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The Fracture Analysis of Collinear Periodic Cracks in an Infinite Piezoelectric Plate

Wenbin Zhao, Xuexia Zhang, Hailing Xie, Zhiyao Zhao and Yutong Guo

School of Applied Science, Taiyuan University of Science and Technology, Taiyuan 030024, P. R. China
Email: zwbhappy@aliyun.com; xuexiaz@126.com

Abstract. The fracture problem of collinear periodic cracks in an infinite transversely isotropic piezoelectric plate subjected to the anti-plane shear stress and the in-plane electric load at infinity is studied. Using the complex function method, the mechanical problem is turned into the boundary value problem of partial differential equations. The solutions of the boundary value problem of partial differential equation are obtained by undetermined coefficients method. Then, considering the periodicity of cracks, the stress intensity factors and the electric displacement intensity factors for mode III near the right tip of every crack are defined, the expressions of the stress fields, electric displacement fields, displacement fields, electric potential fields and the mechanical strain energy release rate around the crack tip are obtained with the assumption that the surface of the crack is electrically impermeable. Finally, interference effect and scale effect of collinear periodic cracks and the mechanical strain energy release rate are discussed by analysis of examples. It can be seen interference effect of collinear periodic cracks is strong when $1<b/a<2$. The scale effect of the singularity of the stress intensity factors and electric displacement intensity factors in crack tip is obvious. Stress always promotes extension of the cracks, the mechanical strain energy release rate is related to the size and direction of the electric field, the positive electric field can promote the expansion of the cracks, the negative electric field can inhibit the extension of cracks.

1. Introduction

Piezoelectric material is widely used in making transducer, sensor and the brakes and other electronic devices. But piezoelectric material has great brittleness; it inevitably appears defects such as cracks, inclusions, holes. It often leads to failure and even destroy, which affects the performance and reliability of the smart structure. So the fracture problem of piezoelectric materials has very important significance. Fracture mechanics analysis of piezoelectric material has drawn many researchers' attention. Erdogan and Pak [1-3] studied anti-plane elasticity problem of periodic cracks in the functionally graded coated on the surface. Stress intensity factors for mode-III were obtained. By the Stroh formalism and conformal mapping method, Hu, Choi and Gao [4-6] studied fracture problems of anisotropic material with collinear periodic cracks. Hao and Zhou [7- 8] discussed the fracture problem of two symmetrical parallel cracks in piezoelectric material with the assumption that the surface of the cracks was electrically permeable. Chen, Cui and Wang [9-11] discussed, respectively, the anti-plane problem of single crack and periodic cracks in elasticity functionally graded materials. Shi, Guo and Zhao [12-14] considered the anti-plane problems of collinear periodic cracks in the six quasicrystals. The problem can be turned into Riemann-Hilbert problem of periodic analytic function to solve. The closed-form solutions of anti-plane problem were obtained. A strip-electro-mechanical model is proposed for two semi-permeable collinear cracks, symmetrically situated and transversely oriented in a poled piezoelectric strip[15-16]. The remote boundary of the strip is subjected to mode-III stress and in-

plane electric-displacement. By using the integral transform technique, Zhou and Shi [16-17] studies two collinear permeable anti-plane shear or mode-III cracks of equal length lying at the mid-plane of a one-dimensional hexagonal piezoelectric quasicrystal strip. Currently, there is little reference about collinear parallel cracks in piezoelectric material plate which were studied by using the complex function theory and undetermined coefficients method.

In this paper, the fracture problem of collinear periodic cracks in an infinite transversely isotropic piezoelectric plate under the anti-plane shear stress and the in-plane electric load at infinity is studied. By introducing proper Wester gaard's stress function and electric displacement function, the solutions of the boundary value problem of partial differential equation are obtained by undetermined coefficients method and the complex function method. The expressions of the stress fields, electric displacement fields, displacement fields, electric potential fields and the mechanical strain energy release rate around the crack tip are obtained with the assumption that the surfaces of the cracks were electrically impermeable. Finally, interference effect and scale effect of collinear periodic cracks and the mechanical strain energy release rate are discussed by analysis of examples.

2. Mechanical model

Consider an infinite transversely isotropic piezoelectric plate containing collinear periodic crack of length $2a$ as is shown in Figure 1. The distance between cracks is $2b$. The plate is subjected to the anti-plane shear stress σ and the in-plane electric load D at infinity. In a fixed rectangular coordinate system $x_j\left(j=1,2,3\right)$, x_1ox_2 plane is transversely isotropic plane, x_3 axis is perpendicular to the x_1ox_2 plane.

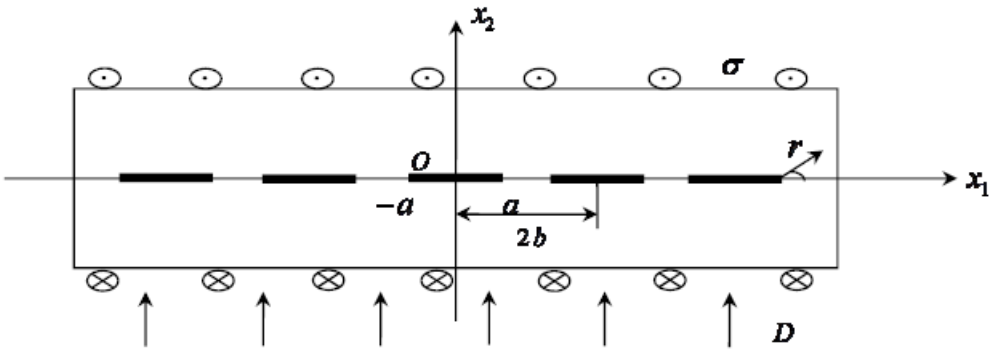

Figure 1. The anti-plane problem of piezoelectric plate with collinear periodic cracks

Here we only consider the general two-dimensional piezoelectric boundary value problem. The constitutive equations can be obtained as follows:

$$\sigma_{3k}=c_{44}\frac{\partial u_3}{\partial x_k}+e_{15}\frac{\partial \phi}{\partial x_k},\ D_k=e_{15}\frac{\partial u_3}{\partial x_k}-\varepsilon_{11}\frac{\partial \phi}{\partial x_k}\ (k=1,2) \tag{1}$$

where $u_3=u_3\left(x_1,x_2\right)$ is the out-of-plane displacement, ϕ is the electric potential.

In the absence of body forces and free charge, the static equilibrium equation and Maxwell equation under static electricity are as follows:

$$\frac{\partial \sigma_{31}}{\partial x_1}+\frac{\partial \sigma_{32}}{\partial x_2}=0,\frac{\partial D_1}{\partial x_1}+\frac{\partial D_2}{\partial x_2}=0 \tag{2}$$

Substituting (1) into (2), the equilibrium equations of piezoelectric material are as follows:

$$c_{44}\nabla^2 u_3+e_{15}\nabla^2\phi=0,\ e_{15}\nabla^2 u_3-\varepsilon_{11}\nabla^2\phi=0 \tag{3}$$

where ∇^2 is the two-dimensional Laplace operator in the variables x_1 and x_2.

For general piezoelectric material, the elastic constant c_{44}, piezoelectric constant e_{15} and dielectric constant ε_{11} satisfy the relation: $c_{44}\varepsilon_{11}+e_{15}{}^2\neq 0$. So the equilibrium equation (3) is simplified into two independent harmonic equations as follows:

$$\nabla^2 u_3 = 0, \quad \nabla^2 \varphi = 0, \tag{4}$$

Assuming that the permittivity of air was far less than that of piezoelectric material, the electric field inside cracks can be ignored. So the conduction boundary conditions are adopted as follows:

$$x_2 \to \infty : \sigma_{32} = \sigma, \quad D_2 = D \tag{5}$$

$$x_2 = 0, \quad 2nb - a < x_1 < 2nb + a : \sigma_{32} = 0, \quad D_2{}^+ = D_2{}^- = 0 \tag{6}$$

Therefore, the anti-plane problem of an infinite transversely isotropic piezoelectric plate can be reduced to solve the solutions of the boundary value problem of partial differential equations (4 to 6).

3. Stress function and electric displacement function

Harmonic equations (4) have real analytical solution, which is based on the knowledge of complex function theory. We may assume:

$$u_3 = a_1 \operatorname{Re}\left(\overline{U_1}\right) + b_1 \operatorname{Im}\left(\overline{U_1}\right), \phi = a_2 \operatorname{Re}\left(\overline{U_2}\right) + b_2 \operatorname{Im}\left(\overline{U_2}\right) \tag{7}$$

where a_j, b_j ($j = 1,2$)are the real undetermined coefficients, U_1, U_2 are the stress function and electric potential function respectively, and

$$\frac{d\overline{U_1}}{dz_1} = U_1, U_1 = U_1(z_1), \frac{d\overline{U_2}}{dz_1} = U_2, U_2 = U_2(z_1), z_1 = x_1 + ix_2 .$$

Substituting (7) into constitutive equations (1), the elastic stresses and electric displacements can be expressed by using U_1 and U_2 .

Noticing the anti-plane shear stress σ and the in-plane electric load D, the stress function and electric potential function are selected as follows

$$U_1 = \frac{\sigma \sec\frac{\pi a}{2b}\tan\frac{\pi z_1}{2b}}{\sqrt{\tan^2\frac{\pi z_1}{2b} - \tan^2\frac{\pi a}{2b}}}, \qquad U_2 = \frac{D \sec\frac{\pi a}{2b}\tan\frac{\pi z_1}{2b}}{\sqrt{\tan^2\frac{\pi z_1}{2b} - \tan^2\frac{\pi a}{2b}}} \tag{8}$$

Substituting (8) into the boundary condition (6), we can obtaina system of non-homogeneous linear equations in 4 unknowns about coefficients a_j, b_j ($j = 1,2$).

$$\begin{aligned} &c_{44}a_1 \frac{\sigma \sec\frac{\pi a}{2b}\tan\frac{\pi x_1}{2b}}{\sqrt{\tan^2\frac{\pi a}{2b} - \tan^2\frac{\pi x_1}{2b}}} + e_{15}a_2 \frac{D \sec\frac{\pi a}{2b}\tan\frac{\pi x_1}{2b}}{\sqrt{\tan^2\frac{\pi a}{2b} - \tan^2\frac{\pi x_1}{2b}}} = 0, \quad c_{44}b_1\sigma + e_{15}b_2 D = \sigma \\ &e_{15}a_1 \frac{\sigma \sec\frac{\pi a}{2b}\tan\frac{\pi x_1}{2b}}{\sqrt{\tan^2\frac{\pi a}{2b} - \tan^2\frac{\pi x_1}{2b}}} - \varepsilon_{11}a_2 \frac{D \sec\frac{\pi a}{2b}\tan\frac{\pi x_1}{2b}}{\sqrt{\tan^2\frac{\pi a}{2b} - \tan^2\frac{\pi x_1}{2b}}} = 0, \quad e_{15}b_1\sigma - \varepsilon_{11}b_2 D = D \end{aligned} \tag{9}$$

Solving non-homogeneous linear equations (9), the unique solutions of equations can be derived:

$$a_1 = 0, \quad a_2 = 0, \quad b_1 = \frac{\varepsilon_{11}\sigma + e_{15}D}{\left(c_{44}\varepsilon_{11} + e_{15}{}^2\right)\sigma}, \quad b_2 = \frac{e_{15}\sigma - c_{44}D}{\left(c_{44}\varepsilon_{11} + e_{15}{}^2\right)D} \tag{10}$$

4. Intensity factors

Considering the periodicity of cracks, the stress intensity factor and the electric displacement intensity factor of mode III near the right tip of every crack is defined as follows:

$$K_{\mathrm{III}}^{\sigma}=\lim_{z_1\to 2nb+a}\left[2\pi\left(z_1-2nb-a\right)\right]^{\frac{1}{2}}U_1,\quad (n=0,\pm1,\pm2\cdots)$$

$$K_{\mathrm{III}}^{D}=\lim_{z_1\to 2nb+a}\left[2\pi\left(z_1-2nb-a\right)\right]^{\frac{1}{2}}U_2\quad (n=0,\pm1,\pm2\cdots) \tag{11}$$

Substituting (8) into (11), we can obtain

$$K_{\mathrm{III}}^{\sigma}=\sigma\sqrt{2b\tan\frac{\pi a}{2b}}=Y\sigma\sqrt{\pi a},\quad K_{\mathrm{III}}^{D}=D\sqrt{2b\tan\frac{\pi a}{2b}}=YD\sqrt{\pi a} \tag{12}$$

where $Y=\sqrt{2b/(\pi a)\tan\left[\pi a/(2b)\right]}$ is called the shape factor. $K_{\mathrm{III}}^{\sigma}{}'$ and $K_{\mathrm{III}}^{D}{}'$ are, respectively, stress intensity factor and electric displacement intensity factor in an infinite transversely isotropic piezoelectric plate with a centre crack under the anti-plane shear stress and the in-plane electric load at infinity. Then $K_{\mathrm{III}}^{\sigma}{}'=\sigma\sqrt{\pi a}$, $K_{\mathrm{III}}^{D}{}'=D\sqrt{\pi a}$. When $a=1$, the variation curves of Y, $K_{\mathrm{III}}^{\sigma}$, K_{III}^{D} with b are given as shown in Figure 2. It can be seen that from Figure 2shape factor Y, $K_{\mathrm{III}}^{\sigma}$, K_{III}^{D} decrease with the increase of the distance between cracks. When the cracks spacing is greater than 2, the shape factor Y, $K_{\mathrm{III}}^{\sigma}$, K_{III}^{D} reach a steady state.

Near the cracks tip, from the equations (11), it can be seen that

$$U_1=\frac{K_{\mathrm{III}}^{\sigma}}{\left[2\pi(z_1-2nb-a)\right]^{\frac{1}{2}}},\qquad U_2=\frac{K_{\mathrm{III}}^{D}}{\left[2\pi(z_1-2nb-a)\right]^{\frac{1}{2}}} \tag{13}$$

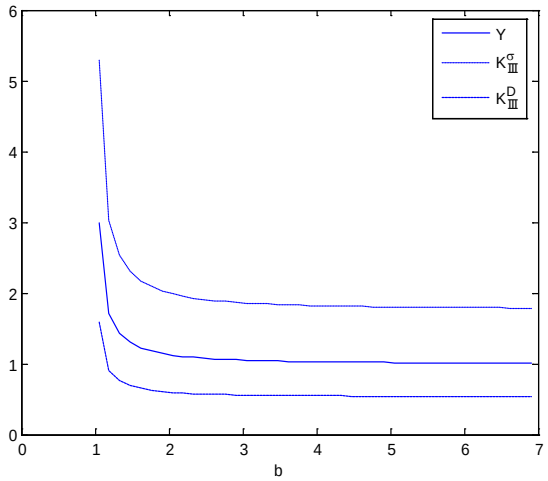

Figure 2. The variation curves of Y, $K_{\mathrm{III}}^{\sigma}$, K_{III}^{D} with the cracks spacing b

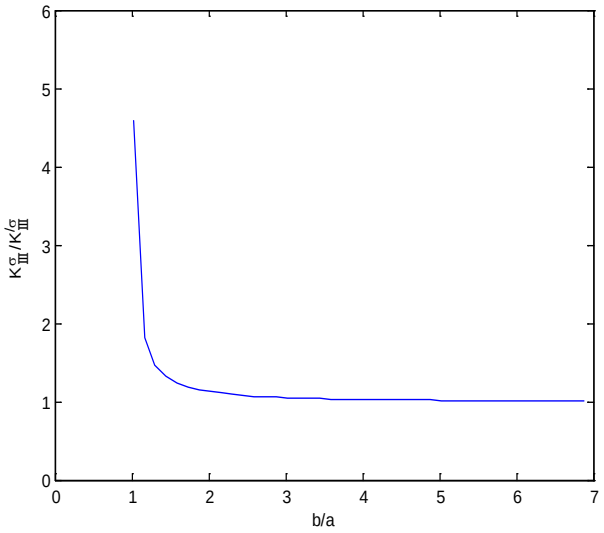

Figure 3. Interaction of collinear periodic cracks

So, the stress field and electric displacement field near the mode- III collinear periodic cracks ($z_1 \to a+2nb$) in an infinite transversely isotropic piezoelectric plate can be written as follows:

$$\sigma_{31}=\frac{\left(c_{44}\varepsilon_{11}\sigma D+c_{44}e_{15}D^2\right)K_{III}^{\sigma}+\left(e_{15}{}^2\sigma^2-c_{44}e_{15}\sigma D\right)K_{III}^{D}}{\sqrt{2\pi}\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)\sigma D}\,\mathrm{Im}\left[\frac{1}{(z_1-2nb-a)^{\frac{1}{2}}}\right] \tag{14}$$

$$\sigma_{32}=\frac{\left(c_{44}\varepsilon_{11}\sigma D+c_{44}e_{15}D^2\right)K_{III}^{\sigma}+\left(e_{15}{}^2\sigma^2-c_{44}e_{15}\sigma D\right)K_{III}^{D}}{\sqrt{2\pi}\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)\sigma D}\,\mathrm{Re}\left[\frac{1}{(z_1-2nb-a)^{\frac{1}{2}}}\right] \tag{15}$$

$$D_1=\frac{\left(e_{15}\varepsilon_{11}\sigma D+e_{15}{}^2D^2\right)K_{III}^{\sigma}-\left(e_{15}\varepsilon_{11}\sigma^2-c_{44}\varepsilon_{11}\sigma D\right)K_{III}^{D}}{\sqrt{2\pi}\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)\sigma D}\,\mathrm{Im}\left[\frac{1}{(z_1-2nb-a)^{\frac{1}{2}}}\right] \tag{16}$$

$$D_2=\frac{\left(e_{15}\varepsilon_{11}\sigma D+e_{15}{}^2D^2\right)K_{III}^{\sigma}-\left(e_{15}\varepsilon_{11}\sigma^2-c_{44}\varepsilon_{11}\sigma D\right)K_{III}^{D}}{\sqrt{2\pi}\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)\sigma D}\,\mathrm{Re}\left[\frac{1}{(z_1-2nb-a)^{\frac{1}{2}}}\right] \tag{17}$$

From constitutive equations (1), letting $k=1$, we can obtain as follow:

$$\frac{\partial u_3}{\partial x_1}=\frac{\varepsilon_{11}\sigma+e_{15}D}{\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)\sigma}\cdot\frac{K_{III}^{\sigma}}{\sqrt{2\pi}}\cdot\mathrm{Im}\left[\frac{1}{(z_1-2nb-a)^{\frac{1}{2}}}\right] \tag{18}$$

$$\frac{\partial \phi}{\partial x_1}=\frac{e_{15}\sigma-c_{44}D}{\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)D}\cdot\frac{K_{III}^{D}}{\sqrt{2\pi}}\cdot\mathrm{Im}\left[\frac{1}{(z_1-2nb-a)^{\frac{1}{2}}}\right] \tag{19}$$

Integrating over x1 about Eq.(18), the displacement can be obtained as follow:

$$u_3=\frac{\left(\varepsilon_{11}\sigma+e_{15}D\right)K_{III}^{\sigma}\sqrt{2}}{\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)\sigma\sqrt{\pi}}\cdot\mathrm{Im}\left[\left(z_1-2nb-a\right)^{\frac{1}{2}}\right] \tag{20}$$

Integrating over x1about Eq.(19), the electric potential can be also obtained as follow:

$$\phi=\frac{\left(e_{15}\sigma-c_{44}D\right)\cdot\sqrt{2}K_{III}^{D}}{\left(c_{44}\varepsilon_{11}+e_{15}{}^2\right)D\sqrt{\pi}}\cdot\mathrm{Im}\left[\left(z_1-2nb-a\right)^{\frac{1}{2}}\right] \tag{21}$$

5. The mechanical strain energy release rate

Let's K_{III}^{s} denotes the strain intensity factor at the mode-III collinear periodic cracks tip, K_{III}^{s} can be written as follow:

$$K_{\mathrm{III}}^{s}=\frac{e_{15}K_{\mathrm{III}}^{D}+\varepsilon_{11}K_{\mathrm{III}}^{\sigma}}{e_{15}^{2}+c_{44}\varepsilon_{11}} \tag{22}$$

Substituting (12) into (22), respectively, the strain intensity factor at the mode-Ⅲ collinear periodic crack-tip is obtained as follow:

$$K_{\mathrm{III}}^{s}=\frac{Y\sqrt{\pi a}\left(e_{15}D+\varepsilon_{11}\sigma\right)}{e_{15}^{2}+c_{44}\varepsilon_{11}} \tag{23}$$

The mechanical strain energy release rate can be expressed as:

$$G_{m}=\frac{K_{\mathrm{III}}^{\sigma}K_{\mathrm{III}}^{s}}{2} \tag{24}$$

Substituting (12) and (23) into (24), the mechanical strain energy release rate at the mode-Ⅲ collinear crack-tip is obtained as follow:

$$G_{m}=\frac{Y^{2}\sigma\pi a\cdot\left(\varepsilon_{11}\sigma+e_{15}D\right)}{2\left(c_{44}\varepsilon_{11}+e_{15}{}^{2}\right)} \tag{25}$$

6. Numerical results

In this section, we study interference effect and scale effect between cracks, and mechanical strain energy release rate of collinear periodic cracks in an infinite piezoelectric plate. We choose the piezoelectric ceramics PZT-5H as the experimental materials, which parameters are as follows:

$$c_{44}=3.53\times10^{10}\,N/m^{2},e_{15}=17.0C/m^{2},\varepsilon_{11}=151\times10^{-10}\,C/Vm,G_{cr}=5.0J/m^{2},$$

where Gr is critical mechanical strain energy release rate.

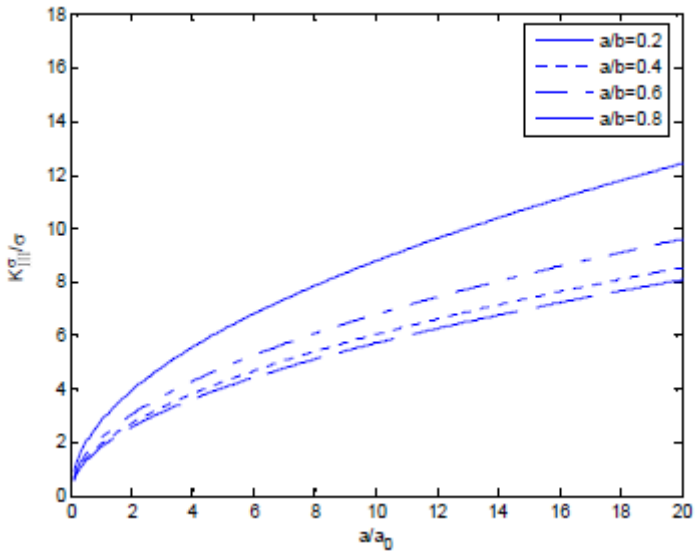

Figure 4. Scale effect of collinear periodic cracks

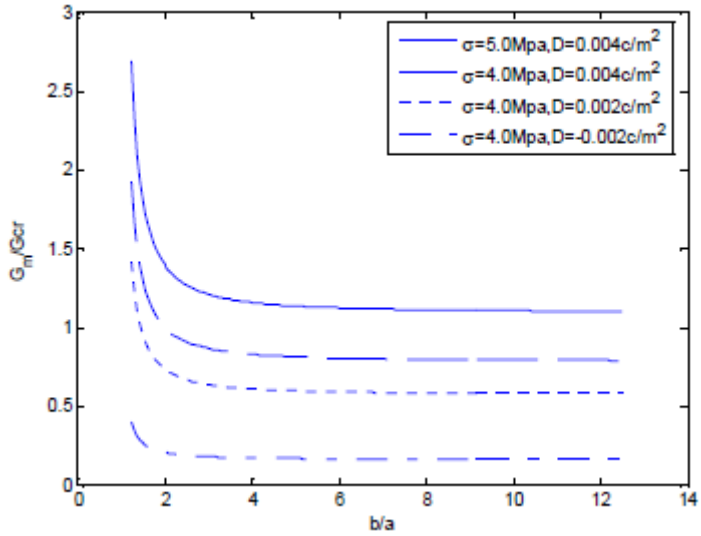

Figure 5. The variation of G_m / G_{cr} with b/a

In order to study the interference effect between the cracks, the variation curves of $K_{\mathrm{III}}^{\sigma} / K_{\mathrm{III}}^{\sigma'}$ with b/a is given as shown in figure 3. When $1 < b/a < 2$, $K_{\mathrm{III}}^{\sigma} / K_{\mathrm{III}}^{\sigma'}$ decreases obviously with the increase of b/a, thus it can be seen that interference of collinear periodic cracks is strong. When $b/a > 2$, $K_{\mathrm{III}}^{\sigma} / K_{\mathrm{III}}^{\sigma'}$ remains the same with the increase of b/a, so the interference of collinear periodic cracks becomes smaller.

The scale effect of collinear periodic cracks with the assumption that the surfaces of the cracks are electrically impermeable is calculated as shown in figure 4. Form figure 4, we take $a_0 = 1$ as reference length of cracks. It can be seen from figure 4, $K_{\mathrm{III}}^{\sigma} / \sigma$ increases with the increase of a / a_0, at the same time, when a / a_0 remains constant, $K_{\mathrm{III}}^{\sigma} / \sigma$ also increases with the increase of a / b. From (12), stress intensity factor and electric displacement intensity factor of collinear periodic cracks under electrically impermeable crack surface conditions have the same rules. The result shows that the singularity of the stress intensity factor and electric displacement intensity factor in cracks tip have obvious the scale effect.

Figure 5 shows the mechanical strain energy release rate changes along with b/a under different mechanical load and electrical load. From figure 5, it can be seen mechanical strain energy release rate decreases with the increase of b/a. When $1 < b/a < 4$, the change of mechanical strain energy release rate with the ratio b/a is larger, when $b/a > 4$, it has less effect on the mechanical strain energy release rate. From figure 5, it also shows that mechanical strain energy release rate increases with the increase of stress with the same in-plane electric field; Mechanical strain energy release rate is not only dependent on the size of the electric load, but also associated with the direction of the electric load. It increases with the increase of positive electric field. When the positive electric field can be changed to negative electric fields with the same size, the mechanical strain energy release rate decreased. So positive electric field can promote the expansion of the cracks, the negative electric field can inhibit extension of the cracks.

7. Conclusions

By using the complex variable function method and undetermined coefficients method, the anti-plane problem of the collinear periodic cracks in an infinite transversely isotropic piezoelectric plate is studied in this paper. The analytic solutions of the stress, displacement, electric displacement, electric potential and the mechanical strain energy release rate at the cracks tip are obtained. The results indicated that: (1) When $1 < b/a < 2$, interference of collinear periodic cracks is strong; (2) The

singularity of the stress intensity factor and electric displacement intensity factor in cracks tip have obvious the scale effect; (3) Stress always promotes extension of the cracks; (4) The mechanical strain energy release rate is related to the size and direction of the electric field, the positive electric field can promote the expansion of the cracks, the negative electric field can inhibit the extension of cracks.

8. Acknowledgments

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