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On the Diophantine equations $x^2 - 47y^2 = 1$ and $y^2 - Pz^2 = 49$

Jianhong Zhao, Lixing Yang*

Department of Teachers and Education, Lijiang Teachers College,Lijiang, Yunnan 674199, China

*Corresponding author's E-mail: 375734990@qq.com

Abstract. The integer solution of diophantine equations $x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z)$ and $y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z)$ is a matter of great concern. Researchers study for different m,n and D_1, D_2, and obtained some correlation results as follows.
When $m = 1$,$n = 1$, the diophantine equations turns into $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 1$.At present, there are only a few conclusions on it, see Ref [1] and [2].When $m = 1$,$n = 4$, the integer solution see Ref [3] - [9].When $m = 1$,$n = 16$, the previous conclusions see Ref [10].
When $m = 1$,$n = 49$, the diophantine equations turns into $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 49$. In this case, $D_1 = 47$, D_2 can be expressed as $2^t{p_1}^{a_1}{p_2}^{a_2}{p_3}^{a_3}{p_4}^{a_4}$ where $a_i = 0$ or 1 for $1 \le i \le 4$, and $t \in Z^+$, $p_s(1 \le s \le 4)$ are different odd primes. Up to now, there is no relevant result on the integer solution of $x^2 - 47y^2 = 1$ and $y^2 - Pz^2 = 49$, this paper mainly discusses the integer solution of it.

1. Introduction

The integer solution of diophantine equations

$$x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z) \text{ and } y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z) \tag{1}$$

is a matter of great concern. Researchers study for different m,n and D_1, D_2, and obtain some correlation results as follows.

When $m = 1$,$n = 1$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1 \text{and } y^2 - D_2z^2 = 1 \tag{2}$$

At present, there are only a few conclusions on (2), see Ref [1] and [2].

When $m = 1$,$n = 4$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1 \text{and } y^2 - D_2z^2 = 4 \tag{3}$$

For even numbers D_1, D_2, the integer solution of (3), see Ref [3] - [9].

When $m = 1$,$n = 16$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1 \text{and } y^2 - D_2z^2 = 16 \tag{4}$$

The previous conclusions on (4), see Ref [10].

When $m = 1$,$n = 49$, the diophantine equations turns into:

$$x^2 - D_1y^2 = 1 \text{and } y^2 - D_2z^2 = 49 \tag{5}$$

In this case, $D_1 = 47$, D_2 can be expressed as $2^t{p_1}^{a_1}{p_2}^{a_2}{p_3}^{a_3}{p_4}^{a_4}$ where $a_i = 0$ or 1 for $1 \le i \le 4$, and $t \in Z^+$, $p_s(1 \le s \le 4)$ are different odd primes. Up to now, there is no relevant result on the integer solution of $x^2-47y^2=1$ and $y^2-Pz^2=49$, this paper mainly discusses the integer solution of it.

2. Critical lemma

Lemma 1[11] Let p be an odd prime number, there is no integer solution of the diophantine equation $x^4-py^2=1$ except $p=5, x=3, y=4$ and $p=29, x=99, y=1820$.

Lemma 2[12] There is 1 sets of solutions of the diophantine equation $ax^4-by^2=1$ at most when a is a square number which is greater than 1.

Lemma 3[13] Let D be a square-free positive integer, then the equation $x^2-Dy^4=1$ has two sets of positive integer solutions (x,y) at most. Furthermore, the necessary and sufficient condition of it is $D=1785$ or $D=28560$, or $2x_0$ and y_0 are square numbers where (x_0,y_0) is the basic solution of $x^2-Dy^4=1$.

Lemma 4 Suppose that all the integer solution on Pell equation $x^2-7y^2=1$ could be $(x_n,y_n), n\in Z$, for the arbitrary $n\in Z$, it has the following properties on (x_n,y_n):

(I) x_n is a square number if and only if $n=0$.

(II)$\frac{x_n}{48}$ is a square number if and only if $n=\pm1$.

(III) $\frac{y_n}{7}$ is a square number if and only if $n=0$ or $n=1$.

Proof: (I) Let $x_n=a^2$, we will get $a^4-47y^2=1$, from Lemma 1 we can get there are only 2 integer solution $(a,y)=(\pm1,0)$ on $a^4-47y^2=1$, so $x_n=1, n=0$. On the contrary, it also holds.

(II) Let $\frac{x_n}{48}=a^2$, we will get $2304a^4-47y^2=1$, from Lemma 2 we can get there are only 4 integer solution $(a,y)=(\pm1,\pm7)$ on $2304a^4-47y^2=1$, so $x_n=48, n=\pm1$. On the contrary, it also holds.

(III) Let $\frac{y_n}{7}=b^2$, we will get $x^2-2303b^4=1$, from Lemma 3 we can get there are only 6 integer solution $(x,b)=(\pm1,0),(\pm48,\pm1)$ on $x^2-2303b^4=1$, so $y_n=0$ or $y_n=7$. $n=0$ or $n=1$. On the contrary, it also holds.

3. Proof of main theorem

By using elementary method such as congruence, the integer solution of the diophantine equations on $x^2-47y^2=1$ and $y^2-Pz^2=9$ can be obtained.

3.1. Theorem

Let $p_s(1\le s\le 4)$ are diverse odd primes, $P=2^k p_1^{a_1}\cdots p_s^{a_s}(a_i=0\ or\ 1, 1\le i\le 4, k\in Z^+)$, then the diophantine equations

$$x^2-47y^2=1 \text{ and } y^2-Pz^2=49 \qquad (6)$$

(i) has common solution $(x,y,z)=(\pm48,\pm7,0)$ and nontrivial solution $(x,y,z)=(\pm442224,\pm64505,\pm672)$ when $P=2\times17\times271$.

(ii) has common solution $(x,y,z)=(\pm48,\pm7,0)$ and nontrivial solution $(x,y,z)=(\pm442224,\pm64505,\pm336)$ when $P=2^3\times17\times271$.

(iii) has common solution $(x,y,z)=(\pm48,\pm7,0)$ and nontrivial solution $(x,y,z)=(\pm442224,\pm64505,\pm168)$ when $P=2^5\times17\times271$.

(iv) has common solution $(x,y,z)=(\pm48,\pm7,0)$ and nontrivial solution $(x,y,z)=(\pm442224,\pm64505,\pm84)$ when $P=2^7\times17\times271$.

(v) has common solution $(x,y,z)=(\pm48,\pm7,0)$ and nontrivial solution $(x,y,z)=(\pm442224,\pm64505,\pm42)$ when $P=2^9\times17\times271$.

(vi) has common solution $(x,y,z)=(\pm48,\pm7,0)$ and nontrivial solution $(x,y,z)=(\pm442224,\pm64505,\pm21)$ when $P=2^{11}\times17\times271$.

(vii) has only nontrivial solution $(x,y,z)=(\pm48,\pm7,0)$ when $P\neq2^\alpha\times17\times127(\alpha=1,3,5,7,9,11)$.

3.2. Proof of main theorem

3.2.1. Primary analysis.

Let (x_1, y_1) be the basic solution of the Pell equation $x^2 - 47y^2 = 1$, then $(x_1, y_1) = (48,7)$.It means that all solution of the Pell equation $x^2 - 47y^2 = 1$ is:

$$x_n + y_n\sqrt{P} = (48 + 7\sqrt{47})^n, n \in Z.$$

It is easily shown that

(i) $y_n{}^2 - 49 = y_{n+1}y_{n-1}$;

(ii) $y_{2n} = 2x_n y_n$;

(iii) $y_{2n+1} \equiv 1(mod2)$;

(iv) $x_{2n} \equiv 1(mod2), x_{2n+1} \equiv 48(mod96)$;

(v) $\gcd(x_n, y_n) = 1, \gcd(x_{n+1}, y_{n+1}) = 1, \gcd(x_n, x_{n+1}) = 1, \gcd(y_n, y_{n+1}) = 7$;

(vi) $\gcd(x_{2n}, y_{2n+1}) = \gcd(x_{2n+2}, y_{2n+1}) = 1, \gcd(x_{2n+1}, y_{2n}) = \gcd(x_{2n+1}, y_{2n+2}) = 48$.

(vii) $y_{2n+2} = 96y_{n+1} - y_n, y_0 = 0, y_1 = 7; x_{n+2} = 96x_{n+1} - x_n, x_0 = 1, x_1 = 48$.

Suppose that $(x, y, z) = (x_n, y_n, z_n), n \in Z$ is the integer solution of the diophantine equation (6), from(I), we can get:

$$y_n{}^2 - 49 = y_{n+1}y_{n-1} \tag{7}$$

from $y^2 - Pz^2 = 49$ of (6) , we can get:

$$Pz^2 = y_{n+1}y_{n-1} \tag{8}$$

As a result the equation (8) will be:

Case 1 n is an positive odd number.

Case 2 n is an positive even number.

3.2.2. Discusion on Case 1

Let $n = 2m - 1, m \in Z$,(8) is equivalent to:

$$Pz^2 = y_{2(m-1)}y_{2m} \tag{9}$$

from (II),(9) is equivalent to:

$$Pz^2 = 4x_{m-1}y_{m-1}x_m y_m \tag{10}$$

1. m is an positive odd number.

From (Ⅴ), we can get $\gcd(x_{m-1}, y_{m-1}) = \gcd(x_m, y_m) = 1$, $\gcd(x_m, x_{m-1}) = 1$, $\gcd(x_{m-1}, y_m) = 1, \gcd(y_m, y_{m-1}) = 7$, it means $\gcd\left(\frac{y_m}{7}, \frac{y_{m-1}}{7}\right) = 1$. From (VI), we can get $\gcd(x_m, y_{m-1}) = 48$, it means $\gcd\left(\frac{x_m}{48}, \frac{y_{m-1}}{48}\right) = 1$.

Therefore, x_{m-1}, $\frac{y_{m-1}}{336}$, $\frac{x_m}{48}$, $\frac{y_m}{7}$ are pairwise coprime.

1.1 k is an positive odd number.

Let $k = 2l - 1$,(10) is equivalent to:

$$Pz^2 = 8x_{4(l-1)}x_{4l-3}x_{2(l-1)}y_{2(l-1)}y_{4l-3} \tag{11}$$

From (II),(11) is equivalent to:

$$Pz^2 = 16x_{4(l-1)}x_{4l-3}x_{2(l-1)}x_{l-1}y_{l-1}y_{4l-3} \tag{12}$$

From (Ⅴ), we can get $\gcd(x_{l-1}, y_{l-1}) = 1$, it means $x_{4(l-1)}, x_{l-1}, \frac{y_{l-1}}{336}, x_{2(l-1)}, \frac{x_{4l-3}}{48}, \frac{y_{4l-3}}{7}$ are pairwise coprime when l is an odd number, and $x_{4(l-1)}, \frac{x_{l-1}}{48}, \frac{y_{l-1}}{7}, x_{2(l-1)}, \frac{x_{4l-3}}{48}, \frac{y_{4l-3}}{7}$ are pairwise coprime when l is an even number.

From (III), we can get $y_{4l-3} \equiv 1(mod2)$, it means$2 \nmid y_{4l-3}$, so y_{4l-3} is an odd number. From (IV), we can get $x_{4(l-1)}, x_{2(l-1)}, \frac{y_{4l-3}}{48}$ are odd numbers. x_{l-1} is an odd number when l is an odd number and $\frac{x_{l-1}}{48}$ is an odd number when l is an even number. Therefore, $x_{4(l-1)}, x_{l-1}, x_{2(l-1)}, \frac{x_{4l-3}}{48}, \frac{y_{4l-3}}{7}$ are odd numbers when l is an odd number and $x_{4(l-1)}, \frac{x_{l-1}}{48}, x_{2(l-1)}, \frac{x_{4l-3}}{48}, \frac{y_{4l-3}}{7}$ are odd numbers when l is an even number.

From Lemma 4, we can get $x_{4(l-1)}, x_{2(l-1)}, x_{l-1}, \frac{x_{4l-3}}{48}, \frac{y_{4l-3}}{7}$ are square numbers if and only if $l = 1$, $\frac{x_{l-1}}{48}$ is a square number if and only if $l = 2$ or $l = 0$.

So, $x_{4(l-1)}, x_{2(l-1)}, x_{l-1}, \frac{x_{4l-3}}{48}, \frac{y_{4l-3}}{7}$ are non-square numbers when odd number $l \neq 1$, and it has 5 diverse odd primes. Therefore, (12) is impossible, which means (6) have no integer solution.

$x_{4(l-1)}, x_{2(l-1)}, \frac{x_{l-1}}{48}, \frac{x_{4l-3}}{48}, \frac{y_{4l-3}}{7}$ are non-square numbers when even number $l \neq 0,2$, and it has 5 diverse odd primes, which is contradict with $P = 2^t {p_1}^{a_1} {p_2}^{a_2} {p_3}^{a_3} {p_4}^{a_4}$, Therefore, (12) is impossible, which means (6) have no integer solution.

When $l = 1$, (11) is equivalent to: $z^2 = 8{x_0}^2 x_1 y_0 y_1 = 0$, so $z = 0$,it means that diophantine equation (6) has and only has common solution $(x, y, z) = (\pm 48, \pm 7, 0)$.

When $l = 0,2$, (12) is equivalent to: $Pz^2 = 16x_{-4}x_{-3}x_{-2}x_{-1}y_{-1}y_{-3} = 16x_4x_3x_2x_1y_1y_3$, From (IV), we can get x_4, x_2 are odd numbers, from (III), we can get y_1, y_3 are odd numbers, it means $Pz^2 = 2^8 \times 3 \times x_4x_3x_2y_1y_3$, Therefore, the right part of (12) has 5 diverse odd primes at least, which is contradict with $P = 2^t {p_1}^{a_1} {p_2}^{a_2} {p_3}^{a_3} {p_4}^{a_4}$, Therefore, (12) is impossible, which means (6) have no integer solution.

1.2 k is an positive even number.

From (III), we can get y_{k-1}, y_{2k-1} are odd numbers, it means $\frac{y_{k-1}}{7}, \frac{y_{2k-1}}{7}$ are odd numbers too, From (IV), we can get $x_{2k-1}, \frac{x_{k-1}}{48}, \frac{x_{2k-1}}{48}$ are odd numbers.

From Lemma 4, we can get $x_{2(k-1)}, \frac{y_{2k-1}}{7}$ are square numbers if and only if $k = 1$. $\frac{x_{k-1}}{48}$ is a square number if and only if $k = 0$ or $k = 2$. $\frac{x_{2k-1}}{48}$ is a square number if and only if $k = 0$ or $k = 1$. $\frac{y_{k-1}}{7}$ is a square number if and only if $k = 1$ or $k = 2$. So, $x_{2k-1}, \frac{x_{k-1}}{48}, \frac{x_{2k-1}}{48}, \frac{y_{k-1}}{7}, \frac{y_{2k-1}}{7}$ are non-square numbers when even number $k \neq 0,2$, and it has 5 diverse odd primes. Therefore, (12) is impossible, which means (6) have no integer solution.

When $k = 0$, (11) is equivalent to: $Pz^2 = 8x_2{y_1}^2{x_1}^2 = 2^{11} \times 3^2 \times 7^2 \times 17 \times 271$, so $z = 21, P = 2^{11} \times 17 \times 271$ or $z = 42, P = 2^9 \times 17 \times 271$ or $z = 84, P = 2^7 \times 17 \times 271$ or $z = 168, P = 2^5 \times 17 \times 271$ or $z = 336, P = 2^3 \times 17 \times 271$ or $z = 672, P = 2 \times 17 \times 271$, From (6), we can get:

(6) has common solution $(x, y, z) = (\pm 48, \pm 7, 0)$ when $P = 2^\alpha \times 17 \times 127 (\alpha = 1,3,5,7,9,11)$ and has nontrivial solution $(x, y, z) = (\pm 442224, \pm 64505, \pm 672)$ when $P = 2 \times 17 \times 271$. $(x, y, z) = (\pm 442224, \pm 64505, \pm 336)$ when $P = 2^3 \times 17 \times 271$, $(x, y, z) = (\pm 442224, \pm 64505, \pm 168)$ when $P = 2^5 \times 17 \times 271$., $(x, y, z) = (\pm 442224, \pm 64505, \pm 84)$ when $P = 2^7 \times 17 \times 271$, $(x, y, z) = (\pm 442224, \pm 64505, \pm 42)$ when $P = 2^9 \times 17 \times 271$, $(x, y, z) = (\pm 442224, \pm 64505, \pm 21)$ when $P = 2^{11} \times 17 \times 271$.

When $k = 2$, (11) is equivalent to: $Pz^2 = 8x_2{x_1}^2y_1y_3y = 2^{11} \times 7^2 \times 53^2 \times 17 \times 19 \times 97 \times 271$, it has 5 diverse odd primes, which is contradict with $P = 2^t {p_1}^{a_1} {p_2}^{a_2} {p_3}^{a_3} {p_4}^{a_4}$, Therefore, (12) is impossible, which means (6) have no integer solution.

2. m is an positive even number.

Imitating the previous proof of 1 we can get the diophantine equation (7) only has common solution $(x, y, z) = (\pm 48, \pm 7, 0)$.

3.2.3. Discusion on Case 2

From (III), we can get $y_{n-1} \equiv y_{n+1} \equiv 1(mod2)$, it means y_{n-1}, y_{n+1} are odd numbers. Therefore, the left part of (8) is an even number, when it's right is an odd number, it is self-contradiction. Therefore, diophantine equation (6) have no integer solution.

To sum up, the theorem is proved.

4. Conclusion

The integer solution of diophantine equations $x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z)$ and $y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z)$ is a matter of great concern.

By using elementary number theory methods,we solved the common solution and nontrivial solution on the diophantine equation when $m = 1$, $n = 49$, $D_1 = 47$, D_2 can be expressed as $2^t {p_1}^{a_1} {p_2}^{a_2} {p_3}^{a_3} {p_4}^{a_4}$ where $a_i = 0$ or 1 for $1 \leq i \leq 4$, and $t \in Z^+$, $p_s (1 \leq s \leq 4)$ are different odd primes.

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