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Thermal buckling analysis of laminated hybrid beam with variable piezoelectric layer thicknesses

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Abstract. Thermal buckling analysis of a laminated hybrid beam with surface mounted piezoelectric layers is presented in this work. A one dimensional finite element (1D-FE) model based on efficient layerwise (zigzag) theory is used for the analysis. Two noded and three noded 1D elements are utilized for interpolating electromechanical and thermal variables respectively. Piezoelectric layers are bonded to top and bottom surfaces of the elastic substrate of hybrid beam.The beams are subjected to uniform temperature with closed circuit condition at top and bottom surfaces. The thicknesses of piezoelectric layers are varied and its effect on critical buckling temperatures is studied. Results are presented for composite and sandwich beam configurations under simply supported boundary conditions. The 1D-FE results are compared with the 2D-FE results obtained using commercial FE package ABAQUS.

1. Introduction

Temperature variations cause significant shape distortions and thermal stresses in laminated composite and sandwich structures. Electro thermal load induce pre-buckling transverse normal strain. This may have significant effect on the buckling temperature depending on the material properties. Designing and analysis of such practical smart structures inevitably require the use of finite element (FE) method. Khdeir [1] studied thermal buckling of cross-ply laminated composite beams. He developed the state space concept in conjunction with Jordan canonical form. Metin Aydogdu [2] presented the thermal buckling analysis of cross-ply laminated composite beams subjected to different boundary conditions on the basis of a unified three degree of freedom shear deformable beam theory. Kapuria and Alam [3] developed an efficient electromechanically coupled geometrically nonlinear zigzag theory for buckling analysis of hybrid piezoelectric beams, under electro-thermo-mechanical loads. Zhong et al. [4] investigated the thermal buckling and postbuckling behaviour of composite plates with embedded SMA wire actuators using a finite element analysis. Chen and Chen [5] investigated thermal buckling of laminated cylindrical shells and determined the critical temperature under clamped and simply supported end conditions.

A one dimensional finite element (1D-FE) model of hybrid piezoelectric beam is presented for buckling analysis under thermal load. The coupled efficient layerwise (zigzag) theory [3] is used for making the model. The thicknesses of piezoelectric layers bonded to the top and bottom surfaces of the beam are varied and its effects on critical buckling temperatures is studied. Hybrid composite and

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sandwich beam configurations under simply supported boundary conditions are considered for the analysis. 1D-FE results are compared with the 2D-FE results obtained using commercial FE package ABAQUS.

2. One dimensional Finite element model of hybrid piezoelectric beam

Consider a hybrid beam subjected to thermal load, transverse load and actuation potentials applied to some piezoelectric layers. The load distribution is independent of the width coordinate. The potential, thermal and displacement field variables ϕ, θ, u_x and u_z are assumed [3] in terms of the three primary variables of the reference plane, u_{x_0}, u_{z_0}, ψ_0 the potential variables, ϕ^l and the thermal variable, θ^j as

$$\phi(x,z,t) = \xi_\phi^l(z)\phi^l(x,t) \tag{1}$$

$$\theta(x,z,t) = \xi_\theta^j(z)\theta^j(x,t) \tag{2}$$

$$\begin{aligned} u_x(x,z,t) = u_{x_0}(x,t) - z\frac{\partial u_{z_0}}{\partial x}(x,t) + S^k(z)\psi_0(x,t) \\ +S^{kl}(z)\frac{\partial \phi^l}{\partial x}(x,t) + \bar{S}^{kj}(z)\frac{\partial \theta^j}{\partial x}(x,t) \end{aligned} \tag{3}$$

$$u_z(x,z,t) = u_{z_0}(x,t) - \bar{\xi}_\phi^l(z)\phi^l(x,t) + \bar{\xi}_\theta^j(z)\theta^j(x,t) \tag{4}$$

where $\xi_\phi^l(z)$ and $\xi_\theta^j(z)$ are linear interpolation functions for through the thickness variation of potential field ϕ and thermal variable θ; $\bar{\xi}_\phi^l(z)$ and $\bar{\xi}_\theta^j(z)$ are piecewise linear and quadratic functions respectively for electric and thermal fields; $S^k(z), S^{kl}(z)$ and $\bar{S}^{kj}(z)$ are the cubic functions of z. The axial displacement, u_x and transverse shear stress, σ_{zx} are continuous at the layer interfaces and the displacement field ensures shear traction free condition at the top and bottom surfaces of beam

The variational equation, using the extended Hamilton's principle [6] for the beam, is obtained as:

$$\begin{aligned} \int_0^a \left[\left\{\delta\tilde{\varepsilon}_1^T \quad \delta\tilde{\varepsilon}_5^T \quad \frac{\partial \delta\phi^l}{\partial x} \quad \delta\phi^l\right\}\bar{F} - \delta\tilde{u}_2^T g_{u\phi} - \delta\tilde{u}_2^T\{\hat{q}_2 \quad -\hat{q}_4^l\}^T + \left(\frac{\partial \delta u_{z_0}}{\partial x}N_x\frac{\partial u_{z_0}}{\partial x}\right)\right]dx - \\ \left[N_x^*\delta u_{x_0}^* + V_x^*\delta u_{z_0}^* - M_x^*\frac{\partial \delta u_{z_0}^*}{\partial x} + P_x^*\delta\psi_0^* + \left(H^{l^*} - V_\phi^{l^*}\right)\delta\phi^{l^*} + R_x^{l^*}\frac{\partial\delta\phi^{l^*}}{\partial x}\right]\Bigg|_0^a \end{aligned} \tag{5}$$

where $\bar{F}$ is the generalized stress vector and $g_{u\phi}$ is the electromechanical load vector of the beam; $\hat{q}_2$ and $\hat{q}_4^l$ are the damping loads; $\tilde{u}_2$ is the generalised displacement vector; $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_5$ are generalised beam mechanical strains; N_x, M_x, P_x, R_x^l and V_x, V_ϕ^l are the beam stress resultants of σ_x and σ_{zx} respectively; H^l and G^l (discussed later) are the beam electric displacement resultants. The superscript * means values at the ends.

A two noded beam element [7] is considered for electromechanical variables and three noded element for thermal variable [8]. The primary variables $u_{x_0}, u_{z_0}, \psi_0, \phi^l$ and θ^j are interpolated in an element of length l as

$$\begin{Bmatrix} u_{x_0} \\ u_{z_0} \\ \psi_0 \\ \phi^l \end{Bmatrix} = \begin{bmatrix} \tilde{N} & 0 & 0 & 0 \\ 0 & \hat{N} & 0 & 0 \\ 0 & 0 & \tilde{N} & 0 \\ 0 & 0 & 0 & \tilde{N} \end{bmatrix} \begin{Bmatrix} u_{x_0}^e \\ u_{z_0}^e \\ \psi_0^e \\ \phi^{l^e} \end{Bmatrix}, \qquad \theta^j = \bar{N}\theta^{j^e} \tag{6}$$

where $\tilde{N}$ is a vector of linear Lagrange interpolation functions, $\hat{N}$ is a vector of cubic Hermite interpolation functions and $\bar{N}$ is a vector of quadratic Lagrange Interpolation functions; $u_{x_0}^e, u_{z_0}^e, \psi_0^e, \phi^{l^e}$ and θ^{j^e} are vectors of nodal values.The element generalized displacement vector, d^e is defined as

$$d^{e^T} = \left\{u_{x_0}^{e^T} \quad u_{z_0}^{e^T} \quad \psi_0^{e^T} \quad \phi^{l^{e^T}}\right\} \tag{7}$$

The contribution T^e of one element of length l to the integral from 0 to l in Eq. (5) for the case of static inplane electro-thermo-mechanical load is obtained as

$$T^e = \int_0^l \left\{ \delta\tilde{\varepsilon}_1^T F_1 + \delta\tilde{\varepsilon}_5^T F_5 + \left\{ \frac{\partial \delta\phi^l}{\partial x} \quad \delta\phi^l \right\} \{H^l \quad G^l\}^T + \delta\tilde{u}_2^T q_4^{l\,T} \right\} dx \tag{8}$$

where F_1 and F_5 are the vectors of beam stress resultant of σ_x and σ_{zx} respectively and q_4^l is the electrical load.

For buckling analysis, the transverse loads applied on the bottom and top surfaces of the beam and hence total transverse mechanical load are considered as zero. With this consideration the Eq. (8) may be simplified as:

$$T^e = \int_0^l \left[\delta\bar{\varepsilon}^T \left(\bar{D}\bar{\varepsilon} + g_{\theta_2}^j \frac{\partial^2 \theta^j}{\partial x^2} + g_{\theta_1}^j \frac{\partial \theta^j}{\partial x} + g_{\theta_0}^j \theta^j \right) - \delta\tilde{u}_2^T (\bar{g}_{u\phi}) \right] dx \tag{9}$$

where only electric load is considered to obtain the vector $\bar{g}_{u\phi}$as

$$\bar{g}_{u\phi} = \{0 \quad -\bar{q}_4^l\}^T \tag{10}$$

with

$$\bar{q}_4^l = b\left[D_{z_N}\delta_{ln_\emptyset} - D_{z_0}\delta_{l1} + q_{li}\delta_{ll_i}\right] \tag{11}$$

$\bar{D}$ is the generalized stiffness matrix of the beam, $g_{\theta_2}^j, g_{\theta_1}^j, g_{\theta_0}^j$ are the generalized stress-temperature coefficient matrices.

The generalised strain, $\bar{\varepsilon}$ and the displacement vector, $\tilde{u}_2$ are interpolated as

$$\bar{\varepsilon} = \bar{B}d^e, \quad \tilde{u}_2 = \bar{B}_{m_2}d^e \tag{12}$$

Here $\bar{B}$ and $\bar{B}_{m_2}$are strain displacement and displacement interpolation matrices respectively.

Substituting the expressions of θ^j ,$\bar{\varepsilon}$ and $\tilde{u}_2$ from Eqs. (6) and (12) respectively, we get

$$\begin{aligned} T^e &= \int_0^l \left[\delta d^{eT} \bar{B}^T \left(\bar{D}\bar{B}d^e + g_{\theta_2}^j \frac{\partial^2 \bar{N}}{\partial x^2} \theta^{je} + g_{\theta_1}^j \frac{\partial \bar{N}}{\partial x} \theta^{je} + g_{\theta_0}^j \bar{N} \theta^{je} \right) - \delta d^{eT} \bar{B}_{m_2}^T (\bar{g}_{u\phi}) \right] dx = \\ &\int_0^l \delta d^{eT} \left[\bar{B}^T \bar{D} \bar{B} d^e + \bar{B}^T \left(g_{\theta_2}^j \frac{\partial^2 \bar{N}}{\partial x^2} + g_{\theta_1}^j \frac{\partial \bar{N}}{\partial x} + g_{\theta_0}^j \bar{N} \right) \theta^{je} - \bar{B}_{m_2}^T \bar{g}_{u\phi} \right] dx = \delta d^{eT} [K^e d^e + \\ &\bar{P}_\theta^e - \bar{P}^e] \end{aligned} \tag{13}$$

where K^e, $\bar{P}^e$ and $\bar{P}_\theta^e$ are given by

$$K^e = \int_0^l \bar{B}^T \bar{D} \bar{B} dx\,, \;\; \bar{P}^e = \int_0^l \bar{B}_{m_2}^T \bar{g}_{u\phi} dx, \;\; \bar{P}_\theta^e = \left(\int_0^l \bar{B}^T \left(g_{\theta_2}^j \frac{\partial^2 \bar{N}}{\partial x^2} + g_{\theta_1}^j \frac{\partial \bar{N}}{\partial x} + g_{\theta_0}^j \bar{N} \right) dx \right) \theta^{je} \tag{14}$$

Thus the general equation after integration is obtained as:

$$K^e d^e = \bar{P}^e - \bar{P}_\theta^e \tag{15}$$

where

$$\bar{P}^e = \left\{0 \quad 0 \quad a_{18}\bar{q}_\phi^{le}\right\}^T \tag{16}$$

with

$$\bar{q}_\phi^{le} = \left\{\bar{q}_{\phi_1}^{le} \quad \bar{q}_{\phi_2}^{le}\right\}^T, \qquad a_{18} = b\int_0^l \hat{N}^T \tilde{N} dx \tag{17}$$

3. Results and discussion

3.1. Validation

The present 1D-FE formulation of zigzag theory is first validated for buckling response under thermal loading with simply supported hybrid piezoelectric beam of Ref. [3]. Beam configurations (b) and (c) and pre-buckling load case 1 of Ref. [3] is considered for validation. The results are compared (Table-1) for span to thickness ratio $S = l/h = 20$.

Critical temperature, $\bar{\theta}_{cr}$ for fundamental buckling mode, $n = 1$						
S	Beam b			Beam c		
	1D FE	Ref [3]	% error	1D FE	Ref [3]	% error
20	17.415	16.9378	2.817	51.550	49.5434	4.050

Table 1: Validation of critical temperature, $\bar{\theta}_{cr}$ with simply supported beams of Ref. [3]

The %age of difference is 4.05 which shows the correctness of the present (1D-FE) formulation.

3.2. Numerical Example

The hybrid beams (d) and (e) [9] with composite and sandwich substrate respectively are considered for the analysis. Piezoelectric layers (PZT-5A) of thicknesses h_p are bonded at the top and bottom surfaces of the substrate in both the beams. h_s is the thickness of elastic substrate and h is the total beam thickness. +z is considered to be the poling direction for piezoelectric layers. The surfaces of the substrate where PZT-5A layers are bonded are grounded. The composite substrate (d) has four numbers of $0.25h_s$ thick graphite epoxy layers with layup$[0^o/90^o/90^o/0^o]$. The sandwich substrate (e) has three layers, an intermediate soft core and top-bottom face sheets of thicknesses $0.08h_s/0.84h_s/0.08h_s$. h_p is varied from $0.1h$ to $0.3h$ keeping h as constant .

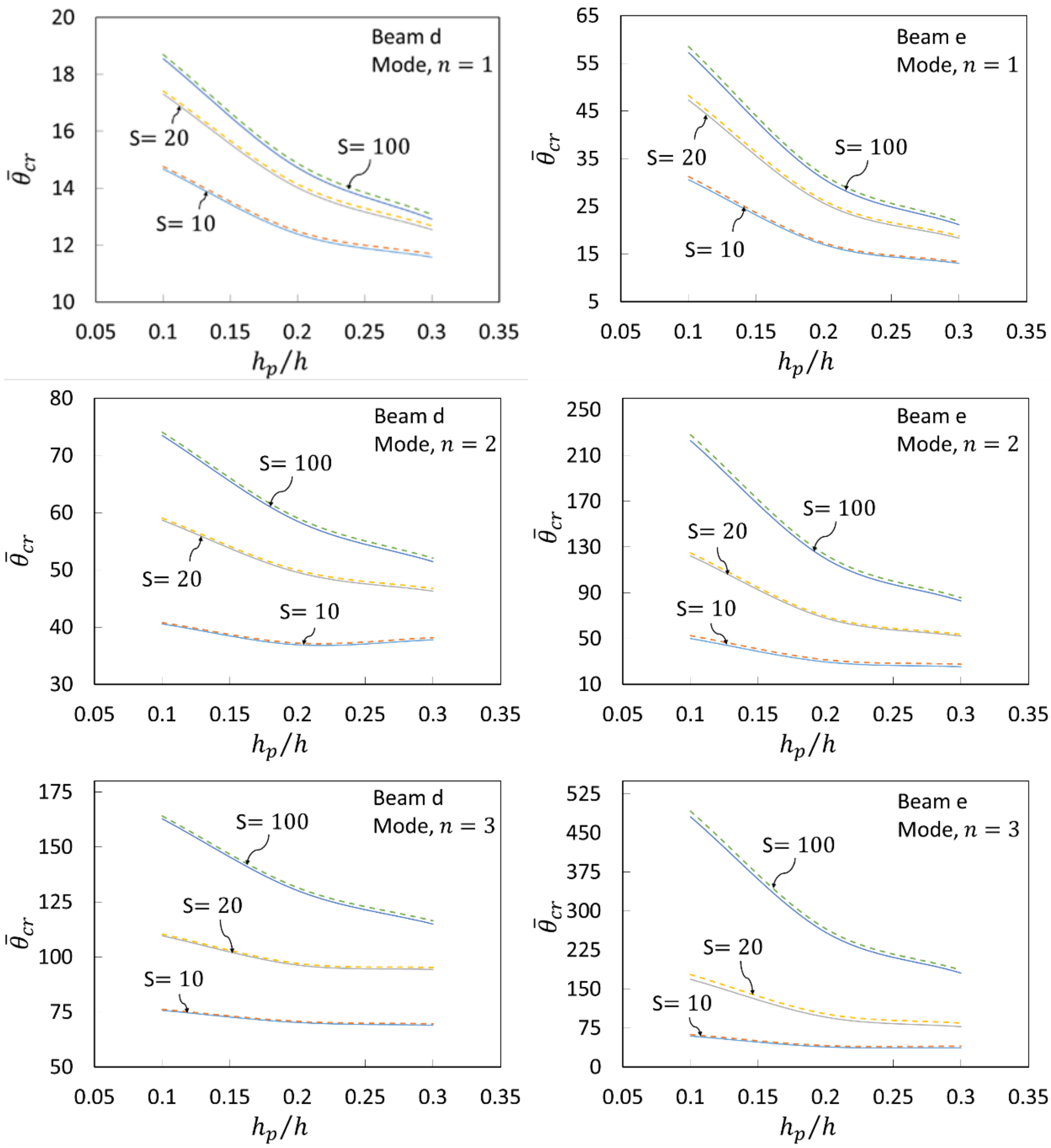

Figure 1: Variation of critical temperature $\bar{\theta}_{cr}$ with thickness ratio of piezoelectric layers

The following pre-buckling load case is considered

1. Uniform temperature rise of the beam with zero potential at top and bottom surfaces and immovable beam ends. The critical temperature for buckling is expressed as θ_{cr} which is non-dimensionalised as $\bar{\theta}_{cr} = \alpha_0\theta_{cr}S^2$, where $S = l/h$ and α_0 is taken as $22.5 \times 10^{-6}K^{-1}$.

The critical buckling temperatures are obtained for hybrid beams under simply supported boundary conditions. The converged 1D-FE results are obtained using forty elements of equal size. The 1D-FE results are compared with 2D-FE (ABAQUS) results where the beam is first analyzed in heat transfer step which is then followed by buckling step. The temperature distribution as obtained in heat transfer step is used as a predefined field in the next step of linear perturbation buckling procedure to obtain the buckling response under thermal load. The beam has been discretized into more than 600 elements for span to thickness ratio $S = 10$. This number increases substantially for larger span to thickness ratios of hybrid beam.
The variation of critical buckling temperature for first three buckling modes $(n = 1,2,3)$ with thickness ratio of piezoelectric layer, h_p/h for both the hybrid beams is shown in Fig.1.The results are presented for different span to thickness ratios, S. The critical buckling temperature increases as the beams are made thinner for the same span length. It may be observed that for the same value of S there is substantial increase in values of critical buckling temperature for higher buckling modes.
The critical temperature, $\bar{\theta}_{cr}$ decreases with increase in thickness ratio of piezoelectric layers for beam d as well as beam e. $\bar{\theta}_{cr}$ is maximum for $h_p = 0.1h$ and exponentially reduces as the thickness of piezoelectric layer increases and become $0.3h$. Further the critical temperature corresponding to the fundamental buckling mode for beam e is higher in comparison to beam d for all the thickness ratios (h_p/h) considered. The 1D-FE and 2D-FE plots are in good agreement. The critical temperatures are obtained ignoring the effect of pre-buckling deformation.

4. Conclusions
The thickness of piezoelectric layers in smart beams is significant as it results in different buckling response under thermal loading. A 1D-FE model to assess the effect of thicknesses of piezoelectric layers on thermal buckling response of a hybrid beam is presented. The results are obtained for simply supported boundary conditions. The critical temperature decreases with increase in thickness ratio of piezoelectric layers for both hybrid composite and sandwich beams. For all the thickness ratios, the critical temperature corresponding to the fundamental buckling mode is higher for sandwich beam in comparison to the composite beam. These observations may be used as a basis for experimental investigation and design for composite and sandwich hybrid structures.

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