

Analysis of Laminated and FGM Beams using Various Theories

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Abstract- In this work, we present a comparison of some recently developed higher order theories for static and free vibration responses of laminated and functionally graded material (FGM) beams. The equations of motion are derived using Hamilton's principle. Analytical Fourier series solution taking first n terms is derived for beams having simply supported end conditions. A computer code in MATLAB has been developed and the results obtained are compared with 2D elasticity/finite element solution for laminated beam to assess the accuracy of various theories. A parametric study has been presented for deflection, stresses and natural frequencies of FG beams.

Keywords: Functionally Graded Beams, Higher Order Theories, Voigt's ROM, Analytical Solution.

1. Introduction

Laminated/functionally graded material (FGM) beams have been extensively used as structural member in many engineering applications. The popularity of these structures increases day by day due to their tailorable material properties which has intrigued researchers in the last few decades. In FGM beams, the material properties are smoothly varying along a particular direction (preferably transverse direction) which results in continuous normal stresses and makes them utile for the components operating in severe thermal environment where the laminated composite structures fail due to delamination. The concept of FGM was given by group of material scientists in Japan around mid-eighties. Sankar [1] presented static analysis of FG beam using Euler-Bernoulli beam theory. This theory is suitable for thin beams but the discrepancies evolve as the beam gets thicker. Li [2] presented static and dynamic analysis of FG beams considering rotary inertia and shear deformation effects. Sina et.al [3] developed a refined first order shear deformation theory to analyze free vibration of FG beams. These theories, although consider the shear deformation effects, yet they yield unnatural shear stress distribution across the thickness. Higher order theories (HOTs) considering third or higher order polynomial/function of thickness coordinate have been developed for FG beams. These theories yield accurate results and the distributions of shear stresses are comparable to 2D elasticity solution. Thai and Thuc [4] employed various higher order shear deformation theories for bending and free vibration analysis of FG beams. Simsek [5] presented natural frequencies of FG beam using classical, first order and higher order shear deformation theories. Vo et. al [6] proposed a quasi-3D polynomial shear and normal deformation theory taking into account the effects of transverse shear and normal deformation



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in the displacement field. Kadoli et.al [7] carried out finite element analysis based on third order shear deformation theory of FG beams to study their static response.

In this work, we present a comparison of various HOTs for the static and free vibration responses of laminated and FGM beams. The effective properties are calculated using Voight's rule of mixtures (ROM). The Fourier series solution has been obtained considering first 85 odd terms. The results obtained are validated with existing literature and commercially available FE package ABAQUS. Deflection, stresses and natural frequencies are presented for laminated and FGM beams of different a/h values.

2. Problem formulation

Consider an FGM beam of length L width b and thickness h having metallic top surface and ceramic bottom surface. The concentration of metal and ceramic is smoothly varying with the variation of ceramic volume fraction with power law

$$V_c = \left(0.5 + \frac{z}{h}\right)^p \quad (1)$$

The geometry of the beam and the coordinate axes has been shown in figure 1. The bottom surface of the beam lies at $z = -h/2$ and top surface lies at $z = h/2$. The reference plane lies at $z = 0$. The beam is loaded with transverse pressure P_z at top surfaces without any variation along y axis. For the beam having plane stress condition ($\sigma_y = \tau_{yz} = \tau_{xy} = 0$) and considering $\sigma_z \approx 0$, the material constitutive behaviour for FGM beam is expressed as

$$\sigma_x = Q_{11}(z)\varepsilon_x \quad \tau_{zx} = Q_{55}(z)\gamma_{zx} \quad (2)$$

Where σ_x, τ_{zx} axial and transverse shear stresses and $\varepsilon_x, \gamma_{zx}$ are axial and transverse shear strains, $Q_{11}(z) = Y(z)$ and $Q_{55} = Y(z)/2(1 + \nu(z))$ are reduced stiffness coefficients. The effective values of Young's modulus and Poisson's ratio are obtained using ROM

$$E(z) = V_c Y_c + V_m Y_m \quad \nu(z) = V_c \nu_c + V_m \nu_m \quad (3)$$

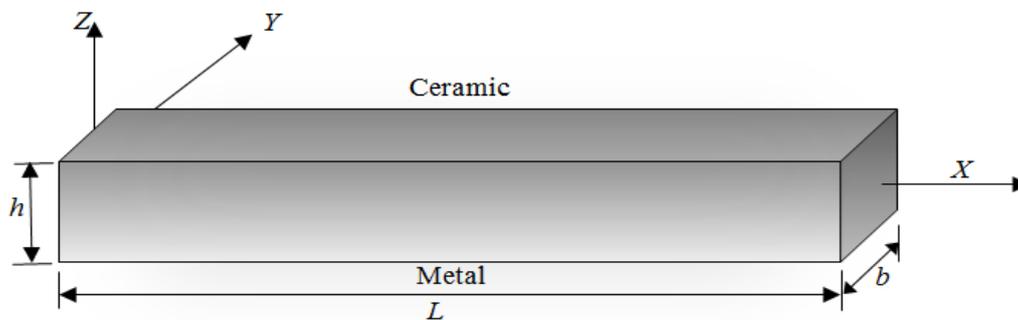


Figure 1. Geometry and coordinate FG beam

For laminated beams, $Q_{11}(z) = Y(z)$ is taken independent of z and its transformed value from lamina coordinate system to the laminate coordinate system has been obtained using transformation equation

$$\frac{1}{Y} = \frac{\cos^4 \theta}{Y_{11}} + \left(\frac{1}{G_{12}} - \frac{2\nu_{21}}{Y_{22}}\right) \sin^2 \theta \cos^2 \theta + \frac{\sin^4 \theta}{Y_{22}} \quad \frac{1}{G_{31}} = \frac{\cos^2 \theta}{G_{31}} + \frac{\sin^2 \theta}{G_{23}} \quad (4)$$

Where θ is the fiber angle with respect to the x - axis. The longitudinal and shear strain in the beam are expressed in terms of displacements as

$$\varepsilon_x = u_{,x} \quad \gamma_{zx} = w_{,x} + u_{,z} \quad (5)$$

In present analysis, the kinematic relations of beams are obtained using higher order theories (HOTs). The displacement field for the HOTs [8] can be expressed as

$$u = f(z)\bar{u} \quad w = w_0 \quad (6)$$

Where $\bar{u} = [u_0 \ w_{0,x} \ \psi_0]^T$ is the displacement vector and $f(z)$ is a higher order function of z in matrix form which takes the following form

$$f(z) = \begin{cases} [1 \ -z \ z - 4z^3/3h^2] & \text{(TOT)} \\ [1 \ -z \ \sin(\pi z/h)] & \text{(Sinusoidal theory)} \\ [1 \ -z \ (z - (h/\pi) \sinh(\pi z/h))/(\cosh(\pi/2) - 1)] & \text{(Hyperbolic theory)} \\ [1 \ -z \ z \exp(-2z^2/h^2)] & \text{(Exponential theory)} \end{cases} \quad (7)$$

Using equations Eq. (5) and Eq. (6), the strain ε_x and γ_{zx} are expressed in functional form as

$$\varepsilon_x = f_1(z) \bar{\varepsilon}_1 \quad \gamma_{zx} = f_2(z) \bar{\varepsilon}_2 \quad (8)$$

Where $\bar{\varepsilon}_1 = [u_{0,x} \ w_{0,xx} \ \psi_{0,x}]$, $\bar{\varepsilon}_2 = \psi_0$ and $f_1(z) = f(z)$ and

$$f_2(z) = \begin{cases} 1 - 4z^2/h^2 & \text{(TOT)} \\ (\pi/h) \cos(\pi z/h) & \text{(Sinusoidal theory)} \\ \cosh(\pi z/h) - 1/(\cosh(\pi/2) - 1) & \text{(Hyperbolic theory)} \\ 1 - 4(z^2/h^2) \exp(-2z^2/h^2) & \text{(Exponential theory)} \end{cases} \quad (9)$$

The equations of motion based on different HOTs of Eq. (7) are derived using Hamilton's principle.

Using $\langle \dots \rangle = \sum_{k=1}^L \int_{z_{k-1}^+}^{z_k^-} (\dots) b dz$, the Hamilton's principle takes the following form

$$\int_x [(\rho^k \ddot{u} \delta u + \rho^k \ddot{w} \delta w + \sigma_x \delta \varepsilon_x + \tau_{zx} \delta \gamma_{zx}) - b P_z \delta w(x, h/2, t)] dx - \langle \sigma_x \delta u + \tau_{zx} \delta w \rangle |_x = 0 \quad (10)$$

$\forall \delta u_0, \delta w_0$ and $\delta \psi_0$, where $(\ddot{}) = ()_{,tt}$. Substituting the displacements and strains and integrating Eq. (10) by parts yields the following equations of motion

$$\begin{aligned} I_{11} \ddot{u}_0 + I_{12} \ddot{w}_{0,x} + I_{13} \ddot{\psi}_0 - N_{x,x} &= 0 \\ I_{21} \ddot{u}_{0,x} - I_{22} \ddot{w}_{0,xx} + I_{23} \ddot{\psi}_{0,x} + \hat{I} \ddot{w}_0 - M_{x,xx} &= P_2 \\ I_{31} \ddot{u}_0 + I_{32} \ddot{w}_{0,x} + I_{33} \ddot{\psi}_0 - P_{x,x} + Q_x &= 0 \end{aligned} \quad (11)$$

Where I_{ij} and \hat{I} are the inertia parameters N_x, M_x, P_x and Q_x are the force, moment, higher order moment and shear resultants respectively and P_2 is the load vector. These resultants are expressed in terms of field variables as

$$\begin{aligned} N_x &= A_{11} u_{0,x} + A_{12} w_{0,xx} + A_{13} \psi_{0,x}, & M_x &= A_{12} u_{0,x} + A_{22} w_{0,xx} + A_{23} \psi_{0,x} \\ P_x &= A_{31} u_{0,x} + A_{32} w_{0,xx} + A_{33} \psi_{0,x}, & Q_x &= \bar{A} \psi_0 \end{aligned} \quad (12)$$

Where the beam stiffnesses A_{ij} and the beam inertia coefficients I_{ij} are defined as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \langle f^T(z) f(z) Q_{11} \rangle \quad \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} = \langle \rho^k f^T(z) f(z) \rangle \quad (13)$$

Substituting the Eq. (12) in Eq. (11), yield the equations of motion

$$\bar{L} \ddot{\bar{U}} + L \bar{U} = \bar{P} \quad (14)$$

Where $\bar{U} = [u_0 \ w_0 \ \psi_0]^T$, $\bar{P} = [0 \ P_2 \ 0]^T$. \bar{L} and L are symmetric matrices containing differential operators.

We derive exact solution for the coupled differential equations of motion having simply supported boundary condition. Considering n terms of Fourier series, the primary and secondary variables are expressed as function of sine and cosine as

$$\begin{bmatrix} u_0 \\ w_0 \\ \psi_0 \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} (u_o)_n \cos \bar{n}x \\ (w_o)_n \sin \bar{n}x \\ (\psi_o)_n \cos \bar{n}x \end{bmatrix} \quad \begin{bmatrix} N_x \\ M_x \\ P_x \\ Q_x \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} (N_x)_n \sin \bar{n}x \\ (M_x)_n \sin \bar{n}x \\ (P_x)_n \sin \bar{n}x \\ (Q_x)_n \cos \bar{n}x \end{bmatrix} \quad P_2 = (P_2)_n \sin \bar{n}x \quad (15)$$

Substituting the solution in Eq. (14) yield the equation of motion,

$$M\ddot{U}^n + KU^n = \bar{P}^n \quad (16)$$

Where $\bar{U}^n = [u_0 \ w_0 \ \psi_0]_n^T$ is the displacement vector for the n^{th} Fourier term, $\bar{P}^n = [0 \ -P_2 \ 0]^T$ is the corresponding load vector. M and K are the mass and stiffness matrices. For performing the free vibration analysis, \bar{P}^n is set to zero to transform Eq. (16) to a generalized eigenvalue problem.

3. Results and Discussions

For assessing the different HOTs presented above for their accuracy, four layers symmetric $0^\circ/90^\circ/90^\circ/0^\circ$ and antisymmetric $[0^\circ/90^\circ/0^\circ/90^\circ]$ laminated beam have been considered. Each lamina of the laminated beam has equal thickness. The values of engineering constants are:

Young's moduli $[Y_1 \ Y_2 \ Y_3] = [181 \ 10.3 \ 10.3]$ GPa, shear moduli $[G_{23} \ G_{31} \ G_{12}] = [2.87 \ 7.17 \ 7.17]$ GPa and Poisson's ratios $[\nu_{23} \ \nu_{31} \ \nu_{12}] = [0.33 \ 0.25 \ 0.25]$. The uniformly distributed load P_z is applied on the top surface of the beam for obtaining the static response by different HOTs. Non-dimensionalised formulae used are: transverse displacement ($\bar{w}_0 = 100w_0Y_2/hS^4 P_z$), longitudinal shear stress $\bar{\sigma}_x = \sigma_x/S^2 P_z$ and transverse shear stress $\bar{\tau}_{zx} = \tau_{zx}/SP_z$.

Table 1. Comparison of results for symmetric and antisymmetric laminated composite beams.

		$0^\circ/90^\circ/90^\circ/0^\circ$				$0^\circ/90^\circ/0^\circ/90^\circ$			
S	Method	$\bar{W}_0(0.5l)$	$\bar{\sigma}_x^t(0.5l)$	$\bar{\sigma}_x^b(0.5l)$	$\bar{\tau}_{zx}^c(0)$	$\bar{W}_0(0.5l)$	$\bar{\sigma}_x^t(0.5l)$	$\bar{\sigma}_x^b(0.5l)$	$\bar{\tau}_{zx}^c(0)$
5	Exact [9]	-2.6748	-1.0602	1.0711	-0.5688	-3.7943	-1.4862	0.1582	-0.7667
	TOT	-2.5530	-1.0097	1.0097	-0.5168	-3.2514	-1.4345	0.1270	-0.7236
	Sine	-2.5815	-1.0221	1.0221	-0.5427	-3.2559	-1.4430	0.1280	-0.7449
	Hyperbolic	-1.1633	-0.7875	0.7875	-0.0000	-2.1297	-1.2153	0.1099	-0.0000
	Exponential	-2.6045	-1.0341	1.0341	-0.5693	-3.2568	-1.4508	0.1289	-0.7663
10	Exact [9]	-1.4343	-0.9031	0.9059	-0.6093	-2.4461	-1.3442	0.1268	-0.8242
	TOT	-1.3975	-0.8901	0.8901	-0.5374	-2.2976	-1.3307	0.1188	-0.7444
	Sine	-1.4054	-0.8932	0.8932	-0.5669	-2.2991	-1.3328	0.1191	-0.7685
	Hyperbolic	-1.0473	-0.8346	0.8346	-0.0000	-2.0162	-1.2759	0.1146	-0.0000
	Exponential	-1.4120	-0.8962	0.8962	-0.5972	-2.2996	-1.3348	0.1193	-0.7928
20	Exact [9]	-1.1152	-0.8636	0.8641	-0.6295	-2.0958	-1.3081	0.1188	-0.8529
	TOT	-1.1056	-0.8602	0.8602	-0.5477	-2.0579	-1.3048	0.1168	-0.7547
	Sine	-1.1076	-0.8610	0.8610	-0.5789	-2.0583	-1.3053	0.1169	-0.7802
	Hyperbolic	-1.0178	-0.8463	0.8463	-0.0000	-1.9875	-1.2911	0.1157	-0.0000
	Exponential	-1.1093	-0.8617	0.8617	-0.6112	-2.0584	-1.3058	0.1169	-0.8060

Deflection and stresses of laminated beam with different HOTs are enumerated in Table 1. Results display almost similar accuracy with exponential shear deformation theory yielding closer results to 2D elasticity solution [9]. To further elaborate the layer-wise mechanics, through-the-thickness variations of longitudinal and transverse shear stresses are presented in Figure.2, for symmetric and antisymmetric laminates along with the 2D FE results obtained using ABAQUS. Unlike other theories, the hyperbolic shear deformation theory yields zero shear stress at mid-plane and non-zero values at top and bottom of the beams.

Figure.3 exhibits the variation of effective properties namely volume fraction and Young's modulus with varying power-law exponent ($p = 0.25, 0.5, 1, 2, 4$) for the FG beam having aluminium at the bottom [$Y = 70 \text{ GPa}, \nu = 0.3$] and alumina at the top [$Y = 380 \text{ GPa}, \nu = 0.3$] [10]. ROM is used as homogenization scheme to evaluate the effective properties.

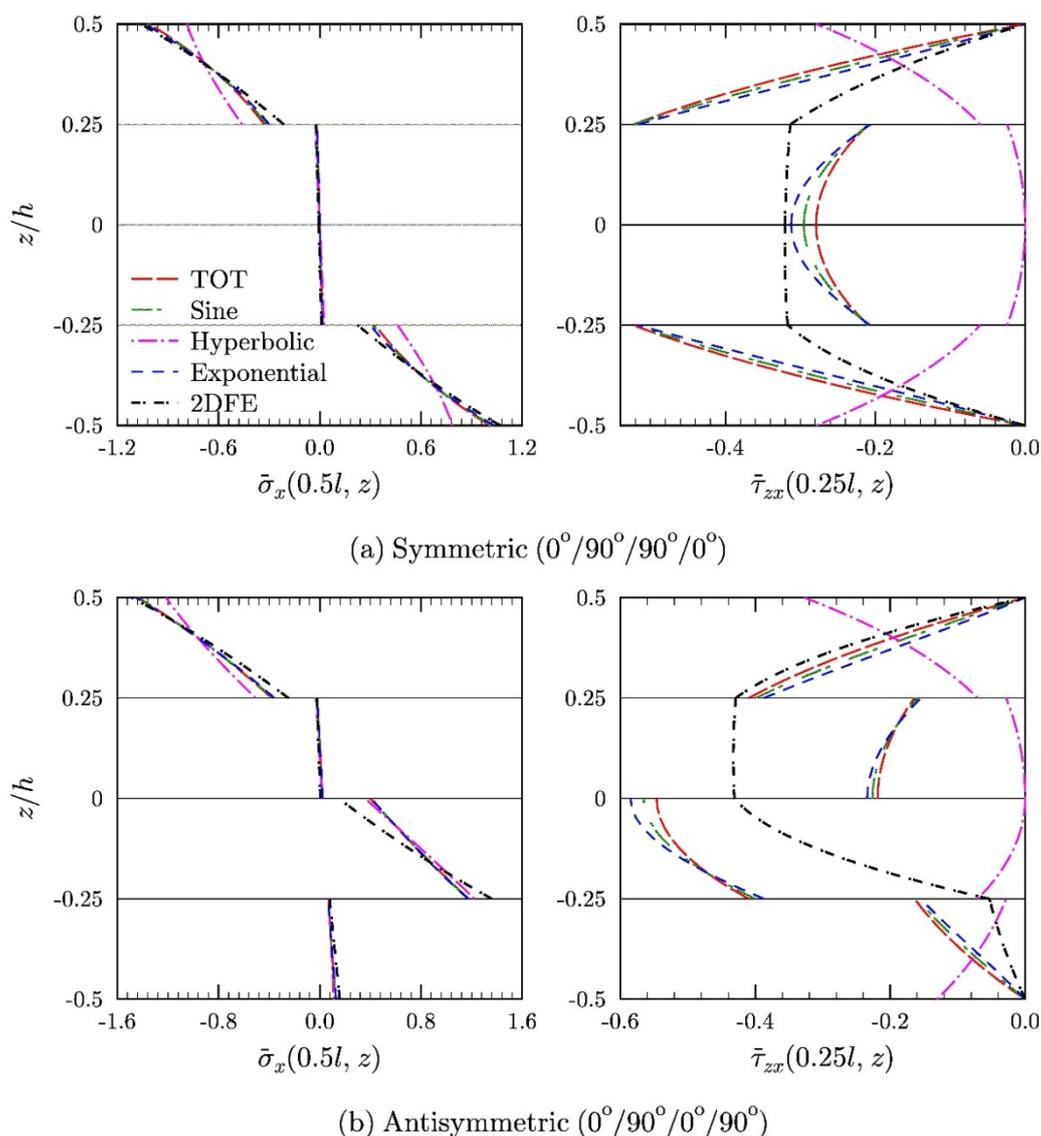


Figure 2. Variation of stresses in laminated beams across the thickness (S=5).

Now we analyze FG beam having aluminum at the top and zirconia at the bottom [10] using different theories for static response. The beam is simply supported and under a uniform pressure P_z at the top surface. Results for non dimensionalised transverse deflection ($w_0 = 348w_0Y_{Al}/60hS^4P_z$) at

the mid span of the beam using different theories are presented in Table 2. Different span to thickness ratios ($S = a/h = 4, 16$) are considered for the analysis. It can be observed that the \bar{w}_0 decreases with increase in the power-law exponent and the trend are validated with the existing literature [10]. Also, for higher span to thickness ratio i.e. $S = 16$, results are in close proximity with the results available in the existing literature [10]. Through the thickness variation for the different stresses developed under uniform loading in FG beam are presented in Figure.4. As predicted earlier for laminated composites, here also, hyperbolic shear deformation theories display distant behavior for the variation of transverse shear stress. However minimal difference is found in longitudinal stresses for the theories presented.

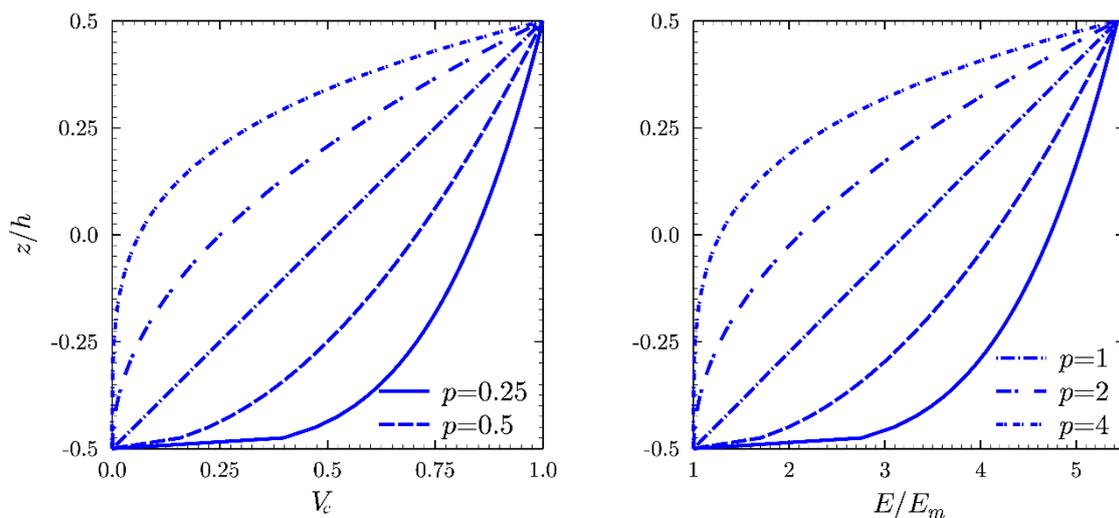


Figure 3. Variation of effective properties across thickness of FG Al/Al₂O₃ beam.

Table 2. Non-dimensional central deflection of FG beams for various values of power law exponent.

S	Theory	Power-law exponent "p"						
		0	0.2	0.5	1	2	5	∞
4	FSDT[10]	1.1300	0.8486	0.7147	0.6293	0.5616	0.4918	0.3955
	HSBT[4]	1.1558	0.8710	0.7326	0.6427	0.5714	0.4998	0.4045
	HSDT[10]	1.1558	0.8710	0.7325	0.6427	0.5714	0.4998	0.4045
	TOT	1.1558	0.8710	0.7326	0.6427	0.5714	0.4998	0.4045
	Sine	1.1555	0.8708	0.7324	0.6426	0.5712	0.4996	0.4044
	Hyperbolic	1.0231	0.7664	0.6483	0.5739	0.5142	0.4501	0.3581
	Exponential	1.1547	0.8703	0.7320	0.6422	0.5708	0.4993	0.4042
	16	FSDT[10]	1.0081	0.7555	0.6394	0.5661	0.5072	0.4439
HSDT[4]		1.0098	0.7570	0.6406	0.5670	0.5078	0.4444	0.3534
HSDT[10]		1.0097	0.7569	0.6406	0.5670	0.5078	0.4444	0.3534
TOT		1.0098	0.7569	0.6406	0.5670	0.5078	0.4444	0.3534
Sine		1.0097	0.7569	0.6406	0.5670	0.5078	0.4444	0.3534
Hyperbolic		1.0014	0.7504	0.6353	0.5627	0.5042	0.4413	0.3505
Exponential		1.0097	0.7569	0.6406	0.5670	0.5078	0.4444	0.3534

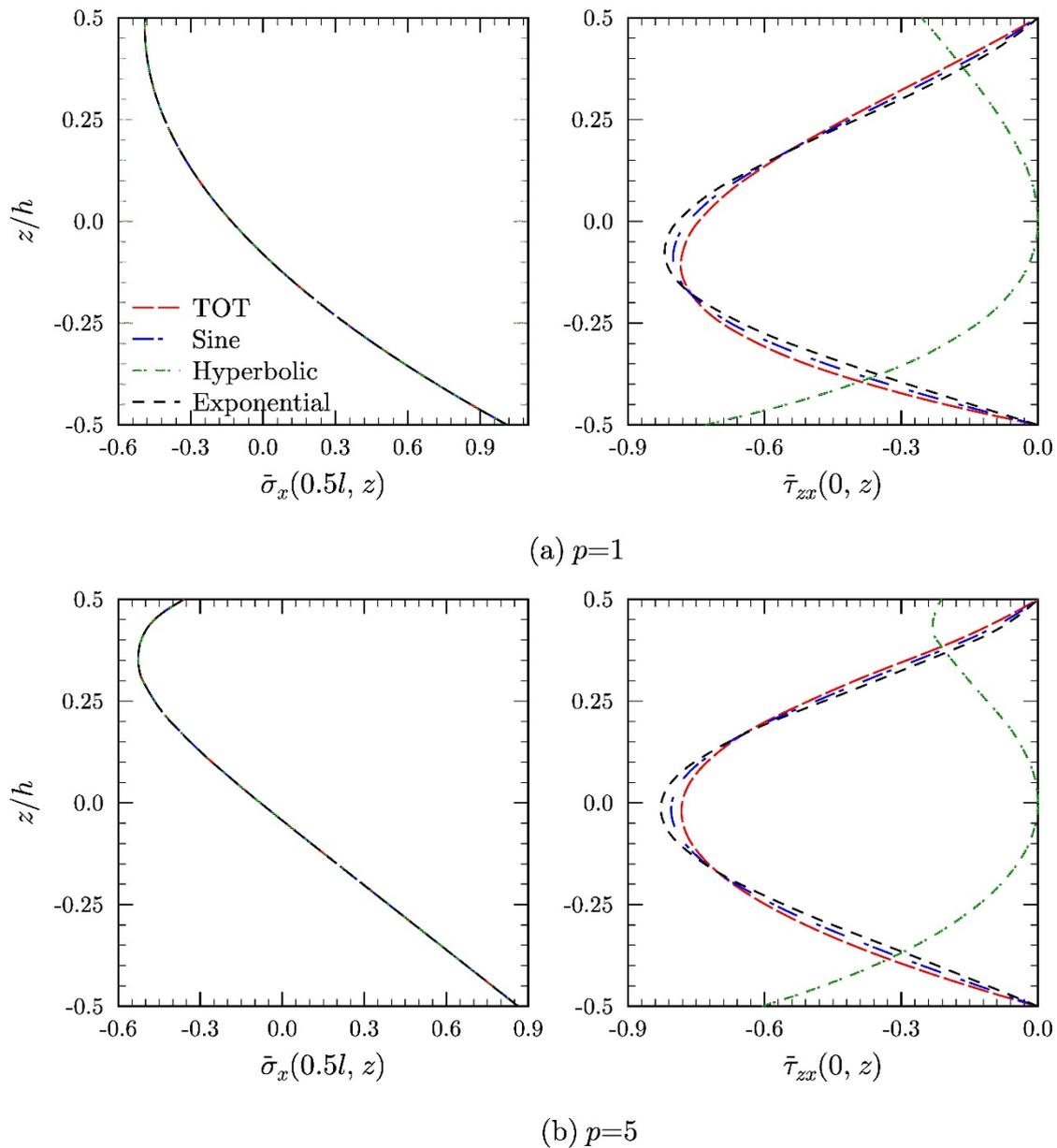


Figure 4. Variation of stresses in FG beams through thickness for (S=4).

Next, free vibration analysis is carried out for FG beam having aluminium at the bottom and alumina at the top [10]. Fundamental frequencies are evaluated for different theories against varying power-law exponent for two different span to thickness ratios ($S = 5, 20$). Results for non dimensionalised fundamental frequencies ($\bar{\omega} = \omega h S^2 \sqrt{\rho_{Al}/Y_{Al}}$) of FG beam for simply-supported end conditions from different theories are presented in Table 3. It is very much evident from Table 3 that the results obtained from the present formulation for the fundamental frequencies are coherent with the results in the literature [10]. In addition, results from the theories presented here are displaying the same trend for the variation of fundamental frequencies with power-law exponent as displayed in Table 3.

Table 3. Comparison of the non-dimensional fundamental natural frequencies of FG beams with various values of power-law exponent.

S	Theory	Power-law exponent " p "						
		0	0.2	1	2	5	10	∞
5	FSDT[10]	5.1526	4.8033	3.9711	3.6050	3.4025	3.2963	2.6773
	HSBT[4]	5.1528	4.8081	3.9904	3.6264	3.4012	3.2816	2.6773
	HSDT[10]	5.1528	4.8059	3.9716	3.5979	3.3743	3.2653	2.6773
	TOT	5.1527	4.8081	3.9904	3.6264	3.4012	3.2816	2.6773
	Sine	5.1531	4.8084	3.9907	3.6263	3.3998	3.2811	2.6775
	Hyperbolic	5.3573	4.9866	4.1238	3.7577	3.5731	3.4687	2.7836
	Exponential	5.1542	4.8093	3.9914	3.6267	3.3990	3.2813	2.6781
20	FSDT[10]	5.4603	5.0812	4.2039	3.8349	3.6490	3.5405	2.8371
	HSDT[4]	5.4603	5.0815	4.2051	3.8361	3.6485	3.5390	2.8371
	HSDT[10]	5.4603	5.0814	4.2039	3.8343	3.6466	3.5379	2.8371
	TOT	5.4603	5.0815	4.2051	3.8361	3.6485	3.5390	2.8371
	Sine	5.4603	5.0815	4.2051	3.8361	3.6484	3.5389	2.8371
	Hyperbolic	5.4751	5.0944	4.2147	3.8457	3.6613	3.5530	2.8448
	Exponential	5.4604	5.0816	4.2051	3.8361	3.6483	3.5390	2.8372

4. Conclusions

This study dealt with static and free vibration analysis of simply supported laminated and functionally graded (FG) beam using various theories. It is found that exponential shear deformation theory yields closer results to 2D elasticity solution. Unlike other theories, the hyperbolic shear deformation theory yields zero shear stress at mid-plane and non-zero values at top and bottom of the laminated and FG beam. Also, for higher span to thickness ratio i.e. $S = 16$, results are in close proximity with the results available in the existing literature.

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