

# Stochastic optimization of reinforced concrete elements' design

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**Abstract.** Main goal of the paper is to present an algorithm for stochastic optimization of design of steel-reinforced concrete element's cross section. Firstly, the deterministic problem is introduced and described, followed by the description of uncertainties involved in the process and stochastic reformulation of the problem. Afterwards, the algorithm itself is introduced. This algorithm is based on internal cycle of deterministic optimization using reduced gradient method and external cycle of stochastic optimization using regression analysis. The probability is assessed on orthogonal grid via modified bisection method. The paper concludes with presentation of the performed calculations and their results.

## 1. Introduction

It is possible to see various applications of mathematical optimization in civil engineering (structural design, reconstruction of transportation networks etc.) Initially, deterministic approaches have been introduced to solve these issues (see [1]). But despite their complexity, these approaches are insufficient to comprehend the probabilistic nature of mentioned problems and thus provide only suboptimal solutions. Hence the effort prevails to reconsider these deterministic approaches and deal with uncertainties involved in said issues in less straightforward way (see [2, 3]).

Paper focuses on optimization of the structural design. There are various approaches to the optimization. The difference between deterministic and stochastic optimization was already mentioned. This paper is aimed on stochastic optimization. Further differences derive from the subject of optimization. In this case, it is the shape of the cross section.

It is also possible to distinguish various optimization algorithms. There are analytic algorithms (e.g. [4]) and simulation algorithms (e.g. [5]). Analytic algorithms are based on probabilistic analysis of the problem and it is known, that they cannot be used to achieve a high precision. Simulation algorithms are based on discretization of probability into particular scenarios followed by simulation using these scenarios (every particular scenario is actually deterministic). Since the precision that needs to be achieved can be very high (even  $10^{-6}$ ) it seems that the only possibility to achieve such precision is to use simulation and heuristic algorithm simultaneously.

This paper aims to introduce an optimization algorithm, which works in two cycles. The internal cycle is deterministic and based on reduced gradient method. The external cycle deals with probability and as mentioned before it is heuristic, based on simulation and regression analysis. It works in the space of stochastic variables rather than the space of the design variables. The algorithm is aimed to reach 99.99 % probability that the structural element satisfies ultimate and serviceability limit states.



## 2. Deterministic approach

It is necessary to begin by introducing the problem in its deterministic form. In deterministic approach, all probability inputs are replaced with their design values. Furthermore, the principle of safety coefficients is applied. Simplified definition of the optimization task therefore is:

$$\min_{\mathbf{x}} f(\mathbf{x}), \text{ subject to:} \quad (1)$$

$$L(\mathbf{x}) \leq R(\mathbf{x}), \quad (2)$$

$$w(\mathbf{x}) \leq w_{\max}, \quad (3)$$

$$\mathbf{x} \in \mathbf{C}. \quad (4)$$

All conditions and variables are described in more detail in the following text. It is necessary to mention, that besides the vector of design variables  $\mathbf{x}$ , there are another inputs into the task (1-4), namely prescribed loads and material characteristics.

### 2.1. Vector of design variables $\mathbf{x}$ .

This paper restricts itself to one-dimensional construction elements – beams. The length of the beam is obviously set and cannot change. Vector  $\mathbf{x}$  in task (1-4) therefore contains variables, which describe shape of the beam's cross section, including the area of reinforcing steel (see figure 1 as an example).

### 2.2. Objective function

The objective function  $f(\mathbf{x})$  in (1) can represent anything from simple cost of materials to multi-criteria function describing environmental aspects through construction, maintenance and disposal of the structure. To maintain simplicity, the objective function, which calculates the cost of materials, is used in this paper:

$$f(\mathbf{x}) = V_c(\mathbf{x}) \times C_{cv} + W_s(\mathbf{x}) \times C_{sw}, \quad (5)$$

where  $V_c$  is volume of used concrete,  $C_{cv}$  is cost of concrete per unit of volume,  $W_s$  is weight of used steel and  $C_{sw}$  is cost of steel per unit of weight.

### 2.3. Ultimate limit state

The condition (2) represents ULS in task (1-4) –  $L(\mathbf{x})$  is the effect of load and  $R(\mathbf{x})$  is the structural resistance. This description of ULS is of course very simplified. In fact, ULS consists of many equations and a condition limiting the values of strain of concrete and steel.

In order to evaluate ULS it is necessary to calculate the vector of deformations first. This is achieved using matrix calculations, which are in principle based on solving the following differential equation via the finite element method:

$$EIw^{(4)} = q, \quad (6)$$

where  $E$  is the Young modulus of elasticity,  $I$  is the moment of inertia,  $w$  is the deflection and  $q$  is the prescribed uniformly distributed load over unit of length. Solving this equation via matrix calculations based on FEM gives:

$$\mathbf{KU} = \mathbf{F}, \quad (7)$$

where  $\mathbf{K}$  is the stiffness matrix,  $\mathbf{U}$  is the vector of deformations and  $\mathbf{F}$  is the vector of load.

Using this equation, the internal forces are calculated. From them, the so-called strain parameters (see [5]) and subsequently the strains of concrete and steel in decisive points are computed and they are compared to their limiting values.

### 2.4. Serviceability limit state

As it is obvious from condition (3), the SLS in this paper is restricted to limiting deflection. The condition (3) is again very simplified and actually represents large number of equations. The evaluation of SLS is similar to evaluation of ULS with the difference, that the possible formation of cracks must be taken into account. These cracks then affect the bending stiffness  $B_i$ :

$$B_i = EI_i, \quad (8)$$

which affects the moment of inertia  $I_i$  – one of the inputs into the differential equation and therefore the matrix calculation. Whether the cracks form is determined from the vector of internal forces. It is therefore necessary to find the state of equilibrium (of variables' values in the mentioned equations, see [5]).

### 2.5. Design variables' boundaries

Condition (4) define boundaries of the design variables' values. It also represents conditions following mutual relations among these variables – e.g. maximal and minimal area of steel reinforcement in relation to area of the concrete cross section.

### 2.6. Optimization algorithm

Optimization itself uses the reduced gradient method, specifically the solver CONOPT from the optimization software GAMS. This sets some rules for the performed calculations. All relations (equations) and their first derivatives must be continuous. In some cases (stress-strain diagrams, mentioned stiffness in SLS), this is achieved using Hermit interpolation (see [5] again).

## 3. Stochastic approach

The designing process involves a lot of uncertainties. Among these are:

- randomness of physical quantities used in the design (as its natural characteristic)
- statistical uncertainties during the description of a quantity caused by a lack of data,
- model uncertainties caused by inaccuracies in the calculation model in comparison with real structural behavior,
- uncertainties caused by inaccuracies of the limit states definitions,
- human element deficiencies within the design procedure and execution and usage of the structure.

In the deterministic approach all these uncertainties are included in the calculation by using design values and/or via using various safety coefficients. It is obvious, that for desired probability of structure failure, the stochastic approach provides more precise and improved solution.

By embodying uncertainties, the original task (1-4) is reformulated to its stochastic form:

$$\min_x M(f(\mathbf{x}, \xi)), \text{ subject to:} \quad (9)$$

$$P(L(\mathbf{x}, \xi) \leq R(\mathbf{x}, \xi)) \leq \alpha, \quad (10)$$

$$P(w(\mathbf{x}, \xi) \leq w_{\max}) \leq \beta, \quad (11)$$

$$\mathbf{x} \in \mathbf{C}, \quad (12)$$

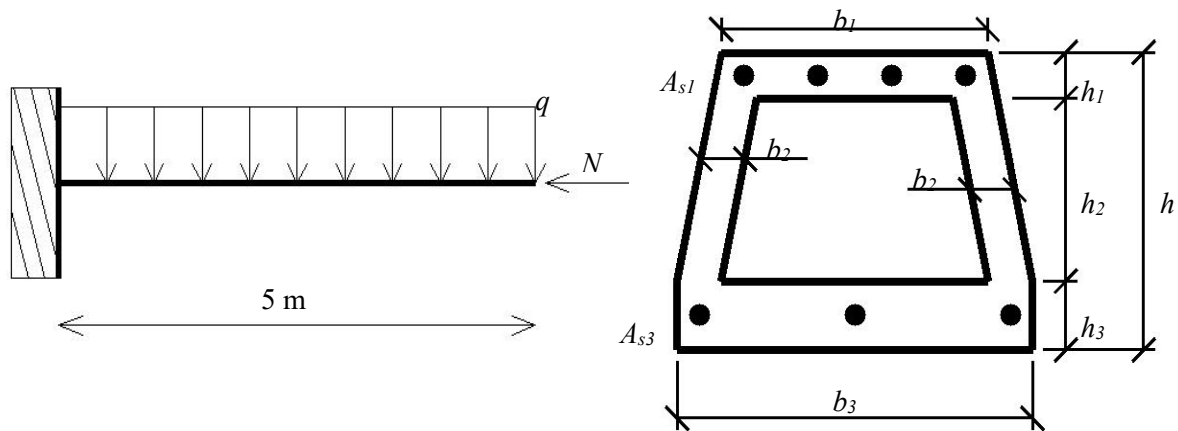
where  $M$  is mean,  $\xi$  represents uncertainty,  $\alpha$  is a desired probability that the ULS condition holds and  $\beta$  is a desired probability that the SLS condition holds.

## 4. Example

Proposed solution is presented directly on a simple example.

The task is to design 5 m long cantilever beam loaded with uniformly distributed load  $q$  and normal force  $N$  as shown on figure 1.

Cross section of the beam is described by 8 variables  $b_1, b_2, b_3, h_1, h_2, h_3, A_{s1}, A_{s3}$  – 6 of them describe the shape of cross section, while 2 describe the area of reinforcing steel (see figure 1).



**Figure 1.** Scheme of the cantilever,  $\mathbf{x} = (b_1, b_2, b_3, h_1, h_2, h_3, A_{s1}, A_{s3})$ .

In the FEM calculation, the cantilever is divided into 10 elements  $e_1, \dots, e_{10}$ , each 0.5 m in length.

The distributed load  $q$  [kN/m] and normal force  $N$  [kN] are random variables with gamma distribution. Probability distributions of  $q$  and  $N$  are:

$$q \sim \Gamma(1500; 0.015), \quad N \sim \Gamma(148; 0.05). \quad (13)$$

In order to better understand the distributions, the 0.05 quantile of distribution of  $q$  is 21.6 kN/m, 0.5 quantile is 22.5 kN/m and 0.95 quantile is 23.5 kN/m. Analogically, 0.05 quantile of distribution of  $N$  is 6.4 kN, 0.5 quantile is 7.4 kN and 0.95 quantile is 8.4 kN. All these values are approximate.

To maintain simplicity, the probabilities  $\alpha$  and  $\beta$  from conditions (10) and (11) are replaced with only one probability  $\gamma$ . The mentioned conditions are thus replaced with following condition:

$$P(L(\mathbf{x}, \xi) \leq R(\mathbf{x}, \xi) \wedge w(\mathbf{x}, \xi) \leq w_{\max}) \leq \gamma. \quad (14)$$

The value  $\gamma$  is set to 99.99 %.

There are some other restrictions regarding design variables. The overall height of cross section is limited to 350 mm. Each of the heights  $h_1, h_2, h_3$  has to be minimally 100 mm. The width  $b_2$  is at least 100 mm,  $b_1$  and  $b_3$  at least 200 mm but no more than 900 mm. For widths  $b_1, b_3$  the following condition needs to be fulfilled:

$$b_k \geq 2 \times 50 + A_{sk} / (\pi \times 20^2 / 4) \times (50 + 20). \quad (15)$$

This relation represents, that the given width has to accommodate required area of reinforcing steel; reinforcing steel is composed of bars 20 mm in diameter, minimum distance between the surface of any steel bar and the concrete surface and between any two steel bars is 50 mm.

The considered costs are 2500 CZK per 1 m<sup>3</sup> of concrete and 30 CZK per 1 kg of steel.

## 5. Solution algorithm

The algorithm uses regression analysis to iterate towards the suitable solution of the task. In this chapter, the algorithm is described, including the method of probability assessment.

### 5.1. Probability assessment

The probability is assessed on an orthogonal mesh via method, that can be described as modified bisection method. The principle of this method is to find the boundary between scenarios, which satisfies deterministic conditions of the task, and scenarios, which do not.

Orthogonal mesh in its maximum size (bisection divides maximum mesh into smaller ones) consists of 514×514 points. The mesh layout is thicker toward the expected location of the mentioned boundary.

### 5.2. Calculation initialization

The calculation is initialized by selecting given number of scenarios of random variables. For each scenario the deterministic optimization is performed and probability, that thus acquired solution holds ULS and SLS conditions, is assessed (assessment of the left hand side of condition (14)). In this paper, the example defined in chapter 4 is initialized by 16 scenarios, which are made of combinations of values of  $q$  and  $N$  as follows:

$$q \in \{20.25; 21.75; 23.25; 24.75\}, \quad (16)$$

$$N \in \{5.25; 6.75; 8.25; 9.75\}. \quad (17)$$

### 5.3. Regression analysis

Acquired points are fitted with a polynomial function via least squares method – for probability it is the 3<sup>rd</sup> degree polynomial, for objective function it is the 1<sup>st</sup> degree polynomial. Degrees of polynomials were chosen appropriately in regard to values of probability and objective function observed on a greater sample than just the mentioned 16 scenarios.

Regression analysis is weighted. The error in each point  $i$  is multiplied by  $1/|p_i - \gamma|$ , where  $p_i$  is the assessed left hand side of condition (14) for point  $i$ . This way, the regression resembles interpolation while near the solution.

### 5.4. Iteration

After initialization, the iterative process follows. According to regression analysis, the point that satisfies condition (14) and at the same time has the minimal value of objective function is selected as a possible solution. For this scenario the deterministic optimization is performed and probability that thus acquired solution satisfies deterministic conditions is assessed.

If the resulting probability is greater and also sufficiently close to value  $\gamma$ , the algorithm ends and the current deterministic solution is the solution of the whole task.

In the opposite case, this newly assessed point is added as another point to the regression analysis. Regression, new point selection, deterministic optimization and probability assessment are all performed again. This iterative procedure continues, until the assessed probability is higher and also sufficiently close to value  $\gamma$ .

### 5.5. Heuristic algorithm

Proposed algorithm actually does not solve minimization task (9) but the task:

$$\min_{\xi} \min_x f(x, \xi). \quad (18)$$

Therefore, the algorithm is heuristic and provides only suboptimal solution.

## 6. Calculation results

Table 1 contains the values of design variables, which were found as a solution to the example from chapter 4.

**Table 1.** Values of design variables  $b_1, b_2, b_3, h_1, h_2, h_3$  [mm] and  $A_{s1}, A_{s2}$  [mm<sup>2</sup>] in finite elements  $e_1, \dots, e_{10}$  as a solution to example defined in chapter 4.

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$	$e_8$	$e_9$	$e_{10}$
$b_1$	900.0	781.9	655.5	546.1	445.1	353.0	900.0	470.1	200.0	200.0
$b_2$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$b_3$	900.0	900.0	899.6	712.5	542.3	392.1	402.0	200.0	200.0	200.0
$h_1$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$h_2$	150.0	150.0	150.0	150.0	150.0	150.0	150.0	150.0	100.0	100.0
$h_3$	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$A_{s1}$	3590.4	3060.2	2493.0	2002.2	1548.9	1135.6	360.7	196.4	92.8	60.0
$A_{s2}$	210.0	195.8	180.6	152.5	126.8	103.7	170.2	102.4	60.0	60.0

The probability, that solution satisfies ULS and SLS was assessed as 99.99017 %. This probability was subsequently tested by the Monte Carlo simulation. From  $10^5$  scenarios, 9 failed the deterministic conditions which is in accordance with previous probability assessment.

The solution was found for the scenario  $q = 24.6767$  kN/m,  $N = 4.7635$  kN. Its objective function value is 3575.08 CZK.

Evidently, the values from table 1 are not suitable for an actual design purposes. They serve rather as a guide to arrange the cantilever's cross section over its length.

Table 1 contains one peculiar value ( $b_l$  in  $e_7$ ). This is caused by the deterministic optimization algorithm – the GRG method. Since the only function of width  $b_l$  is to accommodate reinforcement area  $A_{sl}$ , this value can obviously be lowered. Nevertheless, the optimization algorithm kept the current value, because the minimum (defined as a reduced gradient sufficiently close to zero) was already reached.

## 7. Conclusions

The algorithm to stochastically optimize design of reinforced concrete structural elements was developed. This algorithm was successfully tested on a simple example. Probability assessment of proposed algorithm was successfully tested by Monte Carlo method. The future work should focus on reaching higher precision of probability of failure as well as on using the algorithm in issues with more random variables.

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