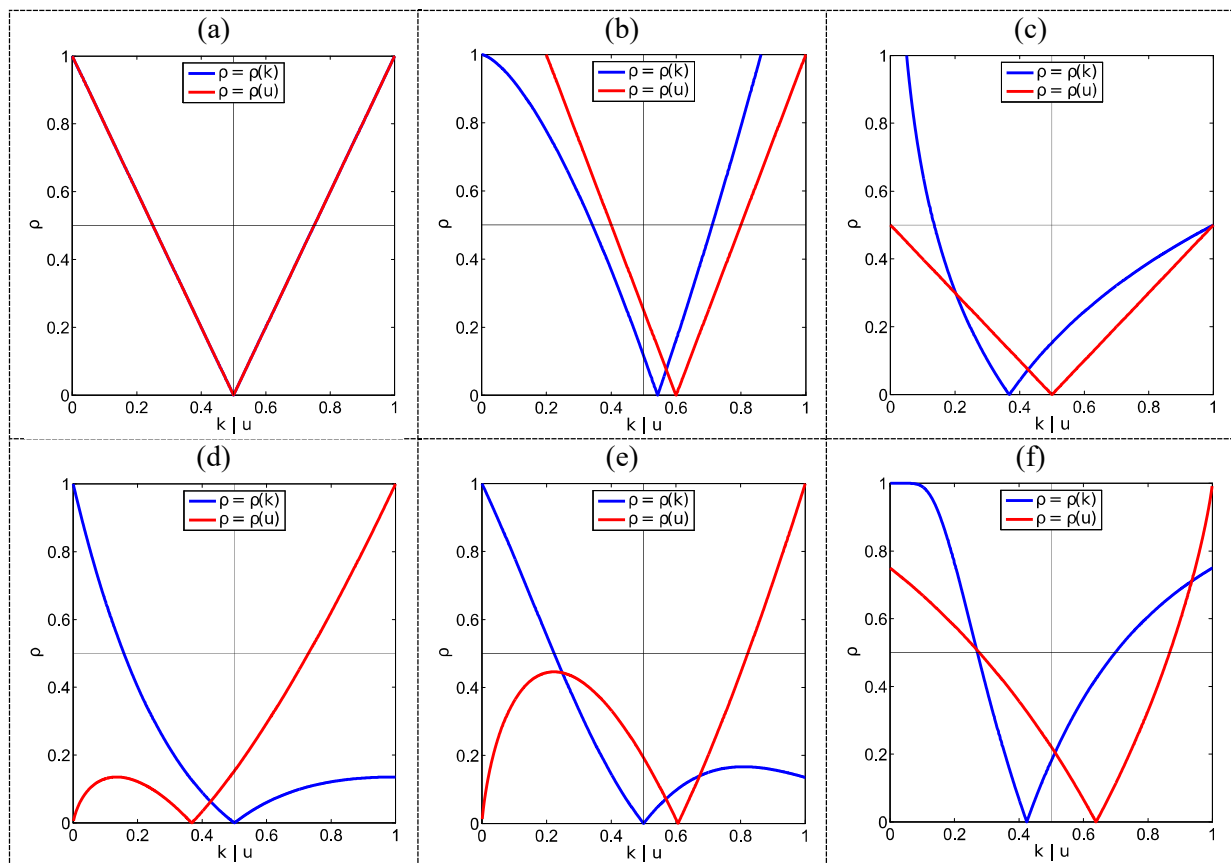
The relationships  $k = k_e(u)$  for considered static models are presented in figure 3, and in figure 4 the relationships for the eigenvalues of corresponding LWR models  $\lambda = \lambda(u)$  are shown – all as a function of a normalized speed  $u$ .



### 5. Spectral radiuses for the same LWR models formulated for density and for speed

Generally, the spectral radius of a square matrix is the largest absolute value of its eigenvalues. In the case of the one-equation LWR models, the eigenvalue is only one, and the spectral radius is just its absolute value  $\rho = |\lambda|$ . Because the LWR models are quasi-linear, the spectral radius can be treated as a function of any variable, i.e. the density  $k$  or the speed  $u$ . It depends on how the given model is formulated. Having therefore two expressions on spectral radius, it is possible to determine which is more favorable (smaller), and therefore what form of model versions is potentially more advantageous in the context of its solution.

The spectral radiuses  $\rho$  for each considered LWR model, in its two formulations (for  $k$  and for  $u$ ), are presented in the figure 5.



**Figure 5.** The spectral radiuses for various formulations of the LWR model, i.e. with different static models: (a) the Greenshields model, (b) the Pipes–Munjial model, (c) the Greenberg model, (d) the Underwood model, (e) the Northwestern model, and (f) the Newell model.

### 6. Conclusions

Comparisons of the LWR models formulated on the basis of different static models (i.e. for assumed relationships between density  $k$  and speed  $u$ ) were carried out to determine what formulation of a given LWR model, for density or velocity, is more favorable. For this purpose, spectral radiuses of these models had to be compared. Because the considered equations were quasi-linear, spectral radiuses  $\rho$  could be presented in the function of variables,  $k$  or  $u$ . On the basis of graphs showing the course of spectral radiuses (figure 5), one can determine what formulation of the one-equation LWR model may be potentially more beneficial due to the solution issues.



Generally, the differences are not great, and there is no difference for the Greenshields model. Moreover, it should be noted that although the dependencies  $\rho = \rho(k)$  and  $\rho = \rho(u)$  are different for individual LWR models, the relations  $\rho = \rho(k)$  and  $\rho = \rho[u_e(k)]$ , however, will be identical. Similarly, the results for relationships  $\rho = \rho(u)$  and  $\rho = \rho[k_e(u)]$  will be also identical.

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### References

- [1] Greenshields B D 1935 A study of traffic capacity *Highway Research Board Proceedings* **14** pp 448–477
- [2] Greenberg H 1959 An analysis of traffic flow *Operations Research* **7** pp 78–85
- [3] Underwood R T 1961 Speed, volume, and density relationships: quality and theory of traffic flow *Yale Bureau of Highway Traffic* pp 141–188
- [4] Drake J S, Schofer J L and May A D 1967 A statistical analysis of speed density hypotheses *Highway Research Record* **154** pp 53–87
- [5] Pipes L A 1967 Car-following models and the fundamental diagram of road traffic *Transportation Research* **1** pp 21–29
- [6] Drew D R 1968 *Traffic Flow Theory and Control* (New York: McGraw-Hill Book Company)
- [7] Newell G F 1961 Nonlinear effects in the dynamics of car following *Operations Research* **9** pp 209–229
- [8] Jayakrishnan A C R, Tsai W K and Chen A 1995 A dynamic traffic assignment model with traffic flow relationships *Transportation Research C* **3** pp 51–72
- [9] Ardekani S A, Ghandehari M and Nepal S M 2011 Macroscopic speed–flow models for characterization of freeway and managed lanes, *Buletinul Institutului Politehnic din Iasi LVII (LXI)* pp 149–159
- [10] Kerner B S and Konhäuser P 1994 Structure and parameters of clusters in traffic flow *Physical Review E* **50** pp 54–83
- [11] Del Castillo F G B J M 1995 On the functional form of the speed–density relationship – I: general theory *Transportation Research B* **29** pp 373–389
- [12] Del Castillo F G B J M 1995 On the functional form of the speed–density relationship – II: empirical investigation *Transportation Research B* **29** pp 391–406
- [13] Van Aerde M 1995 Single regime speed–flow–density relationship for congested and uncongested highways *The 74th TRB Annual Conference* (Washington: Transportation Research Board)
- [14] MacNicholas M J 2008 A simple and pragmatic representation of traffic flow *Symposium on The Fundamental Diagram: 75 years* (Washington: Transportation Research Board)
- [15] Edie L C 1961 Car-following and steady-state theory for noncongested traffic *Operations Research* **9** pp 66–76
- [16] May A D 1990 *Traffic Flow Fundamentals* Englewood Cliffs: Prentice Hall)
- [17] Lu Sun J Z 2005 Development of multiregime speed–density relationship by cluster analysis *Transportation Research Record* **193** pp 64–71
- [18] Lighthill M J and Whitham G B 1955 On kinematic waves: II. A theory of traffic flow on long crowded roads *Proceedings of the Royal Society A* **229** (London: Royal Society) pp 317–345
- [19] Richards P I 1956 Shockwaves on the highway *Operations Researches* **4** pp 42–51