

The effect of non-zero initial displacement value on displacement jump and wave form based on the cubic nonlinear spring interface model

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Abstract. The effect of non-zero initial displacement value on displacement jump and wave form based on the cubic nonlinear spring interface model between two semi-infinite plates is studied. The wave forms of reflected and transmitted pulses are sensitive to the initial value. The phenomenon for non-zero initial displacement value is much different from the counterpart for zero initial displacement value. First, the displacement jump is larger in the case of linear spring than that of nonlinear spring, but the displacement jump goes across the curve for linear spring with the initial displacement value increasing. Second, when the initial displacement value is not zero, for reflected pulse, the curve for nonlinear spring interface is on the top of the curve for linear spring interface, but for transmitted wave, the curve for linear spring interface is on the top of the curve for nonlinear spring interface. This is opposite for zero initial displacement. The initial displacement value plays a great role in elastic wave reflecting. The results in this paper will be of help to nondestructive evaluation of the interface.

1. Introduction

The interface/interphase exists widely in composite materials. It is very hard to give an accurate model to determine the interface/interphase because of the complexity of various purpose and environment. When the interface phase is thin enough, the inertia effect of the interface can be neglected, the interface phase is usually equalized as distributed springs which are regularly used for an interface between two solids[1-3]. Biwa et al.[4] made a thorough investigation of the second harmonic generation at a contact interface theoretically and experimentally. Pecorari[5] studied the nonlinear interaction between an acoustic wave and an interface, and the spring boundary condition was employed to describe its acoustic properties.

Nonlinear parameter is more sensitive than the linear one, especially at an early stage of the degradation of materials[6-8]. The first nonlinear spring interface model was proposed by Achenbach[9]. Recently, An [10] analyzed the parameters that affect the amplitudes of the second harmonics generated at the interface by using a nonlinear spring model, providing the theoretical foundation as well as validate the applicability of the model by Achenbach.

The model has been employed in nondestructive evaluation of an adhesive bonding interface widely. However, few literatures can be found about the effect of the non-zero initial displacement value. This problem is discussed in particular in this paper.

2. Cubic Nonlinear Spring Interface Model

Assume two semi-infinite plates are bonded together. The ultrasonic pulse of S wave propagates in the positive y -direction with the displacement in the x -direction. The incident wave is reflected and



transmitted by the interface. The shear moduli and mass densities are μ_1, ρ_1 for $y > 0$ and μ_2, ρ_2 for $y < 0$, respectively. The thickness of the interface is h . Normally, h is much less than the wave length. The interface is thin enough and the mass density is small, the inertial effect of the interface can be ignored. The function of the interface is just to link different material components together. Considering the essential nonlinearity of the spring, a cubic nonlinear interface model is as follows[9].

$$\sigma_{yx}|_{y=0^+} = \sigma_{yx}|_{y=0^-} = \sigma_{yx}|_{y=0}, \tag{1}$$

$$\sigma_{yx}(0) = \alpha\Delta - \beta\Delta^3, \tag{2}$$

Where $\Delta = u|_{y=0^+} - u|_{y=0^-}$ denotes the jump of displacement across the interface. $\beta = 0$ indicates the linear case, but $\beta \neq 0$ indicates the nonlinear case.

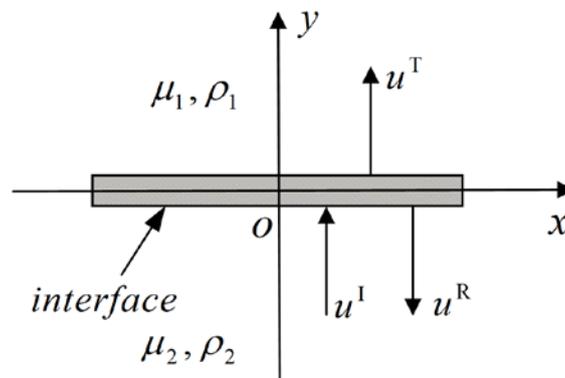


Figure 1. elastic wave motion incident on an interface

3. Analytical Solution

Assume the pulse is normally incident from the underlying solid on the interface. Without loss of generality, the form of incident pulse is assumed as $u^I(y,t) = f(t - y/c_2)$. Where $f(t) = 0$ for $t \leq 0$ and $t \geq t_0$, t_0 is the duration of the incident pulse. The reflected and transmitted displacement pulses are $u^R(y,t) = g(t + y/c_2)$, $u^T(y,t) = h(t - y/c_1)$.

The displacement jump is defined by

$$\Delta = h(t) - f(t) - g(t) + \Delta_{ini}, \tag{3}$$

where Δ_{ini} is the initial displacement jump of the interface. Obviously, $f(0) = g(0) = h(0) = 0$, thus the assumption of $\Delta(0) = 0$ seems entirely plausible to us. However, $h(t) - f(t) - g(t) < 0$ may occur in theoretical calculation, which means one side of the interface is embedded in the other side. This phenomenon is unreasonable since it is contrary to the physical facts. We modify the initial condition to $\Delta(0) = \Delta_{ini}$, and suppose $\Delta_{ini} > 0$. It means that the spring is assumed to be lengthened in advance. Even if $h(t) - f(t) - g(t) < 0$, it can be understood that the compression is on the basis of a pre-stretched length. The contradiction of the imbedding phenomenon is reasonably avoided in part.

According to the equality of the shear stresses across the interface and the continuity of the displacement at the boundary, we can obtain

$$\mu_1 \left. \frac{\partial u^T}{\partial y} \right|_{y=0} = \mu_2 \left. \frac{\partial u^I}{\partial y} \right|_{y=0} + \mu_2 \left. \frac{\partial u^R}{\partial y} \right|_{y=0}, \tag{4}$$

Substitution of u^I, u^R and u^T into Eq.(4) and Substitution of Δ yield $\dot{h}(t) = m[2\dot{f}(t) + \dot{\Delta}]$, where $m = \frac{c_1\mu_2}{c_2\mu_1} / \left(1 + \frac{c_1\mu_2}{c_2\mu_1}\right)$, and the dot denotes differentiation of time t . Combining with $\sigma_{yx}(0) = -\dot{h}(t)\mu_1/c_1$, we can introduce the nonlinear ordinary differential equation of Δ

$$\dot{\Delta} - \frac{\beta c_1}{m\mu_1} \Delta^3 + \frac{\alpha c_1}{m\mu_1} \Delta = -2\dot{f}(t), \tag{5}$$

The solution must satisfy the initial condition $\Delta(0) = \Delta_{ini}$.

In the range of $0 < t \leq t_0$, if $\beta = 0$ (linear case), Eq.(5) degenerate into the first order linear nonhomogeneous ordinary differential equation, which can be solved analytically. If $\beta \neq 0$ (nonlinear case), Eq.(5) can be solved numerically. In the range of $t \geq t_0$, the solution of the initial value problem of ordinary differential equation (5) is

$$\Delta = \begin{cases} \left(D_1 e^{\frac{2\alpha c_1 t}{m\mu_1}} + \frac{\beta}{\alpha} \right)^{-\frac{1}{2}}, & \beta \neq 0, \\ D_2 e^{-\frac{\alpha c_1 t}{m\mu_1}}, & \beta = 0. \end{cases} \tag{6}$$

Where the unknown coefficient D_1 and D_2 can be obtained by $\Delta(t_0)$ which can be computed by solving the nonhomogeneous ordinary differential equation (5) when $0 < t \leq t_0$. Once Δ is obtained, the transmitted pulse $h(t)$ and the reflected pulse $g(t)$ can also be derived as

$$h(t) = m(2f(t) + \Delta - \Delta_{ini}), \quad g(t) = \frac{1}{1 + (c_1\mu_2)/(c_2\mu_1)} \left[((c_1\mu_2)/(c_2\mu_1) - 1)f(t) - \Delta \right] - m\Delta_{ini}.$$

When the input $f(t)$ is given, the response can be calculated.

4. Numerical Calculation and Discussion

For the sake of convenience, the variables can be nondimensionalized by the thickness of the interface h , we introduce $\bar{\Delta} = \Delta/h$, $\bar{f} = f/h$, $\bar{t} = t/(h/c_1)$, $\bar{\alpha} = \alpha h/\mu_1$, $\bar{\beta} = \beta h^3/\mu_1$, $\bar{f} = f/h$, $\bar{\Delta}_{ini} = \Delta_{ini}/h$, then Eq.(5) is simplified as

$$\dot{\bar{\Delta}} - \bar{\beta}/m\bar{\Delta}^3 + \bar{\alpha}/m\bar{\Delta} = -2\dot{\bar{f}}, \tag{7}$$

where the dot denotes differentiation of \bar{t} . The initial condition is simplified as $\bar{\Delta}(0) = \bar{\Delta}_{ini}$.

For the convenience of comparison with the reference [9], we consider the same incident wave pulse of the form $f(t) = \frac{1}{2} \frac{A}{\Delta_{ini}} [1 - \cos(2\pi t/t_0)]$, $0 \leq t \leq t_0$. Introducing the dimensionless variables

$\bar{A} = A/h$, $\bar{t} = t/(h/c_1)$, and supposing $\bar{\alpha} = 1$, $m = 0.5$, $(h/c_1)/t_0 = 0.5$, $A/\Delta_{ini} = \bar{A}/\bar{\Delta}_{ini} = 0.5$, then we obtain $\bar{f} = [1 - \cos(\pi\bar{t})]/4$. Eq.(6) is simplified as

$$\bar{\Delta} - 2\bar{\beta}\bar{\Delta}^3 + 2\bar{\Delta} = -\pi \sin(\pi\bar{t})/2, 0 \leq \bar{t} \leq 2. \tag{8}$$

The solution of Eq.(8), the reflected wave and the transmitted wave are as follows.

(1) $\bar{\beta} = 0$ (linear spring)

$$\bar{\Delta} = \begin{cases} \bar{\Delta}_{ini}e^{-2\bar{t}} - \frac{\pi^2}{2\pi^2 + 8}e^{-2\bar{t}} - \frac{\pi}{2\pi^2 + 8}[2\sin(\pi\bar{t}) - \pi\cos(\pi\bar{t})], & 0 < \bar{t} \leq 2, \\ \bar{\Delta}_{ini} + \frac{\pi^2(e^4 - 1)}{2\pi^2 + 8}e^{-2\bar{t}}, & \bar{t} > 2. \end{cases}$$

(2) $\bar{\beta} = 2$ (nonlinear spring), let $\bar{\Delta}(2) = \bar{\Delta}_0$, then $\bar{\Delta} = \left(2 - (2\bar{\Delta}_0^2 - 1)e^{4\bar{t}-8} / \bar{\Delta}_0^2\right)^{\frac{1}{2}}, \bar{t} > 2$.

$$\bar{g}(t) = -\Delta/2 - \Delta_{ini}/2,$$

$$\bar{h}(t) = h(t)/h = (2\bar{f}(t) + \bar{\Delta} - \bar{\Delta}_{ini})/2.$$

Fig.2 shows $\bar{\Delta}$ versus \bar{t} in linear spring interface model ($\bar{\beta} = 0$) and nonlinear spring interface model ($\bar{\beta} = 2$). It is found that, for nonlinear spring interface, the curve of $\bar{\Delta}$ is sensitive to the initial displacement value $\bar{\Delta}_{ini}$. The change of the curves for linear spring interface is very slight. In the case of $\bar{\Delta}_{ini} = 0.05$ (smaller initial value), the curve for $\bar{\beta} = 2$ is underneath the curve for $\bar{\beta} = 0$, which is similar to reference[9]. But when $\bar{\Delta}_{ini} = 0.5$ (larger initial value), the curve goes across the curve for linear spring, and the amplitude varies greatly, which is much different from reference [9].

Fig.3 displays the reflected and transmitted pulses for nonlinear spring interface model. It is found that the initial displacement has great effects on the wave form of reflected pulse and transmitted pulse.

Fig.4 describes the effects of $\bar{\beta}$ on the reflected and transmitted pulses when $\bar{\Delta}_{ini} = 0.8$. The effects are mainly concentrated on the range of $[0, 2]$ which is the duration of incident pulse. For reflected pulse, the curve for nonlinear spring interface is on the top of the curve for linear spring interface, but for transmitted wave, the result is the opposite. This is also different from existing results in reference [9].

5. Conclusions

A cubic nonlinear spring interface model is used to investigate the effect of non-zero initial displacement value on the elastic wave reflection. Changing the initial condition of ordinary differential equation, we obtained a more general analytical solution. For different initial value, the forms of the reflected pulse and transmitted pulse have obvious differences. How to apply the results to the actual nondestructive evaluation is worth further discussing.

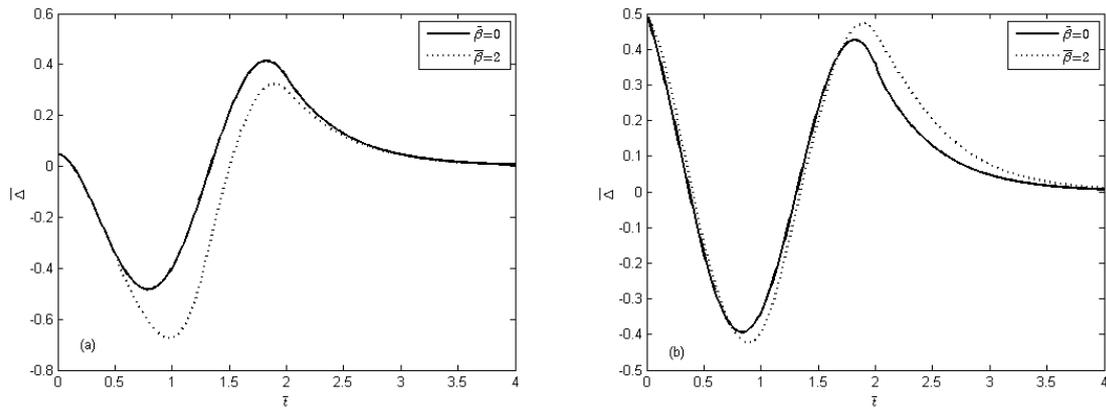


Figure 2. $\bar{\Delta}$ versus \bar{t} in linear spring interface model($\bar{\beta} = 0$) and nonlinear spring interface model($\bar{\beta} = 2$)(a),(b) correspond to $\bar{\Delta}_{ini} = 0.05, 0.5$, respectively)

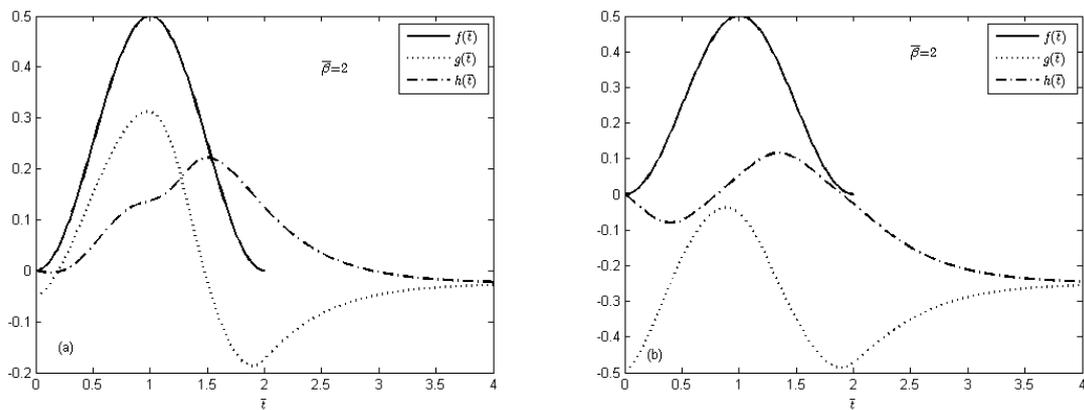


Figure 3. The reflected and transmitted pulses for soft nonlinear spring interface model($\bar{\beta} = 2$),(a),(b) correspond to $\bar{\Delta}_{ini} = 0.05, 0.5$, respectively)

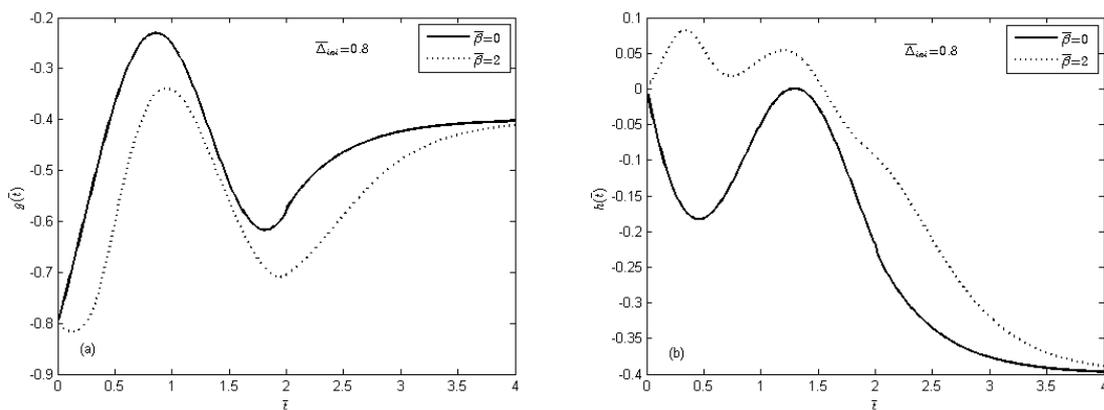


Figure 4. The reflected and transmitted pulses in linear spring interface model($\bar{\beta} = 0$) and nonlinear spring interface model ($\bar{\beta} = 2$)(a),(b) correspond to $g(t), h(t)$, respectively)

Acknowledgement

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