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Novel condition for exponential stability of linear system with mixed non-differential time-varying delay with feedback control

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Abstract. The problem of exponential stability for linear system with mixed time-varying delay is investigated. The stability of linear system are proposed with feedback control. Based on the constructing of improved Lyapunov-Krasovskii functionals combined with Leibniz-Newton's formula and by utilizing the integral inequality, new sufficient condition for the stabilization of the systems is first established in terms of LMIs. A numerical example is given to illustrate that the effectiveness of the proposed feedback control.

1. Introduction

The stability analysis of dynamic systems with time delays has been one of the most attention in the field of control theory, because time-delay systems occur in various areas including chemical engineering systems, neural networks, biological systems etc. The stability analysis for linear time-delay systems has been discussed widely and various approaches to such problems have been proposed and the references therein. Time-varying delays systems have been an interesting topic in recent year such as interval time-varying delays [1-2], distributed time-varying delay [3], mixed time-varying delays [4] and so on. The stability analysis and control of time-varying delay systems have received considerable attention widely for the last few decades. Some delay-dependent conditions for designing stabilizing feedback control [4-9] and intermittent feedback control [4,5], [9,10] are studied in various system. Both of them have been considered see in [4] which is more advantageous in the field of the stability of the control theory. In this paper, the problem of exponential stabilization of linear systems with mixed time-varying delay with feedback control is studied. The time delay is a continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is not necessary to be differentiable. By utilizing the construction of improved Lyapunov-Krasovskii functionals combined with Leibniz-Newton's formula and using the technique of dealing with some integral terms, new delay-dependent sufficient conditions for the exponential stabilization of the those systems are first established in terms of LMIs without introducing any free-weighting matrices. A numerical example is given to demonstrate the effectiveness of the obtained results.

2. Network model and mathematical preliminaries

Let us consider the linear system as equation (1)

$$\dot{x}(t) = Ax(t) + Bx(t-h(t)) + C\int_{t-k_1(t)}^{t} x(s)ds + U(t),\ t \geq 0\ , \tag{1}$$

$$x(t) = \phi(t),\ t \in \left[-\tau_{\max}, 0\right],\ \tau_{\max} = \max\left\{h_2, d, k_1, k_2\right\},$$

where $x(t) = \left(x_1(t), ..., x_n(t)\right)^T \in \Re^n$ is the state vector; A, B and C are known real constant matrices, $U(t) \in \Re^m$ is the control input. In order to stabilize the origin of linear system as equation (1) by means of the state feedback controller $U(t)$ satisfying as equation (2),

$$U(t) = D_1u(t) + D_2u(t-d(t)) + D_3\int_{t-k_2(t)}^{t} u(s)ds,\ \forall t \geq t_0, \tag{2}$$

where $D_i, i = 1, 2, 3$ are given matrices of appropriate dimensions, $u(t) = Kx(t)$ and K is a constant matrix control gain, and n is a non-negative integer. Then, substituting $U(t)$ into linear system as equation (1),

$$\dot{x}(t) = Ax(t) + Bx(t-h(t)) + C\int_{t-k_1(t)}^{t} x(s)ds + D_1u(t) + D_2u(t-d(t)) + D_3\int_{t-k_2(t)}^{t} u(s)ds. \tag{3}$$

It is clear that, if the zero solution of linear system as equation (3) is globally exponentially stable, the exponential stability of the controlled linear system as equation (1) is achieved. The time-varying delay functions $h(t), d(t),\ k_1(t)$ and $k_2(t)$ satisfy the conditions as equation (4).

$$0 \leq h_1 \leq h(t) \leq h_2,\ 0 \leq d(t) \leq d,\ 0 \leq k_1(t) \leq k,\ 0 \leq k_2(t) \leq k_2\ . \tag{4}$$

The initial condition function $\phi(t)$ denotes a continuous vector-valued initial function of $t \in \left[-\tau_{\max}, 0\right]$.

3. Exponential stability of linear delay system via feedback control

Let us denote

$$\|\phi(t)\| = \|x(0)\|, \|\varphi(t)\| = \sup_{-\tau_{\max} \leq s \leq 0} \|x(s)\|, K = -LP^{-1}, \gamma = \lambda_{\min}\left(P^{-1}\right), \beta = \frac{h_2 - h(t)}{h_2 - h_1}, N = N_1\|\phi(t)\|^2 + N_2\|\varphi(t)\|^2,$$

$$N_1 = \lambda_{\max}\left(P^{-1}\right) + \left[2h_2\lambda_{\max}\left(P^{-1}RP^{-1}\right) + h_2\lambda_{\max}\left(P^{-1}UP^{-1}\right)\right]\left(\frac{1-e^{-2\alpha h_2}}{2\alpha}\right) + d\lambda_{\max}\left(P^{-1}L^TT^{-1}LP^{-1}\right)\left(\frac{1-e^{-2\alpha d}}{2\alpha}\right),$$

$$\begin{aligned} N_2 = &\left[2\lambda_{\max}\left(P^{-1}QP^{-1}\right) + 2h_2\lambda_{\max}\left(P^{-1}RP^{-1}\right) + h_2\lambda_{\max}\left(P^{-1}UP^{-1}\right)\right]\left(\frac{1-e^{-2\alpha h_2}}{2\alpha}\right) \\ &+ k_1\lambda_{\max}\left(P^{-1}SP^{-1}\right)\left(\frac{1-e^{-2\alpha k_1}}{2\alpha}\right) + d\lambda_{\max}\left(P^{-1}L^TT^{-1}LP^{-1}\right)\left(\frac{1-e^{-2\alpha d}}{2\alpha}\right) + k_2\lambda_{\max}\left(P^{-1}L^TW^{-1}LP^{-1}\right)\left(\frac{1-e^{-2\alpha k_2}}{2\alpha}\right). \end{aligned}$$

Theorem 1 For some given scalars $\alpha > 0$, the linear system as equation (3) with time-varying delay satisfying equation (4) are exponentially stable if there exist symmetric positive definite matrices $P > 0, Q > 0, R > 0, S > 0,\ U > 0, T > 0, W > 0$ and a matrix L appropriately dimensioned such that the following symmetric linear matrix inequality holds, equations (5)-(9):

$$\Sigma_1 = \Sigma - \begin{bmatrix}0 & 0 & I & -I & 0\end{bmatrix}^T e^{-2\alpha h_2}U\begin{bmatrix}0 & 0 & I & -I & 0\end{bmatrix} < 0, \tag{5}$$

$$\Sigma_2 = \Sigma - \begin{bmatrix}0 & 0 & 0 & I & -I\end{bmatrix}^T e^{-2\alpha h_2}U\begin{bmatrix}0 & 0 & 0 & I & -I\end{bmatrix} < 0, \tag{6}$$

$$\Sigma_3 = \begin{bmatrix} -0.5\left(e^{-2\alpha h_1}+e^{-2\alpha h_2}\right)R & 2k_1CP & k_2L^T & 2L^T \\ * & -2k_1e^{-2\alpha k_1}S & 0 & 0 \\ * & * & -k_2W & 0 \\ * & * & * & -2de^{2\alpha d} \end{bmatrix} < 0, \tag{7}$$

$$\Sigma_4 = \begin{bmatrix} -0.5P & 2k_1CP & d^2L^T \\ * & -2k_1e^{-2\alpha k_1}S & 0 \\ * & * & -d^2T \end{bmatrix} < 0, \tag{8}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} \\ * & \Sigma_{22} & 0 & \Sigma_{24} & 0 \\ * & * & \Sigma_{33} & \Sigma_{34} & 0 \\ * & * & * & \Sigma_{44} & \Sigma_{45} \\ * & * & * & * & \Sigma_{55} \end{bmatrix}, \tag{9}$$

$$\Sigma_{11} = P^T\left(A+\alpha I\right)+\left(A+\alpha I\right)^T P - D_1L - L^TD_1^T + 3e^{2\alpha d}D_2^TTD_2 + 2k_2e^{2\alpha k_2}D_3^TWD_3 + 2Q$$
$$+k_1S - 0.5\left(e^{-2\alpha h_1}+e^{-2\alpha h_2}\right)R,$$
$$\Sigma_{12} = PA^T - L^TD_1^T,\ \Sigma_{13} = e^{-2\alpha h_1}R,\ \Sigma_{14} = BP,\ \Sigma_{15} = e^{-2\alpha h_2}R,$$
$$\Sigma_{22} = h_1^2R + h_2^2R + \eta^2U - 1.5P + 3e^{2\alpha d}D_2^TTD_2 + 2k_2e^{2\alpha k_2}D_3^TWD_3,\ \eta = h_2 - h_1$$
$$\Sigma_{24} = BP, \Sigma_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U, \Sigma_{34} = e^{-2\alpha h_2}U,$$
$$\Sigma_{44} = -2e^{-2\alpha h_2}U, \Sigma_{45} = e^{-2\alpha h_2}U, \Sigma_{55} = -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U,$$

then, the linear system as equation (3) has an exponential stability. Moreover, the feedback control is equation (10),

$$u(t) = -LP^{-1}x(t) \tag{10}$$

and the solution $x(t,\phi)$ satisfies $\left\|x(t,\phi)\right\| = \sqrt{\frac{N}{\gamma}}e^{-\alpha t}, \forall t \geq 0.$

Proof Let $Y = P^{-1}$ and $y(t) = Yx(t)$. By using the feedback control equation (10), let us consider the following Lyapunov-Krasovskii functional as equation (11) show:

$$V\left(x(t)\right) = \sum_{i=1}^{9} V_i(t) \tag{11}$$

where as equation (12) show

$$\begin{aligned} &V_1(t) = x^T(t)Yx(t), V_2(t) = \int_{t-h_1}^{t} e^{2\alpha(s-t)}x^T(s)YQYx(s)ds, V_3(t) = \int_{t-h_2}^{t} e^{2\alpha(s-t)}x^T(s)YQYx(s)ds, \\ &V_4(t) = h_1\int_{-h_1}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YRY\dot{x}(\tau)d\tau ds, V_5(t) = h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YRY\dot{x}(\tau)d\tau ds, \\ &V_6(t) = \eta\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YUY\dot{x}(\tau)d\tau ds,\ V_7(t) = \int_{-k_1}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}x^T(\tau)YSYx(\tau)d\tau ds, \\ &V_8(t) = d\int_{-d}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)K^TT^{-1}K\dot{x}(\tau)d\tau ds, V_9(t) = \int_{-k_2}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}x^T(\tau)K^TW^{-1}Kx(\tau)d\tau ds. \end{aligned} \tag{12}$$

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It is easy to check that equation (13)

$$\gamma\|x(t)\|^2 \le V(x(t)) \tag{13}$$

By taking the derivatives of $V_i(t), i=1,...,9$ along the trajectories of linear system as equation (3), and then apply Jensen's inequality, Cauchy inequality and Leibniz-Newton formula, we have the following equations (14)-(22):

$$\begin{aligned}\dot{V}_1(t) \le{}& y^T(t)\left[PA + A^T P\right]y(t) + 2y^T(t)BPy\left(t-h(t)\right) - 2y^T(t)D_1Ly(t) + 2k_1e^{2\alpha k_1}y^T(t)CPS^{-1}PC^Ty(t)\\ &+\frac{1}{2}e^{-2\alpha k_1}\int_{t-k_1(t)}^{t} y^T(s)Sy(s)ds + 3e^{2\alpha d}y^T(t)D_2TD_2{}^Ty(t) + \frac{1}{3}e^{-2\alpha d}u^T\left(t-d(t)\right)T^{-1}u\left(t-d(t)\right)\\ &+2k_2e^{2\alpha k_2}y^T(t)D_3TD_3{}^Ty(t) + \frac{1}{2}e^{-2\alpha k_2}\int_{t-k_2(t)}^{t} u^T(s)W^{-1}u(s)ds,\end{aligned} \tag{14}$$

$$\dot{V}_2(t) = -2\alpha V_2 + y^T(t)Qy(t) - e^{-2\alpha h_1}y^T\left(t-h_1\right)Qy\left(t-h_1\right), \tag{15}$$

$$\dot{V}_3(t) = -2\alpha V_3 + y^T(t)Qy(t) - e^{-2\alpha h_2}y^T\left(t-h_2\right)Qy\left(t-h_2\right), \tag{16}$$

$$\dot{V}_4(t) \le -2\alpha V_4 + h_1^2\dot{y}^T(t)R\dot{y}(t) - e^{-2\alpha h_1}\left[y(t)-y\left(t-h_1\right)\right]^T R\left[y(t)-y\left(t-h_1\right)\right], \tag{17}$$

$$\dot{V}_5(t) \le -2\alpha V_5 + h_2^2\dot{y}^T(t)R\dot{y}(t) - e^{-2\alpha h_2}\left[y(t)-y\left(t-h_2\right)\right]^T R\left[y(t)-y\left(t-h_2\right)\right], \tag{18}$$

$$\begin{aligned}\dot{V}_6(t) \le{}& -2\alpha V_6 + \eta^2\dot{y}^T(t)U\dot{y}(t) - e^{-2\alpha h_2}\left[y\left(t-h(t)\right)-y\left(t-h_2\right)\right]^T U\left[y\left(t-h(t)\right)-y\left(t-h_2\right)\right]\\ &-e^{-2\alpha h_2}\left[y\left(t-h_1\right)-y\left(t-h(t)\right)\right]^T U\left[y\left(t-h_1\right)-y\left(t-h(t)\right)\right]\\ &-\left(1-\beta\right)e^{-2\alpha h_2}\left[y\left(t-h(t)\right)-y\left(t-h_2\right)\right]^T U\left[y\left(t-h(t)\right)-y\left(t-h_2\right)\right]\\ &-\beta e^{-2\alpha h_2}\left[y\left(t-h_1\right)-y\left(t-h(t)\right)\right]^T U\left[y\left(t-h_1\right)-y\left(t-h(t)\right)\right],\end{aligned} \tag{19}$$

$$\dot{V}_7(t) \le -2\alpha V_7 + k_1y^T(t)Sy(t) - e^{-2\alpha k_1}\int_{t-k_1(t)}^{t} y^T(s)Sy(s)ds, \tag{20}$$

$$\dot{V}_8(t) \le -2\alpha V_8 + d^2\dot{y}^T(t)L^TT^{-1}L\dot{y}(t) + 2e^{-2\alpha d}y^T(t)L^TT^{-1}Ly(t) - \frac{2}{3}e^{-2\alpha d}u^T\left(t-d(t)\right)T^{-1}u\left(t-d(t)\right), \tag{21}$$

$$\dot{V}_9(t) \le -2\alpha V_9 + k_2y^T(t)L^TW^{-1}Ly(t) - e^{-2\alpha k_2}\int_{t-k_2(t)}^{t} y^T(s)W^{-1}y(s)ds. \tag{22}$$

By using the identity relation and then multiplying its by $2\dot{y}^T(t)$ gives as equation (23)

$$\begin{aligned}&-2\dot{y}^T(t)P\dot{y}(t) + 2\dot{y}^T(t)APy(t) + 2\dot{y}^T(t)BPy(t-h(t)) + 2\dot{y}^T(t)CP\int_{t-k_1(t)}^{t} y(s)ds\\ &\quad -2\dot{y}^T(t)D_1Ly(t) + 2\dot{y}^T(t)D_2u(t-d(t)) + 2\dot{y}^T(t)D_3\int_{t-k_2(t)}^{t} u(s)ds = 0\end{aligned} \tag{23}$$

Hence, according to equations (14) - (22) and adding the zero items of equation (12), we have equation (24)

$$\dot{V}\left(x(t)\right) + 2\alpha V\left(x(t)\right) \le \xi^T(t)\left[\left(1-\beta\right)\Sigma_1 + \beta\Sigma_2\right]\xi(t) + y^T(t)M_1y(t) + \dot{y}^T(t)M_2\dot{y}(t), \tag{24}$$

where Σ_1 and Σ_2 are defined as in equation (5) and equation (6), respectively, and

$$\xi^T(t) = \begin{bmatrix} y^T(t) & \dot{y}^T(t) & y^T(t-h_1) & y^T(t-h(t)) & y^T(t-h_2) \end{bmatrix},$$

$$M_1 = -0.5\left(e^{-2\alpha h_1} + e^{-2\alpha h_2}\right)R + 2k_1 e^{2\alpha k_1} CPS^{-1}PC^T + k_2 L^T W^{-1} L + 2e^{-2\alpha d} L^T T^{-1} L,$$

$$M_2 = -0.5P + 2k_1 e^{2\alpha k_1} CPS^{-1}PC^T + +dL^T T^{-1} L.$$

By $(1-\beta)\Sigma_1 + \beta\Sigma_2 < 0$ as in equation (24) holds if and only if $\Sigma_1 < 0$ and $\Sigma_2 < 0$ as in equation (5) and (6). Apply Schur complement, the inequalities $M_1 < 0$ and $M_2 < 0$ are equivalent to $\Sigma_3 < 0$ and $\Sigma_4 < 0$ as in equation (7) and equation (8), respectively. It follows from equations (5)-(9) and equation (24), we obtain equation (25)

$$\dot{V}(x(t)) + 2\alpha V(x(t)) \le 0, \forall t \ge 0. \tag{25}$$

Integrating both sides of equation (25) from 0 to t, and using the condition equation (13), we have equation (26)

$$\left\|V(x(t))\right\| \le \sqrt{\frac{V(x(0))}{\gamma}} e^{-\alpha t}, \forall t \ge 0. \tag{26}$$

Finally, by estimating $V(x(0))$, we have that $\left\|V(x(t))\right\| \le \sqrt{\frac{N}{\gamma}} e^{-\alpha t}, \forall t \ge 0,$ which implies the linear system as equation (3) is globally exponential stable under the controller $U(t)$ as in equation (2), then exponential stability of the controlled linear system as equation (3) is achieved.

4. Numerical example

Example 1 Consider the linear feedback control (3) with the parameters

$$A = \begin{bmatrix} -0.3 & -0.1 \\ 0 & 0.2 \end{bmatrix}, B = \begin{bmatrix} -0.3 & 0 \\ -0.1 & -0.1 \end{bmatrix}, C = \begin{bmatrix} -0.1 & 0.1 \\ 0.2 & -0.3 \end{bmatrix}, D_1 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, D_2 = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}, D_3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \tag{27}$$

where $h_1 = 0.1, h_2 = 0.4, d = 0.3, k_1 = 0.1, k_2 = 0.3$ and $\alpha = 0.1$. Then, using the MATLAB LMI control toolbox to solve in (5)-(9), we obtain the solution as follows:

$$P = \begin{bmatrix} 3.0305 & 0.7147 \\ 0.7147 & 0.6970 \end{bmatrix}, Q = \begin{bmatrix} 0.0754 & 0.0584 \\ 0.0584 & 0.0513 \end{bmatrix}, R = \begin{bmatrix} 0.6304 & 0.2910 \\ 0.2910 & 0.3079 \end{bmatrix}, U = \begin{bmatrix} 0.86\ 20 & 0.1010 \\ 0.10\ 10 & 0.2662 \end{bmatrix},$$

$$T = \begin{bmatrix} 0.0093 & -0.0344 \\ -0.0344 & 0.1513 \end{bmatrix}, W = \begin{bmatrix} 0.0565 & -0.0350 \\ -0.0350 & 0.1104 \end{bmatrix}, S = \begin{bmatrix} 1.5603 & 0.3950 \\ 0.3950 & 0.4117 \end{bmatrix}, K = \begin{bmatrix} -0.0064 & 0.0317 \\ 0.0266 & -0.1727 \end{bmatrix}.$$

The numerical simulation of linear system as equation (3) with the parameters as equation (27) with time-varying delays $h(t) = 0.1 + 0.3|\cos t|, k_1(t) = 0.1|\sin t|$.

The initial condition $\phi(t) = [0.5\cos s, -0.5\cos s], \forall s \in [-0.4, 0]$ and without feedback control is represented in figure 1**,** which shows that the linear system as equation (3) is unstable**.** Figure 2 shows the trajectories of $x_1(t)$ and $x_2(t)$ of linear system as equation (3) with time-varying delays $d(t) = k_2(t) = 0.3|\cos t|$ via feedback control**.** The method proposed in [2] is not applicable for the case that the delay function is not necessary to be differentiable and the improved control input.

5. Conclusion

This study has investigated the stability of dynamical system with mixed time-varying delays via feedback control. Based on the constructing of an improved Lyapunov-Krasovskii functionals combined with Leibniz-Newton's formula and using the integral inequality technique of dealing with some integral terms, new delay-dependent sufficient condition for the exponential stabilization of the these systems are first established in terms of LMIs without introducing any free-weighting matrices.

The delay feedback controllers designed can guarantee the exponential stability of the dynamical system. A numerical example is provided to show the advantages of the present results.

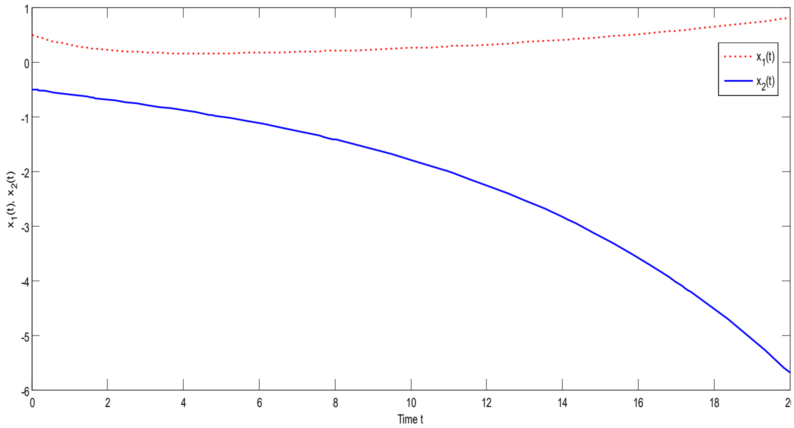

Figure 1. The trajectories of $x_1(t)$ and $x_2(t)$ of the uncertain linear system with interval time-varying delay and without feedback control activated.

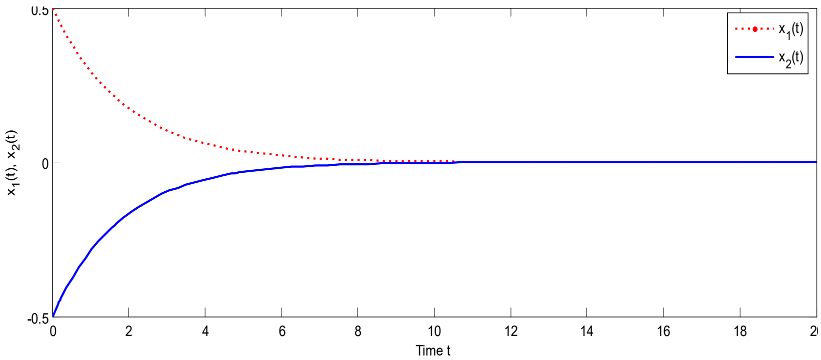

Figure 2. The trajectories of $x_1(t)$ and $x_2(t)$ of the uncertain linear system with interval time-varying delay and feedback control activated.

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