

About the sensitivity to hydrostatic pressure of Mohr-Coulomb criterion with circular failure envelope, dedicated to ductile materials

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Abstract. Knowledge of material properties, in particular under hydrostatic pressure, is essential, both to select the proper material for a specific engineering application and to predict accurately its behaviour using mathematical models. Many criteria were proposed to describe the behaviour of the various engineering materials under complex state of stress. Both materials and criteria can be sensitive or insensitive to hydrostatic pressure. This work demonstrates that modified Mohr-Coulomb criterion, with circular failure envelope, applied for ductile materials, is insensitive to hydrostatic pressure.

1. Introduction

Classical theory of plasticity is based on the hypotheses such as: the material is isotropic and homogeneous; plastic deformation proceeds under constant volume; tensile and compressive yield strengths are equal; the yielding is uninfluenced by the hydrostatic pressure etc. and has shown great success in various applications. However, these assumptions are suitable for ductile metallic materials, but most times inaccurate for other materials (polymers, ceramics, brittle metals and alloys, metallic glasses etc.). The last two hypotheses determine that tensile and compressive stress-strain behaviours are identically treated in this theory.

For the common ductile metals and alloys, only small differences between tensile and compressive yield strengths have been reported. However these differences can be appreciable in other materials (martensitic steels, high strength alloys, cast iron, ceramics as well as in polymers). In other words, it is said that the materials in the second category present a strength-differential (SD) effect. *D. C. Drucker* [1] appreciated that: „A precise but not quite equivalent definition of the presence or absence of a true SD effect is obtained by asking whether or not the application of hydrostatic pressure (which adds equal compression normal to each plane) raises the magnitude of the yield or flow strength in shear (or in tension or compression).”

Yielding may occur in a point of a solid body under complex state of stress when a specific function of stress reaches a critical value. On the other hand, this function and the magnitude of this critical value may be influenced or not by the hydrostatic pressure [2]. The influence of hydrostatic pressure on the yield strength must be considered, within the scope of the mathematical plasticity theory as well as for engineering applications.



The influence of hydrostatic pressure on the yield stress of many ductile solid metals and alloys is negligible, at least in the range of pressure of plus or minus the yield stress. There are failure criteria for both categories of material (sensitive or insensitive to hydrostatic pressure). The most popular criteria insensitive to hydrostatic pressure are *von Mises* and *Tresca* (this is a particular case of the *Mohr-Coulomb* criterion), while the criteria sensitive to hydrostatic pressure are *Mohr-Coulomb* and *Drucker-Prager* [3], [4], [5]. *Drucker-Prager* is an extension of the *von Mises* criterion for pressure-dependent materials **Error! Reference source not found..**

All points where the yielding occurrence is imminent form the yield surface. It is usually expressed in terms of principal stresses $F(\sigma_1, \sigma_2, \sigma_3)=0$ or stress invariants $F(I_1, J_2, J_3)=0$ etc. The *Cauchy* stress invariants are denoted above by I_1, I_2, I_3 , and the deviatoric invariants by J_1, J_2, J_3 :

$$\begin{cases} I_1 = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 = \sigma_1\sigma_2\sigma_3 \end{cases} \quad (1)$$

$$\begin{cases} J_1 = 0 \\ J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\ J_3 = \frac{1}{27}(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_1 - \sigma_3)(2\sigma_3 - \sigma_1 - \sigma_2) \end{cases} \quad (2)$$

Between the two groups of invariants there are the following relations [7]:

$$\begin{cases} J_2 = \frac{1}{3}I_1^2 - I_2 \\ J_3 = \frac{2}{27}I_1^3 - \frac{1}{3}I_1I_2 + I_3 \end{cases} \quad (3)$$

All the expressions of yield surface mentioned above are the same **Error! Reference source not found.**, but, because only the first invariant I_1 is responsible for sensitivity to hydrostatic pressure [8], the function in terms of invariants is preferred in order to appreciate the sensitivity of materials to hydrostatic pressure. But this is not always easy to achieve.

Many empirical criteria were proposed to describe the behaviour of the various engineering materials. Some of these are listed in Table 1, where the material parameter $C = \sqrt{2}\sigma_y/3$ and σ_y is the yield stress for uniaxial tension. Of all these criteria, only *Krenk* is written in terms of deviatoric invariants, does not contain axiatoric invariant I_1 , and so is insensitive to hydrostatic pressure.

Table 1. Different criteria written in terms of invariants **Error! Reference source not found..**

No.	Criterion	Year	Equation
1	<i>Matsuoka and Nakai</i>	1974	$F = \frac{I_1 I_2}{I_3} = C$
2	<i>Lade and Duncan</i>	1975	$F = \frac{I_1^2}{I_3} = C$

3	<i>Lade</i>	1977	$F = \left(\frac{I_1 I_2}{I_3} - 27 \right) \left(\frac{I_1}{p_a} \right)^m = C$
4	<i>Yin et al.</i>	1987	$F = J_2 - a + b I_1 J_3^{1/3} = C$
5	<i>Krenk</i>	1996	$F = J_3 + c J_2 - (1 - \eta) c^3 = 0$

Although *Mohr-Coulomb* criterion has a wide application to brittle materials (especially to geomaterials), its particular form *Tresca* criterion have only a limited application to ductile materials [9]. An improved criterion for ductile materials, based on the *Mohr's* theory, has been proposed by *P. Barsanescu* [10]. It use a circular failure envelope, tangent to the limit *Mohr's* circles for uniaxial tension/compression and pure shear. For ductile materials in a biaxial state of stress ($\sigma_2=0$) and for failure envelope described above, has been derived the following equation:

$$\sigma_y = \frac{1}{2} \left(\sigma_1 - \sigma_3 + \sqrt{\sigma_1^2 + \sigma_3^2 - 2 \frac{2 - 4k_{ST} + k_{ST}^2}{k_{ST}^2} \sigma_1 \sigma_3} \right) \quad (4)$$

where the material coefficient k_{ST} is

$$k_{ST} = \frac{\tau_y}{\sigma_y} \quad (5)$$

and τ_y is the yield stress for pure shear.

The equation (4) exhibit a good correlation with experimental results obtained for different ductile materials under biaxial state of stress.

This paper aims to show whether this new criterion is sensitive or not to hydrostatic pressure.

2. Sensitivity to hydrostatic pressure

Since writing the equation (4) in terms of invariants is difficult, the following demonstration will use the *von Mises* and *Tresca* criteria as functions of principal stresses and of invariants, respectively. As we will see below, this is only algebrae and does not mean the mixing these criteria.

In terms of principal stresses, *von Mises* criterion for biaxial state of stress may be expressed as:

$$\sigma_y^2 = \sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3 \quad (6)$$

The same criterion can be written in terms of invariants:

$$\sigma_y^2 = 3J_2 \quad (7)$$

Combining the equations (6) and (7) it is obtained:

$$\sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3 = 3J_2 \quad (8)$$

It should be noted that equation (8) is no longer conditioned by the initial hypotheses of *von Mises* theory. It does not contain any characteristic of material, it is only mathematics. This statement can be argued as follows: equation (8) can be obtained directly from the equation (2), written for the biaxial state of stress, without involvement of *von Mises* criterion.

It is much more difficult to write the difference of the principal stresses in terms of invariants. For this reason, the *Tresca* criterion is used, which may be expressed in terms of principal stresses and in terms of invariants, respectively [11], [12]:

$$\sigma_y = \sigma_1 - \sigma_3 \quad (9)$$

$$4J_2^3 - 27J_3^2 - 9\sigma_{yt}^2 J_2^2 + 6\sigma_{yt}^4 J_2 = \sigma_{yt}^6 \quad (10)$$

Replacing $x = \sigma_y^2$ in equation (10), the following cubic equation is obtained:

$$x^3 - 6J_2 x^2 + 9J_2^2 x + 27J_3^2 - 4J_2^3 = 0 \quad (11)$$

By solving the above cubic equation, are obtained successively the roots:

$$\sigma_y = \frac{C - 3J_2}{\sqrt{3C}} \quad (12)$$

where

$$C = \frac{3}{\sqrt[3]{2}} \sqrt{27J_3^2 - 2J_2^3 \pm 3\sqrt{3}J_3 \sqrt{27J_3^2 - 4J_2^3}} \quad (13)$$

Combining the equations (9) and (12) it is obtained that

$$\sigma_1 - \sigma_3 = \frac{C - 3J_2}{\sqrt{3C}} \quad (14)$$

For equation (14) a similar observation with the one that refers to equation (8) can be made: it no longer implies the initial assumptions of *Tresca* criterion, do not contain any material characteristic and it is only a mathematical relationship.

In the biaxial state of stress $\sigma_2 = 0$ and it follows that

$$\begin{aligned} I_2 &= \sigma_1 \sigma_3 \\ I_3 &= 0 \end{aligned} \quad (15)$$

Replacing the second of equation (15) in equation (3) results the incomplete cubic equation

$$I_2^3 - 3J_2 I_2^2 + 4J_2^3 - 81J_3^2 = 0 \quad (16)$$

The roots of equation (11) are:

$$I_2 = J_2 \left(\frac{3J_2}{C_1} - 1 \right) \quad (17)$$

where

$$C_1 = \frac{3}{\sqrt[3]{2}} \sqrt{2J_2^3 - 81J_3^2 \pm 9J_3 \sqrt{81J_3^2 - 4J_2^3}} \quad (18)$$

Replacing the first of equation (15) in equation (17) it results that

$$\sigma_1 \sigma_3 = J_2 \left(\frac{3J_2}{C_1} - 1 \right) \quad (19)$$

Equation (4) can be re-written as

$$\sigma_y = \frac{1}{2} \left[\sigma_1 - \sigma_3 + \sqrt{(\sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3) - \frac{k_{ST}^2 - 8k_{ST} + 4}{k_{ST}^2} \sigma_1 \sigma_3} \right] \quad (20)$$

Replacing (in this order) equation (14), (8) and (19) in equation (20), the yielding function σ_y only in terms of stress deviator invariants is obtained.

According to D.W.A. Rees [13], “...when a yielding function is formulated from the stress deviator invariants, it assumes that initial yielding is unaffected by the magnitude of hydrostatic stress.”

The equation (4) was determined for ductile metallic materials, without SD effect. For these materials, we can suppose that the constant k_{ST} is a material parameter, which is insensitive to hydrostatic pressure. This assumption about k_{ST} appears to be very reasonable for ductile metals and alloys. So, the modified M-C criterion described by the equation (4) is pressure insensitive, if k_{ST} coefficient is also pressure insensitive.

3. Conclusions

In order to know if a criterion is sensitive or insensitive to hydrostatic pressure, the yield function must be formulated in terms of stress invariants, but this is often difficult. For this reason, the mathematical demonstration presented in this paper uses the *von Mises* and *Tresca* criteria written in terms of deviatoric invariants. It has been shown so that the modified *Mohr-Coulomb* criterion with circular failure envelope, applied for ductile materials, is insensitive to hydrostatic pressure. It is once again noted that different criteria above were not „mixed up” during this demonstration.

4. References

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